Matching Relational Structures using the Edge-Association Graph

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Matching Relational Structures using the Edge-Association Graph

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Abstract

The matching of relational structures is a problem that pervades computer vision and pattern recognition research. A classic approach is to reduce the matching problem into one of search of a maximum clique in an auxiliary structure: the association graph. The approach has been extended to incorporate vertex-attributes by reducing it to a weighted clique problem, but the extension to edge-attributed graphs has proven elusive. However, in vision problems, quite often the most relevant information is carried by edges. For example, when the graph abstracts scene layout, the edges can represent the relative position of the detected features, which abstracts the geometry of the scene in a way that is invariant to rotations and translations. In this paper, we provide a generalization of the association graph framework capable of dealing with attributes on both vertices and edges. Experiments are presented which demonstrate the effectiveness of the proposed approach.

1. Introduction

Graph-based representations have long been used with considerable success in computer vision and pattern recognition in the abstraction and recognition of objects and scene structure. Concrete examples include the use of shock graphs to represent shape-skeletons [8, 13, 14], the use of trees to represent articulated objects [7, 17] and the use of Delanuay graph to represent the distribution of features in a scene [16]. The attractive feature of structural representations is that they concisely capture the relational arrangement of object primitives, in a manner which can be invariant to changes in object viewpoint. Using this framework we can transform a recognition problem into a relational matching problem.

The problem of how to measure the similarity or distance of pictorial information using graph abstractions has been a widely researched topic of over twenty years. Early work on the topic includes Barrow and Burstall's idea [2] of locating matches by searching for maximum common subgraphs and Shapiro and Haralick's idea [12] of locating the isomorphism that minimizes the weight of unmapped nodes. A common approach transforms the combinatorial problem into a continuous optimization problem and then uses the wide range of available optimization algorithms available to find an approximate solution. In [9] Kittler, Christmas, and Petrou use relaxation labeling to label nodes in the data graph with the corresponding node in the model graph and use graph connectivity to combine evidence. Relaxation labeling, however, does not guarantee a one-to-one correspondence between nodes. In order to guarantee a one-to-one assignment, Gold and Ragaranjan [5] introduced the "graduated assignment" method. This is an evidence combining model that guarantees two way constraints. Evidence combining methods like relaxation labeling and graduated assignment give a very interesting framework to iteratively improve on our initial estimate, but they are critically dependent on a good consistency model and a reliable initialization.

Another classic approach pioneered by Ambler et al. [1] is to reduce the matching problem into one of search of a maximum clique in an auxiliary structure: the association graph. This graph is defined over a vertex-set that is the Cartesian product of the vertex-sets of the original structures, and the edges represent the compatibility of two maps between the original graphs. Optimal matches between the two structures are then in a one-to-one relationship with maximum cliques on the association graph. By re-casting the search for the maximum common subgraph as a max clique problem [2], we can tap into a diverse array of powerful heuristics and theoretical results available for solving the max clique problem. An important development in that direction is reported by Pelillo [10] who, using the Motzkin-Straus theorem [54] transforms the max clique problem into a continuous quadratic programming problem, and shows how relaxation labeling can be used to find an approximate solution.

By adding a weight to the vertices of the association graph, the similarity between vertices can be taken into account. This allows us to match vertex-attributed graphs by searching for cliques of maximum weight [11]. However, in vision problems, quite often the most relevant information is carried by edges. For example, when the graph abstracts scene layout, the edges can represent the relative position of the detected features, which abstracts the geometry of the scene in a way that is invariant to rotations and translations. Furthermore, any deformation of the scene will induce changes in the relative distances of the features, leaving the actual features mostly unchanged. Examples of such representations include the use of Delanuay graphs for scene registration [16] and the use of skeletons for shape recognition [14]. In these structural representations it is the edge that takes center stage, and, hence, it is essential for a matching algorithm to deal with edge and vertex attributes in a uniform and well founded way. To this end we propose to extend the association graph framework to deal with edge associations directly, without having to infer them from vertex associations. This way edge similarity information can be incorporated into the matching process.

A drawback of this approach is that the size of the association graph increases with the product of the number of edges in the two graphs, making the space and time requirements of the Motzkin-Straus formulation too demanding. For this reason we opted for the use of Reactive local search (RLS) [3], a search-based clique heuristic which can find a candidate solution quickly and then refine it as requested. This characteristic, which is common to most search-based heuristics, make the approach usable for range queries and any-time queries in a structural database, i.e, queries where only an upper bound on the actual distance is required and interactive queries that can be stopped as soon as the user is satisfied with the results. However, the size requirement still limits us to sparse graphs. Fortunately, most scene abstractions use planar graph and, hence, are sparse.

2. Edge Association Graph

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set $E \subset V^2$, any time there is an arc between nodes $u, v \in V$ we say that the nodes are *adjacent* and we write $u \sim v$. If $G = (V, E)$ is a directed graph, with the notation $u \rightsquigarrow v$ we indicate that there is an edge from node $u \in V$ to node $v \in V$; furthermore, we call a *self loop* an arc from one node to itself. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$[1] be two directed graphs, a subgraph isomorphism f between G_1 and G_2 is a partial injective function from V_1 to V_2 that respects the adjacency of the two graph, i.e.,

$$\forall u, v \in \mathrm{dom}(f),\, u \rightsquigarrow v \Leftrightarrow f(u) \rightsquigarrow f(v)\,,$$

where $\mathrm{dom}(f)$ is the subset of V_1 where f is defined.

[1] Here, as in the remainder of the paper, subscripts indicate the graph of origin of the vertices or edges.

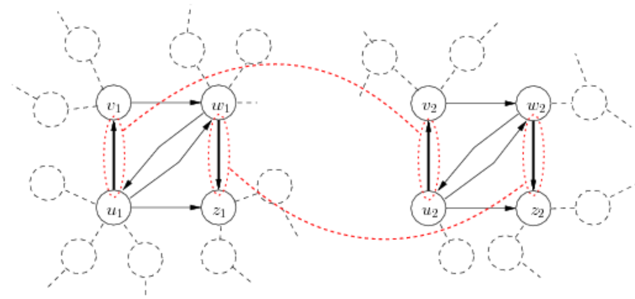

Figure 1. Two edge associations are compatible if they induce a subgraph isomorphism.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two directed graphs without self loops, we define the set of correspondences between edges $V_e = E_1 \times E_2$. As each vertex can be bijectively mapped to a self loop, we can extend V_e to include vertex-correspondences by defining a generalized edge association set $V_a = V_e \cup \hat{V}_1 \times \hat{V}_2$, where $\hat{V} = \{(v, v) | v \in V\}$ is the space of self loops. This way we guarantee uniformity in notation, since every entity, be it an edge or a vertex, is represented by a (possibly equal) pair of vertices.

Any subset $S \subseteq V_a$ represent a relation between edges and vertices in G_1 and edges and vertices in G_2. We define a map $\phi : \mathcal{P}(V_a) \to \mathcal{P}(V_1 \times V_2)$ from edge-relations to vertex-relation as follows:

$$\phi(X) = \{(v_1, v_2) \in V_1 \times V_2 \,|\, \exists u_1 \in V_1, u_2 \in V_2\,, \\ ((v_1, u_1), (v_2, u_2)) \in X \vee ((u_1, v_1), (u_2, v_2)) \in X\}\,, \quad (1)$$

ϕ is not invertible, as it is not injective, but it has a right partial inverse $\phi^{-1} : \mathcal{P}(V_1 \times V_2) \to \mathcal{P}(V_a)$ defined as follows:

$$\phi^{-1}(Y) = \{((u_1, v_1), (u_2, v_2)) \in V_a \,| \\ (u_1, u_2) \in Y \wedge (v_1, v_2) \in Y\}\,. \quad (2)$$

It is easy to show that $Y = \phi(\phi^{-1}(Y))$ and $X \subseteq \phi^{-1}(\phi(X))$. The last condition implies that for each vertex-relation there is a maximal edge-association that contains all the edge-associations inducing the same vertex-relations. Our goal is to find a set of edge associations that induce through ϕ a subgraph isomorphism between G_1 and G_2.

Let $ea1 = ((u_1, v_1), (u_2, v_2)) \in V_a$ and $ea2 = ((w_1, z_1), (w_2, z_2)) \in V_a$ be two edge-associations, $ea1$ and $ea2$ are said to be *compatible* if and only if $\phi(\{ea1, ea2\})$ is a subgraph isomorphism between G_1 and G_2, that is if the relation mapping u_1 to u_2, v_1 to v_2, w_1 to w_2, and z_1 to z_2 is a partial injective function that respects the adjacency conditions between the subgraphs of G_1 and G_2 obtained restricting them to the vertices

$\{u_1, v_1, w_1, z_1\}$ and $\{u_2, v_2, w_2, z_2\}$ respectively. More formally, the edge-associations $((u_1, v_1), (u_2, v_2)) \in V_a$ and $((w_1, z_1), (w_2, z_2)) \in V_a$ are compatible if they if they satisfy the following relations:

$$\begin{aligned} u_1 = w_1 &\Leftrightarrow u_2 = w_2\,, && \text{(map1)}\\ v_1 = z_1 &\Leftrightarrow v_2 = z_2\,, && \text{(map2)}\\ u_1 = z_1 &\Leftrightarrow u_2 = z_2\,, && \text{(map3)}\\ v_1 = w_1 &\Leftrightarrow v_2 = w_2\,; && \text{(map4)} \end{aligned}$$

and

$$\begin{aligned} u_1 \rightsquigarrow w_1 \Leftrightarrow u_2 \rightsquigarrow w_2 \wedge w_1 \rightsquigarrow u_1 \Leftrightarrow w_2 \rightsquigarrow u_2\,, && \text{(iso1)}\\ v_1 \rightsquigarrow w_1 \Leftrightarrow v_2 \rightsquigarrow w_2 \wedge w_1 \rightsquigarrow v_1 \Leftrightarrow w_2 \rightsquigarrow v_2\,, && \text{(iso2)}\\ u_1 \rightsquigarrow z_1 \Leftrightarrow u_2 \rightsquigarrow z_2 \wedge z_1 \rightsquigarrow u_1 \Leftrightarrow z_2 \rightsquigarrow u_2\,, && \text{(iso3)}\\ v_1 \rightsquigarrow z_1 \Leftrightarrow v_2 \rightsquigarrow z_2 \wedge z_1 \rightsquigarrow v_1 \Leftrightarrow z_2 \rightsquigarrow v_2\,, && \text{(iso4)}\\ v_1 \rightsquigarrow u_1 \Leftrightarrow v_2 \rightsquigarrow u_2\,, && \text{(iso5)}\\ z_1 \rightsquigarrow w_1 \Leftrightarrow z_2 \rightsquigarrow w_2\,. && \text{(iso6)} \end{aligned}$$

The first four relations guarantee that ϕ induces a partial injective map, while the last six guarantee that the map induces a subgraph isomorphism. Note that the existence of the edge-associations already guarantees that $u_1 \rightsquigarrow v_1$, $u_2 \rightsquigarrow v_2$, $w_1 \rightsquigarrow z_1$, and $w_2 \rightsquigarrow z_2$, hence the lack of symmetry in the last two isomorphism conditions. Furthermore, the missing map conditions $u_1 = v_1 \Leftrightarrow u_2 = v_2$ and $u_1 = v_1 \Leftrightarrow u_2 = v_2$ are enforced by the way V_a is constructed.

Given the notion of compatibility between edge association, we define the *edge-association graph* as the undirected graph $G_a = (V_a, E_a)$ where two edge associations are adjacent if and only if they are compatible.

Proposition 1 *Let $X \subseteq V_a$, then X is a clique of G_a if and only if $\phi(X)$ is a subgraph isomorphism between G_1 and G_2.*

Sketch of proof. Let $S \subseteq V_a$ such that S is not a clique, then there must be two edge-associations $ea1, ea2 \in S$ such that $\phi(\{ea1, ea2\})$ is not a subgraph isomorphism. This implies that $\phi(S)$ is not a subgraph isomorphism and hence, by contrappositive, that the fact that $\phi(X)$ is a subgraph isomorphism implies that X is a clique of G_a. Conversely, Let $S \subseteq V_a$ such that $\phi(S)$ is not a subgraph isomorphism, then there must be two distinct vertex relations (u_1, w_2) and (v_1, w_2) that either prevent the relation from being a partial injective function, i.e., $u_1 = w_1$ or $u_2 = w_2$, or that do not respect the adjacency relations. Since $\phi(S)$ is induced by S, the two vertex relation must be induced by some edge relations, i.e., there must exist $v_1, z_1 \in V_1$ and $v_2, z_2 \in V_2$ such that either $((u_1, v_1), (u_2, v_2)) \in S$ or $((v_1, u_1), (v_2, u_2)) \in S$, and either $((w_1, z_1), (w_2, z_2)) \in S$ or $((z_1, w_1), (z_2, w_2)) \in S$, but it can be easily shown that any of these conditions must break one of the compatibility relations. QED

Theorem 1 *If $X \subseteq V_a$ is a maximal clique of G_a, then $\phi(X)$ is a maximal subgraph isomorphism between G_1 and G_2. Conversely, If f is a maximal subgraph isomorphism between G_1 and G_2, then $\phi^{-1}(f)$ is a maximal clique of G_a.*

Proof. Assume that $\phi(X)$ is not maximal, but that the relation (u_1, u_2) can be added, then we can add to X the edge relation between the self loops (u_1, u_1) and (u_2, u_2), hence X cannot be maximal. Conversely, assume that we can add the edge-association $((u_1, v_1), (u_2, v_2))$ to $\phi^{-1}(f)$ then either (u_1, u_2) or (v_1, v_2) must not be in f or else $((u_1, v_1), (u_2, v_2))$ would be in $\phi^{-1}(f)$. Hence, $g = f \cup \{(u_1, u_2), (v_1, v_2)\}$ is a subgraph isomorphism and $f \subset g$, hence f is not maximal. QED

This result provides us with a strong relation between maximal cliques in the edge-association graph and maximal subgraph isomorphism, note however that the relation is not a bijection, since there might be non-maximal cliques inducing a maximal isomorphism; however, these must be subsets of a maximal clique inducing the same isomorphism.

3. Weighted isomorphism

Let $\omega_e : E_1 \times E_2 \to \mathbb{R}_+$ be a similarity function between two edges in G_1 and G_2 and $\omega_v : V_1 \times V_2 \to \mathbb{R}_+$ a similarity function between two vertices in G_1 and G_2, we define the weight of an isomorphism f as

$$\Omega(f) = \sum_{\substack{(u_1,v_1) \in E_1\\ u_1, v_1 \in \mathrm{dom}(f)}} \omega_e((u_1, v_1), (f(u_1), f(v_1))) + \sum_{u_1 \in \mathrm{dom}(f)} \omega_v(u_1, f(u_1))\,, \quad (3)$$

Similarly, we define a weight on a vertex $((u_1, v_1), (u_2, v_2)) \in V_a$ as

$$\omega((u_1, v_1), (u_2, v_2)) = \begin{cases} \omega_v(u_1, u_2) & \text{if } u_1 = v_1\,,\\ \omega_e((u_1, v_1), (u_2, v_2)) & \text{otherwise.} \end{cases} \quad (4)$$

With these weights, we can define a weighted association graph $G_a = (V_a, E_a, \omega)$. The weight of a subset of vertices $X \subseteq V_a$ is:

$$\Omega(X) = \sum_{((u_1,v_1),(u_2,v_2)) \in X} \omega((u_1, v_1), (u_2, v_2))\,. \quad (5)$$

We can now prove the following:

Proposition 2 *If f is a subgraph isomorphism between G_1 and G_2, then $\Omega(f) = \Omega(\phi^{-1}(f))$. Conversely, if $X \subseteq V_a$ is a maximal clique of G_a, then $\Omega(X) = \Omega(\phi(X))$.*

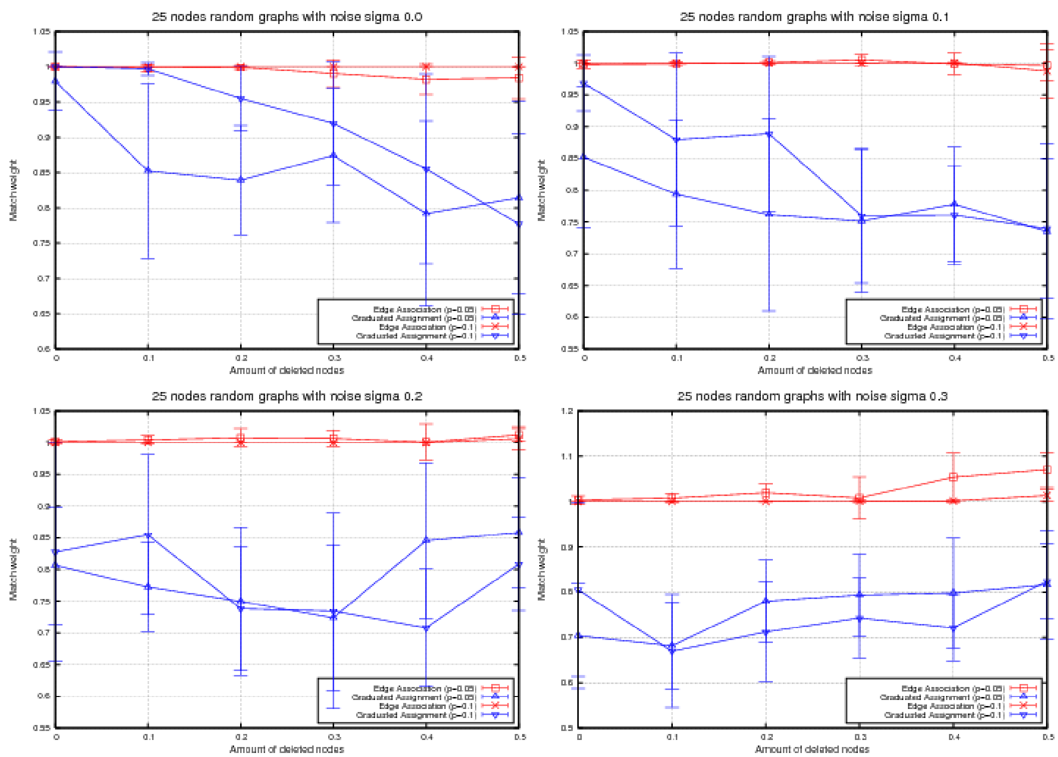

Figure 2. Comparison with Graduated Assignment on synthetic p-random graphs

Proof. If f is an isomorphism, for each $(u_1, v_1) \in E_1, u_1, v_1 \in \mathrm{dom}(f)$ we have $((u_1, v_1), (f(u_1), f(u_2))) \in \phi^{-1}(f)$, and for each $u_1 \in \mathrm{dom}(f)$ we have $((u_1, u_1), (f(u_1), f(u_1))) \in \phi^{-1}(f)$. Hence each element in the sum in (3) will be present in one and only one node of $\phi^{-1}(f)$, which implies that $\Omega(f) = \Omega(\phi^{-1}(f))$. The second part derives from the fact that if X is maximal, then $X = \phi^{-1}(\phi(X))$. QED

Finally, we obtain:

Theorem 2 *$X \subseteq V_a$ is a maximum weight clique of G_a if and only if $\phi(X)$ is a maximum weight isomorphism between G_1 and G_2. Furthermore, $\Omega(X) = \Omega(\phi(X))$.*

Proof. This derives directly from Theorem 1 and Proposition 2. QED

This result allows us to cast the search for the maximum weight subgraph isomorphism between vertex- and edge-attributed graphs into an instance of the weighted clique problem in the edge-assoction graph.

4. Experimental Results

In order to asses the usefulness of our matching approach in recognition tasks using graph-based representations, we performed a set of experiments with synthetic as well as real-world data. In our first set of experiments we compare matches obtained through the proposed approach with matches obtained with Graduated Assignment (GA) [5] on random graphs. For each experiment we randomly generate graph with 25 nodes and 5% and 10% densities. Vertices and edges where given real-valued attributes uniformly distributed between 0 and 1 and the similarity between two

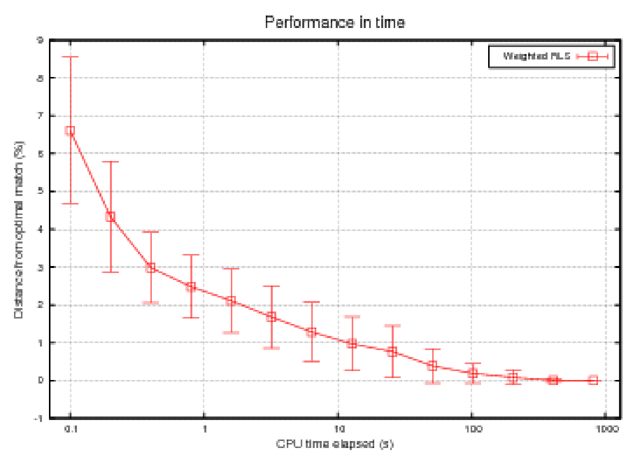

Figure 3. Quality of match found by early stop

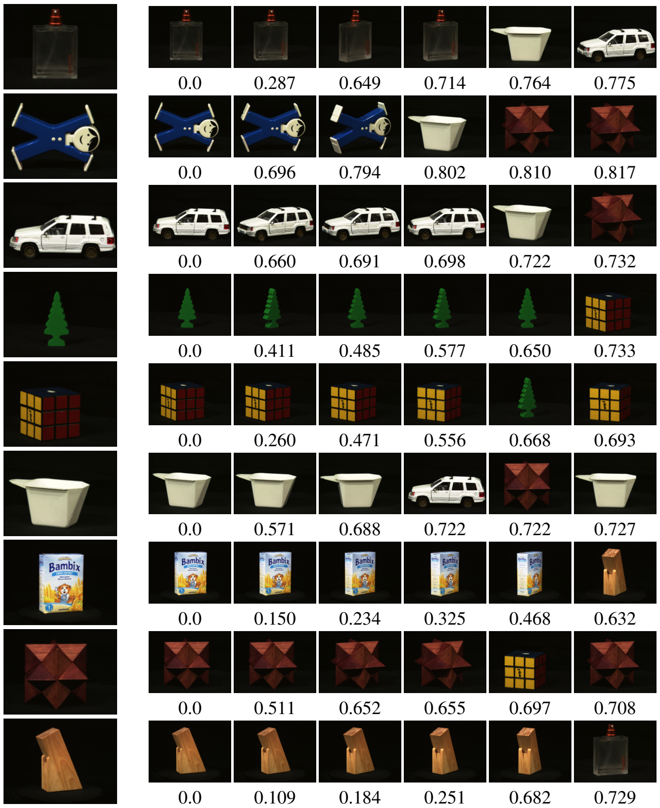

Figure 4. Retrieval with Delaunay graphs

Figure 5. Retrieval with 5-neighbor graphs

such attributes was an exponential decay of the abslute difference of the values. Each graph was perturbed by adding Gaussian noise to the attributes and by randomly removing nodes. Figure 2 compares the performance of the proposed approach with the results obtained by GA as both the amount of structural and of attribute noise is increased. The plot shows the ratio of the weight of the match obtained by the two algorithms over the weight of the ground-truth correspondences. Here the proposed approach clearly outperforms GA.

Next we tried to characterize the time versus performance behavior of the algorithm. To do this we generated 100 random graph graphs of size 25 and density 10% and ran our algorithm on them sampling at various times the best match found. To measure the overall quality of the match we used a metric proposed in [15]:

$$d(G_1, G_2) = 1 - \frac{\Omega(f)}{|G_1| + |G_2| - \Omega(f)}\,. \qquad (6)$$

In Figure 3 we report the error of the best solution at time t relative to the best obtained letting the search algorithm run for 15 minutes. The experiments were run on PC clocked at 1.6 Ghz. From these results we see that in just 1 second the match is within 2.5% of the optimum.

We were also interested in assessing the performance of our algorithm with real-world classification problems. To this end we used a collection of 45 images from the ALOI[4] database, showing 9 objects from different viewpoints. Starting from feature-points extracted using the Harris corner detector [6], We generated attributed graphs in two different ways: The Delaunay triangulation of the points and a 5-neighbor graph, in which each vertex is connected to 5 the nearest vertices. In both cases the distances between points where used as attributes for the edges. Figures 4 and 5 show the 6 best matches for several test images as well as the corresponding value of the metric (6).

From the Figures we can see that the approach almost always selects similar images top matches, although, due to the instability of the representations with respect to changes in viewpoints, the algorithm was often presented with graphs with significant structural deformation.

Finally we tested the robustness of our approach with respect to random occlusions on the images. To this aim we generated random occlusions ranging from 5% to 50% of the total area and ran our algorithm matching the original the graphs geenrated from the origianl image with those obtained from the occluded images. The results of this set of experiments is summarized in Figure 6. Here we can see that the distances from the original graph increases

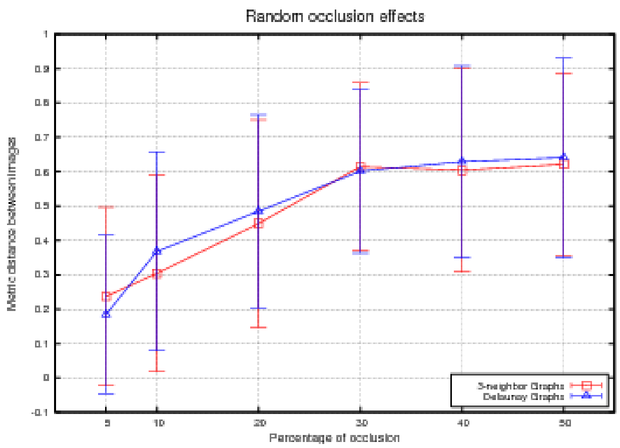

Figure 6. Effect of occlusions on distance

smoothly with the amount of occlusion, assigning reasonably small distances to images with small to moderate occlusion.

5. Conclusions

In this paper we presented a extension of the association graph framework which deals with edge associations directly, without having to infer them from vertex associations. This way edge similarity information can be incorporated into the matching process allowing the framework to deal with vertex- and edge-attributed graphs in an uniform way. The experimental results showed that the approach is robust to structural and geometric noise, and capable of dealing with several relational abstractions of scene in a robust way. However, the scaling behavior of the association graph rendered the space requirements of our implementation very demanding. This severely limited the approach in dealing only with sparse graphs. Note, however, that holding the full edge-association graph explicitly in memory is not necessary. Future work include the reformulation of the problem as an implicit search on the edge-association space in order to make the approach applicable to larger and denser graphs.

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