# Line Integral and Rotation Examples

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## 1 Line Integrals Examples

### 1.1 Questions

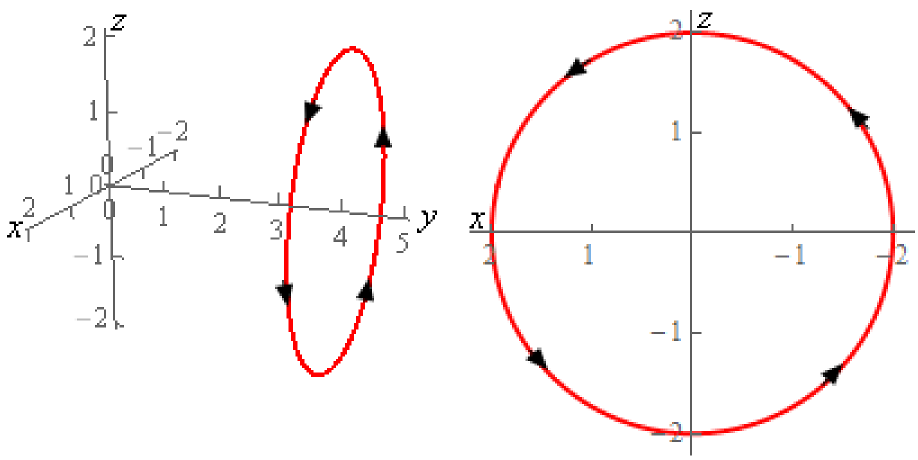
1. Evaluate $\int\limits_C 3x^2 - 2y\, ds$ where C is the line segment from $(3, 6)$ to $(1, -1)$

2. Evaluate $\int\limits_C 2yx^2 - 4x\, ds$ where C is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.

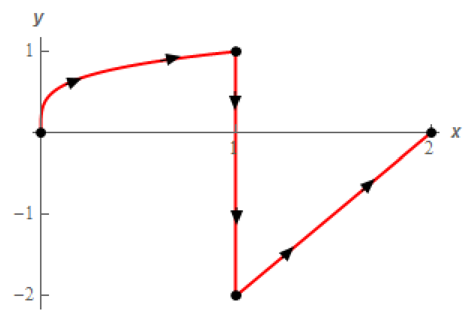
3. Evaluate $\int\limits_C 6x\, ds$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of C is in the direction of increasing x.

4. Evaluate $\int\limits_C xy - 4z\, ds$ where C is the line segment from $(1, 1, 0)$ to $(2, 3, -2)$

5. Evaluate $\int\limits_C x^2y^2\, ds$ where C is the circle centered at the origin of radius 2 centered on the $y$-axis at $y = 4$. See the sketches below for orientation.




6. Evaluate $\int\limits_C 16y^5\, ds$ where C is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment form $(1, 1)$ to $(1, -2)$ which in turn is followed by the line segment from $(1, -2)$ to $(2, 0)$. See the sketch below for the direction.



### 1.2 Solutions

1. Using the point-slope formula, we have $(y+1) = \dfrac{7}{2}(x-1)$

$\therefore 2y = 7x - 9 \implies 2dy = 7dx \implies \dfrac{dy}{dx} = \dfrac{7}{2}$

$\therefore ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \left(\dfrac{7}{2}\right)^2} = \dfrac{\sqrt{53}}{2}\, dx$

$$\therefore \int_C 3x^2 - 2y\, ds = \int_1^3 \left(3x^2 - 7x + 9\right) \frac{\sqrt{53}}{2}\, dx = \frac{\sqrt{53}}{2}\left[x^3 - \frac{7x^2}{2} + 9x\right]_1^3 = \boxed{8\sqrt{53}}$$

2. Let $x = 3\cos\theta$ and $y = 3\sin\theta$

$\therefore ds = \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2} d\theta = 3\sqrt{\sin^2\theta + \cos^2\theta} d\theta = 3\, d\theta$

$$\therefore \int_C 2yx^2 - 4x\, ds = 3\int_\pi^{2\pi} (54\sin\theta\cos^2\theta - 12\cos\theta)\, d\theta = -162\int_{-1}^{1} u^2 du + \cancel{12\left[\sin\theta\right]_\pi^{2\pi}} = \boxed{-108}$$

3. $y = x^2 \implies \dfrac{dy}{dx} = 2x$

$\therefore ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} dx = \sqrt{1+4x^2}\, dx$

$$\therefore \int_C 6x\, ds = \int_{-1}^{2} 6x\sqrt{1+4x^2}\, dx = \frac{3}{4}\int_5^{17} \sqrt{u}\, du = \boxed{\frac{17^{1.5} - 5^{1.5}}{2}}$$

4. First we have to calculate the direction cosines of the line.

$\therefore l = 1, m = 2, n = -2$

$\therefore$ The required equation of the line is $\dfrac{x-1}{1} = \dfrac{y-1}{2} = \dfrac{z-0}{-2}$

$\therefore z = -2x + 2$ and $y = 2x - 1$

$\implies \dfrac{dz}{dx} = -2$ and $\dfrac{dy}{dx} = 2$

$\therefore ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2 + \left(\dfrac{dz}{dx}\right)^2} dx = \sqrt{1 + 2^2 + (-2)^2} dx = 3dx$

$$\therefore \int_C xy - 4z\, ds = 3\int_1^2 x(2x-1) + 8(x-1) dx = 3\left[\frac{x\cdot(4x^2 + 21x - 48)}{6}\right]_1^2 = \boxed{\frac{43}{2}}$$

5. Let $x = 2\cos\theta$ and $z = 2\sin\theta$

$\therefore ds = \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dz}{d\theta}\right)^2} = 2\sqrt{\sin^2\theta + \cos^2\theta} d\theta = 2d\theta$

$$\therefore \int_C x^2y^2\, ds = 64\int_0^{2\pi} 2\sin^2\theta d\theta = 64\left(\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 2\theta d\theta\right) = 128\pi - \cancel{32\left[\sin 2\theta\right]_0^{2\pi}} = \boxed{128\pi}$$

6. $\displaystyle\int_C 16y^5\, ds = \int_{C_1} 16y^5\, ds + \int_{C_2} 16y^5\, ds + \int_{C_3} 16y^5\, ds$

$C_1 \equiv x = y^4, [0 \le y \le 1], ds = \sqrt{1 + (4y^3)^2} dy = \sqrt{1 + 16y^6} dy$

$C_2 \equiv x = 1, [-2 \le y \le 1], ds = \sqrt{1 + 0^2} dy = dy$

$C_3 \equiv x = \dfrac{y}{2} + 2, [-2 \le y \le 0], ds = \sqrt{1 + \left(\dfrac{1}{2}\right)^2} dy = \dfrac{\sqrt{5}}{2} dy$

$$\int_C 16y^5\,ds = \int_0^1 16y^5\sqrt{1+16y^6}\,dy + 16\int_{-2}^{1} y^5\,dy + 8\sqrt{5}\int_{-2}^{0} y^5\,dy$$

$$= \frac{1}{6}\int_1^{17}\sqrt{u}\,du + \frac{8}{3}\left[y^6\right]_{-2}^{1} + \frac{4\sqrt{5}}{3}\left[y^6\right]_{-2}^{0} = \frac{1}{9}\left[u^{1.5}\right]_1^{17} - 168 - \frac{256\sqrt{5}}{3}$$

$$= \frac{17^{1.5}-1}{9} - 168 - \frac{256\sqrt{5}}{3} = \boxed{-351.1341}$$

# 2 Rotation of Coordinate Systems

## 2.1 Rotation Matrices

The 2D Rotation Matrix is $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ where $\theta$ is the angle of rotation.

The 3D Rotation Matrices are:-

About X-Axis $R_x(\theta)=\begin{bmatrix}1 & 1 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$

About Y-Axis $R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$

About Z-Axis $R_z(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

## 2.2 Example

If the Axes are rotated through 60° in the anticlockwise direction, find the transformed form of the equation $x^2 - y^2 = a^2$

**Soln:-** Let the transformed coordinates be $X$ and $Y$.

$$\therefore \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\cos 60 & -\sin 60\\ \sin 60 & \cos 60\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix} = \frac{1}{2}\begin{bmatrix}X-\sqrt{3}Y\\ \sqrt{3}X+Y\end{bmatrix}$$

Putting the values of x and y in the given equation we get

$$\left(X-\sqrt{3}Y\right)^2 - \left(\sqrt{3}X+Y\right)^2 = 4a^2$$

$$\implies 2Y^2 - 2X^2 - 4\sqrt{3}XY = 4a^2 \implies \boxed{Y^2 - X^2 - 2\sqrt{3}XY = 2a^2}$$