Properties of Arithmetic Progression

- **General Term: -** $T_n = a + d(n-1)$, where n= index no. of term, a=first term, d=common difference.
- Sum of 2 terms having same index number from both sides is constant & always equal to the sum of first & last terms of the AP. **Example: - Consider the AP: - 2,5,8,11,14,17 where a=2, d=3. If we add 5 & 14, we get 19. If we add 8 & 11, we get 19. If we add 2 & 17, we get 19.**
- **Summation of n terms: -** $S_n = \dfrac{\mathbf{n(First\ Term + nth\ Term)}}{\mathbf{2}}$.
- If we multiply each term of an AP by a number (say n), then the resultant series is also an AP with common difference nd.
- **Considering Terms in AP: -** If we need to consider odd number of terms in AP (say 3 terms), then we consider them as **a-d, a, a+d.** Similarly, for 5 terms, **a-2d, a-d, a, a+d, a+2d**. If we need to consider even number of terms (say 4 terms), **a-3d, a-d, a+d, a+3d.** [Where the common difference is 2d]

Properties of Geometric Progression

- **General Term: -** $T_n = ar^{(n-1)}$, where a= first term, r=common ratio.
- **Summation of n terms: - If** r>1, $S_n = \dfrac{a(r^n-1)}{r-1}$. If r<1, $S_n = \dfrac{a(1-r^n)}{1-r}$. If r<1, then $S_\infty = \dfrac{a}{1-r}$.
- Product of 2 terms having same index number from both sides is constant & always equal to the product of first & last terms of the GP. **Example: - Consider the GP: - 3,9,27,81,243 where a=3, r=3. If we multiply 9 & 81, we get 729. If we multiply 3 & 243, we get 729. If we multiply 27 & 27, we get 729.**
- **Considering Terms in GP: -** If we need to consider odd number of terms in GP (say 3 terms), then we consider them as ar^{-1}, **a** , **ar**. Similarly, for 5 terms, ar^{-2}, ar^{-1}, **a** , **ar** , ar^2. If we need to consider even number of terms (say 4 terms), ar^{-3}, ar^{-1}, **ar**, ar^3.
- If we multiply or raise each term of GP with some number the resultant sequence still remains a GP.

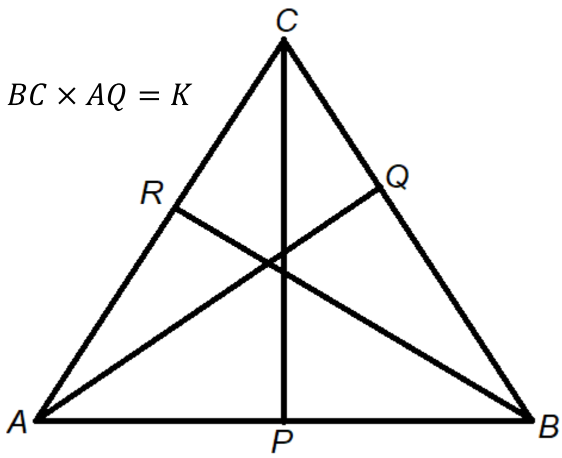
Harmonic Progression

If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \ldots, \frac{1}{n}$ are in AP, then a, b, c, …, n are in HP.

Some important Results

1. In a triangle, if the altitudes are in arithmetic progression, then the sides are in harmonic progression.

Proof



Area of $\Delta ABC = \frac{1}{2} \times AC \times BR = \frac{1}{2} \times AB \times PC = \frac{1}{2} \times AC \times BR = \frac{1}{2} \times BC \times AQ = K$

$\therefore AB = \frac{K}{PC}, BC = \frac{K}{AQ}, AC = \frac{K}{BR}$. Since PC, AQ, BR are in AP,

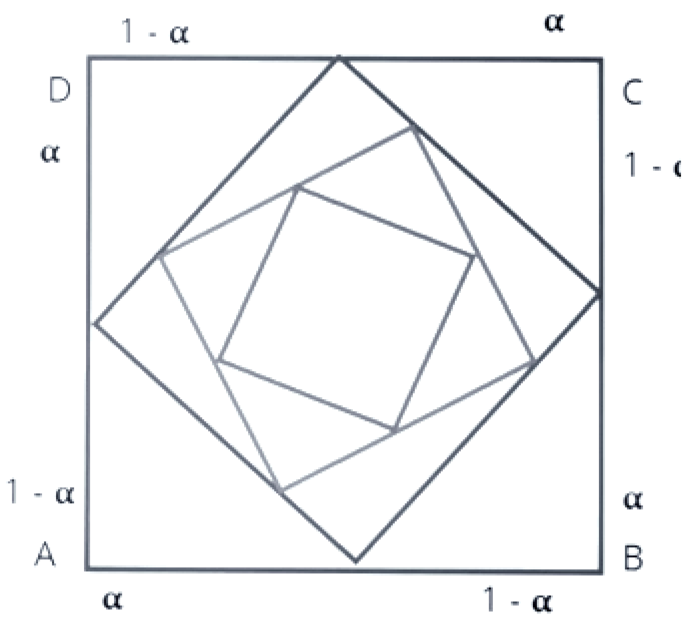
$\therefore$ by definition, AB, BC, CA are in AP (since K is a constant).

2. RMS ≥ AM ≥ GM ≥ HM

Or,

$$\sqrt{\frac{a^2+b^2+c^2+\cdots+n^2}{n}} \geq \frac{a+b+c+\cdots+n}{n} \geq \sqrt[n]{abc \ldots n} \geq \frac{n}{1/a + 1/b + 1/c + \cdots + 1/n}$$

Q1. Find the value for α for which $\sum_{n=1}^{\infty} A_n = \frac{8}{3}$, where A_n= *Area of the nth square*. Area of the first square = 1m^2 & the sides of all the squares are divided in the ratio $\alpha : (1-\alpha)$



Soln. Side of the second square = $\sqrt{(1-\alpha)^2+\alpha^2}$

$\therefore$ Area second square = $(1-\alpha)^2+\alpha^2 = 2\alpha^2-2\alpha+1$

Side of third square =

$$\sqrt{\left(\alpha\sqrt{(1-\alpha)^2+\alpha^2}\right)^2+\left((1-\alpha)\sqrt{(1-\alpha)^2+\alpha^2}\right)^2}$$

$$= \sqrt{(1-\alpha)^2+\alpha^2} \times \sqrt{(1-\alpha)^2+\alpha^2} = 2\alpha^2-2\alpha+1$$

$\therefore$ Area of third square = $(2\alpha^2-2\alpha+1)^2$

Similarly, area of nth square = $(2\alpha^2-2\alpha+1)^{(n-1)}$

$$\sum_{n=1}^{\infty} A_n = 1+(2\alpha^2-2\alpha+1)+(2\alpha^2-2\alpha+1)^2+\cdots+(2\alpha^2-2\alpha+1)^{(n-1)}$$

$$\Rightarrow \frac{1}{1-(2\alpha^2-2\alpha+1)} = \frac{8}{3} \Rightarrow\ 2\alpha^2-2\alpha+1 = \frac{5}{8}$$

Solving this quadratic equation, we get, $\alpha = \frac{1}{4}, \frac{3}{4}$.

Q2. Find the minimum value of $\dfrac{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}{ABCD}$ where *A,B,C,D >0*

Soln. By AM $\geq$ GM,

$$\frac{(A^2+A+1)+(B^2+B+1)+(C^2+C+1)+(D^2+D+1)}{4} \geq \sqrt[4]{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}$$

$$\left(\frac{A^2+B^2+C^2+D^2}{4}\right)+\left(\frac{A+B+C+D}{4}\right)+1 \geq \sqrt[4]{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}$$

$$\left(\frac{A+B+C+D}{4}\right) \geq \sqrt[4]{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}-\left(\frac{A^2+B^2+C^2+D^2}{4}\right)-1 (i)$$

$$\sqrt{\frac{A^2+B^2+C^2+D^2}{4}} \geq \left(\frac{A+B+C+D}{4}\right) \ldots (ii) \qquad \sqrt{\frac{A^2+B^2+C^2+D^2}{4}} \geq \sqrt[4]{ABCD} \ldots (iii)$$

Let $x = \sqrt{\frac{A^2+B^2+C^2+D^2}{4}}$

$\therefore$ from i, ii & iii, we get,

$$\sqrt[4]{\frac{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}{ABCD}} \geq \frac{x^2+x+1}{x}$$

We can evaluate the minimum value of $\frac{x^2+x+1}{x}$ using maxima-minima.

$$\frac{d}{dx}\left(1+x+\frac{1}{x}\right) = 1-\frac{1}{x^2} = 0 \quad \Rightarrow x = \pm 1$$

$$\frac{d^2}{dx^2}\left(1-\frac{1}{x^2}\right)=\frac{2}{x^3}$$

∴ Putting x=1, we get 2 which is greater than 0

$$\boldsymbol{\frac{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}{ABCD} \geq 3^4 = 81}$$

Alter:- Let's consider the series $\left(A+\frac{1}{A}+1\right),\left(B+\frac{1}{B}+1\right),\left(C+\frac{1}{C}+1\right),\left(D+\frac{1}{D}+1\right)$

From AM ≥ GM,

$$\frac{A+\frac{1}{A}+1}{3} \geq \sqrt[3]{A\times\frac{1}{A}\times 1} \Rightarrow \frac{A^2+A+1}{A} \geq 3$$

Similarly,

$$\frac{B^2+B+1}{B} \geq 3 \qquad \frac{C^2+C+1}{C} \geq 3 \qquad \frac{D^2+D+1}{D} \geq 3$$

Multiplying all these inequalities we get,

$$\boldsymbol{\frac{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}{ABCD} \geq 3^4 = 81}$$

Q3. For a,b,c ∈ R & a,b,c ≠ 0, prove that $(a+1)^7(b+1)^7(c+1)^7>7^7a^4b^4c^4$

<u>Soln.</u> $(a+1)(b+1)(c+1) = abc+ab+bc+ca+a+b+c+1$

From AM ≥ GM,

$$\frac{abc+ab+bc+ca+a+b+c}{7} \geq \sqrt[7]{abc\times ab\times bc\times ca\times a\times b\times c}$$

$$\Rightarrow \frac{abc+ab+bc+ca+a+b+c}{7}+\frac{1}{7} > \sqrt[7]{a^4b^4c^4}$$

$$\Rightarrow \frac{(a+1)(b+1)(c+1)}{7} > \sqrt[7]{a^4b^4c^4} \quad \Rightarrow \left\{\frac{(a+1)(b+1)(c+1)}{7}\right\}^7 > a^4b^4c^4$$

$$\therefore \mathbf{(a+1)^7(b+1)^7(c+1)^7 > 7^7a^4b^4c^4}$$

<u>Some important Summations</u>

$$\boldsymbol{\sum_{i=1}^{n} n = \frac{n(n+1)}{2}} \qquad \boldsymbol{\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}} \qquad \boldsymbol{\sum_{i=1}^{n} n^3 = \left\{\frac{n(n+1)}{2}\right\}^2}$$

Q4. The sum of first n term of a series is cn^2 .Find the sum of the squares of these n-terms.

<u>Soln.</u> We know, $T_n = S_n - S_{n-1}$

∴ $T_n = cn^2 - c(n-1)^2 = c(1-2n)$

∴ $T_n^2 = c^2(1-2cn)^2 = c^2(1-4n+4n^2)$

$$\therefore \sum_{i=1}^{n} T_n^2 = c^2\left(\sum_{i=1}^{n} -4\sum_{i=1}^{n} n + 4\sum_{i=1}^{n} n^2\right) \Rightarrow \boldsymbol{S_n = \frac{c^2n}{3}(4n^2-1)}$$

Method of Difference (Basic)

Example:- 1 Find the sum of the series 3 + 7 + 13 + 21 + 31 up to n-terms

Soln. If we subtract each term from its next term, we get an AP. Since we are getting AP in the very first step, ∴ K=1.

	3	7	13	21	31
K=1	4	6	8	10	

∴ T_n is a polynomial of degree (K+1) since it's an AP

$\therefore\ T_n = an^2 + bn + c$

Putting n=1,2,3 we get these 3 equations

$$a + b + c = 3 \qquad 4a + 2b + c = 7 \qquad 9a + 3b + c = 13$$

Upon solving these equations, we get, $a = b = c = 1$

$\therefore\ \boldsymbol{T_n = n^2 + n + 1}$

$$\therefore \sum_{i=1}^{n} T_n = \sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} n + \sum_{i=1}^{n} 1 \Rightarrow \boldsymbol{S_n} = \boldsymbol{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n}$$

Example:- 2 Find the sum of the series 1 + 4 + 10 + 22 + 46 up to n-terms

Soln. If we subtract each term from its next term, we get a GP. Since we are getting GP in the very first step, ∴ K=1.

	1	4	10	22	46
K=1	3	6	12	24	

$\therefore\ T_n = ar^{(n-1)} + (\text{a polynomial of degree k} - 1)$, where a=3 & r=2 since it's a GP

$\Rightarrow \boldsymbol{T_n = 3 \times 2^{(n-1)} - 2}$

$$\therefore \sum_{i=1}^{n} T_n = \frac{3(2^n - 1)}{2 - 1} - 2\sum_{i=1}^{n} 1 \Rightarrow \boldsymbol{S_n = 3(2^n - 1) - 2n}$$

Example:- 3 Find the sum of the series 2 + 12 + 36 + 80 + 150 + 252 + 392 up to n-terms

Soln. This time we get an AP at the second step. ∴ K=2 & T_n is a polynomial of degree (K+1)

	2	12	36	80	150	252
	10	24	44	70	102	
K=2	14	20	26	32		

$\therefore \quad T_n = an^3 + bn^2 + cn + d$

Putting n=1,2,3,4 we get the following equations

$a + b + c + d = 2 \quad 8a + 4b + 2c + d = 12 \quad 27a + 9b + 3c + d = 36 \quad 64a + 16b + 4c + d = 80$

Solving these equations, we get a=b=1, c=d=0

$$\therefore \; \boldsymbol{T_n = n^3 + n^2}$$

$$\therefore \sum_{i=1}^{n} T_n = \sum_{i=1}^{n} n^3 + \sum_{i=1}^{n} n^2 \Rightarrow \boldsymbol{S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)(2n+1)}{6}}$$

Method of Difference (Advanced Telescoping series)

Find the sum up to n-terms: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$

Soln. $\sum_{i=1}^{n} T_n = S_n = \frac{(2-1)}{1.2} + \frac{(3-2)}{2.3} + \frac{(4-3)}{3.4} + \cdots + \frac{(n+1)-n}{n(n+1)}$

$$\Rightarrow S_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1} \quad \Rightarrow S_n = 1 - \frac{1}{n+1}$$

Find the sum up to n-terms: $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$

Soln. Like the previous one, here we can observe that the denominators of each terms are in an AP with common difference 3.

$$\therefore \sum_{i=1}^{n} T_n = S_n = \frac{1}{3}\left\{\frac{(4-1)}{1.4} + \frac{(7-4)}{4.7} + \frac{(10-7)}{7.10} + \cdots + \frac{\{4+3(n-1)\} - \{1+3(n-1)\}}{\{1+3(n-1)\}\{4+3(n-1)\}}\right\}$$

$$\Rightarrow S_n = \frac{1}{3}\left\{1 - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{10}} + \cdots + \cancel{\frac{1}{1+3(n-1)}} - \frac{1}{4+3(n-1)}\right\}$$

$$\Rightarrow \boldsymbol{S_n = \frac{1}{3}\left\{1 - \frac{1}{4+3(n-1)}\right\}}$$

TYPE-II

Find the sum up to n-terms: 1.2.3 + 2.3.4 + 3.4.5 + …

Soln. $T_n = n(n+1)(n+2)$

Let's multiply a modifier & see what happens, $\text{Modifier} = \frac{(\text{Next Factor} - \text{Previous Factor})}{\text{Constant}}$

$\therefore \text{Modifier} = \frac{(n+3)-(n-1)}{4} \qquad \therefore T_n = n(n+1)(n+2)\left\{\frac{(n+3)-(n-1)}{4}\right\}$

$$\Rightarrow T_n = \frac{n(n+1)(n+2)(n+3)}{4} - \frac{(n-1)n(n+1)(n+2)}{4}$$

$\Rightarrow T_n = \frac{1}{4}(V_n - V_{n-1}) \qquad \Rightarrow S_n = \frac{1}{4}(\cancel{V_1} - V_0 + \cancel{V_2} - \cancel{V_1} + \cancel{V_3} - \cancel{V_2} + \cdots + V_n - \cancel{V_{n-1}})$

$$\Rightarrow \boldsymbol{S_n = \frac{1}{4}(V_n - V_0) = \frac{n(n+1)(n+2)(n+3)}{4}}$$

ALTER: - We can also solve this problem as follows:-

$T_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$

$$\therefore \sum_{i=1}^{n} T_n = \sum_{i=1}^{n} n^3 + 3\sum_{i=1}^{n} n^2 + 2\sum_{i=1}^{n} n = \boldsymbol{\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1)}$$

Find the sum up to n-terms: 1.4.7.10 + 4.7.10.13 + 7.10.13.16 + 10.13.16.19

Soln. $T_n = \{1+3(n-1)\}\{4+3(n-1)\}\{7+3(n-1)\}\{10+3(n-1)\}$

$\Rightarrow T_n = (3n-2)(3n+1)(3n+4)(3n+7)$

$\therefore \text{Modifier} = \dfrac{(3n+10)-(3n-5)}{15}$ $\quad\therefore T_n = (3n-2)(3n+1)(3n+4)(3n+7)\left\{\dfrac{(3n+10)-(3n-5)}{15}\right\}$

$$\therefore T_n = \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (3n-5)(3n-2)(3n+1)(3n+4)(3n+7)\}$$

$\therefore T_n = \dfrac{1}{15}(V_n - V_{n-1})$ $\quad\Rightarrow S_n = \dfrac{1}{15}(\cancel{V_1} - V_0 + \cancel{V_2} - \cancel{V_1} + \cancel{V_3} - \cancel{V_2} + \cdots + V_n - \cancel{V_{n-1}})$

$$\Rightarrow \boldsymbol{S_n = \frac{1}{15}(V_n - V_0) = \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}}$$

TYPE-III

Find the sum up to n-terms: $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + \cdots$

Soln. $T_n = \dfrac{1}{n(n+1)(n+2)}$

Let's multiply a modifier & see what happens, $Modifier = \dfrac{(Last\ Factor - First\ Factor)}{Constant}$

$\therefore \text{Modifier} = \dfrac{(n+2)-n}{2}$ $\quad\therefore T_n = \dfrac{1}{n(n+1)(n+2)}\left\{\dfrac{(n+2)-n}{2}\right\}$

$\Rightarrow T_n = \dfrac{1}{2}\left\{\dfrac{1}{n(n+1)} - \dfrac{1}{(n+1)(n+2)}\right\}$ $\quad\therefore T_n = \dfrac{1}{15}(V_n - V_{n+1})$

$\Rightarrow S_n = \dfrac{1}{2}(V_1 - \cancel{V_2} + \cancel{V_2} - \cancel{V_3} + \cancel{V_3} - \cancel{V_4} + \cdots + \cancel{V_n} - V_{n+1})$ $\quad\Rightarrow S_n = \dfrac{1}{2}(V_1 - V_{n+1})$

$$\Rightarrow \boldsymbol{S_n = \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right\} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}}$$

Find the sum up to n-terms: $\dfrac{1}{1.3.5} + \dfrac{1}{3.5.7} + \dfrac{1}{5.7.9} + \cdots$

Soln.

$$T_n = \frac{1}{\{1+2(n-1)\}\{3+2(n-1)\}\{5+2(n-1)\}} = \frac{1}{(2n-1)(2n+1)(2n+3)}$$

$\text{Modifier} = \dfrac{(2n+3)-(2n-1)}{4}$ $\quad\therefore T_n = \dfrac{1}{(2n-1)(2n+1)(2n+3)}\left\{\dfrac{(2n+3)-(2n-1)}{4}\right\}$

$T_n = \dfrac{1}{4}\left\{\dfrac{1}{(2n-1)(2n+1)} - \dfrac{1}{(2n+1)(2n+3)}\right\}$ $\quad\Rightarrow T_n = \dfrac{1}{4}(V_n - V_{n+1})$

$S_n = \dfrac{1}{4}(V_1 - \cancel{V_2} + \cancel{V_2} - \cancel{V_3} + \cancel{V_3} - \cancel{V_4} + \cdots + \cancel{V_n} - V_{n+1})$

$$\Rightarrow \boldsymbol{S_n = \frac{1}{4}\left\{\frac{4}{15} - \frac{1}{(2n+1)(2n+3)}\right\}}$$