Saptarshi's Math Notes

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1 Inverse Hyperbolic Trigonometric Functions

Pre-requisites/Synopsis :-

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{1}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{2}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{3}$$

Let's assume $\sinh^{-1}(x) = \ln|f(x)|$ where $f(x)$ is some function of x

$$\therefore \frac{e^{\ln|f(x)|} - e^{-\ln|f(x)|}}{2} = x$$

$$\implies f(x) - \frac{1}{f(x)} = 2x$$

$$\implies f^2(x) - 2xf(x) - 1 = 0$$

Solving this quadratic equation, we get,

$$f(x) = x \pm \sqrt{1 + x^2}$$

If $f(x) = x - \sqrt{1 + x^2}$, then $f(x) < 0$ for all $x \in (-\infty, 0)$
But $f(x)$ can't be -ve since $\ln|f(x)|$ will be an complex number. $\therefore f(x) = x + \sqrt{1 + x^2}$

$$\therefore \boxed{\sinh^{-1}(x) = \ln|x + \sqrt{1 + x^2}|} \tag{4}$$

Similarly, we can prove,

$$\boxed{\cosh^{-1}(x) = \ln|x + \sqrt{x^2 - 1}|} \tag{5}$$

Let $\tanh^{-1}(x) = \ln|f(x)|$

$$\therefore \frac{e^{\ln|f(x)|} - e^{-\ln|f(x)|}}{e^{\ln|f(x)|} + e^{-\ln|f(x)|}} = x$$

$$\implies \frac{f(x) - \frac{1}{f(x)}}{f(x) + \frac{1}{f(x)}} = x$$

Using componendo-dividendo,

$$\frac{1}{f^2(x)} = \frac{1 - x}{1 + x}$$

$$\implies f(x) = \sqrt{\frac{1 + x}{1 - x}}$$

$$\therefore \boxed{\tanh^{-1}(x) = \ln(\sqrt{\frac{1 + x}{1 - x}}) = \frac{1}{2}\ln(\frac{1 + x}{1 - x})} \tag{6}$$

2 Integration Problems

$$1. \int \frac{x^2}{(x\sin x+\cos x)^2}dx$$

$$= \int x\sec x \frac{x\cos x}{(x\sin x+\cos x)^2}dx$$

$$= x\sec x \int \frac{x\cos x}{(x\sin x+\cos x)^2}dx + \int \frac{\sec x + x\tan x}{x\sin x+\cos x}dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{(\sec x + x\tan x)\times\cos^2 x}{(x\sin x+\cos x)\times\cos^2 x}dx + C_1$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{\cancel{(x\sin x+\cos x)}}{\cancel{(x\sin x+\cos x)}\cos^2(x)}dx + C_1$$

$$\boxed{\tan x - \frac{x\sec x}{x\sin x+\cos x} + C}$$

$$2. \int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$$

Let $a = x - 1$

$$\therefore I = \int a^{\frac{-3}{4}}(a+3)^{\frac{-5}{4}}da$$

$$\therefore I = \int a^{\frac{1}{4}}a^{-1}(a+3)^{\frac{-1}{4}}(a+3)^{-1}da$$

$$\therefore I = \int (\frac{a}{a+3})^{\frac{1}{4}}(a^2+3a)^{-1}da$$

$$\therefore I = \int e^{\frac{1}{4}\ln(\frac{a}{a+3})}(a^2+3a)^{-1}da$$

Let $u = \ln(\frac{a}{a+3})$

$$\therefore \frac{du}{da} = \frac{\cancel{a+3}}{a} \times \frac{(\cancel{a}+3)-\cancel{a}}{(a+3)^{\cancel{2}}}$$

$$du = 3(a^2+3a)^{-1}da$$

$$\therefore I = \frac{1}{3}\int e^{\frac{u}{4}}du$$

$$\implies I = \frac{4}{3}\int e^{\frac{u}{4}}d[\frac{u}{4}] = \frac{4}{3}e^{\frac{u}{4}} + C$$

Plugging in the substitutions we get,

$$\therefore I = \frac{4}{3}(\frac{a}{a+3})^{\frac{1}{4}} + C = \boxed{\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + C}$$

Leibinz's Formula

$$f(x) = \int_{\phi}^{\psi} g(t)\,dt$$

$$f'(x) = g(\psi)\frac{d\psi}{dx} - g(\phi)\frac{d\phi}{dx}$$