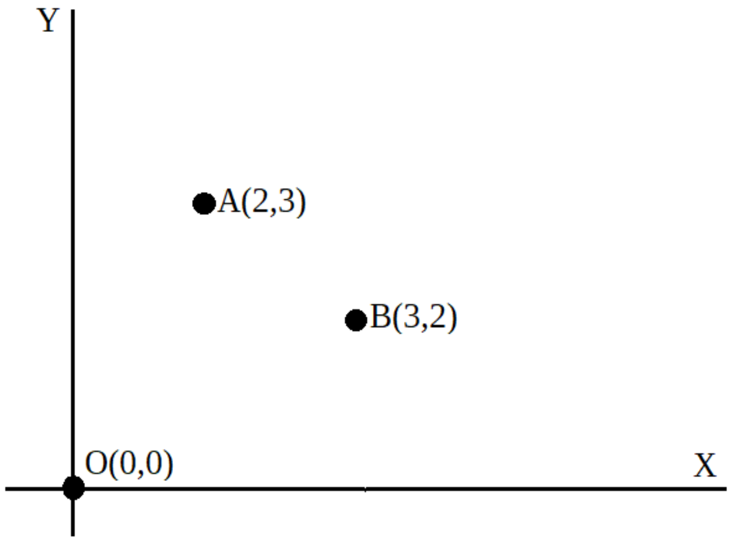
CENTER OF MASS

Centre of Mass of a body or mass distribution is defined as the sum of all the product of the all the masses and their distance from the origin. It is that unique point where a force can be applied to cause a translational motion without causing rotational motion. Mathematically,

Centre of Mass (CM) $=\sum_{i=0}^{n} m_i r_i = \frac{1}{M}\int m\,dr$ [Where m=point mass & M=total mass of the body]

It is a hypothetical point where entire mass of an object may be assumed to be concentrated to visualise its motion. In other words, the center of mass is the particle equivalent of a given object for application of Newton's laws of motion. Example:-

1. Find the Centre of Mass of the Given body.



Soln:- Let $O(0,0)$ be the origin. Given:-

A=2kg B=3kg O=5kg.

$$\therefore CM=\sum_{i=0}^{n} m_i r_i$$

$$\Rightarrow CM(x)=\frac{m_A x_A+m_B x_B+m_O x_O}{m_A+m_B+m_O}$$

$$\Rightarrow CM(x)=\frac{2*2+3*3+5*0}{2+3+5}=\frac{13}{10}=1.3$$

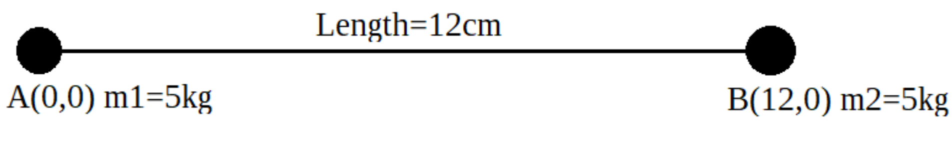
Similarly, $CM(y)=\frac{m_A y_A+m_B y_B+m_O y_O}{m_A+m_B+m_C}$

$$\Rightarrow CM(y)=\frac{2*3+3*2+5*0}{2+3+5}=\frac{12}{10}=1.2$$

$\therefore$ Coordinates of COM are $C(1.3,1.2)$.

The Centre of Mass of a symmetrical body is located at the geometrical centre of the body. Let’s see an example:-

2. Find the COM of the given figure.



Soln:- Let $A(0,0)$ be the origin.

$$\therefore CM(x)=\frac{m_A x_A+m_B x_B}{m_A+m_B}$$

$$\Rightarrow CM(x)=\frac{5*0+5*12}{5+5}=\frac{60}{10}=6\,cm$$

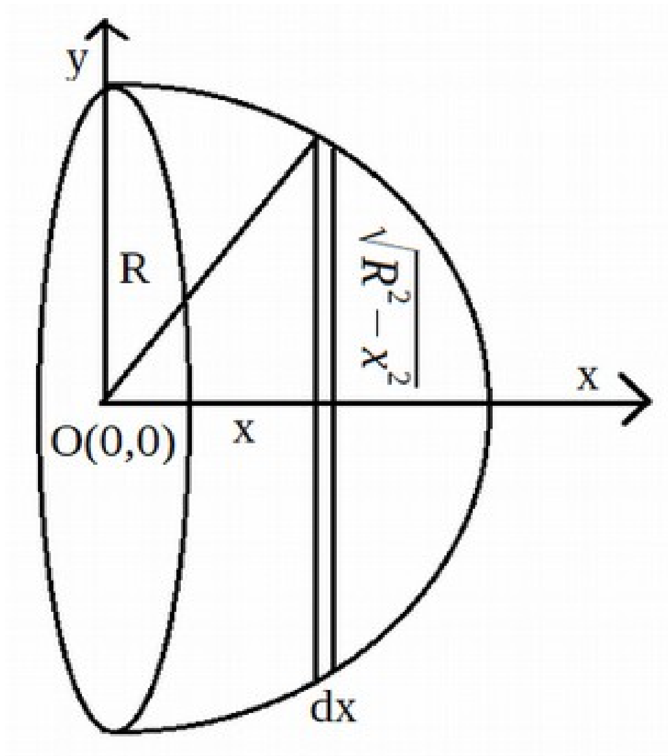
Similarly, $CM(y)=\frac{m_A y_A+m_B y_B}{m_A+m_B}$

$\Rightarrow CM(y)=\frac{5*0+5*0}{5+5}=0\,cm$ [If all the points lie on the same line, then COM also lies on the same line.]

$\therefore$ The coordinates of COM are $C(6,0)$, or it can be said/concluded that $COM(x)=L/2=12/2=6$cm.

Centre of Mass of a body does not depend upon the origin we choose. No matter whichever origin we may choose, we will always get the same result.

COM of a Solid Hemisphere



Let's take the centre of the hemisphere as the origin.

$$\therefore X_{COM}=\frac{1}{M}\int xdm$$ [Where dm is point mass & x is distance from origin.]

Let ρ be the density of the solid hemisphere.

$\therefore dm=\rho\Pi r^2 dx$ [Where $r=\sqrt{R^2-x^2}$]

$$\therefore M=\frac{2}{3}\rho\Pi R^3$$

$$\therefore X_{COM}=\frac{1}{\frac{2}{3}\rho\Pi R^3}\int_0^R x\rho\Pi(R^2-x^2)dx$$

$$\Rightarrow X_{COM}=\frac{3}{2R^3}\int_0^R(xR^2-x^3)dx$$

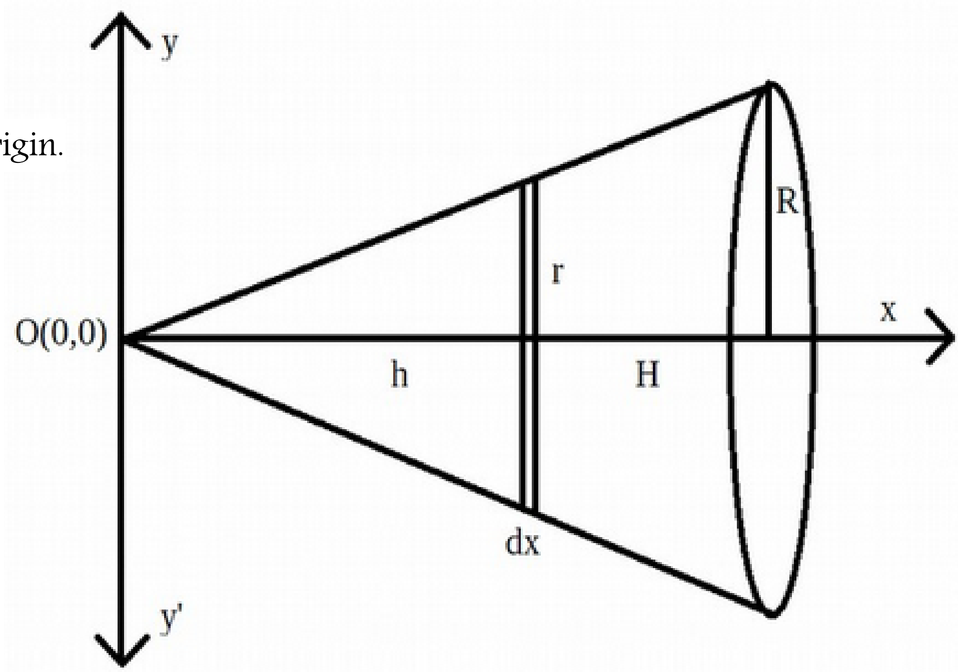
$$\Rightarrow X_{COM}=\frac{3}{2R^3}\left[\frac{x^2R^2}{2}-\frac{x^4}{4}\right]_0^R$$

$$\Rightarrow X_{COM}=\frac{3}{2R^3}\times\frac{R^4}{4}=\frac{3R}{8}$$

[Where R is the radius of the hemisphere.] $Y_{COM}=0$ $Z_{COM}=0$.

∴ The coordinates of COM are C(3R/8,0,0) from the origin.

COM of a Solid Cone



Let's take the tip of the cone as the origin.

$$\therefore X_{COM}=\frac{1}{M}\int xdm$$

Let ρ be the density of the cone.

$$\therefore dm=\Pi\rho r^2 dx$$

$$M=\frac{1}{3}\Pi\rho R^2 H$$

$$\therefore X_{COM}=\frac{1}{\frac{1}{3}\Pi\rho R^2 H}\int_0^H x\Pi\rho r^2 dx$$

We know, $\frac{x}{r}=\frac{H}{R}\Rightarrow r=\frac{xR}{H}$

$$\therefore X_{COM}=\frac{3}{R^2H}\int_0^H\frac{x^3R^2}{H^2}dx=\frac{3}{H^3}\int x^3dx=\frac{3}{H^3}\left[\frac{x^4}{4}\right]_0^H=\frac{3}{H^3}\times\frac{H^4}{4}=\frac{3H}{4}$$

[Where H =Height of the Cone] $Y_{COM}=0$, and $Z_{COM}=0$.

∴ The coordinates of COM are C(3H/4,0,0) from the origin.

MOMENT OF INERTIA

Moment of Inertia is a quantity expressing a body's tendency to resist angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation. It plays the same role as mass does in linear motion. Units:- kg m^2.

In linear motion we have,

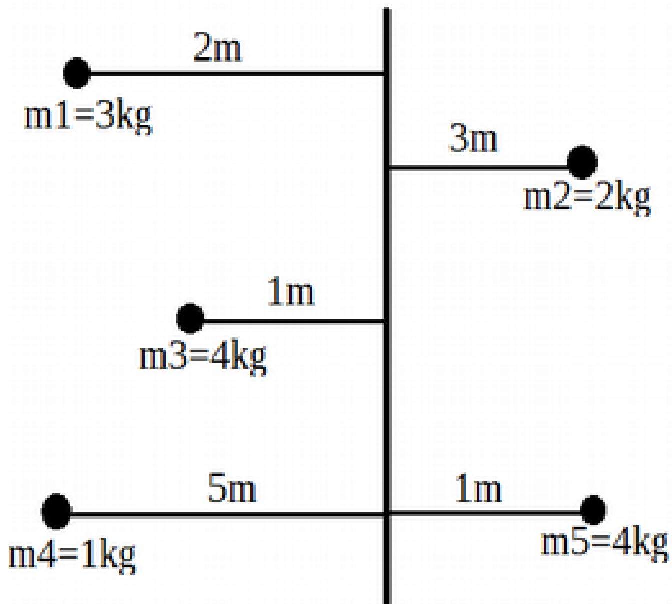
$$F = m\frac{dv}{dt} = ma$$ [F= Force m=mass a=acceleration]

In rotational motion we have,

$$\tau = I\frac{d\omega}{dt} = I\alpha$$ [τ=Torque I=Moment of Inertia α=Angular acceleration]

Mathematically, $I = \sum_{i=0}^{n} m_i r_i^2 = \int r^2 dm$ [m and dm both are point masses and r is the perpendicular distance from the axis of rotation]

1. Find the moment of Inertia of the following body.

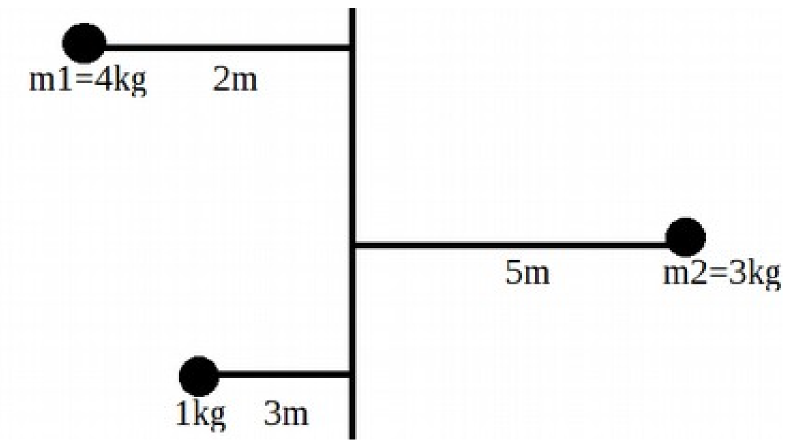


Soln:- We know that $I = \sum_{i=0}^{n} m_i r_i^2$

$$\therefore I = 3*2^2 + 2*3^2 + 4*1^2 + 1*5^2 + 4*1^2$$

$$\Rightarrow I = 12 + 18 + 4 + 25 + 4 = 63\, kg\, m^2$$

2. Find the moment of Inertia of the given body.



Soln:- We know $I = \sum_{i=0}^{n} m_i r_i^2$

$$\therefore I = 4*2^2 + 3*5^2 + 3*1^2\, kg\, m^2$$

$$\Rightarrow I = 16 + 75 + 3 = 94\, kg\, m^2$$

Perpendicular Axis Theorem

The perpendicular axis theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its own plane. The axes must all pass through a single point on the plane.

Proof:- $I_X = mx^2 \qquad I_Y = my^2$

$$I_Z = m\left(\sqrt{x^2+y^2}\right)^2$$

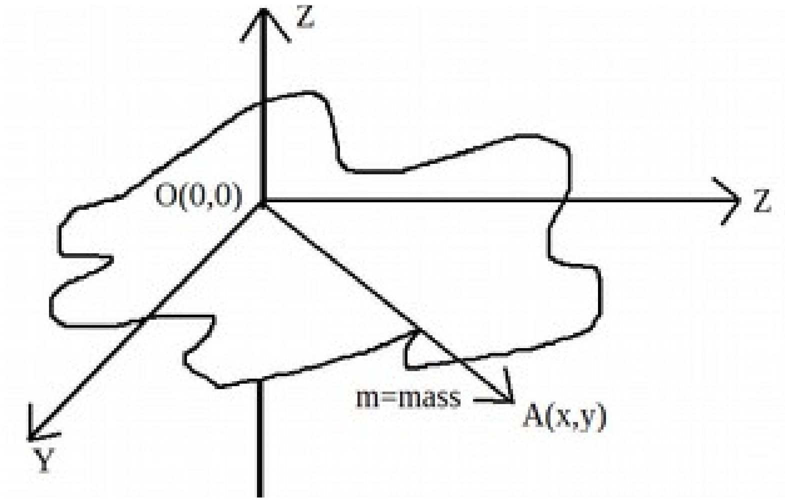
$$\Rightarrow I_Z = m\left(x^2+y^2\right) = mx^2 + my^2$$

$$\boxed{\Rightarrow I_Z = I_X + I_Y}$$

Where I_X=Moment of Inertia about x-axis.

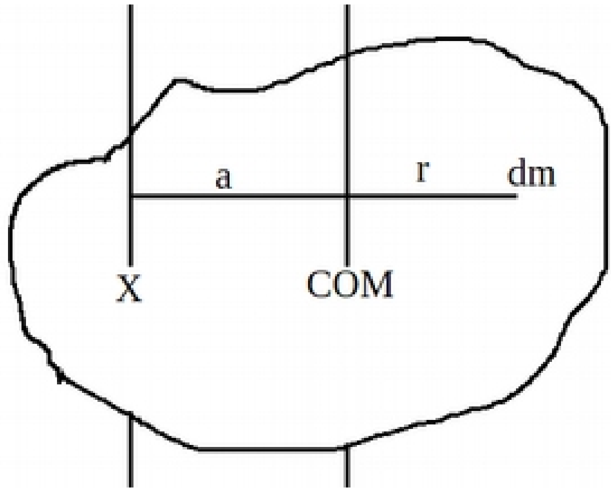
I_Y=Moment of Inertia about y-axis.

I_Z=Moment of Inertia about z-axis.



Parallel Axis Theorem

The theorem of parallel axes states that the moment of inertia of a rigid body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.



$$dI_{COM} = r^2\,dm \Rightarrow I_{COM} = \int r^2\,dm = Mr^2$$

$$dI = (r+a)^2\,dm$$

$$dI = dm\left(r^2 + a^2 + 2ar\right) = r^2\,dm + a^2\,dm + 2ar\,dm$$

$$\Rightarrow \int dI = \int r^2\,dm + \int a^2\,dm + \int 2ar\,dm$$

$$\Rightarrow I = Mr^2 + Ma^2 + 2a\int r\,dm$$

Now, $2a\int r\,dm = 2aM\dfrac{\int r\,dm}{M}$

Since we are taking COM as the origin,

$$\frac{\int r\,dm}{M} = 0 \Rightarrow 2a\int r\,dm = 0 \qquad Mr^2 = I_{COM}$$

$$\boxed{\therefore I = I_{COM} + Ma^2}$$

There are some necessary conditions for this theorem to work:-

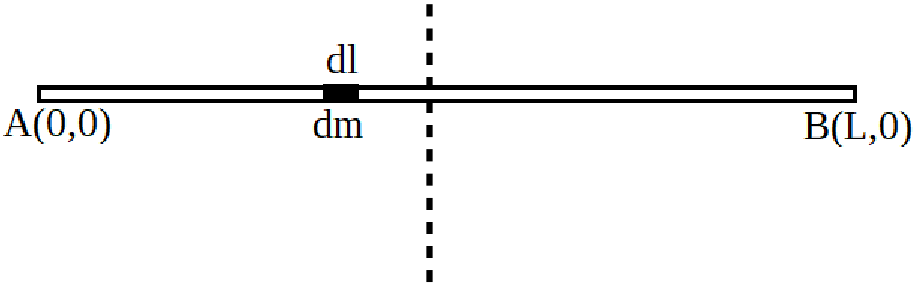
i. Both the axes must be parallel to each other.

ii. One of the axes must pass through COM.

iii. The distance 'a' is the perpendicular distance between the axes.

NOTE:- Parallel axis theorem can only be applied to 2D objects whereas Perpendicular axes theorem can be applied to all the bodies including 1D and 3D.

Moment of Inertia of a Horizontal ROD about axis passing through centre



Let σ be mass per unit length (M/L).

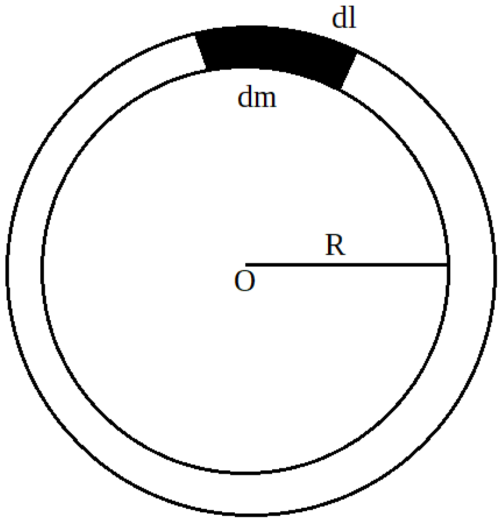
$$I=\int x^2 dm$$

Now, $dm=\sigma\, dl=\sigma\, dx$

$$\therefore I=2\int_0^{L/2}\sigma x^2 dx=2\sigma\left[\frac{x^3}{3}\right]_0^{L/2}$$

$$\Rightarrow I=2\frac{M}{L}\times\frac{L^3}{3\times 8}=\frac{ML^2}{12} \qquad \boxed{\therefore I=\frac{ML^2}{12}}$$

Moment of Inertia of a Circular RING about the center



$$I=\int x^2 dm$$

Now, $I=\int R^2 dm=R^2\int dm=MR^2$

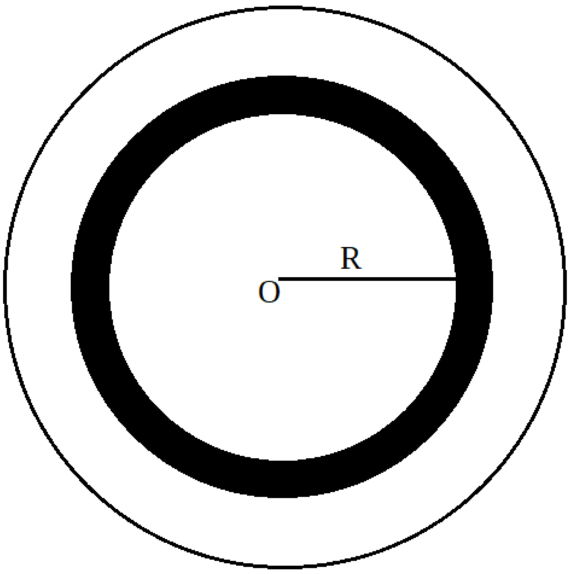
$$\boxed{\therefore I=MR^2}$$

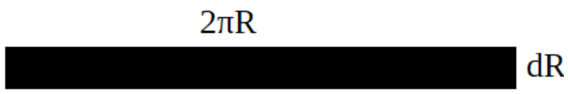
Applying Perpendicular Axis Theorem, we can also fin the Moment of Inertia about the axis passing through the centre on the plane of the ring.

$$I_X+I_Y=I_Z\Rightarrow 2I_X=MR^2$$

$$\boxed{\Rightarrow I_x=I_Y=\frac{MR^2}{2}}$$

Moment of Inertia of a Circular DISC about its center





$I=\int x^2 dm$ Let σ be the mass per unit area.

$$\therefore dm=\sigma\, 2\pi\, R dR$$

$$\therefore I=\int_0^R \sigma\, 2\pi\, R^3 dR=\sigma\, 2\pi\int_0^R R^3 dR$$

$$\Rightarrow I=\sigma\, 2\pi\left[\frac{R^4}{4}\right]_0^R=\frac{M}{\pi R^2}2\pi\times\frac{R^4}{4}$$

$$\boxed{\therefore I=\frac{MR^2}{2}}$$

We can apply the Perpendicular axes theorem here to fin I_X :– $I_X+I_Y=I_Z$

$$\Rightarrow 2I_X=\frac{MR^2}{2} \qquad \boxed{\therefore I_X=I_y=\frac{MR^2}{4}} \qquad \left[\text{Since } I_Z=\frac{MR^2}{2}\right]$$