Some Interesting Problems

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1 Equations with Complex Solutions

$$\sin(x)^{\sin(x)} = 2$$
$$\implies \sin(x)\ln(\sin(x)) = \ln(2)$$

Let's assume $\sin(x) = e^a$

$$\therefore ae^a = \ln(2)$$
$$\implies a = W(\ln(2))$$

W is the Lambert W function

$$\therefore \sin(x) = e^{W(\ln(2))}$$
$$\sin(x) = e^{W(0.6931)} = 1.5596 \approx 1.56$$

Clearly this has no real solution. But there might be a complex one:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = 1.56$$

Let's assume $u = e^{ix}$

$$\therefore u - u^{-1} = 3.12i$$
$$\implies u^2 - 3.12iu - 1 = 0$$

Taking $a = 1, b = -3.12i, c = -1$

$$\therefore u = \frac{3.12i \pm \sqrt{-9.7344 + 4}}{2}$$
$$\therefore u = \frac{3.12i \pm 2.4i}{2} = 1.56i \pm 1.2i$$
$$\therefore e^{ix} = 1.56i \pm 1.2i$$
$$ix = \ln(1.56i \pm 1.2i) = \ln(i) + \ln(1.56 \pm 1.2)$$

We know, $i = e^{i\pi/2}$

$$\therefore ix = \frac{\pi}{2}i + \ln(1.56 \pm 1.2)$$
$$\implies \boxed{x = \frac{\pi}{2} - \ln(1.56 \pm 1.2)i}$$

2 Feynman's Technique of Integration

2.1 Evaluate the following using Feynman's Technique

1. $\displaystyle\int_0^\infty e^{-x^2}\cos(5x)dx$

2. $\displaystyle\int_0^1 \frac{\sin(\ln(x))}{\ln(x)}dx$

2.2 Solution

1. Let $\mathrm{I}(\alpha) = \int\limits_0^\infty e^{-x^2}\cos(\alpha x)dx$

$$\implies \mathrm{I}'(\alpha) = \int\limits_0^\infty \frac{\delta}{\delta\alpha}\left(e^{-x^2}\cos(\alpha x)\right)dx = -\int\limits_0^\infty xe^{-x^2}\sin(\alpha x)\;dx$$

Integrating by parts

$$\mathrm{I}'(\alpha) = \left[\frac{\sin(\alpha x)}{2e^{x^2}}\right]_0^\infty - \frac{\alpha}{2}\int\limits_0^\infty e^{-x^2}\cos(\alpha x)dx \implies \mathrm{I}'(\alpha) = 0 - \frac{\alpha}{2}\mathrm{I}(\alpha)$$

$$\therefore \int \frac{d\left[\mathrm{I}(\alpha)\right]}{\mathrm{I}(\alpha)} = \frac{-1}{2}\int \alpha\;d\alpha \implies \ln(\mathrm{I}(\alpha)) = \frac{-\alpha^2}{4} + C' \implies \mathrm{I}(\alpha) = Ce^{-\alpha^2/4}$$

We know that $\mathrm{I}(0) = \int\limits_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ (The Gaussian Integral)

$$\therefore C = \frac{\sqrt{\pi}}{2} \;\therefore \mathrm{I}(\alpha) = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4} \text{ and } \mathrm{I}(5) = \boxed{\frac{\sqrt{\pi}}{2}e^{-25/4}}$$

2. Using the complex form of the sin function we get

$$\int\limits_0^1 \frac{\sin(\ln(x))}{\ln(x)}\;dx = \int\limits_0^1 \frac{x^i - x^{-i}}{2i\ln(x)}\;dx$$

Let $\mathrm{I}(b) = \int\limits_0^1 \frac{x^{bi} - x^{-i}}{2i\ln(x)}dx$

$$\therefore \mathrm{I}'(b) = \int\limits_0^1 \frac{\delta}{\delta b}\left(\frac{x^{bi} - x^{-i}}{2i\ln(x)}\right)dx = \int\limits_0^1 \frac{x^{bi}\cancel{i\ln(x)}}{2\cancel{i\ln(x)}}\;dx = \frac{1}{2(1+bi)}\left[x^{1+bi}\right]_0^1 = \frac{1}{2(1+bi)}$$

$$\therefore \mathrm{I}(b) = \frac{1}{2}\int \frac{db}{1+bi} = \frac{1}{2i}\ln(1+bi) + C$$

Let $b = -1$

$$\therefore \mathrm{I}(-1) = \int\limits_0^1 \frac{x^{-i} - x^{-i}}{2i\ln(x)}\;dx = 0$$

$$\therefore \frac{1}{2i}\ln(1-i) + C = 0 \implies C = -\frac{1}{2i}\ln(1-i)$$

$$\therefore \mathrm{I}(b) = \frac{\ln(1+bi) - \ln(1-i)}{2i} = \frac{1}{2i}\ln\left(\frac{1+bi}{1-i}\right)$$

$$\therefore \mathrm{I}(1) = \frac{1}{2i}\ln\left(\frac{1+i}{1-i}\right) = \frac{1}{2i}\ln\left(e^{i\pi/2}\right) = \boxed{\frac{\pi}{4}}$$