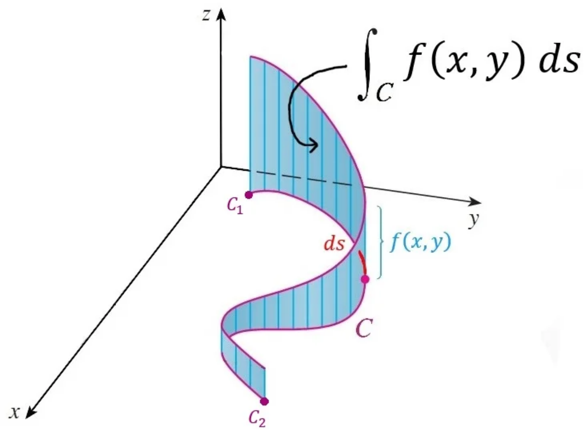
Line Integral and Linear Transformations

Saptarshi Dey

March 21, 2020

1 Theory

A line integral is a way of integrating a quantity along a curve instead of across an interval.



What we actually calculate

2 Line Integrals Examples

2.1 Questions

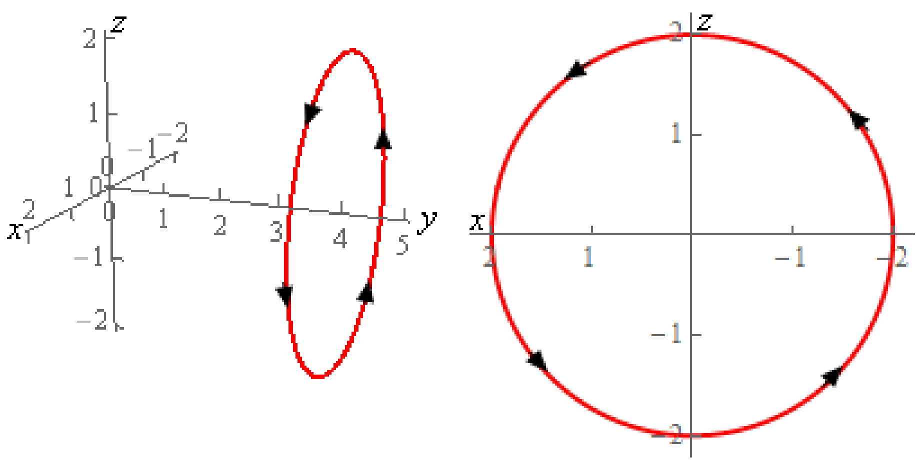
1. Evaluate $\int_C 3x^2 - 2y\,ds$ where C is the line segment from $(3,6)$ to $(1,-1)$

2. Evaluate $\int_C 2yx^2 - 4x\,ds$ where C is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.

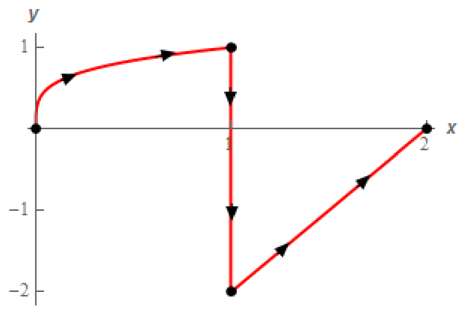
3. Evaluate $\int_C 6x\,ds$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of C is in the direction of increasing x.

4. Evaluate $\int_C xy - 4z\,ds$ where C is the line segment from $(1,1,0)$ to $(2,3,-2)$

5. Evaluate $\int_C x^2y^2\,ds$ where C is the circle centered at the origin of radius 2 centered on the y-axis at $y = 4$. See the sketches below for orientation.



6. Evaluate $\int\limits_C 16y^5\,ds$ where C is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment form $(1,1)$ to $(1,-2)$ which in turn is followed by the line segment from $(1,-2)$ to $(2,0)$. See the sketch below for the direction.



2.2 Solutions

1. Using the point-slope formula, we have $(y+1) = \frac{7}{2}(x-1)$

$\therefore 2y = 7x - 9 \implies 2dy = 7dx \implies \frac{dy}{dx} = \frac{7}{2}$

$\therefore ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{7}{2}\right)^2} = \frac{\sqrt{53}}{2}\,dx$

$\therefore \int\limits_C 3x^2 - 2y\,ds = \int\limits_1^3 \left(3x^2 - 7x + 9\right)\frac{\sqrt{53}}{2}\,dx = \frac{\sqrt{53}}{2}\left[x^3 - \frac{7x^2}{2} + 9x\right]_1^3 = \boxed{8\sqrt{53}}$

---

2. Let $x = 3\cos\theta$ and $y = 3\sin\theta$

$\therefore ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = 3\sqrt{\sin^2\theta + \cos^2\theta}d\theta = 3\,d\theta$

$\therefore \int\limits_C 2yx^2 - 4x\,ds = 3\int\limits_\pi^{2\pi}(54\sin\theta\cos^2\theta - 12\cos\theta)\,d\theta = -162\int\limits_{-1}^{1} u^2du + \cancel{12\,[\sin\theta]_\pi^{2\pi}} = \boxed{-108}$

---

3. $y = x^2 \implies \frac{dy}{dx} = 2x$

$\therefore ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + 4x^2}\,dx$

$\therefore \int\limits_C 6x\,ds = \int\limits_{-1}^{2} 6x\sqrt{1+4x^2}\,dx = \frac{3}{4}\int\limits_5^{17}\sqrt{u}\,du = \boxed{\frac{17^{1.5} - 5^{1.5}}{2}}$

---

4. First we have to calculate the direction cosines of the line.

$\therefore l = 1, m = 2, n = -2$

$\therefore$ The required equation of the line is $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-0}{-2}$

$\therefore z = -2x + 2$ and $y = 2x - 1$

$\implies \frac{dz}{dx} = -2$ and $\frac{dy}{dx} = 2$

$\therefore ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\,dx = \sqrt{1 + 2^2 + (-2)^2}dx = 3dx$

$$\therefore \int\limits_C xy - 4z\, ds = 3\int\limits_1^2 x(2x-1) + 8(x-1)dx = 3\left[\frac{x\cdot(4x^2+21x-48)}{6}\right]_1^2 = \boxed{\frac{43}{2}}$$

---

5. Let $x = 2\cos\theta$ and $z = 2\sin\theta$

$$\therefore ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} = 2\sqrt{\sin^2\theta + \cos^2\theta}d\theta = 2d\theta$$

$$\therefore \int\limits_C x^2y^2\, ds = 64\int\limits_0^{2\pi} 2\cos^2\theta d\theta = 64\left(\int\limits_0^{2\pi} d\theta + \int\limits_0^{2\pi}\cos 2\theta d\theta\right) = 128\pi + \cancel{32\left[\sin 2\theta\right]_0^{2\pi}} = \boxed{128\pi}$$

---

6. $\int\limits_C 16y^5\, ds = \int\limits_{C_1} 16y^5\, ds + \int\limits_{C_2} 16y^5\, ds + \int\limits_{C_3} 16y^5\, ds$

$C_1 \equiv x = y^4, \left[0 \le y \le 1\right], ds = \sqrt{1+(4y^3)^2}dy = \sqrt{1+16y^6}dy$

$C_2 \equiv x = 1, \left[-2 \le y \le 1\right], ds = \sqrt{1+0^2}dy = dy$

$C_3 \equiv x = \frac{y}{2} + 2, \left[-2 \le y \le 0\right], ds = \sqrt{1+\left(\frac{1}{2}\right)^2}dy = \frac{\sqrt{5}}{2}dy$

$$\int\limits_C 16y^5\, ds = \int\limits_0^1 16y^5\sqrt{1+16y^6}\, dy + 16\int\limits_{-2}^1 y^5\, dy + 8\sqrt{5}\int\limits_{-2}^0 y^5\, dy$$

$$= \frac{1}{6}\int\limits_1^{17}\sqrt{u}\, du + \frac{8}{3}\left[y^6\right]_{-2}^1 + \frac{4\sqrt{5}}{3}\left[y^6\right]_{-2}^0 = \frac{1}{9}\left[u^{1.5}\right]_1^{17} - 168 - \frac{256\sqrt{5}}{3}$$

$$= \frac{17^{1.5}-1}{9} - 168 - \frac{256\sqrt{5}}{3} = \boxed{-351.1341}$$

---

3 Rotation of Coordinate Systems

3.1 Rotation Matrices

The 2D Rotation Matrix is $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ where θ is the angle of rotation.

The 3D Rotation Matrices are:-

About X-Axis $R_x(\theta)$= $\begin{bmatrix}1 & 1 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$

About Y-Axis $R_y(\theta)$= $\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$

About Z-Axis $R_z(\theta)$= $\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

3.2 Example

If the Axes are rotated through 60° in the anticlockwise direction, find the transformed form of the equation $x^2 - y^2 = a^2$

Soln:- Let the transformed coordinates be X and Y.

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X - \sqrt{3}Y \\ \sqrt{3}X + Y \end{bmatrix}$$

Putting the values of x and y in the given equation we get

$$\left(X - \sqrt{3}Y\right)^2 - \left(\sqrt{3}X + Y\right)^2 = 4a^2$$

$$\implies 2Y^2 - 2X^2 - 4\sqrt{3}XY = 4a^2 \implies \boxed{Y^2 - X^2 - 2\sqrt{3}XY = 2a^2}$$

4 Change of Basis Vectors

Usually we represent vector as a linear combination of the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$. Hence this basis is written as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (the identity matrix). Let's understand this with an example.

Suppose we have a basis A given by the vectors $(2\hat{i} + 3\hat{j})$ and $(4\hat{i} - \hat{j})$ and we want to represent the vector $\vec{v} = (3\hat{i} + 2\hat{j})$ in basis A. Let $\vec{v}$ be represented as $< X, Y >$ in basis A.

$$\therefore X(2\hat{i} + 3\hat{j}) + Y(4\hat{i} - \hat{j}) = (3\hat{i} + 2\hat{j})$$

$$\implies (2X + 4Y)\hat{i} + (3X - Y)\hat{j} = (3\hat{i} + 2\hat{j})$$

This can also be written as

$$\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.79 \\ 0.36 \end{bmatrix}$$

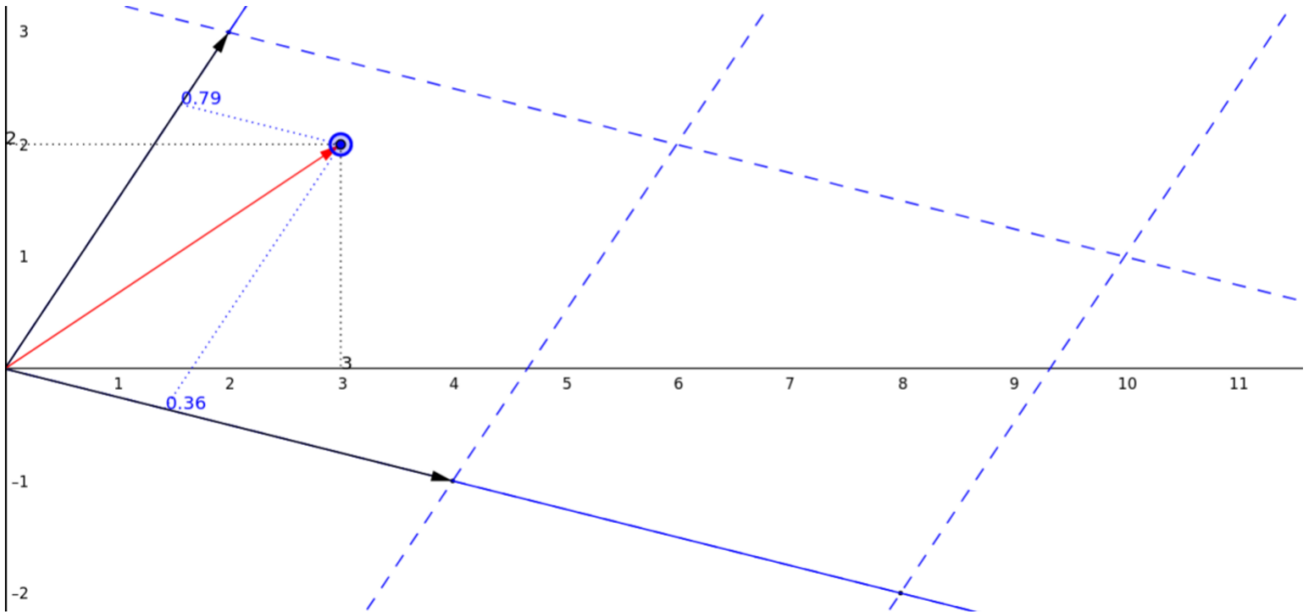


Figure 1: Representing vector $\vec{v}$ in basis A

If you want to play around with this tool (which I strongly suggest) then click here.

4.1 Questions

Consider the basis $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \right\}$ and $B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.Both basis are for $\mathbb{R}^3$

Find $P_{B \to A}$ (the matrix that maps a vector from basis B to A)

4.2 Solutions

Let vector $\vec{v}$ be $< a, b, c >$ in basis A and $< x, y, z >$ in basis B

$$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\implies \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ \frac{1}{3} & \frac{2}{3} & 4 \\ \frac{1}{3} & \frac{-1}{3} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore P_{B \to A} = \begin{bmatrix} 2 & 0 & -3 \\ \frac{1}{3} & \frac{2}{3} & 4 \\ \frac{1}{3} & \frac{-1}{3} & -1 \end{bmatrix}$$
