Lagrange Multipliers

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1 Vector calculus basics

1.1 Divergence

Measures how much a vector field is "spreading out" from a point. It takes a vector field $\vec{F}(x, y, z)$ as input and gives a scalar field as output.

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

1.2 Gradient

Gives the direction and rate of steepest increase of a scalar field. It takes a function $f(x, y, z)$ as input and gives a vector field as output.

$$\vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

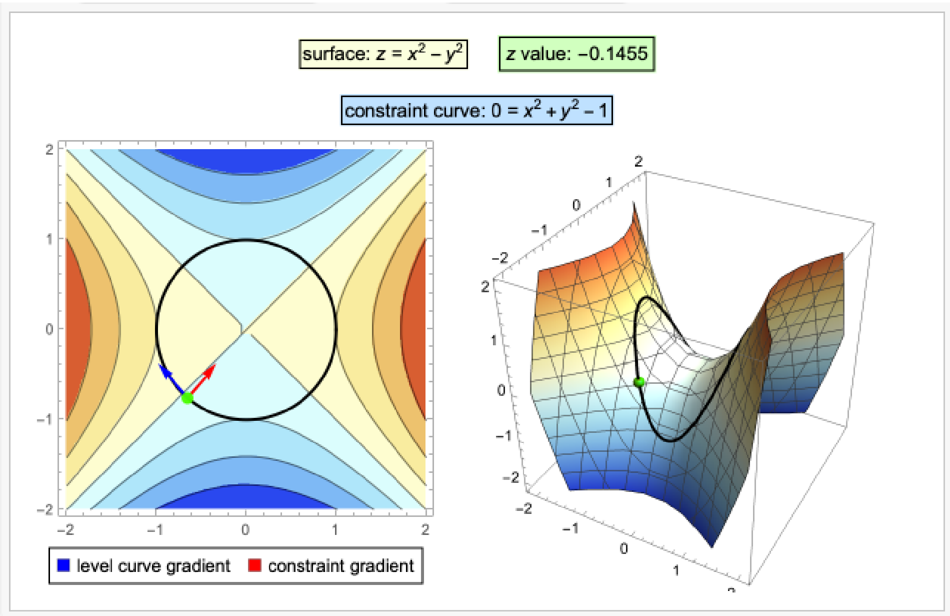
1.3 Curl

Measures the tendency of a vector field to rotate around a point. It takes a vector field $\vec{F}(x, y, z)$ as input and gives another vector field as output.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

2 Theory

Lagrange multipliers are used to find the maxima or minima of a function $f(x, y, z)$ subject to a constraint such as $g(x, y, z) = 0$.



Example illustration

Geometrically, at an extremum under a constraint, the surface (or level set) of f is just touching the constraint surface — meaning they are tangent to each other and do not cross. When two surfaces just touch, their normals are parallel. Since the gradient vector represents the normal direction of a level surface, this gives the essential condition:

$$\vec{\nabla} F = \lambda \vec{\nabla} g$$

This yields all candidate points where f may achieve a maximum or minimum while staying on the constraint curve.

3 Questions

1. Objective function $f(x, y) = 4xy$. Constraint: $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.
2. Objective function $f(x, y) = x^2 y$. Constraint: $x^2 + 2y^2 = 6$.
3. Objective function $f(x, y) = x^2 + y^2 + 2x - 2y + 1$. Constraint: $x^2 + y^2 = 2$.
4. Objective function $f(x, y) = xy$. Constraint: $x^2 + 2y^2 = 4$.
5. Objective function $f(x, y) = x^2 + y^2$. Constraint: $xy = 1$.
6. Objective function $f(x, y, z) = x + 3y - z$. Constraint: $x^2 + y^2 + z^2 = 4$.
7. Objective function $f(x, y, z) = xyz$. Constraint: $x^2 + 2y^2 + 3z^2 = 6$.

4 Solutions

1. $f(x, y) = 4xy$ and $g(x, y) = \dfrac{x^2}{9} + \dfrac{y^2}{16} - 1$
$\dfrac{\partial}{\partial x} f(x, y) = \lambda \dfrac{\partial}{\partial x} g(x, y) \implies 4y = \lambda \dfrac{2}{9} x \implies x = \dfrac{18y}{\lambda}$
$\dfrac{\partial}{\partial y} f(x, y) = \lambda \dfrac{\partial}{\partial y} g(x, y) \implies 4x = \dfrac{1}{8} x \implies x = \lambda \dfrac{y}{32}$
$\therefore \dfrac{18\cancel{y}}{\lambda} = \lambda \dfrac{\cancel{y}}{32} \implies \lambda^2 = 18 \times 32 \implies \lambda = \pm 24$
Putting the value of λ in any of the previous equations yields the relation: $x = \pm \dfrac{3}{4} y$.
Putting the value of x in $g(x, y)$ we get $\dfrac{\cancel{9}y^2}{\cancel{9} \times 16} + \dfrac{y^2}{16} = 1 \implies y^2 = 8 \implies y = \pm 2\sqrt{2}$.
$\therefore x = \pm \dfrac{3}{\cancel{2\sqrt{2}} \times \sqrt{2}} \times \cancel{2\sqrt{2}} = \pm \dfrac{3}{\sqrt{2}}$.
$\therefore$ We get 4 points: $\left(\dfrac{3}{\sqrt{2}}, 2\sqrt{2}\right)$, $\left(-\dfrac{3}{\sqrt{2}}, 2\sqrt{2}\right)$, $\left(\dfrac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ and $\left(-\dfrac{3}{\sqrt{2}}, -2\sqrt{2}\right)$. To get the critical points, we have to check which of these points correspond to those points where the gradient of $f(x, y)$ and $g(x, y)$ are parallel.

$$\vec{\nabla} f(x, y) = 4(y\hat{\imath} + x\hat{\jmath})$$
$$\vec{\nabla} g(x, y) = \frac{2}{9} x\hat{\imath} + \frac{1}{8} y\hat{\jmath}$$

at $\left(\dfrac{3}{\sqrt{2}}, 2\sqrt{2}\right)$, $\vec{\nabla} f(x, y) = 8\sqrt{2}\hat{\imath} + 6\sqrt{2}\hat{\jmath} = 2\sqrt{2}(4\hat{\imath} + 3\hat{\jmath})$ and $\vec{\nabla} g(x, y) = \dfrac{\sqrt{2}}{3}\hat{\imath} + \dfrac{1}{2\sqrt{2}}\hat{\jmath} = \dfrac{1}{6\sqrt{2}}(4\hat{\imath} + 3\hat{\jmath})$.
These 2 vectors are parallel. $\therefore \left(\dfrac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ is a critical point.

at $\left(-\dfrac{3}{\sqrt{2}}, 2\sqrt{2}\right)$, $\vec{\nabla} f(x, y) = 2\sqrt{2}(4\hat{\imath} - 3\hat{\jmath})$ and $\vec{\nabla} g(x, y) = -\dfrac{1}{6\sqrt{2}}(4\hat{\imath} - 3\hat{\jmath})$. These 2 vectors are parallel. $\therefore \left(-\dfrac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ is a critical point.

at $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$, $\vec{\nabla}f(x,y) = -2\sqrt{2}(4\hat{\imath} - 3\hat{\jmath})$ and $\vec{\nabla}g(x,y) = \frac{1}{6\sqrt{2}}(4\hat{\imath} - 3\hat{\jmath})$. These 2 vectors are parallel. $\therefore \left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ is a critical point.

at $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$, $\vec{\nabla}f(x,y) = -2\sqrt{2}(4\hat{\imath} + 3\hat{\jmath})$ and $\vec{\nabla}g(x,y) = -\frac{1}{6\sqrt{2}}(4\hat{\imath} + 3\hat{\jmath})$. These 2 vectors are parallel. $\therefore \left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ is a critical point.

$\therefore$ Max. $f(x,y) = 24$ at $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ and $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$

and Min. $f(x,y) = -24$ at $\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$

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2. $f(x,y) = x^2y$ and $g(x,y) = x^2 + 2y^2 - 6$

$\frac{\partial}{\partial x}f(x,y) = \lambda\frac{\partial}{\partial x}g(x,y) \implies \cancel{2x}y = \lambda\cancel{2x} \implies y = \lambda$

$\frac{\partial}{\partial y}f(x,y) = \lambda\frac{\partial}{\partial y}g(x,y) \implies x^2 = \lambda 4y \implies x^2 = 4\lambda^2 \implies x = \pm 2\lambda$

Substituting the values of x and y in $g(x,y)$ we get

$4\lambda^2 + 2\lambda^2 = 6 \implies \cancel{6}\lambda^2 = \cancel{6} \implies \lambda = \pm 1$

$\therefore y = \pm 1$ and $x = \pm 2$

The points are $(2,1), (-2,1), (2,-1)$ and $(-2,-1)$.

$$\vec{\nabla}f(x,y) = 2xy\hat{\imath} + x^2\hat{\jmath}$$

$$\vec{\nabla}g(x,y) = 2(x\hat{\imath} + 2y\hat{\jmath})$$

at $(2,1)$, $\vec{\nabla}f(x,y) = 4(\hat{\imath} + \hat{\jmath})$ and $\vec{\nabla}g(x,y) = 4(\hat{\imath} + \hat{\jmath})$. $\therefore (2,1)$ is a critical point.
at $(-2,1)$, $\vec{\nabla}f(x,y) = -4(\hat{\imath} - \hat{\jmath})$ and $\vec{\nabla}g(x,y) = -4(\hat{\imath} - \hat{\jmath})$. $\therefore (-2,1)$ is a critical point.
at $(2,-1)$, $\vec{\nabla}f(x,y) = -4(\hat{\imath} - \hat{\jmath})$ and $\vec{\nabla}g(x,y) = 4(\hat{\imath} - \hat{\jmath})$. $\therefore (2,-1)$ is a critical point.
at $(-2,-1)$, $\vec{\nabla}f(x,y) = 4(\hat{\imath} + \hat{\jmath})$ and $\vec{\nabla}g(x,y) = -4(\hat{\imath} + \hat{\jmath})$. $\therefore (-2,-1)$ is a critical point.

$\therefore$ Max. $f(x,y) = 4$ at $(2,1)$ and $(-2,1)$ and Min. $f(x,y) = -4$ at $(2,-1)$ and $(-2,-1)$

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3. $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ and $g(x,y) = x^2 + y^2 - 2$

$\frac{\partial}{\partial x}f(x,y) = \lambda\frac{\partial}{\partial x}g(x,y) \implies \cancel{2}(x+1) = \lambda\cancel{2}x \implies x = \frac{1}{\lambda - 1}$

$\frac{\partial}{\partial y}f(x,y) = \lambda\frac{\partial}{\partial y}g(x,y) \implies \cancel{2}(y-1) = \lambda\cancel{2}y \implies y = \frac{-1}{\lambda - 1} = -x$

$\therefore$ Putting $y = -x$ in $g(x,y)$ we get

$\cancel{2}y^2 = \cancel{2} \implies y = \pm 1$ and $x = \mp 1$. $\therefore$ The points are $(1,-1)$ and $(-1,1)$.

$$\vec{\nabla}f(x,y) = 2\{(x+1)\hat{\imath} + (y-1)\hat{\jmath}\}$$

$$\vec{\nabla}g(x,y) = 2(x\hat{\imath} + y\hat{\jmath})$$

at $(1,-1)$, $\vec{\nabla}f(x,y) = 4(\hat{\imath} - \hat{\jmath})$ and $\vec{\nabla}g(x,y) = 2(\hat{\imath} - \hat{\jmath})$. $\therefore (1,-1)$ is a critical point.
at $(-1,1)$, $\vec{\nabla}f(x,y) = 0$ and a null vector is mathematically parallel to any vector.
So we need not calculate $\vec{\nabla}g(x,y)$. $\therefore (-1,1)$ is a critical point.

$\therefore$ Max. $f(x,y) = 7$ at $(1,-1)$ and Min. $f(x,y) = -1$ at $(-1,1)$

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4. $f(x,y)=xy$ and $g(x,y)=x^2+2y^2-4$
$\frac{\partial}{\partial x}f(x,y)=\lambda\frac{\partial}{\partial x}g(x,y) \implies y=\lambda 2x$
$\frac{\partial}{\partial y}f(x,y)=\lambda\frac{\partial}{\partial y}g(x,y) \implies x=\lambda 4y$
Substituting $x=\lambda 4y$ in $y=\lambda 2x$, we get $\cancel{y}=8\lambda^2\cancel{y} \implies \lambda=\pm\frac{1}{2\sqrt{2}}$
$\therefore y=\pm\frac{x}{\sqrt{2}} \implies y^2=\frac{x^2}{2} \implies 2y^2=x^2$
Substituting the value of $2y^2$ in $g(x,y)$ we get $2x^2=4 \implies x=\pm\sqrt{2}$ and $y=\pm 1$
$\therefore$ The points are $(\sqrt{2},1),(-\sqrt{2},1),(\sqrt{2},-1)$ and $(-\sqrt{2},-1)$

$$\vec{\nabla}f(x,y)=y\hat{\imath}+x\hat{\jmath}$$

$$\vec{\nabla}g(x,y)=2(x\hat{\imath}+2y\hat{\jmath})$$

at $(\sqrt{2},1)$, $\vec{\nabla}f(x,y)=\hat{\imath}+\sqrt{2}\hat{\jmath}$ and $\vec{\nabla}g(x,y)=2\sqrt{2}(\hat{\imath}+\sqrt{2}\hat{\jmath})$. $\therefore(\sqrt{2},1)$ is a critical point.
at $(-\sqrt{2},1)$, $\vec{\nabla}f(x,y)=\hat{\imath}-\sqrt{2}\hat{\jmath}$ and $\vec{\nabla}g(x,y)=-2\sqrt{2}(\hat{\imath}-\sqrt{2}\hat{\jmath})$. $\therefore(-\sqrt{2},1)$ is a critical point.
at $(\sqrt{2},-1)$, $\vec{\nabla}f(x,y)=-\hat{\imath}+\sqrt{2}\hat{\jmath}$ and $\vec{\nabla}g(x,y)=-2\sqrt{2}(-\hat{\imath}+\sqrt{2}\hat{\jmath})$. $\therefore(\sqrt{2},-1)$ is a critical point.
at $(-\sqrt{2},-1)$, $\vec{\nabla}f(x,y)=-(\hat{\imath}+\sqrt{2}\hat{\jmath})$ and $\vec{\nabla}g(x,y)=-2\sqrt{2}(\hat{\imath}+\sqrt{2}\hat{\jmath})$. $\therefore(-\sqrt{2},-1)$ is a critical point.

$\therefore$ Max. $f(x,y)=2\sqrt{2}$ at $(\sqrt{2},1)$ and $(-\sqrt{2},-1)$
and Min. $f(x,y)=-2\sqrt{2}$ at $(-\sqrt{2},1)$ and $(\sqrt{2},-1)$

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5. $f(x,y)=x^2+y^2$ and $g(x,y)=xy-1$
$\frac{\partial}{\partial x}f(x,y)=\lambda\frac{\partial}{\partial x}g(x,y) \implies 2x=\lambda y \implies \lambda=\frac{2x}{y}$
$\frac{\partial}{\partial y}f(x,y)=\lambda\frac{\partial}{\partial y}g(x,y) \implies 2y=\lambda x \implies \lambda=\frac{2y}{x}$
From these 2 equations we get $\frac{\cancel{2}x}{y}=\frac{\cancel{2}y}{x} \implies x^2=y^2 \implies x=y$
Using this relation in $g(x,y)$ we get $x=y=\pm 1$. $\therefore$ The critical points are (1,1) and (-1,-1).

$$\vec{\nabla}f(x,y)=2(x\hat{\imath}+y\hat{\jmath})$$
$$\vec{\nabla}g(x,y)=y\hat{\imath}+x\hat{\jmath}$$

at $(1,1)$, $\vec{\nabla}f(x,y)=2(\hat{\imath}+\hat{\jmath})$ and $\vec{\nabla}g(x,y)=\hat{\imath}+\hat{\jmath}$. $\therefore(1,1)$ is a critical point.
at $(-1,-1)$, $\vec{\nabla}f(x,y)=-2(\hat{\imath}+\hat{\jmath})$ and $\vec{\nabla}g(x,y)=-(\hat{\imath}+\hat{\jmath})$. $\therefore(-1,-1)$ is a critical point.

$\therefore$ Max. $f(x,y)=2$ at $(1,1)$ and $(-1,-1)$.

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6. $f(x,y,z)=x+3y-z$ and $g(x,y,z)=x^2+y^2+z^2-4$.
$\frac{\partial}{\partial x}f(x,y,z)=\lambda\frac{\partial}{\partial x}g(x,y,z) \implies 1=\lambda 2x \implies x=\frac{1}{2\lambda}$
$\frac{\partial}{\partial y}f(x,y,z)=\lambda\frac{\partial}{\partial y}g(x,y,z) \implies 3=\lambda 2y \implies y=\frac{3}{2\lambda}=3x$
$\frac{\partial}{\partial z}f(x,y,z)=\lambda\frac{\partial}{\partial z}g(x,y,z) \implies -1=\lambda 2z \implies z=-\frac{1}{2\lambda}=-x$
Substituting $y=3x$, and $z=-x$ in $g(x,y,z)$ we get
$x^2+9x^2+x^2=4 \implies 11x^2=4 \implies x=\pm\frac{2}{\sqrt{11}}$, $y=\pm\frac{6}{\sqrt{11}}$ and $z=\mp\frac{2}{\sqrt{11}}$

$\therefore$ The points are $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

$$\vec{\nabla} f(x,y) = \hat{\imath} + 3\hat{\jmath} - \hat{k}$$

$$\vec{\nabla} g(x,y) = 2(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

at $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$, $\vec{\nabla} g(x,y) = \frac{4}{\sqrt{11}}(\hat{\imath} + 3\hat{\jmath} - \hat{k})$. $\therefore$ $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ is a critical point.
at $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$, $\vec{\nabla} g(x,y) = -\frac{4}{\sqrt{11}}(\hat{\imath}+3\hat{\jmath}-\hat{k})$. $\therefore$ $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ is a critical point.

$\therefore$ Max. $f(x,y,z) = 2\sqrt{11}$ at $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and Min. $f(x,y,z) = -2\sqrt{11}$ at $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

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7. $f(x,y,z) = xyz$ and $g(x,y,z) = x^2 + 2y^2 + 3z^2 - 6$.
$\frac{\partial}{\partial x} f(x,y,z) = \lambda \frac{\partial}{\partial x} g(x,y,z) \implies yz = \lambda 2x \implies \lambda = \frac{yz}{2x}$
$\frac{\partial}{\partial y} f(x,y,z) = \lambda \frac{\partial}{\partial y} g(x,y,z) \implies xz = \lambda 4y \implies \lambda = \frac{xz}{4y}$
$\frac{\partial}{\partial z} f(x,y,z) = \lambda \frac{\partial}{\partial z} g(x,y,z) \implies xy = \lambda 6z \implies \lambda = \frac{xy}{6z}$
From the first 2 relations we get $\frac{y\cancel{z}}{2x} = \frac{x\cancel{z}}{4y} \implies x^2 = 2y^2 \implies x = \pm\sqrt{2}y$
From the second and third relations we get $\frac{\cancel{x}z}{4y} = \frac{\cancel{x}y}{6z} \implies 3z^2 = 2y^2 \implies z = \pm\sqrt{\frac{2}{3}}y$
Using these relations in $g(x,y)$ we get $6y^2 = 6 \implies y = \pm 1$, $x = \pm\sqrt{2}$ and $z = \pm\frac{2}{3}$.
$\therefore$ The points are $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ and $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$.

$$\vec{\nabla} f(x,y) = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$$

$$\vec{\nabla} g(x,y) = 2(x\hat{\imath} + 2y\hat{\jmath} + 3z\hat{k})$$

at $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x,y) = \sqrt{\frac{2}{3}}(\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x,y) = 2\sqrt{2}(\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$. $\therefore$ $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x,y) = -\sqrt{\frac{2}{3}}(\hat{\imath} + \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x,y) = 2\sqrt{2}(\hat{\imath} + \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$. $\therefore$ $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x,y) = -\sqrt{\frac{2}{3}}(\hat{\imath} - \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x,y) = 2\sqrt{2}(\hat{\imath} - \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$. $\therefore$ $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{\imath} - \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{\imath} - \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$.

$\therefore \left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{\imath} - \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{\imath} - \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{\imath} - \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{\imath} - \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{\imath} + \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{\imath} + \sqrt{2}\hat{\jmath} - \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ is a critical point.

$\therefore$ Min. $f(x, y, z) = -\frac{2}{\sqrt{3}}$ at $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ and $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$.

and Max. $f(x, y, z) = \frac{2}{\sqrt{3}}$ at $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ and $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$.
