

Lagrange Multipliers

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1 Vector calculus basics

1.1 Divergence

Measures how much a vector field is "spreading out" from a point. It takes a vector field $\vec{F}(x, y, z)$ as input and gives a scalar field as output.

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

1.2 Gradient

Gives the direction and rate of steepest increase of a scalar field. It takes a function $f(x, y, z)$ as input and gives a vector field as output.

$$\vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

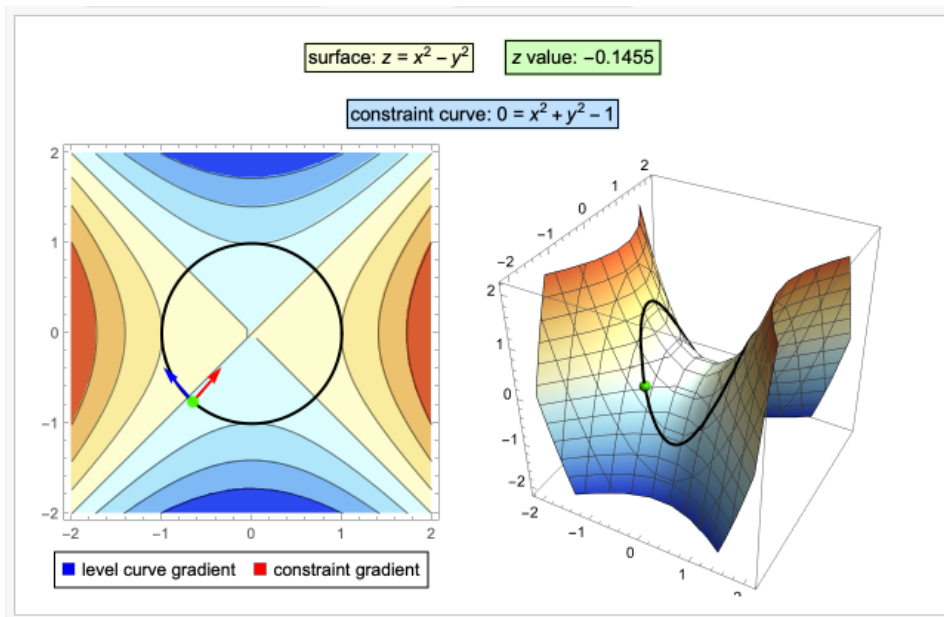
1.3 Curl

Measures the tendency of a vector field to rotate around a point. It takes a vector field $\vec{F}(x, y, z)$ as input and gives another vector field as output.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

2 Theory

Lagrange multipliers are used to find the maxima or minima of a function $f(x, y, z)$ subject to a constraint such as $g(x, y, z) = 0$.



Example illustration

Geometrically, at an extremum under a constraint, the surface (or level set) of f is just touching the constraint surface — meaning they are tangent to each other and do not cross. When two surfaces just touch, their normals are parallel. Since the gradient vector represents the normal direction of a level surface, this gives the essential condition:

$$\vec{\nabla}F = \lambda \vec{\nabla}g$$

This yields all candidate points where f may achieve a maximum or minimum while staying on the constraint curve.

3 Questions

1. Objective function $f(x, y) = 4xy$. Constraint: $\frac{x^2}{9} + \frac{y^2}{16} = 1$.
2. Objective function $f(x, y) = x^2y$. Constraint: $x^2 + 2y^2 = 6$.
3. Objective function $f(x, y) = x^2 + y^2 + 2x - 2y + 1$. Constraint: $x^2 + y^2 = 2$.
4. Objective function $f(x, y) = xy$. Constraint: $x^2 + 2y^2 = 4$.
5. Objective function $f(x, y) = x^2 + y^2$. Constraint: $xy = 1$.
6. Objective function $f(x, y, z) = x + 3y - z$. Constraint: $x^2 + y^2 + z^2 = 4$.
7. Objective function $f(x, y, z) = xyz$. Constraint: $x^2 + 2y^2 + 3z^2 = 6$.

4 Solutions

$$\begin{aligned} 1. \quad & f(x, y) = 4xy \text{ and } g(x, y) = \frac{x^2}{9} + \frac{y^2}{16} - 1 \\ & \frac{\partial}{\partial x}f(x, y) = \lambda \frac{\partial}{\partial x}g(x, y) \implies 4y = \lambda \frac{2}{9}x \implies x = \frac{18y}{\lambda} \\ & \frac{\partial}{\partial y}f(x, y) = \lambda \frac{\partial}{\partial y}g(x, y) \implies 4x = \lambda \frac{1}{8}y \implies x = \lambda \frac{y}{32} \\ \therefore & \frac{18y}{\lambda} = \lambda \frac{y}{32} \implies \lambda^2 = 18 \times 32 \implies \lambda = \pm 24 \end{aligned}$$

Putting the value of λ in any of the previous equations yields the relation: $x = \pm \frac{3}{4}y$.

Putting the value of x in $g(x, y)$ we get $\frac{9y^2}{9 \times 16} + \frac{y^2}{16} = 1 \implies y^2 = 8 \implies y = \pm 2\sqrt{2}$.

$$\therefore x = \pm \frac{3}{2\sqrt{2} \times \sqrt{2}} \times 2\sqrt{2} = \pm \frac{3}{\sqrt{2}}.$$

\therefore We get 4 points: $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right), \left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right), \left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ and $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$. To get the critical points, we have to check which of these points correspond to those points where the gradient of $f(x, y)$ and $g(x, y)$ are parallel.

$$\vec{\nabla}f(x, y) = 4(y\hat{i} + x\hat{j})$$

$$\vec{\nabla}g(x, y) = \frac{2}{9}x\hat{i} + \frac{1}{8}y\hat{j}$$

at $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$, $\vec{\nabla}f(x, y) = 8\sqrt{2}\hat{i} + 6\sqrt{2}\hat{j} = 2\sqrt{2}(4\hat{i} + 3\hat{j})$ and $\vec{\nabla}g(x, y) = \frac{\sqrt{2}}{3}\hat{i} + \frac{1}{2\sqrt{2}}\hat{j} = \frac{1}{6\sqrt{2}}(4\hat{i} + 3\hat{j})$.

These 2 vectors are parallel. $\therefore \left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ is a critical point.

at $\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$, $\vec{\nabla}f(x, y) = 2\sqrt{2}(4\hat{i} - 3\hat{j})$ and $\vec{\nabla}g(x, y) = -\frac{1}{6\sqrt{2}}(4\hat{i} - 3\hat{j})$. These 2 vectors are parallel. $\therefore \left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ is a critical point.

at $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$, $\vec{\nabla}f(x, y) = -2\sqrt{2}(4\hat{i} - 3\hat{j})$ and $\vec{\nabla}g(x, y) = \frac{1}{6\sqrt{2}}(4\hat{i} - 3\hat{j})$. These 2 vectors are parallel. $\therefore \left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ is a critical point.

at $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$, $\vec{\nabla}f(x, y) = -2\sqrt{2}(4\hat{i} + 3\hat{j})$ and $\vec{\nabla}g(x, y) = -\frac{1}{6\sqrt{2}}(4\hat{i} + 3\hat{j})$. These 2 vectors are parallel. $\therefore \left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ is a critical point.

\therefore Max. $f(x, y) = 24$ at $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ and $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$
and Min. $f(x, y) = -24$ at $\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$

2. $f(x, y) = x^2y$ and $g(x, y) = x^2 + 2y^2 - 6$

$$\frac{\partial}{\partial x}f(x, y) = \lambda \frac{\partial}{\partial x}g(x, y) \implies 2xy = \lambda 2x \implies y = \lambda$$

$$\frac{\partial}{\partial y}f(x, y) = \lambda \frac{\partial}{\partial y}g(x, y) \implies x^2 = \lambda 4y \implies x^2 = 4\lambda^2 \implies x = \pm 2\lambda$$

Substituting the values of x and y in $g(x, y)$ we get

$$4\lambda^2 + 2\lambda^2 = 6 \implies 6\lambda^2 = 6 \implies \lambda = \pm 1$$

$\therefore y = \pm 1$ and $x = \pm 2$

The points are $(2, 1), (-2, 1), (2, -1)$ and $(-2, -1)$.

$$\vec{\nabla}f(x, y) = 2xy\hat{i} + x^2\hat{j}$$

$$\vec{\nabla}g(x, y) = 2(x\hat{i} + 2y\hat{j})$$

at $(2, 1)$, $\vec{\nabla}f(x, y) = 4(\hat{i} + \hat{j})$ and $\vec{\nabla}g(x, y) = 4(\hat{i} + \hat{j})$. $\therefore (2, 1)$ is a critical point.

at $(-2, 1)$, $\vec{\nabla}f(x, y) = -4(\hat{i} - \hat{j})$ and $\vec{\nabla}g(x, y) = -4(\hat{i} - \hat{j})$. $\therefore (-2, 1)$ is a critical point.

at $(2, -1)$, $\vec{\nabla}f(x, y) = -4(\hat{i} - \hat{j})$ and $\vec{\nabla}g(x, y) = 4(\hat{i} - \hat{j})$. $\therefore (2, -1)$ is a critical point.

at $(-2, -1)$, $\vec{\nabla}f(x, y) = 4(\hat{i} + \hat{j})$ and $\vec{\nabla}g(x, y) = -4(\hat{i} + \hat{j})$. $\therefore (-2, -1)$ is a critical point.

\therefore Max. $f(x, y) = 4$ at $(2, 1)$ and $(-2, 1)$ and Min. $f(x, y) = -4$ at $(2, -1)$ and $(-2, -1)$

3. $f(x, y) = x^2 + y^2 + 2x - 2y + 1$ and $g(x, y) = x^2 + y^2 - 2$

$$\frac{\partial}{\partial x}f(x, y) = \lambda \frac{\partial}{\partial x}g(x, y) \implies 2(x + 1) = \lambda 2x \implies x = \frac{1}{\lambda - 1}$$

$$\frac{\partial}{\partial y}f(x, y) = \lambda \frac{\partial}{\partial y}g(x, y) \implies 2(y - 1) = \lambda 2y \implies y = \frac{-1}{\lambda - 1} = -x$$

\therefore Putting $y = -x$ in $g(x, y)$ we get

$$2y^2 = 2 \implies y = \pm 1 \text{ and } x = \mp 1. \therefore \text{The points are } (1, -1) \text{ and } (-1, 1).$$

$$\vec{\nabla}f(x, y) = 2\{(x + 1)\hat{i} + (y - 1)\hat{j}\}$$

$$\vec{\nabla}g(x, y) = 2(x\hat{i} + y\hat{j})$$

at $(1, -1)$, $\vec{\nabla}f(x, y) = 4(\hat{i} - \hat{j})$ and $\vec{\nabla}g(x, y) = 2(\hat{i} - \hat{j})$. $\therefore (1, -1)$ is a critical point.

at $(-1, 1)$, $\vec{\nabla}f(x, y) = 0$ and a null vector is mathematically parallel to any vector.

So we need not calculate $\vec{\nabla}g(x, y)$. $\therefore (-1, 1)$ is a critical point.

\therefore Max. $f(x, y) = 7$ at $(1, -1)$ and Min. $f(x, y) = -1$ at $(-1, 1)$

$$4. f(x, y) = xy \text{ and } g(x, y) = x^2 + 2y^2 - 4$$

$$\frac{\partial}{\partial x} f(x, y) = \lambda \frac{\partial}{\partial x} g(x, y) \implies y = \lambda 2x$$

$$\frac{\partial}{\partial y} f(x, y) = \lambda \frac{\partial}{\partial y} g(x, y) \implies x = \lambda 4y$$

$$\text{Substituting } x = \lambda 4y \text{ in } y = \lambda 2x, \text{ we get } y = 8\lambda^2 y \implies \lambda = \pm \frac{1}{2\sqrt{2}}$$

$$\therefore y = \pm \frac{x}{\sqrt{2}} \implies y^2 = \frac{x^2}{2} \implies 2y^2 = x^2$$

$$\text{Substituting the value of } 2y^2 \text{ in } g(x, y) \text{ we get } 2x^2 = 4 \implies x = \pm\sqrt{2} \text{ and } y = \pm 1$$

$$\therefore \text{The points are } (\sqrt{2}, 1), (-\sqrt{2}, 1), (\sqrt{2}, -1) \text{ and } (-\sqrt{2}, -1)$$

$$\vec{\nabla} f(x, y) = y\hat{i} + x\hat{j}$$

$$\vec{\nabla} g(x, y) = 2(x\hat{i} + 2y\hat{j})$$

$$\text{at } (\sqrt{2}, 1), \vec{\nabla} f(x, y) = \hat{i} + \sqrt{2}\hat{j} \text{ and } \vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j}). \therefore (\sqrt{2}, 1) \text{ is a critical point.}$$

$$\text{at } (-\sqrt{2}, 1), \vec{\nabla} f(x, y) = \hat{i} - \sqrt{2}\hat{j} \text{ and } \vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j}). \therefore (-\sqrt{2}, 1) \text{ is a critical point.}$$

$$\text{at } (\sqrt{2}, -1), \vec{\nabla} f(x, y) = -\hat{i} + \sqrt{2}\hat{j} \text{ and } \vec{\nabla} g(x, y) = -2\sqrt{2}(-\hat{i} + \sqrt{2}\hat{j}). \therefore (\sqrt{2}, -1) \text{ is a critical point.}$$

$$\text{at } (-\sqrt{2}, -1), \vec{\nabla} f(x, y) = -(\hat{i} + \sqrt{2}\hat{j}) \text{ and } \vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j}). \therefore (-\sqrt{2}, -1) \text{ is a critical point.}$$

$$\therefore \text{Max. } f(x, y) = 2\sqrt{2} \text{ at } (\sqrt{2}, 1) \text{ and } (-\sqrt{2}, -1)$$

$$\text{and Min. } f(x, y) = -2\sqrt{2} \text{ at } (-\sqrt{2}, 1) \text{ and } (\sqrt{2}, -1)$$

$$5. f(x, y) = x^2 + y^2 \text{ and } g(x, y) = xy - 1$$

$$\frac{\partial}{\partial x} f(x, y) = \lambda \frac{\partial}{\partial x} g(x, y) \implies 2x = \lambda y \implies \lambda = \frac{2x}{y}$$

$$\frac{\partial}{\partial y} f(x, y) = \lambda \frac{\partial}{\partial y} g(x, y) \implies 2y = \lambda x \implies \lambda = \frac{2y}{x}$$

$$\text{From these 2 equations we get } \frac{2x}{y} = \frac{2y}{x} \implies x^2 = y^2 \implies x = y$$

$$\text{Using this relation in } g(x, y) \text{ we get } x = y = \pm 1$$

$$\vec{\nabla} f(x, y) = 2(x\hat{i} + y\hat{j})$$

$$\vec{\nabla} g(x, y) = y\hat{i} + x\hat{j}$$

$$\text{at } (1, 1), \vec{\nabla} f(x, y) = 2(\hat{i} + \hat{j}) \text{ and } \vec{\nabla} g(x, y) = \hat{i} + \hat{j}. \therefore (1, 1) \text{ is a critical point.}$$

$$\text{at } (-1, 1), \vec{\nabla} f(x, y) = 2(-\hat{i} + \hat{j}) \text{ and } \vec{\nabla} g(x, y) = \hat{i} - \hat{j}. \therefore (-1, 1) \text{ is not a critical point.}$$

$$\text{at } (1, -1), \vec{\nabla} f(x, y) = 2(\hat{i} - \hat{j}) \text{ and } \vec{\nabla} g(x, y) = -\hat{i} + \hat{j}. \therefore (1, -1) \text{ is not a critical point.}$$

$$\text{at } (-1, -1), \vec{\nabla} f(x, y) = -2(\hat{i} + \hat{j}) \text{ and } \vec{\nabla} g(x, y) = -(\hat{i} + \hat{j}). \therefore (-1, -1) \text{ is a critical point.}$$

$$\therefore \text{Max. } f(x, y) = 2 \text{ at } (1, 1) \text{ and } (-1, -1).$$

$$6. f(x, y, z) = x + 3y - z \text{ and } g(x, y, z) = x^2 + y^2 + z^2 - 4.$$

$$\frac{\partial}{\partial x} f(x, y, z) = \lambda \frac{\partial}{\partial x} g(x, y, z) \implies 1 = \lambda 2x \implies x = \frac{1}{2\lambda}$$

$$\frac{\partial}{\partial y} f(x, y, z) = \lambda \frac{\partial}{\partial y} g(x, y, z) \implies 3 = \lambda 2y \implies y = \frac{3}{2\lambda} = 3x$$

$$\frac{\partial}{\partial z} f(x, y, z) = \lambda \frac{\partial}{\partial z} g(x, y, z) \implies -1 = \lambda 2z \implies z = -\frac{1}{2\lambda} = -x$$

Substituting $y = 3x$, and $z = -x$ in $g(x, y, z)$ we get

$$x^2 + 9x^2 + x^2 = 4 \implies 11x^2 = 4 \implies x = \pm \frac{2}{\sqrt{11}}, y = \pm \frac{6}{\sqrt{11}} \text{ and } z = \mp \frac{2}{\sqrt{11}}$$

\therefore The points are $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

$$\vec{\nabla} f(x, y) = \hat{i} + 3\hat{j} - \hat{k}$$

$$\vec{\nabla} g(x, y) = 2(x\hat{i} + y\hat{j} + z\hat{k})$$

at $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$, $\vec{\nabla} g(x, y) = \frac{4}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$. $\therefore \left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ is a critical point.

at $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$, $\vec{\nabla} g(x, y) = -\frac{4}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$. $\therefore \left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ is a critical point.

\therefore Max. $f(x, y, z) = 2\sqrt{11}$ at $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and Min. $f(x, y, z) = -2\sqrt{11}$ at $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

7. $f(x, y, z) = xyz$ and $g(x, y, z) = x^2 + 2y^2 + 3z^2 - 6$.

$$\frac{\partial}{\partial x} f(x, y, z) = \lambda \frac{\partial}{\partial x} g(x, y, z) \implies yz = \lambda 2x \implies \lambda = \frac{yz}{2x}$$

$$\frac{\partial}{\partial y} f(x, y, z) = \lambda \frac{\partial}{\partial y} g(x, y, z) \implies xz = \lambda 4y \implies \lambda = \frac{xz}{4y}$$

$$\frac{\partial}{\partial z} f(x, y, z) = \lambda \frac{\partial}{\partial z} g(x, y, z) \implies xy = \lambda 6z \implies \lambda = \frac{xy}{6z}$$

From the first 2 relations we get $\frac{yz}{2x} = \frac{xz}{4y} \implies x^2 = 2y^2 \implies x = \pm\sqrt{2}y$

From the second and third relations we get $\frac{xz}{4y} = \frac{xy}{6z} \implies 3z^2 = 2y^2 \implies z = \pm\sqrt{\frac{2}{3}}y$

Using these relations in $g(x, y)$ we get $6y^2 = 6 \implies y = \pm 1$, $x = \pm\sqrt{2}$ and $z = \pm\sqrt{\frac{2}{3}}$.

\therefore The points are $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ and $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$.

$$\vec{\nabla} f(x, y) = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\vec{\nabla} g(x, y) = 2(x\hat{i} + 2y\hat{j} + 3z\hat{k})$$

at $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$. \therefore

$\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$.

$\therefore \left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ is a critical point.

at $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$, $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$.

$\therefore \left(\sqrt{2}, -1, \sqrt{\frac{2}{3}} \right)$ is a critical point.

at $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}} \right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$.

$\therefore \left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}} \right)$ is a critical point.

at $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}} \right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}} \right)$ is a critical point.

at $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}} \right)$, $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}} \right)$ is a critical point.

at $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}} \right)$, $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}} \right)$ is a critical point.

at $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}} \right)$, $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ and $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$.

$\therefore \left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}} \right)$ is a critical point.

$\therefore \text{Min. } f(x, y, z) = -\frac{2}{\sqrt{3}} \text{ at } \left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}} \right), \left(\sqrt{2}, -1, \sqrt{\frac{2}{3}} \right), \left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}} \right) \text{ and } \left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}} \right).$

and $\text{Max. } f(x, y, z) = \frac{2}{\sqrt{3}} \text{ at } \left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}} \right), \left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}} \right), \left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}} \right) \text{ and } \left(\sqrt{2}, 1, \sqrt{\frac{2}{3}} \right).$
