Calculus II and III Notes

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1 Inverse Hyperbolic Trigonometric Functions

Pre-requisites/Synopsis :-

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{1}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{2}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{3}$$

Let's assume $\sinh^{-1}(x) = \ln|f(x)|$ where $f(x)$ is some function of x

$$\therefore \frac{e^{\ln|f(x)|} - e^{-\ln|f(x)|}}{2} = x$$

$$\implies f(x) - \frac{1}{f(x)} = 2x$$

$$\implies f^2(x) - 2xf(x) - 1 = 0$$

Solving this quadratic equation, we get,

$$f(x) = x \pm \sqrt{1+x^2}$$

If $f(x) = x - \sqrt{1+x^2}$, then $f(x) < 0$ for all $x \in (-\infty, 0)$
But $f(x)$ can't be -ve since $\ln|f(x)|$ will be an complex number. $\therefore f(x) = x + \sqrt{1+x^2}$

$$\therefore \boxed{\sinh^{-1}(x) = \ln|x + \sqrt{1+x^2}|} \tag{4}$$

Similarly, we can prove,

$$\boxed{\cosh^{-1}(x) = \ln|x + \sqrt{x^2-1}|} \tag{5}$$

Let $\tanh^{-1}(x) = \ln|f(x)|$

$$\therefore \frac{e^{\ln|f(x)|} - e^{-\ln|f(x)|}}{e^{\ln|f(x)|} + e^{-\ln|f(x)|}} = x$$

$$\implies \frac{f(x) - \frac{1}{f(x)}}{f(x) + \frac{1}{f(x)}} = x$$

Using componendo-dividendo,

$$\frac{1}{f^2(x)} = \frac{1-x}{1+x}$$

$$\implies f(x) = \sqrt{\frac{1+x}{1-x}}$$

$$\therefore \boxed{\tanh^{-1}(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)} \tag{6}$$

2 Basic Integration Problems

2.1 Evaluate

1. $\displaystyle\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx$
2. $\displaystyle\int \frac{x^2}{(x\sin x+\cos x)^2}dx$
3. $\displaystyle\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

2.2 Solutions

1. $\displaystyle \text{I} = \int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2-1}\sqrt{x^2+1}}dx$

$\displaystyle = \int \frac{dx}{\sqrt{x^2-1}} - \int \frac{dx}{\sqrt{x^2+1}} = \boxed{\cosh^{-1}(x) - \sinh^{-1}(x) + C}$

2. $\displaystyle \text{I} = \int x\sec x \frac{x\cos x}{(x\sin x+\cos x)^2}dx$

$\displaystyle = x\sec x \int \frac{x\cos x}{(x\sin x+\cos x)^2}dx + \int \frac{\sec x + x\tan x}{x\sin x+\cos x}dx$

$\displaystyle = -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{(\sec x + x\tan x)\times\cos^2 x}{(x\sin x+\cos x)\times\cos^2 x}dx + C_1$

$\displaystyle = -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{\cancel{(x\sin x+\cos x)}}{\cancel{(x\sin x+\cos x)}\cos^2(x)}dx + C_1$

$\displaystyle = \boxed{\tan x - \frac{x\sec x}{x\sin x+\cos x} + C}$

3. Let $a = x-1 \implies da = dx$

$\displaystyle \therefore \text{I} = \int a^{\frac{-3}{4}}(a+3)^{\frac{-5}{4}}da$

$\displaystyle = \int a^{\frac{1}{4}}a^{-1}(a+3)^{\frac{-1}{4}}(a+3)^{-1}da$

$\displaystyle = \int \left(\frac{a}{a+3}\right)^{\frac{1}{4}}(a^2+3a)^{-1}da$

$\displaystyle = \int e^{\frac{1}{4}\ln\left(\frac{a}{a+3}\right)}(a^2+3a)^{-1}da$

Let $\displaystyle u = \ln\left(\frac{a}{a+3}\right)$

$\displaystyle \therefore \frac{du}{da} = \frac{\cancel{a+3}}{a}\times\frac{(\cancel{a}+3)-\cancel{a}}{(a+3)^{\cancel{2}}} \implies du = 3(a^2+3a)^{-1}da$

$\displaystyle \therefore \text{I} = \frac{1}{3}\int e^{\frac{u}{4}}du \implies I = \frac{4}{3}\int e^{\frac{u}{4}}d\left[\frac{u}{4}\right] = \frac{4}{3}e^{\frac{u}{4}} + C$

Plugging in the substitutions we get,

$\displaystyle \therefore \text{I} = \frac{4}{3}\left(\frac{a}{a+3}\right)^{\frac{1}{4}} + C = \boxed{\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + C}$

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3 Partial Fraction Decomposition

3.1 Evaluate

1. $\displaystyle\int \frac{1+x^3}{1+x^2}dx$
2. $\displaystyle\int \frac{x^2-3x-7}{(x^2+x+2)(2x-1)}dx$
3. $\displaystyle\int \frac{x}{(x-3)^2(2x+1)}dx$

3.2 Solutions

1. I $\displaystyle = \int \frac{x(1+x^2)+(1-x)}{1+x^2}dx$

$$= \int x\ dx + \int \frac{dx}{1+x^2} - \int \frac{x}{1+x^2}dx = \boxed{\frac{x^2-\ln(1+x^2)}{2} + \arctan(x) + C}$$

2. $\displaystyle\frac{x^2-3x-7}{(x^2+x+2)(2x-1)} = \frac{Ax+B}{x^2+x+2} + \frac{C}{2x-1}$

$\implies (Ax+B)(2x-1) + C(x^2+x+2) = x^2-3x-7$

Putting $x = -\frac{1}{2}$

$C\left(\frac{1}{4}+\frac{1}{2}+2\right) = \left(\frac{1}{4}-\frac{3}{2}-7\right) \implies C = -\frac{33}{11} = -3$

Putting $x = 0$

$-B+2C = -7 \implies B = 7+2C = 7-6 = 1$

Putting $x = 1$

$(A+B)+4C = -9 \implies (A+1)-12 = -9 \implies A = 2$

$$\therefore I = \int \frac{2x+1}{x^2+x+2}dx + \int \frac{3}{2x-1}dx = \boxed{\ln(x^2+x+2) + \frac{3}{2}\ln(2x-1) + C}$$

3. $\displaystyle\frac{x}{(x-3)^2(2x+1)} = \frac{A}{2x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

$\implies A(x-3)^2 + B(x-3)(2x+1) + C(2x+1) = x$

Putting $x = 3$, we get $C = \frac{3}{7}$

Putting $x = -\frac{1}{2}$, we get $\frac{49}{4}A = -\frac{1}{2} \implies A = -\frac{2}{49}$

Putting $x = 0$, we get $\frac{-18}{49} - 3B + \frac{3}{7} = 0 \implies B = \frac{1}{49}$

$$\therefore \text{I} = \frac{-1}{49}\int \frac{2}{2x+1}dx + \frac{1}{49}\int \frac{dx}{x-3} + \frac{3}{7}\int \frac{dx}{(x-3)^2}$$

$$\implies I = \boxed{\frac{1}{49}\ln\left(\frac{x-3}{2x+1}\right) - \frac{3}{7(x-3)} + C}$$

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4 Change of order in double integrals

4.1 Evaluate the following by changing the order

1. $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{1+y^4} dy\ dx$

2. $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 e^{-x^2} dx\ dy$

4.2 Solutions

1. $\displaystyle \text{I} = \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{1+y^4} dy\ dx = \int_0^2 \int_0^{y^3} \frac{1}{1+y^4} dy\ dx$

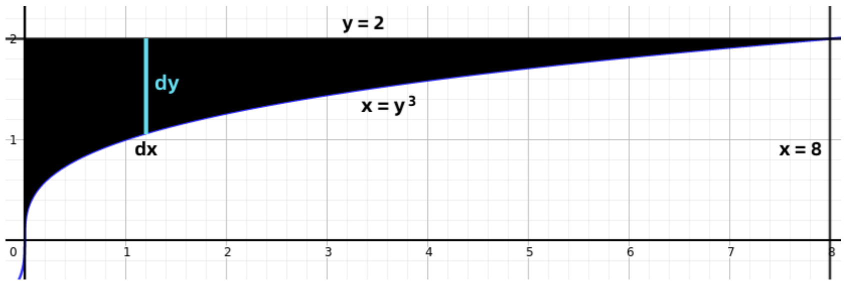

Figure 1: Integrating in the order dy dx

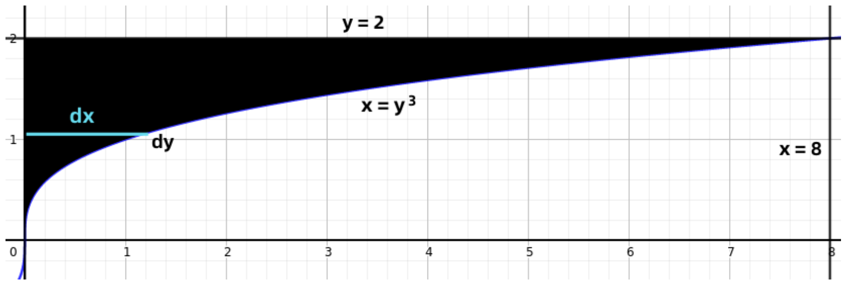

Figure 2: Integrating in the order dx dy

$$\therefore \text{I} = \int_0^2 \frac{y^3}{1+y^4} dx = \frac{1}{4}\left[\ln(|1+y^4|)\right]_0^2 = \boxed{\frac{\ln(17)}{4}}$$

2. $\displaystyle \text{I} = \int_0^2 \int_{\frac{y}{2}}^1 e^{-x^2} dx\ dy = \int_0^1 \int_0^{2x} e^{-x^2} dy\ dx$

Let $u = -x^2 \implies du = -2x\ dx$

$$\therefore \text{I} = -\int_0^{-1} e^u du = -\left[e^u\right]_0^{-1} = \boxed{1 - \frac{1}{e}}$$

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5 Weierstrass Substitution Technique

We take t as a dummy variable such that $t = \tan\left(\frac{x}{2}\right) \implies dx = \frac{2}{1+t^2}dt$

$\therefore \sin(x) = \frac{2t}{1+t^2}$ and $\cos(x) = \frac{1-t^2}{1+t^2}$. The proof is left as an exercise for the reader

5.1 Evaluate

1. $\displaystyle\int \frac{1}{2+\cos(x)}dx$
2. $\displaystyle\int \frac{1}{1+\sin(x)}dx$
3. $\displaystyle\int_0^\pi \frac{\sin(x)}{1+\sin(x)}dx$

5.2 Solutions

1. $\mathrm{I} = \displaystyle\int \frac{1}{2+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\ dt = \int \frac{2}{3+t^2}\ dt$ where $t = \tan(\frac{x}{2})$

$$\therefore \mathrm{I} = \frac{2}{3}\int \frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right) + C$$

Plugging in previous substitutions, I = $\boxed{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x)}{\sqrt{3}}\right) + C}$

2. $\mathrm{I} = \displaystyle\int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\ dt = \int \frac{2}{(1+t)^2}\ dt$, where $t = \tan(\frac{x}{2})$

$$\therefore \mathrm{I} = -\frac{2}{1+t} + C$$

Plugging in previous substitutions, I = $\boxed{-\frac{2}{1+\tan(\frac{x}{2})} + C}$

3. $\mathrm{I} = \displaystyle\int_0^\pi 1 - \frac{1}{1+\sin(x)}\ dx = \pi - \int_0^\pi \frac{dx}{1+\sin(x)}$

Substituting $t = \tan\left(\frac{x}{2}\right)$ and changing the limits

$$\mathrm{I} = \pi - \int_0^\infty \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\ dt = \pi + 2\int_0^\infty \frac{-1}{(1+t)^2}\ dt$$

$$\mathrm{I} = \pi + 2\left[\frac{1}{1+t}\right]_0^\infty = \pi + 2(0-1) = \boxed{\pi - 2}$$
