

# Lagrange Multipliers

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## 1 Vector calculus basics

### 1.1 Divergence

Measures how much a vector field is "spreading out" from a point. It takes a vector field  $\vec{F}(x, y, z)$  as input and gives a scalar field as output.

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

### 1.2 Gradient

Gives the direction and rate of steepest increase of a scalar field. It takes a function  $f(x, y, z)$  as input and gives a vector field as output.

$$\vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

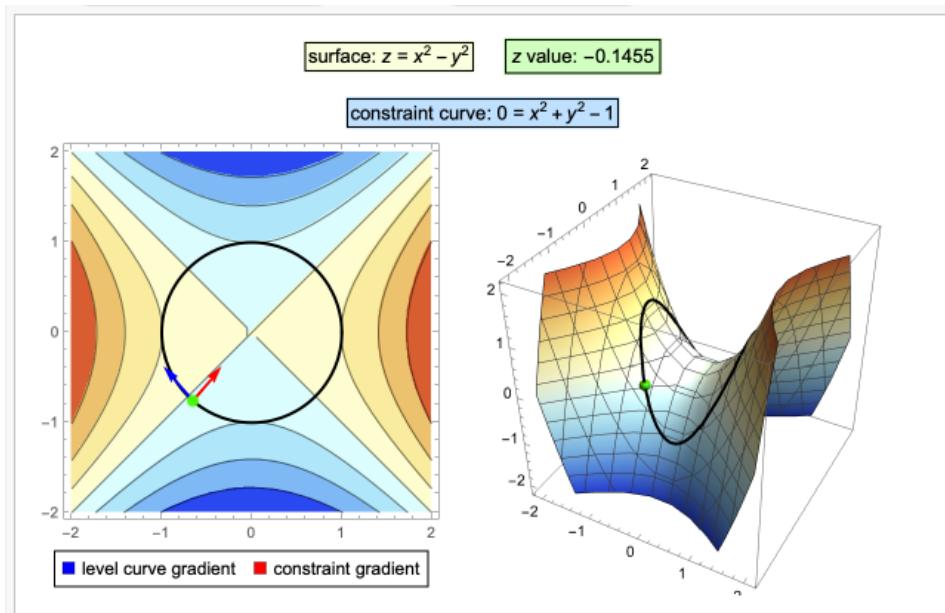
### 1.3 Curl

Measures the tendency of a vector field to rotate around a point. It takes a vector field  $\vec{F}(x, y, z)$  as input and gives another vector field as output.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

## 2 Theory

Lagrange multipliers are used to find the maxima or minima of a function  $f(x, y, z)$  subject to a constraint such as  $g(x, y, z) = 0$ .



Example illustration

Geometrically, at an extremum under a constraint, the surface (or level set) of  $f$  is just touching the constraint surface — meaning they are tangent to each other and do not cross. When two surfaces just touch, their normals are parallel. Since the gradient vector represents the normal direction of a level surface, this gives the essential condition:

$$\vec{\nabla}F = \lambda \vec{\nabla}g$$

This yields all candidate points where  $f$  may achieve a maximum or minimum while staying on the constraint curve.

### 3 Questions

1. Objective function  $f(x, y) = 4xy$ . Constraint:  $\frac{x^2}{9} + \frac{y^2}{16} = 1$ .
2. Objective function  $f(x, y) = x^2y$ . Constraint:  $x^2 + 2y^2 = 6$ .
3. Objective function  $f(x, y) = x^2 + y^2 + 2x - 2y + 1$ . Constraint:  $x^2 + y^2 = 2$ .
4. Objective function  $f(x, y) = xy$ . Constraint:  $x^2 + 2y^2 = 4$ .
5. Objective function  $f(x, y) = x^2 + y^2$ . Constraint:  $xy = 1$ .
6. Objective function  $f(x, y, z) = x + 3y - z$ . Constraint:  $x^2 + y^2 + z^2 = 4$ .
7. Objective function  $f(x, y, z) = xyz$ . Constraint:  $x^2 + 2y^2 + 3z^2 = 6$ .

### 4 Solutions

$$1. f(x, y) = 4xy \text{ and } g(x, y) = \frac{x^2}{9} + \frac{y^2}{16} - 1$$

$$\frac{\partial}{\partial x}f(x, y) = \lambda \frac{\partial}{\partial x}g(x, y) \implies 4y = \lambda \frac{2}{9}x \implies x = \frac{18y}{\lambda}$$

$$\frac{\partial}{\partial y}f(x, y) = \lambda \frac{\partial}{\partial y}g(x, y) \implies 4x = \frac{1}{8}x \implies x = \lambda \frac{y}{32}$$

$$\therefore \frac{18y}{\lambda} = \lambda \frac{y}{32} \implies \lambda^2 = 18 \times 32 \implies \lambda = \pm 24$$

Putting the value of  $\lambda$  in any of the previous equations yields the relation:  $x = \pm \frac{3}{4}y$ .

Putting the value of  $x$  in  $g(x, y)$  we get  $\frac{\cancel{y}y^2}{\cancel{y} \times 16} + \frac{y^2}{16} = 1 \implies y^2 = 8 \implies y = \pm 2\sqrt{2}$ .

$$\therefore x = \pm \frac{3}{2\sqrt{2} \times \sqrt{2}} \times 2\sqrt{2} = \pm \frac{3}{\sqrt{2}}$$

$\therefore$  We get 4 points:  $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ ,  $\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ ,  $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$  and  $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ . To get the critical points, we have to check which of these points correspond to those points where the gradient of  $f(x, y)$  and  $g(x, y)$  are parallel.

$$\vec{\nabla}f(x, y) = 4(y\hat{i} + x\hat{j})$$

$$\vec{\nabla}g(x, y) = \frac{2}{9}x\hat{i} + \frac{1}{8}y\hat{j}$$

$$\text{at } \left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right), \vec{\nabla}f(x, y) = 8\sqrt{2}\hat{i} + 6\sqrt{2}\hat{j} = 2\sqrt{2}(4\hat{i} + 3\hat{j}) \text{ and } \vec{\nabla}g(x, y) = \frac{\sqrt{2}}{3}\hat{i} + \frac{1}{2\sqrt{2}}\hat{j} = \frac{1}{6\sqrt{2}}(4\hat{i} + 3\hat{j}).$$

These 2 vectors are parallel.  $\therefore \left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$  is a critical point.

at  $\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$ ,  $\vec{\nabla}f(x, y) = 2\sqrt{2}(4\hat{i} - 3\hat{j})$  and  $\vec{\nabla}g(x, y) = -\frac{1}{6\sqrt{2}}(4\hat{i} - 3\hat{j})$ . These 2 vectors are parallel.  $\therefore \left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$  is a critical point.

at  $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ ,  $\vec{\nabla}f(x, y) = -2\sqrt{2}(4\hat{i} - 3\hat{j})$  and  $\vec{\nabla}g(x, y) = \frac{1}{6\sqrt{2}}(4\hat{i} - 3\hat{j})$ . These 2 vectors are parallel.  $\therefore \left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$  is a critical point.

at  $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$ ,  $\vec{\nabla}f(x, y) = -2\sqrt{2}(4\hat{i} + 3\hat{j})$  and  $\vec{\nabla}g(x, y) = -\frac{1}{6\sqrt{2}}(4\hat{i} + 3\hat{j})$ . These 2 vectors are parallel.  $\therefore \left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$  is a critical point.

$\therefore$  Max.  $f(x, y) = 24$  at  $\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$  and  $\left(-\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$

and Min.  $f(x, y) = -24$  at  $\left(-\frac{3}{\sqrt{2}}, 2\sqrt{2}\right)$  and  $\left(\frac{3}{\sqrt{2}}, -2\sqrt{2}\right)$

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$$2. f(x, y) = x^2y \text{ and } g(x, y) = x^2 + 2y^2 - 6$$

$$\frac{\partial}{\partial x}f(x, y) = \lambda \frac{\partial}{\partial x}g(x, y) \implies 2xy = \lambda 2x \implies y = \lambda$$

$$\frac{\partial}{\partial y}f(x, y) = \lambda \frac{\partial}{\partial y}g(x, y) \implies x^2 = \lambda 4y \implies x^2 = 4\lambda^2 \implies x = \pm 2\lambda$$

Substituting the values of  $x$  and  $y$  in  $g(x, y)$  we get

$$4\lambda^2 + 2\lambda^2 = 6 \implies 6\lambda^2 = 6 \implies \lambda = \pm 1$$

$$\therefore y = \pm 1 \text{ and } x = \pm 2$$

The points are  $(2, 1), (-2, 1), (2, -1)$  and  $(-2, -1)$ .

$$\vec{\nabla}f(x, y) = 2xy\hat{i} + x^2\hat{j}$$

$$\vec{\nabla}g(x, y) = 2(x\hat{i} + 2y\hat{j})$$

at  $(2, 1)$ ,  $\vec{\nabla}f(x, y) = 4(\hat{i} + \hat{j})$  and  $\vec{\nabla}g(x, y) = 4(\hat{i} + \hat{j})$ .  $\therefore (2, 1)$  is a critical point.

at  $(-2, 1)$ ,  $\vec{\nabla}f(x, y) = -4(\hat{i} - \hat{j})$  and  $\vec{\nabla}g(x, y) = -4(\hat{i} - \hat{j})$ .  $\therefore (-2, 1)$  is a critical point.

at  $(2, -1)$ ,  $\vec{\nabla}f(x, y) = -4(\hat{i} - \hat{j})$  and  $\vec{\nabla}g(x, y) = 4(\hat{i} - \hat{j})$ .  $\therefore (2, -1)$  is a critical point.

at  $(-2, -1)$ ,  $\vec{\nabla}f(x, y) = 4(\hat{i} + \hat{j})$  and  $\vec{\nabla}g(x, y) = -4(\hat{i} + \hat{j})$ .  $\therefore (-2, -1)$  is a critical point.

$\therefore$  Max.  $f(x, y) = 4$  at  $(2, 1)$  and  $(-2, 1)$  and Min.  $f(x, y) = -4$  at  $(2, -1)$  and  $(-2, -1)$

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$$3. f(x, y) = x^2 + y^2 + 2x - 2y + 1 \text{ and } g(x, y) = x^2 + y^2 - 2$$

$$\frac{\partial}{\partial x}f(x, y) = \lambda \frac{\partial}{\partial x}g(x, y) \implies 2(x+1) = \lambda 2x \implies x = \frac{1}{\lambda-1}$$

$$\frac{\partial}{\partial y}f(x, y) = \lambda \frac{\partial}{\partial y}g(x, y) \implies 2(y-1) = \lambda 2y \implies y = \frac{-1}{\lambda-1} = -x$$

$\therefore$  Putting  $y = -x$  in  $g(x, y)$  we get

$$2y^2 = 2 \implies y = \pm 1 \text{ and } x = \mp 1. \therefore$$
 The points are  $(1, -1)$  and  $(-1, 1)$ .

$$\vec{\nabla}f(x, y) = 2\{(x+1)\hat{i} + (y-1)\hat{j}\}$$

$$\vec{\nabla}g(x, y) = 2(x\hat{i} + y\hat{j})$$

at  $(1, -1)$ ,  $\vec{\nabla}f(x, y) = 4(\hat{i} - \hat{j})$  and  $\vec{\nabla}g(x, y) = 2(\hat{i} - \hat{j})$ .  $\therefore (1, -1)$  is a critical point.

at  $(-1, 1)$ ,  $\vec{\nabla}f(x, y) = 0$  and a null vector is mathematically parallel to any vector.

So we need not calculate  $\vec{\nabla}g(x, y)$ .  $\therefore (-1, 1)$  is a critical point.

$\therefore$  Max.  $f(x, y) = 7$  at  $(1, -1)$  and Min.  $f(x, y) = -1$  at  $(-1, 1)$

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4.  $f(x, y) = xy$  and  $g(x, y) = x^2 + 2y^2 - 4$

$$\frac{\partial}{\partial x} f(x, y) = \lambda \frac{\partial}{\partial x} g(x, y) \implies y = \lambda 2x$$

$$\frac{\partial}{\partial y} f(x, y) = \lambda \frac{\partial}{\partial y} g(x, y) \implies x = \lambda 4y$$

Substituting  $x = \lambda 4y$  in  $y = \lambda 2x$ , we get  $y = 8\lambda^2 y \implies \lambda = \pm \frac{1}{2\sqrt{2}}$

$$\therefore y = \pm \frac{x}{\sqrt{2}} \implies y^2 = \frac{x^2}{2} \implies 2y^2 = x^2$$

Substituting the value of  $2y^2$  in  $g(x, y)$  we get  $2x^2 = 4 \implies x = \pm\sqrt{2}$  and  $y = \pm 1$

$\therefore$  The points are  $(\sqrt{2}, 1), (-\sqrt{2}, 1), (\sqrt{2}, -1)$  and  $(-\sqrt{2}, -1)$

$$\vec{\nabla} f(x, y) = y\hat{i} + x\hat{j}$$

$$\vec{\nabla} g(x, y) = 2(x\hat{i} + 2y\hat{j})$$

at  $(\sqrt{2}, 1)$ ,  $\vec{\nabla} f(x, y) = \hat{i} + \sqrt{2}\hat{j}$  and  $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j})$ .  $\therefore (\sqrt{2}, 1)$  is a critical point.

at  $(-\sqrt{2}, 1)$ ,  $\vec{\nabla} f(x, y) = \hat{i} - \sqrt{2}\hat{j}$  and  $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j})$ .  $\therefore (-\sqrt{2}, 1)$  is a critical point.

at  $(\sqrt{2}, -1)$ ,  $\vec{\nabla} f(x, y) = -\hat{i} + \sqrt{2}\hat{j}$  and  $\vec{\nabla} g(x, y) = -2\sqrt{2}(-\hat{i} + \sqrt{2}\hat{j})$ .  $\therefore (\sqrt{2}, -1)$  is a critical point.

at  $(-\sqrt{2}, -1)$ ,  $\vec{\nabla} f(x, y) = -(\hat{i} + \sqrt{2}\hat{j})$  and  $\vec{\nabla} g(x, y) = -2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j})$ .  $\therefore (-\sqrt{2}, -1)$  is a critical point.

$\therefore$  Max.  $f(x, y) = 2\sqrt{2}$  at  $(\sqrt{2}, 1)$  and  $(-\sqrt{2}, -1)$

and Min.  $f(x, y) = -2\sqrt{2}$  at  $(-\sqrt{2}, 1)$  and  $(\sqrt{2}, -1)$

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5.  $f(x, y) = x^2 + y^2$  and  $g(x, y) = xy - 1$

$$\frac{\partial}{\partial x} f(x, y) = \lambda \frac{\partial}{\partial x} g(x, y) \implies 2x = \lambda y \implies \lambda = \frac{2x}{y}$$

$$\frac{\partial}{\partial y} f(x, y) = \lambda \frac{\partial}{\partial y} g(x, y) \implies 2y = \lambda x \implies \lambda = \frac{2y}{x}$$

From these 2 equations we get  $\frac{2x}{y} = \frac{2y}{x} \implies x^2 = y^2 \implies x = y$

Using this relation in  $g(x, y)$  we get  $x = y = \pm 1$

$$\vec{\nabla} f(x, y) = 2(x\hat{i} + y\hat{j})$$

$$\vec{\nabla} g(x, y) = y\hat{i} + x\hat{j}$$

at  $(1, 1)$ ,  $\vec{\nabla} f(x, y) = 2(\hat{i} + \hat{j})$  and  $\vec{\nabla} g(x, y) = \hat{i} + \hat{j}$ .  $\therefore (1, 1)$  is a critical point.

at  $(-1, 1)$ ,  $\vec{\nabla} f(x, y) = 2(-\hat{i} + \hat{j})$  and  $\vec{\nabla} g(x, y) = \hat{i} - \hat{j}$ .  $\therefore (-1, 1)$  is not a critical point.

at  $(1, -1)$ ,  $\vec{\nabla} f(x, y) = 2(\hat{i} - \hat{j})$  and  $\vec{\nabla} g(x, y) = -\hat{i} + \hat{j}$ .  $\therefore (1, -1)$  is not a critical point.

at  $(-1, -1)$ ,  $\vec{\nabla} f(x, y) = -2(\hat{i} + \hat{j})$  and  $\vec{\nabla} g(x, y) = -(\hat{i} + \hat{j})$ .  $\therefore (-1, -1)$  is a critical point.

$\therefore$  Max.  $f(x, y) = 2$  at  $(1, 1)$  and  $(-1, -1)$ .

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6.  $f(x, y, z) = x + 3y - z$  and  $g(x, y, z) = x^2 + y^2 + z^2 - 4$ .

$$\frac{\partial}{\partial x} f(x, y, z) = \lambda \frac{\partial}{\partial x} g(x, y, z) \implies 1 = \lambda 2x \implies x = \frac{1}{2\lambda}$$

$$\frac{\partial}{\partial y} f(x, y, z) = \lambda \frac{\partial}{\partial y} g(x, y, z) \implies 3 = \lambda 2y \implies y = \frac{3}{2\lambda} = 3x$$

$$\frac{\partial}{\partial z} f(x, y, z) = \lambda \frac{\partial}{\partial z} g(x, y, z) \implies -1 = \lambda 2z \implies z = -\frac{1}{2\lambda} = -x$$

Substituting  $y = 3x$ , and  $z = -x$  in  $g(x, y, z)$  we get

$$x^2 + 9x^2 + x^2 = 4 \implies 11x^2 = 4 \implies x = \pm \frac{2}{\sqrt{11}}, y = \pm \frac{6}{\sqrt{11}} \text{ and } z = \mp \frac{2}{\sqrt{11}}$$

$\therefore$  The points are  $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$  and  $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ .

$$\vec{\nabla} f(x, y) = \hat{i} + 3\hat{j} - \hat{k}$$

$$\vec{\nabla} g(x, y) = 2(x\hat{i} + y\hat{j} + z\hat{k})$$

at  $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ ,  $\vec{\nabla} g(x, y) = \frac{4}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$ .  $\therefore \left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$  is a critical point.

at  $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ ,  $\vec{\nabla} g(x, y) = -\frac{4}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$ .  $\therefore \left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$  is a critical point.

$\therefore$  Max.  $f(x, y, z) = 2\sqrt{11}$  at  $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$  and Min.  $f(x, y, z) = -2\sqrt{11}$  at  $\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ .

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7.  $f(x, y, z) = xyz$  and  $g(x, y, z) = x^2 + 2y^2 + 3z^2 - 6$ .

$$\frac{\partial}{\partial x} f(x, y, z) = \lambda \frac{\partial}{\partial x} g(x, y, z) \implies yz = \lambda 2x \implies \lambda = \frac{yz}{2x}$$

$$\frac{\partial}{\partial y} f(x, y, z) = \lambda \frac{\partial}{\partial y} g(x, y, z) \implies xz = \lambda 4y \implies \lambda = \frac{xz}{4y}$$

$$\frac{\partial}{\partial z} f(x, y, z) = \lambda \frac{\partial}{\partial z} g(x, y, z) \implies xy = \lambda 6z \implies \lambda = \frac{xy}{6z}$$

From the first 2 relations we get  $\frac{yz}{2x} = \frac{xz}{4y} \implies x^2 = 2y^2 \implies x = \pm\sqrt{2}y$

From the second and third relations we get  $\frac{xz}{4y} = \frac{xy}{6z} \implies 3z^2 = 2y^2 \implies z = \pm\sqrt{\frac{2}{3}}y$

Using these relations in  $g(x, y)$  we get  $6y^2 = 6 \implies y = \pm 1$ ,  $x = \pm\sqrt{2}$  and  $z = \pm\frac{2}{3}$ .

$\therefore$  The points are  $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ ,  $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ ,  $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ ,  $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ ,  $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ ,  $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ ,  $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$  and  $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ .

$$\vec{\nabla} f(x, y) = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\vec{\nabla} g(x, y) = 2(x\hat{i} + 2y\hat{j} + 3z\hat{k})$$

at  $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla} f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$  and  $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ .  $\therefore$

$\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$  and  $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ .

$\therefore \left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla} f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$  and  $\vec{\nabla} g(x, y) = 2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ .

$\therefore \left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla}f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$  and  $\vec{\nabla}g(x, y) = 2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ .

$\therefore \left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla}f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$  and  $\vec{\nabla}g(x, y) = -2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ .

$\therefore \left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla}f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$  and  $\vec{\nabla}g(x, y) = -2\sqrt{2}(\hat{i} - \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ .

$\therefore \left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla}f(x, y) = -\sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$  and  $\vec{\nabla}g(x, y) = -2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} - \sqrt{3}\hat{k})$ .

$\therefore \left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$  is a critical point.

at  $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ ,  $\vec{\nabla}f(x, y) = \sqrt{\frac{2}{3}}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$  and  $\vec{\nabla}g(x, y) = -2\sqrt{2}(\hat{i} + \sqrt{2}\hat{j} + \sqrt{3}\hat{k})$ .

$\therefore \left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$  is a critical point.

$\therefore \text{Min. } f(x, y, z) = -\frac{2}{\sqrt{3}}$  at  $\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$ ,  $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ ,  $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$  and  $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ .

and Max.  $f(x, y, z) = \frac{2}{\sqrt{3}}$  at  $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$ ,  $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$ ,  $\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$  and  $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$ .