Spherical Shells

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A gravitational force on a point mass inside a uniform spherical shell is zero. Reference [1] gives the derivation that is often seen by technical students by performing an integration. The derivation applies to the case of a point mass either inside or outside of the shell. This is used to demonstrate that a spherical shell with uniform mass density behaves (as far as its gravitational force on the exterior point mass is concerned) as if the shell was concentrated at a point at the sphere's center. This textbook derivation requires calculus, so it leaves out students who haven't studied that subject yet (for a web-based derivation using calculus, see [4]). But one of the problems in the chapter on gravitation in reference [1] asks for another derivation that doesn't require calculus and reference [2] gives the result in a short derivation . It's an elegant argument worth seeing.

This derivation also applies to a spherical shell with a uniform electrical charge on it, attracting or repelling a point charge inside the shell. In fact, it applies to any inverse-square law force where the force lies on the line connecting the two point objects.

We require the notion of a solid angle, which we'll first have to discuss. The general definition of the solid angle requires a vertex of the angle and a surface. Then the definition of the solid angle subtended by the surface S from the vertex point is the surface area of the surface S projected onto the unit sphere whose center is the vertex [5].

Suppose we have a unit sphere and a small circle on the sphere. The circle's area subtends a solid angle Ω with respect to the center of the sphere. This solid angle is entirely analogous to the angle defined in plane geometry. If you recall the plane geometry definition, the angle in radians is the arc length of the corresponding arc of the circle divided by the circle's radius. This results in a dimensionless measure of angle which is given the name of radians.

For the solid angle Ω in three dimensions, we have the following picture:

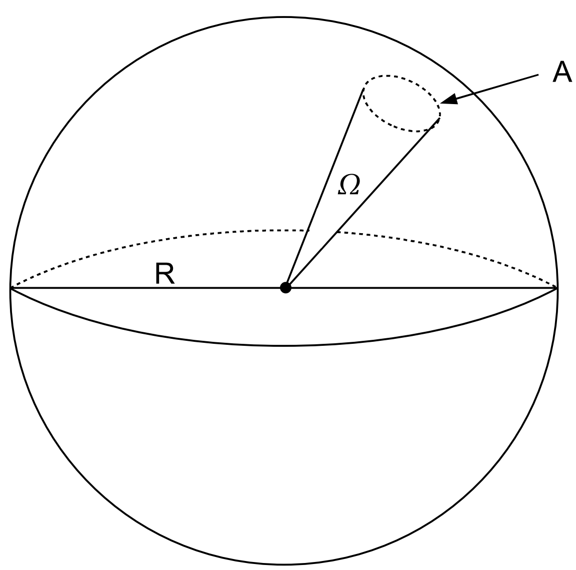

Figure 1

A small circle is on the surface of the sphere of radius R. The solid angle is then

$$\Omega = \frac{A}{R^2}$$

where A is the area of the spherical cap with the circle as its border. A solid angle is a dimensionless number; it is also given a name, the *steradian*. If you make the circle a great circle of the sphere, you can see that the solid angle is 2π and it's 4π for the whole sphere. If you make the area of the cap equal to R^2, then you have a unit solid angle. You can convert steradians to square degrees by multiplying by $(180/\pi)^2$, analogous to converting from radians to degrees by multiplying radians by $180/\pi$,

Suppose I have an area A that subtends a solid angle Ω from point S as shown in Figure 2. If I rotate the area A about the axis PQ by an angle θ, the solid angle subtended by the circle from point S is reduced to $\Omega\cos\theta$.

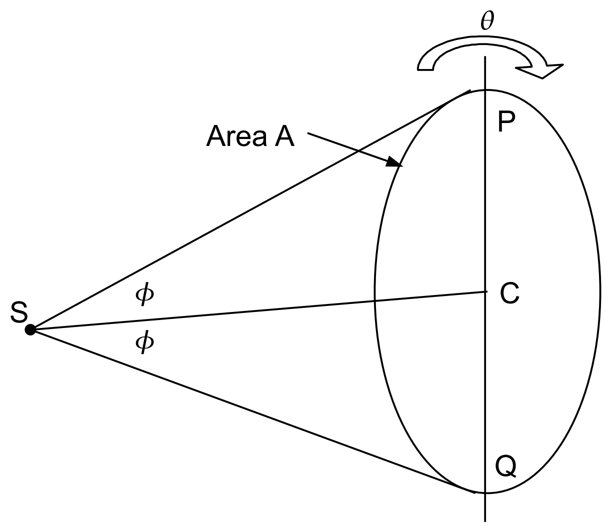

Figure 2

You can see the reasonableness of this from Figure 2. Suppose the area A is a unit circle and your eye is at point S. If you rotate the circle by angle θ about the axis PQ, you'll now see an ellipse with a major radius of 1 and a minor radius of $\cos\theta$. The area of this ellipse is π times the major radius times the minor diameter, or $\pi\cos\theta$. Now, convert this elliptical area to a circular area in the same plane as the original circle and concentric with it, then project this circle onto the unit sphere used in defining the solid angle. Since the newly-projected circle's area is smaller by $\cos\theta$, the solid angle has fallen by the same amount. It's the area of the surface's projection on the unit sphere, not the shape of the surface that's important in determining the solid angle.

If ϕ is the angle subtended by the radius of the circle and the plane of the circle is perpendicular to SC in Figure 2, the solid angle subtended by the cone with vertex at S and intersecting the circle's circumference is

$$\Omega = 2\pi(1-\cos\phi)$$

We won't need this relation for our discussion, but it is often handy.

Now, we're ready to derive the gravitational force on a point mass inside a spherical shell. In Figure 3, assume the point mass is at point P. The center of the spherical shell is at C and the shell is shown in cross section as a circle, as we consider it to be of small thickness.

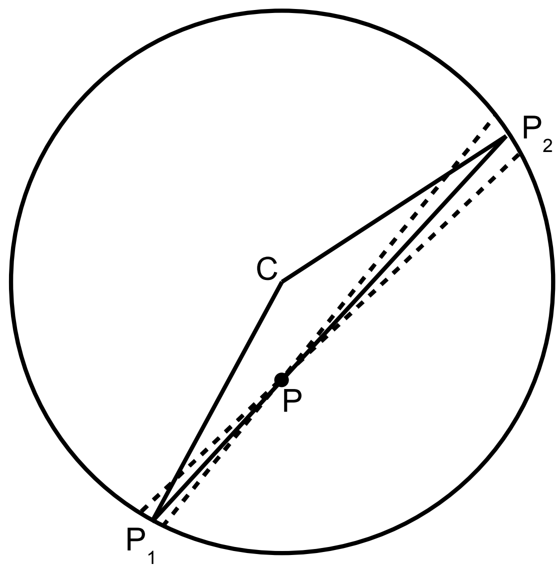

Figure 3

Consider a small solid angle Ω that is formed by a narrow cone (indicated by the dashed lines) with vertex at P and intersecting the spherical shell around the points P_1 and P_2. Let θ be the angle CP_1P_2, which is also equal to the angle CP_2P_1. We have

$$\Omega = \frac{S_1 \cos\theta}{r_1^2} = \frac{S_2 \cos\theta}{r_2^2} \quad \text{where } r_1 = PP_1 \text{ and } r_2 = PP_2 \tag{1}$$

where S_1 is the small area on the shell at P_1 where the cone intersects the shell and correspondingly for S_2. Note we had to include the $\cos\theta$ term because the area made up of the intersection of the thin shell and the cone is tilted with respect to the line P_1P_2 precisely by the angle θ.

The cosine terms in equation (1) cancel out and we can multiply the equation by any constant k. Thus, we have the general relation

$$k\frac{S_1}{r_1^2} = k\frac{S_2}{r_2^2} \tag{2}$$

One choice for the constant k is $Gm\sigma$ where σ is the uniform areal mass density of the shell, m is the mass of a particle at P, and G is the universal gravitational constant. Then $\sigma S_1 = M_1$ which is the mass of the small piece of shell at P_1; similarly for P_2. Then we have

$$\frac{GmM_1}{r_1^2} = \frac{GmM_2}{r_2^2}$$

Each side, of course, is Newton's universal law of gravitation. This equation shows the equality in the magnitude of the gravitational attraction of the small areas of the shell on the particle of mass m. In addition, we know from the vector form of the law of gravitation that the forces on the particle from each of the small chunks of the shell are in opposite directions and both lie on the line P_1P_2. **Thus, these two forces exactly cancel**. It's then not difficult to see that all such pairs of forces from small areas on the spherical shell will cancel out as the line P_1P_2 assumes all directions in space while passing through point P. We conclude that the particle of mass m experiences no net gravitational force inside the spherical shell. It's a simple and elegant argument[1].

Often in doing such physical problems, one has to sum up the contributions of lots of little forces like

1 I don't know who first gave it, but reference [2] states that Newton gave a different derivation. I'd hazard a guess that it was probably given in the early 1800's, if not the late 1700's.

we just discussed; this turns into an integration problem when one has studied calculus. It's interesting that in this problem, however, the small terms cancel out. Thus, no fancier math is needed.

Note the magnitude of the solid angle of the cone was never needed. However, out of habit, we assumed the cone was a skinny cone, leading to a small solid angle. We could make the same argument for a substantial solid angle -- just sum the contributions from each of the skinny solid angles that make up the substantial solid angle. But there's no need, as they all neatly cancel in corresponding pairs.

Another choice for the constant k is for the case of the particle at P being a point electrical charge and the sphere having a uniform electrical charge density (i.e., an application of Coulomb's Law). In both cases of the point charge being equal or opposite to the charge on the shell, you can see that the net force on the point charge is zero.

Feather [2] relates that Joseph Priestly had heard from Benjamin Franklin about a "curious observation" that Franklin made that a small cork ball suspended on a fine thread would be attracted to the outside of an electrically charged metal cup. However, when the ball was lowered into the cup, it was not attracted to the cup. Priestly repeated the experiments with various modifications and made the insightful observation in 1767 that the electrical force must be an inverse square law force, precisely because of the result known at the time that a spherical shell of mass would not attract a point mass inside the shell. Priestly did not back this statement up with careful measurements.

However, Henry Cavendish did. He quantitatively determined that, if the electrostatic force between two point charges was the inverse square law, then the exponent could not differ by more than $2 \pm \frac{1}{59}$ (his ingenious work lay essentially unknown for 100 years before being published by Maxwell). Cavendish's experiment is described in Feather [2]; let's look at it because it is very clever.

Suppose we change Figure 3 into Figure 4:

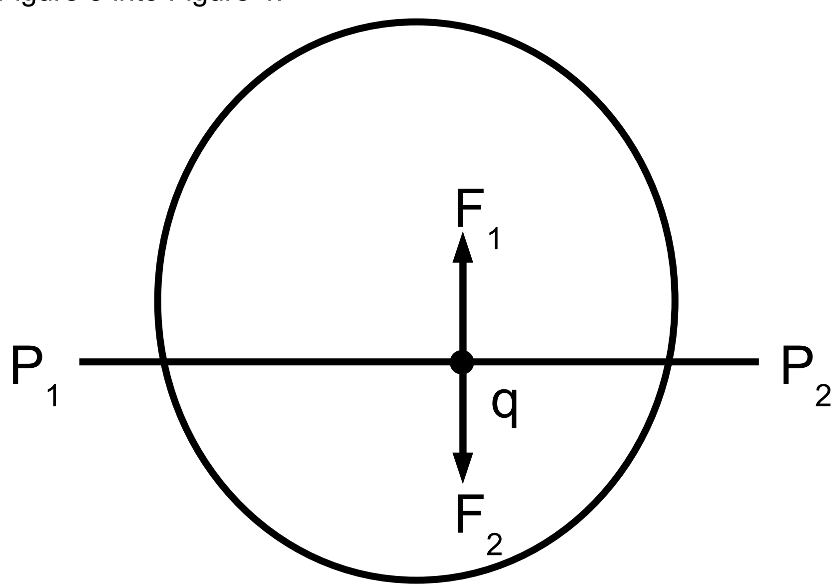

Figure 4

P_1P_2 is a plane that passes through an internal point charge q, dividing the spherical shell into two parts. Each part produces a force on the charge q, shown by F_1 and F_2 (which force belongs to which shell is not important for this discussion). We have that $F_1 = F_2$ for an arbitrary location of q inside the shell.

In the comments after equation (2) above for the electrostatic case, we implicitly assumed the force law was the inverse square law (since we recognized the mathematical form as Coulomb's Law). But assume the exponent of r in the law is not 2. Then the forces will no longer balance. If the exponent is greater than 2, the force due to the section of the shell below P_1P_2 will be larger and vice versa if the exponent is less than 2.

Cavendish's brilliant experimental design was to encase the first metal shell within a second metal shell; this removed the requirement for a test charge q (you'll see why in a moment). The outer shell was initially uncharged while the inner one was charged. Then the two shells were connected by a conducting wire. Imagine you're a small charge sitting at the junction of the wire and the inner shell. Do you feel a force? You do, because you're just like lots of other like charges on the inner shell[2]. There thus will be a current flow from the inner shell to the outer shell until the forces on these charges balance.

Now here's the key: **if the force law is proportional to $1/r^2$, then this current will flow until there is no charge left on the inner shell**. Why? Because we showed that there is no net force on the charge from the outer shell, regardless of the charge on it. But there will always be a force on the charge as long as any charge remains on the inner shell because these like charges repel each other[3].

If the force law is not proportional to $1/r^2$, then some charge will be left on the inner shell; its sign will tell you whether the exponent is larger or smaller than 2. What an elegant experimental design!

When Cavendish ran his experiments, he was not able to detect any charge left on the inner shell. He estimated the sensitivity of his pith-ball electroscope and thus was able to put the error bound of 1/59 on the measured exponent. This is beautiful science indeed.

Maxwell repeated Cavendish's experiment 100 years later and found the exponent to be $2 \pm 4.6\times10^{-5}$. Another experiment in 1936 got $2 \pm 1.0\times10^{-9}$. Modern measurements indicate the exponent is 2 within 10^{-15} (see [3]).

Of course, today we know this electrostatic law as Coulomb's Law. Coulomb verified Cavendish's measurements by using a torsion balance (Cavendish would use the torsion balance in 1798 to measure the gravitational force between masses in the laboratory, a notoriously fickle measurement). Feather [2] points out that Coulomb's investigations also verified that these electrostatic forces depended on the product of the amount of charges present. He was able to do this from the known phenomenon that two identical conducting spheres, one initially uncharged, will share the charge equally when they are connected together by a conductor and hence was able to work with amounts of charge related by powers of two. Thus, Coulomb verified experimentally the law named after him

$$F \propto \frac{q_1 q_2}{r^2}$$

References

[1] Resnick, R. and Halliday, D., *Physics*, Part 1, Wiley, 1966.

[2] Feather, N., *Electricity and Matter*, Aldine Publishing, Chicago, 1968.

[3] http://en.wikipedia.org/wiki/Inverse_square_law

[4] http://en.wikipedia.org/wiki/Shell_theorem

[5] http://mathworld.wolfram.com/SolidAngle.html

2 Note this assumes the existence of small charges or a charged "fluid" that are free to move under these electrostatic forces.

3 An interesting question is whether there will be one charge left behind because there's no other charge to apply a force to it. Logically, there has to be if charge is quantized, but I don't know if anyone has ever measured this experimentally.