RMS Measurement

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This is an article aimed at hobbyists who want to know a little bit more about RMS (root mean square) measurements. It's motivated by a need to quantify the power in an electrical waveform that changes over time. I assume you've had a smattering of a few math/science courses at the college level to understand things in depth; however, the math knowledge is not essential to get the main points of this paper.

Table of Contents

Motivation

Let's do a simple Gedankenexperiment[1]. Take a resistor and hook it up in series with a battery. Most folks know that the resistor will heat up[2]. Now, connect the same resistor to an alternating voltage; say, a low-voltage sine wave. Most of us know the resistor will also heat up -- this is how the elements on our electric stove helps us cook our meals. But, if you think about it, it tells you something rather deep about electrical current flow in materials -- you'd be led to an atomic theory of charge carriers converting some their kinetic energy to thermal vibrations of the atomic lattice. You could quickly get mired down in quantum statistical mechanics trying to explain things.

We don't need to go that deep, but being quantitative beings, we want to say something about *how much* the resistor will heat up. Experiment ultimately leads us to the RMS value of a waveform because it's useful in predicting the time-averaged macroscopic power of a changing voltage or current:

$$P_{\text{time averaged}} = V_{rms} i_{rms} \tag{1}$$

This is desirable because it looks like the same formula for the instantaneous power and is thus easy to remember.

This document will look at some of the issues with measuring RMS electrical values. The problem is not quite as simple as measuring unchanging values like DC measurements.

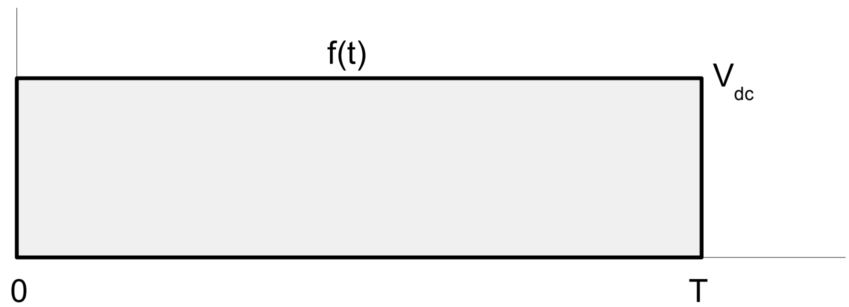
Background

To use formula (1), we need to quantify the "amount" of a waveform in a single number. Thus, given a waveform f(t) that's a function of the time t over the interval of 0 to T, we want a thingy $\boldsymbol{F}(f(t), 0, T)$ to produce a number that characterizes how "big" the waveform is. This thingy $\boldsymbol{F}$ that maps a function to a number is called a **functional** (and the study of the extrema of functionals

1 A thought experiment. See http://en.wikipedia.org/wiki/Gedankenexperiment.
2 If you want to try this for real, make sure you don't dissipate too much power in the resistor, as you can burn yourself (the author remembers seeing the white burn marks on his finger pads from touching hot resistors when he was young ☺).

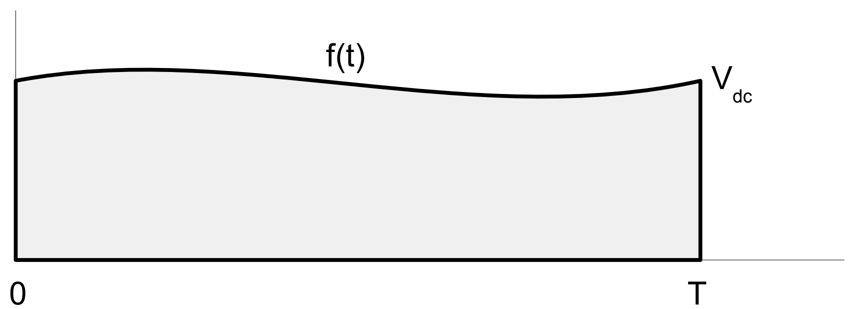
leads to an interesting area of mathematics -- see reference [gf]).

One common functional that measures a quantity $f(t)$ that varies over time t is the average. Let's motivate this with a geometrical argument. Suppose we have a DC voltage $f(t) = V_{dc}$. Clearly, its average value is V_{dc}. Here's a plot of $f(t)$ vs. t:



The area of the gray rectangle under the constant curve $f(t) = V_{dc}$ is $V_{dc} T$. Thus, if we divide the area under the "curve" $f(t)$ by T, the width of the interval, we get the average voltage $f_{avg} = V_{dc}$.

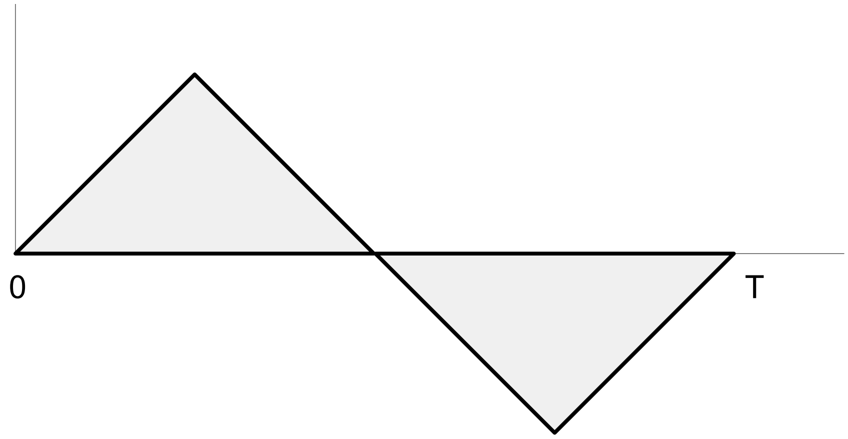
This geometrical picture suggests how to extend the definition to non-constant functions. Let's "perturb" the constant waveform a bit so it's no longer constant:



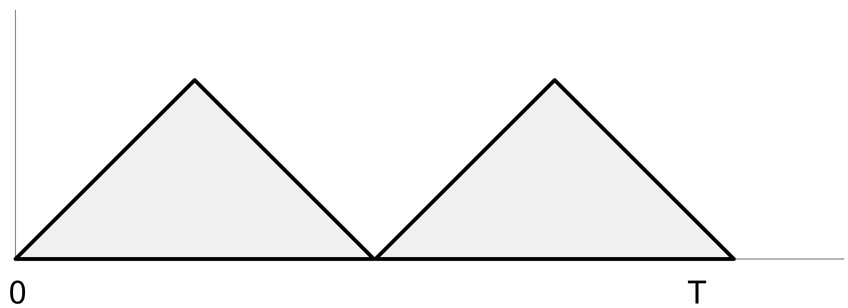
Now, we'll have to specify the functional form $f(t)$ to specify the waveform. But we can still use the concept of the area under the waveform divided by the width T to calculate the average. This area is an integral straight from basic calculus:

$$f_{avg} = \frac{1}{T} \int_0^T f(t) dt$$

While we might think we've found a good single number to quantify the "amount" in a waveform, an example destroys this notion -- a triangle wave:



If we calculate f_{avg} for this waveform, it will come out zero because the portion below the t axis contributes a negative area. A little reflection shows how to fix this -- we can take the absolute value of $f(t)$:



In fact, this is a perfectly acceptable definition of an average value:

$$f_{avg} = \frac{1}{T}\int_0^T |f(t)|\,dt$$

For an odd[3] periodic function such as we've shown (or, such as the sine), we could instead calculate the average of the first half of the waveform from 0 to $T/2$ and double the result -- we'd get the same answer. Let's calculate the average of the sine wave with unity amplitude (just do the integration for the positive half of the waveform):

$$f_{avg} = \frac{1}{\pi/2}\int_0^{\pi/2} \sin t\,dt = \frac{2}{\pi}\left[-\cos t\right]_0^{\pi/2} = \frac{2}{\pi}\left[0-(-1)\right] = \frac{2}{\pi} = 0.6366$$

We could stick with this definition of an average -- and that's what average-responding meters measure. Their measured value is multiplied by $\frac{\pi}{2\sqrt{2}} = 1.1107$ to get the equivalent RMS value of a sine wave (see the integration below that gets the RMS value of a sine wave). However, it is known from experimentation that **the average doesn't predict the heating power of a waveform**. For this, experiment shows that the RMS current squared times the resistance gives the power being dissipated in a resistance. An average-responding meter gives the wrong measurement if the input is not a sine wave.

The name "root mean square" tells you how the RMS calculated -- it's the square root of the mean of the square of the function $f(t)$:

3 Odd in the mathematical sense, not "strange". An odd function f(t) has the property that f(t) = -f(-t) for all t.

$$\text{RMS} = \sqrt{\frac{1}{T}\int_0^T [f(t)]^2 dt} \tag{2}$$

The square of the function has the nice property that it's everywhere positive or zero. Thus, the RMS value is always positive or zero -- and it's only zero if the function $f(t)$ is identically equal to zero on the interval [0, T].

For periodic functions, the RMS value is calculated over one period -- and in the majority of cases when we're measuring things, we're interested in periodic waveforms.

Let's calculate the RMS value of a unity-amplitude sine wave:

$$\text{RMS}^2 = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 t\, dt = \frac{1}{2\pi}\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{1}{2}2\pi\right] = \frac{1}{2}$$

Thus, the RMS value of a sine wave is $\frac{1}{\sqrt{2}} = 0.707$. This yields the useful rule **for a sine wave** that you can multiply the peak-to-peak amplitude by $\frac{1}{2\sqrt{2}} = 0.354$ to get the RMS value and multiply the RMS value by $2\sqrt{2} = 2.83$ to get the peak-to-peak amplitude.

The RMS value of a discrete function is analogously defined to be[4]

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{n} [f(x_i)]^2}{n}} \tag{3}$$

There's no requirement that the samples be evenly spaced, but in practice they usually are. Both expressions (2) and (3) are functionals.

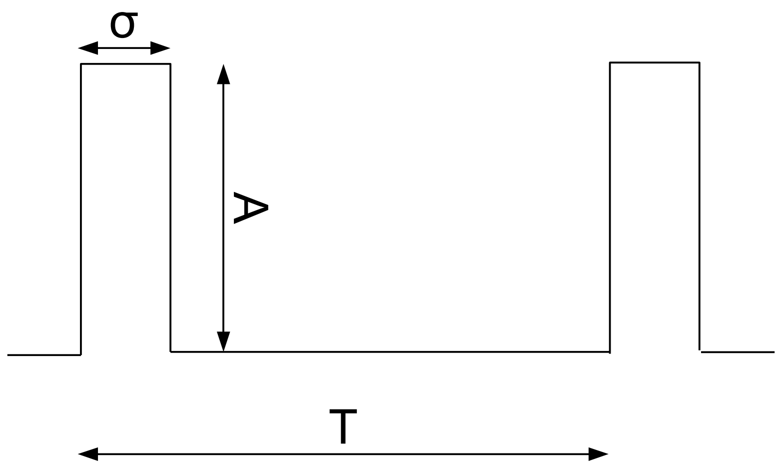
For a sampled waveform, if a signal has no DC component, the AC-coupled RMS value is equal to the population standard deviation[5] statistic.

An important measure of a waveform for practical RMS measurements is the **crest factor**. This is defined to be the peak value of the waveform divided by the RMS value with the DC component removed (what we'll call the AC-coupled RMS value). The larger the crest factor is, the more difficult it is for circuits to measure the RMS value. Practically, we'll find that the less expensive instruments can handle waveforms with crest factors in the region of 3 to 5; the more expensive lab-quality instruments can handle crest factors of 10 or higher.

Let's calculate the RMS value for a pulse waveform of period T as shown in the following diagram:

4 You could instead define f(x) in the continuous-case definition to be a sum of suitably-weighted Dirac delta functions and the continuous definition could be used for the discrete case.

5 The population standard deviation uses the sample size n in the denominator rather than the more familiar $n-1$ in the sample standard deviation.



Let the duty cycle of this waveform be D so that $\sigma = DT$. From the definition of RMS (equation (2)), we have

$$\text{RMS value} = \sqrt{\frac{\int_0^\sigma A^2 dt}{T}} = A\sqrt{D}$$

Since the crest factor is defined as the ratio of the peak value to the RMS value, we've derived that **for a pulse, the crest factor is the reciprocal of the square root of the duty cycle**.

Using python and numpy, you can calculate the RMS values of waveforms using the above discrete definition (3) without having to do any integration. Here's a chunk of code that prints the RMS value of a sine wave:

```
from numpy import *

n, period = 100, 2*pi
t = arange(0, period, period/n)
f = sin(t)
print "rms = %.3f", sqrt(sum(f*f)/n)
print "average = %.3f", sum(abs(f))/n
```

When run, the output is

```
rms = 0.707
average = 0.636
```

The RMS value is $1/\sqrt{2}$. The average value is $0.707/0.636 = 1.111$ lower than the RMS value. If you run the script for other waveforms, you'll see that this factor doesn't work to get the RMS value; hence, **average-responding meters won't correctly display the RMS value of non-sinusoids**.

It's a simple matter to use a script to calculate the RMS value of a pulse train. The Agilent document at http://cp.literature.agilent.com/litweb/pdf/5988-6916EN.pdf discusses RMS measurements and on page 5 gives the example of a 1.984 Vpp pulse train with a 2% duty cycle. The following script calculates the RMS value of this waveform:

```
from numpy import *

n, duty_cycle = 100, 0.02
t = arange(0, 1, 1./n)
f = zeros(n)
for i in xrange(int(n*duty_cycle)):  # Set the non-zero points
    f[i] = 1.984
rms = sqrt(sum(f*f)/n)
print "rms =", rms, "  crest factor =", max(f)/rms
print "average =", sum(abs(f))/n
```

When run, the output is

```
rms = 0.280579970775   crest factor = 7.07106781187
average = 0.03968
```

You can see the average is only 14% of the RMS value -- an average-responding meter would be seriously in error. The crest factor of this pulse train is $\frac{1}{\sqrt{0.02}}$ or about 7.

When you're experimenting, you may want to increase the number of points n by a factor of 10 to ensure you're not seeing sampling size effects.

Instruments

In this section, we look at some of the instruments that can be had to make RMS measurements. Since RMS measurements are inherently more complicated than average measurements, you should not be surprised that RMS-capable instruments are more complicated than averaging meters -- and, thus, more expensive.

The RMS-measuring instruments are divided into two categories: the lightweights and the heavyweights. These terms not only refer to the approximate mass of the instrument, but also to their capabilities. The lightweights are the numerous digital multimeters sold as **true RMS** instruments. The heavyweights are the more expensive lab-quality meters; these usually have higher bandwidths, crest factor specs, and accuracies -- and they often let you measure either **true RMS** or **AC+DC**.

I have only used a few of the instruments on this list and I emphasize that it isn't intended to be complete in any manner. I know there are other instruments out there I don't even know about. These lists are just what I could track down with a reasonable amount of effort.

First are the heavyweights. These instruments tend to be able to measure waveforms with crest factors of 10 or greater and the meters have wide bandwidths. They are the lab-quality instruments that are intended to be used for serious scientific and engineering use -- and the new instruments are priced way beyond what the typical hobbyist can afford. However, a hobbyist can find these lab-quality instruments used on places like ebay. While there is some risk that you'll buy a broken instrument, the benefit is that you can also get a working instrument for a fraction of what a new one would cost. My advice is to be patient, get to know what things are worth, and look for a seller who offers the ability to return the instrument if you're not happy with it.

The models with a gray background are obsolete, but can still be found used. Prices are estimates of what you'll find on ebay; used equipment companies will also sell these instruments, but they are usually 5 to 10 times the prices shown, as a reputable seller will recondition the instrument and make sure it meets its specifications.

Manufacturer	Model	Price, $	Comments
Agilent	34401A	1100	Digital, 10 Hz to 20 kHz (up to 100 kHz at reduced accuracy)
Ballantine	323	2500	Analog, 2 Hz to 25 MHz
Boonton	93A	?	Analog, 10 Hz to 20 MHz, crest factor 6 at full scale, accuracy 1.5%.
Fluke	8920A	50-150	Digital, 10 Hz to 20 MHz, adjustable dBm reference resistance
HP	3456A	100-300	Digital, 20 Hz to 100 kHz
HP	3403A	100-200	Digital, DC to 100 MHz

Manufacturer	Model	Price, $	Comments
HP	3400A/B	50-150	Analog, 10 Hz to 10 or 20 MHz
Rohde & Schwarz	URE3	?	Digital, 0.01 Hz to 30 MHz. Also reads peak values.

The Agilent 34401A is a descendant of the HP 3456A; both instruments are general purpose digital multimeters. The other instruments are (primarily) dedicated RMS instruments.

Here are the lightweights. In general, the crest factors are in the neighborhood of 3 to 5. DMM is "digital multimeter". Note that accuracy is generally specified in multiple parts with different values in different frequency bands, so quoting a single number is meant to be an estimate. Prices gotten from web in July 2012; street prices may be somewhat cheaper.

Manufacturer	Model	Price, $	Comments
Agilent	U1251B	390	DMM, 1% accuracy over 45 Hz to 5 kHz (up to 30 kHz at reduced accuracy), crest factor < 3 at full scale, < 5 at half scale.
B&K Precision	2709B	105	DMM, 50-500 Hz, accuracy probably in the 1.5% to 2% range, crest factor < 3.
Fluke	87V	400	DMM, 50 Hz to 20 kHz, 2% accuracy over this range. Crest factor up to 3 at full scale; up to 6 at half scale. The specs say add (2% of reading + 2% of full scale) for non-sinusoidal waveforms, so it's at best around 5% measurement for non-sinusoids.

I've never used it, but the B&K 2709B intrigues me because it is capable of measuring both **true RMS** and **AC+DC** values in a DMM that sells for a bit more than $100.

Making RMS measurements

There are four types of measurements you run into from a practical point of view: average, AC-coupled RMS, RMS, and peak. I won't discuss peak measurements.

For DC and sinusoids, the average-responding meter works just fine. If you're just measuring "normal" line voltages and currents and DC values, this is all the meter you need. I used quotes around "normal" to alert you that all's not well. When you get around loads with reactive components, loads with switching going on (e.g., a triac or SCR), or lots of noise, an average-responding meter may give you erroneous readings -- **and you won't be aware of this fact**. In a previous section you saw why -- for a high crest factor waveform like a pulse, the average is nowhere near the RMS value. In addition, if you're calculating power, a non-unity power factor can also mess up your calculations, regardless what type of meter you're using.

Because of this, you want to make the measurement with an RMS-responding meter. There's another trap for the unwary -- you need to know the type of RMS measurement your meter makes.

There are two types: **AC-coupled RMS** and **RMS**. I'm using these terms because they are descriptive, but the marketing people have unfortunately used less clear terms. I'm going to show the marketing terms in **this font** to distinguish them from the previous two more sensible terms.

A majority of the meters called **true RMS** meters in fact only measure the **AC-coupled RMS** value of a waveform. I assume you've used an oscilloscope before and know what AC coupling is -- the signal is connected to the instrument through a low impedance capacitor. Since this eliminates the

DC component of a signal, you won't have a number that represents the heating value of a waveform.

If you have an AC-coupled RMS meter, you'll need to measure the DC value of the waveform and combine it in quadrature with the AC-coupled RMS value to get the RMS value of the waveform:

$$\text{RMS} = \sqrt{V_{dc}^2 + V_{AC-coupled}^2}$$

The marketers have apparently decided to use the term **AC+DC** to designate those meters that measure the "real" RMS value of a waveform -- i.e., the heating equivalent, what I'm just calling the RMS value here. Users would probably be less confused if the industry standardized on "AC-coupled RMS" and "RMS", but we're probably stuck with the silly **true RMS** and **AC+DC**.

Fortunately, it's pretty easy to tell which type of meter you have. First, read the documentation and it will hopefully tell you. If you can't figure it out from the documentation, a quick measurement is in order. Connect your RMS meter to a source of DC voltage. If the meter reads the DC voltage level properly, it's an **AC+DC** type of meter. Otherwise, it's likely an AC-coupled RMS meter. You can check this with a function generator -- set the function generator to a 1 volt peak-to-peak 60 Hz square wave. Your RMS meter should read $\frac{1}{2\sqrt{2}} = 0.356$ volts RMS. If you then put a DC offset on the output of the function generator and the RMS meter's reading doesn't change, you have an AC-coupled RMS (**true RMS**) meter.

If you have a meter that can read both RMS and AC-coupled RMS values, beware that it can be easy to forget which you have selected and thus make an incorrect measurement.

An alternative: a scope

For a hobbyist on a budget, you might want to consider whether you need an RMS meter at all, because there's an alternative: a digital oscilloscope. Modern digital oscilloscopes typically have the ability to measure the RMS value of a waveform. The ones I have used measure the RMS value when they are DC coupled and the AC-coupled RMS value when they are AC-coupled. Thus, these scopes can give you either the **true RMS** or **AC+DC** value of a waveform. In addition, these scopes usually give you many more statistics and functionals associated with the waveform.

Besides giving the measurements, a modern dual channel digital scope is a powerful general purpose tool. You can do things like display the FFT of a waveform and multiply the two channels -- thereby getting the power waveform if you have the current and voltage waveform. Since oscilloscope bandwidths can go into the GHz region, you can make these RMS measurements on waveforms beyond the capabilities of the typical RMS meter. Thus, for a hobbyist, in general I would say a scope is probably a better investment than an RMS meter. Of course, a scope may be more money, but it's probably the most useful general-purpose measurement tool to have. If I could have only one electrical measurement tool, it would be a modern digital scope with a built-in function generator.

The typical digital scope has 8 bits of voltage resolution, so if you need RMS measurements with better resolution, you'll have to look for a scope with higher resolution (or get a dedicated RMS meter). 8 bits gives you roughly 0.4% resolution, which is better than the typical 3-5% resolution you can get from the scope's screen.

References

[gf] O. M. Gelfand and S. V. Fomin, *The Calculus of Variations*, Prentice-Hall, 1963. Dover has reprinted this excellent book written by two highly-regarded Russian mathematicians. It contains a nice explanation of a beautiful theorem: Noether's theorem, which connects mathematical symmetry with things that don't change in the time evolution of a physical system (i.e., conservation laws).

Emmy Noether was a remarkable woman mathematician in the early 1900's. See http://en.wikipedia.org/wiki/Noether%27s_theorem.

[hp3400] http://www.hpl.hp.com/hpjournal/pdfs/IssuePDFs/1964-01.pdf Article about the HP 3400A RMS voltmeter.

[hpbb] http://www.hparchive.com/Bench_Briefs/HP-Bench-Briefs-1973-11-12.pdf 1973 article on analog meters and RMS measurements.

[schw] M. Schwartz, *Information Transmission, Modulation, and Noise*, 2nd. ed., McGraw-Hill, 1970.