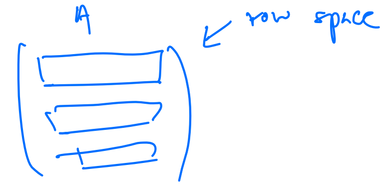
$\boxed{A = m \times n \text{ matrix}}$

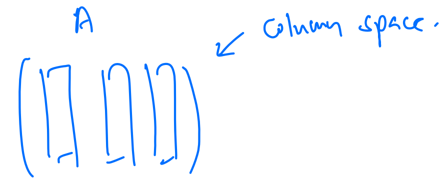
Row space

→ Subspace of $\mathbb{R}^n$ spanned by rows of $A$.

↳ each row has $n$ coordinates.

→ $\text{Row}(A) = \text{span}\{(a_{11}\ a_{12} \cdots a_{1n}), \cdots (a_{m1}\ a_{m2} \cdots a_{mn})\}$




→ Row space is preserved after row operations.

↳ updating the rows or moving around the rows do not modify the linear relationships between columns

↳ e.g. $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \xrightarrow{2R_1} \begin{pmatrix} 2 & 4 & 6 \end{pmatrix}$

×2 no change ×2

$\text{Row}(A) = \text{Row}(B)$ if $A$ and $B$ are row equivalent.

→ If $R$ is in RREF form, the non-zero rows form a basis for its row space.

↳ non-zero rows in RREF are linearly independent.

↳ non-zero rows of RREF of $A$ form the basis for $\text{Row}(A)$

→ since $A$ is row equivalent to RREF of itself

Column space

→ Subspace of $\mathbb{R}^m$ spanned by columns of $A$.

↳ each column has $m$ coordinates.

→ $\text{Col}(A) = \text{span}\left\{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1}\end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn}\end{pmatrix}\right\}$

→ Basis for $\text{Col}(A)$ = columns of $A$ corresponding to pivot columns in RREF of $A$.

↳ So long as RREF of selected columns of $A$ contains no non-pivot columns and they span $\mathbb{R}^m$, they form a basis for $\text{Col}(A)$

linear combinations of each other

↳ e.g. $\begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

pivot columns

any other columns other than the first 2 columns is sufficient to form the basis for $\text{Col}(A)$

→ Column space is NOT preserved by row operations.

↳ linear relations between rows are modified by row operations.

→ Checking if a vector is in $Col(A)$

↳ vector $v$ is in $Col(A)$ iff:

→ $v$ can be expressed as a linear combination of the columns of $A$. i.e.

$c_1 u_1 + c_2 u_2 + \dots + c_k u_k = v$

$\Leftrightarrow$ solving for $(u_1\ u_2\ \dots\ u_k)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = v \Rightarrow Ax = v$

$\Leftrightarrow$ solving $(u_1\ u_2\ \dots\ u_k \mid v)$

Null space

Null (A) is subspace of $\mathbb{R}^n$

→ $Null(A) = \{ v \in \mathbb{R}^n \mid Av = 0 \}$

→ similar to solution space → Solution set obtained by finding solution for $Ax = b$, solving $(A \mid v)$

→ Null (A) is obtained by finding solution for $Ax = 0$, solving $(A \mid 0)$

→ nullity $(A) = \dim(Null(A))$ = no. of non-pivot columns in RREF.

→ row space always perpendicular to nullspace ⇒ row vector · unknown vector = 0

orthogonal complement

→ $\dim(\text{row space}) + \dim(\text{null space}) = \dim(\text{matrix})$

→ $Null(A) = Null(A^T A)$ → $A^T A u = A^T(0) = 0$ ⇒ $u$ also in $Null(A^T A)$

($\Rightarrow$) If $u$ in $Null(A)$, $Au = 0$ ⇒ $A^T A u = A^T(0) = 0$ ⇒ $(Au)^T(Au) = 0$

($\Leftarrow$) If $u$ in $Null(A^T A)$, $A^T A u = 0$ ⇒ $u^T A^T A u = u^T(0) = 0$

⇒ $(Au) \cdot (Au) = 0$ ⇒ $Au = 0$ ⇒ $u$ also in $Null(A)$.

Rank

→ $rank(A) = \dim(Col(A))$

= no. of pivot columns in RREF

= no. of leading entries in RREF

= no. of non-zero rows in RREF

$= \dim(Row(A))$

→ $rank(A) = rank(A^T)$

↳ $\dim(Col(A^T)) = \dim(Row(A))$

$= \dim(Col(A))$

$= \dim(Row(A^T))$

→ $rank(0) = 0$ as zero vector contains no non-zero rows.

→ $Ax = b$ is consistent iff $\text{rank}(A) = \text{rank}(A|b)$

↳ rank = no. of pivot columns in RREF

↳ if $\text{rank}(A) < \text{rank}(A|b) \Rightarrow b$ is a pivot column in RREF ⇒ system is inconsistent.

Properties of Rank

→ $\text{Col}(AB)$ is a subspace of $\text{Col}(A)$

↳ $\text{Col}(AB) \subseteq \text{Col}(A)$

↳ $AB = A(b_1\ b_2 \cdots b_k) = (Ab_1\ Ab_2 \cdots Ab_k)$

↳ Since $Au \in \text{Col}(A)$ for all $u$, $Ab_i \in \text{Col}(A)$ for $i = 1, \ldots, k$

↳ So $\text{Col}(AB) = \text{span}(Ab_1, Ab_2 \cdots Ab_k) \subseteq \text{Col}(A)$

→ $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

→ Show that if A and B are row equivalent, $\text{rank}(A) = \text{rank}(B)$

↳ $A = PB$, where $P = E_k \cdots E_2 E_1$

$\text{rank}(PB) \leq \min\{\text{rank}(P), \text{rank}(B)\}$

since P is a product of elementary matrices, it is always invertible,

So P will be full rank.

$\min\{\text{rank}(P), \text{rank}(B)\}$ depends on $\text{rank}(B)$

so $\text{rank}(PB) \leq \text{rank}(B) \Rightarrow \text{rank}(A) \leq \text{rank}(B)$.

↳ $B = P^{-1}A$, where $P^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

$\text{rank}(P^{-1}A) \leq \min\{\text{rank}(P^{-1}), \text{rank}(A)\} \Rightarrow \text{rank}(B) \leq \text{rank}(A)$

↳ since $\text{rank}(A) \geq \text{rank}(B)$ and $\text{rank}(B) \leq \text{rank}(A)$,

$\text{rank}(A) = \text{rank}(B)$.

Rank-Nullity Theorem

→ $\underbrace{\text{rank}(A)}_{\text{no. of pivot columns in RREF}} + \underbrace{\text{nullity}(A)}_{\text{no. of non-pivot columns in RREF}} = $ no. of columns in A. = no. of columns in RREF

Full Rank

→ $\text{rank}(A) = \min\{m, n\}$, where $m$ = no. of rows, $n$ = no. of columns.

If $A$ is a $m \times n$ matrix where $m \neq n$

→ If full rank = no. of columns = $n$, the statements are equivalent.

RREF = $\begin{pmatrix} I \\ \text{zero rows} \end{pmatrix}$

1. rank $(A) = n$
2. Row $(A) = \mathbb{R}^n$, rows of $A$ spans $\mathbb{R}^n$
3. Columns of $A$ are linearly independent.
4. $Ax = 0$ only has the trivial solution ⇒ Null $(A) = \{0\}$
5. $A^T A$ is invertible. → Null $(A)$ = Null $(A^T A)$ ⇒ $A^T A x = 0$ also only has the trivial solution ⇒ Since $A^T A$ is square matrix, $A^T A$ is invertible (by statements of invertibility)
6. $A$ has a left inverse → $I = (A^T A)^{-1}(A^T A) = \underbrace{((A^T A)^{-1} A^T)}_{\text{left inverse of } A.} A$
7. The transformation $T_A$ represented by $A$ is injective.

→ If full rank = no. of rows = $m$.

1. rank $(A) = m$
2. Col $(A) = \mathbb{R}^m$, columns of $A$ span $\mathbb{R}^m$ ⇒ there exist no non-zero rows.
3. Rows of $A$ are linearly independent.
4. $Ax = b$ is consistent for every $b \in \mathbb{R}^m$ → all rows have leading entries ⇒ $Ax = 0$ is consistent ⇒ $Ax = b$ is consistent for any $b$.
5. $AA^T$ is invertible
6. $A$ has a right inverse. → $I = (AA^T)(AA^T)^{-1} = A \underbrace{(A^T (AA^T)^{-1})}_{\text{right inverse of } A.}$
7. The transformation $T_A$ represented by $A$ is surjective.