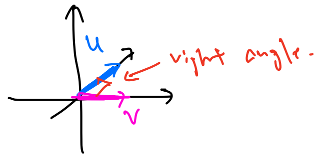
Orthogonal



→ $u \cdot v = 0$

↳ Case ① : $u = 0$ or $v = 0$

↳ Case ② : $\cos(\theta) = \frac{u \cdot v}{\|u\|\,\|v\|} = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ u and v are perpendicular

→ Orthogonality is not affected by scalar multiplication

↳ Given $u \cdot v = 0$

↳ Explanation ① :

$(su) \cdot (tv) = (st)\underbrace{(u \cdot v)}_{0} = st(0) = 0$

↳ Explanation ② :

scalar multiplication only affects the length of the vectors, they do not change the angle of the vectors.

So, $(su) \cdot (tv) = 0$

→ orthogonal set can contain zero vector. ⇒ orthogonal sets may not be linearly independent.

→ Pairwise orthogonal

↳ $v_i \cdot v_j = 0$ for every $i \neq j$ in $S = \{v_1, v_2, \ldots, v_k\}$

→ Orthonormal

↳ $v_i \cdot v_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ ← $v_i \cdot v_i = \|v_i\| =$ norm $= 1$ ⇒ unit vector

↳ Standard basis is always orthonormal.

↳ Orthogonal sets not containing the zero vector can be normalized into orthonormal sets.

→ $\{u_1, u_2, \ldots, u_k\} \Rightarrow \left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \ldots, \frac{u_k}{\|u_k\|}\right\}$ where $\|u_1\|, \|u_2\|, \ldots, \|u_k\| \neq 1$

→ Orthonormal sets are always linearly independent

Orthogonal to Subspace

→ $n \perp V$ iff $n \cdot v = 0$ for all $v \in$ subspace V

↳ $0 \perp V$ as $0 \cdot v = 0$ for all $v \in V$

↳ n can be called the normal to plane V

→ Subspace can be expressed in terms of non-zero orthogonal vectors :

↳ $V = \{v \in \mathbb{R}^n \mid v \cdot n = 0\}$

↳ if $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $v \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax + by + cz = 0$

Checking for Orthogonal to Subspace

$\to$ vector $w \perp$ subspace V iff $w \in Null(A^T)$, where $A = \underbrace{(u_1\ u_2 \cdots u_k)}_{\text{spans } V}$

$\hookrightarrow$ $w \perp V$ iff $w \cdot v = v^T w = 0$ for all $v \in V$

$\hookrightarrow$ $A^T w = \underline{(u_1\ u_2 \cdots u_k)^T} w = \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_k^T \end{pmatrix} w = \begin{pmatrix} u_1^T w \\ u_2^T w \\ \vdots \\ u_3^T w \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$

$\hookrightarrow$ solving for $w \to$ solving $(A^T | 0) \Rightarrow$ finding $Null(A^T)$

Orthogonal Complement

$\to$ the set of all vectors orthogonal to subspace V.

$\to$ $V^\perp = \{ w \in \mathbb{R}^n \mid w \cdot v = 0 \text{ for all } v \in V \}$

$\to$ if $S = \{u_1, u_2, \cdots, u_k\}$ such that $span(S) = V$, and $A = (u_1\ u_2 \cdots u_k)$,

$V^\perp = Null(A^T)$

$\to$ Prove $Row(A)^\perp = Null(A)$

$\hookrightarrow$ $Null(A) \Rightarrow$ solution for $Ax = 0$

$Ax = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix} x = 0 \Rightarrow u_1 \cdot x = u_2 \cdot x = \cdots = u_k \cdot x = 0$

Rows of A is orthogonal to $\underline{\underline{x}}$ $\swarrow$ solution to $Ax = 0 \Rightarrow Null(A)$

$\therefore Row(A)^\perp = Null(A)$

Orthogonal & Orthonormal Basis

$\to$ S is an orthogonal / orthonormal basis for subspace V if S is an orthogonal / orthonormal set.

Coordinates relative to Orthogonal Basis.

$\to$ $S = \{u_1, u_2, \cdots, u_k\}$ is an orthogonal basis for V. $v = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$.

Recall: $\|u\| = \sqrt{u \cdot u}$

$\|u\|^2 = u \cdot u$

$\hookrightarrow$ $u_i \cdot v = u_i \cdot (c_1 u_1 + c_2 u_2 + \cdots + c_k u_k)$

$= c_1 u_1 \cdot u_i + \cdots + c_i u_i \cdot u_i + \cdots + c_k u_k \cdot u_i$

$= c_1(0) + \cdots + c_i \|u_i\|^2 + \cdots + c_k(0)$

$= c_i \|u_i\|^2$

$c_i = \dfrac{u_i \cdot v}{\|u_i\|^2}$

$[v]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} \frac{u_1 \cdot v}{\|u_1\|^2} \\ \vdots \\ \frac{u_k \cdot v}{\|u_k\|^2} \end{pmatrix}$

→ if S is an orthonormal basis, $\|u_i\|^2 = 1$

↳ $[v]_S = \begin{pmatrix} u_1 \cdot v \\ \vdots \\ u_k \cdot v \end{pmatrix}$

Suppose $S = \{u_1, u_2 \cdots u_k\}$, $Q = (u_1\ u_2 \cdots u_k)$

→ $Q^T Q = \begin{pmatrix} u_1^T \\ \vdots \\ u_k^T \end{pmatrix} (u_1 \cdots u_k) = \begin{pmatrix} u_1 \cdot u_1 & \cdots & u_1 \cdot u_k \\ \vdots & \ddots & \vdots \\ u_k \cdot u_1 & \cdots & u_k \cdot u_k \end{pmatrix}$

→ if $S = \begin{cases} \text{orthogonal} \\ \text{orthonormal} \end{cases}$ ⟺ $Q^T Q$ is $\begin{cases} \text{diagonal matrix} \leftarrow \text{orthogonal} \Rightarrow u_i \cdot u_i \neq 0 \\ \text{identity matrix} \leftarrow \text{orthonormal} \Rightarrow u_i \cdot u_i = 1 \end{cases}$

Orthogonal Matrices

→ Square matrix A is orthogonal if $A^T = A^{-1}$

↳ $A^T A = I = A A^T$

→ for any square matrix A, the following are equivalent.

1. A is an orthogonal matrix
2. Columns of A form an orthonormal basis for $\mathbb{R}^n$
3. Rows of A form an orthonormal basis for $\mathbb{R}^n$

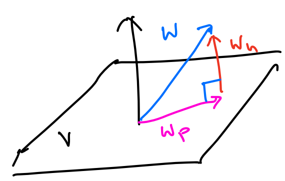
Orthogonal Projection

→ Every vector w in $\mathbb{R}^n$ can be written uniquely as : $w = w_p + w_n$, where $w_n \perp V$, $w_p \in V$

→ if $S = \{u_1, u_2, \cdots, u_k\}$ is an orthogonal basis for V,

$$w_p = \frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k$$

projection of w onto u_i



Best Approximation Theorem

w_p is the closest vector in V to w.

→ if w_p is the projection of w onto V,

→ $\|w - w_p\| \le \|w - v\|$ for any $v \in V$

distance between w and w_p ↑ ↑ distance between w and v for any $v \in V$

Gram-Schmidt Process.

→ Converting a basis S to an orthonormal basis. ← normalized $\{v_1, v_2, \cdots, v_k\}$

$\text{span}\{u_1, u_2, \cdots, u_k\} = \text{span}\{v_1, v_2, \cdots, v_k\}$

→ $v_1 = u_1$

$v_2 = u_2 - \left(\frac{v_1 \cdot u_2}{\|v_1\|^2}\right) v_1$ ← projection of u_2 onto v_1

$\vdots$

$v_k = u_k - \left(\frac{v_1 \cdot u_k}{\|v_1\|^2}\right) u_k - \cdots - \left(\frac{v_{k-1} \cdot u_k}{\|v_{k-1}\|^2}\right) v_{k-1}$

← projection of u_k onto v_1 ← projection of u_k onto v_{k-1}

→ $\{v_1, v_2, \cdots, v_k\}$ = orthogonal set

$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \cdots, \frac{v_k}{\|v_k\|} \right\}$ = orthonormal set.

→ $S = \{u_1, u_2, u_3, u_4\}$, After Gram-Schmidt Process, $v_4 = 0$ and $v_3 \neq 0$. Why?

$v_4 = u_4 -$ projection of u_4 onto span $\{v_1, v_2, v_3\}$

span $\{v_1, v_2, v_3\}$ = span $\{u_1, u_2, u_3\}$

$u_4 =$ projection of u_4 onto span $\{u_1, u_2, u_3\}$

$u_4 \in$ span $\{u_1, u_2, u_3\}$ ⟺ u_4 is a linear combination of u_1, u_2, u_3

⟺ S is linearly dependent.

QR factorization.

→ $A = QR$, where

↳ $A_{m\times n}$ has linearly independent columns ⟹ rank $(A) = n$

↳ $Q_{m\times n}$ where $Q^TQ = I$

↳ R is an invertible upper triangular matrix with positive diagonal entries.

→ Suppose $A = (a_1\ a_2 \cdots a_n)$

↳ Perform Gram-Schmidt Process to get $Q = (q_1\ q_2 \cdots q_n)$ where $\{q_1, q_2, \cdots, q_k\}$ forms an orthonormal set.

↳ $a_1 = r_{11} q_1 = (q_1\ q_2 \cdots q_n)\begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$a_2 = r_{12} q_1 + r_{22} q_2 = (q_1\ q_2 \cdots q_n)\begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vdots$

$a_n = r_{1n} q_1 + \cdots + r_{nn} q_n = (q_1 \cdots q_n)\begin{pmatrix} r_{1n} \\ \vdots \\ r_{nn} \end{pmatrix}$

↳ $A = (a_1\ a_2 \cdots a_n)$

$= \underbrace{(q_1\ q_2 \cdots q_n)}_{\text{orthonormal columns}} \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & r_{nn} \end{pmatrix}$ ← upper triangular square matrix

$= QR$

↳ $A = QR$ ⟹ $Q^TA = Q^TQR = R$

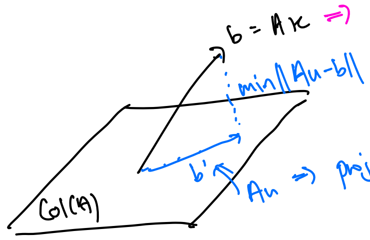
→ Unique least square solution to $Ax = b$

↳ $x = R^{-1}Q^Tb$ ⟹ Solve for $Rx = Q^Tb$

Least Square Approximation

→ u is the least square solution of $Ax=b$ if $\|Au-b\| \leq \|Av-b\|$

dist of b' to b — dist of other vectors $\in Col(A)$ to b



$\Rightarrow$ b is not in $Col(A)$ $\Rightarrow$ $Ax=b$ is inconsistent

$Au \Rightarrow$ projection of b onto $Col(A)$

$\Rightarrow$ u is least square solution to $Ax=b$ $\Leftrightarrow$ Au is the projection of b onto $Col(A)$

$\Leftrightarrow$ u is the solution to $A^TAx = A^Tb$

↳ $Au-b \perp Col(A)$

$\Rightarrow (Au-b) \in Null(A^T)$

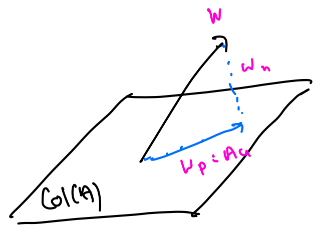
$\Rightarrow Au-b$ is a solution to $A^Tx = 0$

$\Rightarrow A^T(Au-b) = 0$

$\Rightarrow A^TAu = A^Tb$

→ Projection Au is unique

→ least square solution u may not be unique.



Relooking at Orthogonal Projection

→ $w_p = Au$, where u is the least square solution.

→ Since A is linearly independent, A^TA is invertible.

↳ $u = (A^TA)^{-1}A^TAu = (A^TA)^{-1}A^Tw$, where $Ax=w$ is inconsistent