Matrix

→ size : row × columns

Types of Matrices.

→ Vectors

↳ Column vectors : $n \times 1$ matrix $\begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}$

↳ Row vectors : $1 \times n$ matrix $\begin{pmatrix} x & x & x & x \end{pmatrix}$

→ zero matrix : all entries are 0.

→ square matrix

↳ no. of rows = no. of columns

↳ size : order $n$ , $n \in \mathbb{R}$

→ Diagonal matrix

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_n \end{pmatrix}$$

where $d_1, d_2, \cdots, d_n \in \mathbb{N}$

↳ $d_1, d_2, \dots d_n$ can be different values.

→ Scalar matrix

$$\begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & & \vdots \\ \vdots & & \ddots & \vdots \\ \cdot & \cdots & & c \end{pmatrix}$$

where $c \in \mathbb{N}$

↳ diagonal entries are the same.

↳ $cI$ → $I$ is identity matrix

→ Identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ \vdots & \cdots & & 1 \end{pmatrix}$$

diagonal entries are 1.

→ Upper Triangular Matrix

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & * \end{pmatrix}$$

$a_{ij} = 0$ for all $i > j$

→ Strictly upper Triangular Matrix

$$\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & & 0 \end{pmatrix}$$

$a_{ij} = 0$ for all $i \geq j$

→ Lower Triangular Matrix

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & & \vdots \\ \vdots & & \ddots & \vdots \\ * & \cdots & \cdots & * \end{pmatrix} \qquad a_{ij} = 0 \text{ for all } i < j$$

→ Strictly Lower Triangular Matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ * & \cdots & \cdots & 0 \end{pmatrix} \qquad a_{ij} = 0 \text{ for all } i \leq j$$

→ Symmetric matrices

↳ $a_{ij} = a_{ji}$

↳ $A = A^T$

Matrices Equality

→ $A = B$ if size is the same and entries are the same.

Matrices Addition

→ $$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

Scalar Multiplication

→ $$c \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{21} & \cdots & ca_{1m} \\ ca_{21} & & & \vdots \\ \vdots & & \ddots & \\ ca_{n1} & \cdots & & ca_{nm} \end{pmatrix}$$

Matrix Multiplication

→ $$\underset{\underline{2} \times \underline{3}}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{\underline{3} \times \underline{2}}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} = \underset{\underline{2} \times \underline{2}}{\begin{pmatrix} \boxed{2} & \boxed{1} \\ 8 & 7 \end{pmatrix}}$$

→ not commutative : $AB \neq BA$

→ Pre-multiply A to B : $AB$

→ Post-multiply A to B : $BA$

## Zero Divisors

→ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

→ If $AB = 0$, either $A$ or $B$ is $0$, OR $A$ & $B \neq 0$

## Power of Square Matrices ← only for square matrices.

→ $A^0 = I$

→ $A^n = AA^{n-1}$, for $n \geq 1$

→ $A^{n+m} = A^n A^m$

## Transpose

→ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

→ With transpose, row becomes column, column becomes row

→ $(A^T)^T = A$ $\qquad$ → $(A+B)^T = A^T + B^T$

→ $(cA)^T = cA^T$ $\qquad$ → $(AB)^T = B^T A^T$

## Matrix Equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \ddots & & \vdots \\ a_{n1} & \cdots & & a_{nm} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \Rightarrow Ax = b$$

↑ Coefficient matrix $\qquad$ ↑ Variable Vector $\qquad$ ↑ Constant vector

## Vector Equation

$$x_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_m\begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \Rightarrow x_1 a_1 + x_2 a_2 + \cdots + x_m a_m = b$$

↑ ↑ ↑

coefficient vector

## Solutions to Homogenous & Non-Homogenous Linear Systems

→ If $v$ is sol$^n$ to $Ax = b$, $u$ is sol$^n$ to $Ax = 0$

then $u + v$ is sol$^n$ to $Ax = b$

↳ $Av = b$, $Au = 0$

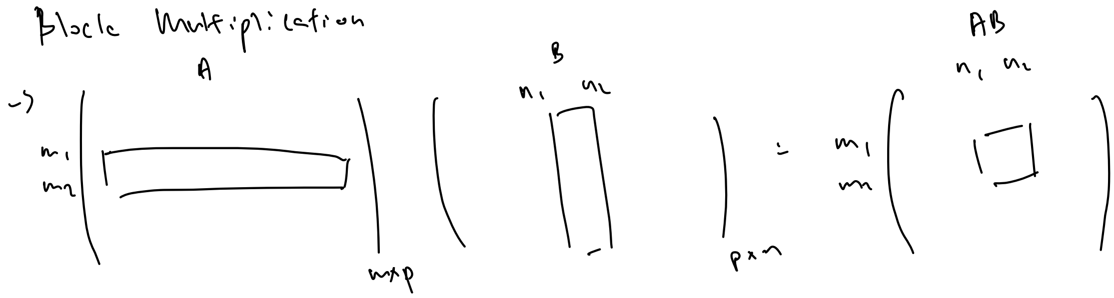
$$A(u+v) = Au + Av = 0 + b = b$$

→ If $v_1$ & $v_2$ are solutions to $Ax = b$, $v_1 - v_2$ is solution to $Ax = 0$

↳ $Av_1 = b$, $Av_2 = b$

$$A(v_1 - v_2) = Av_1 - Av_2 = b - b = 0$$

## Block Multiplication




→ $AB = A(b_1 \; b_2 \; b_3 \dots b_n) = (Ab_1 \; Ab_2 \dots Ab_n)$

→ $AB = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} B = \begin{pmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{pmatrix}$

## Solving for Combined Augmented Matrices.

→ $\left(\begin{array}{cccc|c|c|c|c} a_{11} & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & & b_{1p} \\ a_{21} & & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & & & & & & & \\ a_{m1} & & \dots & a_{mn} & b_{m1} & b_{m2} & & b_{mp} \end{array}\right)$

## Invertible Square matrices. → Non-invertible matrices = Singular.

↳ A is singular if $Ax = 0$ has a non-trivial solution.

→ $AB = I = BA \Rightarrow AA^{-1} = A^{-1}A$

↳ Size of $A$ = Size of $B$

→ only square matrices are invertible.

→ Inverse is unique.

→ If $A$ is invertible, $Ax = b$ has a unique solution

* → If $A$ is invertible, $Ax = 0$ has the trivial solution.

→ $(A \mid b) \xrightarrow{RREF} (I \mid A^{-1}b)$

↳ $A$ is invertible iff RREF of $A$ is $I$

→ Computing Inverse

↳ $(A \mid I) \xrightarrow{RREF} (I \mid A^{-1})$ ← $A^{-1}I = A^{-1}$

Cancellation law for Invertible Matrices

→ Left Cancellation : $AB = AC \Rightarrow B = \underbrace{A^{-1}A}_{I}B = \underbrace{A^{-1}A}_{I}C = C$

→ Right Cancellation : $BA = CA \Rightarrow B = B\underbrace{AA^{-1}}_{I} = C\underbrace{AA^{-1}}_{I} = C$

Properties of Inverse

→ $(A^{-1})^{-1} = A$ → $A^{-n} = (A^{-1})^{n}$

→ Inverse of $(aA) = (aA)^{-1} = \frac{1}{a}A^{-1}$, for $a \neq 0$

→ Inverse of $A^{T} = (A^{T})^{-1} = (A^{-1})^{T}$ → $A$ is invertible if $A^{T}$ is

* caution.

→ If $B$ is invertible, Inverse of $(AB) = (AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}$

↳ $(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = \underbrace{B^{-1}B}_{I} = I = \underbrace{AA^{-1}}_{I} = A\underbrace{(BB^{-1})}_{I}A^{-1} = (AB)(B^{-1}A^{-1})$

↳ If $A$ is invertible, $A \times A \times A \times \ldots$ is invertible with $(A \times A \times A \ldots)^{-1}$

→ If $AB$ is invertible, $A$ & $B$ are invertible

Proof

1. Suppose $A$ is invertible, $B$ is not invertible..
   1.1. $Ax = 0$, $x = 0$ (by def of invertible)
   1.2. $Bx = 0$, $x \neq 0$ (by def of not invertible / singular)
   1.3. $ABx = 0$, $x = 0$ (by def of invertible)
   1.4. $A(Bx) = 0$, $Bx = 0$ (from 1.1)
   1.5. So $x = 0$ (trivial sol'n) is solution for $Bx = 0$, contradicts line 1.2.
   1.6 So $B$ is invertible.
2. Repeat 1.1 to 1.6 for when $A$ is not invertible & $B$ is invertible.

→ Inverse of an invertible symmetric matrix is symmetric

Proof: $A^{-1} = (A^{-1})^T$

$AA^{-1} = (AA^{-1})^T$ as $I = I^T$

$= (A^{-1})^T A^T$

as $AA^{-1} = I = A^{-1}A$

$A^{-1}A = (A^{-1})^T A$ as $A = A^T$ (by def of symmetric)

$A^{-1}\underline{AA^{-1}} = (A^{-1})^T \underline{AA^{-1}}$ (post-multiply by $A^{-1}$)

$A^{-1} = (A^{-1})^T$

Elementary Matrices

→ Row operation corresponding to ERO

→ $A \xrightarrow[E_1]{r_1} \xrightarrow[E_2]{r_2} \cdots \xrightarrow[E_k]{r_k} B$ ⇒ $A \xrightarrow{r_1} E_1 A \xrightarrow{r_2} E_2 E_1 A \cdots$

↳ $B = E_k \cdots E_2 E_1 A$

Inverse of Elementary Matrices

→ $B \xrightarrow{r_k'} \cdots \xrightarrow{r_2'} \xrightarrow{r_1'} A$

↳ $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} B$

→ $E : R_i + cR_j$ ⇒ $E^{-1} : R_i - cR_j$

→ $E : R_i \leftrightarrow R_j$ ⇒ $E^{-1} : R_i \leftrightarrow R_j$

→ $E : cR_j$ ⇒ $E^{-1} : \frac{1}{c} R_j$

→ Elementary matrices are always invertible.

↳ they are row equivalent to Identity matrices.

* Equivalent Statements of Invertibility on another document

Suppose $A$ is $n \times m$ matrix

Left Inverse : $B_{m\times n}$ is left inverse of $A$ iff $BA = I_m$ ⇒ $A$ is right inverse of $B$

Right Inverse : $B_{m\times n}$ is right inverse of $A$ iff $AB = I_n$ ⇒ $A$ is left inverse of $B$.

# LU Factorization

→ $A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_k^{-1}}_{L} \times \underbrace{REF}_{U}$

always be unit lower triangular matrix ⇒ diagonal entries = 1

⇒ ERO consist only addition & subtraction ($R_i + cR_j$, for $i > j$), no row swaps or scalar multiplication to maintain unit lower triangular matrix.

→ $Ax = b$ ⇒ $LUx = b$

Steps:

↳ let $Ux = y$ ⇒ $Ly = b$

$L$ is unit lower triangular matrix, so $Ly = b$ is always consistent.

↳ solve for $Ux = y$

→ Not all matrices are LU factorizable.

# Determinant ⇒ "area" of the matrix

→ $\det(A)$ or $|A|$

→ Determinant by cofactor expansion

↳ Finding cofactor. take note of the sign (they are alternating)

e.g. $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\begin{pmatrix} + & - & + & \cdots \\ - & + & \cdots & \\ + & \cdots & \ddots & \\ \vdots & \vdots & & \end{pmatrix}$

$A_{11} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(3) = -5$

$A_{12} = -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = -[(-1)(1) - (3)(3)] = 10$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -[(1)(2) - (3)(2)] = 4$

↳ cofactor expansion along row or column

→ Determinant is invariant under transpose : $\det(A) = \det(A^T)$

→ Tip : cofactor expand along the row / column with the most 0's

→ Determinant of Triangular matrix = product of its diagonal entries.

→ Determinant by Reduction.

↳ Aim : since det (triangular matrices) = $\Pi$ of diagonal entries, we reduce to triangular matrix and multiply by determinant of elementary matrices used.

↳ $E : R_i + aR_j \Rightarrow \det(B) = \det(A)$ $\Rightarrow \det(E) = 1$

↳ $E : aR_i \Rightarrow \det(B) = a\det(A)$ $\Rightarrow \det(E) = a$

↳ $E : R_i \leftrightarrow R_j \Rightarrow \det(B) = -\det(A)$ $\Rightarrow \det(E) = -1$

↳ $R = E_k \cdots E_2 E_1 A \Rightarrow \det(R) = \det(E_k) \cdots \det(E_2)\det(E_1)\det(A)$

→ if $R$ is an upper triangular matrix,

$$\det(A) = \frac{d_1 \times d_2 \times \cdots \times d_n}{\det(E_k)\cdots\det(E_2)\det(E_1)}$$

← det (R) as det of triangular matrix is $\Pi$ of diagonal entries.

→ If $A$ and $B$ are same size, $\det(AB) = \det(A) \times \det(B)$

→ If $A$ is invertible : $\det(A^{-1}) = \det(A)^{-1} = \frac{1}{\det(A)}$

→ $\det(cA) = c^n \times \det(A)$

↳ $cA = (cI)(A) = \begin{pmatrix} c & \cdots & \\ \vdots & c & \\ & & c \end{pmatrix} A$

↳ $\det(cA) = \det(cI) \times \det(A) = c^n \times \det(A)$

Adjoint Formula

→ $\operatorname{adj}(A)$ = transpose of cofactors of $A$.

cofactors of $A$..

$$= \begin{pmatrix} A_{11} & \underline{A_{12}} & \cdots & \underline{A_{1n}} \\ A_{21} & \cdot & & \cdot \\ \vdots & & \cdot & \vdots \\ A_{n1} & \cdot & \cdot & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & \cdot & & \vdots \\ \vdots & & \cdot & \vdots \\ A_{1n} & \cdots & & A_{nn} \end{pmatrix}$$

→ $A(\operatorname{adj}(A)) = \det(A) I$

↳ true for all square matrices.

→ If $A$ is invertible : $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

Cramer's Rule

→ If $A$ is invertible, unique solution $x$:

$$x = \frac{1}{\det(A)} \begin{pmatrix} \det(A_1(b)) \\ \det(A_2(b)) \\ \vdots \\ \det(A_n(b)) \end{pmatrix}$$ where $A_i(b)$ is the $i^{th}$ column of $A \times b$.