Eigenvalues & Eigenvectors

if $v = 0$, any number can be eigenvalue

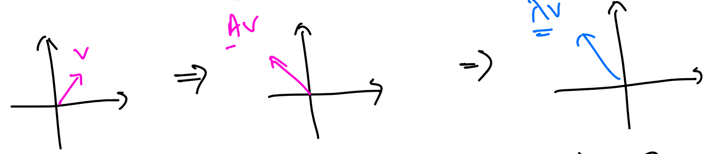
→ $Av = \lambda v$, for $v \neq 0$ and v in $\mathbb{R}^n$

eigenvalue. eigenvector

matrix that scales v

real number that denotes the scalar transformation

→ Geometrically :



→ $Av = \lambda v \Rightarrow Av = \lambda I v \Rightarrow \lambda I v - Av = 0 \Rightarrow (\lambda I - A)v = 0 \Rightarrow Bx = 0$

↳ $v \neq 0 \Rightarrow v$ is a non-trivial solution to $(\lambda I - A)x = 0$

↳ λ is an eigenvalue of $A \iff (\lambda I - A)x = 0$ has non-trivial solutions

↳ $(\lambda I - A)$ is singular $\Rightarrow \det(\lambda I - A) = 0$

$\iff \lambda$ is a root of $\det(xI - A)$

↳ values of a variable that makes polynomial $= 0$

↳ values of λ such that $\det(xI - A) = 0$

→ λ can be 0

↳ $Av = \lambda v = 0(v) = 0$, since $v \neq 0$, $Ax = 0$ has a non-trivial solution.

↳ A is singular / not invertible if $\lambda = 0$ (added to equivalent statements of invertibility)

$v \in \text{Null}(A)$

→ quick way to calculate eigenvalues of 2×2 matrices.

↳ given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 2 & 6 \end{pmatrix}$

↳ mean $(m) = \frac{a+d}{2} = 7$, product $(p) = ad - bc = 40$

↳ $\lambda = m \pm \sqrt{m^2 - p} = 7 \pm \sqrt{7^2 - 40} = 7 \pm 3 = 4, 10$

Characteristic Polynomial

→ Characteristic Polynomial of square matrix $A_{n \times n}$ is the degree n polynomial

→ $\text{char}(A) = \det(xI - A)$

→ e.g. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}$, $\det(xI - A) = \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x & -2 \\ 0 & -3 & x-1 \end{vmatrix} = (x-1)[x(x-1) - 6] = (x-1)(x+2)(x-3)$

↳ $\lambda = 1, -2, 3$

→ invariant under transpose : $\text{char}(A) = \text{char}(A^T) \Rightarrow \lambda$ for $A = \lambda$ for A^T

Algebraic Multiplicity

→ largest power $r\lambda$ of $\det(xI - A) = (x - \lambda)^{r\lambda} p(x)$, for some polynomial $p(x)$

→ $\det(xI - A) = (x - \lambda_1)^{r_1} (x - \lambda_2)^{r_2} \cdots (x - \lambda_k)^{r_k}$

↳ algebraic multiplicity of $\lambda_i = r_i$ for $i = 1, \ldots, k$.

→ $\det(xI - A) = (x - 1)(x^2 + 1)$

↳ only 1 real eigenvalue $\Rightarrow \lambda = 1$, $r_1 = 1$

Eigenvalues of Triangular Matrices.

→ λ_i = diagonal entries , r_i = no. of times λ_i appeared

→ recall det of triangular matrices = product of triangular matrices.

→ Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & a_{nn} \end{pmatrix}$.

$$\det(xI - A) = \begin{pmatrix} x-a_{11} & x-a_{12} & \cdots & x-a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & x-a_{nn} \end{pmatrix} = (x-a_{11})(x-a_{22})\cdots(x-a_{nn})$$

Eigenspace (E_λ)

→ Solution set of non-trivial solutions to $(\lambda I - A)x = 0$

→ $E_\lambda = \{ v \in \mathbb{R}^n \mid Av = \lambda v \} = \text{Null}(\lambda I - A)$

→ Geometric multiplicity : dimension of E_λ ⇒ range : $1 \le \dim(E_\lambda) \le r_\lambda$

↳ $r_1 + r_2 + \cdots + r_k \le$ order $\underline{\underline{n}}$

↳ $\dim(E_\lambda)$ = nullity $(\lambda I - A)$

↳ $\dim(E_{\lambda_i})$ of A = $\dim(E_{\lambda_i})$ of A^T

→ Note : E_{λ_i} and E_{λ_j} may not be the same for all $i, j \in \mathbb{R}$

✗ → Row operations do not preserve eigenvalues and eigenvectors.

↳ eigenvalues records the transformation of eigenvectors relative to a matrix.

↳ eigenvalues records the linear relationship between columns

↳ row operations do not preserve linear relationship between columns

↳ so, a different set of eigenvalues & eigenvectors are needed to record the transformation relative to the row equivalent matrix.

Diagonalization

→ $A_{n\times n}$ is diagonalizable iff there exist an invertible P such that $P^{-1}AP = D$ is a diagonal matrix → diagonal entries may not be distinct.

→ $P^{-1}AP = D$ ⇒ $A = PDP^{-1}$

→ zero square matrices are diagonalizable : $0 = I0I^{-1}$

→ identity matrices are diagonalizable : $I = PIP^{-1}$ for any invertible P

→ diagonal matrices are diagonalizable : $D = IDI^{-1}$

→ $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{2} & 0 & 0 \\ 0 & \underline{2} & 0 \\ 0 & 0 & \underline{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$

eigenvectors   eigenvalues

→ the diagonal entries of D are the eigenvalues to the respective columns (eigenvectors) in P.

↳ $A_{n\times n}$ is diagonalizable

⟺ A has n linearly independent eigenvectors.

⟺ there exist a $D_{n\times n}$ and $P = (u_1\ u_2 \cdots u_n)$ where the diagonal entries of D are the eigenvalues associated to $u_1, u_2, \dots, u_n$.

⟺ there exist a basis $\{u_1, u_2, \dots, u_k\}$ of eigenvectors of A.

⟺ $\det(xI - A) = (x-\lambda_1)^{r_{\lambda_1}} \cdots (x-\lambda_k)^{r_{\lambda_k}}$, where $\dim(E_{\lambda_i}) = r_{\lambda_i}$

↳ e.g.

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

① Check if $\det(xI - A)$ can be split into linear factors completely.

$$\det(xI - A) = \begin{vmatrix} x-3 & -1 & 1 \\ -1 & x-3 & 1 \\ 0 & 0 & x-2 \end{vmatrix} = (x-2)\left[(x-3)^2 - 1\right] = (x-2)(x^2-6x+8)$$
$$= (x-2)(x-2)(x-4)$$
$$= (x-2)^2(x-4)$$

② Check if Geometric multiplicity of each eigenvalue equals its algebraic multiplicity.

Algebraic multiplicity of $\lambda = 4 \to 1 \Rightarrow \dim(E_4) = 1$

Algebraic multiplicity of $\lambda = 2 \to 2 \Rightarrow 1 \le \dim(E_2) \le 2$

$$2I - A = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(E_2) = 2 = r_2$$

③ Find basis for the eigenspaces.

$$2I - A \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Basis for } E_2 = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$4I - A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Basis for } E_4 = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

④ Compute the diagonalization

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \overset{(P)}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \overset{(D)}{\begin{pmatrix} \boxed{2} & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{4} \end{pmatrix}} \overset{(P^{-1})}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}}$$

basis for E_2 basis for E_4

$\Leftrightarrow$ A has n distinct eigenvalues.

↳ $1 \leq \dim(E_\lambda) \leq r_\lambda$

↳ n distinct eigenvalues $\Rightarrow r_{\lambda_i} = 1$ for $i = 1, \dots, n$

$\Rightarrow 1 \leq \dim(E_{\lambda_i}) \leq 1$

$\Rightarrow \dim(E_{\lambda_i}) = 1 = r_{\lambda_i}$

→ $A_{n \times n}$ is not diagonalizable

$\Leftrightarrow$ $\det(xI - A)$ does not split into linear factors.

$\Leftrightarrow$ there exist an eigenvalue such that $\dim(E_\lambda) < r_\lambda$

→ If square matrix A only has 1 eigenvalue, then A is diagonalizable iff $A = \lambda I_n$, A is a scalar matrix

$\Rightarrow$ All non-scalar matrices with 1 eigenvalues are not diagonalizable.

Eigenspaces are linearly independent

→ $\lambda_1 \neq \lambda_2$, $\{u_1, u_2, \dots, u_k\} \subseteq E_{\lambda_1}$ and $\{v_1, v_2, \dots, v_m\} \subseteq E_{\lambda_2}$ where $\{u_1, u_2, \dots, u_k\}$ and $\{v_1, v_2, \dots, v_m\}$ are linearly independent subsets.

↳ $\{u_1, u_2, \dots, u_k, v_1, \dots, v_m\}$ is linearly independent.

→ $\lambda_1 \neq \lambda_2 \Rightarrow Av_1 = \lambda_1 v_1 \neq \lambda_2 v_1$

Orthogonally Diagonalizable

→ square matrix P where $P^T = P^{-1} \Rightarrow$ P is orthogonal

↳ rows | cols of P is an orthonormal basis.

→ A is orthogonally diagonalizable if $A = PDP^T \Rightarrow A = PDP^{-1}$

Spectral Theorem

→ A is orthogonally diagonalizable $\Leftrightarrow$ A is symmetric

↳ $A = PDP^T \Rightarrow A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$

diagonal matrices are symmetric.

Equivalent Statements for Orthogonally Diagonalizable.

→ $A_{n \times n}$ is orthogonally diagonalizable.

→ there exist an orthonormal basis $\{u_1, u_2, \dots, u_n\}$ of eigenvectors of A.

→ A is symmetric matrix.

need to perform Gram-Schmidt Process only on eigenspaces with more than 1 vector in its basis, since $E_{\lambda_i} \perp E_{\lambda_j}$ for $i \neq j$ in symmetric matrices.

Power of Diagonalizable Matrices.

→ $A = PDP^{-1} \Rightarrow A^m = PD^mP^{-1}$

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_n \end{pmatrix} \Rightarrow D^m = \begin{pmatrix} d_1^m & 0 & \cdots & 0 \\ 0 & d_2^m & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_n^m \end{pmatrix}$$

for $m > 0$

if A is invertible, $m \in \mathbb{Z}$
⇒ m can be -ve / +ve

Markov Chain

→ Probability vector : a vector with non-negative coordinates whose coordinates add to 1

↳ e.g. $\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$, counterexamples. $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

→ Stochastic matrix : a square matrix P where the columns are probability vectors.

↳ regular stochastic matrix : P^k has all positive entries (> 0) for $k > 0$

→ Markov Chain : $x_1 = Px_0$, $x_2 = Px_1$, $\ldots$, $x_k = Px_{k-1}$, $\ldots$

↳ probability vectors

↳ AKA state vectors as the coordinates of x_k denotes the probability of a state happening.

↳ Stochastic matrix

↳ AKA probability transition matrix of the Markov chain

↳ $x_k = P^{k-1}x_0 \Rightarrow x_2 = Px_1 = P(Px_0) = P^2x_0$

$\Rightarrow x_k = Px_{k-1} = P(P^{k-1}x_0) = P^kx_0$

Equilibrium vector / Steady-state vector.

for a stochastic matrix.

→ a probability vector that is an eigenvector with $\lambda = 1$

→ if the Markov chain converges, it will converge to an equilibrium vector.

→ if the stochastic matrix in Markov chain is a regular stochastic matrix,

↳ if the stochastic matrix in markov chain is a regular stochastic matrix, the equilibrium vector will be unique.

→ Deriving the Equilibrium vector.

① find eigenvector u associated to $\lambda = 1$ ⇒ solve $(I - P)x = 0$

② write $u = (u_i)$, equilibrium vector $v = \dfrac{1}{\sum_{k=1}^n u_k} u = \left(\sum_{k=1}^n u_k\right)^{-1} u$

Singular Values

→ All nonsquare matrices $A_{m\times n}$ can be represented as

$$\underset{m\times n}{A} = \underset{m\times m}{U} \underset{m\times n}{\Sigma} \underset{n\times n}{V^T}$$

$$\Sigma = \begin{pmatrix} D & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}$$ where $D = r\times r$ matrix and $r \le \min\{m, n\}$

$\rightarrow$ A = $m \times n$ matrix

$\Rightarrow$ $A^T A$ is order n symmetric matrix

$\Rightarrow$ $A^T A$ is orthogonally diagonalizable.

$\Rightarrow$ $\{v_1, v_2, \cdots, v_n\}$ be orthonormal basis where $v_1, \cdots, v_n$ are the eigenvectors of $A^T A$, let μ_i be the eigenvalues associated to v_i

* Note : μ_i may not be distinct.

* Note : μ_i is non-negative.

$\Rightarrow$ Arranging μ_i in descending order : $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$

Singular value : $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0$ where $\theta_i = \sqrt{\mu_i}$

$$\Sigma = \begin{pmatrix} D & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{pmatrix} \text{ where } D = \begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \theta_r \end{pmatrix}$$

$$\Rightarrow \Sigma^T \Sigma = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \mu_n \end{pmatrix}$$

$\Rightarrow$ $Av_i \neq 0$ for $i \leq r$, $Av_i = 0$ for $i > r$

Singular Value Decomposition

$\rightarrow$ Suppose $\theta_1 \geq \theta_2 \geq \cdots \geq \theta$