Euclidean n-space : $\mathbb{R}^n$

→ Collection of all n-vectors.

→ $\mathbb{R}^n = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| \; v_i \in \mathbb{R} \text{ for } i = 1, \dots, n \right\}$

Vector Axioms

1. u, v in V, $u+v$ in V
2. $u + v = v + u$
3. $u + (v + w) = (u+v) + w$
4. 0 vector in V, $0 + v = v$
5. $u + (-u) = 0$
6. αv in V, for $\alpha \in \mathbb{R}$
7. $\alpha(u+v) = \alpha u + \alpha v$, for $\alpha \in \mathbb{R}$
8. $(a+b)u = au + bu$, for $a, b \in \mathbb{R}$
9. $a(bu) = (ab)u$, for $a, b \in \mathbb{R}$
10. $1u = u$.

closed under addition

contains zero vector.

closed under scalar multiplication.

Dot Product. (Inner Product)

→ $u \cdot v = u^T v = (u_1 \; u_2 \; \cdots \; u_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

Norm ⇒ length / magnitude of vector.

→ $\|u\| = \sqrt{u \cdot u}$

Unit Vector

→ Norm : $\|u\| = 1$

Properties of Norm & Dot Product.

→ Symmetric : $u \cdot v = v \cdot u$

→ Scalar multiplication : $cu \cdot v = (cu) \cdot v = u \cdot (cv)$

→ Distributive : $u \cdot (av + bw) = au \cdot v + bu \cdot w$

→ Positive definite : $u \cdot u \geq 0$ iff $u = 0$

→ $\|cu\| = |c| \, \|u\|$

absolute value

Normalizing a vector.

→ $u \times \frac{1}{\|u\|}$

→ Norm of $\frac{u}{\|u\|}$: $\left(\frac{u}{\|u\|}\right) \cdot \left(\frac{u}{\|u\|}\right) = \frac{u \cdot u}{\|u\|^2} = \frac{u \cdot u}{(\sqrt{u \cdot u})^2} = \frac{u \cdot u}{u \cdot u} = 1$

Distance between vectors.

→ $d(u,v) = \|u-v\| = \left\| \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \right\|$

Angle between vectors.

→ $\cos(\theta) = \dfrac{u \cdot v}{\|u\| \times \|v\|}$, for $u, v \neq 0$

↳ $\theta = \cos^{-1}\left(\dfrac{u \cdot v}{\|u\| \times \|v\|} \right)$

Linear Span

→ Span of $u_1, u_2, \ldots, u_k$ is a subset of $\mathbb{R}^n$ containing all linear combinations of $u_1, u_2, \ldots, u_k$.

→ $\text{span}\{u_1, u_2, \ldots, u_k\} = \{ c_1 u_1 + c_2 u_2 + \ldots + c_k u_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R} \}$

↳ v is in $\text{span}\{u_1, u_2, \ldots, u_k\}$ iff v can be represented as $\underline{d_1 u_1 + \ldots + d_k u_k}$ (linear combination of $u_1, u_2, \ldots, u_k$)

→ Checking for span

↳ e.g. $c_1 u_1 + c_2 u_2 + c_3 u_3 = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{v}$

$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Solve for $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. If system is consistent, i.e. there exist a solution for $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, v can be represented as a linear combination of the vectors in spanning set, hence v is in the span.

↳ $v \in \text{span}\{u_1, u_2, \ldots, u_k\}$ iff $(u_1 \ u_2 \ \cdots \ u_k \mid v)$ is consistent.

→ Checking if All vectors ($\mathbb{R}^n$) is in the span

↳ same as asking if $\text{span}(S) = \mathbb{R}^n$, $S = \{u_1, u_2, \ldots, u_k\}$ in $\mathbb{R}^n$

↳ RREF of S, if consistent ⇒ all vectors in span

if inconsistent ⇒ not all vectors in span

→ $\text{span}(A) = \mathbb{R}^n \iff Ax = v$ is consistent for all v

$\iff$ RREF of A has no zero rows.

↳ $\left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & 2 & 3 & y \\ 1 & 1 & 2 & z \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2x - y \\ 0 & 1 & 1 & -x + y \\ 0 & 0 & 0 & -x + z \end{array}\right)$

↳ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \text{span}(S) \iff z - x = 0$

Properties of linear span

- $\to$ zero vector in span (S)
- $\to$ span is closed under scalar multiplication
- $\to$ span is closed under addition

} closed under linear combination,

Checking for Set Relations between spans.

- $\to$ $S := \{u_1, u_2, \dots, u_k\}$, $T = \{v_1, v_2, \dots, v_m\}$
- $\to$ $\text{span}(T) \subseteq \text{span}(S) \iff \left(\text{"}S\text{"} \,\middle|\, \text{"}T\text{"}\right) = \left(u_1\ u_2 \dots u_k \,\middle|\, v_1 \,\middle|\, v_2 \,\middle|\, \dots \,\middle|\, v_m\right)$ is consistent.
- $\to$ $\text{span}(T) = \text{span}(S) \iff \text{span}(T) \subseteq \text{span}(S)$ & $\text{span}(S) \subseteq \text{span}(T)$
  - ↳ if $\text{span}(S) \subseteq \text{span}(T)$ but $\text{span}(T) \not\subseteq \text{span}(S) \Rightarrow \text{span}(T) > \text{span}(S)$

Solution Sets to Linear System

- $\to$ solution set to $Ax = b$ (U) is a subset of $\mathbb{R}^n$
  - ↳ empty set if system is inconsistent.
- $\to$ Implicitly : $V = \{u \in \mathbb{R}^n \mid Au = b\}$
  - ↳ e.g. $\begin{cases} x + y = 0 \\ z = 1 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$ $\uparrow_s$ $\begin{array}{l} x = -s \\ z = 1 \end{array} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
  - ↳ $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x = -y,\ z = 1 \right\}$
- $\to$ Explicitly : $V = \{u + s_1 v_1 + s_2 v_2 + \dots + s_k v_k \mid s_1, s_2, \dots, s_k \in \mathbb{R}\}$,
  where $u + s_1 v_1 + s_2 v_2 + \dots + s_k v_k \in \mathbb{R}$ is the general solution.
  - ↳ $V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$

- $\to$ Solution Set to $Ax = 0$
  - ↳ in the form : $s_1 v_1 + s_2 v_2 + \dots + s_k v_k$, $s_1, s_2, \dots, s_k \in \mathbb{R}$
  - ↳ Explicitly : $V = \{s_1 v_1 + s_2 v_2 + \dots + s_k v_k \mid s_1, s_2, \dots, s_k \in \mathbb{R}\}$
  - ↳ $\text{span}\{v_1, v_2, \dots, v_k\}$
  - ↳ $x - y + z = 0$

subspace

$$V = \left\{ s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Subspace

* → V is a subspace if:
  1. contains zero vector. ⇒ V is non-empty.
  2. closed under scalar multiplication } closed under linear combinations.
  3. closed under addition

→ $V = \{u \mid Au = b\}$ to $Ax = b$ is a subspace iff $b = 0$

↳ solution set is subspace ⟺ $Ax = 0$

↳ solution space ← obtained by solving $(A \mid b)$

→ $S = \{u_1, u_2, \dots, u_k\} \in \mathbb{R}^n$, $V \subseteq \mathbb{R}^n$ is subspace iff $V = \text{span}(S)$

→ Steps to check if V is subspace

1. look for span(S) such that $V = \text{span}(S)$
2. V satisfies the 3 properties of subspace.

→ Steps to check if V is not subspace.

1. V does not satisfy one of the 3 properties of subspace.

→ $\mathbb{R}^n$ can have $n-1, n-2, \dots, 2, 1, 0$ dimension subspaces.

2 ↑ plane, 1 ↑ line, 0 ↑ dot (zero vector)

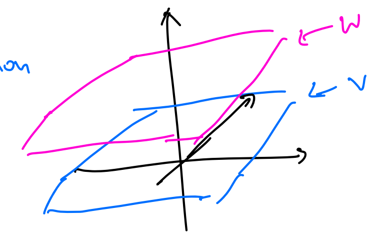
Affine space

→ Solution set to $Ax = b$, $b \neq 0$

→ solution set $W = \{w \mid Aw = b\} = u + V := \{u + v \mid v \in V\}$, where $V = \{v \mid Av = 0\}$

↳ V is solution set to $Ax = 0$

↳ W is solution set to $Ax = b$ ⇒ W is V shifted along $n-1$ dimension



Linear Independence

→ $\{u_1, u_2, \dots, u_k\}$ is linearly independent iff the only solution to

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = 0$$ is the trivial solution.

↳ $c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0$ ⇒ $c_1 + c_2 + \cdots + c_k = 0$ is the only solution.

↳ $\left(\begin{array}{cccc|c} u_1 & u_2 & \cdots & u_k & 0 \end{array}\right) \xrightarrow{\text{RREF}}$ non-pivot columns exist ⇒ linearly dependent.
no non-pivot columns ⇒ linearly independent.

↳ non-pivot columns are a linear combination of the pivot columns.

→ k = no. of columns , n = no. of rows

↳ if $k > n$, there exist $k-n$ no. of vectors that are non-pivot

↳ sets with k columns & n rows are linearly dependent.

→ Special Cases.

↳ $\{0\}$ is always linearly dependent ⇒ only column that exist is a non-pivot column.

↳ $\{v\}$ where $v \neq 0$ is always linearly independent. ⇒ the only column is a pivot column.

↳ $\{u, v\}$ where u is a scalar multiple of v is always linearly dependent

↳ $\{\}$, empty set, is always linearly independent ⇒ vacuously true.

→ if $\{u_1, u_2, \ldots, u_k\}$ is linearly dependent, $\{u_1, u_2, \ldots, u_k, u\}$ is also linearly dependent

for any vector u.

↳ set is already dependent, adding any vectors won't change that fact.

→ if $\{u_1, u_2, \ldots, u_k\}$ is linearly independent, $\{u_1, u_2, \ldots, u_k, u\}$ is linearly independent

iff u is not a linear combination of $u_1, u_2, \ldots, u_k$

→ if $\{u_1, u_2, \ldots, u_k\}$ is linearly independent, any subset of it is also linearly independent.

Basis

→ $S = \{u_1, u_2, \ldots, u_k\} \subseteq$ subspace V is a basis iff

① $\text{span}(S) = V$

② S is linearly independent

→ If S is a basis for V, any vectors $v \in V$ can be expressed as a linear combination of the vectors in S uniquely

as $\text{span}(S) = V$

↳ S is linearly independent ⇒ unique coefficients for different linear combinations.

→ $V = \{u \mid Au = 0\}$ is the solution space for $Ax = 0$, then $S = \{u_1, u_2, \ldots u_k\}$ is a basis for subspace V.

→ Basis is NOT unique., but if S & T spans V, size of S = size of T (dimension are the same)

→ Basis for $\{0\}$ is the empty set $\{\}$ or $\emptyset$

↳ Recall : def of span of S : smallest subspace V such that $S \subseteq V$

→ $V = \text{span}(S)$ if $V \subseteq W$ for all subspaces W containing S.

↳ zero space is the smallest subspace containing the empty set, so span of empty set is zero space

→ No relation between linear independence and spanning a subspace

Coordinates Relative to Basis.

→ $S = \{u_1, u_2, \dots, u_k\}$ basis for V. , $v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$

↳ $[v]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ ← coefficients such that fulfill

coordinates of v relative to basis S.

dim / size = no. of vectors in basis S.

↳ $[v]_S$ is unique iff S is linearly independent.

→ $[v]_S \neq [v]_T$ if $S \neq T$

→ Finding $[v]_S$ ⇒ solving for $(u_1\ u_2\ \dots\ u_k)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = v$

⇒ solving $(u_1\, u_2 \cdots u_k \mid v)$

→ Properties

↳ $u = v$ iff $[u]_S = [v]_S$

↳ $[c_1 v_1 + c_2 v_2 + \dots + c_k v_k]_S = c_1 [v_1]_S + c_2 [v_2]_S + \dots + c_k [v_k]_S$

↳ $v_1, v_2, \dots, v_k$ is linearly independent / dependent iff $[v_1]_S, [v_2]_S, \dots, [v_k]_S$ is linearly independent / dependent.

↳ $\{v_1, v_2, \dots, v_k\}$ spans V iff $\{[v_1]_S, [v_2]_S, \dots, [v_k]_S\}$ spans $\mathbb{R}^m$, where $|S| = m$

$(v_1\ v_2 \cdots v_k \mid u)$ have the same properties as $([v_1]_S\ [v_2]_S \cdots [v_k]_S \mid [u]_S)$

Dimension

→ independent degree of freedom of movement. 3D →, 2D →

→ If S & T are bases for V, then no. of vectors in S = no. of vectors in T

→ $\dim(V)$ = no. of vectors in any basis of V.

→ dim (solution space) = no. of non-pivot columns in RREF

= no. of vectors in basis of solution space.

→ Suppose S is a subset of solution space V

↳ if no. of columns in S > $\dim(V)$, S is linearly dependent.

↳ if no. of columns in S < $\dim(V)$, S will not span the whole of V.

Spanning Set Theorem

→ Suppose $\text{span}(S) = V$. If $V \neq \{0\}$, there must be a subset of S that forms the basis for V.

↳ the linear independent subset of S forms a basis for V since $\text{span}(S) = V$.

Condition ② of basis

Condition ① of basis.

Linear Independence Theorem.

-> $S = \{u_1, u_2, \dots, u_k\}$ is a linearly independent subset of $V \Rightarrow S \subseteq V$

-> there must be a set T, $S \subseteq T$, such that T is a basis for V.

↳ $S \subseteq V$, so if $\text{span}(S) = V$, S is basis for V. $\Rightarrow S$ can be renamed as T, where $S \subseteq T$.

↳ if $\text{span}(S) \neq V$, S don't span the whole of V, will need to add n no. of vectors to S such that the new set, called T, spans V. If T is linearly independent, T is basis for V.

↳ $|T| = \dim(V)$

Dimension & Subspaces

-> U & V are subspaces. If $U \subseteq V$, $\dim(U) \leq \dim(V)$

-> if $U \subsetneq V$, $\dim(U) < \dim(V)$

$\boxed{\text{Equivalent Methods to check for Basis}}$

-> ① $\text{span}(S) = V$

② S is linearly independent.

B1 -> ① $|S| = \dim(V)$ $\dim(\text{span}(S)) \Rightarrow$ no. of vectors in S.

② $S \subseteq V$

③ S is linearly independent

B2 -> ① $|S| = \dim(V)$

② $V \subseteq \text{span}(S)$ ← can think of it as $U = \text{span}(S)$

Transition Matrices.

-> $S = \{u_1, u_2, \dots, u_k\}$, $T = \{v_1, v_2, \dots, v_k\}$ are bases for V.

-> transition matrix from T to S -> $P = \left([v_1]_S \ [v_2]_S \ \cdots \ [v_k]_S \right)$

-> for any vector w in V, $[w]_S = P[w]_T$

Finding Transition Matrices

-> $\left(\text{"}S\text{"} \mid \text{"}T\text{"} \right) = \left(u_1 \ u_2 \cdots u_k \mid v_1 \ v_2 \cdots v_k \right) \xleftrightarrow{\text{RREF}} \left(\begin{array}{c|ccc} I_k & [v_1]_S & [v_2]_S & \cdots \ [v_k]_S \\ \text{zero rows} & & \text{zero rows} & \end{array} \right)$

$= \left(\begin{array}{c|c} I_k & P \left(\begin{smallmatrix}\text{from}\\ T \text{ to } S\end{smallmatrix}\right) \\ \text{zero rows} & \text{zero rows} \end{array} \right)$

-> $\underset{A}{P}\,\underset{x}{[w]_T} = \underset{b}{[w]_S} \Rightarrow Ax = b$

↳ to solve for $[w]_T \Rightarrow$ solving $\left(P \mid [w]_S \right)$

Inverse of Transition Matrix

→ P is transition matrix from T to S

→ P^{-1} is transition matrix from S to T

→ If P is invertible ⇒ $(P \mid I) \xrightarrow{RREF} (I \mid P^{-1})$

→ If P is not invertible

↳ $(S \mid T) \xrightarrow{E_1} \xrightarrow{E_2} \cdots \xrightarrow{E_k} \left(\begin{array}{c|c} I & P \\ \text{zero rows} & \text{zero rows} \end{array} \right)$

↳ $P \xrightarrow{E_k^{-1}} \cdots \xrightarrow{E_2^{-1}} \xrightarrow{E_1^{-1}} T$

↳ $(T \mid S) \xrightarrow{RREF} \left(\begin{array}{c|c} I & P^{-1} \\ \text{zero rows} & \text{zero rows} \end{array} \right)$