Trigonometry Identities

add: formula of tangent and normal

$a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$

$\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$

$\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

$\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$

$\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

Limit Laws/Results

1. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
2. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
3. $\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c}(g(x))\right)$
4. $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\ \left[\lim_{x\to c} g(x) \neq 0\right]$
5. If g is continuous at point b and $\lim_{x\to c} f(x) = b$,

then $\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right)$

If $\lim_{x\to c} g(x) = 0$,

$\lim_{x\to c}\frac{\sin(g(x))}{g(x)} = \lim_{x\to c}\frac{g(x)}{\sin(g(x))} = 1$

$\lim_{x\to c}\frac{\tan(g(x))}{g(x)} = \lim_{x\to c}\frac{g(x)}{\tan(g(x))} = 1$

Squeeze Theorem

Suppose $g(x) \le f(x) \le h(x)$,

If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$,

then $\lim_{x\to c} f(x) = L$

Replacement Law

$f(x) = g(x)$ for $x \neq a$.

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ OR
2. Limit does not exists

L'Hopital's Rule

$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}\ \left(\frac{0}{0}/\frac{\infty}{\infty}\right)$

Use $e^{\ln}$ for other forms $(1^\infty, \infty^0, 0^0)$

Differentiation Misc

$y = f(x)^{g(x)}$

$\ln y = g(x)\ln f(x$

Change of base

$\log_a x = \frac{\ln x}{\ln a}$, $a > 0$ and $a \neq 1$

Fundamental Theorem of Calculus

$\int_a^b f(t)dt = F(b) - F(a)$

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)$

Functions Laws

$(f \pm g)(x) = f(x) \pm g(x)$

$(fg)(x) = f(x)g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\ [g(x) \neq 0$

Derivative Rules

$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$

	First Derivative Test	Second Derivative Test
Local Maximum	$f'(x) > 0$ before $x = c$ $f'(x) < 0$ after $x = c$	$f'(c) = 0$ and $f''(c) < 0$
Local Minimum	$f'(x) < 0$ before $x = c$ $f'(x) > 0$ after $x = c$	$f'(c) = 0$ and $f''(c) > 0$

Series Test

→ $\lim_{n\to\infty} S_n = L \Rightarrow$ Convergent / $= \pm\infty$ / DNE $\Rightarrow$ D

n-th Term Test (take limit) / Divergence Test

→ $\lim_{n\to\infty} a = 0 \Rightarrow$ inconclusive

$\lim_{n\to\infty} a_n \neq 0$ or does not exist $\to \sum_{n=1}^{\infty} a_n$ diverges

Integral Test

$f'(x) = 0$, find

$f(x) = a_n$ and $f(x)$ is continuous, positive, decreasing left & right

$\int_1^\infty f(x)dx$ converge/diverge $\implies \sum_{n=1}^{\infty} a_n$ converge/diverge

Comparison Test (by creating 2nd series b)

$\sum_{n=1}^{\infty} b_n$ converge, $a_n \le b_n$ for all $n \implies \sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} b_n$ diverge, $a_n \ge b_n$ for all $n \implies \sum_{n=1}^{\infty} a_n$ diverge

Geometric Series Test

$\sum_{n=1}^{\infty} ar^{n-1}$ converge $\iff |r| < 1 \Rightarrow \sum_{n=1}^{\infty} ar^{n-1} = \frac{a_1}{1-r}$

$|r| \ge 1 \Rightarrow$ Diverges

p-series

$\sum_{n=1}^{\infty}\frac{1}{n^p}$, $p > 1 \implies$ converge; $p \le 1 \implies$ diverge;

Ratio Test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$< 1 \implies$ converge; $> 1 \implies$ diverge; $1 \implies$ inconclusive

Root Test

$\lim_{n\to\infty}\sqrt[n]{|a_n|}$

$< 1 \implies$ converge; $> 1 \implies$ diverge; $1 \implies$ *inconclusive*

Alternating Series Test

$b_n \ge 0$, ~~b_n increasing~~ $b_{n+1} \le b_n$, $\lim_{n\to\infty} b_n = 0 \implies \sum_{n=1}^{\infty}(-1)^n b_n$ converges

Useful Info

$\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent(even though $\lim_{n\to\infty}\frac{1}{n} = 0$)

Partial Fractions

$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

$\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Power Series (about a)

Power Series $\implies \sum_{n=0}^{\infty} c_n(x-a)^n$

Radius of Convergence (R, use Ratio Test)

1. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0$, "for all x", $R = \infty$, $I(-\infty, \infty)$
2. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \infty$, converge $\Rightarrow x = c$, $R = 0$, $I = \{c\}$
3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{R}|x-c| < 1 \Rightarrow |x-c| < R$

↳ Endpoints included or not, need to test the individual endpoints for convergence

Interval of Convergence (I)

$-R < x - c < R \implies I = (-R+c, R+c)$

Taylor/Maclaurin Series

Taylor $\implies \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$; about a

Maclaurin $\implies \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$; about $a = 0$

Common Maclaurin Series (centered at 0)

For $-1 < x < 1$ and $p \ge 1$

$\frac{1}{1-x^p} = \sum_{n=0}^{\infty} x^{pn} = 1 + x^p + x^{2p} + \dots$

$\frac{1}{1+x^p} = \sum_{n=0}^{\infty}(-1)^n x^{pn} = 1 - x^p + x^{2p} - \dots$

$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots$

$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2}(2 + 6x + 12x^2 + \dots)$

$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n = 1 - 2x + 3x^2 - \dots$

For $-\infty < x < +\infty$

$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

3-D Vector Geometry (+ Formulas)

Dot Product and Cross Product

Dot Product: $a \cdot b = ||a||\,||b||\cos\theta$

Cross Product: $a \times b = ||a||\,||b||\sin\theta\,\hat{n}$, $\hat{n}$: unit vector

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$

Area between 2 vectors: $||a \times b||$

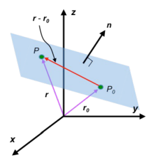
Magnitude: $||a|| = \sqrt{x^2+y^2+z^2}$

Equation of Plane + Plane Formulas

with normal <a, b, c> and point (x_0, y_0, z_0),

1. $(r - r_0). < a, b, c >= 0$
2. $ax + by + cz + d = 0$
3. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Dist from point to plane $= \left|\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}}\right|$



Equation of Line + Line Formulas

with point (x_0, y_0, z_0) on line and direction vector <a, b, c>

1. $r(t) = (x_0, y_0, z_0) + t < a, b, c >$
2. $x = x_0 + ta;\ y = y_0 + tb;\ z = z_0 + tc$

Tangent Vector (use differentiation)

With vector function $\vec{r}(t) =< i(t), j(t), k(t) >$

$r'(t) =< i'(t), j'(t), k'(t) >$

Unit Tangent : $\vec{T}'(t) = \frac{r'(t)}{||r'(t)||}$

Other Formulas

Distance between $P_1(x_1, y_1, z_1)$ & $P_2(x_2, y_2, z_2)$,

$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Unit Vector $\vec{u} = \frac{\vec{a}}{||\vec{a}||}$, $||\vec{a}|| = \sqrt[2]{i^2+j^2+k^2}$

Vector between $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$,

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$

Angle between vectors : $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\,||\vec{b}||}\right)$

Equation of Sphere with centre C(h,k,l):

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Arclength: $\int_a^b\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2 + 1}\,dt$

Integration by parts

$\int u\,dv = uv - \int v\,du$

Try to differentiate in order (highest to lowest priority):

$\ln x, x^n, e^x, e^{-x}, \sin x, \cos x$

Areas and Volumes

$V_{about\ x} = \pi\int_a^b [f(x)]^2 - [g(x)]^2 dx$

$V_{about\ y} = \pi\int_a^b [f(y)]^2 - [g(y)]^2 dy$

$A = \int_a^b f(x) - g(x)dx$; where f(x) > g(x)

$V_{cone} = \frac{1}{3}\pi r^2$ / $A_{cone} = \pi r\sqrt[2]{r^2+h^2} + \pi r^2$

$V_{cylinder} = \pi r^2 h$ / $A_{cylinder} = 2\pi rh + 2\pi r^2$

$V_{sphere} = \frac{4}{3}\pi r^3$ / $A_{sphere} = 4\pi r^2$

$V_{bounded\ by\ x\ rotated\ about\ y} = \int_a^b 2\pi x f(x)dx$

$V_{bounded\ by\ y\ rotated\ about\ x} = \int_a^b 2\pi y f(y)dx$

Formula for Tangent :

$y - f(x_0) = m(x - x_0)$

Formula for Normal :

$y - f(x_0) = -\frac{1}{m}(x - x_0)$

Direct Comparison :

$0 \le a_n < b_n$

Limit Comparison :

$\lim_{n\to\infty}\frac{a_n}{b_n}$ = constant

Absolute Series Test :

if $\sum|a_n| \to C$, $\sum a_n \to C$

↳ Absolutely Convergent

if $\sum|a_n| \to D$, $\sum a_n \to C$

↳ Conditionally Convergent

Misc

$y = f(x)^{g(x)}$

$\ln y = g(x)\ln(f(x))$

$a^3 - b^3 = (a-b) \times (a^2 + ab + b^2)$

Applications of Differentiation

Increasing / Decreasing Functions

f is increasing on [a,b] if $f'x > 0$

f is decreasing on [a,b] if $f'x < 0$

Concave Upward / Downward Functions

Let f be differentiable on (a, b), $c \in (a, b)$,

f is concave upward if $f''(c) > 0$ at $(c, f(c))$

f is concave downward if $f''(c) < 0$ at $(c, f(c))$

f has a point of inflection at $(c, f(c))$ if $f''(c) = 0$

Partial Differentiation

For function $z = f(x, y)$,

$f_x = \frac{\partial z}{\partial x}$, $f_y = \frac{\partial z}{\partial y}$

Clairaut's Theorem: $f_{xy}(a, b) = f_{yx}(a, b)$

Gradient

$\nabla f(a, b) = f_x(a, b)i + f_{y(a,b)}j$

$\nabla f(a, b)\cdot u = D_u f(a, b)$

Maximum value of $D_u f(x, y)$,

$||\nabla f(x, y)|| = \sqrt{f_x(a, b)^2 + f_y(a, b)^2}$

Chain Rule

For function $w = f(x,y,z)$,
where $x = x(t), y = y(t), z = z(t)$:
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$
where $x = x(s,t), y = y(s,t), z = z(s,t)$:
$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Directional Derivative

Given $f(x,y)$ at point (a,b),
with unit vector $u = u_1 i + u_2 j$,
$D_u f(a,b) = f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$

Second Derivative Test

(a,b) is a critical point if,
$f_x(a,b) = 0$ and $f_y(a,b) = 0$
$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$
Local Maximum (a,b): If $D > 0$ and $f_{xx}(a,b) < 0$
Local Minimum (a,b): If $D > 0$ and $f_{xx}(a,b) > 0$
Saddle Point (a,b): If $D < 0$
No Conclusion: If $D = 0$
f_{yy} can replace f_{xx}

Tangent Plane (2 / 3 variable)

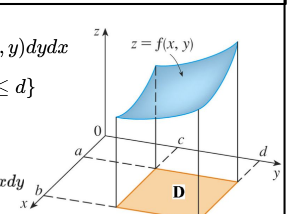
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
$\nabla f(a,b,c)\cdot\langle x-a, y-b, z-c\rangle = 0$

Implicit Partial Differentiation

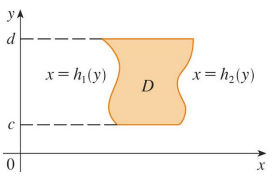
$$f(x,y,z) = 0 \qquad \frac{\partial z}{\partial x} = -\frac{f_x(x,y,z)}{f_z(x,y,z)} \qquad \frac{\partial z}{\partial y} = -\frac{f_y(x,y,z)}{f_z(x,y,z)}$$

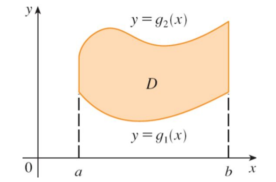
Double Integrals

$$\iint_D f(x,y)dA \implies \int_a^b\int_c^d f(x,y)dydx$$
$D = \{(x,y) : a \le x \le b, c \le y \le d\}$
Note: $z = f(x,y)$
Fubini's Theorem:
$$\int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$$



Type 1 / Type 2 regions

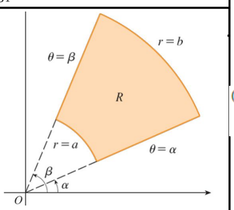




$D = \{h_1 \le x \le h_2,\ c \le y \le d\}$ $\quad$ $D = \{a \le x \le b,\ g_1 \le y \le g_2\}$
$$\int_c^d\int_{h_1}^{h_2} f(x,y)dxdy \qquad \int_a^b\int_{g_1}^{g_2} f(x,y)dydx$$

Polar Form (dA => r dr dθ)

$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$
$$\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)(r)\,dr\,d\theta$$
$r^2 = x^2 + y^2 \qquad x = r\cos\theta$
$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad y = r\sin\theta$



Surface Area (let f(x,y) = z)

$$\iint_D \sqrt[2]{(f_x)^2 + (f_y)^2 + 1}\,dA$$

Integration by substitution

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Ordinary Differential Equation

Separable Equation $\quad$ y' =g(y/x)

Given $\frac{dy}{dx} = f(x)g(y)$
$\frac{dy}{g(y)} = f(x)\,dx$
$$\int\frac{1}{g(y)}\,dy = \int f(x)\,dx + C$$

Put $y = vx$
$$\int\frac{dv}{g(v) - v} = \int\frac{dx}{x} + C$$

Linear Equation

With $y' + P(x)y = Q(x)$,
$I(x) = e^{\int P(x)dx}$, $I(x)y' = I(x)Q(x)$
$$y = \frac{\int Q(x)I(x)\,dx}{I(x)}$$

Bernoulli Equation

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$y' + P(x)y = Q(x)y^n \qquad y^{1-n} = \frac{1}{I(x)}\left[\int (1-n)Q(x)I(x)\,dx + C\right]$

$I(x) = e^{\int (1-n)P(x)dx}$

(b) Consider the function f given by
$$f(x) = \frac{x^3}{9} + \int_0^{\sqrt{x+3}}\cos(\sin(t-2))\,dt.$$
Find the value of $f'(1)$. Justify your answer.

(b) Let $u = \sqrt{x+3}$. By the Second Fundamental Theorem of Calculus (FTC2) and chain rule, we have
$$f'(x) = \frac{d}{dx}\left(\frac{x^3}{9}\right) + \frac{d}{du}\left(\int_0^u \cos(\sin(t-2))\,dt\right)\cdot\frac{du}{dx}$$
$$= \frac{x^2}{3} + \cos(\sin(u-2))\cdot\frac{1}{2\sqrt{x+3}} = \frac{x^2}{3} + \frac{\cos(\sin(\sqrt{x+3}-2))}{2\sqrt{x+3}}$$
Hence we have
$$f'(1) = \frac{1}{3} + \frac{\cos(\sin(\sqrt{1+3}-2))}{2\sqrt{1+3}} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

(a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{(4x+3)^{2n+1}}{(-3)^n}$. Justify your answer.

(b) Let $f(x)$ be a function such that $f''(x)$ is continuous on $\mathbb{R}$. It is given that $f(\pi) = 2$, and
$$\int_0^\pi (f(x) + f''(x))\sin x\,dx = 5.$$
Find the value of $f(0)$. Justify your answer.

Solution. (a) For each $n \ge 1$, let $u_n = \frac{(4x+3)^{2n+1}}{(-3)^n}$, so that $\sum_{n=1}^{\infty}\frac{(4x+3)^{2n+1}}{(-3)^n} = \sum_{n=1}^{\infty}u_n$. We apply the Ratio Test, and compute
$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{\left|\frac{(4x+3)^{2n+3}}{(-3)^{n+1}}\right|}{\left|\frac{(4x+3)^{2n+1}}{(-3)^n}\right|} = \frac{|4x+3|^2}{3}.$$
By the Ratio Test, the series converges absolutely if
$$L < 1 \iff \frac{|4x+3|^2}{3} < 1 \iff |4x+3| < \sqrt{3} \iff \left|x + \frac{3}{4}\right| < \frac{\sqrt{3}}{4}.$$
Also, the series diverges if $L > 1 \iff \frac{|4x+3|^2}{3} > 1 \iff \left|x+\frac{3}{4}\right| > \frac{\sqrt{3}}{4}$.
Hence the radius of convergence of the power series is $\frac{\sqrt{3}}{4}$.

(b) We are given that $f(\pi) = 2$, and
$$5 = \int_0^\pi (f(x) + f''(x))\sin x\,dx = \int_0^\pi f(x)\sin x\,dx + \int_0^\pi f''(x)\sin x\,dx. \qquad (*)$$
Using integration by parts twice, we have
$$\int_0^\pi f''(x)\sin x\,dx = \Big[f'(x)\sin x\Big]_0^\pi - \int_0^\pi f'(x)\cos x\,dx$$
$$= 0 - 0 - \left(\Big[f(x)\cos x\Big]_0^\pi - \int_0^\pi f(x)(-\sin x)\,dx\right)$$
$$= f(\pi) + f(0) - \int_0^\pi f(x)\sin x\,dx$$
$$\implies \int_0^\pi f(x)\sin x\,dx + \int_0^\pi f''(x)\sin x\,dx = f(\pi) + f(0) = 2 + f(0).$$
Together with $(*)$, we have $5 = 2 + f(0) \implies f(0) = 3$.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$ $\quad$ (b) $\sum_{n=2}^{\infty}\frac{(\ln n)^n}{n!}$

Solution. (a) The series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1} = \sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series with each $b_n = \frac{n}{n^2+1} > 0$. Also, for each $n \ge 1$, we have
$$b_n - b_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} = \frac{n((n+1)^2+1) - (n+1)(n^2+1)}{(n^2+1)((n+1)^2+1)} = \frac{n^2+n-1}{(n^2+1)((n+1)^2+1)} > 0$$
since $n^2 + n - 1 \ge 1^2 + 1 - 1 = 1 > 0$. We also have
$$\lim_{n\to\infty}b_n = \lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0.$$
Thus by the Alternating Series Test, the series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$ converges.

(b) We apply the Root Test. Let $a_n = \frac{(\ln n)^n}{n!}$. For each $n \ge 2$, we have $0 \le |a_n|^{1/n} = \frac{\ln n}{(n!)^{1/n}}$.
Note that for each $n \ge 2$, by truncating the smaller factors of $n!$, we have
$$n! = n\cdot(n-1)\cdots 2\cdot 1 \ge \begin{cases} n\cdot(n-1)\cdots\frac{n+2}{2} \ge \left(\frac{n+2}{2}\right)^{n/2} \ge \left(\frac{n}{2}\right)^{n/2} & \text{if } n \text{ is even,}\\ n\cdot(n-1)\cdots\frac{n+1}{2} \ge \left(\frac{n+1}{2}\right)^{(n+1)/2} \ge \left(\frac{n}{2}\right)^{n/2} & \text{if } n \text{ is odd.}\end{cases}$$
Thus for all $n \ge 2$, we have $n! \ge \left(\frac{n}{2}\right)^{n/2}$, and thus
$$0 \le |a_n|^{1/n} \le \frac{\ln n}{\sqrt{n/2}}$$
By L'Hôpital's Rule, we have $\lim_{n\to\infty}\frac{\ln n}{\sqrt{n/2}} = \lim_{x\to\infty}\frac{\ln x}{\sqrt{x/2}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot x^{-1/2}} = \lim_{x\to\infty}\frac{2\sqrt{2}}{\sqrt{x}} = 0.$
Hence, by squeeze theorem, we have $\lim_{n\to\infty}|a_n|^{1/n} = 0 < 1$.
Hence by the Root Test, the series $\sum_{n=2}^{\infty}a_n = \sum_{n=2}^{\infty}\frac{(\ln n)^n}{n!}$ converges absolutely, and thus it converges.

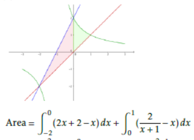
(a) The region bounded by the curve $y = \frac{1}{x^2(1+\ln x)^3}$, the line $y = 0$, the line $x = e$ and the line $x = e^2$ is revolved completely about the line $x = 0$. Find the volume of the solid generated. Justify your answer, and give your answer in terms of π.

(b) Find the area of the region bounded by the curve $y = \frac{2}{x+1}$, the line $y = x$ and the line $y = 2x + 2$. Justify your answer, and give your answer correct to two decimal places.

[You may use the following information freely: The points of intersection of the curve $y = \frac{2}{x+1}$ and the line $y = x$ are $(-2,-2)$ and $(1,1)$. The points of intersection of the curve $y = \frac{2}{x+1}$ and the line $y = 2x + 2$ are $(-2,-2)$ and $(0,2)$. The point of intersection of the two lines $y = x$ and $y = 2x + 2$ is $(-2,-2)$.]

Solution. (a) Using the shell method, we have
$$\text{Volume} = \int_e^{e^2} 2\pi x\cdot\frac{1}{x^2(1+\ln x)^3}\,dx = \int_e^{e^2}\frac{2\pi}{x(1+\ln x)^3}\,dx.$$
Let $u = \ln x$. Then $\frac{du}{dx} = \frac{1}{x}$, so that $\frac{dx}{x} = du$.
When $x = e$, $u = 1$. When $x = e^2$, $u = 2$. Thus we have
$$\text{Volume} = \int_{u=1}^{u=2}\frac{2\pi}{(1+u)^3}\,du = \left[\frac{2\pi(1+u)^{-2}}{-2}\right]_{u=1}^{u=2} = \left[\frac{-\pi}{(1+u)^2}\right]_{u=1}^{u=2} = \frac{-\pi}{(1+2)^2} + \frac{\pi}{(1+1)^2} = \frac{5\pi}{36}.$$

(b) By using the information given, a sketch of the bounded region is given as follows:



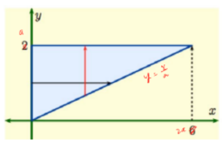
$$\text{Area} = \int_{-2}^{0}(2x+2-x)\,dx + \int_0^1\left(\frac{2}{x+1} - x\right)dx$$
$$= \left[\frac{x^2}{2} + 2x\right]_{-2}^{0} + \left[2\ln|x+1| - \frac{x^2}{2}\right]_0^1 = 0 - (2-4) + (2\ln 2 - \frac{1}{2}) - 0 = \frac{3}{2} + 2\ln 2 \approx 2.89.$$

5. (10 marks)

(a) Let a be a positive number. Find the value of the iterated integral
$$\int_0^{2a}\int_{x/a}^{2} x\sqrt{1+y^3}\,dy\,dx.$$
Justify your answer, and give your answer in terms of a.

(b) Consider the solid region in $\mathbb{R}^3$ given by
$$W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 25 \text{ and } x^2 + y^2 \ge 9\}.$$
Find the volume of W. Justify your answer, and give your answer in terms of π.

Solution. (a) Consider the region in $\mathbb{R}^2$ given by
$$D = \{(x,y) \in \mathbb{R}^2, : 0 \le x \le 2a,\ \frac{x}{a} \le y \le 2\}$$
$$= \{(x,y) \in \mathbb{R}^2, : 0 \le y \le 2,\ \ 0 \le x \le ay\}.$$



By Fubini's Theorem,
$$\int_0^{2a}\int_{x/a}^{2} x\sqrt{1+y^3}\,dy\,dx = \iint_D x\sqrt{1+y^3}\,dA$$
$$= \int_0^2\int_0^{ay} x\sqrt{1+y^3}\,dx\,dy$$
$$= \int_0^2\left[\frac{x^2}{2}\sqrt{1+y^3}\right]_0^{ay}dy$$
$$= \int_0^2\frac{a^2y^2}{2}\sqrt{1+y^3}\,dy = \left[\frac{a^2}{9}(1+y^3)^{3/2}\right]_0^2 = \frac{a^2(27-1)}{9} = \frac{26a^2}{9}.$$

(b) Note that
$$x^2+y^2+z^2 \le 25 \iff z^2 \le 25 - x^2 - y^2 \iff -\sqrt{25-x^2-y^2} \le z \le \sqrt{25-x^2-y^2},$$
and the inequalities are valid only when $x^2 + y^2 \le 25$. Thus we may regard W as the solid region between the graphs of the functions $z = \sqrt{25-x^2-y^2}$ and $z = -\sqrt{25-x^2-y^2}$ over the annular domain
$$D = \{(x,y) \in \mathbb{R}^2 : 9 \le x^2 + y^2 \le 25\}$$
$$= \{(r,\theta) : 3 \le r \le 5,\ \ 0 \le \theta \le 2\pi\} \quad \text{(in polar coordinates).}$$
Thus
$$\text{Volume}(W) = \iint_D\left(\sqrt{25-x^2-y^2} - (-\sqrt{25-x^2-y^2})\right)dx\,dy$$
$$= \int_0^{2\pi}\int_3^5 2\sqrt{25-r^2}\cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[-\frac{2(25-r^2)^{3/2}}{3}\right]_3^5 d\theta = \int_0^{2\pi}-\frac{2(0-64)}{3}\,d\theta = \frac{128}{3}\int_0^{2\pi}d\theta = \frac{128}{3}\cdot 2\pi = \frac{256\pi}{3}.$$

6. (10 marks) Let A and B be two particles moving in $\mathbb{R}^3$. At time $t \ge 0$, the coordinates of A are given by $(t^{3/2}, \cos t, \sin t)$, and the coordinates of B are given by $(\pi^{3/2} - \pi + t, -1 + \pi - t, -\pi + t)$. Here t is measured in seconds, and the coordinates x, y, z in $\mathbb{R}^3$ are measured in metres.

(a) Consider the function $g(t) = t - \pi - \sin t$. Show that the equation $g(t) = 0$ has exactly one solution $t = \pi$ in the interval $[0,\infty)$. Hence deduce that for time $t \ge 0$, A and B meet only at $t = \pi$.

(b) Find the distance that A has travelled from the time $t = 0$ to the time when A and B meet. Justify your answer, and give your answer in metres and correct to two decimal places.

Solution. (a) Consider the function $g(t) = t - \pi - \sin t$. Clearly, for $t \ge 2\pi$, one has $g(t) \ge 2\pi - \pi - 1 > 0$. Now on the interval $[0, 2\pi$, one has $g'(t) = 1 - \cos t$, and thus $g'(t) > 0$ on the interval $(0, 2\pi)$. Hence $g(t)$ is increasing on the interval $[0, 2\pi]$. Also, $g(\pi) = \pi - \pi - \sin\pi = 0$. Hence $g(t) < 0$ for all $t \in [0,\pi)$, and $g(t) > 0$ for all $t \in [\pi, 2\pi)$. Hence the equation $g(t) = 0$ has exactly one solution in the interval $[0,\infty)$.

From the above result, one knows that for $t \ge 0$ the z coordinates of A and B are equal only at $t = \pi$. Hence A and B can meet at most once at time $t = \pi$. Now one checks that the coordinates of A at time $t = \pi$ are $(\pi^{3/2}, \cos\pi, \sin\pi) = (\pi^{3/2}, -1, 0)$. Similarly, the coordinates of B at time $t = \pi$ are $(\pi^{3/2} - \pi + \pi, -1 + \pi - \pi, -\pi + \pi) = (\pi^{3/2}, -1, 0)$. Thus A and B meet at time $t = \pi$, since they have the same coordinates at $t = \pi$.

In summary, for $t \ge 0$, A and B meet only at $t = \pi$.

(b) A traces out the curve in $\mathbb{R}^3$ given by $\mathbf{r}(t) = (t^{3/2}, \cos t, \sin t)$, $\quad t \ge 0$.

Note that $\mathbf{r}'(t) = (\frac{3}{2}t^{1/2}, -\sin t, \cos t)$. From part (a), we know that A and B meet at time $t = \pi$. Distance travelled by A is equal to the length of the above curve from $t = 0$ to $t = \pi$. Thus
$$\text{Distance} = \int_0^\pi\|\mathbf{r}'(t)\|\,dt = \int_0^\pi\sqrt{(\tfrac{3}{2}t^{1/2})^2 + \sin^2 t + \cos^2 t}\,dt = \int_0^\pi\sqrt{\frac{9t}{4} + 1}\,dt$$
$$= \int_0^\pi\frac{3}{2}\cdot(t + \frac{4}{9})^{1/2}dt = \left[(t+\frac{4}{9})^{3/2}\right]_0^\pi = (\pi + \frac{4}{9})^{3/2} - \frac{8}{27} \approx 6.49..$$

7. (10 marks) A manufacturer makes a rectangular box of volume 32 cubic metres. The rectangular box is of length x metres, width y metres and height z metres. The top and bottom rectangular faces of the rectangular box are made of a material costing 3 dollars per square metre, while the four vertical rectangular faces of the rectangular box are made of another material costing 6 dollars per square metre. Let C, measured in dollars, be the total material cost for the six faces of the rectangular box. Express C as a function of the two variables x and y, and determine a local minimum value of C, if any. Justify your answers.

Solution. Since the rectangular box is of volume 32, and has length x, width y and height z, it follows that
$$\text{Volume} = xyz = 32. \qquad (*)$$
In particular, we must have $x > 0$, $y > 0, z > 0$. Also, both the top and bottom faces are of area xy, while the four vertical faces are of area xz, yz, xz, yz. Then the total material cost is given by
$$C = 3\cdot 2\cdot xy + 6\cdot(2xz + 2yz) = 6xy + 12(x+y)\cdot\frac{32}{xy} \quad (\text{by } (*))$$
$$\implies C = 6xy + \frac{384}{x} + \frac{384}{y}. \qquad (**)$$
Then
$$C_x = 6y - \frac{384}{x^2}, \quad C_y = 6x - \frac{384}{y^2}. \qquad (***)$$
To find critical points of C, we solve
$$\begin{cases} C_x = 0 \\ C_y = 0 \end{cases} \implies \begin{cases} 6y - \frac{384}{x^2} = 0 \\ 6x - \frac{384}{y^2} = 0 \end{cases} \implies \begin{cases} x^2y = 64, & (1) \\ xy^2 = 64. & (2) \end{cases}$$
Since $x \ne 0$, $y \ne 0$, it follows from (1) and (2) that
$$x^2y = xy^2 \implies xy(x-y) = 0 \implies x - y = 0 \implies x = y.$$
Together with (1), we have
$$x^3 = 64 \implies x = 4 \text{ and } y = x = 4 \implies (x,y) = (4,4).$$
Thus $C(x,y)$ has a critical point at $(x,y) = (4,4)$. From (***), we have
$$C_{xx} = \frac{1152}{x^3}, \quad C_{xy} = 6, \quad C_{yy} = \frac{1152}{y^3}.$$
Thus $C_{xx}(4,4) = \frac{1152}{4^3} = 18 > 0$, and the discriminant of C at $(4,4)$ is given by
$$D(4,4) = C_{xx}(4,4)C_{yy}(4,4) - (C_{xy}(4,4))^2 = \frac{1152}{4^3}\cdot\frac{1152}{4^3} - 6^2 = 18\cdot 18 - 36 = 288 > 0.$$
Thus by the second derivative test, C has a local minimum point at $(4,4)$, and by (**), the corresponding local minimum value of C is
$$C(4,4) = 6\cdot 4\cdot 4 + \frac{384}{4} + \frac{384}{4} = 288.$$

9. (10 marks) Let $y(x)$ be a function such that $y(0) = e^2$, and
$$\frac{dy}{dx} = \frac{3y\ln y}{2x+2} - \frac{3y}{\ln y}.$$
Find the value of $y(1)$. Justify your answer, and give your answer correct to two decimal places.

[Suggestion: Make the substitution $w = \ln y$.]

Solution. Note that the given differential $\boxed{y(1) = e^{\sqrt{3(1+1)+(1+1)^3}} = e^{\sqrt{14}} = 42.17.}$
$$\frac{1}{y}\frac{dy}{dx} = \frac{3\ln y}{2x+2} - \frac{3}{\ln y}, \quad y(0) = e^2. \qquad (1)$$
Now we let $w = \ln y$. Then by chain rule, we have $\frac{dw}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$. So (1) gives
$$\frac{dw}{dx} = \frac{3w}{2x+2} - \frac{3}{w}, \quad w(0) = \ln(e^2) = 2$$
$$\implies \frac{dw}{dx} - \frac{3}{2x+2}w = -3w^{-1}, \quad w(0) = 2. \qquad (2)$$
This is a Bernoulli equation with $n = -1$. Let $u = w^{1-(-1)} = w^2$, so that $\frac{du}{dx} = 2w\frac{dw}{dx}$.

10. (10 marks) Consider the surface in $\mathbb{R}^3$ given by
$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 36,\ \ z \ge 0,\ \ x^2 + (y-3)^2 \le 9 \text{ and } x + y \ge 0\right\}.$$
Find the surface area of S. Justify your answer, and give your answer correct to two decimal places.

Solution. Note that
$$x^2 + y^2 + z^2 = 36 \ \&\ z \ge 0 \iff z = \sqrt{36 - x^2 - y^2}.$$
Thus S is the graph of the function $f(x,y) = \sqrt{36-x^2-y^2}$ over the domain
$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-3)^2 \le 9 \text{ and } x + y \ge 0\}.$$
First we write D in polar coordinates (r,θ) with $r \ge 0$ and $-\pi \le \theta \le \pi$. In polar coordinates, the inequality $x^2 + (y-3)^2 \le 9$ becomes
$$r^2\cos^2\theta + (r\sin\theta - 3)^2 \le 9 \iff r^2 - 6r\sin\theta \le 0 \iff 0 \le r \le 6\sin\theta,\ 0 \le \theta \le \pi. \qquad (1)$$
(Here the last inequality $0 \le \theta \le \pi$ follows from the fact that for $-\pi \le \theta \le \pi$, one has $\sin\theta \ge 0 \iff 0 \le \theta \le \pi$.) In polar coordinates, the line $x + y = 0$ corresponds to the two rays $\theta = \frac{3\pi}{4}$ and $\theta = -\frac{\pi}{4}$. Then one easily checks that
$$x + y \ge 0 \iff -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}. \qquad (2)$$
From (1) and (2), one sees that in polar coordinates,
$$D = \{(r,\theta) | 0 \le \theta \le \frac{3\pi}{4} \text{ and } 0 \le r \le 6\sin\theta\}.$$
Next we have
$$f_x = \frac{1}{2\sqrt{36-x^2-y^2}}\cdot(-2x) = \frac{-x}{\sqrt{36-x^2-y^2}}, \qquad f_y = \frac{1}{2\sqrt{36-x^2-y^2}}\cdot(-2y) = \frac{-y}{\sqrt{36-x^2-y^2}}.$$
Then
$$\text{Area}(S) = \iint_D\sqrt{1 + (f_x)^2 + (f_y)^2}\,dA = \iint_D\sqrt{1 + \frac{x^2+y^2}{36-x^2-y^2}}\,dA = \iint_D\sqrt{\frac{36}{36-x^2-y^2}}\,dA$$
$$= \int_0^{3\pi/4}\int_0^{6\sin\theta}\frac{6}{\sqrt{36-r^2}}r\,dr\,d\theta = \int_0^{3\pi/4}\left[-6\sqrt{36-r^2}\right]_0^{6\sin\theta}d\theta$$
$$= \int_0^{3\pi/4}-6(\sqrt{36-36\sin^2\theta} - 6)\,d\theta = 36\int_0^{3\pi/4}(1 - |\cos\theta|)\,d\theta$$
$$= 36\left(\int_0^{3\pi/4}d\theta - \int_0^{\pi/2}\cos\theta\,d\theta + \int_{\pi/2}^{3\pi/4}\cos\theta\,d\theta\right)$$
$$= 36\left(\frac{3\pi}{4} - \Big[\sin\theta\Big]_0^{\pi/2} + \Big[\sin\theta\Big]_{\pi/2}^{3\pi/4}\right) = 36\left(\frac{3\pi}{4} - (1-0) + (\frac{1}{\sqrt{2}} - 1)\right) = 27\pi - 72 + 18\sqrt{2} = 38.28$$

8. (10 marks) Consider the two surfaces in $\mathbb{R}^3$ given by the equations
$$2x^2 - y^2 + 2z^2 - 25 = 0 \quad\text{and}\quad x^2 - y^2 + z^2 = 0$$
respectively. Here the coordinates x, y, z in $\mathbb{R}^3$ are measured in cm. It is given that C is a curve in $\mathbb{R}^3$ lying on both of the above surfaces, and C passes through the point $(3,5,4)$. The temperature T at any point (x,y,z) in $\mathbb{R}^3$ is given by
$$T(x,y,z) = (x-1)^2 + y^2 + (z-3)^2,$$
measured in °C. An insect flies along the curve C at a constant speed of 3 cm per second and in such a way that it experiences an increase in temperature as time increases. Find the rate of change of temperature experienced by the insect at the instant when the insect passes through the point $(3,5,4)$. Justify your answer, and give your answer in °C per second.

[Suggestion: First find a tangent vector to the curve C at $(3,5,4)$.]

Solution. Let $f(x,y,z) = 2x^2 - y^2 + 2z^2 - 25$, and $g(x,y,z) = x^2 - y^2 + z^2$, so that the two surfaces are given by the equations $f(x,y,z) = 0$ and $g(x,y,z) = 0$ respectively. Then we have
$$\nabla f(x,y,z) = \langle 4x, -2y, 4z\rangle, \quad \nabla g(x,y,z) = \langle 2x, -2y, 2z\rangle.$$
Then we get normal vectors to the two surfaces at $(3,5,4)$ given by
$$\mathbf{n}_1 = \nabla f(3,5,4) = 12\mathbf{i} - 10\mathbf{j} + 8\mathbf{k} \quad\text{and}$$
$$\mathbf{n}_2 = \nabla g(3,5,4) = 6\mathbf{i} - 10\mathbf{j} + 8\mathbf{k}$$
respectively. Since the curve C lies on both surfaces, it follows that a tangent vector to C at $(3,5,4)$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$ and thus it can be given by
$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -10 & 16 \\ 6 & -10 & 8 \end{vmatrix} = 80\mathbf{i} - 60\mathbf{k}$$
Then a unit tangent vector to C at $(3,5,4)$ can be given by
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{80\mathbf{i} - 60\mathbf{k}}{\sqrt{80^2 + 60^2}} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{k} = \langle\frac{4}{5}, 0, -\frac{3}{5}\rangle.$$
Given the temperature function $T(x,y,z) = (x-1)^2 + y^2 + (z-3)^2$, it follows that
$$\nabla T(x,y,z) = \langle 2x-2, 2y, 2z-6\rangle \implies \nabla T(3,5,4) = \langle 4, 10, 2\rangle.$$
Let the position vector of the insect at time t be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, and let t_o be the time at which the insect passes through the point $P(3,5,4)$, so that $\mathbf{r}(t_o) = \langle 3,5,4\rangle$. Since the insect flies along C, it follows that $\mathbf{r}'(t_o)$ is parallel to $\mathbf{u}$. Since the insect flies at a constant speed of 3 cm per second, it follows that $\|\mathbf{r}'(t_o)\| = 3$. Hence we have
$$\mathbf{r}'(t_o) = \|\mathbf{r}'(t_o)\|\mathbf{u} = 3\mathbf{u} \quad\text{or}\quad \mathbf{r}'(t_o) = -\|\mathbf{r}'(t_o)\|\mathbf{u} = -3\mathbf{u}.$$