Combinatiorics Homework 02

DarkSharpness

2023.09.17

目录

Problem 1

(a)

$$((4, blue), 4)$$

(b)

$$\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$$

Problem 2

$$3^n$$

Problem 3

$$6^n - 5^n - 5^n + 4^n$$

Problem 4

$$n(1 - \frac{1}{2^k})(2^n)^k$$

Problem 5

$$\binom{2022}{42}$$

Problem 6

$$\begin{aligned} LHS &= \sum_{k=1}^{n} (\binom{n}{k} - \binom{n-1}{k}) \binom{n+m}{k}^{-1} \\ &= \frac{n!m!}{(n+m)!} \sum_{k=1}^{n} \frac{(n+m-k)!}{(n-k)!m!} - \frac{(n-1)!(m+1)!}{(n+m)!} \sum_{k=1}^{n} \frac{(n+m-k)!}{(n-k-1)!(m+1)!} \end{aligned}$$

而

$$\sum_{k=1}^{n} \frac{(n+m-k)!}{(n-k)!m!} = \sum_{t=0}^{n-1} \frac{(m+t)!}{m!t!} = \sum_{t=0}^{n-1} \binom{m+t}{t} = \binom{m+n}{m+1}$$

所以

$$\begin{aligned} LHS &= \binom{n+m}{m}^{-1} \binom{n+m}{m+1} - \binom{n+m}{m+1}^{-1} \binom{n+m}{m+2} \\ &= \frac{n}{m+1} - \frac{n-1}{m+2} \\ &= \frac{n+m+1}{(m+1)(m+2)} \end{aligned}$$

Problem 7

$$(\frac{1}{3})^{50} (\frac{2}{3})^{50} \binom{100}{50}$$

Problem 8

$$\begin{aligned} ANS &= \int_0^1 p^{50} (1-p)^{50} \binom{100}{50} dp \\ &= \int_0^{\frac{\pi}{2}} \frac{(\sin 2x)^{100}}{2^{100}} \binom{100}{50} dx \\ &= \frac{\binom{100}{50}}{2^{100}} \frac{99!!}{100!!} \frac{\pi}{2} \end{aligned}$$