# Solution to the Rational Function Model Based on the Levenberg-Marquardt Algorithm

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***Abstract*—Conventional method of solving Rational Function Coefficients is based on the Least Squares Estimation. When there are large number of coefficients or the control points are not well-distributed, the normal equation will become ill-conditioned and the Least Squares Estimation cannot get reliable solution. A new method for solving the Rational Polynomial Coefficients (RPCs) is proposed, which substitutes the Least Squares Estimation with the Levenberg-Marquardt (LM) algorithm. The LM algorithm is a standard technique for nonlinear least-squares problems and can be thought of as a combination of steepest descent and the Gauss-Newton method. It can overcome the ill-condition of the normal equation and is very efficient in calculation. In this paper, we implement the LM algorithm with SPOT5 imagery registration and compare it with the Ridge Estimation method which is widely used to improve the condition number of the normal equation and the Rigorous Physical model. The empirical results have verified that LM algorithm is reliable and valid for solving the RPCs.**

***Keywords-component; rational polynomial coefficients; Levenberg-Marquardt algorithm; ridge estimation; L-curve method ; geometric correction***

## I. Introduction

The sensor model represents the geometric relationship between the image space and the ground object space. Different kinds of sensors have different sensor models because of their different characteristics of the imaging geometry. Generally, the sensor models are divided into two categories, physical sensor models and generalized sensor models. The sensor parameters are needed to know first when using the physical sensor models. Furthermore, different sensors have different sensor parameters and the mushrooming new satellites make it difficult to develop various sensor models [1]. The Rational Function Model (RFM) is a new generalized sensor model. It can achieve a high accuracy that compatible to the rigorous sensor models. Some high resolution satellite imagery providers keep the sensor parameters confidential for commercial reasons, and it becomes difficult to develop a physical sensor model. For such reason, more and more researchers divert their attention to the RFM.

The RFM is a nonlinear model and it is essentially a generic form of polynomial models. The key to use RFM to process the satellite image is to find the precise rational polynomial coefficients. At present, the main method to solve RPCs is the Least Squares Estimation. But when the number of the Coefficients is large or the distribution of the control points is not well, the normal equation will become ill-conditioned and the Least Squares Estimation cannot get reliable solution. A lot of methods have been proposed to solve the ill-condition of the normal equation including the ridge estimation and the Artificial Intelligence. The Ridge Estimation is a widely used method, it is a revised biased estimation based on the Least Squares Estimation and the main work of it is to find the appropriate ridge parameter [2]. In this paper, the L-curve method is used to obtain the ridge parameter.

The Levenberg-Marquardt algorithm is first created by Levenberg, and modified by Marquardt. It is a nonlinear optimization method. LM algorithm is good at dealing with nonlinear problems and can overcome the ill-condition problem of the equation [3]. Considering its outstanding performances in handling nonlinear problems and its efficient calculation capability, it is introduced to solve the nonlinear RFM model. In this paper, the Least Squares Estimation is substituted by the LM algorithm in solving the RPCs. The SPOT5 image is as experiment data to do the geometry correction experiment. Finally, different algorithms are compared among the geometry correction precision they achieved.

The remainder of this paper is organized as follows. The RFM is briefly introduced in Section2, conventional methods of solving the RFM including the method to solve the RPCs without initial value and the Ridge Estimation are described in Section3, the LM algorithm is discussed in Section4, experiment implementation and result analysis are provided in Section5. Finally, conclusions are drawn in Section6.

## II. Rational Function Model

In the rational function model, the pixel coordinates $(r, c)$ are expressed as the ratio of polynomials containing the ground coordinates$(X, Y, Z)$. It can be expressed as the following equation:

$$r_n = \frac{p_1(X_n, Y_n, Z_n)}{p_2(X_n, Y_n, Z_n)}$$
$$c_n = \frac{p_3(X_n, Y_n, Z_n)}{p_4(X_n, Y_n, Z_n)} \tag{1}$$

Change (1) to the polynomial form:

$$r_n = \frac{\sum_{i=0}^{m1}\sum_{j=0}^{m2}\sum_{k=0}^{m3} a_{ijk} X_n^i Y_n^j Z_n^k}{\sum_{i=0}^{n1}\sum_{j=0}^{n2}\sum_{k=0}^{n3} b_{ijk} X_n^i Y_n^j Z_n^k}$$
$$c_n = \frac{\sum_{i=0}^{m1}\sum_{j=0}^{m2}\sum_{k=0}^{m3} c_{ijk} X_n^i Y_n^j Z_n^k}{\sum_{i=0}^{n1}\sum_{j=0}^{n2}\sum_{k=0}^{n3} d_{ijk} X_n^i Y_n^j Z_n^k} \tag{2}$$

Where

$$\begin{aligned} p_1(X_n, Y_n, Z_n) &= a_0 + a_1 Z + a_2 Y + a_3 X + a_4 ZY + a_5 ZX \\ &+ a_6 YX + a_7 Z^2 + a_8 Y^2 + a_9 X^2 + a_{10} ZYX \\ &+ a_{11} Z^2 Y + a_{12} Z^2 X + a_{13} Y^2 Z + a_{14} Y^2 X \\ &+ a_{15} ZX^2 + a_{16} YX^2 + a_{17} Z^3 \\ &+ a_{18} Y^3 + a_{19} X^3 \end{aligned}$$

In (1), $r_n$ and $c_n$ are the normalized pixel coordinates, $X_n$, $Y_n$ and $Z_n$ are the normalized ground point coordinates. They can be transferred from the normalization equations:

$$\begin{aligned} r_n &= \frac{r - r_0}{r_s} \\ c_n &= \frac{c - c_0}{c_s} \\ X_n &= \frac{X - X_0}{X_s} \\ Y_n &= \frac{Y - Y_0}{Y_s} \\ Z_n &= \frac{Z - Z_0}{Z_s} \end{aligned} \tag{3}$$

In (3), $(r_0, c_0, X_0, Y_0, Z_0)$ are the standardized translation parameters, $(r_s, c_s, X_s, Y_s, Z_s)$ are the standardized scaling parameters. The multinomial coefficients are the RPCs needed to be solved.

The RFM has nine different forms due to its denominator polynomials, they are described in the TABLE I.

The number of polynomial coefficients needed to be solved and the minimum GCPs needed under the nine different RPC models are shown in TABLE I. When the denominators of the RPC model are the same and constant for 1 (cases 7∼9), the RPC model becomes the conventional polynomial model, but in the case of $p_2 = p_4$ with the first order (case 4 in TABLE I), it degrades into the direct linear transformation model.

TABLE I
THE NINE DIFFERENT FORMS OF THE RFM.

| Case | Denominator | Order of polynomials | Number of RFCs | Min. number of GCPs |
|---|---|---|---|---|
| 1 | $p_2 \neq p_4$ | 1 | 14 | 7 |
| 2 | | 2 | 38 | 19 |
| 3 | | 3 | 78 | 39 |
| 4 | $p_2 = p_4$ | 1 | 11 | 6 |
| 5 | | 2 | 29 | 15 |
| 6 | | 3 | 59 | 30 |
| 7 | $p_2 = p_4 = 1$ | 1 | 8 | 4 |
| 8 | | 2 | 20 | 10 |
| 9 | | 3 | 40 | 20 |

## III. Conventional Methods for Solving the RFM

### A. Solution to the RFM without Initial Value

In order to get rid of the restriction of the initial value, we change the nonlinear rational function model into linear form which don't need the iteration process [5].

Transform (1) to the following linear equation:

$$\begin{cases} F_{r_n} = p_1(X_n, Y_n, Z_n) - r_n p_2(X_n, Y_n, Z_n) = 0 \\ F_{c_n} = p_3(X_n, Y_n, Z_n) - c_n p_4(X_n, Y_n, Z_n) = 0 \end{cases} \tag{4}$$

And the error equation is: $V = BX - L$, where

$$B = \begin{bmatrix} \frac{\partial F_X}{\partial a_i} & \frac{\partial F_X}{\partial b_j} & \frac{\partial F_X}{\partial c_i} & \frac{\partial F_X}{\partial d_j} \\ \frac{\partial F_Y}{\partial a_i} & \frac{\partial F_Y}{\partial b_j} & \frac{\partial F_Y}{\partial c_i} & \frac{\partial F_Y}{\partial d_j} \end{bmatrix}, L = \begin{bmatrix} -F_X^0 \\ -F_Y^0 \end{bmatrix};$$

$$(i = 1, \cdots, 20; j = 2, \cdots, 20); X = \begin{bmatrix} a_i & b_j & c_i & d_j \end{bmatrix}^T$$

According to the Least-Square principle, the estimation of the coefficients can be approached by the following equation:

$$X = (B^T W B)^{-1} B^T W L \tag{5}$$

$W$ is the weight matrix.

### B. Ridge Estimation Method

The Ridge Estimation is an improved biased estimation based on the Least Square estimation. It is a common method to manage the ill-condition problems. The narrow estimation of parameter $X$ can be expressed as:

$$X(k) = (B^T P B + kE)^{-1} B^T P L \tag{6}$$

Where $k$ is the ridge parameter and it is usually a positive decimal value, $E$ is a unit matrix, $X(k)$ is the ridge estimation value of $X$ at the point $k$.

It has been proved that there is always a $k$ value makes MSE$(X(k))$ <MSE$(X)$, where MSE$(X)$ means the mean square error of $X$. For a linear model, there is a $k$ value makes the Ridge Estimation value $X(k)$ outperforms the Least Square Estimation value $X$. So the most important part of the Ridge Estimation is to find the appropriate value of $k$.

Recently, the ways to find the $k$ value are mainly four kinds: the ridge mark Method, the GCV method, the $L$-curve method and the empirical formula method.The $L$-curve method is proposed by Hansen, it is good at dealing with ill-posed problems [4]. Its core is to positioning the $L$-curve maximum curvature point. In this paper, the $L$-curve method is used to obtain the ridge parameter $k$ for it is intuitional and precise.

## IV. LM Algorithm to Solve the RPC Model

The LM algorithm is specialized in handling non-linear least squares problem. It combines the steepest decent method and the Gauss-Newton method and inherits the global-search of gradient descent as well as the local-fast-converge of Gauss-Newton. When the current solution is far from the correct one, the algorithm behaves like a steepest descent method: slow, but guaranteed to converge. When the current solution is close to the correct solution, it becomes a Gauss-Newton method [6,7].

For a nonlinear error equation:

$$V = f(X) - L \tag{7}$$

Suppose $X_{(k)}$ is the $k$-th iterative solution vector, then recurrence formula can be described as follow:

$$\begin{cases} X_{(k+1)} = X_{(k)} + \Delta X_{(k)} \\ \Delta X_{(k)} = -(\nabla^2 S_{(k)})^{-1} \nabla S_{(k)} \end{cases} \tag{8}$$

where $S_{(k)} = V_{(k)}^T V_{(k)}$ is the error criterion function,

$V_{(k)}$ is the error vector in the $k$-th iteration,

$X_{(k)}$ is the solution vector of LM algorithm in the $k$-th iteration,

$\nabla^2 S_{(k)}$ is the Hessian matrix of $S_{(k)}$,

$\nabla S_{(k)}$ is the gradient vector of $S_{(k)}$.

In the $k$-th iteration, the solution of LM Algorithm is (D.W. Marquardt, 1963) [8]

$$\Delta X_{(k)} = -({J^T}_{(k)} J_{(k)} + \mu(k) I)^{-1} {J^T}_{(k)} V \tag{9}$$

Where $J_{(k)}$ is the Jacobian matrix of $S_{(k)}$, $\mu_{(k)} > 0$ is the damping coefficient in the $k$-th iteration, $I$ is the identity matrix. In this paper, $\mu_{(k)}$ is calculated by the formula $\mu_{(k)} = ||J_{(k)}^T V||$ and such $\mu_{(k)}$ ensures the convergence of LM algorithm.

Taking SPOT5 data as an example, the process of using LM algorithm to solve the RPCs is shown in the following steps:

STEP 1 Use the method of solving the RPC model without initial value to get the initial RPCs, take it as the initial value of the LM algorithm.

STEP 2 Calculate the coefficient matrix $A$, error vector $V$, weight matrix $P$ and the normal matrix $A^T PA$.

STEP 3 Find the damping coefficient $\mu$ with equation $\mu = ||J^T V||$.

STEP 4 Calculate the LM solution with equation $\Delta X = -(A^T PA + \mu I)^{-1} AV$ and update the model.

STEP 5 Check whether $|\Delta X|$ is larger than the convergence threshold, if it is, repeat step $1-4$; if not, accept the value $X$.

Some researchers have used the LM algorithm to solve the Rigorous Physical Model and add the equivalent weights method to the LM, and make it into the robust LM estimation. It shows that the robust LM estimator can efficiently dismiss the influence of the outlier and can get the robust parameters of the rectification mode [9, 10].

## V. Test and Result Analysis

### A. Data and Experiment

In order to test and verify the validation of solution to the RFM based on the LM algorithm, we choose 2.5 meter resolution SPOT5 satellite image as the experiment data. The data were collected in Beijing, China on 2nd June, 2004. The area is 60kilometers$\times$60 kilometers and the image size is $24000 \times 24000$ pixels. The test region includes both mountain areas and flat resident areas as shown in Fig.1. The elevation of ground control points ranges from 150 meters to 879.74 meters.

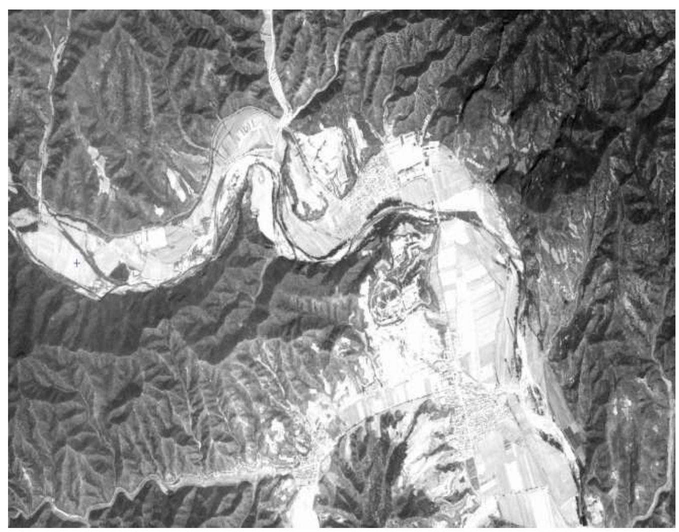

Fig. 1. The test data view.

There are two approaches to solve the RFM, known as terrain-dependent and terrain-independent. When the physical model parameters are known, the RFM can be solved by the terrain-independent method which won't be affected by the terrain condition. In the absence of physical model, we can use the GCPs to solve the RFM. Thus, its solution is dependent on the number and distribution of the GCPs. Because there are no available physical model parameters, we use terrain-dependent method to solve the RFM in this experiment. Eighty GCPs were collected from an ortho-rectified SPOT5 image with 2.5 meters resolution by using the PCI software, and another 12 checkpoints were also collected from it. The 3-D view of the distribution of these control points (marked by blue dot "o") and checkpoints (marked by red cross "+") is given in Fig.2.

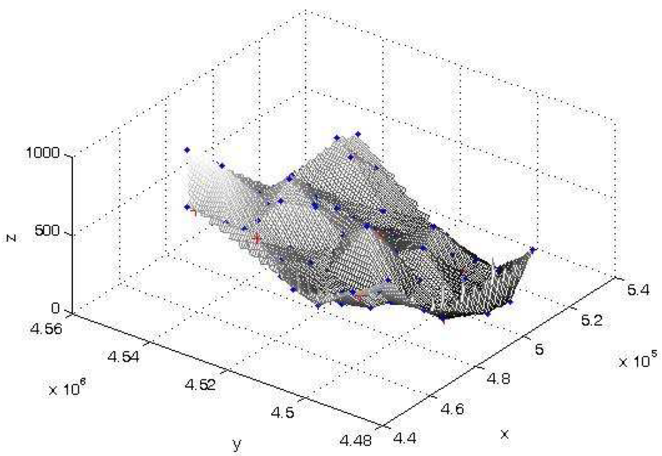


Fig. 2. The 3-D view of the terrain surface and the distribution of GCPs(●) and CPs(+).

As discussed before, the $L$-curve method is used to find the ridge estimation parameter $k$. The shape of $L$-curve can be seen from Fig.3. The ordinate axis represents $log||X||$ and the horizontal axis represents $log||BX - L||$. The turning point is obvious in the $L$-curve, so it is very easy to locate the point of the maximum curvature.

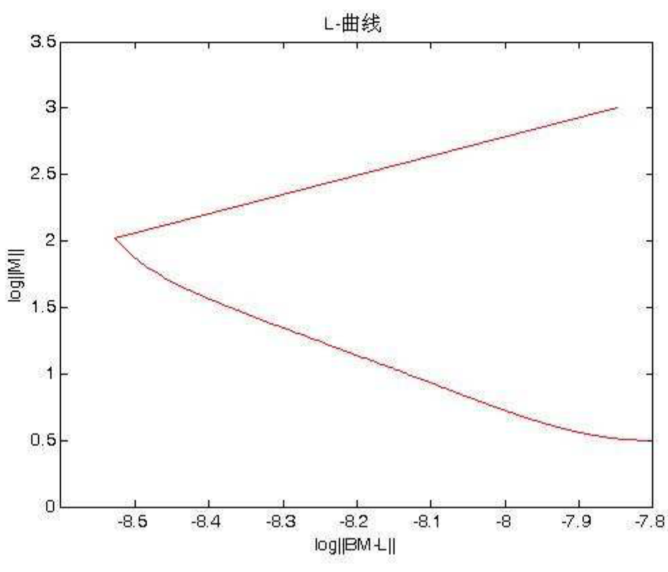


Fig. 3. Illustration of determining ridge parameter using L-curve method.

By using the curvature formula, we can find the $k$ value which makes the maximum $\kappa$ [4].

$$\kappa = 2\frac{\eta\rho}{\eta'}\frac{k^2\eta'\rho + 2k\eta\rho + k^4\eta\eta'}{(k^2\eta^2 + \rho^2)^{\frac{3}{2}}} \tag{10}$$

In this paper, the appropriate $k$ value is $8.76 \times 10^{(-7)}$. And it is applied in the Ridge Estimation to solve the RFM.

The LM algorithm is used to solve the RFM with 80 GCPs. The initial value can be solved by using the method of solution without initial value that is introduced before. The final satisfactory value can be approached through iteration. Then we can use the RPCs it obtains to solve the image coordinates $(r, c)$ of the checkpoints. The final precision of the rectification can be determined by the root mean square error between the solved image coordinates $(r, c)$ and the initial coordinates $(r_0, c_0)$. The whole process is shown in Fig.4.

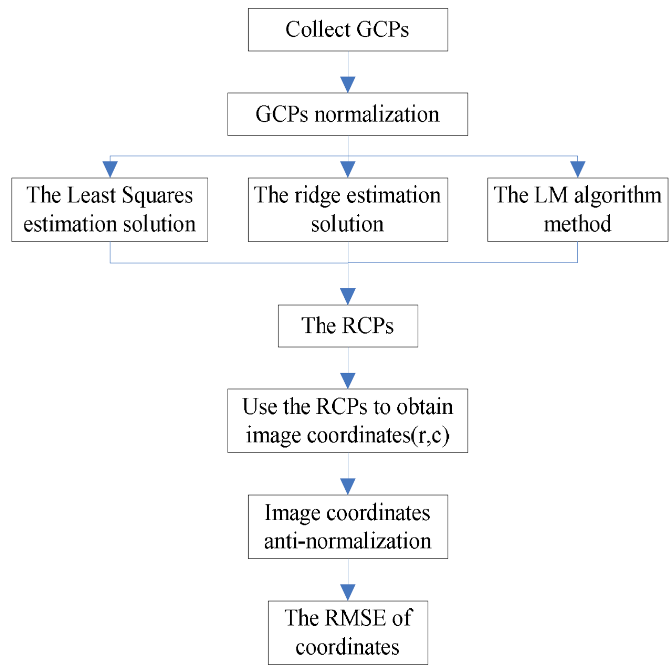


Fig. 4. The process of the experiment.

### B. Results and Analysis

In this paper, three different methods, the Least Squares Estimation, ridge estimation and the LM algorithm, are used to solve the RFM. The RFCs solved by them are applied to the SPOT5 image geometric correction experiment. Comparisons have been made among the three different methods. At the same time, the Rational Function Model is compared to the Rigorous Physical Model in geometric correction precision.

The geometric correction precision of the four different methods can be seen from TABLE II and TABLE III. And a comprehensive conclusion can be draw from them.

1) Rigorous Physical Model achieves a high precision in the SPOT5 image geometric correction. The RFM also gets a high precision which is very close to the rigorous model except for the RFM solved by the Least-Squares estimation.

2) When use the RPCs solved by the Least Squares Estimation to do the geometric correction, the precision is 69.4257 pixels which is very low from the needed precision. It demonstrates that the RPCs are far away from its true value. So the Least Squares Estimation is not a sound solution for

RPCs and it cannot get over the ill-condition of the RFM. But the other two methods are of very high precision and can handle the ill-condition problem.

3) The Ridge Estimation gives a good performance in solving the RPCs. Its precision is much higher than the Least Squares Estimation. And it has the capability to overcome the ill-condition problems. What's more, it also shows that the L-curve method is a valid method to obtain the ridge parameter $k$.

4) The LM algorithm introduced in this paper also shows a good capability of solving the RFM. The precision it achieves is comparable to the ridge estimation which is under 1 pixel. It is a great progress towards the conventional Least Squares Estimation. Compared to Ridge Estimation, it do not need to find a ridge parameter which affects the estimation precision.

TABLE II
THE RMS ERROR OF CONTROL POINTS (UNIT: PIXELS).

| Methods | RMSE of Row Coordinate | RMSE of Column Coordinate | RMSE |
|---|---|---|---|
| Rigorous Physical Model | 0.3810 | 0.3302 | 0.5042 |
| The Least Squares Estimation | 5.1080 | 22.2637 | 22.8422 |
| Ridge Estimation | 0.4444 | 0.3157 | 0.5451 |
| LM algorithm | 0.4104 | 0.2524 | 0.4818 |

TABLE III
THE RMS ERROR OF CHECKPOINTS (UNIT: PIXELS).

| Methods | RMSE of Row Coordinate | RMSE of Column Coordinate | RMSE |
|---|---|---|---|
| Rigorous Physical Model | 0.2905 | 0.4120 | 0.5041 |
| The Least Squares Estimation | 1.7845 | 69.4257 | 69.4257 |
| Ridge Estimation | 0.5703 | 0.5090 | 0.7644 |
| LM algorithm | 0.5010 | 0.6953 | 0.8570 |

## VI. CONCLUSION

When the number of the parameters is large or the distribution of the control points is not well, the conventional Least Squares Estimation solution of the RFM is not stable and the geometric correction precision is low. In this paper, the LM algorithm was introduced to solve the RPCs instead of the Least Squares Estimation. Through the experiment of solving the 3-order RFM with different denominators, and compared to the Ridge Estimation solution and the rigorous model, it testify the soundness of LM algorithm and the capability of it to overcome the ill-condition problem. And the precision is also very high when applied to Spot5 data geometric correction. Though there are still problems in using the LM algorithm to solve the RFM, such as it is a biased estimation, it is still a good attempt to add new source to the RFM solution and can be used to help to find more sound and precise solution.


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## REFERENCES


[1] C.Vicent Tao and Yong Hu, *Use of the Rational Function Model for Image Rectification*. Canadian journal of remote sensing, 27(6), pp. 593-602, 2001.

[2] Yuan Xiaoxiu and Lin Xianyong, *A Method for Solving Rational Polynomial Coefficients Based on Ridge Estimation*. Geomatics and Information Science of Wuhan University, 33(11), pp. 1130-1133, November 2008.

[3] More Jorge, *The Levenberg-Marquardt Algorithm: Implementation and Theory*. Numerical analysis, PP. 105-116, 1978.

[4] Wang Zhenjie and OU Jikun, *Determining the Ridge Parameter in a Ridge Estimation Using L-curve Method*. Geomatics and Information Science of Wuhan University, 29(3), pp. 235-238, March 2004.

[5] Qin Xuwen, Tian Shufang, Hong Youtang and Zhang Guo, *The Algorithm for Parameters of RPC Model Without Initial Value*. Remote Sensing for Land & Resources, No. 4, December 2005.

[6] K. Levenberg. *A Method for the Solution of Certain Non-linear Problems in Least Squares*. Quarterly of Applied Mathematics, 2(2), pp. 164-168, July 1944.

[7] Manolis I. A. Lourakis, *A Brief Description of the Levenberg-Marquardt Algorithm Implemented by Levmar*. Institute of Computer Science Foundation for Research and Technology, February 2005.

[8] Marquardt D.W., *An Algorithm for Least-Squares Estimation of Nonlinear Parameters*. Journal of the society for Industrial and Applied Mathematics, 11(2), pp. 431-441, 1963.

[9] Chen Pengshan, Jiao Weili, Jia Xioupeng and Wang Wei, *Robust LM Algorithm and Its Application on Rigorous Physical Model*. Science Technology and Engineering, 9(16), pp. 4614-4618, August 2009.

[10] Long Tengfei, Jiao Weili and Jia Xiupeng, *A New Method for Automatic Gross Error Detection in Remote Sensing Image Geometric Correction*. The Seventh International Conference on Fuzzy Systems and Knowledge Discovery (FSKD), 2010.