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Quantum Particles and Classical Particles

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The relation between wave mechanics and classical mechanics is reviewed, and it is stressed that the latter cannot be regarded as the limit of the former as $\hbar \to 0$. The motion of a classical particle (or ensemble of particles) is described by means of a Schrödinger-like equation that was found previously. A system of a quantum particle and a classical particle is investigated (1) for an interaction that leads to stationary states with discrete energies and (2) for an interaction that enables the classical particle to act as a measuring instrument for determining a physical variable of the quantum particle.

1. INTRODUCTION

A great deal has been written in the effort to understand quantum mechanics. By "to understand" is usually meant to understand in terms of the concepts of classical physics. Up to now such attempts at understanding have not been very successful. It may be that in the future quantum mechanics will be replaced by some new theory. However, such a theory is likely to be more abstract and even harder to understand in classical terms than the present theory.

It appears therefore that it is desirable to change one's approach. One should try to understand quantum mechanics in its own terms and not those of classical physics. In other words, starting with the formalism of quantum mechanics, one should develop concepts that are in conformity with this formalism, and one should investigate its various implications. From this point of view it would be more appropriate to try to understand classical mechanics in terms of quantum mechanics rather than the other way around. The purpose of the present work is to go in this direction.

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2. CLASSICAL WAVE MECHANICS

Let us begin by reviewing briefly some earlier work.[1,2] If one starts with the Schrödinger equation for a particle of mass m in a potential-energy field $V(x, y, z)$,

$$i\hbar\dot{\psi} = -(\hbar^2/2m)\nabla^2\psi + V\psi \tag{1}$$

with $\dot{\psi} = \partial\psi/\partial t$, and one writes

$$\psi = Re^{iS/\hbar} \tag{2}$$

with R and S real functions, one gets

$$-\dot{S} = (1/2m)(\nabla S)^2 + V - (\hbar^2/2m)\nabla^2 R/R \tag{3}$$

$$\dot{\rho} = -(1/m)\nabla\cdot(\rho\nabla S) \tag{4}$$

where

$$\rho = R^2 = |\psi|^2 \tag{5}$$

Let us assume that ψ is normalized so that

$$\int \rho\, d^3x = 1 \tag{6}$$

Equation (3) looks like the classical Hamilton–Jacobi equation for a particle moving in a potential-energy field $V + V_Q$ with

$$V_Q = -(\hbar^2/2m)\nabla^2 R/R \tag{7}$$

so that the particle momentum and velocity are, respectively,

$$\mathbf{p} = \nabla S, \qquad \mathbf{v} = \nabla S/m \tag{8}$$

Equation (4) can be written

$$\dot{\rho} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{9}$$

which has the form of the continuity equation for a fluid of density ρ and velocity $\mathbf{v}$. Hence one can interpret ψ as describing an ensemble of particles behaving like a fluid in irrational flow. The above equations have been used to give a classical interpretation of quantum mechanics.[3-5]

The motion of a classical particle is described with the help of the ordinary Hamilton–Jacobi equation

$$-\dot{S} = (1/2m)(\nabla S)^2 + V \tag{10}$$

From (8) and (10) one sees that

$$\frac{d\mathbf{p}}{dt} = \frac{\partial \nabla S}{\partial t} + (\mathbf{v} \cdot \nabla)\, \nabla S = -\nabla V \tag{11}$$

so that Newton's equations of motion hold. One can also adopt (4) or (9) as the continuity equation for an ensemble of classical particles distributed in space with some probability density ρ.

How does one go from quantum mechanics as described by (1), or (3) and (4), to classical mechanics described by (10) and (4)? It is often said that classical mechanics is the limit of quantum mechanics for $\hbar \to 0$. However, one can raise objections to this point of view:

(a) In quantum mechanics the superposition principle holds. It apparently does not hold in classical mechanics, although it should, if the latter is the limit of quantum mechanics.

(b) The quantum potential energy V_Q given by (7) should go to zero as $\hbar \to 0$. In many cases it does, but it is easy to find cases in which it does not,[1] so that (3) does not always go over into (10) as $\hbar \to 0$.

It appears therefore that classical mechanics has to be regarded as something essentially different from quantum mechanics and not as its limit. However, in spite of this, one can try to incorporate classical mechanics into the framework of quantum mechanics. The Schrödinger equation (1) led to (3) and (4). One can look for a corresponding equation that leads to (10) and (4). Obviously this is given by

$$i\hbar\dot{\psi} = -(\hbar^2/2m)\, \nabla^2 \psi + (V - V_Q)\, \psi \tag{12}$$

which can also be written

$$i\hbar\dot{\psi} = [-(\hbar^2/2m)\, \nabla^2 + (\hbar^2/2m)\, \nabla^2 |\psi| / |\psi| + V]\, \psi \tag{13}$$

If one makes use of (2), then (13) gives (10) and (4). It will be noted that $\hbar$, while appearing in (13) and (2), is absent in (10) and (4).

We can regard (13) as the "classical Schrödinger equation." Since it is nonlinear, it does not satisfy the superposition principle in general; that is, if ψ_1 and ψ_2 are solutions, their sum will in general not be a solution. The exceptations are: (a) ψ_1 and ψ_2 are nonoverlapping (i.e., their supports have no points in common), and (b) ψ_1 is a multiple of ψ_2. Since (13) is equivalent to (10) and (4), it is clear that it differs from (1) in that there is no quantization of energy levels.

This "classical wave mechanics" differs from the Schrödinger mechanics also in that there is no uncertainty principle holding. If one has a solution of (10) for S, one can form a particle trajectory

$$\mathbf{x} = \boldsymbol{\xi}(t) \tag{14}$$

with $\mathbf{x} = (x, y, z)$, $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, such that at each point

$$\dot{\boldsymbol{\xi}}(t) = (1/m)(\nabla S)_{\mathbf{x} = \boldsymbol{\xi}(t)} \tag{15}$$

that is, $\mathbf{x} = \boldsymbol{\xi}(t)$ is a solution of Newton's equations of motion. To describe the motion of a single particle along this trajectory (or of an ensemble of particles in identical motion), one can take

$$\rho = \delta^{(3)}(\mathbf{x} - \boldsymbol{\xi}(t)) \tag{16}$$

where $\delta^{(3)}(\mathbf{x})$ is the three-dimensional Dirac delta function. The function ψ in this case does not spread in the course of time as does the Schrödinger ψ.

As was remarked above, one cannot in general combine two solutions by superposition to get another solution. If one wants to combine them, one can only do this in a way that corresponds to forming a mixed state in quantum mechanics. Suppose we have two solutions,

$$\psi_1 = R_1 e^{(i/\hbar)S_1}, \qquad \psi_2 = R_2 e^{(i/\hbar)S_2} \tag{17}$$

We think of them as describing two noninteracting ensembles of particles. Let us now combine them with weights w_1 and w_2, such that

$$w_1, w_2 > 0, \qquad w_1 + w_2 = 1 \tag{18}$$

If we write $\rho_1 = R_1^2$, $\rho_2 = R_2^2$, then at a given point we have the combined density

$$\rho = w_1 \rho_1 + w_2 \rho_2 \tag{19}$$

where $w_1 \rho_1$ is the probability density of finding a particle having a velocity $\mathbf{v}_1 = (1/m) \nabla S_1$, and $w_2 \rho_2$ that of finding a particle with velocity $\mathbf{v}_2 = (1/m) \nabla S_2$. The mean velocity is

$$\bar{\mathbf{v}} = (1/\rho)(w_1 \rho_1 \mathbf{v}_1 + w_2 \rho_2 \mathbf{v}_2) \tag{20}$$

but the particles have only the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. One can think of the two ensembles as forming ideal fluids in irrotational flow, interpenetrating each other and moving independently.

Using the formalism of quantum mechanics, one can describe the situation with the help of a density matrix,

$$\boldsymbol{\rho}(\mathbf{x}', \mathbf{x}'') = w_1 \psi_1(\mathbf{x}')\, \psi_1^*(\mathbf{x}'') + w_2 \psi_2(\mathbf{x}')\, \psi_2^*(\mathbf{x}'') \tag{21}$$

One then has

$$\rho(\mathbf{x}') = \boldsymbol{\rho}(\mathbf{x}', \mathbf{x}') \tag{22}$$

and

$$\rho(\mathbf{x}')\, \bar{v}(\mathbf{x}') = (-i\hbar/2m)[(\nabla_{\mathbf{x}'} - \nabla_{\mathbf{x}''})\, \boldsymbol{\rho}(\mathbf{x}', \mathbf{x}'')]_{\mathbf{x}''=\mathbf{x}'} \tag{23}$$

The above can be generalized to a mixture of any number of states.

Before ending this section let us consider the case of one-dimensional motion, so that $\psi = \psi(x, t)$. Then (10) and (4) become

$$-\dot{S} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) \tag{24}$$

$$-\dot{\rho} = \frac{1}{m}\frac{\partial}{\partial x}\left(\rho\frac{\partial S}{\partial x}\right) \tag{25}$$

Let us assume that

$$\dot{S} = -E \qquad (=\text{const}) \tag{26}$$

Hence

$$S = \pm\int^x [2m(E-V)]^{1/2}\, dx - Et \tag{27}$$

in the region where $V \leqslant E$. In this region we can get a solution of (25) in the form

$$\rho = \left|\frac{\partial S}{\partial x}\right|^{-1} f^2(\tau), \qquad \tau = t - m\int^x \left(\frac{\partial S}{\partial x}\right)^{-1} dx \tag{28}$$

where f is an arbitrary function of its argument subject to the normalization condition (6). Here

$$\frac{\partial S}{\partial x} = \pm[2m(E-V)]^{1/2} \tag{29}$$

We can now write

$$\psi = \left|\frac{\partial S}{\partial x}\right|^{-1/2} f e^{(i/\hbar)S} \tag{30}$$

In the case of a free particle ($v = 0$), the situation is simple. Let us write

$$S = px - (p^2/2m)\, t \tag{31}$$

with p the particle momentum. Then we can take

$$\psi = R(x - pt/m)\, e^{(i/\hbar)S} \tag{32}$$

where R is an arbitrary normalized function of its argument.

3. TWO-PARTICLE SYSTEM

Let us consider a system consisting of two quantum particles. To simplify matters, let us assume they have one-dimensional motion. If we denote their masses by m_1, m_2 and their coordinates by x_1, x_X, the Schrödinger equation describing their motion is

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\Psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2\Psi}{\partial x_2^2} + V\Psi \tag{33}$$

where we take the potential energy V to have the form

$$V = V(x_1 - x_2) \tag{34}$$

We can introduce a relative coordinate y and a center-of-mass coordinate Y,

$$y = x_1 - x_2, \qquad Y = (m_1 x_1 + m_2 x_2)/(m_1 + m_2) \tag{35}$$

and put (33) into the form

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\Psi}{\partial y^2} - \frac{\hbar^2}{2M}\frac{\partial^2\Psi}{\partial Y^2} + V(y)\,\Psi \tag{36}$$

with the total mass M and the reduced mass μ,

$$M = m_1 + m_2, \qquad \mu = m_1 m_2/(m_1 + m_2) \tag{37}$$

One can now take as a solution

$$\Psi = \psi(y, t)\,\phi(Y, t) \tag{38}$$

so that (36) splits into two equations,

$$i\hbar\dot{\phi} = -\frac{\hbar^2}{2M}\frac{\partial^2\phi}{\partial Y^2} \tag{39}$$

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial y^2} + V\psi \tag{40}$$

The first equation has as a solution

$$\phi = e^{(i/\hbar)(-Et+PY)} \tag{41}$$

with E the energy and P the momentum of the center-of-mass motion, so that

$$E = P^2/2M \tag{42}$$

The second equation can be solved by taking

$$\psi = e^{-(i/\hbar)E_n t}\psi_n(y) \tag{43}$$

where ψ_n satisfies the equation

$$E_n\psi_n = -\frac{\hbar^2}{2\mu}\frac{d^2\psi_n}{dy^2} + V\psi_n \tag{44}$$

If we assume that the allowed values of the internal energy E_n are discrete, we can now write as the general solution for a given momentum P,

$$\Psi = e^{(i/\hbar)(-Et+PY)}\sum c_n\, e^{-(i/\hbar)E_n t}\psi_n(y) \tag{45}$$

with the constants c_n satisfying the relation

$$\sum |c_n|^2 = 1 \tag{46}$$

provided the ψ_n are normalized.

Now let us consider a similar system with two classical particles. Corresponding to (33) we now have

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\Psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2\Psi}{\partial x_2^2} + V(x_1 - x_2) + \left(\frac{\hbar^2}{2m_1}\frac{\partial^2|\Psi|}{\partial x_1^2} + \frac{\hbar^2}{2m_2}\frac{\partial^2|\Psi|}{\partial x_2^2}\right)\frac{\Psi}{|\Psi|} \tag{47}$$

which, with

$$\Psi = R\, e^{(i/\hbar)S}, \qquad \rho = R^2 \tag{48}$$

gives

$$-\dot{S} = \frac{1}{2m_1}\left(\frac{\partial S}{\partial x_1}\right)^2 + \frac{1}{2m_2}\left(\frac{\partial S}{\partial x_2}\right)^2 + V \tag{49}$$

$$\dot{\rho} = -\frac{1}{m_1}\frac{\partial}{\partial x_1}\left(\rho\frac{\partial S}{\partial x_1}\right) - \frac{1}{m_2}\frac{\partial}{\partial x_2}\left(\rho\frac{\partial S}{\partial x_2}\right) \tag{50}$$

If we go over to the coordinates of (35) and the masses of (37), the equations become

$$-\dot{S} = = \frac{1}{2\mu}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2M}\left(\frac{\partial S}{\partial Y}\right)^2 + V(y) \tag{51}$$

$$\dot{\rho} = -\frac{1}{\mu}\frac{\partial}{\partial y}\left(\rho\frac{\partial S}{\partial y}\right) - \frac{1}{M}\frac{\partial}{\partial Y}\left(\rho\frac{\partial S}{\partial Y}\right) \tag{52}$$

Let us write

$$S = -(E + E')\,t + PY + \tilde{S}(y) \tag{53}$$

with E given by (42). Then (51) becomes

$$E' = \frac{1}{2\mu}\left(\frac{d\tilde{S}}{dy}\right)^2 + V \tag{54}$$

which is the equation determining the internal motion of the system.

If we assume that $\rho = \rho(y)$, (52) gives

$$\rho\frac{d\tilde{S}}{dy} = \text{const} \tag{55}$$

which determines ρ or $R(y)$.

As in the one-particle case, it is clear that the superposition principle is not valid here.

The two cases considered above are well known. Let us now consider a system consisting of a quantum particle with mass m_1 and coordinate x_1 and a classical particle with mass M_2 and coordinate X_2, the interaction being given by $V = V(x_1 - X_2)$. The wave equation in this case has the form

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\Psi}{\partial x_1^2} - \frac{\hbar^2}{2M_2}\frac{\partial^2\Psi}{\partial X_2^2} + V\Psi + \frac{\hbar^2\Psi}{2M_2|\Psi|}\frac{\partial^2|\Psi|}{\partial X_2^2} \tag{56}$$

Again we introduce relative and center-of-mass coordinates (y, Y) and total and reduced masses (M, μ),

$$y = x_1 - X_2, \qquad Y = (m_1 x_1 + M_2 X_2)/(m_1 + M_2) \tag{57}$$

$$M = m_1 + M_1, \qquad 1/\mu = 1/m_1 + 1/M_2 \tag{58}$$

so that (56) becomes

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\Psi}{\partial y^2} - \frac{\hbar^2}{2M}\frac{\partial^2\Psi}{\partial Y^2} + \frac{\hbar^2\Psi}{2M_2 R}\frac{\partial^2 R}{\partial X_2^2} + V(y)\,\Psi \tag{59}$$

where we have used (48).

Now let us take

$$\Psi = \psi(y, t)\, e^{(i/\hbar)(-Et + PY)} \tag{60}$$

with E again given by (42). In view of (57) we have

$$\frac{\partial^2 R}{\partial X_2^2} = \frac{\partial^2 R}{\partial y^2} \tag{61}$$

Hence (59) becomes

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial y^2} + \frac{\hbar^2\psi}{2M_2 R}\frac{\partial^2 R}{\partial y^2} + V\psi \tag{62}$$

If we assume

$$\psi = R(y)\, e^{-(i/\hbar)E't} \tag{63}$$

and use (58), we get

$$E'R = -\frac{\hbar^2}{2m_1}\frac{d^2 R}{dy^2} + VR \tag{64}$$

The form of (64) is similar to that of (44) except for the fact that m_1 occurs in place of μ. However, from (58) it follows that, if $M_2 \gg m_1$ (which will be the case for a classical particle and a quantum particle), μ will be very nearly equal to m_1, and we shall have $E' = E_n$, $R = \psi_n$.

We see that the situation here is similar to that in the case of two quantum particles as regards stationary states of the form (43). However, we encounter a fundamental difference when we consider a superposition of such states, as in (45). In the present case a superposition of two solutions of the form (63) with different values of E' will in general not be a solution of (62). To combine such solutions one must resort to a mixture like that considered in the previous section.

Finally it should be noted that, if one wants a more general solution of (62) which could also describe an unbound state, one can write

$$\psi = R\, e^{(i/\hbar)\tilde{S}}, \qquad \rho = R^2 \tag{65}$$

Then we get from (62), taking (58) into account,

$$-\dot{\tilde{S}} = \frac{1}{2\mu}\left(\frac{\partial \tilde{S}}{\partial y}\right)^2 + V - \frac{\hbar^2}{2m_1 R}\frac{\partial^2 R}{\partial y^2} \tag{66}$$

$$\dot{\rho} = -\frac{1}{\mu}\frac{\partial}{\partial y}\left(\rho \frac{\partial \tilde{S}}{\partial y}\right) \tag{67}$$

so that both μ and m_1 appear in the equations.

4. MEASUREMENT

Les us consider again the last system of two particles, with quantum particle I having mass m_1 and coordinate x_1 and classical particle II having mass M_2 and coordinate X_2. However, this time let us assume the interaction between them is such as to enable particle II to act as a measuring instrument for determining the value of some physical variable associated with particle I.

In the absence of interaction the Schrödinger equation for particle I can be written

$$i\hbar\dot{\psi}_n = H_1 \psi_n \tag{68}$$

where the ψ_n are the energy eigenfunctions so that

$$H_1 \psi_n = E_n \psi_n \tag{69}$$

and the energy levels E_n are assumed to be discrete.

Let us assume that without the interaction particle II is free so that its wave equation is

$$i\hbar\dot{\phi} = H_2 \phi \tag{70}$$

where

$$H_2 = -\frac{\hbar^2}{2M_2}\frac{\partial^2}{\partial X_2^2} + \frac{\hbar^2}{2M_2|\phi|}\frac{\partial^2|\phi|}{\partial X_2^2} \tag{71}$$

and that it is in a state corresponding to (32), i.e.,

$$\phi = R\, e^{(i/\hbar)S} \tag{72}$$

with

$$R = R(X_2 - Pt/M_2), \qquad S = PX_2 - (P^2/2M_2)\, t \tag{73}$$

Since R is an arbitrary function of its argument, we can take, as a special case,

$$R^2 = \delta(X_2 - Pt/M_2) \tag{74}$$

This can be interpreted as describing either a single particle or an ensemble of particles in identical motion. For the present purpose the latter interpretation is preferable.

Suppose we now have a physical variable g associated with I that we wish to measure. Let us assume that ψ_n is also an eigenfunction of g,

$$g\psi_n = g_n \psi_n \tag{75}$$

i.e., that g commutes with H_1. Let us now write a wave function for the two-particle system,

$$i\hbar\dot{\Psi} = [H_1 + H_2 - i\hbar Z(t)\, g\, \partial/\partial X_2]\, \Psi \tag{76}$$

where $Z(t)$ is defined by

$$\left.\begin{aligned} Z(t) &= 0, \qquad t < 0 \\ &= A, \qquad 0 \leqslant t \leqslant T \\ &= 0, \qquad t > T \end{aligned}\right\} \tag{77}$$

with A a constant. If we take Ψ as

$$\Psi_n = \phi_n(X_2, t)\, \psi_n(x_1, t) \tag{78}$$

and make use of the (75) then, with ψ_n satisfying (68), ϕ_n satisfies the equation

$$i\hbar\dot{\phi}_n = H_2\phi_n - i\hbar Z(t)\, g_n\, \partial\phi_n/\partial X_2 \tag{79}$$

For $t<0$, $Z(t)=0$, and (79) has the same form as (70). Hence we can take $\phi_n=\phi$ for $t<0$. For $0\leqslant t\leqslant T$, one sees that the solution can be taken in the form

$$\phi_n = \phi(X_2 - Ag_n t, t) \tag{80}$$

that is,

$$\phi_n = R_n\, e^{(i/\hbar)S_n} \tag{81}$$

with

$$R_n = R(X_2 - Ag_n t - Pt/M_2), \qquad S_n = PX_2 - Ag_n Pt - (P^2/2M_2)\, t \tag{82}$$

For $t>T$ (79) again looks like (70), but for continuity we now take

$$R_n = R(X_2 - Ag_n T - Pt/M_2), \qquad S_n = PX_2 - Ag_n PT - (P^2/2M_2)\, t \tag{83}$$

We see that the form of ϕ_n after the interaction corresponds to a rigid displacement of ϕ by a distance $Ag_n T$ so that, for a given value of A, one can determine g_n. Hence particle II serves as a measuring instrument for determining the value of g. To be sure, we could have carried out a similar procedure using a quantum particle in the place of II. However, if a quantum particle acted as a measuring instrument, the question would arise: how do we determine its state? For this another measurement would be required, and in this way one would get involved in an endless chain of measurements. In our case one should be able to determine the displacement of the classical particle (which could be a needle on a dial) by inspection (reading the dial), in particular, if one takes R as in (74) and $P=0$.

Let us now consider what happens if, before the interaction begins ($t<0$), particle I is in a state ψ which is a superposition of the states ψ_n, i.e.,

$$\psi = \sum c_n \psi_n \tag{84}$$

with $\sum |c_n|^2 = 1$. In the general case, just after the interaction sets in ($t>0$), no corresponding superposition of the combined wave functions Ψ_n given by (78) will satisfy (76). What one can do is to go over to a mixture of states described by the density matrix

$$\rho(x_1', X_2'; x_1'', X_2'') = \sum w_n \Psi_n(x_1', X_2') \Psi_n^*(x_1'', X_2'') \tag{85}$$

with

$$w_n = |c_n|^2 \tag{86}$$

each Ψ_n satisfying (76).

There is an exceptional case, that for which R is given by (74). In that case, if we consider R_m and R_n with $m \neq n$, for $t > 0$ they either do not overlap (if $g_m \neq g_n$) or they are the same functions (if $g_m = g_n$). As was pointed out in Sec. 2, under these conditions a superposition is possible, so that

$$\Psi = \sum c_n \Psi_n \tag{87}$$

will be a solution of (76). However, because there is no overlap of terms with different values g_n, the state is equivalent to a mixture. If one tried by means of some experimental arrangement to bring about interference between terms in (87), i.e., to create overlapping, the superposition would no longer satisfy (76).

5. DISCUSSION

In the present work it is emphasized that classical mechanics is not to be regarded as the limit of wave mechanics as $\hbar \to 0$. However one can set up an equation somewhat similar to that of the Schrödinger equation to describe the motion of a classical particle. One would expect this equation to hold for a large particle mass, while the ordinary Schrödinger equation holds for a small mass. This suggests that there should be a more general equation holding for all values of the mass and going over into these equations for large and small masses. Such an equation would presumably be based on a theory that would replace the present quantum theory and the nature of which one cannot foresee at present.

It was suggested[1] that the transition from wave mechanics to classical mechanics takes place near a mass of the order of the Planck mass, $m_p = (\hbar c/G)^{1/2} = 2.17 \times 10^{-5} gm$. This is plausible since it is the only quantity having the dimension of mass that can be formed from the fundamental constants c, $\hbar$, and G. If this conjecture is valid, the fact that this mass depends on G, the gravitational constant, implies that the more fundmental theory referred to above must also involve the general theory of relativity.

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