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Integrable magnetic geodesic flows on 2-surfaces

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The problem of integrability of Hamiltonian systems usually reduces to searching for the first integrals of motion. In the talk we will consider geodesic flows (including magnetic ones) on 2-surfaces and will discuss some questions related to local and global existence of such integrals.

Topological meaning of the kolmogorov spectrum of magnetic turbulence

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Magneto hydro dynamics equations describe magnetic fields in a conductive liquid. Explicit solutions are complicated and are known in exceptional cases. In a generic case one assume that solutions are randomly distributed over magnetic and hydrodynamic spectra. We will consider a question: "Is it possible to achieve an exponent -1.7 for a turbulent spectrum of the magnetic energy?" We get an affirmative answer, assuming that quasi-periodic magnetic fields are distributed free over the scale. Our answer uses asymptotic Hopf invariants, and the M-invariant (a numerical measure of knottiness of magnetic lines).

Algorithms of Hamiltonian neural networks

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I consider the invariants of dynamical systems from the view of neural networks. Specifically, the invariant is the Hamiltonian of the system.

We need to predict the evolution of a dynamic system, and the basic algorithms of neural networks have some problems. For example, a single pendulum in $\mathbf{R}^2$ is a simple dynamical system that has a Hamiltonian. The basic neural network predicts the future position of the pendulum in space (that is, two-dimensional vector (p, q)).

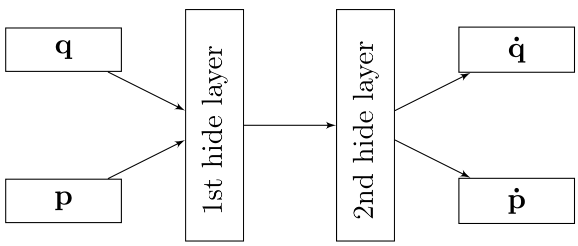

Figure 1: Scheme of a Base neural network

Then the Hamiltonian neural network predicts a function, we differentiate this function and calculate the position of the pendulum.

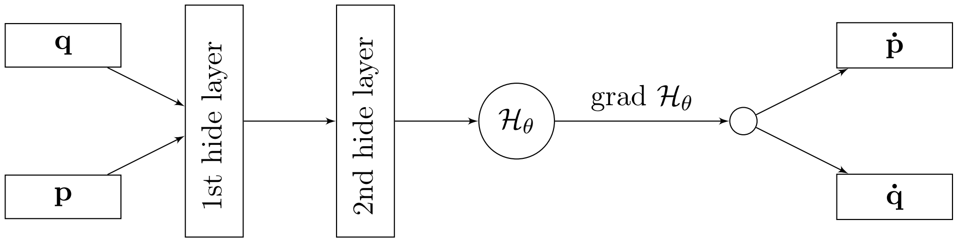

Figure 2: Scheme of a Hamiltonian neural network

Here we use the property of the Hamiltonian: $(\dot{q}, -\dot{p}) = \left(\frac{\mathrm{d}q}{\mathrm{d}t}, -\frac{\mathrm{d}p}{\mathrm{d}t}\right) = \left(\frac{\partial \mathcal{H}}{\partial p}, -\frac{\partial \mathcal{H}}{\partial q}\right)$

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Densely branching trees as models for AGH University of Science and Technology Hénon-like and Lozi-like attractors

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Inspired by a recent work of Crovisier and Pujals on strongly dissipative diffeomorphisms of the plane, we show that H enon-like and Lozi-like maps on their strange attractors are conjugate to natural extensions (a.k.a. shift homeomorphisms on inverse limits) of maps on metric trees with dense set of branch points. In consequence, these trees very well approximate the topology of the attractors, whereas the maps on them give good models of the dynamics. To the best of our knowledge, these are the first examples of canonical two-parameter families of attractors in the plane for which one is guaranteed such a 1-dimensional locally connected model tying together topology and dynamics of these attractors. For H enon maps this applies to Benedicks-Carleson positive Lebesgue measure parameter set, and sheds more light onto the result of Barge from 1987, who showed that there exist parameter values for which H enon maps on their attractors are not natural extensions of any maps on branched 1-manifolds. For Lozi maps the result applies to an open set of parameters given by Misiurewicz in 1980. Our results can be seen as a generalization to the non-uniformly hyperbolic world of a classical result of Williams from 1967.

Hyperelliptic sigma functions,
Adler–Moser polynomials,
and polynomial dynamical systems

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In [1], for each $g > 0$, a system of $2g$ multidimensional heat equations in a nonholonomic frame was constructed. The sigma function of the universal hyperelliptic curve of genus g is a solution of this system. In the talk we present explicit expressions for the Schrödinger operators that define the equations of the system considered. These expressions were published in [2].

This result has numerous interesting applications.

In the problem of constructing the series expansion for the genus g hyperelliptic sigma function, it allows to obtain the necessary initial conditions for this expansion. Namely, we show that the condition that the initial condition of the system considered is polynomial determines the solution of the system up to a constant factor.

We give an explicit description of the connection of such solutions to well-known Burchnall–Chaundy polynomials and Adler–Moser polynomials. For each g we find a system of linear second-order differential equations that determines the corresponding Adler–Moser polynomial.

These systems are closely related to a Lie subalgebra of the Witt algebra, where the generators are the second-order differential operators A_{2k} for $k = 0, 1, 2, \ldots$, where

$$A_{2k} = -\frac{1}{2}\sum_{s=1}^{k} \partial_{2s-1}\partial_{2k+1-2s} - \sum_{s=1}^{\infty}(2s-1)z_{2s-1}\partial_{2s+2k-1}.$$

In the problem finding the Lie algebra of derivations of the field of genus g hyperelliptic functions these results allow to obtain the explicit expressions for some of the generators of this Lie algebra for any genus g. A construction given in [3] allows to obtain the corresponding polynomial vector fields that lead to graded homogeneous polynomial dynamical systems in the complex $3g$-dimensional space. One of these polynomial dynamical systems

$$\frac{\partial}{\partial \tau_1} x_{i,j} = x_{i+1,j}, \quad i = 1, 2, \quad j = 1, 3, \ldots, 2g-1,$$

$$\frac{\partial}{\partial \tau_1} x_{3,j} = 4(2x_{1,1}x_{2,j} + x_{2,1}x_{1,j} + x_{2,j+2}), \quad j = 1, 3, \ldots, 2g-1,$$

where $x_{2,2g+1} = 0$, in the rational limit $\lambda = 0$ gives the Korteweg–de Vries equation

$$4\partial_3\widehat{\wp} = \partial_1(\partial_1^2\widehat{\wp} - 6\widehat{\wp}^2)$$

for the function $\widehat{\wp} = -\partial_1\partial_1 \ln \widehat{\sigma}$, where $\widehat{\sigma}$ is the rational limit of the genus g hyperelliptic sigma function.

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Structure of the Kolmogorov set near separatrices in a Hamiltonian system with two degrees of freedom

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Consider a Hamiltonian system with 2 degrees of freedom with Hamiltonian H and second independent integral F. This system may possess hyperbolic invariant objects such that hyperbolic periodic solutions and saddle-focus, saddle, and hyperbolic knot fixed points. Assume that Hamiltonian H and the integral F are independent at each of these objects, which is expressed in terms of their second differentials as follows:

- d^2H and d^2F are independent at each hyperbolic fixed point,
- $d^2F \neq \lambda d^2H$ at same (and, consequently, at each) point of hyperbolic periodic solution where a number λ is such that $dF = \lambda dH$ at this point.

Under such the condition, a common critical level Σ of both integrals H and F corresponding to either a hyperbolic periodic solution or a saddle-focus is constituted by the separatrices (invariant manifolds) of the latter. In contrary, the set Σ is built of four or three smooth manifolds in the vicinity of a saddle or knot hyperbolic fixed point, respectively. Hence, hyperbolic periodic solutions and saddle-focus points may possess homo- and heteroclinic connections (and such examples are easily constructed), while both a saddle and knot fixed point may be connected only with a point of the same type.

Now let the system experience an autonomous perturbation of small order ε and become non-integrable. We are interested in the structure of the associated Kolmogorov set of invariant two-dimensional tori that survive under the perturbation, assuming that the system is sufficiently smooth. The set of all the point located outside some small given neighborhoods of the hyperbolic invariant objects under consideration and at distance not greater than μ from the set Σ will be called a μ-neighborhood of Σ. In the sequel, $\mu \geqslant C\varepsilon$, where C is a positive constant, i.e. the order of the perturbation parameter ε doesn't exceed the order of the distance parameter μ. We will describe the structure of the part the Kolmogorov set located at the neighborhood of Σ. However, we will need to exclude from consideration the $c\mu$-neighborhood of some manifold S of codimension 1 that correspond to the violation of twist property of an associated separatrix map, the $c > 0$ being arbitrarily small.

Theorem Under the above conventions and some non-degeneracy conditions (that guarantee a non-degeneracy of the separatrix map), the following results are valid:

1) a part of destroyed invariant curves is $O(\sqrt{\mu/\varepsilon})$ at a distance of order μ from the set Σ;

2) preserved invariant curves experience deformation $O(\sqrt{\mu\varepsilon})$;

3) the measure of complement of invariant tori in μ-neighborhood of Σ is $O(\sqrt{\mu\varepsilon})$;

4) the closest invariant tori restricting the set Σ from both sides (if they exist) have mutual distance $O(\varepsilon)$ from one to another;

5) on a distance of order μ from Σ each trajectory remains bounded by invariant tori that have mutual distance $O(\sqrt{\mu\varepsilon})$ from one to another.

Earlier, this result has been established by the author [1] for invariant curves of a nearly-integrable exact canonical two-dimensional mapping. It has been explained the well-known picture observed in numerical calculations and described for a long time in a number of physical works, which is related to the birth of the so-called "stochastic layer". Hence, analogous phenomenon of stochastic layer holds in the system under discussion.

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Geometric interpretation of entropy for the Dyck shift

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We begin with some notation.

Definition. Let A be a final set of symbols and let T be the right shift on $A^{\mathbb{Z}}$, then ($A^{\mathbb{Z}}$, T) is called the full shift over the alphabet A.

Definition. If $w \in A^n$ for some $n \in \mathbb{N}$, then w is called a (finite) word over the alphabet A and n is the length of the word.

Suppose that $x \in A^{\mathbb{Z}}$ and w is a word of length n over the alphabet A. If there exists l, such that $w = (x(l+1), ..., x(l+n))$, then we say that w occurs in x.

For x, $y \in A^{\mathbb{Z}}$, $x \neq y$, let $d(x,y) = (1/2)^{m(x,y)}$, where $m(x,y) = \max\{m \in N_0\colon\ x(m) = y(m)$ and $x(-m) = y(-m)\}$. Let $d(x,x) = 0$. Then d is a metric and d induces the topology on $A^{\mathbb{Z}}$.

Definition. If X is a closed T-invariant subset of $A^{\mathbb{Z}}$, then (X,T) is called a symbolic system or a shift space. By $W(X)$ we denote the language of X, i.e. the set of all finite words in the alphabet A which can occur in an element of X.

Remark. Let X be a metric and topological subspace of $A^{\mathbb{Z}}$. Suppose x is a point of X and C is a subset of X. By $O_\epsilon(x)$ and $O_\epsilon(C)$ we denote open ϵ-neighbourhoods of x and C respectively, i.e. $O_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$, $O_\epsilon(C) = \bigcup_{x\in C} O_\epsilon(x)$.

Let's introduce the special class of symbolic systems called synchronized. Recall that a shift space X is transitive when X contains an element x whose forward orbit $\{T^j(x) : j \in N\}$ is dense in X.

Definition. A synchronized system is a transitive shift space X which contains a synchronizing (magic) word, i.e. a word $v \in W(X)$ with a following property: if $uv, vw \in W(X)$, then $uvw \in W(X)$.

Let A be (,), [,]

Definition. A word w over the alphabet A is called balanced if it respects standard bracket rules. A subword of a balanced word is called correct.

Definition. The symbolic system which language consists of all correct words is called the Dyck shift.

The Dyck shift is transitive, but not synchronized.

Definition. A function n from $\mathbb{R}^+$ to $\mathbb{N}$ is called slowly growing if it satisfies the following conditions: $n(\epsilon) \to \infty$ and $n(\epsilon) = o(|\log(\epsilon)|)$ as $\epsilon \to 0$. The class of all slowly growing functions is denoted by N.

Let X be the shift space (not necessarily synchronized), let $M(X)$ be the set of shift-invariant Borel probability ergodic measures on X and let $h_\mu(X, T)$ be the metric entropy.

Definition. Suppose that $n \in N$, $\mu \in M(X)$ and $\epsilon > 0$ are fixed. The function $P^{\epsilon}_{X,T,\mu}(x)$ from X to $\mathbb{Z}^+$ defined by expression

$$\frac{1}{n(\epsilon)} \log \left(\frac{\mu\left(O_\epsilon\left(T^{n(\epsilon)}\left(O_\epsilon\left(x\right)\right)\right)\right)}{\mu\left(O_\epsilon\left(x\right)\right)} \right)$$

is called ϵ-boundary deformation rate of measure μ. This function is well defined a.e. and belongs to $L_1(X, \mu)$.

Now we can formulate the main hypothesis (MH):

Hypothesis. For every $n \in N$: $P^{\epsilon}_{X,T,\mu} \xrightarrow[\epsilon \to 0]{} h_\mu(X, T)$ in $L_1(X, \mu)$.

If (MH) is true for some measurable shift space (X,T,μ), then we call (X,T,μ) (MH)-friendly and representation from the (MH) is called geometric interpretation of entropy. A possibility of geometric interpretation was well-studied for synchronized and related systems by Komech [1] and me (the paper is in the process of publication). In my report we consider an opportunity of geometric approach for Dyck shifts, which are not synchronized.

The main result of this report is the following:

Theorem. The sufficient condition for a measurable Dyck shift (X,T,μ) to be (MH)-friendly is $\mu\{(\} \neq \mu\{)\}$ or $\mu\{[\} \neq \mu\{]\}$.

We also show that condition from theorem 1 is not necessary.

This report is a part of my paper "Geometric interpretation of entropy: new results", which is accepted for publication in "Problems of Information Transmission".

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Numerical Solution for Hyperbolic-Type PDE

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Here $Q=(0,l)\times(0,T)$. The function $z(x,t)\in C^2(Q)\cap C^1(\overline{Q})$ is solution of a standard hyperbolic equation of the 2nd order

$$z''_{tt}(x,t)=a^2z''_{xx}(x,t)+b(x)z(x,t),\quad (x,t)\in Q, \tag{1}$$

with the second-type boundary conditions and initial conditions

$$z'_x(0,t)=z'_x(l,t)=0, \tag{2}$$

$$z(x,0)=\varphi(x),\quad z'_t(x,0)=\psi(x). \tag{3}$$

The function $b(x)$ is a continuously differentiable function. The initial functions are $\varphi(x)\in C^3[0,l]$, $\psi(x)\in C^2[0,l]$. In addition, the function $\varphi(x)$ and the function $\psi(x)$ satisfy conditions based on (2) for $t=0$ and $\varphi(x)$ is a positive function. The unique solution of problem (1)–(3) exists and one can be found by the separation of variables method. The author received this task as an auxiliary one in the work [1].

The system of cosines: $v_k(x)=\cos(\lambda_k x)$, $\lambda_k=\pi k/l$, $k=\overline{0,+\infty}$, is a complete orthogonal system on the segment $[0,l]$, for functions satisfying the second-type boundary conditions. We will decompose the functions $z(x,t)$, $b(x)$, $\varphi(x)$, $\psi(x)$ by the system of cosines: $z(x,t)=\sum\limits_{k=0}^{+\infty}\xi_k(t)v_k(x)$, $b(x)=\sum\limits_{k=0}^{+\infty}b_kv_k(x)$ and $\varphi(x)=\sum\limits_{k=0}^{+\infty}\varphi_kv_k(x)$, $\psi(x)=\sum\limits_{k=0}^{+\infty}\psi_kv_k(x)$. The series of Fourier coefficients of a continuously differentiable function $b(x)$, decomposed in the system of cosines, converges absolutely, (see, for example [1]).

After standard transformations the equation (1) is written as an infinite-dimensional system of the linear second-order differential equations with constant coefficients and initial conditions

$$\xi''_{tt}(t)=(C-\Lambda)\xi(t),\quad \xi_k(0)=\varphi_k,\quad \xi'_{kt}(0)=\psi_k,\quad k=\overline{0,+\infty}, \tag{4}$$

where $\xi(t)$ is infinite-dimensional vector-function with components $\xi_i(t)$, $i=\overline{0,+\infty}$. Matrix C is stationary matrix of infinite dimensions with elements: $c_{00}=b_0$, $c_{0j}=0.5b_j$, $j=\overline{1,+\infty}$, $c_{ii}=b_0+0.5b_{2i}$, $i=\overline{1,+\infty}$, $c_{ij}=0.5(b_{i+j}+b_{|i-j|})$, $i=\overline{1,+\infty}$, $j=\overline{0,+\infty}$. Matrix $\Lambda=\mathrm{diag}(0,a^2\lambda_1^2,...,a^2\lambda_j^2,...)$ is infinite-dimensional stationary and diagonal matrix. Let C_N be the corner minor of the (N+1)th order of the matrix $C-\Lambda$. The system of differential equations (4) will be rewritten in a shortened form as

$$\xi^{N''}_{tt}(t)=C_N\xi^N(t),\quad \xi^N_k(0)=\varphi^N_k,\quad \xi^{N'}_{kt}(0)=\psi^N_k,\quad k=\overline{0,+\infty}, \tag{5}$$

where $\xi^N(t)$ is the (N+1)th dimensional vector-function with components $\xi^N_i(t)$, $i=\overline{0,N}$, $\varphi^N=(\varphi_0,...,\varphi_N)^T$, $\psi^N=(\psi_0,...,\psi_N)^T$.

Let us denote $\zeta_n=\xi^N(n\tau)$, $T=s\tau$, s is a natural number, $\zeta_s=\xi^N(T)$; $\zeta_0=\varphi^N$, $\tau^{-1}(\zeta_1-\zeta_0)=\psi^N$, (we can apply formula $(2\tau)^{-1}(-\zeta_2+4\zeta_1-3\zeta_0)=\psi^N$, it is used for the second-order of approximation), then from (5) we get the difference equations

$$\frac{\zeta_{n+1}-2\zeta_n+\zeta_{n-1}}{\tau^2}=C_N\zeta_{n+1},\quad (E-\tau^2C_N)\zeta_{n+1}=2\zeta_n-\zeta_{n-1}.$$

As a result, we get the difference equation

$$\zeta_{n+1}=(E-\tau^2C_N)^{-1}(2\zeta_n-\zeta_{n-1}). \tag{6}$$

For a small τ the matrix $(E-\tau^2C_N)$ be diagonally dominant matrix. Due to the estimation of the norm of the inverse matrix from [2] the inequality $||E-\tau^2C_N)^{-1}||_\infty<1+\widetilde{c}_0\tau$ is true, $\widetilde{c}_0$ is some τ-independent constant. The necessary spectral condition of Neumann stability for difference scheme (6) holds.

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Riemannian gradient for manifolds of density matrices and isometric matrices to solve optimization problems of quantum information theory

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In the present report we are going to review some recent findings that relate Riemannian geometry on one side and quantum information science on the other side [1].

One connection is due to the complex Stiefel manifold that is a Riemannain manifold $V_{n,p}$ consisting of all $n \times p$ isometric matrices, $n \geq p$. If $n = p$, then $V_{n,n}$ is a manifold of unitary $n \times n$ matrices. Unitary matrices describe the time evolution of a closed quantum system as the evolution operator $U = \exp(-iHt)$ is unitary provided the Hamiltonian H is Hermitian. Therefore, the complex Stiefel manifold $V_{n,n}$ is of great use in the circuit quantum computation utilizing unitary gates. Coherent quantum control also operates with unitary transformations, so optimization algorithms on the Stiefel manifold of unitary matrices are of great need to efficiently manipulate quantum systems and design quantum algorithms. General $n \times p$ isometric matrices provide a useful parameterization of quantum channels (completely positive and trace preserving maps) via the Stinespring dilation theorem. Thanks to the Stinespring dilation, optimization methods on the Stiefel manifold of isometric matrices provide new efficient ways to perform optimization on the set of quantum channels, for instance, to perform the maximum likelihood estimation of quantum channels (process tomography) and reconstruct both Markovian and non-Markovian open system dynamics [2]. A wide area for application of isometric and unitary matrices is quantum tensor networks such as the multiscale entanglement-renormalization ansatz [1].

Another frequently used manifold is the set of density matrices, i.e., Hermitian positive-semidefinite operators with unit trace. Density operators are natural elements of dynamical optimization problems as initial states (say, in the problem of entanglement robustness against a specific noise [3]). The cone of positive-semidefinite matrices comprises both the unnormalized quantum states and the effects of positive operator-valued measures (POVMs), so optimization problems involving denstity operators and effects can be reduced to the optimization problems on the cones of positive-semidefinite operators. There exist other important problems of quantum information science, where the optimal density operator is mixed (not a pure state). For instance, the maximum of the coherent information $I_c(\Phi) = S(\Phi[\rho]) - S(\widetilde{\Phi}[\rho])$ gives the single-letter quantum capacity of the quantum channel Φ, and this quantity is strictly greater than 0 only if ρ is a mixed density operator ($\widetilde{\Phi}$ denotes the complementary channel). It was shown recently [4] that the two-letter quantum capacity can exceed the single-letter quantum capacity, which makes that optimization problem even more relevant. We believe that the optimization on the cone of positive definite matrices has a number of possible applications.

In the report, we overview all the technicalities related with the first-order Riemannian optimization methods over corresponding manifolds [1]. An open-source library [5] allows one to perform optimization with many natural 'quantum' constraints including the constraints of positive-definiteness and isometry.

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Uncountably many pairwise disjoint compacta in Euclidean space

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It is known that each non-empty perfect compact subset of S^{n-1} can be embedded in R^n in uncountably many inequivalent ways. We would like to embed uncountably many copies of a compactum simultaneously, so that they are mutually exclusive. Concentric spheres of arbitrary radii form a family of cardinality continuum; in contrast to this, R.H. Bing proved [1] that it is impossible to place an uncountable collection of pairwise disjoint *wild* closed surfaces in R^3.

Definition 1. A subset $P \subset R^n$ homeomorphic to a polyhedron is called tame if there exists a homeomorphism h of R^n onto itself such that $h(P)$ is a subpolyhedron in R^n; otherwise, P is wild.

J. Stallings constructed [2] a family of $\mathfrak{c}$ (continuum cardinality) pairwise disjoint wild 2-disks in R^3. J. Martin proved that all except countably many disks in such a collection must be locally tame except on their boundaries, hence must lie on 2-spheres [3]. In [4], R.B. Sher modified Stallings' construction so that no two disks of the family are ambiently homeomorphic in the following sense:

Definition 2. Two subsets $A, B \subset R^n$ are called ambiently homeomorphic if there exists a homeomorphism of R^n onto itself such that $h(A) = B$.

For higher dimensions, partial generalizations of Bing's non-embedding result are given in [5, Thm. 1, 2], [6, Thm. 10.5], [7, p.383, Thm. 3C.2]. We get the following generalization of Stallings' and Sher's results (here $\mathcal{C}$ is the Cantor set) [8]:

Theorem. *For each $n \geqslant 3$ and any non-empty perfect compact set $M \subset I^{n-1}$, there exists an embedding $F : M \times \mathcal{C} \to R^n$ such that for each $s \in \mathcal{C}$ the image $F(M \times \{s\}) =: M_s$ is non-locally-hyperplanar, and for $s \neq t$ the sets M_s, M_t are ambiently incomparable. If, in addition, M is a polyhedron, then each M_s is wild.*

Corollary. *There exists an embedding f of the Cantor fence $[0;1] \times \mathcal{C}$ into R^3 such that its image has the following property. If $H : R^3 \to R^3$ is a homeomorphism satisfying $H(f([0;1] \times \mathcal{C})) \subset f([0;1] \times \mathcal{C})$, then $H(f([0;1] \times \{s\})) \subset f([0;1] \times \{s\})$ for each $s \in \mathcal{C}$.*

In the planar case, the phenomenon described in the Corollary is impossible by [9].

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Existence and stability of the traveling waves in fully connected networks of nonlinear oscillators

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We consider a fully connected system of nonlinear oscillators

$$\dot{x}_j = F(x_j, u_j), \quad j = 1, 2, \ldots, m. \tag{1}$$

where $m \geq 2$, $x_j = x_j(t) \in \mathbb{R}^n$, $n \geq 2$, $u_j = \sum_{s=1,\, s\neq j}^{m} G(x_s)$,and the vector functions $F(x, u)$, $G(x)$ with values in $\mathbb{R}^n$ are infinitely differentiable with respect to their variables $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The partial system

$$\dot{x} = F(x, 0) \tag{2}$$

admits an exponentially orbitally stable cycle. We consider the situation when m of the same oscillators (2) interact with each other according to the principle "each with all". In particular, when $F(x, u) = F(x) + D(x)u$, where $D(x)$ is a square matrix of size $n \times n$, the system (1) takes the form

$$\dot{x}_j = F(x_j) + D(x_j) \sum_{s=1,\, s\neq j}^{m} G(x_s), \quad j = 1, 2, \ldots, m. \tag{3}$$

This situation is of the interest, since systems (3) arise in the mathematical modeling of fully connected neural and gene networks (see [1, 2, 3, 4]).

We will be interested in the problems of existence and stability for the system (1) of canonical traveling waves, which are special periodic solutions of the form

$$x_j = x(t + (j-1)\Delta), \quad j = 1, 2, \ldots, m. \tag{4}$$

Taking into account that the system (1) is invariant under a change of variables of the form $(x_1, x_2, \ldots, x_m) \to (x_{j_1}, x_{j_2}, \ldots, x_{j_m})$, any canonical traveling wave (4) generates a whole family U_k of induced traveling waves. The number of cycles of the family U_k is equal to $(m-1)!$

The construction of the family of traveling waves U_k is reduced to finding the canonical cycle (4). In solving this problem, we need an auxiliary equation with delays

$$\dot{x} = F(x, u_\Delta), \quad u_\Delta = \sum_{s=1}^{m-1} G(x(t - s\Delta)), \tag{5}$$

where $x = x(t) \in \mathbb{R}^n$, $\Delta = \text{const} > 0$. We assume that on some interval $\Delta \in (\Delta_1, \Delta_2) \subset (0, +\infty)$ the equation (5) admits a periodic solution $x = x(t, \Delta)$ of period $T = T(\Delta) > 0$. In this case, the following statement is true.

Theorem 1. *Suppose that there is a natural number $k : 1 \leq k \leq m-1$ for which the equation*

$$T(\Delta) = m\Delta/k \tag{6}$$

has the root $\Delta = \Delta_{(k)} \in (\Delta_1, \Delta_2)$. Then in the original system (1) this root corresponds to the cycle (canonical traveling wave)

$$C_k: \quad x_j = x_{(k)}(t + (j-1)\Delta_{(k)}), \quad j = 1, 2, \ldots, m \tag{7}$$

of the period $T_{(k)} = m\Delta_{(k)}/k$, where $x_{(k)}(t) = x(t, \Delta)|_{\Delta=\Delta_{(k)}}$.

Theorem 1 and theorems on the stability of the obtained solutions provide a certain general method for studying periodic solutions of the traveling wave type in fully connected networks of nonlinear oscillators. Indeed, the question of the existence of a canonical traveling wave (4) is reduced to finding the cycle $x(t, \Delta)$ of the auxiliary equation with delays (5) and to finding the roots of the equations (6).

It is clear that the problems of analysis of the auxiliary equations (5), which underlie our method, are generally nonlocal. But, nevertheless, in some situations when it is possible to apply any asymptotic methods, the indicated problems can be solved.

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On the dynamics of difference approximations of the logistic equation with delay

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We consider the logistic equation with delay

$$\frac{du}{dt} = r[1 - u(t-1)]u \tag{1}$$

or the Hutchinson equation. Here, the non-negative function $u(t)$ models the normalized population density, the positive parameter r characterizes its growth rate, and the delay in the right part — the age structure of the population.

A family of maps was constructed from the equation (1) by replacing the time derivative with the divided one "central" the difference $k\big(u(t+1/k) - u(t-1/k)\big)/2$, where the value k is the number of points into which the segment of length one is divided. Assuming $t = n/k$, $n \in \mathbb{Z}$ and denoting $u_n = u(n/k)$, we arrive at a difference equation of the order $k+1$

$$u_{n+1} = u_{n-1} + \frac{2r}{k}\,(1 - u_{n-k})u_n, \;\; n \geqslant 0. \tag{2}$$

The difference equation (2) approximates the solutions of the equation (1) and as the time step decreases, the accuracy of the calculations increases. However, nothing guarantees that the map (2) retains the same dynamic properties as the equation (1). In particular, for $r = \pi/2 + \varepsilon$, where $0 < \varepsilon \ll 1$ in the equation (1), an Andronov–Hopf bifurcation occurs and a stable cycle branches off from the unit state of equilibrium. In the case of maps, the birth of a cycle corresponds to the birth of a stable invariant curve ([1, 2, 3]), and the so-called Neumark-Saker bifurcation occurs. It is shown that in the difference scheme (2) the loss of stability of a single equilibrium state does not lead to the creation of a stable invariant curve. Thus, the dynamic properties of the scheme (2) and the equations (1) are fundamentally different. It is interesting to note that this mapping property (2) is due to the fact that the derivative on the left side of the equation (1) is approximated by the central divided difference. Therefore, when choosing a difference scheme that simulates the dynamics of the Hutchinson equation, it is not always possible to obtain a difference equation that preserves the dynamic properties of the original problem.

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Topological conjugacy rough circle's transformations n-ary product

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The results were obtained in collaboration with O.V. Pochinka and are devoted to the topological classification of rough circle's transformations n-ary products.

It is known from the 1939 work [1] of A.G. Mayer that rough transformations of the circle are limited to the diffeomorphisms of Morse-Smale. A topological conjugacy class of orientation-preserving diffeomorphism is entirely determined by its rotation number and the number of its periodic orbits, while for orientation-changing diffeomorphism the topological invariant will be only the number of periodic orbits. Thus, any such diffeomorphism is topologically conjugate to some model transformation $\psi : \mathbb{S}^1 \to \mathbb{S}^1$. This study examines rough model circle's transformations n-ary products on n-torus. The main result of the work is the proof of the following theorem.

Theorem. Diffeomorphisms $\phi,\ \phi' :\ \mathbb{T}^n \to \mathbb{T}^n$ are topologically conjugate if and only if there is a substitution $\xi = \begin{pmatrix} 1 & 2 & \dots & n \\ \xi_1 & \xi_2 & \dots & \xi_n \end{pmatrix}$ of indices from the set $\{1,\ 2, \dots,\ n\}$, $\xi(i) = \xi_i$, such that diffeomorphisms-components ϕ_i and ϕ'_{ξ_i} of n-ary products $\phi,\ \phi'$, respectively, are topologically conjugate.[1]

[1]for $n = 2$ the results arises from the work [2].

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On topological structure of ambient surface for A-diffeomorphism with non-wandering set consisting of one-dimensional basic sets

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In [1], it was introduced a class $\mathbb{G}(M^2)$ of A-diffeomorphisms of closed orientable surfaces such that their non-wandering set consists of one-dimensional basic sets. Also in [1] topological classification of diffeomorphisms from this class was obtained. Examples of diffeomorphisms from the class $\mathbb{G}(M^2)$ can be constructed on any closed orientable surface of genus $g \geq 2$. The primary mission of this report is to establish the topological structure of the ambient surfaces for diffeomorphisms from the class $\mathbb{G}(M^2)$ and to study the properties of these diffeomorphisms.

Let M^2 be a smooth closed orientable surface, $f : M^2 \to M^2$ be a diffeomorphism satisfying axiom A, $NW(f)$ be a non-wandering set of f.

An arbitrary one-dimensional basic set of diffeomorphism $f : M^2 \to M^2$ is either an attractor or a repeller and has the local structure of the direct product of the interval and the Cantor set. Periodic point p belonging to a one-dimensional attractor (repeller) Λ of diffeomorphism $f : M^2 \to M^2$ is called s-boundary (u-boundary) periodic point if one of the connected components of the set $W^s(p)\backslash p$ ($W^u(p)\backslash p$) does not intersect with Λ and both connected components of the set $W^u(p)\backslash p$ ($W^s(p)\backslash p$) intersect with Λ. For a one-dimensional attractor (repeller), the set of s-boundary (u-boundary) periodic points is not empty and is finite.

It is known that for a one-dimensional attractor (repeller) Λ accessible from inside boundary of the set $M^2\backslash\Lambda$ decays uniquely into a finite number of bunches. A bunch b of the attractor (repeller) Λ is the union of the maximum number h_b of the unstable (stable) manifolds of the s-boundary (u-boundary) periodic points $p_1, ..., p_{h_b}$ of the set Λ accessible from some (the same for all) point $x \in (M^2\backslash\Lambda)$. The number h_b is said to be the degree of the bunch.

Any basic set Λ of diffeomorphism f is uniquely expressed as the finite union of the compact subsets: $\Lambda = \Lambda_1 \cup ... \cup \Lambda_q,\ q \geq 1$, such that $f^q(\Lambda_j) = \Lambda_j, f(\Lambda_j) = \Lambda_{j+1}, j \in \{1, ..., q\}$

($\Lambda_{q+1} = \Lambda_1$) (these subsets Λ_j, $j \in \{1, ..., q\}$, are called the C-dense components of the set Λ). For every point x of a C-dense component Λ_j the set $W^s_x \cap \Lambda_j$ ($W^u_x \cap \Lambda_j$) is dense in Λ_j.

For a C-dense component Λ_i of the one-dimensional attractor (repeller) Λ, we denote by m_{Λ_i} the number of bunches belonging to Λ_i and by r_{Λ_i} the sum of the degrees of these bunches. For an arbitrary C-dense component Λ_i of the one-dimensional attractor (repeller) Λ of diffeomorphism $f : M^2 \to M^2$, there exists a submanifold N_{Λ_i} (canonical support) with the following properties: $N_{\Lambda_i} \cap NW(f) = \Lambda_i$; N_{Λ_i} is a compact orientable surface of genus $q_{\Lambda_i} = 1 + \frac{r_{\Lambda_i}}{4} - \frac{m_{\Lambda_i}}{2}$ with m_{Λ_i} boundary components and negative Euler characteristic.

Let $f : M^2 \to M^2$ be a diffeomorphism from the class $\mathbb{G}(M^2)$. We denote by k_f the number of all C-dense components of all basic sets of the diffeomorphism f, by κ_f the number of all bunches belonging to these C-dense components, by g_i ($i \in \{1, ..., k_f\}$) genus of the canonical support of the i-th C-dense component. For the number $g \geq 0$, we denote by M^2_g a closed orientable surface of genus g. The main results of the report are the following theorems.

Theorem. Let $f \in \mathbb{G}(M^2)$. Then the surface M^2 is homeomorphic to the connected sum:

$$M^2_{g_1}\#...\#M^2_{g_{k_f}}\#\underbrace{\mathbb{T}^2\#...\#\mathbb{T}^2}_{m_f},$$

where $m_f = \frac{\kappa_f}{2} - k_f + 1$.

Theorem. Let $f \in \mathbb{G}(M^2)$. Then f is Ω-stable, but is not structurally stable.

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Large bifurcation supports

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In the study of global bifurcations of vector fields on S^2, it is important to distinguish a set "where the bifurcation actually occurs", – the bifurcation support. Hopefully, it is sufficient to study the bifurcation in a neighborhood of the support only.

The first definition of bifurcation support was proposed by V.Arnold. However this set appears to be too small. In particular, the newly discovered effect, an open domain in the space of three-parametric families on S^2 with no structurally stable families , is not visible in a neighborhood of the bifurcation support.

In this talk, we give a new definition of "large bifurcation support" that accomplishes the task. Roughly speaking, if we know the topological type of the phase portrait of a vector field, and we also know the bifurcation in a neighborhood of the large bifurcation support, then we know the bifurcation on the whole sphere.

On ill-posed problems for systems of equations of elliptic type of the first order

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In this paper, we are talking about a regularized solution of the Cauchy problem for matrix factorization of the Helmholtz equation in multidimensional bounded and unbounded domains. Such problems naturally arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.). In general, the theory of ill-posed problems for elliptic systems of equations has been sufficiently formed thanks to the works of A.N. Tikhonov, V.K. Ivanov, M.M. Lavrent'ev, Sh. Yarmukhamedov, N.N. Tarkhanov of many other famous mathematicians. Among them, the most important for applications are the so-called conditionally well-posed problems, characterized by stability in the presence of additional information about the nature of the problem data. One of the most effective ways to study such problems is to construct regularizing operators. For example, this can be the Carleman-type formulas (as in complex analysis) or iterative processes (the Kozlov-Maz'ya-Fomin algorithm, etc.).

It is known that the Cauchy problem for elliptic equations is unstable with respect to a small change in the data, i.e. incorrect (example of Hadamard). Consequently, the theory of the solvability of such problems is significantly more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see for instance [1]).

In many well-posed problems for a system of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz, the calculation of the value of the vector function on the whole boundary is inaccessible. Therefore, the problem of reconstructing, solving a system of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz (see, for instance [2, 3, 4, 5, 6, 7, 8, 9] and [10]) is one of the topical problems in the theory of differential equations.

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Von Neumann's ergodic theorem and Fejér sums for signed measures on the circle

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The Fejer sums for measures on the circle and the norms of the deviations from the limit in von Neumann's ergodic theorem are calculated, in fact, using the same formulas (by integrating the Fejer kernels) — and so, this ergodic theorem is a statement about the asymptotics of the Fejer sums at zero for the spectral measure of the corresponding dynamical system [1].

It made it possible, having considered the integral Holder condition for signed measures, to prove a theorem that unifies both following well-known results: classical S.N. Bernstein's theorem on polynomial deviations of the Fejer sums for Holder functions — and theorem about polynomial rates of convergence in von Neumann's ergodic theorem. On the way, a new proof of the Bernstein's theorem was obtained [2].

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Topology of order convergence and properties of solutions to some nonlinear problems of mathematical physics

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We consider in the work some classes of problems of mathematical physics that can be studied using methods of the theory of ordered spaces [1]– [5], in which topology is directly related to the concept of order convergence and the metric properties are determined by positive estimates that have the meaning of the energy of the system. Nonlinear problems of transport theory for integro-differential systems of equations of the theory of radiation transport [6] are considered as specific problems.

For various formulations of nonlinear problems, issues of order and metric stabilization of solutions are investigated. The results presented in this work are a development of the approaches presented in [7]– [10] and essentially use theorems on fixed points of isotone operators acting in complete and conditionally complete lattices [1, 5].

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Asymptotics of dynamic Andronov-Hopf bifurcation

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The main object under consideration is a system of two differential equations with a small parameter $0 < \varepsilon \ll 1$

$$\varepsilon \frac{d\mathbf{x}}{d\tau} = \mathbf{f}(\mathbf{x}; \mathbf{a}); \quad \mathbf{x} \in \mathbf{R^2}, \ \tau \in \mathbf{R^1}.$$

The right-hand side depends on a parameter $\mathbf{a} \in \mathbf{R^n}$. There is an equilibrium state $x \equiv \mathbf{p}(\mathbf{a})$ taken from functional equation

$$\mathbf{f}(\mathbf{x}; \mathbf{a}) = 0.$$

The Andronov-Hopf bifurcation takes place [1] on the surface $S \subset \mathbf{R}^n$. Let be $\mathbf{a} = \mathbf{A}(\tau)\}$ a smooth line, which crosses the bifurcation surface at some moment: $\mathbf{A}(0) \in S$.

Problem for the non autonomous system with parameter depending on the slow time $\mathbf{a} = \mathbf{A}(\tau)$ is considered. We study the solutions, which in the leading order term of the asymptotics in small parameter coincide with zero of the right-hand side: $\mathbf{x} \equiv \mathbf{p}(\mathbf{A}(\tau))$ for $\tau < 0$. An asymptotic solution as $\varepsilon \to 0$ on a time interval $\tau \in (-\delta, \delta)$, including the bifurcation moment $\tau = 0$, is the purpose of the report.

Delay of the loss of stability occurs in the case $\delta = \text{const} > 0$. This problem was investigated by A.Neishtadt in [2]. We consider the case, when sturting point is close to bifurcation moment $\delta = \mathcal{O}(\sqrt{\varepsilon})$. The leading order term of the dynamic bifurcation is described by a solution of non autonomous Bernoulli equation.

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The Influence of Coupling on the Dynamics of Three Delayed Oscillators

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The purpose of this study is to construct the asymptotics of the relaxation modes of a system of differential equations with delay

$$\begin{cases} \dot{u}_j + u_j = \lambda F(u_j(t-T)) + \gamma(u_{j-1} - 2u_j + u_{j+1}), \quad (j = 1, \dots, 3) \\ u_0 \equiv u_3, \quad u_4 \equiv u_1, \end{cases}$$

which simulates three diffusion-coupled oscillators with nonlinear compactly supported delayed feedback F under the assumption that the factor in front of the feedback function λ is large enough. Also, the purpose is to study the influence of the coupling between the oscillators on the nonlocal dynamics of the model.

We construct the asymptotics of solutions of the considered model with initial conditions from a special set. From the asymptotics of the solutions, we obtain an operator of the translation along the trajectories that transforms the set of initial functions into a set of the same type. The main part of this operator is described by a finite-dimensional mapping. The study of its dynamics makes it possible to refine the asymptotics of the solutions of the original model and draw conclusions about its dynamics.

It follows from the form of the constructed mapping that for positive coupling parameters of the original model, starting from a certain moment of time, all three oscillators have the same main part of the asymptotics – the oscillators are "synchronized". At negative values of the coupling parameter, both inhomogeneous relaxation cycles and irregular regimes are possible. The connection of these modes with the modes of the constructed finite-dimensional mapping is described.

From the results of the work it follows that the dynamics of the model under consideration is fundamentally influenced by the value of the coupling parameter between the oscillators.

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Endless process of direct and inverse bifurcations in systems with large delay

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The talk is devoted to phenomenon of endless process of direct and inverse bifurcations in singular perturbed dynamical system with large delay as the delay time tends to infinity.

Consider a differential equation with one ore multiply delays

$$\dot{x} = f(x(t), x(t-T_1), \dots, x(t-T_m)) \quad (x \in R^n),$$

where $T_1 > \dots > T_m > 0$. Let $f(0, 0, \dots, 0) = 0$. Let study the dynamics in neighbourhood of equilibrium state $x = 0$. The main assumption is that considered system is singular perturbed, so let $T_1 \gg 1$. Using time substitution come to equivalent form of initial system

$$\varepsilon\dot{x} = f(x(t), x(t-h_1), \dots, x(t-h_m)), \quad \varepsilon = T_1^{-1} \ll 1, \quad h_i = T_i/T_1. \tag{1}$$

Characteristic equation has form

$$\det(a_0 + a_1 e^{-\lambda} + \ldots + a_m e^{-h_m \lambda} - \varepsilon\lambda I) = 0, \tag{2}$$

where $a_0 = f'_x(0)$, $a_j = f'_{x(t-h_j)}(0)$.

Consider situation when there are no roots of (2) with positive (and separated from zero for small ε) real part and at the same time there are roots which located arbitrary close to imaginary axis for sufficiently small ε. So, system is near point of bifurcation. Consider main part of (2) as equation for real values δ and Ω

$$\det(a_0 + a_1 e^{-i\Omega} + \ldots - i\delta I) = 0.$$

This system should have a root (Ω, δ) (otherwise this is not point of bifurcation). The bifurcation effect of interest to us may appear if $\delta > 0$. In this case system (1) in the neighborhood of $x = 0$ may be reduced to so-called quasi-normal form

$$\frac{\partial u}{\partial \tau} = -c_2 \frac{\partial^2 u}{\partial r^2} + d_1(\theta + \Omega)) i \frac{\partial u}{\partial r} + (d_2(\theta + \Omega)^2 + d_3)u + d_4 u|u|^2, \quad u(\tau, r+1) \equiv u(\tau, r). \tag{3}$$

Here $\theta = \theta(\varepsilon) \in [0, 2\pi)$ such that $\delta\varepsilon^{-1} + \theta$ is proportional to 2π. As $\varepsilon \to 0$ it takes all its values infinite number of times, so solutions (3) and their stability may change infinite number of times. This allows us to conclude about the possibility of an infinite process of direct and inverse bifurcations in (1) when $\varepsilon \to 0$. For example, it can be series of bifurcations of birth and death of a stable cycle as a result of super- and subcritical Andronov-Hopf bifurcations.

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Rovella attractor at the homoclinic butterfly bifurcation with a neutral saddle equilibrium

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Let X_μ be a three-parameter family of $(n+1)$-dimensional smooth vector fields. Suppose that X_μ has a saddle equilibrium O with a single positive eigenvalue γ, while other eigenvalues λ_i, $i = 1, \ldots, n$, have negative real parts and, moreover, one of them, the nearest to 0, is real, i.e.,

$$\gamma > 0 > \lambda_1 > \operatorname{Re}\lambda_i, \quad i = 2, \ldots, n.$$

Also, we assume that for X_0, the following holds:

(A1) one-dimensional separatrices Γ_1 and Γ_2 of the equilibrium O tend to O as $\to +\infty$ along the leading direction touching each other;

(A2) $\rho = \lambda_1 + \gamma = 0$;

(A3) $\operatorname{Re}\lambda_i < 2\lambda_1, \quad i = 2, \ldots, n.$

L. P. Shilnikov was the first to prove that the Lorenz attractor (containing the saddle O with $\rho > 0$) appears (under certain additional assumptions) by perturbation of the vector field X_0.[1] Morales, Pacifico and Martin in [2,3] proved this fact assuming that the separatrix values A_1 and A_2 of the vector filed X_0 satisfies the inequality $\frac{1}{A_1} + \frac{1}{A_2} > 1$. In the case when $\frac{1}{A_1} + \frac{1}{A_2} < 1$, the authors showed that the perturbation of the vector field X_0 may lead to the appearance of the so-called Rovella attractor, i.e., chaotic attractor containing the saddle O with $\rho < 0$. In this talk we present the following generalization of this result.

Theorem. *Let X_μ be a family of vector fields as above such that the unperturbed vector filed X_0 satisfy the assumptions A1-A3. Assume that the separatrix values A_1 and A_2 of the vector filed X_0 satisfy the following condition*

$$A_1, A_2 > 0, \quad \max(A_1, A_2) > 1.$$

Then, in the parameter space, there exists a set D_{RA} with positive Lebesgue measure such that $0 \in D_{LA}$ and X_μ has the Rovella attractor for all $\mu \in D_{LA}$.

As a result we obtain the following trichotomy:

- if $A_1, A_2 > 0$ and $\frac{1}{A_1} + \frac{1}{A_2} \leq 1$, then the Rovella attractor appears by perturbations of the vector field X_0;
- if $A_1, A_2 > 0$, $\frac{1}{A_1} + \frac{1}{A_2} > 1$ and $\max(A_1, A_2) > 1$, the both the Rovella and Lorenz attractors appear by perturbations of the vector field X_0;
- if $0 < A_1, A_2 < 1$, then appears the Lorenz attractor only by perturbations of the vector field X_0.

We also show that the theorem is true for two-parameter family of vector fields possessing the so-called Lorenz-like symmetry. We discuss the application of our result to some systems of differential equations including the Shimizu-Morioka and Lybimov-Zaks models.[4]

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The Symplectic Structure for Renormalization of Circle Diffeomorphisms with Breaks

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In the first part of the talk I'll present the main results of the renormalization theory for circle homeomophisms with one break. We then shall discuss the symplectic structure related to renormalisation of circle maps with many breaks. The invariant symplectic form which we construct is related to the symplectic form introduced by Goldman back in 1984. The talk is based on a recent paper joint with Selim Ghazouani.

Escape velocity and massiveness of the Besicovitch set

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Let $T_\rho\colon \mathbb{T} \to \mathbb{T}$ be an irrational rotation of a circle: $T_\rho x = x + \rho \pmod 1$, and $f\colon \mathbb{T} \to \mathbb{R}$ be a continuous function with a zero mean. We consider a cylindrical transformation $T_{\rho,f}\colon \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ with a cocycle f:

$$T_{\rho,f}(x,y) = (T_\rho x, y + f(x)).$$

The iterations of a point (x,y) are described by

$$T^n_{\rho,f}(x,y) = (T^n_\rho x, y + f^n(x)), \quad n \in \mathbb{Z},$$

where $f^n(x)$ is the n-th Birkhoff sum

$$f^n(x) = \begin{cases} f(x) + f(Tx) + \dots + f(T^{n-1}x) & \text{for } n > 0,\\ 0 & \text{for } n = 0,\\ -f^{|n|}(T^n x) & \text{for } n < 0. \end{cases}$$

A.S. Besicovitch [1] showed that for any irrational circle rotation T_ρ, there exists a continuous f such that $T_{\rho,f}$ is topologically transitive and has orbits tending to infinity, which means the existence of (x,y) such that

$$\lim_{n\to\infty} f^n(x) = \infty.$$

The *Besicovitch set* of points in the circle $\mathbb{T}\times\{0\}$ having discrete orbits, has a null Lebesgue measure, but may have a positive, and even full, Hausdorff dimension [2], [3].

The question regarding the escape velocity of trajectories to infinity was formulated by A. B. Antonevich. This question is related to the spectral properties of the weighted shift operator.

By the ergodicity T_ρ, the velocity of orbit escaping to infinity tends to zero.

It is shown that this tendency can be arbitrarily slow, and the subset of the points of the circle $\mathbb{T}\times\{0\}$ with such escape velocity may have the Hausdorff dimension equal to 1.

Theorem. *For any irrational ρ and any infinitesimal sequence $\sigma_n \in (0,1)$ there exist a function f with a unit norm and a zero mean, and a set $B \in \mathbb{T}$ such that for any $x \in B$ and $n \in \mathbb{N}$*

$$f^{\pm n}(x) > n\sigma_n - 2.$$

In addition,

$$\dim_H(D) = 1.$$

Note that a similar theorem for a single point is contained in [4], which is being prepared for publication.

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Entropy and the boundary distortion growth in synchronized and more general systems

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A connection between the deformation rate of a small set boundary in the phase space of a dynamical system and the metric entropy of the system was claimed (not too rigorously) in physics literature [4], [5] and later studied for discrete time Markov shifts and synchronized systems in the mathematical papers [1], [3]. Further it was established for Anosov diffeomorphisms [2] and suspension flows (continuous time systems).

Let $X = A^{\mathbb{Z}}$ be the space of two-sided infinite sequences over a finite alphabet A, i.e., $X = \{x : \mathbb{Z} \to A\}$, equipped with a usual metric ρ. The shift transformation S on X is defined by $(Sx)_i = x_{i+1}, \quad x = (x_i, \ i \in \mathbb{Z})$.

Let $B(x, \varepsilon)$ denote the open ball of radius ε in X centered at x, and $O_\varepsilon(X_1)$ denote the ε-neighborhood of a set $X_1 \subset X$.

Every closed S-invariant set $Y \subset X$ and the restriction of S to Y together define the symbolic system (Y, S), which is often referred to as a *subshift* of the *full shift* (X, S).

A more general class of symbolic systems than Markov shift (*Subshift of Finite Type*) is formed by synchronized systems. To define them we recall that a word w in the alphabet A is said to be a Y-word if there is a sequence $y = (y_i, y_i \in A) \in Y$ such that $(y_k \dots y_l) = w$ for some $k, l \in \mathbb{Z}$, $k \leq l$.

Definition. A subshift (Y, S) is called *transitive* if for every pair of Y-words u and v there is a Y-word w such that uwv is also a Y-word (here uwv is obtained by successive writing u, v, and w).

A transitive subshift is called *synchronized* (a *synchronized system*) if there exists an Y-word w (*a magic word*) such that if uw and wv are Y-words, then uwv is a Y-word as well.

Consider an S-invariant probability measure μ on X, concentrated on Y. Such a measure can be naturally identified with its restriction to Y. All balls and cylinders we deal with are treated as living in X, but their measures, as well as the measures of all measurable subsets of X, coincide with the measures of their intersections with Y.

In the symbolic systems the following L_1 convergence holds (see [1], [3]):

$$\lim_{\varepsilon\to 0}\frac{1}{t(\varepsilon)}\ln\frac{\mu(O_\varepsilon(S^t(Y\cap B(y,\varepsilon))))}{\mu(S^tB(y,\varepsilon))}=h_\mu(S), \tag{1}$$

where $h_\mu(S)$ is the entropy and $t\in\mathbb{N}$: $\lim\limits_{\varepsilon\to 0}t(\varepsilon)=\infty$, $\lim\limits_{\varepsilon\to 0}t(\varepsilon)/\log\varepsilon=0$.

Despite the assumption $\mu(Y)=1$, the presence of the set Y on the left-hand side of (**??**) is of use. The reason is that the numerator of the fraction in (**??**) is the measure of a set in X, and, generally speaking, there can exist points $x\in S^tB(y,\varepsilon)$ and $y'\in Y$ such that $\rho(x,y')<\varepsilon$, while $\rho(y',Y\cap S^tB(y,\varepsilon))>\varepsilon$. If we delete Y from (**??**), then the measure of such points y' can be positive and will contribute into the numerator, but this would contradict our intention to consider only the dynamics on Y. Note that the described picture is impossible if Y is a Markov set and ε is sufficiently small.

Recently in the paper by Grigory Dvorkin it was proved that (**??**) holds true for any closed S-invariant set Y if there is synchronized subset $Y'\subset Y$ of the full measure. His arguments resembled the complicated proof from [3]. We can establish latter result in a much simpler manner. Our arguments based on the convergence (**??**) for the cases Y' and full shift X.

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Berezin-Toeplitz quantization and spectral analysis of Bochner Laplacians on symplectic manifolds

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The talk is devoted to the theory of Berezin-Toeplitz quantization of compact symplectic manifolds. We describe constructions of Berezin-Toeplitz quantizations, whose quantum spaces are eigenspaces of Bochner Laplacians associated with high tensor powers of a prequantum line bundle on a given manifold. We also discuss generalizations to noncompact manifolds and orbifolds.

On topological classification of Morse-Smale flows in sense of conjugacy

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Two flows $f^t, f'^t \colon M \to M$ on a manifold M are called *topologically equivalent* if there exists a homeomorphism $h \colon M \to M$ sending trajectories of f^t into trajectories of f'^t preserving orientations of the trajectories. Two flows are called *topologically conjugate* if h sends trajectories into trajectories preserving not only directions but in addition the time of moving. To find an invariant showing the class of topological equivalence or topological conjugacy of flows in some class means to get a *topological classification* for one. Note, that for some classes their classifications in sense of equivalence and conjugacy coincide; for other classes these classifications completely differ.

The *Morse-Smale flows* were introduced on the plane for the first time in the classical paper of A.A. Andronov and L.S. Pontryagin in [1]. The non-wandering set of such flows consists of a finite number of hyperbolic fixed points and finite number of hyperbolic limit cycles, besides, saddle separatrices cross-sect only transversally, which means on surfaces that there is no a trajectory connecting saddle points. The most important for us combinatorial invariants for Morse-Smale flows are the *Leontovich-Maier's scheme* [2], [3] for flows on the plane, the *Peixoto's directed graph* [4] for Morse-Smale flows on any closed surface and the *Oshemkov-Sharko's molecule* [5] for Morse-Smale flows on any closed surface.

J. Palis in [6] proved that the class of topological equivalence of a flow can contain any volume of topological conjugacy classes, describing by parameters called *moduli of stability*. For example, a modulus appears when a flow has a separatrix common for two saddle points.

Obviously, any limit cycle generates a modulus equals to the period of one. Additionally, in [7] it was proved that the presence of a region bounded by limit cycles gives infinite number of moduli connected with the uniqueness of invariant foliation in the basin of the limit cycle.

Theorem 1. A Morse-Smale surface flow has finite number of moduli iff it has no a trajectory going from one limit cycle to another.

Second, we use the complete topological classification in sense of equivalence for Morse-Smale surface flows [5], [8] by means of an *equipped graph* $\Upsilon^*_{\phi^t}$ describing dynamics of ϕ^t.

To distinguish topological conjugacy classes we add to the equipped graph an information on the periods of the limit cycles. It gives a new equipped graph $\Upsilon^{**}_{\phi^t}$. In this way we get the following result.

Theorem 2. Morse-Smale surface flows ϕ^t, ϕ'^t without trajectories going from one limit cycle to another one are topologically conjugate iff the equipped graphs $\Upsilon^{**}_{\phi^t}$ and $\Upsilon^{**}_{\phi'^t}$ are isomorphic.

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Invariant Measures for Interval Translation Maps

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We study the so-called interval translation maps of the segment $[0,1]$ without flips [1]. More precisely, let $0 = t_0 < t_1 < \ldots < t_n = 1$, $0 \le t_{k-1} + c_k < t_k + c_k \le 1$.

Let a piecewise continuous map $S : [0,1) \to [0,1)$ be defined by the formula $S(t) = t + c_k$ if $t \in I_k := [t_{k-1}, t_k)$. Then, S is called an oriented interval translation map (OITM).

Theorem 1. Any OITM admits a Borel probability non-atomic invariant measure.

We discuss various generalizations of the above statement.

Open question 1. Given an interval translation map with a non-atomic Borel probability invariant measure, is it metrically conjugated to an interval exchange map with the Lebesgue measure?

It is known that if the ITM without periodic points admits at most n ergodic invariant measures [2].

Open question 2. Does any OITM have at most $[n/2]$ non-atomic ergodic invariant measures, like interval exchange maps do [3]?

Also, we discuss, how the set of invariant measures depends on parameters.

Theorem 2 (joint with Ilya Kassikhin). There is a residual set Ξ at the space of admissibe parameters such that any point of Ξ is a continuity point for the set of invariant measures.

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Guiding functional families and the existence of Poisson bounded solutions [1]

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We consider an arbitrary system of differential equations of n variables

$$\frac{dx}{dt} = F(t,x), \quad F(t,x) = (F_1(t,x), \ldots, F_n(t,x))^T, \tag{1}$$

where the functions of right-hand side are defined and continuous in $\mathbb{R}^+ \times \mathbb{R}^n$, where $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$. It is assumed that the function $F(t,x)$ satisfied Lipschitz condition.

Each increasing numerical sequence $\tau = (\tau_i)$, where $\tau_i \geqslant 0$, $i \geqslant 1$, $\lim\limits_{i\to\infty} \tau_i = +\infty$ we will call $\mathcal{P}$-sequence. For each $\mathcal{P}$-sequence $\tau = (\tau_i)$ let $M(\tau)$ we denote the set $\bigcup\limits_{i=1}^{\infty} [\tau_{2i-1}; \tau_{2i}]$.

The solution of system (1) is said to be Poisson bounded [1], if for this solution there exist $\mathcal{P}$-sequence and number $\beta > 0$ such that the condition $\|x(t,t_0,x_0)\| \leqslant \beta$ for all $t \in R^+(t_0) \bigcap M(\tau)$ holds, where $\mathbb{R}^+(t_0) = \{t \in \mathbb{R} \mid t \geqslant t_0\}$.

Continuously differentiable function $G : \mathbb{R}^n \to \mathbb{R}$ is called [2] guiding function or, more precisely, the r_0-guiding function for the system (1), if the following condition is met: $(\mathrm{grad}G(x), F(t,x)) > 0$, $t \geqslant 0$, $\|x\| \geqslant r_0$.

Each r_0-guiding function G for the system (1) defines vector field $\mathrm{grad}G : B^n_{r_0} \to \mathbb{R}^n$, where $B^n_{r_0} = \{x \in \mathbb{R}^n \mid \|x\| \leqslant r_0\}$. This vector field defines a continuous mapping

$$T : S^{n-1}_{r_0} \to S^{n-1}_{r_0}, \quad T(x) = r_0 \cdot \frac{\mathrm{grad}G(x)}{\|\mathrm{grad}G(x)\|}, \quad x \in S^{n-1}_{r_0} = \partial B^n_{r_0}.$$

Rotation $\gamma(\mathrm{grad}G, S^{n-1}_{r_0})$ of the vector field $\mathrm{grad}G : B^n_{r_0} \to \mathbb{R}^n$ is the degree $\deg(T) \in \mathbb{Z}$ of mapping $T : S^{n-1}_{r_0} \to S^{n-1}_{r_0}$.

The index of the r_0-non-degenerate function G for the system (1) is an integer $\mathrm{ind}(G)$, defined by $\mathrm{ind}(G) = \gamma(\mathrm{grad}G, S^{n-1}_{r_0})$.

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Further, each set of r_0-guiding functions $G_0, G_1, \ldots, G_q$ for (1), where $q \geqslant 1$ we will call r_0-guiding functional family for system (1). It should be noted that, unlike the notion of a complete set of r_0-guiding functions [3], there is not required to satisfy the condition $\lim_{\|x\|\to+\infty}(|G_0(x)| + \cdots + |G_q(x)|) = +\infty$.

The following statement is a sufficient condition for the existence of Poisson-bounded solutions for the system (1) in terms of Lyapunov vector functions and guiding function families.

Theorem. Let for the system (1) there exist $\mathcal{P}$-sequence $\tau = (\tau_i)$, non-increasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$, $b(r) \to +\infty$ for $r \to +\infty$, and the vector Lyapunov function $V(t, x)$ with the comparison system $\dot{\xi} = f(t, \xi)$, where $f(t, \xi)$ satisfies the local Lipschitz condition by $\xi \in \mathbb{R}^k$ and the Vazhevsky condition, that for any $(t, x) \in M(\tau) \times \mathbb{R}^n$ the inequality $b(\|x\|) \leqslant \sum\limits_{i=1}^{k} V_i(t, x)$ holds. In addition, let there exist numbers $r_1 > r_0$ and a r_0-guiding functional family $G_0, G_1, \ldots, G_q$ for the system

$$\frac{d\varrho}{dt} = g(t, \varrho), \quad (t, \varrho) \in \mathbb{R}^+ \times \mathbb{R}^k, \quad g(t, \varrho) = f(t, \varrho + \overline{r}_1), \quad \overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^k,$$

where $f(t, \xi)$ is the right-hand side of the comparison system for the system (1), which satisfy the following conditions:

1) $\mathrm{ind}(G_0) \neq 0$.

2) $\sum\limits_{i=0}^{q} |G_i(\varrho)| > \sum\limits_{i=0}^{q} (|m_i| + |M_i|)$ for all $\varrho \in \mathbb{R}^k$, $\|\varrho\| = r_1$, where

$$m_i = \min_{\|\varrho\| \leqslant r_0} G_i(\varrho), \quad M_i = \max_{\|\varrho\| \leqslant r_0} G_i(\varrho), \quad 0 \leqslant i \leqslant q.$$

3) $B^k_{r_1}(\overline{r}_1) = \{\xi \in \mathbb{R}^k \mid \|\xi - \overline{r}_1\| \leqslant r_1\} \subset \mathrm{Im}\,(V : \{0\} \times \mathbb{R}^n \to \mathbb{R}^k)$.

Then the system (1) has at least one Poisson bounded solution. ■

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Partially Hyperbolic Symplectic Automorphisms of 6-Torus

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We study topological properties of automorphisms of 6-dimensional torus generated by an integer matrices being symplectic either with respect to the standard symplectic structure in $\mathbf{R^6}$ or w.r.t. a nonstandard symplectic structure generated by an integer non-degenerate skew-symmetric unimodular matrix. Such symplectic matrix generates a partially hyperbolic automorphism of the torus, if its eigenvalues are either a pair of real numbers outside the unit circle and a two pairs of conjugate complex numbers on the unit circle or two pairs of real numbers outside the unit circle and a pair of complex conjugate numbers on the unit circle. The classification is defined by the topology of a foliation generated by unstable (stable) leaves of the automorphism and its action on the center manifold.

For symplectic automorphisms on T^6 we have symplectic partially hyperbolic integer matrices with

- $dim\, W^c = 4, \quad dim\, W^s,\ W^u = 1;$
- $dim\, W^c = 2, \quad dim\, W^s,\ W^u = 2.$

So, we have more possibilities in comparison with the case of a four-dimensional torus studied in [1]: the problem becomes more complicated since dimensions of stable/unstable foliations and a center sub-manifold and can vary even for the fixed dimension of the torus.

In the first case we study the one-dimensional foliations on the torus generated by the projection on the torus of unstable and stable eigen-lines. We prove that the related foliation on the torus T^6 can be either transitive or decomposable into 4-tori or decomposable into 2-tori. Each case requires a special investigation for its classification. Automorphisms realizing all possible cases are provided. In the second case stable and unstable leaves are two-dimensional. We prove that such foliation can be either transitive (each leaf of the foliation is dense in T^6) or the foliation is decomposable: the closure of any leaf is a four-dimensional torus, the union of these 4-tori form smooth invariant foliation of the 6-torus in the sense that any 4-torus is transformed by the automorphism onto another (or the same) such 4-torus. The orbit behavior of automorphisms on their center sub-manifold is also studied.

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Persistence of Heterodimensional Cycles

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A heterodimensional cycle is an invariant set of a dynamical system consisting of two hyperbolic periodic orbits with different dimensions of their unstable manifolds and a pair of orbits that connect them. For systems which are at least C^2, we show that bifurcations of a coindex-1 heterodimensional cycle within a generic 2-parameter family always create robust heterodimensional dynamics, i.e., chain-transitive sets which contain coexisting orbits with different numbers of positive Lyapunov exponents and persist for an open set of parameter values. In particular, we solve the so-called C^r-stabilization problem for the coindex-1 heterodimensional cycles in any regularity class $r = 2, \dots, \infty, \omega$. The results are based on the observation that arithmetic properties of moduli of topological conjugacy of systems with heterodimensional cycles determine the emergence of Bonatti-Diaz blenders.

Realization of combinatorial symmetries of smooth functions by symplectomorphisms

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Let M be a compact connected surface and P is a connected one-dimensional manifold without boundary, i.e. either the real line R^1 or the circle S^1. Denote by $\mathcal{D}(M)$ the group of all smooth (C^∞) diffeomorphisms of M. This group acts from the right on the space $C^\infty(M, P)$ by the following rule: if $h \in \mathcal{D}(M)$ and $f \in C^\infty(M, P)$, then the result of the action of h on f is the composition map $f \circ h : M \to P$. For $f \in C^\infty(M, P)$ let Σ_f be the set of its critical points, and

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\},$$

$$\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$$

be respectively the *stabilizer* and the *orbit* of f under that action. Endow these spaces with C^∞ topologies and denote by $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{S}_{\mathrm{id}}(f)$ the corresponding path components of id_M in $\mathcal{D}(M)$ and $\mathcal{S}(f)$, and by $\mathcal{O}_f(f)$ the path component of $\mathcal{O}(f)$ containing f. We will omit X from notation whenever it is empty.

Notice that $\mathcal{S}_{\mathrm{id}}(f)$ is a normal subgroup of $\mathcal{S}(f)$, and the quotient:

$$\pi_0\mathcal{S}(f) := \mathcal{S}(f)/\mathcal{S}_{\mathrm{id}}(f)$$

is the group of path components of $\mathcal{S}(f)$. This group is an analogue of mapping class group for f-preserving diffeomorphisms.

Let $\mathcal{F}(M, P)$ be a subset of $C^\infty(M, P)$ consisting of maps satisfying the following two axioms:

(B) The map f takes a constant value at each connected component of ∂M and has no critical points in ∂M;

(L) For every critical point z of f, there are local coordinates in which f is a homogeneous polynomial $R^2 \to R$ of degree ≥ 2 without multiple factors.

Evidently, $\mathcal{F}(M, P)$ contains all Morse maps.

For $f \in \mathcal{F}(M, P)$ the homotopy types of $\mathcal{S}_{\mathrm{id}}(f)$ and orbits were described by S. Maksymenko, and the homotopy types of connected components of orbit $\mathcal{O}(f)$ by S. Maksymenko, E. Kudryavtseva (for Morse maps and for smooth functions $f : M \to R$ with *simple singularities* which are not homogeneous but quasi-homogeneous), B. Feshchenko, I. Kuznietsova, Yu. Soroka, A. Kravchenko.

Theorem 1. *Let $f \in \mathcal{F}(M,P)$. Then the natural map $p : \mathcal{S}(f) \to \pi_0\mathcal{S}(f)$ has a section:*

$$s : \pi_0\mathcal{S}(f) \to \mathcal{S}(f),$$

so s is a homomorphism such that $p \circ s = \mathrm{id}_{\pi_0\mathcal{S}_{\mathrm{id}}(f)}$.

Moreover, if M is orientable, then there exists a symplectic structure, i.e. a non-degenerate differential 2-form ω, on M, such that the image $s\big(\pi_0\mathcal{S}(f)\big)$ consists of symplectic diffeomorphisms with respect to ω.

Local dynamics of a second-order equation with a delay at the derivative

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Consider a second-order differential equation with delayed feedback, which is an implementation of the modified Ikeda equation with a time delay:

$$\varepsilon\frac{d^2y}{dt^2} + \frac{dy}{dt} + \delta y = F\left(\frac{dy}{dt}(t-\tau)\right). \tag{1}$$

Here ε and δ are small and proportional parameters $0 < \varepsilon \ll 1,\quad \delta = k\varepsilon$, τ is a delay parameter, real and positive. The function F is sufficiently smooth, without limiting generality, we can assume that $F(0) = 0$. Thus, equation (1) has a zero equilibrium state. The problem under consideration is singularly perturbed.

The characteristic quasi-polynomial of the linearized at zero equation (1) has the form:

$$\varepsilon\lambda^2 + \lambda + k\varepsilon = \lambda\beta_1 e^{-\lambda}. \tag{2}$$

It is shown that for $|\beta_1| < 1$, the zero equilibrium state is stable, and for $|\beta_1| > 1$, it is unstable. In critical cases $\beta_1 = \pm1$, the characteristic equation has an infinite number of roots tending to the imaginary axis at $\varepsilon \to 0$. Thus, the critical cases have infinite dimension.

To study the behavior of solutions in the case of a close $\beta_1 = -1$, the solution of problem (1) is reduced in the case under consideration to a partial differential equation (3) with boundary conditions (4):

$$\frac{\partial V}{\partial\tau} = \frac{1}{2}\frac{\partial^2 V}{\partial^2 t} + kV - \frac{k^2}{2}J^2(V) + \beta V + \beta_2 J\left(U_1\frac{\partial V}{\partial t}\right) + \beta_3 J\left(\left(\frac{\partial V}{\partial t}\right)^3\right), \tag{3}$$

$$\int_0^1 V(\tau,t)dt = 0,\ V(\tau,t) \equiv -V(\tau,t+1). \tag{4}$$

To study the behavior of solutions in the case of a close $\beta_1 = 1$, the solution of the problem (1) is reduced in the case under consideration to the differential equation (5) and the partial differential equation (6) with boundary conditions (7):

$$\begin{aligned}
W^\tau(\tau) &= (\varepsilon\beta - \frac{k}{4})W(\tau) - \frac{2}{3}\beta_2^2\sqrt{k}W^2(\tau)\overline{W}(\tau) + \frac{3}{2}\beta_3\varepsilon^{\frac{1}{2}}kW^2(\tau)\overline{W}(\tau),\\
\overline{W}^\tau(\tau) &= (\varepsilon\beta - \frac{k}{4})\overline{W}(\tau) + \frac{2}{3}\beta_2^2\sqrt{k}W(\tau)\overline{W}^2(\tau) - \frac{3}{2}\beta_3\varepsilon^{\frac{1}{2}}kW(\tau)\overline{W}^2(\tau),
\end{aligned} \tag{5}$$

for $\varepsilon\beta < \dfrac{k}{4}$, the solution of equation (5) tends to zero.

$$\frac{\partial V}{\partial \eta} = \frac{1}{2}\frac{\partial^2 V}{\partial^2 t} + kV - \frac{k^2}{2}J^2(V) + \beta V + \beta_3 J\left(\left(\frac{\partial V}{\partial t}\right)^3\right), \tag{6}$$

$$\int\limits_0^1 V(\eta,t)dt = 0,\ V(\eta,t) \equiv V(\eta,t+1). \tag{7}$$

$J(V)$ denotes the primitive function V with a zero mean:

$$J^2(V) = J(J(V)),\ (J(V))'_t \equiv V.$$

Quasinormal forms are constructed — special nonlinear parabolic equations that do not contain small parameters, the solutions of which give the main part of the solutions of equation (1) that are asymptotic with respect to the residual uniformly over $t \geq 0$.

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Total absolute curvature of real irreducible representations of compact Lie groups and upper bounds for the Chern integral

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Recall that Chern-Lashof [1] proved the following formula for the absolute total curvature $\tau(M, f, \mathbb{R}^N)$ of an immersion $f : M \to \mathbb{R}^N$ of closed oriented manifold M into the Euclidean space $\mathbb{R}^N$:

$$\tau(M, f, R^N) = \frac{1}{\sigma_{N-1}}\int_M \left(\int_{S_m} |\det A_\xi| d\sigma\right) dm,$$

where S_m is the unit sphere in the normal space immersion at $m \in M$, A_ξ shape operator of immersion corresponding normal vector ξ and σ_{N-1} – volume of the unit sphere in space $\mathbb{R}^N$. According to Kuiper [2], $\tau(M, f, \mathbb{R}^N)$ is equal to the mean value of the number of critical points of Morse functions of height $h_l(m) =< l, f(m) >$ of the immersion f. Averaging is over invariant measure on the unit sphere $< l, l >= 1$ euclidean space $\mathbb{R}^N$. The integral on the right-hand side of the above equality for $\tau(M, f, \mathbb{R}^N)$ is often called the Chern-Lashof integral (see [1], [2]).

Our goal consists in application above approach in a situation where as isometric immersion taken nontrivial real irreducible linear representation ρ connected compact simple group G. The space of height functions here is the space of matrix elements of the representation. Due to the homogeneity of the group manifold G and the bi-invariance of the Riemannian metric on it be true following

Proposition. *Under the conditions indicated above, for a nontrivial irreducible real linear representation ρ of a group G in the space V, the absolute total curvature for $\tau(G, \rho, \operatorname{End} V)$ is equal to*

$$\tau(G, \rho, \operatorname{End} V) = \frac{\operatorname{vol} G}{\sigma_{N-1}}\int_S |\det A_\xi| d\sigma,$$

where S is the unit sphere in the orthogonal complement $d\rho(\mathfrak{G})^{\perp}$ to the image of the Lie algebra G in the N-dimensional space $\operatorname{End} V$ *and* $\operatorname{vol}(G)$ *is the volume of the group with respect to the bi-invariant metric.*

Due to bi-invariant metric on the group G with the previously given formula for the operator A_{ξ} and triviality of the normal bundle ρ we have the relation $\det A_{\xi} = \det A_{\rho(g)\xi}$ for any normal vector $\xi \in d\rho(\mathfrak{G})^{\perp}$ and all $g \in G$. Hence, by virtue of the invariance of the Riemannian Haar measure on G, the above equality for the Chern-Lashof integral follows. Now let us state our main result related to an upper bound for this integral.

Theorem. *The values of the Chern-Lashoff integral of real irreducible linear representations $\rho : G \to GL(V)$ of a connected compact simple n-dimensional Lie group G satisfy the following inequalities:*

$$\tau(G, \rho, \operatorname{End} V) \leq \frac{\sigma_{N-n-1}}{\sigma_{N-1}} (\lambda/n)^{1/2} \operatorname{vol}(G),$$

where σ_k is the volume of the unit k-dimensional Euclidean sphere, λ be eigenvalue of the Casimir operator representation ρ, and $\operatorname{vol}(G)$ *is the volume of n-dimensional group G taken with respect to the Riemannian metric induced by the immersion $\rho : G \to GL(V) \subset \operatorname{End}(V)$ into the Euclidean space* $\operatorname{End}(V)$ *of dimension N equipped with an inner product (A, B) proportional to* $\operatorname{Tr}(AB^*)$.

Formulation of the above theorem complements discussed earlier in [3] estimates from below integrals Chern-Lashof irreducible real representations.

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Open problems related to nilpotent approximation of geometric distributions

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Nilpotent approximations had gradually paved their way into geometric control theory and nonholonomic analysis, via works of Hermes, Gamkrelidze, Agrachev, Stefani, Sussmann, Bellaiche – to name but a few. They feature, as a rule, simpler local geometries, retaining some basic properties of the germ of a nonholonomic distribution. For instance the small flag at the reference point. Most notably – the small growth vector at the reference point. We will 'advertise' four particular problems related to the nilpotent approximation, all of them open at present.

(1) After the discovery, in the years 1997-98, of numerical moduli in the world of Goursat distributions (or, the same thing, 1-flags), in 2000 Agrachev asked a natural question whether

those moduli survived passing to the level of nilpotent approximations. (Recalling, general finite-dimensional nilpotent algebras very soon – when their dimensions grow – feature numerical moduli.) To-date only disparate partial negative results are known. There emerges a feeling that generally the answer could be NO.

(2) Numerical moduli are also aplenty among so-called special multi-flags (natural generalizations of Gursat flags). However, extending Agrachev's question to them seems simply premature, because very few concrete examples of moduli in the local geometry of special multi-flags are known to-date.

(3) In the same year 2000 there has emerged the notion of *strong nilpotency* of a distribution germ. That is, the equivalence of the nilpotent approximation and the initial distribution germ. As contrasted to *weak nilpotency* = previous local nilpotency of a distribution in the sense of Sussmann. (The existence of local nilpotent basis of sections of a distribution.) All Goursat distributions are locally nilpotentizable in the Sussmann's sense – that is, are weakly nilpotent. Yet only very few among them appear to be strongly nilpotent. A challenging and hot issue is to find them all in the world of Goursat distributions.

(4) Similar to (3), but for special multi-flags, which are also everywhere locally nilpotentizable, or weakly nilpotent. Strong nilpotency among them is even rarer than among the 1-flags. The issue is difficult even after specifying 'multi' = 2. Which germs of special 2-flags are strongly nilpotent?

On the passage of an invariant torus through a degenerate resonance zone in a nearly Hamiltonian system under quasiperiodic perturbations

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Quasiperiodic perturbations of two–dimensional nearly Hamiltonian systems with a limit cycle are considered. Solutions behavior in a small neighborhood of a degenerate resonance phase curve is studied. Special attention is paid to the synchronization problem. Bifurcations of quasiperiodic solutions that arise when the limit cycle passes through the neighborhood of the resonance trajectory are investigated. The study is based on an analysis of an autonomous pendulum–type system, which is obtained by the method of averaging and determines the dynamics in the resonance zone. Two possible topological structures of the unperturbed averaged system are distinguished. For each case, the intervals of a control parameter that correspond to oscillatory synchronization are found. The results are applied to a Duffing–Van der Pol–type equation.

Averaging and passage through resonances in two-frequency systems near separatrices

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We obtain realistic estimates for the accuracy of averaging method for general time-periodic perturbations of one-frequency Hamiltonian systems with separatrix crossing:

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} + \varepsilon f_q(p, q, z, t, \varepsilon), \\
\dot{p} &= -\frac{\partial H}{\partial q} + \varepsilon f_p(p, q, z, t, \varepsilon), \\
\dot{z} &= \varepsilon f_z(p, q, z, t, \varepsilon), \\
H &= H(p, q, z), \ (p, q) \in \mathbb{R}^2, z \in \mathbb{R}^n.
\end{aligned}$$

Functions f_q, f_p, f_z are 2π-periodic in time t. Small parameter ε characterises the strength of the perturbation.

For $\varepsilon = 0$ we get an unperturbed system. We assume that for all values of z on the phase plane of this system there are domains of closed trajectories separated by separatrices. Phase angle on the closed phase trajectories of the unperturbed system and time are two angle variables, and resonances between their frequencies are possible.

The perturbation produces an evolution which may force phase points to cross separatrices. To approximately describe the dependence of H and z on time the averaging method prescribes to average the rates of changes of these variables over two angle variables in the problem.

Such systems were studied far from separatices [1, 2] (see also review [3] and references therein). Accuracy of averaging method $O(\sqrt{\varepsilon}|\ln \varepsilon|)$ was obtained for majority of conditions over times $\sim \varepsilon^{-1}$. The exceptional set has measure $O(\sqrt{\varepsilon})$, it exists due to the fact that some trajectories can be captured into a resonance, remaining near the resonance for time $\sim \varepsilon^{-1}$. We show that the same estimate for the accuracy of averaging method holds near separatrices, but our estimate on the measure of the exceptional set is a bit worse: $O(\sqrt{\varepsilon}|\ln^5 \varepsilon|)$. The accuracy of averaging method near separatrices was earlier estimated for one-frequency systems (i.e., when the perturbation does not depend on the time, [4] and references therein).

To prove this result, we split the phase space into resonant zones near resonances, non-resonant zones between resonant zones, and a small neighborhood of separatrices with width $\sim \varepsilon|\ln^5 \varepsilon|$. Let us note that far from separatrices the number of resonances such that capture is possible is finite, but these resonances may accumulate on separatrices. In non-resonant zones we use the standard coordinate change used to justify the averaging method, taking into account that many functions describing this coordinate change are unbounded near the separatrices. In the resonant zones we average over time and obtain auxiliary system describing passage through resonances. Far from separatrices a single step of averaging method is enough, but near separatrices single step gives insufficient accuracy and we have to use many steps similarly to [5]. Finally, the neighborhood of separatrices is covered by a general argument (first used in [6]) based on the fact that the measure of this neighborhood is very small, without analyzing the dynamics in this zone.

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Stability analysis of apsidal alignment in double averaged restricted elliptic three body problem

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We are dealing with the averaged model used to study the secular effects in the motion of a body of the negligible mass in the context of a spatial restricted elliptic three-body problem. It admits a two-parameter family of equilibria (stationary solutions) corresponding to the motion of the third body in the plane of primaries' motion, so that the apse line of the orbit of this body is aligned with the apse lines of the primaries' orbits. The aim of our investigation is to analyze the stability of these equilibria. We start by proving their stability in the linear approximation. Then Arnold-Moser stability theorem is applied to obtain the sufficient conditions under which the stability in a nonlinear sense takes place. As it turned out, these conditions are satisfied for all parameters of the problem, with the exception of parameters from some finite set of analytic curves in space of parameters. These exceptional values of parameters correspond to resonances 1:1 and 2:1 between frequencies of oscillations of the apse line in the plane of primaries' motion and across this plane and to a degeneration of 4th order Birkhoff normal form of the problem's Hamiltonian. We have shown that in the case of 2:1 resonance apsidal alignment is unstable. In other cases, the violation of the conditions of Arnold-Moser theorem does not lead to instability.

Our results hopefully will be useful for studying the dynamics of exoplanetary systems Many exoplanets move in orbits with large eccentricities and inclinations. Therefore, an understanding of possible mechanisms of instability of planar orbits is important in this context.

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Topology of Integrable Systems with Separating Variables

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One of the main characteristics of a finite-dimensional integrable Hamiltonian system is its *Liouville fibration.* This is a singular fibration whose fibers are connected common level sets of the first integrals of the system. In the case of two degrees of freedom, A. T. Fomenko and his colleagues suggested [1] a set of invariants describing the topology of the Liouville fibration. One the most important invariant is a *marked molecule* (or *Fomenko–Zieschang invariant*) describing the topology of the Liouville fibration on a regular invariant 3-dimensional submanifold. This invariant has the structure of a graph whose edges correspond to one-parameter families of regular fibers whereas vertices correspond to singularities of the fibration, so-called *atoms.*

The calculation of the Fomenko–Zieschang invariant for an arbitrary integrable system is not an algorithmic procedure and often turns out to be quite a non-trivial task. However, in some cases it can be substantially simplified. One of such cases is a class of *algebraically integrable systems.* These systems admit the separation of variables in the sense that the Hamiltonian equations on each fiber can be written in the form

$$\dot{u}_i = \sqrt{P(u_i)} \quad \text{or} \quad \dot{u}_i = \frac{\sqrt{P(u_i)}}{u_1 - u_2}, \quad i = 1, 2 \quad (P \text{ is a polynomial}).$$

Moreover, all the phase coordinates are expressed as rational functions on the radicals of the form $\sqrt{u_i - \alpha_j}$ where u_1, u_2 are the variables of separation, $\alpha_j \in \mathbb{R}$ (the constants α_j and the polynomial P depend on the values of the first integrals).

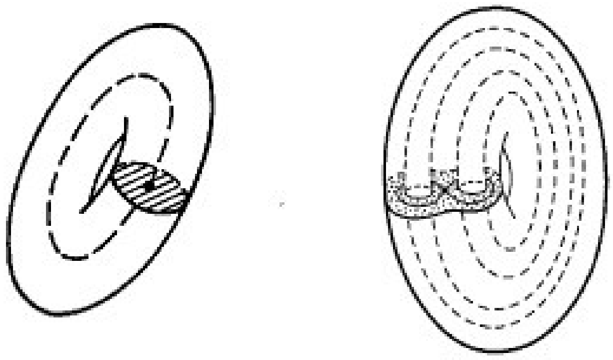

Figure 1: Atoms A and B

The first systematic approach to the study of the topology of algebraically integrable systems was made by M. P. Kharlamov [2]. He noticed that the expressions of the phase coordinates via radicals as mentioned above permits to understand the topological properties of the projection of each fiber onto the plane $\mathbb{R}^2(u_1, u_2)$. We follow this idea and suggest (under some natural conditions) an algorithm for calculating the Fomenko–Zieschang invariant for algebraically integrable systems. As a corollary, we can list the types of all bifurcations that occur in such systems.

Theorem. Any elementary (i.e. corresponding to the coincidence of a unique pair of constants α_k, α_l) compact non-degenerate 3-dimensional singularity of an algebraically integrable system has the type of one of the following atoms:
$A, B, C_2, P_4, D_1, A^*, A^{**}$.

Remark. The exact description of the atoms (=bifurcation types) mentioned in the theorem may be found in [1]. For instance, the atom A denotes the disappearance of a 2D-torus via the critical circle (Fig. 1, left), the atom B stands for the transformation of a torus into two tori via the direct product of the figure "eight" and the cirle (Fig. 1, right).

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On non-compact bifurcation in one integrable model of vortex dynamics

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Since the time of Helmholtz [1], the model of N point vortices in an ideal fluid with constant intensities Γ_α ($\alpha = 1 \dots N$) is well known. The model of N interacting *magnetic vortices* in ferromagnets [2] is a more general case than the hydrodynamic model: in addition to vorticities Γ_α, there are polarities λ_α which take values ± 1 depending on magnetization directed up or down.

In this talk, we consider a *restricted problem* of three magnetic vortices, where in the system of three magnetic vortices at positions $r_\alpha = (x_\alpha, y_\alpha)$, $\alpha \in \{0, 1, 2\}$, the vortex with vorticity Γ_0 is fixed at point $\mathcal{O}(0, 0)$. Equations of motion in our generalized model have the following complex form:

$$i\lambda_\alpha \dot{z}_\alpha = \frac{1}{\bar{z}_\alpha} + \frac{\Gamma_\beta}{\lambda_\beta} \frac{1}{\bar{z}_\alpha - \bar{z}_\beta}, \quad \alpha \neq \beta \in \{1, 2\},$$

where $z_\alpha = x_\alpha + i y_\alpha$ is a complex coordinate specifying the position of vortex with vorticity Γ_α. The system can be written in Hamiltonian form:

$$\begin{gathered} \Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \qquad \alpha = 1, 2, \\ H = \frac{\Gamma_1}{\lambda_1} \ln \ell_1 + \frac{\Gamma_2}{\lambda_2} \ln \ell_2 + \frac{\Gamma_1}{\lambda_1} \frac{\Gamma_2}{\lambda_2} \ln \ell_{12}, \end{gathered} \tag{1}$$

where $\ell_\alpha = |r_\alpha|$ and $\ell_{\alpha\beta} = |r_\alpha - r_\beta|$. In addition, this system has an integral of the *angular momentum of vorticity* $F = \Gamma_1 \ell_1^2 + \Gamma_2 \ell_2^2$, so it is completely Liouville integrable with two degrees of freedom.

The main role in the study of such dynamical systems is played by the *bifurcation diagram* Σ of the momentum map $\mathcal{F}(\boldsymbol{z}) = (F(\boldsymbol{z}), H(\boldsymbol{z}))$. In the restricted system of three magnetic vortices in the case of a «vortex pair» (encountered in hydrodynamics [3]), a bifurcation diagram contains *non-compact bifurcations* of the $(\mathbb{T}^2 + \mathrm{Cyl}) \to \mathrm{Cyl}$ type (Fig. 1). Here $\mathbb{T}^2$

denotes the presence of a two-dimensional Liouville torus, and Cyl denotes a two-dimensional cylinder.

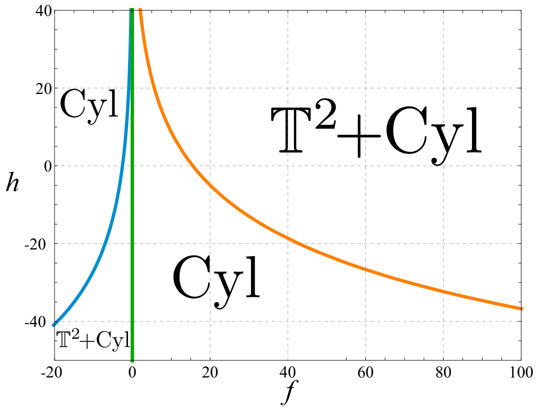

Figure 1: Bifurcation diagram Σ for the «vortex-antivortex» case.

For the system (1) an explicit reduction to a Hamiltonian system with one degree of freedom was performed. Fig. 2 shows typical level lines of the reduced Hamiltonian for the system with non-compact bifurcations.

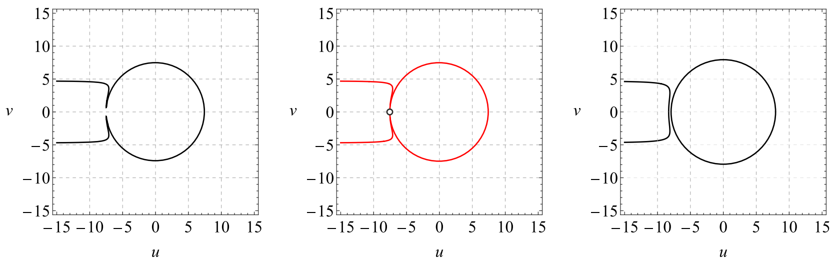

Figure 2: Hamiltonian level lines for the Cyl $\to \left(\text{Cyl} + \mathbb{T}^2\right)$ bifurcation.

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Novikov's problem and tiling billiards

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In the beginning of the 80s, Masur and Veech proved (independently) that a generic interval exchange transformation is uniquely ergodic. At about the same time, on different sides of the iron curtain, Soviet and French mathematicians got interested in the ergodic properties of measured foliations restricted in some parametric classes. Their motivations and constructions were different but the same fractal objet appeared in both. It is called today the Rauzy gasket.

The motivation from the Soviet side came from the question asked by Sergei Novikov related to the conductivity theory of monocrystals. His question concerns minimal foliations defined by plane sections of surfaces inside the 3-torus. The Rauzy gasket brings an answer to Novikov's question for a surface of genus 3 with central symmetry and 2 double saddles.

Recently, we have obtained a solution of Novikov's problem for any centrally symetric surface of genus 3. This is the most interesting case from the point of view of physical motivations. I will tell some aspects of this story which is based on a series of works, the last and most important one of them is a collaboration with Ivan Dynnikov, Pascal Hubert, Paul Mercat and Alexandra Skripchenko.

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Stable manifolds for fractional semilinear equations

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This talk is devoted to the numerical analysis of the abstract semilinear fractional problem

$$D^{\alpha}u(t) = Au(t) + f(u(t)), u(0) = u^0,$$

in a Banach space E [1,2]. We are developing a general approach to establish the existence of stable manifold for fractional equation and then prove the semidiscrete approximation theorem of stable manifolds . The phase space in the neighborhood of the hyperbolic equilibrium can be split in such a way that the original initial value problem is reduced to systems of initial value problems in the invariant subspaces corresponding to positive and negative real parts of the spectrum [3]. We show that such a decomposition of the equation keeps the same structure under general approximation schemes. The main assumption of our results are naturally satisfied, in particular, for operators with compact resolvents and can be verified for finite element as well as finite difference methods.

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On Structural Stability of Axiom A 3-diffeomorphisms with dynamics "one-dimensional surfaced attractor-repeller"

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Let f be a diffeomorphism of a smooth closed n-manifold M^n. One says that f *satisfies axiom A* if its non-wandering set is hyperbolic and the periodic points are dense in it. We say in this case that f is an A-diffeomorphism. For A-diffeomorphisms, the Smale spectral decomposition theorem [7] holds, that is non-wandering set is a union of finite number of pairwise disjoint sets called *basic sets*, each of which is compact, invariant and topologically transitive.

A basic set Λ is called an *attractor* of an A-diffeomorphism f if it has a compact neighborhood U_Λ such that $f(U_\Lambda) \subset int U_\Lambda$ and $\Lambda = \bigcap\limits_{k\geqslant 0} f^k(U_\Lambda)$. U_Λ is called a *trapping neighborhood* of Λ. A *repeller* is defined as the attractor for f^{-1}. By a *dimension* of the attractor (repeller) we mean its topological dimension. The set $\bigcup\limits_{k\in\mathbb{Z}} f^k(U_\Lambda)$ is called a *basin* of the attractor Λ.

By Theorem 3 in [4], every one-dimensional basic set of an A-diffeomorphism on a surface is an attractor or repeller, and $\dim W^u_x = \dim W^s_x = 1$, for $x \in \Lambda$. The trapping neighborhood for such basic set is a surface with a boundary. Therefore it can be naturally included to a three-dimensional dynamics.

A connected one-dimensional attractor A of A-diffeomorphism $f : M^3 \to M^3$ is called a *canonically embedded surface attractor* if:

- A has a trapping neighborhood U_A of the form $V_A \times [-1, 1]$, where $V_A = V_A \times \{0\}$ is a surface with a boundary of a positive genus and $A \subset int\, V_A$;
- V_A is a trapping neighborhood of A as an attractor for the diffeomorphism $\psi_A = f|_{V_A} : V_A \to f(V_A)$;
- diffeomorphism $f|_{U_A} : U_A \to f(U_A)$ is topologically conjugate to the diffeomorphism $f_A(w, z) = (\psi_A(w), z/2) : V_A \times [-1, 1] \to f(V_A) \times [-1/2, 1/2]$.

A one-dimensional repeller is called a *canonically embedded surface repeller* if it is a canonically embedded connected one-dimensional surface attractor for the diffeomorphism f^{-1}.

Denote by G class of 3-diffeomorphisms whose non-wandering sets are union of canonically embedded one-dimensional surface attractor and repeller. Infinitely many pairwise Ω-non-conjugated diffeomorphisms from G were constructed in [2]. Moreover, there a hypothesis was formulated that diffeomorphisms in class G are not structurally stable. Note that the similar "one-dimensional attractor-repeller" dynamics on the surface always is not structurally stable due to results by R. Robinson and R. Williams [5]. The construction of 3-diffeomorphisms with one-dimensional attractor-repeller (not surface) dynamics firstly was suggested in [3], all examples also were not structurally stable. In [1], [6] structurally stable examples with one-dimensional attractor-repeller dynamics were constructed, but due to the following result the basic sets are not canonically embedded in surfaces in that examples.

Theorem. *There are no structurally stable* 3*-diffeomorphisms in the class* G.

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Complicated modes in a ring of oscillators with a unidirectional coupling

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Let us consider a ring network of 3 oscillators with a unidirectional synaptic coupling [1, 2]

$$\dot{u}_j = \lambda \left[F(u_j(t-1)) + G(u_{j-1}(t-h)) \ln(u_*/u_j)\right] u_j, \quad j = 1, 2, 3, \quad u_0 \equiv u_3. \tag{1}$$

Here $u_j > 0$ is a normalized neural membrane potentials, $\lambda \gg 1$ characterises the rate of electric processes in the system, $u_* = \exp(c\,\lambda)$, $c = \text{const} \in \mathrm{R}$, the relay functions $F(u)$ and $G(u)$ have the form

$$F(u) \stackrel{\text{def}}{=} \begin{cases} 1, & 0 < u \le 1, \\ -a, & u > 1, \end{cases} \quad G(u) \stackrel{\text{def}}{=} \begin{cases} 0, & 0 < u \le 1, \\ b, & u > 1. \end{cases}$$

$a, b = \text{const} > 0$.

Suppose that k_1, k_2, k_3 is a set of positive natural numbers with a fixed sum. In this paper, for each $n \ge 3$, we obtain a condition on the parameters a, b, c such that for all tuples k_1, k_2, k_3 such that $k_1 + k_2 + k_3 = n$ and $3k_3 \ge n \ge 3k_1$ modes of system (1) coexist, containing bursts in the j-th component. This means that the system has coexisting complex modes with a bursting effect [1, 2, 3], and the number of these modes accumulates as n grows.

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Chaos generation during nucleation of receptor clusters

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It is well known, that one of key steps in cell's activation is clustering of it's receptors [1]. But the mechanism by which cluster growth is induced by specific ligands remains unclear [2]. It can be explained from the standpoint of heterogeneous nucleation theory via changes in critical cluster size [3]. In the talk it will be discussed what properties of ligands drive these changes in critical cluster size.

Critical cluster size is determined by equal speed of losing and gaining receptors by this cluster [4]. In some cases speed of gaining new receptors can be hindered by recurrence of cluster oscillations, since excess energy on newly formed link between receptors can shortly focus on it once more leading to it's destruction. The conditions of chaotic dynamics which breaks recurrence were widely studied in the framework of KAM theory [5].

The analysis of the conditions of chaotic dynamics was carried out for the case of receptor clusters. The expression for kinetic barrier on energy which is needed to achieve chaotic dynamics was found. Specific ligands can significantly lower this kinetic barrier therefore increasing speed of gaining new receptors by cluster. Approximate expression for nucleation speed increase induced by specific ligands was found which is in a good agreement with experimental data.

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Remark on Coexistence of Migrating Predators and Preys

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Modeling of dynamics of a predator-prey ecosystem in the framework of the following system of ordinary differential equations:

$$\frac{dU}{dt} = f(U,V)\,, \qquad \frac{dV}{dt} = g(U,V)\,, \tag{1}$$

where $U \geq 0$ and $V \geq 0$ are numbers of predators and preys in the ecosystem respectively, is the typical problem of mathematical biology [1].

Model (1) can be modified to take into account migration of animals on the surface of the Earth with variable velocity $\vec{c}(t) = (c_x(t), c_y(t))$ as follows:

$$\frac{\partial U}{\partial t} + \vec{c}(t) \cdot \nabla U = f(U, V)\,, \qquad \frac{\partial V}{\partial t} + \vec{c}(t) \cdot \nabla V = g(U, V)\,, \tag{2}$$

where $\nabla = (\partial/\partial x,\, \partial/\partial y)$.

But in system (2) sense of values U and V varies namely in this case both of them become functions of spatial variables too: $U = U(x, y, t)$ and $V = V(x, y, t)$, these functions denoting spatial densities of numbers of predators and preys, therefore system (2) ought to be provided by initial conditions:

$$U(x, y, 0) = U_0(x, y)\,, \qquad V(x, y, 0) = V_0(x, y)\,. \tag{3}$$

Further let us suppose that point $(U_*\,, V_*)$ is stationary state of input system (1), and in this point the Andronov-Hopf bifurcation of this system happens.

Hence using standard procedure of construction of the normal form [2] one can reduce system (2) with initial conditions (3) to the next Cauchy problem in the vicinity of this point:

$$\begin{array}{c} \frac{\partial u}{\partial t} + \vec{c}(t) \cdot \nabla u = \alpha\, u - \omega\, v + (p\, u - q\, v)\,(u^2 + v^2)\,, \\ \frac{\partial v}{\partial t} + \vec{c}(t) \cdot \nabla v = \omega\, u + \alpha\, v + (q\, u + p\, v)\,(u^2 + v^2)\,, \\ u(x, y, 0) = u_0(x, y)\,, \qquad v(x, y, 0) = v_0(x, y)\,. \end{array} \tag{4}$$

After introduction of new variables $A(x, y, t)$ and $\varphi(x, y, t)$ as follows:

$$u(x, y, t) = A(x, y, t)\, \cos\varphi(x, y, t)\,,\; v(x, y, t) = A(x, y, t)\, \sin\varphi(x, y, t)\,, \tag{5}$$

it is easy to see that the Cauchy problem (4) splits onto two new ones, namely, "master" problem:

$$\frac{\partial A}{\partial t} + \vec{c}(t) \cdot \nabla A = \alpha\, A + p\, A^3\,, \quad A(x, y, 0) = \sqrt{u_0^2(x, y) + v_0^2(x, y)}\,, \tag{6}$$

and "slave" problem:

$$\frac{\partial \varphi}{\partial t} + \vec{c}(t) \cdot \nabla \varphi = \omega + q\, A^2\,, \quad \varphi(x, y, 0) = \arctan \frac{v_0(x, y)}{u_0(x, y)}\,. \tag{7}$$

In the presented report exact solution of the Cauchy problem (4) with initial conditions expressing via Jacobian elliptic function with modulus k: $u_0(x, y) = A_0\, cn(Q_x\, x + Q_y\, y, k)$ and $v_0(x, y) = A_0\, sn(Q_x\, x + Q_y\, y, k)$ is discussed.

In this case exact solution of equation (6) gives one spatially-homogeneous regime with nonstationary amplitude $A(t)$ determining by the well-known logistic equation.

After that it is not difficult to find exact solution of equation (7):

$$\begin{array}{c} \varphi(x, y, t) = am[Q_x\, (x - \int_0^t c_x(\tau) d\tau) + Q_y\, (y - \int_0^t c_y(\tau) d\tau),\, k] + \\ + \omega\, t + q\, \int_0^t A^2(\tau) d\tau\,. \end{array} \tag{8}$$

At last densities of predators and preys are obtained by substitution of expression (8) and function $A(t)$ into formulas (5).

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Bifurcation Diagram of One Model of a Lagrange Top with a Vibrating Suspension Point

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We consider a completely integrable Hamiltonian system with two degrees of freedom that describes the dynamics of a Lagrange top with a vibrating suspension point. For a dynamically symmetric rigid body with the center of mass lying on the axis of dynamic symmetry, the corresponding system of differential equations has the form of generalized Kirchhoff equations

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}} \tag{1}$$

with Hamiltonian function

$$H = \frac{1}{2}\left(M_1^2 + M_2^2 + cM_3^2\right) + a\gamma_3 - \frac{1}{2}b\gamma_3^2.$$

Here, $\boldsymbol{M} = \{M_1, M_2, M_3\}$ and $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$ denote the angular momentum vector and the vertical unit vector in the system of principal axes attached to the rigid body and passing through the point of restraint. The parameters a, b and c can be physically interpreted according to [1], [2]. Specifically, a is related to the location of the center of mass on the vertical axis; in what follows, for a particular study, the sign of a is assumed to be fixed. The parameter b characterizes the difference between the averaged squared projections of the suspension point's velocity onto the OX and OZ axes in the frame $OXYZ$ with the origin at the suspension point; the values of b can be positive or negative. The parameter c is a positive one characterizing the ratio of the principal inertia-tensor components for the dynamically symmetric rigid body.

The phase space $\mathcal{P}$ is specified as the tangent bundle $T\mathbb{S}^2$ of the two-dimensional sphere $\mathbb{S}^2$:

$$\mathcal{P} = \{(\boldsymbol{M}, \boldsymbol{\gamma}) : (\boldsymbol{M}, \boldsymbol{\gamma}) = \ell, \, |\boldsymbol{\gamma}|^2 = 1\}.$$

System (1) has one additional first integral, namely, the Lagrange integral

$$F = M_3$$

The function F and the Hamiltonian H form a complete involute set of integrals of system (1) on $\mathcal{P}$. According to the Liouville-Arnold theorem, a regular level surface of first integrals of a completely integrable Hamiltonian system is a nonconnected union of tori filled with conditionally periodic trajectories. The integral mapping $\mathcal{F} : \mathcal{P} \to \mathbb{R}^2$ is defined by setting $(f, h) = \mathcal{F}(\boldsymbol{x}) = (F(\boldsymbol{x}), H(\boldsymbol{x}))$. Let $\mathcal{C}$ denote the set of all critical points of the integral mapping, i.e., points at which $\operatorname{rank} d\mathcal{F}(\boldsymbol{x}) < 2$. The set of critical values $\Sigma = \mathcal{F}(\mathcal{C} \cap \mathcal{P})$ is called the bifurcation diagram. The type of rank-zero singularities of the integral mapping, which are associated with equilibrium positions, are determined in the paper [3]. In contrast to the classical approach used for stability analysis of the upper equilibrium in [2], an analysis of the type of singularities of the integral mapping revealed relations under which the lower equilibrium position becomes unstable. Additionally, a unique phenomenon is observed in the considered mechanical system, namely, the appearance of a double pinched torus.

The goal of this report is to determine and analyze the bifurcation diagram Σ of the integral mapping $\mathcal{F}$.

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Non-Linear Stability of Equilibrium Solutions

of the Vlasov Equation with a Lennard-Jones type Potential

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We consider the gravitating particles that can collide. Collisions can be described in various ways. We can use the theory of inelastic interaction of solids with Newton's recovery coefficient for the relative velocity of colliding particles. In numerical implementation, the main difficulty of this approach is to track and refine a huge number of time moments of particle collisions. Another approach is to add to the gravitational potential the potential of repulsive forces, similar to the intermolecular Lennard-Jones forces. Numerical experiments show that when the Jacobi stability condition is satisfied, both models lead to a qualitatively identical character of evolution with the possible formation of stable configurations.

As it is known, when pair collisions of an infinitely large number of gravitating particles are taken into account, the probability density function evolves in accordance with the Vlasov-Boltzmann-Poisson system of equations.

We suggest a research method using the Vlasov equation with the Lennard Jones type potential. This allows to take into account the size of the interacting particles, and also take into account not only paired, but also triple or more collisions of the particles. For this dynamical system the existence of a large class of nonlinearly stable equilibrium solutions is proved by the Energy-Casimir method.

Tensor Invariants of Dynamical Systems with Dissipation

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Research into the integrability of autonomous systems on a finite-dimensional configuration manifold M^n leads to the study of systems on the tangent bundle TM^n. A key point, along with the geometry of M^4, is the structure of the force field present in the system. For example, the problem of a $(n+1)$-dimensional pendulum moving on a generalized spherical hinge in a nonconservative force field leads to a dynamical system on the tangent bundle of the n-dimensional sphere with a special metric on it induced by additional symmetry groups [1, 2]. The systems describing the motion of such a pendulum have dissipation of variable sign (referred to as alternating dissipation), and the complete list of first integrals consists of transcendental functions that can be expressed in terms of a finite combination of elementary functions [2, 3].

There are also problems concerning the motion of a point on n-dimensional surfaces of revolution, in the Lobachevsky space of the same dimension, etc. Sometimes, for dissipative systems, it is possible to find a complete list of first integrals consisting of transcendental functions (in the sense of complex analysis), since a complete list of continuous autonomous first integrals fails to be found. These results are important in the context of a nonconservative force field present in the system.

In this activity, we show the integrability in tensor invariants of some classes of homogeneous dynamical systems on tangent bundles of smooth n-dimensional manifolds. The force fields introduced into the systems give rise to alternating dissipation and generalize previously considered fields [2, 3].

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Basic automorphisms of Cartan foliations with integrable Ehresmann connections

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In the theory of foliations with transverse geometries an isomorphism is a diffeomorphism which maps leaves onto leaves and preserves transverse geometries. We study the category $\mathcal{CF}$ of foliations with effective transverse Cartan geometries which are referred to as Cartan foliations.

Denote $Aut(M, F)$ the group of all automorphisms of a Cartan foliation (M, F) in the category $\mathcal{CF}$. The group $A_L(M, F) := \{f \in Aut(M, F) \mid f(L) = L \ \forall L \in F\}$ which we call the group leaf automorphisms of (M, F), is a normal subgroup of the group $A(M, F)$. We say the quotient group $Aut(M, F)/Aut_L(M, F)$ to be the group of basic automorphisms of the Cartan foliation (M, F) denoted $A_B(M, F)$. When investigating Cartan foliations (M, F) it is natural ask if there exists a finite-dimensional Lie group structure for the group of all basic automorphisms of (M, F).

It was proved [1] that if the structure Lie algebra of a Cartan foliation (M, F) is zero, then its group of all basic automorphisms $\mathcal{A}_B(M, F)$ admits a unique structure of a Lie group. This theorem was using in [2].

Now we investigate Cartan foliations (M, F) with an integrable Ehresmann connection and describe their global structure. Obtained results we apply to investigation of the basic automorphism groups of these foliations. A notion of the structure group $\Gamma = \Gamma(M, F)$ is introduced. For Cartan foliations with an integrable Ehresmann connection we present a method for determination and computing the Lie group $\mathcal{A}_B(M, F)$ by means of the structure group Γ of this foliation.

We find some exact estimates of the dimension of the Lie group $\mathcal{A}_B(M, F)$ and construct examples.

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Attractors of direct products

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When a direct product of two dynamical systems is considered, it is tempting to assume that the attractor of the product coincides with the direct product of their attractors. Although this holds, indeed, for so-called maximal attractors (which are simply intersections of the iterates of trapping regions), this is not true in general for several other types of attractors, namely Milnor, statistical, and minimal attractors (these are defined using a reference measure on the phase space), even for systems as nice as smooth vector fields in two dimensions.

The case of Milnor attractors was first considered by P. Ashwin and M. Field [1], who conjectured that for the product of two planar flows with attracting homoclinic saddle loops the Milnor attractor does not contain the whole product of the two loops. Then N. Agarwal, A. Rodrigues, and M. Field [2] proved the conjecture and generalized this result to the case of arbitrary attracting polycycles formed by hyperbolic saddles.

We construct the counterexample for the case of statistical attractors using the so-called modified Bowen example, an attracting biangle formed by a saddlenode and a saddle. V. Kleptsyn [3] showed that, for a flow with this polycycle, the minimal attractor is strictly smaller than the statistical one. It is not too hard to deduce from his results that for the square of such a flow the statistical attractor is smaller than the square of the statistical attractor of the flow itself; however, it took some work to generalize this to the case where one takes the product of two different modified Bowen examples. We also showed that an example of this type cannot be constructed using planar flows of smaller codimension.

Finally, we present the example for minimal attractors which also demonstrates an analogous property for physical measures. This example is obtained using a flow of infinite codimension, and it is, in our opinion, an interesting open problem to find out if the same can be realized for a flow of finite codimension, possibly in a phase space of higher dimension.

This is a joint work with **Stanislav Minkov** (*Brook Institute of Electronic Control Machines, Moscow*).

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Attractors of direct products

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When a direct product of two dynamical systems is considered, it is tempting to assume that the attractor of the product coincides with the direct product of their attractors. Although

this holds, indeed, for so-called maximal attractors (which are simply intersections of the iterates of trapping regions), this is not true in general for several other types of attractors, namely Milnor, statistical, and minimal attractors (these are defined using a reference measure on the phase space), even for systems as nice as smooth vector fields in two dimensions.

The case of Milnor attractors was first considered by P. Ashwin and M. Field [1], who conjectured that for the product of two planar flows with attracting homoclinic saddle loops the Milnor attractor does not contain the whole product of the two loops. Then N. Agarwal, A. Rodrigues, and M. Field [2] proved the conjecture and generalized this result to the case of arbitrary attracting polycycles formed by hyperbolic saddles.

We construct the counterexample for the case of statistical attractors using the so-called modified Bowen example, an attracting biangle formed by a saddlenode and a saddle. V. Kleptsyn [3] showed that, for a flow with this polycycle, the minimal attractor is strictly smaller than the statistical one. It is not too hard to deduce from his results that for the square of such a flow the statistical attractor is smaller than the square of the statistical attractor of the flow itself; however, it took some work to generalize this to the case where one takes the product of two different modified Bowen examples. We also showed that an example of this type cannot be constructed using planar flows of smaller codimension.

Finally, we present the example for minimal attractors which also demonstrates an analogous property for physical measures. This example is obtained using a flow of infinite codimension, and it is, in our opinion, an interesting open problem to find out if the same can be realized for a flow of finite codimension, possibly in a phase space of higher dimension.

This is a joint work with **Stanislav Minkov** (*Brook Institute of Electronic Control Machines, Moscow*).

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Rapidly Converging Chernoff Approximations to Solution of Parabolic Differential Equation on the Real Line

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The report is devoted to a new method for solving the Cauchy problem for a parabolic differential equation on the real line:

$$\begin{cases} u_t'(t,x) = a(x)u_{xx}''(t,x) + b(x)u_x'(t,x) + c(x)u(t,x) \quad t \geq 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where the function a is positive, the functions a, b, c are uniformly continuous and bounded.

It is known that this solution can be represented as a strongly continuous semigroup of operators (otherwise called the C_0 -semigroup) acting on the initial condition. The C_0-semigroup in the Banach space $\mathcal{F}$ is a function defined on the nonnegative semiaxis and taking values in the space of linear bounded operators in $\mathcal{F}$.

The C_0-semigroup can be approximated in different ways. The method used in this report is based on Chernoff theorem. According to it, the C_0-semigroup can be approximated using a special family of operators located on the nonnegative semiaxis (like the semigroup). Such a family is called the Chernoff function. It constructs a sequence of approximations to the solution, which are called Chernoff approximations.

The method for constructing rapidly converging Chernoff approximations, which is used in this report, is based on the Galkin-Remizov theorem. As a rule, ordinary Chernoff functions with sufficient smoothness of the initial condition give the convergence rate $C/n + o(1/n)$ (where C is a real constant, n is the number of a term in the sequence of Chernoff approximations). The main result of the report is as follows. Two Chernoff functions are constructed that converge to the solution at a speed $C/n^{3/2} + o(1/n^{3/2})$ and $C/n^2 + o(1/n^2)$. Further research will be aimed at generalizing the result.

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Bifurcations near resonant hopf-hopf interaction

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In this article, we discuss some unsolved problems of local bifurcations of higher codimensions.The resonans Hopf - Hopf bifurcation are considered.

Let us consider a four - dimensional C^k smooth system (1) depending on parameters. Let us rewrite system (1) in the form

$$\dot{x} = A(\mu)x + G(x, \mu), \tag{3}$$

where $G(x,\mu) = O(|x|^2)$ is a C^k smooth function. Suppose the matrix $A(\mu)$ has two pairs of simple complex - conjugate eigenvalues

$$\lambda_{1,4}(\mu) = \alpha_1(\mu) \pm i\omega_1(\mu),\ \lambda_{2,3} = \alpha_2(\mu) \pm i\omega_2(\mu),$$

for all sufficiently small μ, where $\alpha_{1,2}$ and $\omega_{1,2}$ are smooth functions of μ and

$$\alpha_{1,2}(0) = 0,\ \omega_1(0) > 0,\ .$$

If the ω_2/ω_1 is irrational number, than the classical Hopf bifurcation theorem gives sufficient conditions under which a periodic orbit bifurcates from an equilibrium point (see [4, 5, 9, 10, 13, 17]). The cases in which ω_2/ω_1 is rational are known as the resonant cases. Of these, the so-called strong resonances $\omega_2/\omega_1 = 1, 2, 3, 4$ are the most interesting.

In this paper we consider 2:1 resonance. This problem has been studied by many authors [6, 11, 13, 14, 17, 20]. The most complete previous works to date on the 2:1 resonance are that of [11, 13]. E. Knobloch and R.E. Proctor [11] derived the truncated form of the third order (system (9) our paper). They studied only the bifurcation of pure mode solution of truncated form. V.G. Leblanc and W.F. Langford [13] considered the bifurcation of periodic solutions by the Lyapunov-Schmidt method. Paper [7] is concerned with Hopf-Hopf bifurcation with nonsemisimple 1:1 resonance. Our methods are similar [7], but our truncated form differs from reduction of [15]. For a discussion of of Hopf – Hopf interactions with resonance as well all further references see nice paper [13]. We consider the secondary bifurcation (Neimark - Sacker bifurcation) of mixed - mode solution and dynamics of full system.

We calculates the normal form of the system via the transformation $z = w + P(w,)$, where P is a polynomial of w. After several changes of variables and the introduction of polar coordinates associated with w, we obtains a new four-dimensional system. Changing again the time associated to an angle motion of this system, and introducing a rescaling with a small parameter , a three-dimensional system is derived.

$$\begin{cases} \dot{X} &= \varepsilon(X + \delta Y - Z + 2Y^2), \\ \dot{Y} &= \varepsilon(-\delta X + Y - 2XY), \\ \dot{Z} &= \varepsilon(2(-\gamma + X)Z). \end{cases} \tag{13}$$

The three-dimensional system (12) is the main object of study in the remainder of this paper. In the following we assume that $\theta_0 = \pi$. In this case the system (12) is

The system (13) is isomorphic to the system of equations governing second-harmonic generation in nonlinear optics [1] and system of resonant interaction of waves in a plasma [3, 10, 15, 19]. The equations (13) have been extensively studied. It has been shown that system (13) demonstrate Hopf bifurcation, period–doubling cascades and chaotic attractor [3, 10, 12, 14, 15].

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Modeling of integrable flows on the projective plane and the Klein bottle by billiards with slipping

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In the work [1] by A.T. Fomenko was introduced a new class of billiards with slippage. Let's consider F — the isometry of the boundary of a flat ellipse, which translates the point x into the diametrically opposite point y. Let the material point move rectilinearly through the region and fall into the point x on the ellipse at some angle α. Then it continues its movement from a diametrically opposite point, leaving with the same angle α, 'slipping " along the border. On this basis, the billiard is called a "billiard with slippage".

This system has the same integral as the billiard in an ellipse — the parameter of the sophocus quadric, which is the caustic of the trajectory. This allows us to raise the question of the topology of the Liouville foliation and the calculation of the Fomenko-Tsishang invariants that characterize the closures of the solutions of the system.

As it turned out, billiards with slippage implement some such flows on **undirectable** surfaces. Let be given bounded by two sophocal ellipses from the family and having parameters 0 and $\tilde{a}$, where $0 < \tilde{a} < b < a < \infty$. Having glued two such rings along the inner ellipse, we consider billiards on this area by introducing slippage on the outer ellipses.

Theorem. *Any geodesic flow linear in momentum and a series of quadratically integrable flows on a two-dimensional undirectable manifold (Klein bottle or projective plane) is Liouville equivalent to a suitable billiard with slippage. In this case, the linear and quadratic integrals of geodesic flows are reduced to one canonical linear and one canonical quadratic integrals on the billiard.*

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Construction of energy functions of regular topological flows

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We introduce a class G of continuous flows f^t on M^n that generalize the concept of Morse-Smale flows and called regular flows, they have a hyperbolic (in the topological sense) chain-recurrent set R_{f^t} consisting of a finite number of orbits (*chain components*). Each non-wandering orbit is either a fixed point or a periodic orbit. The dynamics systems of class G was studied in the work [6].

The Lyapunov Function of a dynamical system defined on a closed topological manifold M^n is called a continuous function $\varphi : M \to \mathbf{R}$, which is constant on each chain component of the system and decreases along its orbits outside the chain-recurrent set. Due to the results of C. Conley [1], the Lyapunov function exists for any dynamical system, and the fact of its existence is called the "Fundamental theorem of dynamical systems". Critical values of the φ Function. Conley named the numbers that belong to the image of a chain-recurrent set. However, for a smooth function, its critical value is usually called the image of the critical point (the point where the gradient of the function vanishes), which, generally speaking, does not have to belong to a chain-recurrent set. In this connection, along with the Lyapunov function, the smooth category uses the concept of *energy function*, that is, a smooth Lyapunov function whose set of critical points coincides with the chain-recurrent set of the system.

The first results on the construction of the energy function belong to S. Smale [2], who in 1961 proved the existence of the Morse energy function for gradient-like flows. K. Meyer [3] in 1968 generalized this result by constructing the Morse-Bott energy function for an arbitrary

Morse-Smale flow. O. V. Pochinka and S. Kh. Zinina in the work [4] considered topological flows with a finite hyperbolic chain-recurrent set on closed surfaces and proved that they have an energy (continuous) Morse function, in the work [5] a similar result is proved for topological manifolds of arbitrary dimension. The next step is to prove the existence of Morse Bott functions for flows of the class G. O. V. Pochinka and S. Kh. Zinina proved the following theorem.

Theorem Each flow $f^t \in G$ has a Morse-Bott energy function whose critical points are either nondegenerate or have a degeneracy degree of 1.

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