Ярославский государственный университет им. П.Г. Демидова
P.G. Demidov Yaroslavl State University

2021

Third International Conference on Integrable Systems & Nonlinear Dynamics, and School "Integrable and Nonlinear Days"

2021

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4-8 October 2021

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ТЕНЗОР

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“Centre of Integrable Systems”
Liverpool Hope University, UK
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ABSTRACTS

Yaroslavl, October 4–8, 2021

Ярославский государственный университет
им. П. Г. Демидова

Региональный научно-образовательный
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on Integrable Systems & Nonlinear Dynamics,
and School "Integrable and Nonlinear Days"
ISND–2021
Yaroslavl, October 4–8, 2021

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The P. G. Demidov Yaroslavl State University is hosting an International Scientific Conference on Integrable Systems and Nonlinear Dynamics, and a Scientific School "Integrable and Nonlinear Days" from 4 to 8 October 2021 in the city of Yaroslavl. This is a collection of abstracts of the conference talks. The abstracts published here are edited by the authors.

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ON THE CONSTRUCTION OF EXACT SOLUTIONS IN THE PROBLEM OF INTEGRABLE GEODESIC FLOWS

S. V. Agapov[1,2]

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The talk will be about various methods of integration of quasi-linear systems of PDEs arising in the problems of integrable geodesic and magnetic geodesic flows on 2-surfaces.

EXACT SOLUTION OF ONE NONLINEAR ORDINARY DIFFERENTIAL EQUATION IN BANACH ALGEBRA $l_1(\mathbb{R})$

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Nonlinear denumerable-dimensional systems of ordinary differential equations is known often to arise in different applications (see [1] and references therein).

In the report presented the next Cauchy problem is considered:

$$\dot{w}_n = -\sum_{k=-\infty}^{+\infty} w_k\, w_{n-k}\,, \quad w_n(0) = W_n\,, \quad n \in \mathbb{Z}\,, \tag{1}$$

where dot denotes derivative with respect to time, each function $w_n(t) \in \mathbb{R}$ and

$$w_{-n}(t) = w_n(t)\,. \tag{2}$$

In particular it is proven that if one choose initial condition for the Cauchy problem (1) as follows:

$$W_0 = a\,, \quad W_1 = W_{-1} = \frac{b}{2}\,, \quad 0 < a - b < a + b < 1\,,$$

then exact solution of input equation is equal to:

$$w_0(t) = \frac{a + b\,\zeta(t)}{\sqrt{A^2(t) - B^2(t)}}\,, \qquad w_n(t) = \frac{b\,\zeta^2(t) + 2\,a\,\zeta(t) + b}{2\sqrt{A^2(t) - B^2(t)}}\,\zeta^{n-1}(t)\,,$$

$n \in \mathbb{N}$, where $A(t) = 1 + a\,t$, $B(t) = b\,t$ and

$$\zeta(t) = -\frac{B(t)}{A(t) + \sqrt{A^2(t) - B^2(t)}}\,.$$

System of equations (1) can be rewritten as one ordinary differential equation:

$$\dot{w} = -w * w \tag{3}$$

on linear subspace of Banach algebra $l_1(\mathbb{R})$ [2] of two-sided sequences

$$w = (\ldots, w_{-n}, \ldots, w_0, \ldots, w_n, \ldots)$$

restricted by conditions (2), symbol of multiplication $*$ of sequences from $l_1(\mathbb{R})$ in right hand side of equation (3) corresponding convolution in formula (1).

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EFFICIENT VARIATIONAL METHOD OF CALCULATING LIBRATIONS APPEARING IN TUNNELING PROBLEMS

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It is well-known that in the study of exponentially small "tunneling" effects in quantum mechanics, special trajectories of classical systems – instantons – play an important role. In the simplest problem of this kind – the ground state splitting of the Schrödinger operator with a symmetric "double well" potential – the instanton is a doubly asymptotic trajectory of a classical system with an "inverted" potential $V(x)$ connecting its peaks. Various asymptotic formulas relating the tunneling splitting with the instanton action were derived with or without accurate justification by many authors (e.g. L.D. Landau, E.M. Lifshits, B. Helffer, J. Sjöstrand, B. Simon, E. Harrell, V.P. Maslov). In a series of papers by S.Yu. Dobrokhotov, A.Yu. Anikin, and co-authors (see, for example, [1]) it was shown that the tunneling splitting can be much more efficiently calculated in terms of the action on the "libration" with a large period giving an instanton as a limit, rather than the instanton itself.

By libration we mean a periodic solution whose velocity vanishes twice in a period. The existence of a family of librations near the instanton follows from the results of V.V. Kozlov and S.V. Bolotin [2]. The question of calculating the asymptotics for the tunneling splitting is essentially reduced to seeking some specific libration, called the "tunneling libration", and calculating the value of the action on it, called the "tunneling action". In the case of two degrees of freedom, a tunneling libration can be found by the primitive "shooting method" (as shown in [1]). The efficient calculation of tunneling libration for a larger number of degrees of freedom has not been previously considered.

The goal of the talk is to show that tunneling libration can be efficiently calculated by using a modification of the variational method developed in the papers by I.A. Nosikov, M.V. Klimenko et al. (see, e.g., [3]). Our main example with four degrees of freedom is connected with the quantum problem of rotating dimers (see e.g. [4]). We also consider a simpler two-dimensional example, in which we compare the calculations of the tunneling libration and action by the shooting method and the variational method.

The work is supported by the Russian Science Foundation (Project No. 21-71-30011).

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BILLIARD BOUNDED BY THREE-AXIAL ELLIPSOID IN A HOOKE POTENTIAL FIELD

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Let $\mathcal{E} \subset \mathbb{R}^3$ be an ellipsoid with different semiaxes $a > b > c > 0$. Consider the following dynamical system: a material point (ball) of unit mass moves inside $\mathcal{E}$ under the action of an elastic force (Hooke’s law). We assume that reflection from $\mathcal{E}$ is absolutely elastic. Such system is an integrable Hamiltonian system in a piecewise smooth sense. One of its first integrals is the energy, that is, the function:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{k}{2}(x^2 + y^2 + z^2)$$

Two more first integrals F_1 and F_2, functionally independent of H, can be found using the integrals I_1, I_2 of the billiard without potential. Kozlov V. V. used a similar method in [1]. The involutivity of these integrals is verified. Billiards without potential bounded by confocal quadrics in three-dimensional Euclidean space were studied in [2].

The equations of motion in elliptic coordinates can be rewritten as follows:

$$\dot{\lambda}_i = \pm\frac{4}{\sqrt{2}(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\sqrt{V(\lambda_i)}$$

where $V(z)$ is a 6-th degree polynomial whose coefficients depend only on a, b, c and on values of H, F_1, F_2. Based on the properties of this polynomial, a bifurcation diagram was constructed, the regions of possible motion were found, and the pullback of the points of the moment map were studied.

Classes of homeomorphism of non-singular isoenergy surfaces were also determined. Author proved following theorem:

Theorem 1. *Let h be a non-singular energy level, then:*

1. *if $k > 0$, then the isoenergy surface Q_h is homeomorphic to the sphere S^5.*

2. *if $k < 0$ and*

   - *$h \in \left(\frac{ka}{2}; \frac{kb}{2}\right)$, then the isoenergy surface Q_h is homeomorphic to the disjointed union of two spheres S^5;*
   - *$h \in \left(\frac{kb}{2}; \frac{kc}{2}\right)$, then the isoenergy surface Q_h is homeomorphic to the direct product $S^1 \times S^4$;*
   - *$h \in \left(\frac{kc}{2}; 0\right)$, then the isoenergy surface Q_h is homeomorphic to the direct product $S^2 \times S^3$;*
   - *$h \in \left(0; +\infty\right)$, then the isoenergy surface Q_h is homeomorphic to the sphere S^5.*

The work was done at Moscow State University under the support of the Russian Science Foundation (project 20-71-00155).

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RECENT RESULTS ON INTEGRABLE AND NON-INTEGRABLE LOTKA VOLTERRA SYSTEMS

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In recent years, there has been renewed interest in the study of anti-symmetric Lotka Volterra Hamiltonian (LVH) systems of competing species, $x_i(t)$, satisfying the ODEs

$$dx_i/dt = \sum_{i=1}^{n} a_{ij}x_ix_j = h = const., a_{i,j} = -a_{j,i},\ i,j = 1,2,...,n$$

and preserving the sum $\sum_{i=1}^{n} x_i(t) = h = const.$ [1]. In particular, it is interesting to add linear (or nonlinear) terms to these systems, and either seek to preserve integrability, or investigate the dynamics of ”nearby” nonintegrable systems in the $n-$ dimensional phase space [2]. In this talk, I will first show how new integrable classes of LVH systems were discovered applying the Painlevé property [3], and then demonstrate that “nearby” non-integrable systems typically continue to possess very simple dynamics. Finally, I will discuss some very recent findings revealing possible connections between the Painlevé property and Brenig’s method of integrating polynomial systems of ODEs by reduction to canonical form [4-5].

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SYSTEMS OF DIFFERENCE EQUATIONS AND THEIR SYMMETRIES

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In this talk we consider systems of difference equations which are defined on an elementary quadrilateral of the $\mathbb{Z}^2$ lattice, and propose a systematic method for deriving their lowest order generalised symmetries. For our derivations we exploit the theory of integrability conditions by employing Laurent and Taylor formal series of pseudo-difference operators, derive necessary determining equations, and discuss the dynamical variables which are essential for solving functional equations, such as the determining equations. We demonstrate our derivations by considering three two-component systems and computing their symmetries.

FROM THE KORTEWEG–DE VRIES HIERARCHY TO THE QUANTUM EQUATION OF S.P. NOVIKOV

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The talk is based on the work [1].
Contents:

1. The Korteweg–de Vries equation and different approaches to the definition of the KdV hierarchy;
2. Novikov N-equation and N-KdV hierarchy, a family of N compatible integrable polynomial systems in the graded space $\mathbb{C}^{2N}$;
3. Novikov N-equation on graded free associative algebra;
4. Method of quantization ideals for dynamical systems on graded free associative algebras;
5. Quantum Novikov N-equation and a family of compatible quantum dynamical systems;
6. Quantum Novikov N-equations for $N = 1, 2, 3$ in the form of Heisenberg equations, and their quantum Hamiltonians;
7. Noncanonical Poisson bracket of weight $2N + 3$ in $\mathbb{C}^{2N}$ and Hamiltonian form of the constructed integrable systems.

It will be shown that, in the cases under consideration, the approach based on quantization ideals leads to the same commutation relations as in the Poisson algebra of the KdV N-hierarchy, with the normalized Hamiltonians having quantum corrections.

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ELECTRICAL NETWORKS AND LAGRANGIAN GRASSMANNIANS

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Electrical network is a graph in a disk with inner and outer sets of vertices and positive weight on each edge which means conductance. The (compactified) space E_n of electrical networks embed into the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(n-1, 2n)$. I will talk about the new parametrisation of E_n which defines an embedding into the Grassmannian $\mathrm{Gr}(n-1, V)$, where V is a certain subspace of dimension $2n-2$ and moreover into the nonnegative Lagrangian Grassmannian $\mathrm{LG}_{\geq 0}(n-1) \subset \mathrm{Gr}(n-1, V)$. The latter allows us to connect the combinatorics of E_n to the representation theory of the symplectic group. The talk is based on the joint work with V. Gorbounov, A. Kazakov and D. Talalaev.

SOLVABLE SYSTEMS OF FIRST-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODEs) AND DIFFERENCE EQUATIONS (DEs)

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After a terse discussion of the various possible significances of the term ”solvable”, I will review recent results concerning systems of first-order nonlinear Ordinary Differential Equations, such as special cases of the prototypical system

$$\dot{x}_n(t) = \sum_{m_1, m_2,\ m_1 \geq m_2} \left[c^{(n)}_{m_1 m_2} x_{m_1}(t)\, x_{m_2}(t) \right] , \quad n = 1, 2, ..., N ,$$

and of analogous systems featuring *homogeneous* right-hand sides of *arbitrary* degree M, and also of their variants featuring *non-homogeneous* right-hand sides (including *isochronous* systems).

If time shall permit, I will also report very recent findings on solvable systems of *finite-difference* (rather than *differential*) equations.

Most of these findings have been obtained in collaboration with prof. **Farrin Payandeh**.

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INTEGRABLE BOUNDARY CONDITIONS FOR EQUATIONS ON QUAD-GRAPHS, OPEN BOUNDARY REDUCTIONS AND INTEGRABLE MAPPINGS

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I will present the notion of integrable boundary conditions for partial difference equations formulated on the so-called quad-graphs. It relies on an appropriate adaptation of the established notion of bulk integrability based on the consistency around the cube (or multidimensional consistency). I will then introduce the method of *open (boundary) reductions*, as an alternative to the well-known method of periodic reductions, for constructing discrete integrable mappings and their invariants. The invariants are constructed using Sklyanin’s double-row monodromy matrix and this requires the introduction of the notion of reflection matrix and boundary zero curvature condition in this context. We focus on examples from the Adler-Bobenko-Suris classication and associated integrable boundary equations, and on the simplest case of the lattice $\mathbb{Z}^2$. This presentation is based on joint work with Nicolas Crampé, Peter van der Kamp and Cheng Zhang.

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GEOMETRIC INTERPRETATION OF ENTROPY

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We begin with some notation.

Definition. Let A be a final set of symbols and let T be the right shift on $A^{\mathbb{Z}}$, then $(A^{\mathbb{Z}}, T)$ is called the full shift over the alphabet A.

Definition. If $w \in A^n$ for some $n \in \mathbb{N}$, then w is called a (finite) word over the alphabet A and n is the length of the word.

Suppose that $x \in A^{\mathbb{Z}}$ and w is a word of length n over the alphabet A. If there exists l, such that $w = (x(l+1), ..., x(l+n))$, then we say that w occurs in x.

For x, $y \in A^{\mathbb{Z}}$, $x \neq y$, let $d(x,y) = (1/2)^{m(x,y)}$, where $m(x,y) = \max\{m \in N_0\colon\ x(m) = y(m)$ and $x(-m) = y(-m)\}$. Let $d(x,x) = 0$. Then d is a metric and d induces the topology on $A^{\mathbb{Z}}$.

Definition. If X is a closed T-invariant subset of $A^{\mathbb{Z}}$, then (X,T) is called a symbolic system or a shift space. By $W(X)$ we denote the language of X, i.e. the set of all finite words in the alphabet A which can occur in an element of X.

Remark. Let X be a metric and topological subspace of $A^{\mathbb{Z}}$. Suppose x is a point of X and C is a subset of X. By $O_\epsilon(x)$ and $O_\epsilon(C)$ we denote open ϵ-neighbourhoods of x and C respectively, i.e. $O_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$, $O_\epsilon(C) = \bigcup_{x \in C} O_\epsilon(x)$.

Let's introduce the special class of symbolic systems called synchronized. Recall that a shift space X is transitive when X contains an element x whose forward orbit $\{T^j(x) : j \in N\}$ is dense in X.

Definition. A synchronized system is a transitive shift space X which contains a synchronizing (magic) word, i.e. a word $v \in W(X)$ with a following property: if $uv, vw \in W(X)$, then $uvw \in W(X)$.

Let A be $(,),[,]$ DEFINITION. A word w over the alphabet A is called balanced if it respects standard bracket rules. A subword of a balanced word is called correct.

DEFINITION. The symbolic system which language consists of all correct words is called the Dyck shift.

The Dyck shift is transitive, but not synchronized.

DEFINITION. A function n from $\mathbb{R}^+$ to $\mathbb{N}$ is called slowly growing if it satisfies the following conditions: $n(\epsilon) \to \infty$ and $n(\epsilon) = o(|\log(\epsilon)|)$ as $\epsilon \to 0$. The class of all slowly growing functions is denoted by N.

Let X be the shift space (not necessarily synchronized), let $M(X)$ be the set of shift-invariant Borel probability ergodic measures on X and let $h_\mu(X,T)$ be the metric entropy.

DEFINITION. Suppose that $n \in N$, $\mu \in M(X)$ and $\epsilon > 0$ are fixed. The function $P^\epsilon_{X,T,\mu}(x)$ from X to $\mathbb{Z}^+$ defined by expression

$$\frac{1}{n(\epsilon)} \log \left(\frac{\mu \left(O_\epsilon \left(T^{n(\epsilon)} \left(O_\epsilon \left(x \right) \right) \right) \right)}{\mu \left(O_\epsilon \left(x \right) \right)} \right)$$

is called ϵ-boundary deformation rate of measure μ. This function is well defined a.e. and belongs to $L_1(X,\mu)$.

Now we can formulate the main hypothesis (MH):

Hypothesis. *For every $n \in N$: $P^\epsilon_{X,T,\mu} \underset{\epsilon \to 0}{\longrightarrow} h_\mu(X,T)$ in $L_1(X,\mu)$.*

If (MH) is true for some measurable shift space (X,T,μ), then we call (X,T,μ) (MH)-friendly and representation from the (MH) is called geometric interpretation of entropy. A possibility of geometric interpretation was well-studied for synchronized and related systems by Komech [1] and me (the paper is in the process of publication). In my report we consider an opportunity of geometric approach for Dyck shifts, which are not synchronized.

The main result of this report is the following:

Theorem 1. *The sufficient condition for a measurable Dyck shift (X,T,μ) to be (MH)-friendly is $\mu\{(\} \neq \mu\{)\}$ or $\mu\{[\} \neq \mu\{]\}$.*

We also show that condition from theorem 1 is not necessary.

This report is a part of my paper ”Geometric interpretation of entropy: new results”.

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FEATURES OF THE MOTION OF ULTRACOLD ATOMS IN QUASIPERIODIC POTENTIALS

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We consider here quasiperiodic potentials on the plane, which can serve as a "transitional link" between ordered (periodic) and chaotic (random) potentials. As can be shown, in almost any family of quasiperiodic potentials, depending on a certain set of parameters, it is possible to distinguish a set (in the parameter space) where, according to a certain criterion, potentials with features of ordered potentials arise, and a set where we have potentials with features of random potentials. These sets complement each other in the complete parameter space, and each of them has its own specific structure. The difference between "ordered" and "chaotic" potentials will manifest itself, in particular, in the transport properties at different energies, which we will consider here in relation to systems of ultracold atoms. We also note here that the transport properties of particles in the considered potentials can be accompanied by the phenomena of "partial integrability" inherent in two-dimensional Hamiltonian systems.

DIFFERENT HAMILTONIANS FOR PAINLEVÉ EQUATIONS AND THEIR IDENTIFICATION USING GEOMETRY OF THE SPACE OF INITIAL CONDITIONS

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It is well-known that differential Painlevé equations can be written in a Hamiltonian form [1,2]. However, a coordinate form of such representation is far from unique – there are many very different Hamiltonians that result in the same differential Painlevé equation. In this paper we describe a systematic procedure of finding changes of coordinates connecting different Hamiltonian systems. Our approach is based on the notion of Okamoto space of initial conditions [3] and Sakai's geometric theory of Painlevé equations [4,5] and is motivated by a similar procedure in the discrete case [6]. As an example, we consider the fourth differential $\mathrm{P_{IV}}$ equation and compare Hamiltonians given in the works of Okamoto [2], Jimbo-Miwa [7], Filipuk–Żołądek [8], Kecker [9], and Its-Prokhorov [10]. We explain how geometry makes it easy to find explicit birational change of coordinates transforming one Hamiltonian into another. This approach can be easily adapted to other Painlevé equations as well.

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ON THE NONLINEAR EVOLUTION EQUATIONS RELATED TO THE KAC-MOODY ALGEBRAS $A_5^{(1)}$ AND $A_5^{(2)}$

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This report is a natural extension of Refs. [1, 2, 3]. We outline the derivation of the mKdV equations related to the Kac–Moody algebras $A_5^{(1)}$ and $A_5^{(2)}$. First we formulate their Lax representations and provide details how they can be obtained from generic Lax operators related to the algebra $sl(6)$ by applying proper Mikhailov type reduction groups $\mathbb{Z}_h$. Here h is the Coxeter number of the relevant Kac–Moody algebra. Next we adapt Shabat's method for constructing the fundamental analytic solutions of the Lax operators L. Thus we are able to reduce the direct and inverse spectral problems for L to Riemann–Hilbert problems (RHP) on the union of $2h$ rays l_ν. They start from the origin of the complex λ-plane and close equal angles π/h. To each l_ν we associate a subalgebra $\mathfrak{g}_\nu$ which is a direct sum of $sl(2)$–subalgebras. Thus to each regular solution of the RHP we can associate scattering data of L consisting of scattering matrices $T_\nu \in \mathcal{G}_\nu$ and their Gauss decompositions. The main result of the paper is to extract from T_0 and T_1 related to the rays l_0 and l_1 the minimal sets of scattering data $\mathcal{T}_k$, $k = 1, 2$. We prove that each of the minimal sets $\mathcal{T}_1$ and $\mathcal{T}_2$ allows one to reconstruct both the scattering matrices T_ν, $\nu = 0, 1, \dots 2h$ and the corresponding potentials of the Lax operators L. Following [4] we demonstrate that the the mapping from L to $\mathcal{T}_k$ is a generalized Fourier transform.

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TWO-CLUSTER SYNCHRONIZATION IN A FULLY COUPLED SYSTEM OF QUASILINEAR OSCILLATORS

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We consider a system of m quasilinear weakly coupled oscillators according to the principle “each with all” oscillators (the case of syngular oscillators see in articles [1–4]). It is assumed that the parameters of each of the partial oscillators are close to the critical values at which the Andronov – Hopf bifurcation occurs. In this case, the method of normal forms can be applied to each oscillator, as a result of which, on a stable two-dimensional integral manifold for each oscillator, we have a two-dimensional system, which in the complex form of notation allows the representation

$$\dot{z} = z-(1-i\omega_0)|z|^2 z, \quad z = x+iy, \quad x, y \in \mathbb{R}, \quad \omega_0 = \text{const} > 0. \tag{1}$$

Direct verification shows that this system has an exponentially orbitally stable harmonic cycle $z_0(t) = \exp(i\omega_0 t)$. Consider now a system of m oscillators (1), weakly coupled “each with all ”

$$\dot{z}_j = z_j - (1 - i\omega_0)|z_j|^2 z_j + \frac{\nu\, d}{m}\, \overline{z}_j \sum_{s=1, s\neq j}^{m} z_s^2, \quad j = 1, 2, \ldots, m, \tag{2}$$

where $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $0 < \nu \ll 1$, $d = \text{const} \in \mathbb{C}$, $\mathrm{Re}\,[d(1 + i\omega_0)] > 0$. The specificity of the system under consideration is that it has coexisting two-cluster synchronization modes

$$\begin{aligned} z_j &= \exp(i\omega_0 t), \quad j = 1, \ldots, k, \\ z_j &= -\exp(i\omega_0 t), \quad j = k+1, \ldots, m. \end{aligned} \tag{3}$$

where $1 \geq k \geq m-1$ – integer.

For these two-cluster synchronization modes, stability conditions are found. The nature of the change in the modes (3) when changing the parameter d is investigated by numerical methods.

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TRAVELING-WAVE-TYPE SOLUTIONS OF FULLY COUPLED SYSTEMS OF NONLINEAR OSCILLATORS

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A fully coupled system of nonlinear oscillators

$$\dot{x}_j = F(x_j, u_j), \quad j = 1, 2, \ldots, m. \tag{1}$$

is considered. Here $m \geq 2$, $x_j = x_j(t) \in \mathbb{R}^n$, $n \geq 2$, $u_j = \sum_{s=1,\, s\neq j}^{m} G(x_s)$, and the vector functions $F(x, u)$, $G(x)$ with values in $\mathbb{R}^n$ are infinitely differentiable with respect to their variables $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The partial system $\dot{x} = F(x, 0)$ has an exponentially orbitally stable cycle. We consider the situation when m of the same oscillators interact with each other according to the principle each with all. In particular, when $F(x, u) = F(x) + D(x)u$, where $D(x)$ is a square matrix of size $n \times n$, the system (1) takes the form

$$\dot{x}_j = F(x_j) + D(x_j) \sum_{s=1,\, s\neq j}^{m} G(x_s), \quad j = 1, 2, \ldots, m. \tag{2}$$

This situation is of the interest, since systems (2) arise in the mathematical modeling of fully coupled neural and gene networks (see [1–4]).

We will be interested in the problems of existence and stability for the system (1) of canonical traveling waves, which are special periodic solutions of the form $x_j = x(t+(j-1)\Delta)$, $j = 1, 2, \ldots, m$. Taking into account that the system (1) is invariant under a change of variables of the form $(x_1, x_2, \ldots, x_m) \to (x_{j_1}, x_{j_2}, \ldots, x_{j_m})$, any canonical traveling wave generates a whole family U_k of induced traveling waves. The number of cycles of the family U_k is equal to $(m-1)!$

The construction of the family of traveling waves U_k is reduced to finding the canonical cycle. In solving this problem, we need an auxiliary equation with delays

$$\dot{x} = F(x, u_\Delta), \quad u_\Delta = \sum_{s=1}^{m-1} G(x(t - s\Delta)), \tag{3}$$

where $x = x(t) \in \mathbb{R}^n$, $\Delta = \text{const} > 0$. We assume that on some interval $\Delta \in (\Delta_1, \Delta_2) \subset (0, +\infty)$ the equation (3) has a periodic solution $x = x(t, \Delta)$ of period $T = T(\Delta) > 0$. In this case, the following statement is true.

Theorem 1. *Suppose that there is a natural number $k : 1 \le k \le m-1$ for which the equation $T(\Delta) = m\Delta/k$ has the root $\Delta = \Delta_{(k)} \in (\Delta_1, \Delta_2)$. Then in the original system* (1) *this root corresponds to the cycle* (*canonical traveling wave*)

$$C_k : \quad x_j = x_{(k)}(t + (j-1)\Delta_{(k)}), \quad j = 1, 2, \ldots, m \tag{4}$$

of the period $T_{(k)} = m\Delta_{(k)}/k$, where $x_{(k)}(t) = x(t, \Delta)|_{\Delta=\Delta_{(k)}}$.

Theorem 1 and theorems on the stability of the obtained solutions provide a certain general method for studying periodic solutions of the traveling wave type in fully coupled networks of nonlinear oscillators. Indeed, the question of the existence of a canonical traveling wave is reduced to finding the cycle $x(t, \Delta)$ of the auxiliary equation with delays (3) and to finding the roots $\Delta = \Delta_{(k)}$.

It is clear that the problems of analysis of the auxiliary equations, which underlie our method, are generally nonlocal. But, nevertheless, in some situations when it is possible to apply any asymptotic methods, the indicated problems can be solved.

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LOCAL DYNAMICS OF SINGULARLY PERTUBED SECOND ORDER EQUATION WITH STATE-DEPENDENT DELAY

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Consider second order equation wtih delay:

$$\varepsilon^2\ddot{x} + \varepsilon\sigma\dot{x} + x = ax(t - \varphi(x)) + f(x(t - \varphi(x))), \tag{1}$$

It is singularly pertubed ($0 < \varepsilon \ll 1$ is small parameter) and its delay φ depends on the state x. $a \in \mathbb{R}$, $\sigma > 0$ are parameters. The function φ is analytical at the $x = 0$, positive, bounded by some constant $M > 0$, and $\varphi(0) = 1$. Nonlinear function f is analytical at the $x = 0$ and $f(0) = 0$, $f'(0) = 0$.

Thus, $x \equiv 0$ is the equilibrium state of equation (1), and functions f and φ can be represented in the next form in the neighborhood of $x = 0$:

$$\varphi(x) = 1 + \alpha x + \beta x^2 + o(x^2),$$
$$f(y) = f_2 y^2 + f_3 y^3 + o(y^3).$$

The phase space of the equation (1) is $C^1([-M, 0])$.

The problem is to investigate local dynamics of the equation (1) in some neighborhood (which not depends on ε) of its zero equilibrium state. The method of quasinormal forms (see [1]) is used for solving this problem.

There are four critical cases in this problem:

- $a = \pm 1, \quad \sigma > \sqrt{2}$,
- $a = \pm\sigma\sqrt{1-\sigma^2/4}, \quad \sigma < \sqrt{2}$.

The equilibrium state looses stability when point (a, σ) is going through these curves.

In the case $a = 1, \sigma > \sqrt{2}$ the quasinormal form is:

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2 - 2}{2}\frac{\partial^2 u}{\partial r^2} - \alpha u \frac{\partial u}{\partial r} + a_1 u + f_2 u^2.$$

In the case $a = -1, \sigma > \sqrt{2}$ the quasinormal form is:

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2 - 2}{2}\frac{\partial^2 u}{\partial r^2} - \frac{\alpha^2}{2}u^2\frac{\partial^2 u}{\partial r^2} - \left(\beta + \frac{\alpha f_2}{2}\right)u^2\frac{\partial u}{\partial r} + a_1 u - \left(f_2^2 + f_3\right)u^3.$$

In the cases $a = \pm\sigma\sqrt{1-\sigma^2/4}, \sigma < \sqrt{2}$ the quasinormal form is:

$$\frac{\partial u}{\partial \tau} = \left[2d_1 - i\frac{\sigma}{4}d_2\right]\frac{\partial^2 u}{\partial r^2} + (\Theta + \Omega)\left[\frac{\sigma^2}{2}d_2 + 4id_1\right]\frac{\partial u}{\partial r} +$$
$$+ a_1 u + \left[(\Theta + \Omega)\left\{2d_1 + \frac{d_2\sigma^2}{2} + i\left(4d_1 - \frac{4}{\sigma^2} - \frac{\sigma^2 d_2}{4}\right)\right\} - 2d_1(\Theta + \Omega)^2\right]u +$$
$$+ \left[\frac{\alpha\delta i}{e^{-i\Omega}}\left\{\ell_1\left(e^{i\Omega} - 2e^{-2i\Omega}\right) - \ell_2 e^{-i\Omega}\right\} - \frac{\alpha^2\delta^2}{2e^{-i\Omega}}\left(2e^{-i\Omega} + e^{i\Omega}\right)\right]u|u|^2,$$

where

$$d_1 = \frac{2 - \sigma^2}{2a_0^2}, \quad d_2 = -\frac{4\sqrt{1-\sigma^2/2}}{\sigma a_0^2}.$$

In all cases the dynamics of quasinormal forms determines the dynamics of the initial equation (1).

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ESTIMATES OF LYAPUNOV EXPONENTS SPECTRUM OF SELF-ORGANIZATION MODES IN ONE DISTRIBUTED BIOPHYSICAL MODEL

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We consider a boundary value problem based on a logistic model with delay and diffusion, which describes the population density dynamics in a flat area:

$$\frac{\partial N}{\partial t} = D\Delta N + r\,(1 - N_{t-1})\,N, \qquad \left.\frac{\partial N}{\partial x_1}\right|_{\substack{x_1=0\\ x_1=1}} = 0, \quad \left.\frac{\partial N}{\partial x_2}\right|_{\substack{x_2=0\\ x_2=1}} = 0.$$

Here $N(t,x)$ is the population density at time t and point x of a square area Ω ($\Omega = \{(x_1,x_2) \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$), Δ is a Laplace operator, D is a diffusion coefficient, r is a Malthusian coefficient of linear growth, $N_{t-1} \equiv N(t-1,x)$.

It is shown that there exist two types of solutions (see [1]), the first of which includes solutions inheriting the properties of homogeneous solution, and the second type includes the so-called self-organization modes, that are more complexly distributed over space and have properties that are significantly more preferable in terms of population dynamics.

The solutions that are found in the boundary value problem can be studied using the developed algorithm, which allows to estimate Lyapunov exponents spectrum of attractors in case of dynamical systems with time delay (see [2, 3]). This algorithm is based on the classical Benettin method with renormalization by the Gram–Schmidt algorithm, with the fifth-order Dormand–Prince method (DOPRI54) with variable time step (see [4]) as a numerical method for solving the main system and the systems in variations.

Calculations were carried out for 10 modes at different values of D, including the self-organization modes. The structure of Lyapunov exponents spectrum makes it possible to confirm the multifrequency of the analyzed modes, as well as the presence of quasi-stable behavior (see [5]) at the boundaries of the intervals of diffusion coefficient variation, beyond which the mode loses its stability. Small values of the Lyapunov exponents indicate a slow convergence of solutions to the stable modes in the case of an unsuccessful choice of the initial conditions, which emphasizes the complexity of the problem of finding new attractors in the studied spatially distributed system with time delay.

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ON LIOUVILLE INTEGRABILITY OF YANG-BAXTER MAPS

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We will discuss Yang-Baxter maps related to matrix refactorisation problems and discrete integrability. We will describe invariants of such maps, which are analogues of conserved quantities in Classical mechanics. In addition, we will describe briefly the Poisson structures related to Yang-Baxter maps and their Liouville (or complete) integrability. This generalises the idea of “canonical transformations”.

Along with a brief review of well known discrete maps, we will present recent results about Liouville integrability of integrable maps related to Grassmann-extended discrete nonlinear Schrödinger and discrete derivative nonlinear Schrödinger equations.

NEW INVARIANT OF UNIFORM EQUIVALENCE FOR NON-AUTONOMOUS VECTOR FIELDS ON S^3

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We study non-autonomous vector fields (NVFs) on C^∞-smooth closed manifold M, i.e. uniformly continuous maps $\mathbb{R} \to \mathcal{V}^r(M)$ to the Banach space of C^r-smooth vector fields on M. Graph of any solution $x : \mathbb{R} \to M$ of such NVF v is the infinite curve $\cup_t(x(t), t) \subset M \times \mathbb{R}$ called an integral curve (IC). The union of all ICs defines a foliation $\mathcal{L}_v$ in $M \times \mathbb{R}$. Following [1] we call two NVFs v_1, v_2 *uniformly equivalent*, if there is an equimorphism $h : M \times \mathbb{R} \to M \times \mathbb{R}$ such that h transforms a foliation of v_1 to that of v_2.

The class of NVFs we study is characterized by the features of its foliation $\mathcal{L}_v$. Following [2], we require that every IC of $\mathcal{L}_v$ possesses by an exponential dichotomy of solutions for its linearized system both on semi-axes $\mathbb{R}_+$ and $\mathbb{R}_+$ and both partitions of $M \times \mathbb{R}$ into stable manifolds and unstable manifolds be finite. Each stable (respectively, unstable) manifold W^s (W^u) has its Smale boundary, we require this boundary would consist of whole stable manifolds (resp., unstable manifolds) and has a dimension strictly lesser than the dimension of W^s (W^u). These assumptions allow us to consider on any section $M_t = M \times \{t\}$ the traces of W^s (W^u) being embedded $\mathbb{R}^p$ with $p = 0, 1, \ldots, n$. This class of NVFs (with two additional assumptions imposed) is called *gradient-like NVFs*. For this class of NVFs a finite combinatorial invariant classifying them was introduced for $n = 1$ [6] and for $n = 2$ [2]. We consider $n = 3$ and introduce a new invariant of the uniform equivalence for NVFs on the three-dimensional sphere S^3, which we call the type of wildness. This invariant allow us to construct two NVFs with dynamically the same simple structure but being nevertheless uniformly nonequivalent (see Theorem 1 below).

The main feature of these systems is that they have some completely stable IC γ (with a dichotomy on $\mathbb{R}$ of (3,0) type) being a Smale boundary of a wildly embedded two-dimensional unstable manifold W^u such that their traces on M_0, γ_0 and w_0^u, respectively, give a wildly embedded curve near γ_0. In a sense, the uniform topological structure of such vector field is complicated. On the other hand, such vector field has a rather simple structure of its foliations into ICs in $S^3 \times \mathbb{R}$ from the dynamical point of view.

The construction of such gradient-like NVF exploits two ideas. One belongs to Pixton and was developed further by Bonatti, Grines, Pochinka and others [3], [4]. In [3] the Pixton class of 3-dimensional Morse-Smale diffeomorphisms on S^3 was completely investigated, every such diffeomorphism has, as a non-wandering set, only four fixed points: a source, one saddle of (1,2) type and two sinks. The unstable manifold of the saddle is 1-dimensional and the saddle divides it into two separatrices. It was proved in [3] that one of these separatrices can be wildly embedded and another one is always tamely embedded. The presence of such wildly embedded 1-dimensional separatrix for the saddle fixed point implies wild embedding of its 2-dimensional unstable manifold.

The second idea is borrowed from [5]. It uses the construction of so-called non-autonomous suspension over a diffeomorphism. Namely, suppose $f : M \to M$ be some diffeomorphism of a smooth (C^∞) closed manifold M. Its (usual) suspension is a smooth closed manifold M_f of dimension $\dim M + 1$ with a standard vector field v on M_f. The manifold M_f is a bundle $p : M_f \to S^1$ over circle S^1 with the leaf M and for the flow generated by v any leaf M_θ over $\theta \in S^1$ is a global cross-section. This construction allows one to construct a vector field with the dynamics similar to that for the dynamics of iterations of the mapping f.

The non-autonomous suspension is defined when one takes a covering map $exp : \mathbb{R} \to S^1$ as $s \to e^{2\pi i s}$ and considers the related lifted map $\widetilde{exp} : \widetilde{M}_f \to M_f$ which gives a commutative diagram $exp \circ \tilde{p} = p \circ \widetilde{exp}$. The map $\tilde{p}$ being a bundle over $\mathbb{R}$ says that topologically $\widetilde{M}_f$ is a direct product $M \times \mathbb{R}$ but it can be not a direct product as a uniform space with the uniform structure lifted by the map $\widetilde{exp}$, if f^n is not homotopic to the identity map id_M [5].

Let f belongs to the Pixton class. Any diffeomorphism $f : S^3 \to S^3$ is diffeotopic to the identity. In this case, $\widetilde{M}_f$ can be endowed with the uniform structure of direct product $S^3 \times \mathbb{R}$. Also a foliation into orbits of v in M_f is lifted into a foliation into infinite curves in $S^3 \times \mathbb{R}$. Moreover, this latter foliation defines some non-autonomous vector field V on S^3 such that its foliation $\mathcal{L}_V$ into ICs in $S^3 \times \mathbb{R}$ is uniformly equivalent to the foliation defined. This non-autonomous vector field is periodic in $t \in \mathbb{R}$ and is of the gradient-like type. Such f has wildly embedded separatrices, their wildness type is defined by the type of some knot constructed via the the one-dimensional separatrix near a sink of f.

Theorem 1. *There are NVFs of the gradient-like type which dynamically equivalent but are uniformly nonequivalent, if their types of wildness differ.*

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THE LINEAR INSTABILITY OF THE AKHMEDIEV BREATHER. EXPLICIT UNSTABLE SOLUTIONS AND REGULAR APPROACH h

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The Akhmediev breather (AB) and its M-soliton generalization, hereafter called AB_M, are exact solutions of the focusing NLS equation periodic in space and exponentially localized in time over the constant unstable background; they describe the appearance of M unstable nonlinear modes and their interaction. It is therefore important to establish the stability properties of these solutions under perturbations, to understand if they appear in nature, and in which form. There is the following common believe in the literature: let the NLS background be unstable with respect to the first N modes; then i) if the M unstable modes of the AB_M solution are strictly contained in this set ($M < N$), then the AB_M is unstable; ii) if they coincide with this set ($M = N$), the so-called "saturation of the instability", then the AB_M solution is neutrally stable. In this paper we argue instead that the AB_M solution is always unstable, even in the saturation case $M = N$, and we prove it in the simplest case $M = N = 1$. We prove the linear instability, constructing two examples of x-periodic solutions of the linearized theory growing exponentially in time. Moreover, these solutions grow in both time directions, and using this fact one can explain why

small perturbations near the peak of the breather affects the AB recurrence stronger than the same perturbations of the background (this effect was observed in numerical simulations). In the know literature these solutions were missed.

We also explain how to derive these solutions using the technique developed by Krichever for the KP equation.

A SEMICLASSICAL WKB PROBLEM FOR THE DIRAC OPERAROR WITH A DECAYING POTENTIAL

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We shall be interested in the semiclassical behavior of the *scattering data* of a non-self-adjoint *Dirac operator* with a fairly smooth -but not necessarily analytic- potential decaying at infinity. In particular, using ideas and methods going back to Langer and Olver, we provide a rigorous semiclassical analysis of the *Bohr-Sommerfeld condition* for the location of the eigenvalues, their corresponding *norming constants* and the *scattering coefficients*.

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MOLECULES: THE ROLE OF NONLINEAR DYNAMICS IN BIOLOGICAL MECHANISMS

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By simulating the dynamics of DNA through the well-established mesoscale, nonlinear, Peyrard-Bishop-Dauxois (PBD) model [1], we investigate properties of thermal openings (bubbles) occurring in the double strand. Using the properties of the fitted analytical potentials we establish some important physical properties, including a threshold value for considering a base pair to be separated [2]. Interplay between disorder and the nonlinearity of the potential and inter-base-pair coupling results in longer-lived openings near regions of biologically significant sequences responsible for biological transcription [3]. We study the probabilities and lifetimes of these openings, understood to be a key part in DNA functionality, building both a general understanding of the dynamics of arbitrary sequences and particular viral and bacterial DNA sequences [2,4].

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HERON TRIANGLES WITH TWO RATIONAL MEDIANS AND SOMOS-5 SEQUENCES

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Triangles with integer length sides and integer area are known as Heron triangles. Taking rescaling freedom into account, one can apply the same name when all sides and the area are rational numbers. A perfect triangle is a Heron triangle with all three medians being rational, and it is a longstanding conjecture that no such triangle exists. However, Buchholz and Rathbun showed that there are infinitely many Heron triangles with two rational medians, an infinite subset of which are associated with rational points on an elliptic curve $E(\mathbb{Q})$ with Mordell-Weil group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and they observed a connection with a pair of Somos-5 sequences. Here we make the latter connection more precise by providing explicit formulae for the integer side lengths, the two rational medians, and the area in this infinite family of Heron triangles. The proof relies on elliptic function identities and the real dynamics of certain integrable mappings of the plane (QRT maps), among other things.

HAMILTONIAN DYNAMICS IN ALCUBIERRE AND GÖDEL METRICS: RECURSION OPERATORS AND UNDERLYING MASTER SYMMETRIES

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We study the Hamiltonian dynamics in the background of Alcubierre and Gödel metrics. We derive the Hamiltonian vector fields governing the system evolution, construct and discuss related recursion operators generating the constants of motion. Besides, we characterize relevant master symmetries.

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ALGEBRA AND GEOMETRY OF BÄCKLUND TRANSFORMATIONS FOR (1+1)-DIMENSIONAL PARTIAL DIFFERENTIAL AND DIFFERENTIAL-DIFFERENCE EQUATIONS

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Bäcklund transformations (BTs) are well known to be a powerful tool for constructing exact solutions for integrable nonlinear partial differential and difference equations, including soliton solutions. We present a review of recent results on algebraic and geometric methods in the theory of BTs for (1+1)-dimensional partial differential and differential-difference equations. The main tools in the methods are zero-curvature representations (Lax representations), gauge transformations, jet spaces (jet bundles), Lie algebras (including infinite-dimensional ones), Lie groups, and their actions on manifolds.

The main topics are the following:

1. Algebraic necessary conditions for existence of a Bäcklund transformation (BT) between two given (1+1)-dimensional evolution partial differential equations (PDEs). Here we consider the most general class of BTs, which are not necessarily of Miura type. The obtained necessary conditions allow us to prove non-existence of BTs between two given equations in many cases.

To obtain these conditions, for (1+1)-dimensional evolution PDEs we find a normal form for zero-curvature representations (ZCRs) with respect to the action of the group of local gauge transformations and define, for a given (1+1)-dimensional evolution PDE $\mathcal{E}$, a family of Lie algebras $F(\mathcal{E})$ whose representations classify all ZCRs of the equation $\mathcal{E}$ up to local gauge transformations. Furthermore, these Lie algebras allow us to prove non-existence of any nontrivial ZCRs for some classes of PDEs.

In our approach, ZCRs may depend on partial derivatives of arbitrary order, which may be higher than the order of the PDE. The algebras $F(\mathcal{E})$ are defined in terms of generators and relations and generalize Wahlquist–Estabrook prolongation Lie algebras, which are responsible for a much smaller class of ZCRs.

The structure of the Lie algebras $F(\mathcal{E})$ has been studied for some classes of (1+1)-dimensional evolution PDEs of orders 2, 3, 5, which include Korteweg–de Vries (KdV), modified KdV, Krichever–Novikov, Kaup–Kupershmidt, Sawada–Kotera, nonlinear Schrödinger, and (multicomponent) Landau–Lifshitz type equations. Among the obtained algebras one finds infinite-dimensional subalgebras of Kac–Moody algebras and infinite-dimensional Lie algebras of certain matrix-valued functions on some algebraic curves.

2. A method to construct BTs of Miura type (differential substitutions) for (1+1)-dimensional evolution PDEs, using zero-curvature representations and actions of Wahlquist–Estabrook prolongation Lie algebras. Our method is a generalization of a result of V.G. Drinfeld and V.V. Sokolov [6] on BTs of Miura type for the KdV equation.

3. A method to construct BTs of Miura type for differential-difference (lattice) equations, using Lie group actions associated with Darboux–Lax representations of such equations. The considered examples include Volterra, Narita–Itoh–Bogoyavlensky, Toda, and Adler–Postnikov lattices. Applying our method to these examples, we obtain new integrable nonlinear differential-difference equations connected with these lattices by BTs of Miura type.

Some results of the talk are based on joint works with G. Manno [1–3], with G. Berkeley [4], as well as on the paper [5]. In the study of Bäcklund transformations for PDEs we use the theory of coverings of PDEs developed by A.M. Vinogradov and I.S. Krasilshchik [7, 8].

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DEPENDENCE OF DYNAMICS OF THE MODEL OF COUPLED OSCILLATORS ON THE NUMBER OF OSCILLATORS

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Consider a system of N $(N \geq 4)$ nonlinear differential equations with delay

$$\begin{gathered}\dot{u}_j + u_j = \lambda F(u_j(t-T)) + \gamma(u_{j-1} - 2u_j + u_{j+1}), \quad (j = 1, \ldots, N)\\ u_0 = u_N, \quad u_{N+1} = u_1.\end{gathered} \tag{1}$$

Here u_j are real functions, F is a piecewise smooth bounded compactly supported function (that is, $F(x) \equiv 0$ for $|x| \geq p$, where p is some positive constant), the value of the delay T is a positive constant, the nonzero coupling parameter γ satisfies the inequality $\gamma > -\frac{1}{4}$, the positive parameter λ is large enough ($\lambda \gg 1$). System (1) simulates a ring of oscillators with delayed feedback.

We select a special set of initial conditions in the phase space $C([-T, 0]; \mathbb{R}^N)$ and construct the asymptotics of all solutions of system (1) with initial conditions from this set. As a result, we find that for positive values of the parameter γ, starting from a certain moment in time, all oscillators are synchronized. For negative values of $\gamma > -\frac{1}{4}$, for an even number of oscillators, two-cluster synchronization is observed, and for an odd number of oscillators, both relaxation cycles and irregular oscillations are observed.

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THE LOCAL DYNAMICS OF SINGULAR PERTURBED EQUATIONS WITH DISTRIBUTED DELAY

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Consider differential equation with distributed over the interval $[-T, 0]$ delay:

$$\frac{dx}{dt} + x = \int\limits_{-T}^{0} x(t+s)\, dr(s) + f(x). \qquad (1)$$

Here $T > 0$, $f(x)$ is nonlinear sufficiently smooth function, such that: $f(x) = f_2 x^2 + f_3 x^3 + \dots$.

Let study the local (in the neighbourhood of zero equilibrium) dynamics of such equations in different situations. First, let delay is exponentially distributed:

$$\frac{dr}{ds} = R(s) = a\delta \exp(-\delta(s+T)), \quad \delta > 0.$$

Second, disribution is linear:

$$\frac{dr}{ds} = R(s) = a + \frac{bs}{T}.$$

And in the third case consider periodically distributed delay

$$\frac{dr}{ds} = R(s) = a\cos(\omega s + \varphi) + b.$$

The main assumption of the work is delay is sufficiently large $T \gg 1$. This condition makes possible using of various asymptotic methods.

In the bifurcation cases special nonlinear equations are constructed - normal and quasinormal forms - which do not depend on a small parameter or depend on it regularly. Their solutions determine the main parts of the asymptotic expansion of solutions of (1) [1].

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IRREGULAR SOLUTIONS IN FERMI–PASTA–ULAM PROBLEM

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A so-called irregular solutions of the spatially distributed Fermi–Pasta–Ulam model are considered. These solutions contain components that oscillate rapidly with respect to spatial and temporal variables. The systems of nonlinear equations of the Schrödinger type are constructed which describe the amplitudes of above solutions. It is shown that the constructed system is split into the independent Schrödinger equations when describing the waves moving in opposite directions. The asymptotic expansions of the irregular solutions are given.

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ON NON-ABELIAN QUADRIRATIONAL YANG-BAXTER MAPS

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We introduce four lists of families of non-abelian quadrirational Yang-Baxter maps.

TOPOLOGY OF TYPICAL CORANK-1 SINGULARITIES FOR INTEGRABLE SYSTEMS WITH 3 DEGREES OF FREEDOM

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For $s \in \{1, 2, 4, 5, 6\}$ consider the action $z \to e^{2\pi i/s} z$ of the group $\mathbb{Z}_s$ on $\mathbb{R}^2$ with a coordinate $z = x + iy$ and the $\mathbb{Z}_s$-invariant germs at 0 of the following functions:

$$F_s(z, \vec{a}, \vec{\lambda}) = \begin{cases} \pm x^2 \pm y^2 - \lambda_2 y^4 + \lambda_1 y, & s = 1, \\ \pm x^2 \pm y^6 + \lambda_2 y^4 + \lambda_1 y^2, & s = 2, \\ Re(z^4) + (1+\lambda_2)|z|^4 + a_1 \delta |z|^6 + \lambda_1 |z|^2, & s = 4, \\ a_1 > 0,\ \delta = \pm 1, & \\ Re(z^5) \pm a_1 |z|^6 + \lambda_2 |z|^4 + \lambda_1 |z|^2, & s = 5, \\ a_1 > 0, & \\ Re(z^6) + a_1 |z|^6 \pm a_2 |z|^8 + \lambda_2 |z|^4 + \lambda_1 |z|^2, & s = 6, \\ a_1^2 \neq 1,\ a_1 a_2 \neq 0. & \end{cases}$$

Here $\vec{a} = (a_1, \dots)$ are continuous moduli and $\vec{\lambda} = (\lambda_1, \lambda_2)$ are parameters. Thus we have 2-parameter families of functions $f_s(x, y) = F_s(z, \vec{a}, \vec{\lambda})$ with parameters λ_1, λ_2, cf. [1]. Bifurcations of their phase portraits on $\mathbb{R}^2$ correspond to some corank-1 singular orbits of integrable Hamiltonian systems with 3 degrees of freedom. Integral values correspond to values of the function f_s and parameters λ_1, λ_2, cf. [2] for more details.

Bifurcations in the cases $s = 1, 2$ were studied by L.M. Lerman (see [3] for the case $s = 1$) and later by the authors. In the talk we will describe the case $s = 4$. In dependence on the sign δ before $|z|^4$, we denote f_4 by $f_4^\delta = f_4^\pm$.

Our results are as follows. Bifurcation of the phase portrait of the 2-parameter family of functions $f_4^\delta(x, y)$ in a small neighbourhood $U(0, 0) \subset \mathbb{R}^2(\lambda_1, \lambda_2)$ is described for both $\delta = \pm 1$ and they are different. More specifically, we obtain the following:

- Phase portraits of $f_4^\delta(x,y)$ are described for every $(\lambda_1,\lambda_2)\in U$. Nondegeneracy of critical points is studied.

- The bifurcation diagram Σ in $\mathbb{R}^3(f,\lambda_1,\lambda_2)$ and bifurcation complex $\hat{\Sigma}$ are described for $(\lambda_1,\lambda_2)\in U$ and $f=f_4^\delta$. Sections of Σ and $\hat{\Sigma}$ by the planes $\lambda_2=\pm\varepsilon$ and $\lambda_2=0$ are described. Classes of fiberwise homeomorphism (Fomenko 3-atoms, see [4]) of non-splitting nondegenerate singularities are determined.

- The set U is stratified by strata of dimension 2, 1, 0: the phase portraits of $f_4^\delta(x,y)$ for (λ_1,λ_2) and (λ_1',λ_2') from the same stratum are fiberwise homeomorphic.

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PIECEWISE SMOOTH SOLUTIONS OF QUASI-NORMAL FORM IN THE SIMPLE CRITICAL CASE

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Consider the equation

$$\dot{\xi}=\xi-\xi^3+\gamma M(\xi).$$

with periodic boundary condition

$$\xi(t,x+2\pi)=\xi(t,x).$$

Here $\xi = \xi(t,x)$ is a piecewise smooth function of the variable x for each $t \geq 0$, γ is a constant, $M(\xi) = \frac{1}{2\pi}\int\limits_0^{2\pi} \xi(t,x)dx$.

This boundary value problem is the quasi-normal form of a system of spatially distributed differential equations in the case when the matrix of the linearized system has a simple zero eigenvalue.

In this research, we have proved the existence of two homogeneous equilibrium states and determined the condition of their stability.

The boundary value problem has solutions $\xi = \sqrt{1+\gamma}$ and $\xi = -\sqrt{1+\gamma}$. These solutions are stable if $\gamma > -\frac{2}{3}$.

We proved that this boundary value problem has a one-parameter family of solutions depending on the parameter $\alpha \in (0, 2\pi)$ in the form of step functions

$$\xi(t,x) = \begin{cases} 0, & x = 0, \\ \rho_1, & 0 < x < \alpha, \\ 0, & x = \alpha, \\ \rho_2, & \alpha < x < 2\pi, \\ 0, & x = 2\pi, \end{cases}$$

where ρ_1, ρ_2, α are the solutions of the system

$$\rho_j - \rho_j^3 + \frac{\gamma}{2\pi}(\rho_1\alpha + \rho_2(2\pi - \alpha)), \quad j = 1, 2.$$

The conditions for the stability of these solutions are determined.

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QUADRATIC IN MOMENTA INTEGRALS OF CIRCULATORY SYSTEMS

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The problem of existence of quadratic in velocities (momenta) conservation laws (first integrals) of circulatory systems is considered for the case of nonpotential external forces. Under some assumptions, the equations of motion can be presented in a Hamiltonian form with some symplectic structure and with the Hamiltonian function given by the quadratic integral. In some cases the equations can be presented not in a Hamiltonian but in a conformally Hamiltonian form. The existence of the quadratic integral and its properties allow us to study the stability of equilibria of circulatory systems.

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TOPOLOGICAL ANALYSIS OF MAGNETIC GEODESIC FLOW PROBLEM

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Let $M \simeq S^2$ be a two-dimensional manifold, such that the metric on M is $O(2)$–invariant. As known, this action has only two fixed points (poles), N and S, on M. One can show that this metric has the form $ds^2 = dr^2 + f^2(r)d\varphi^2$ where $f : [0, L] \to \mathbb{R}$ is a smooth function, L is the length of a geodesic between N ans S, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.

Consider the motion of a charged particle on M in the magnetic field given by a 2-form $\Lambda'(r)dr \wedge d\varphi$ where $\Lambda : [0, L] \to \mathbb{R}$ is a smooth function. We will call this system a *magnetic geodesic flow*. It is determined by the pair of functions $(f(r), \Lambda(r))$.

Suppose that $f(r)$, $\Lambda(r)$ satisfy the following conditions:

- $f(r)$ has a smooth odd $2L$-periodic extension to $\mathbb{R}$;
- $\Lambda(r)$ has a smooth even $2L$-periodic extension to $\mathbb{R}$;
- $f(r)$ and $\Lambda(r)$ are Morse functions;
- $f'(0) = 1$, $f'(L) = -1$;
- $(\Lambda'(r))^2 + (f'(r))^2 > 0$.

Introducing the canonical coordinates $(p_r,\, k,\, r,\, \varphi)$ on T^*M, one can prove that this system is Liouville integrable with first integrals $H = \dfrac{p_r^2}{2} + \dfrac{(k-\Lambda(r))^2}{2f^2(r)}$ and $F = k$.

Then the topological analysis of this integrable system is possible. The best tool for it are the Fomenko and the Fomenko–Zieschang invariants. A complete description of this theory is given in [1].

Under the assumption that f and Λ satisfy the conditions from above, we can formulate three theorems.

Theorem 1 (Singularities of ranks 0 and 1)**.**

1. The system has only two singular points of rank 0, namely $(0,\, N)$ *and* $(0,\, S)$. *Their images under the momentum map* $\mathcal{F} = (H,\, F) : T^*M \to \mathbb{R}^2$ *are the points* $(0,\, \Lambda(0))$ *and* $(0,\, \Lambda(L))$. *They are non-degenerate if and only if* $\Lambda''(0) \neq 0$ *and* $\Lambda''(L) \neq 0$, *resp. If this holds then the singular points have the center-center type.*

2. Singular rank-1 points form two families: $\{(0,\, k(r),\, r,\, \varphi) \mid r \in I,\ \varphi \in \mathbb{R}/2\pi\mathbb{Z}\} =: K_1^1$ *where* $k(r) = \Lambda(r) - f(r)\dfrac{\Lambda'(r)}{f'(r)}$, I *is the interval* $(0,\, L)$ *without critical points of* f, *and* $\{(0,\, \Lambda(r),\, r,\, \varphi) \mid r \in (0,\, L),\ \varphi \in \mathbb{R}/2\pi\mathbb{Z}\} =: K_2^1$.

The image of K_1^1 *under* $\mathcal{F}$ *is the curve* $\gamma_1 = \{(h(r),\, k(r)) \mid r \in I\}$, $h(r) = \dfrac{1}{2}\left(\dfrac{\Lambda'}{f'}\right)^2$; *the image of* K_2^1 *under* $\mathcal{F}$ *is the segment* $\gamma_2 = \{(0, \Lambda(r)) \mid r \in (0,\, L)\}$.

Theorem 2. *Suppose* $k(r)$ *is a Morse function on each interval of its domain. Then for any* $r \in (0,\, L)$ *such that* $f'(r) \neq 0$ *the following conditions are equivalent:*

(i) *the critical orbit* $\{(p_r,\, k,\, r,\, \varphi) = (0,\, k(r),\, r,\, \varphi) \mid \varphi \in \mathbb{R}/2\pi\mathbb{Z}\}$ *is non-degenerate,*

(ii) $\dfrac{\partial^2 U_{k(r)}(r)}{\partial r^2} \neq 0$ *where* $U_k(r) = \dfrac{(k - \Lambda(r))^2}{2f^2(r)}$,

(iii) $k'(r)\Lambda'(r) \neq 0$.

If $\dfrac{\partial^2 U_{k(r)}(r)}{\partial r^2} > 0$ *then the type of the critical orbit is elliptic, if* $\dfrac{\partial^2 U_{k(r)}(r)}{\partial r^2} < 0$ *then the type of the critical orbit is hyperbolic.*

Degenerate critical orbits $\{(p_r,\, k,\, r,\, \varphi) = (0,\, k(r),\, r,\, \varphi) \mid \varphi \in \mathbb{R}/2\pi\mathbb{Z}\}$ *from the family* K_1^1 *are described by the following cases:*

- *a parabolic orbit (corresponding to a cusp of* γ_1*, i.e. to* r *such that* $k'(r) = 0$*),*

- *a critical orbit (corresponding to a tangency of* γ_1 *and* γ_2*, i.e. to* r *such that* $\Lambda'(r) = 0$*) whose neighborhood is homeomorphic to the elliptic fork singularity.*

Theorem 3. *The isoenergy manifold* $Q_h^3 = \{H = h\} \subset T^*M$ *is non-singular if and only if* $h \neq 0$. *Any non-singular isoenergy manifold* Q_h^3 *is diffeomorphic to* $\mathbb{R}P^3$. *If* $h > 0$ *does not coincide with the values of* H *at cusps of* γ_1 *then* $F|_{Q_h^3}$ *is a Bott function and the following holds:*

(1) *The Fomenko graph for such a Bott function* $F|_{Q_h^3}$ *is a tree with elliptic "3-atoms"* A *at vertices of degree 1 and saddle 3-atoms* $V_{\eta_1 \ldots \eta_k} \times S^1$ *at the other vertices. The singular fibre of the "2-atom"* $V_{\eta_1 \ldots \eta_k}$ *looks like*  *(see [1] for general definitions of the Fomenko graph and atoms, and [3] for the definition of the 2-atom* $V_{\eta_1 \ldots \eta_k}$*.)*

(2) *The Fomenko-Zieschang topological invariant of the Liouville foliation on* Q_h^3 *is given by this Fomenko graph with marks* $r = 0$, $\varepsilon = 1$ *on edges* $A - V$, *marks* $r = \infty$, $\varepsilon = 1$ *on edges* $V_1 - V_2$. *All saddle atoms form a unique "family" with mark* $n = 2$ *(see [1] for definitions of the Fomenko-Zieschang invariant and a family).*

(3) *If the Fomenko graph is* $A - A$ *then marks on its edge are* $r = 1/2$, $\varepsilon = 1$.

Remark 1. The system has only a potentional field in [2], and it has both magnetic and potentional fields in [3]. So, Theorem 1 and the part of Theorem 2 about characterizing elliptic and hyperbolic singular orbits in fact follow from [3; Propositions 1 and 2 (A)]. But a magnetic field alone (without a potential field) brings some new remarkable properties (e.g. presence of elliptic forks) to the system, so that the rest of Theorem 2 doesn't follow directly from results of [3].

Example. Taking $f(r) = \frac{1}{2\pi}(\sin \pi r + \frac{1}{3}\sin 3\pi r)$, $\Lambda(r) = \frac{1}{8\pi}(\cos \pi r - \cos 3\pi r)$ and $L = h = 1$, we can obtain the Fomenko-Zieschang invariant which looks like

$$\begin{array}{c} A \xrightarrow{r=0,\ \varepsilon=1} \\ A \xrightarrow[r=0,\ \varepsilon=1]{} \end{array} B \xrightarrow[r=\infty,\ \varepsilon=1]{n=2} B \begin{array}{c} \xrightarrow{r=0,\ \varepsilon=1} A \\ \xrightarrow[r=0,\ \varepsilon=1]{} A \end{array}$$

Here the label $n = 2$ is assigned to the 'family' $B - B$.

It coincides with one of the possible Fomenko-Zieschang invariants in the Jukovsky case of the motion of a rigid body. This means that these systems are Liouville equivalent but the magnetic system is much simpler than the Jukovsky case.

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SYMPLECTIC CLASSIFICATION OF STRUCTURALLY STABLE NONDEGENERATE SEMILOCAL SINGULARITIES OF INTEGRABLE SYSTEMS

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Some of the results are obtained in a collaboration with A. Oshemkov.

An integrable Hamiltonian system with n degrees of freedom is given by n functionally independent functions $f_1, \dots, f_n : M \to \mathbb{R}$ pairwise in involution on a symplectic $2n$-manifold (M, ω).

Consider the Hamiltonian $\mathbb{R}^n$-action on M generated by the momentum map $F = (f_1, \dots, f_n) : M^{2n} \to \mathbb{R}^n$. We will call orbits of this action simply *orbits*. Consider the *singular Lagrangian fibration* associated to the integrable Hamiltonian system, whose fibers are connected components of the level sets $F^{-1}(a)$, $a \in \mathbb{R}^n$.

By a *local* (resp. *semilocal*) *singularity* of such a singular fibration, we mean the fibration germ at a singular orbit (resp. fiber).

Definition. Two singularities will be called *equivalent* if there exists a fiberwise homeomorphism of fibration germs at these singularities.

By the Vey theorem [4], in real-analytic case, for each nondegenerate singular point $m_0 \in M$, there exists a real-analytic symplectomorphism

$$\phi = (\lambda_1, \varphi_1, \dots, \lambda_r, \varphi_r, x_1, y_1, \dots, x_{n-r}, y_{n-r}) : (U, \omega) \hookrightarrow (\mathbb{R}^{2n}, \omega_{can})$$

of a neighbourhood U of m_0 and a real-analytic diffeomorphism germ $J = (J_1, \dots, J_n) : (\mathbb{R}^n, F(m_0)) \to (\mathbb{R}^n, 0)$ such that $\phi(m_0) = 0$ and $J \circ F \circ \phi^{-1} = (h_1, \dots, h_n)$ where

$$\begin{array}{ll}
h_s = \lambda_s & \text{for } 1 \le s \le r, \\
h_{r+j} = \frac{1}{2}(x_j^2 + y_j^2) & \text{for } 1 \le j \le k_e, \\
h_{r+j} = x_j y_j & \text{for } k_e + 1 \le j \le k_e + k_h, \\
h_{r+j} = x_j y_j + x_{j+1} y_{j+1} & \text{and} \\
h_{r+j+1} = x_{j+1} y_j - y_{j+1} x_j & \text{for } j = k_e + k_h + 2i - 1,\ 1 \le i \le k_f,
\end{array}$$

$$\omega_{can} = \sum_{s=1}^{r} \mathrm{d}\lambda_s \wedge \mathrm{d}\varphi_s + \sum_{j=1}^{n-r} \mathrm{d}x_j \wedge \mathrm{d}y_j.$$

We introduce *connectedness condition* for nondegenerate semilocal singularities.

Theorem 1 (Semilocal structural stability test)**.** *Suppose a real-analytic integrable system has a compact nondegenerate singular fiber L satisfying the connectedness condition and the non-splitting condition [6; Def. 6.3] (e.g. contains a unique compact orbit). Then the semilocal singularity at the fiber L is structurally stable under real-analytic integrable perturbations (but not necessarily under C^∞ integrable perturbations).*

Our proof of Theorem 1 uses the topological classification by N.T. Zung [6].

As an illustration, we prove that a saddle-saddle singularity of the Kovalevskaya top is structurally stable under real-analytic integrable perturbations, but structurally unstable under smooth integrable perturbations.

Combining ideas of [1], [7; Def. 4.2] and [5; Def. 3.1], we introduce the *Lagrange-Vey class* of a real-analytic semilocal singularity at L (w.r.t. a given reference singularity at L°), which is a 1-cocycle

$$\beta_L \in H^1(L^\circ/(S^1)^{r+k_e+k_f}, Z^1) \cong Hom(H_1(L^\circ/(S^1)^{r+k_e+k_f}), Z^1) \qquad (*)$$

with coefficients in the group Z^1 of converging (near the origin) power series in n variables with real coefficients, vanishing at the origin. Here we assume that the singularity at L is fibrewise homeomorphic to a given singularity at a singular fibre L° of a reference system, and the $(S^1)^{r+k_e+k_f}$-action near L° is generated by the Vey functions $J_s^\circ \circ F^\circ$, $1 \le s \le r + k_e$, and $J^\circ_{r+k_e+k_h+2j} \circ F^\circ$, $1 \le j \le k_f$, at a compact orbit in L°.

The following theorem extends (the real-analytic versions of) the semilocal symplectic classifications of simple Morse functions on compact symplectic surfaces [1] and semitoric systems [2, 3].

Theorem 2 (Semilocal symplectic classification)**.** *Suppose (M_k, ω_k, F_k), $k = 1, 2$, are two integrable real-analytic Hamiltonian systems, and $L_k \subset M_k$ are compact nondegenerate singular fibers satisfying the conditions of Theorem 1. Suppose $U(L_k)$ is a small neighbourhood of L_k in M_k, $k = 1, 2$, and $\psi_k : U(L_k) \to U(L^\circ)$ is a fiberwise homeomorphism being a symplectomorphism near a com-*

pact orbit in L_k*,* $\psi_k(L_k) = L^\circ$*. Then the following conditions are equivalent:*

(i) *there exist neighbourhoods* $U_1(L_k) \subseteq U(L_k)$ *of* L_k*,* $k = 1, 2$*, and a real-analytic fiberwise symplectomorphism* $\psi : U_1(L_1) \to U_1(L_2)$ *isotopic to* $\psi_0 = \psi_2^{-1} \circ \psi_1$ *in the space of fiberwise homeomorphisms;*

(ii) $\beta_{L_2} = \beta_{L_1}$*, where* $\beta_{L_k} \in H^1(L^\circ/(S^1)^{r+k_e+k_f}, Z^1)$ *denotes the Lagrange-Vey class* $(*)$ *of the semilocal singularity at* L_k *w.r.t. the reference singularity at* L°*.*

For each 1-cocycle $[\beta] \in H^1(L^\circ/(S^1)^{r+k_e+k_f}, Z^1)$*, there exists a semilocal singularity* L_1 *topologically equivalent to* L°*, whose Lagrange-Vey class is* $[\beta]$*.*

Theorem 3 (On action variables)**.** *Under the hypothesis of Theorem 2, suppose that there exists a real-analytic diffeomorphism germ* $G : (\mathbb{R}^n, F_1(L_1)) \to (\mathbb{R}^n, F_2(L_2))$ *such that* $G \circ F_1 = F_2 \circ \psi_0$ *(thus,* ψ_0 *is a lift of* G *realizing an equivalence of the singularities at* L_1 *and* L_2*). Suppose that there exists a finite-sheeted covering of a neighbourhood of* L_1 *that is fibrewise homeomorphic to the direct product of several regular, elliptic, hyperbolic and focus-focus semilocal singularities of dimensions* 2, 2, 2 *and* 4*, resp., and all hyperbolic components (2-atoms) of this decomposition have genus 0. Then the following conditions are equivalent:*

(i) *there exists a real-analytic fiberwise symplectomorphism* ψ_1 *which can be joined to* ψ_0 *by a continuous isotopy* ψ_t *such that* $G \circ F_1 = F_2 \circ \psi_t$, $0 \le t \le 1$*;*

(ii) *the map* ψ_0 *preserves the action variables.*

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ON TWO GENERALIZED VARIANTS OF A WEAKLY DISSIPATIVE VERSION OF THE COMPLEX GINZBURG-LANDAU EQUATION

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An equation of the form

$$u_t = u - (1 + ic)u|u|^2 + (a + ib)\Delta u \tag{1}$$

is usually called the complex Ginzburg-Landau equation [1]. In the equation (1) $u = u(t, x_1, \ldots, x_n), c, b \in \mathbb{R}, a \geq 0, t \geq 0, x_j \in \mathbb{R}, j = 1, \ldots, n$. This equation deserves particular attention if $a = 0$. The corresponding applications are also related to nonlinear optics, hydrodynamics [2,3]. Equation (1) is usually called either the weakly dissipative version of the Ginzburg-Landau equation, or the generalized cubic Schrodinger equation. Note that the weakly dissipative version of equation (1) can be included in the class of abstract hyperbolic equations. For $a > 0$ the equation (1) can be interpreted as an equation of parabolic type.

Instead of equation (1), one usually considers its various modifications, including its weakly dissipative version.

In the report is supposed to discuss the following two equations

$$u_t = \alpha u - (d + ic)u|u|^2 - (f + ih)u|u|^4 - ibu_{xx}, \tag{2}$$

and

$$u_t = \alpha u - (d + ic)uV(u) - (f + ih)uV^2(u) - ibu_{xx}. \tag{3}$$

Here $d, b, c, f, h \in \mathbb{R}$, $f > 0$, $d + f = 1$, $\alpha = \pm 1$, $V(u) = \frac{1}{2\pi} \int\limits_0^{2\pi} |u|^2 dx$, $u = u(t, x)$. Both equations are considered together with the boundary conditions

$$u(t, x + 2\pi) = u(t, x). \tag{4}$$

Equation (2) was proposed as a generalization of the standard cubic version of the complex Ginzburg-Landau equation (see, for example, [4]). Equation (3) has an independent interest and is called the nonlocal Ginzburg-Landau equation (see, for example, [5, 6]). It appeared in connection with the description of such a phenomenon as ferromagnetism.

For the boundary value problem (2), (4), the question of stability and local bifurcations of single-mode periodic solutions of the form

$$u_n(t, x) = \exp(i\sigma_n t) \exp(inx),\ \sigma_n = bn^2 - c - h \tag{5}$$

is studied.

In particular, all solutions (5) are stable if $b - 2c - 4h > 0$ and unstable if $b - 2c - 4h < 0$. For $b = b_* = 2c + 4h$ the critical case of the triple zero eigenvalue of the stability spectrum is realized. For $b = b_* - \gamma\varepsilon$, $\gamma = \pm 1$, $\varepsilon \in (0, \varepsilon_0)$, the local bifurcations of solutions (2) are studied. In a general situation, the question of the nature of bifurcations can be reduced to an analysis of the scalar equation $z' = \gamma z + az^3$, i.e. to a normal form corresponding to a fork bifurcation.

For the boundary value problem (3), (4), the existence of a global attractor is shown [7] A_∞ ($dim A_\infty = \infty$), distinguished by the condition $V(u) = 1$. In a general situation, all solutions belonging to the global attractor A_∞ with respect to variable t will be quasiperiodic functions, and the corresponding solutions can be represented by converging function series.

The boundary value problem (3), (4) in the case $f = h = 0$ was considered in the work [8].

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THE DYNAMICS NEAR A SYMMETRIC HOMOCLINIC ORBIT OF A SADDLE-FOCUS IN A REVERSIBLE SYSTEM

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Let M be a smooth (analytic or C^∞) four-dimensional manifold and $R : M \to M$ be a smooth involution, $R^2 = id_M$. A smooth vector field v on M is called to be reversible w.r.t. R, if the identity $DR(v) = -v \circ R$ holds. The set $Fix(R) = \{x \in M|R(x) = x\}$ is supposed to be a smooth two-dimensional sub-manifold. We assume v to have a symmetric equilibrium $p \in Fix(R)$ of the saddle-focus type (with a quadruple of eigenvalues $\pm\alpha \pm i\omega$) and its two-dimensional stable and unstable manifolds intersect each other along a homoclinic orbit Γ being symmetric ($R(\Gamma) = \Gamma$) and non-degenerate. We are interested in the orbit behavior in a neighborhood of Γ.

L.P. Shilnikov was the first who discovered the complicated orbit behavior near a homoclinic orbit to a saddle-focus with a positive saddle value in a 3-dimensional system [1], later this was extended on systems of greater dimension [2]. His results cannot be transferred directly to the reversible and Hamiltonian systems since their saddle values are always zero due to symmetry of the spectrum at the equilibrium. R.Devaney discovered a hyperbolic subset (a suspension over Bernoulli scheme) in a neighborhood of a homoclinic orbit for a Hamiltonian system [3] and found a one-parameter family of symmetric periodic orbits (POs) in a four-dimensional reversible system near a non-degenerate homoclinic orbit to a saddle-focus [4]. The complete orbit behavior on the degenerate level of Hamiltonian and bifurcations under varying the level of a Hamiltonian near a transverse homoclinic loop of a saddle-focus were described in [5].

Denote by $\mathcal{L}$ a Devaney's one-parameter family of symmetric periodic orbits accumulating at Γ. They are parameterized by the period $T \to \infty$ and their types alternate infinitely many times from quasi-hyperbolic to quasi-elliptic and back [4].

Let us consider, for T large enough, a piece of symmetric orientable hyperbolic orbits from $\mathcal{L}$ bounded by two parabolic POs with respectively triple unit multipliers at one end and a simple unit and double -1 multipliers on another end. Such piece always exists in the family. The continuation, as T increases, along $\mathcal{L}$ gives a neighboring piece of quasi-elliptic symmetric POs accompanied by a couple of hyperbolic periodic orbits interchanged by R. Here results of [6] can be applied.

Theorem 1. *For T large enough every branch of symmetric quasi-elliptic POs contains majority of non-degenerate such orbit around which a positive Lebesque measure set of invariant tori exists.*

When moving further along $\mathcal{L}$, increasing T, we come to the neighboring family of symmetric non-orientable POs, then again increasing T to symmetric orientable POs orbits, and so forth. The family of quasi-hyperbolic symmetric POs forms an invariant smooth annulus, being a hyperbolic set, and therefore it has three-dimensional stable and unstable manifolds. The same is true for the families of non-orientable symmetric POs. Consider two nearest pieces of quasi-hyperbolic PO, one is from orientable POs K_1 and another one is from non-orientable POs K_2. Their stable and unstable manifolds intersect each other giving way to the existence of heteroclinic connections between these two families of POs. Moreover, the following assertion holds.

Theorem 2. *For T large enough there are two sub-annulae, each from the separate annulae K_1, K_2, such that every PO γ_1 from one sub-annulus has a heteroclinic orbit $\Gamma_1 = W^u(\gamma_1) \cap W^s(\gamma_2)$ to a PO γ_2 from another sub-annulus and a heteroclinic orbit $\Gamma_2 = W^s(\gamma_1) \cap W^u(\gamma_2)$, giving thus a one-parameter family of symmetric heteroclinic contours. Such a structure generate an invariant subsets being a lamination with one-dimensional leaves over a Markov chain.*

These latter results remind those obtained in [7] near a symmetric homoclinic orbit to a sheet of symmetric POs.

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EXISTENCE OF A STABLE STATIONARY SOLUTION WITH A TWO-SCALE TRANSITION LAYER OF A SYSTEM OF TWO DIFFUSION EQUATIONS WITH QUASIMONOTONICITY CONDITIONS OF DIFFERENT SIGNS

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We consider the question of the conditions for the existence of a stable stationary solution for the system of parabolic equations

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \varepsilon^4 \frac{\partial^2 u}{\partial x^2} = f(u, v, x, \varepsilon), \ \ 0 < x < 1, \ t > 0, \\
&\frac{\partial v}{\partial t} - \varepsilon^2 \frac{\partial^2 v}{\partial x^2} = g(u, v, x, \varepsilon), \ \ 0 < x < 1, \ t > 0, \\
&\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0, \ \ \left.\frac{\partial v}{\partial x}\right|_{x=0} = \left.\frac{\partial v}{\partial x}\right|_{x=1} = 0, \ \ t > 0, \\
&u(x, 0) = u^0(x), \ \ v(x, 0) = v^0(x), \ \ 0 \le x \le 1;
\end{aligned}
$$

where ε is a small parameter, f and g are functions of class C^4.

The main feature of the system is the fulfillment of the inequalities

$$
f_v(u, v, x, \varepsilon) < 0, \ \ g_u(u, v, x, \varepsilon) > 0
$$

in the domain these functions are defined. These inequalities are called "the condition of quasimonotonicity".

Such inequalities are characteristic for the systems of activator-inhibitor type. Similar problem war considered in paper [1] with another quasimonotonicity condition. The existence and stability theorems were proved using the method of the upper and lower solutions [2].

In this paper, due to the quasimonotonicity conditions the algorithm for constructing the upper and lower solutions is more complicated in comparison with the previous work.

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GENERALIZED MISHCHENKO–FOMENKO CONJECTURE FOR LIE ALGEBRAS OF SMALL DIMENSIONS

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Let g be a Lie algebra over an algebraically closed field $\mathfrak{K}$. Accordingly, g^*- is the dual space of the Lie algebra g. Consider at g^* structure which is called the Lie-Poisson bracket:

$$\mathcal{A}_x(x) = (c_{ij}^k x_k),\ x \in g^*.$$

This tensor defines the Lie-Poisson bracket on $C^\infty(g^*)$. The functions $f \in C^\infty$, lying in the kernel of the Lie-Poisson bracket are called Casimir functions.
It is also possible to consider a similar structure which is called a bracket with a frozen argument:

$$\mathcal{A}_a(x) = (c_{ij}^k a_k),\ a, x \in g^*.$$

A Lie algebra is called completely integrable if over this Lie algebra there is a complete set of functions that are in involution. A complete set is considered to contain n functionally independent functions, where n is defined by the formula:

$$n = \frac{1}{2}(dim\ g + ind\ g).$$

The greatest practical interest is sets of polinomials.

- Mishchenko-Fomenko’s conjecture: on the dual space g^* of any Lie algebra g, there is a complete set of polynomials that are in involution.
- Generalized Mishchenko-Fomenko’s conjecture: on the dual space g^* of any Lie algebra g there is a complete set of polynomials in bi-involution, i.e, a set that is simultaneously in involution with respect to $\mathcal{A}_x$ and $\mathcal{A}_a$.

The first hypothesis was proved by Sadetov in 2004, but the sets obtained by him did not always turn out to be in involution with respect to the bracket with a frozen argument. It is worth noting that when applying the general method for constructing sets in bi-involution (Mischenko-Fomenko’s argument shift method), firstly, the resulting sets are not complete for all Lie algebras (as, for example, for semisimple Lie algebras), secondly, not for all values of the parameter a, set of functions are functionally independent.
This report will tell about the verification of the generalized Mishchenko-Fomenko conjecture for nilpotent Lie algebras of low dimension ($dim\ g = 7$), solvable ($dim\ g < 7$). For all such Lie algebras, complete sets of polynomials in bi-involution for all parameters a were constructed by various methods. Parameters for which the argument shift method revealed functionally dependent sets were studied, and other complete sets in bi-involution were provided in these parameters.

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SOLVING INVERSE PROBLEMS FOR NONLINEAR EQUATIONS OF THE REACTION-DIFFUSION-ADVECTION TYPE WITH DATA ON THE POSITION OF A REACTION FRONT

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Problems for nonlinear singularly perturbed reaction-diffusion-advection equations arise in gas dynamics, combustion theory, chemical kinetics, biophysics, medicine, ecology, finance and other fields of science. A specific feature of problems of this type is the presence of processes of different scales. Therefore, the mathematical models of these problems are described by nonlinear parabolic equations with a small parameter at the highest derivative. In this regard, solutions to these problems may contain narrow moving fronts that divide the space into two parts: the disturbed part, through which the front has already passed, and the undisturbed part. The front is a region in which the function describing some characteristic of the medium (temperature, density, etc.) changes quite sharply from the values of the function describing one state of the medium (for example, undisturbed) to the value of the function describing another state. If there is a small parameter with the highest derivative, the width of such a front will be rather small in relation to the size of the entire region. As a consequence, the reaction front can be sometimes distinguished experimentally.

Some applied problems for equations of this type require solving inverse problems for recovering some coefficient in the equation. To formulate the inverse problem, additional information is required, which is usually measured in an experiment. Often, in the formulation of inverse problems for partial differential equations, additional information about the solution on a part of the boundary of the domain is used. However, one of the possible statements of inverse problems for equations of the type under consideration is a statement with additional information about the dynamics of the reaction front motion (see, for example, [1,2,3,4]). Additional data of this type are in demand in practice, since they are most easily to observe in an experiments (the front is an easily distinguishable contrast structure).

The simplest formulation of an inverse problem of this type relates to the case of recovering a function of an argument from experimental observations of a function of the same argument. For example, in [2], an approach was considered to restore the function of a temporary variable argument from the observation data of the function also of the variable argument in time. More complex formulations are formulations in which it is required to restore the function of one argument (for example, spatial) from the observation data of the function of another argument (for example, temporal) [1,3,4]. This class of inverse problems is considered recently — the recovering of the function of the argument of a spatial variable (that determines the properties of the medium) from the data of observations of the function of the argument of the time variable (that determined by the dynamics of the reaction front).

To effectively solve such inverse problems the methods of asymptotic analysis of V. F. Butuzov et al. can be used (sometimes!). In such cases, it is possible to reduce the original inverse problem for a nonlinear singularly perturbed partial differential equation to a much simpler problem with respect to the coefficient to be restored. The resulting simplified problem is called as a reduced statement (formulation) of the inverse problem. However, the reduced formulations of such inverse problems may have special features. It was shown that the reduced formulations can contain 1) algebraic equations for an unknown coefficient (see, for example, [2], 2) differential equations for an unknown coefficient (see, for example, [4], 3) integral equations for an unknown coefficient [1]. The first case is the simplest and allows to restore the unknown function only at those points through which the reaction front passed during its experimental observation. In this case, a pointwise recovering of the unknown coefficient is possible. The second case is more complicated, since additional input information is required for the correct formulation of the inverse problem being solved. The third case is the most difficult.

Often the methods of asymptotic analysis are inapplicable for solving inverse problems of this type (see, for example, [3]). In this case the methods based on minimizing the target functional by the gradient method are used to sole the inverse problem under consideration.

Thus, the features of numerical reconstruction of some coefficients in solving the coefficient inverse problem for a nonlinear singularly perturbed equation of the reaction-diffusion-advection type will be discussed.

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LOCAL DYNAMICS OF A SECOND-ORDER EQUATION WITH A DELAY AT THE DERIVATIVE

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Consider a second-order differential equation with delayed feedback, which is an implementation of the modified Ikeda equation with a time delay:

$$\varepsilon\frac{d^2y}{dt^2}+\frac{dy}{dt}+\delta y=F\left(\frac{dy}{dt}(t-\tau)\right). \tag{1}$$

Here ε and δ are small and proportional parameters $0<\varepsilon\ll 1$, $\delta=k\varepsilon$, τ is a delay parameter, real and positive. The function F is sufficiently smooth, such that $F(0)=0$. Thus, equation (1) has a zero equilibrium state. Let study the local dynamics in the vicinity of the equilibrium state,in the phase space $C^1_{[-1,0]}$. Note, that the problem under consideration is singularly perturbed.

The characteristic quasi-polynomial of the linearized at zero equation (1) has the form:

$$\varepsilon\lambda^2 + \lambda + k\varepsilon = \lambda\beta_1 e^{-\lambda}. \tag{2}$$

It is shown that for $|\beta_1| < 1$, the zero equilibrium state is stable, and for $|\beta_1| > 1$, it is unstable. In critical cases $\beta_1 = \pm 1$, the characteristic equation has an infinite number of roots tending to the imaginary axis at $\varepsilon \to 0$. Thus, the critical cases have infinite dimension.

To study the behavior of solutions in β_1 close to -1, the solution of problem (1) is reduced in the case under consideration to a partial differential equation (3) with boundary conditions (4):

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\frac{\partial^2 V}{\partial^2 t} + kV - \frac{k^2}{2}J^2(V) + \beta V + \beta_2 J\left(U_1\frac{\partial V}{\partial t}\right) + \beta_3 J\left(\left(\frac{\partial V}{\partial t}\right)^3\right), \tag{3}$$

$$\int\limits_0^1 V(\tau,t)dt = 0,\ V(\tau,t) \equiv -V(\tau,t+1). \tag{4}$$

Theorem 1. *Let $V_*(\tau,t)$ be a bounded with its derivatives solution (3) with boundary conditions (4), where $V_*(\tau,t) = \sum\limits_{-\infty}^{\infty} e^{\pi(2n+1)it}V_n(\tau)$, then*

$$y(t) = \varepsilon\sum_{-\infty}^{\infty} e^{iIm(\lambda_{n0}+\varepsilon\lambda_{n1}+\varepsilon^2\lambda_{n2}+\ldots)t}V_n(\varepsilon^2 t) + \varepsilon\frac{\beta_2}{k}\int\limits_0^1 \dot{V}^2(t,\varepsilon^2 t)dt,$$

is asymptotic in the residual with an accuracy of $O(\varepsilon^3)$ uniformly over $t \geq 0$ by the solution of (1).

Problem (3), (4) is an analog of normal form. It's solution determine main terry of asymptotic approximate for solution of (1). To study the behavior of solutions in the case of a close $\beta_1 = 1$, the solution of the problem (1) is reduced in the case under consideration to the differential equation (5) and the partial differential equation (6) with boundary conditions (7):

$$W^{\tau}(\tau) = (\varepsilon\beta - \frac{k}{4})W(\tau) - \frac{2}{3}\beta_2^2\sqrt{k}iW^2(\tau)\overline{W}(\tau) + \frac{3}{2}\beta_3\varepsilon^{\frac{1}{2}}kW^2(\tau)\overline{W}(\tau), \tag{5}$$

$$\frac{\partial V}{\partial \eta} = \frac{1}{2}\frac{\partial^2 V}{\partial^2 t} + kV - \frac{k^2}{2}J^2(V) + \beta V + \beta_3 J\left(\left(\frac{\partial V}{\partial t}\right)^3\right), \tag{6}$$

$$\int\limits_0^1 V(\eta,t)dt = 0,\ V(\eta,t) \equiv V(\eta,t+1). \tag{7}$$

$J(V)$ denotes the primitive function V with a zero mean:

$$J^2(V) = J(J(V)),\ (J(V))'_t \equiv V.$$

For $\varepsilon\beta < \frac{k}{2}$, the solution of equation (5) tends to zero.

Theorem 2. *Let $\varepsilon\beta < \frac{k}{2}$, let $V_*(\eta,t)$ be a bounded with its derivatives solution (6), (7), and $V_*(\eta,t) = \sum\limits_{-\infty}^{\infty} e^{2\pi nit}V_n(\eta)$, then*

$$y(t) = \varepsilon^2\sum_{-\infty}^{\infty} e^{iIm(\lambda_{n0}+\varepsilon\lambda_{n1}+\varepsilon^2\lambda_{n2}+\ldots)t}V_n(\varepsilon^2 t)$$

is asymptotic with respect to the residual with an accuracy of $O(\varepsilon^3)$ uniformly over $t \geq 0$ solution (1).

Theorem 3. *Let $\varepsilon\beta > \frac{k}{2}$, let $W_*(\tau)$ is a stable solution (5), then*

$$y(t) = \varepsilon^{\frac{1}{4}}(W_*(\varepsilon t)e^{i\sqrt{\varepsilon k}t} + \overline{W}_*(\varepsilon t)e^{-i\sqrt{\varepsilon k}t}) + O(\varepsilon^{\frac{1}{2}})$$

is a periodic stable solution (1).

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QUANTISATION OF FREE ASSOCIATIVE DYNAMICAL SYSTEMS. QUANTISATION IDEALS

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In my talk I'll discuss a new approach to the problem of quantisation of dynamical systems, introduce the concept of quantisation ideals and provide meaningful examples. Traditional quantisation theories start with classical Hamiltonian systems with variables taking values in
commutative algebras and then study their non-commutative deformations, such that the commutators of observables tend to the corresponding Poisson brackets as the (Planck) constant of deformation goes to zero. I am proposing to depart from dynamical systems (or maps) defined on a free associative algebra $\mathfrak{A}$. In this approach the quantisation problem is reduced to description of two-sided ideals $\mathfrak{J} \subset \mathfrak{A}$ satisfying two conditions: the ideals have to be invariant with respect to the dynamics of the system and they define commutation relations in the quotient algebras $\mathfrak{A}_{\mathfrak{J}} = \mathfrak{A} \diagup \mathfrak{J}$. Such ideals are called the quantisation ideals. Surprisingly, this idea works rather efficiently and in a number of cases I have been able to quantise the system, i.e. to find commutation relations consistent with the dynamics of the system. A Poisson structure is not assumed in this apprach, but if it is known, then it helps to find quantisation ideals corresponding the deformation of the Poisson strcture.

To illustrate this approach I'll consider the quantisation problem for ODEs on free associative algebra, including N–th Novikov equations and corresponding finite KdV hierarchy. Also, I am going to discuss quantisation of the Bogoyavlensky family of integrable N–chains (in the case $N = 1$ it is a well known Volterra chain)

$$\frac{du_n}{dt} = \sum_{k=1}^{N} (u_{n+k} \cdot u_n - u_n \cdot u_{n-k}), \qquad n \in \mathbb{Z}, \tag{1}$$

and quantisation of their symmetries and modifications. In particular, I will show that odd degree symmetries of the Volterra chain admit two quantisations, one of them corresponds to known quantisation of the Volterra chain, and another one is new and it is not deformational.

The talk is based on:

AVM, *Quantisation ideals of nonabelian integrable systems*, arXiv:2009.01838, 2020 (Published in Russ. Math. Surv. v.75:5, pp. 199–200, 2020)

V.M.Buchstaber and AVM, *KdV hierarchies and quantum Novikov's equations*, arXiv:2109.06357v1, 2021.

THE LIOUVILLE EQUATION AND COMBINATORICS

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Consider a differential operator D that plays an important role in the theory of combinatorial identities [1]

$$D = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \dots + u_{k+1}\frac{\partial}{\partial u_k} + \dots, \tag{1}$$

Consider an arbitrary sulution $u = u(x, y)$ of the Liouville equation $u_{xy} = e^u$ and a composition function $F(u, u_x, u_{xx}, u_{xxx}, \dots) = F(u, u_1, u_2, u_3, \dots)$. Then

$$\frac{\partial}{\partial y}F = u_y\frac{\partial F}{\partial u} + u_{xy}\frac{\partial F}{\partial u_1} + u_{xxy}\frac{\partial F}{\partial u_2} + \dots = u_y\frac{\partial F}{\partial u} + e^u\frac{\partial F}{\partial u_1} + D(e^u)\frac{\partial F}{\partial u_2} + \dots$$

Introduce a polynomial dfferential operator X

$$e^uX = e^u\frac{\partial}{\partial u_1} + D(e^u)\frac{\partial}{\partial u_2} + D^2(e^u)\frac{\partial}{\partial u_2} + \dots = e^u\left(\sum_{k=1}^{\infty} B_{k-1}(u_1, \dots, u_{k-1})\frac{\partial}{\partial u_k}\right)$$

where $B_k(u_1, \dots, u_k) = B_k(u_1, \dots, u_k) = (D + u_1)^k(1)$ are Bell polynomials [1, 3]. For instance $B_1(u_1) = u_1$, $B_2(u_1, u_2) = u_1^2 + u_2$, $B_3(u_1, u_2, u_3) = u_1^3 + 3u_1u_2 + u_3$.

A polynomial $F(u_1, u_2, \dots)$ is called Darboux integral of the Liuoville equation if $\frac{\partial F}{\partial y} = 0$. That is equivalent to $X(F) = 0$. For instance $q_2 = u_2 - \frac{1}{2}u_1^2$ is Darboux integral

$$X(q_2) = \left(\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2}\right)(u_2 - \frac{1}{2}u_1^2) = u_1 - u_1 = 0.$$

Define the grading w of the polynomial algebra $A = \mathbb{K}[u_1, u_2, u_3, \dots]$ by the rule

$$w(u_k) = k, \quad k = 1, 2, 3, \dots$$

Theorem 1. (Shabat, Zhiber, [2]) *The subalgebra $Ker\, X \subset A$ is isomorphic to the polynomial algebra $\mathbb{K}[q_2, q_3, \dots, q_k, \dots]$, where $q_k = D^{k-2}(q_2), k \geq 2$.*

A polynomial F is called higher symmetry of the Liouville equation if it satisfies

$$(D + u_1)XF = XDF = F$$

it means that $F = F(u_1, u_2, \dots)$ is an eigen-vector of the operator XD with $\lambda = 1$.

Theorem 2. [Zhiber, Shabat [2]] *An arbitrary symmetry F (an eigen-vector of the operator XD with $\lambda = 1$) of the Liouville equation can be represented in the following form*

$$F = (D + u_1)(Q),\ Q \in Ker\, X = \mathbb{K}[q_2, q_3, \dots].$$

In [4] the following generalization of Theorems 1,2 was proved.

Theorem 3.[2021, [4]] *1) The operator $(D + u_1)X = XD$ is diagonalizable on the subspace A_n of homogeneous polynomials degree $n \geq 2$;*

2) the set of its eigen-values consists of triangular numbers

$$\lambda_0 = 0, \lambda_1 = 1, \dots, \lambda_{n-2} = \frac{(n-2)(n-1)}{2}, \lambda_n = \frac{n(n+1)}{2};$$

where the value , $\lambda_{n-1} = \frac{(n-1)n}{2}$ is skipped.

3) an arbitrary eigen-vector F which corresponds to the eigen-value λ_k can be written in the form

$$(D + u_1)(D + 2u_1)\dots(D + ku_1)(Q),$$

where Q is some homogeneous polynomial of weight $(n-k)$ in $Ker\, X$.

4) The mapping $V_0(n-k) \to V_{\lambda_k}(n)$ *defined by*

$$F \to (D+u_1)(D+2u_1)\dots(D+ku_1)(F)$$

is an isomorphism.

5) the maximal eigen-value $\lambda_n = \frac{n(n+1)}{2}$ *has the multiplicity one and the corresponding eigen-vector is* $P_n = (D+u_1)(D+2u_1)\dots(D+nu_1)(1)$;

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ON THE BEHAVIOR OF THE GENERALIZED ALIGNMENT INDEX (GALI) METHOD FOR REGULAR MOTION IN MULTIDIMENSIONAL HAMILTONIAN SYSTEMS

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One of the fundamental tasks in the study of dynamical systems is the discrimination between regular and chaotic behavior. Over the years several methods of chaos detection have been developed (see e.g. [1]. Efficient numerical methods like the Smaller (SALI) and Generalized Alignment Index (GALI) [2] can be used for this purpose and are Generalized Alignment Index appropriate for investigating chaos and regular motion in high-dimensional systems [3, 4]. It has been observed that in multidimensional Hamiltonian models the constant values of the GALIs for regular orbits decrease as the order of the index increases [4]. In the presented work, we numerically investigate the behavior of the GALIs in the neighborhood of simple periodic orbits (SPO's) [5] of the well-known Fermi-Pasta-Ulam-Tsingou (FPUT) lattice model [6]. In particular, we study how the values of the GALIs depend on the width of the stability island and the system's energy [7]. We find that the asymptotic GALI values increase when the studied regular orbits move closer to the edge of the stability island for fixed energy, while these indices decrease as the system's energy increases.

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CIRCULAR CRUNODAL CUBICS WITH AN INFINITE REAL INFLEXION POINT IN A MINKOWSKI PLANE

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Crunodal cubics of both Euclidean and non-Euclidean planes can be used in modeling of various geometric objects and physical processes [1–6]. In this regard, we set up a problem of classifying and studying the crunodal cubics in non-Euclidean planes. In this direction, we start with the study of circular cubics in a Minkowski plane R_1^2. Given the variety of types of such curves, at the first stage, we limit ourselves to considering cubics with an inflection point on the absolute of the plane R^2_1. The asymptotes of such cubics are isotropic lines. To simplify calculations when classifying cubics and derive canonical equations, we use both the orthogonal coordinate system of the Minkowski plane, which is an analogue of the Cartesian system of the Euclidean plane, and projective frames (see, for instance, [5]).

A projective plane P_2 with an infinitely removed line l_∞ and a fixed pair of real points K_1, K_2 on it is the Cayley – Klein projective model of the Minkowski plane R^2_1. The set of objects l_∞, K_1, and K_2 is called the *absolute* of the plane R^2_1. The fundamental group of the plane R^2_1 consists of all projective automorphisms of the absolute and is the group of similarity transformations. By choosing the projective frame $R = \{A_1, A_2, A_3, E\}$ of the plane R^2_1 due course, we define the elements of the absolute so that $l_\infty(0 : 0 : 1) = A_1A_2$, $K_1(1 : 1 : 0) = A_1 + A_2$, $K_2(1 : -1 : 0) = A_1 - A_2$. In this case, the quadratic form $(x_1/x_3)^2 - (x_2/x_3)^2$, which corresponds to the absolute setting in the frame R, induces a pseudo-Euclidean metric in the plane R^2_1 and determines the isometries group of this plane.

We prove that each circular crunodal cubic with the infinite real inflection point belongs to one of six base types. These types depend on the location of the cubic with respect to the absolute. We derive the canonical equations for each type of the cubics and find out the geometric meaning of the coefficients of the equations. In each case, we find a generating set (see [5]) and a set of independent invariants of the cubic. Continuing the classification process, we distinguish different classes in four of the six types of the considered cubics. Let us consider, for example, the following canonical equations of the cubics

$$k^2(x^2 - y^2)(x - y) + 2k(x - y)^2 - (x + y) = 0, \quad k \neq 0, \tag{1}$$

$$(x^2 - y^2)(x - y) + axy = 0, \quad a \neq 0. \tag{2}$$

A circular crunodal cubic whose node and real inflection point coincide with the circle-points K_1, K_2 of the plane R^2_1, can be given by the equation (1). Each such cubic is one of the possible analogues of a strophoid of the Euclidean plane and the cubic of stationary curvature in terms of the Burmester theory, that is, the locus of points of the mobile plane whose four corresponding positions lie on circles of the fixed plane. Moreover, the cubic of stationary curvature can be defined as the locus of points whose trajectories have a vanishing differential of the curvature with respect to the natural parameter at zero position

[2]. A circular crunodal cubic whose loop vertice and real inflection point coincide with the circle-points, can be given by the equation (2).

Any cubic of each of presented here two types has a single independent invariant, which determines the distance from the node of the cubic to a point on the asymptote along an axis. In the first case, the axes consist the loop vertice of the cubic and harmonicaly devide isotropic lines in this point. In the second case, the axes are orthogonal tangents of the cubic in its node.

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CALCULATION OF WAVE FIELDS BY RAY METHOD AND VARIATIONAL METHOD OF FINDING RAYS

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For the 2011 Japan tsunami we are calculating the wave fields by the ray method [1]. We pre-smooth bathymetry and do local averaging [2] to exclude the influence of small–scale non–flatness of the bottom on the asymptotic solution. For comparison, the rays are found by two methods: the variational method [3] and using the Hamilton system.

Also we discuss a variational method for finding rays in the case when a ray is reflected from the coast.

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ON A NEW TYPE OF PERIODIC FRONTS IN BURGERS TYPE EQUATIONS

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A new class of singularly perturbed parabolic periodic boundary value problems for Burgers type equations including modular advection is considered. The interior layer type asymptotics and a modification procedure to get asymptotic lower and upper solutions is proposed. By using such lower and upper solutions, we prove the existence of a periodic solutions with an interior layer, estimate the accuracy of its asymptotics and establish the asymptotic stability of this solutions.

ENERGY SPREADING, EQUIPARTITION AND CHAOS IN LATTICES WITH NON-CENTRAL FORCES

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We present a one-dimensional (1D), nonlinear, lattice model which in the linear limit is relevant to the study of bending (flexural) waves. In contrast with the classical 1D mass-spring system, the linear dispersion relation of the considered model has different characteristics in the low frequency limit. With disorder in the masses of the lattice particles, we investigate how different nonlinearities (cubic, quadratic or their combination) lead to energy delocalization, equipartition and chaotic dynamics. We excite the lattice using single site initial momentum excitations corresponding to a strongly localized linear mode and increase the initial energy of excitation. Beyond a certain energy threshold, when the cubic nonlinearity is present, the system is found to reach energy equipartition and total delocalization. On the other hand, when only the quartic nonlinearity is activated, the system remains localized and away from equipartition at least for the energies considered here. However, for sufficiently large energies, we observe chaos either with delocalization or when the system remains localised for the different types of nonlinearities considered. Our results reveal a rich dynamical behaviour and show differences with the relevant Fermi-Pasta-Ulam-Tsingou model. Our findings pave the way for future studies of models relevant to bending (flexural) waves in the presence of nonlinearity and disorder, from which different energy transport behaviours are anticipated.

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TOPOLOGY OF ALGEBRAICALLY SOLVABLE HAMILTONIAN SYSTEMS

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One of the main characteristics of a finite-dimensional integrable Hamiltonian system revealing its qualitative properties is its *Liouville fibration*. This is a singular fibration whose fibers are connected common level sets of the first integrals of the system. In the case of two degrees of freedom, A. T. Fomenko and his colleagues suggested [1] a set of invariants describing the topology of the Liouville fibration. One the most important invariant is a *marked molecule* (or *Fomenko–Zieschang invariant*) describing the topology of the Liouville fibration on a regular invariant 3-dimensional submanifold. This invariant has the structure of a graph whose edges correspond to one-parameter families of regular fibers whereas vertices correspond to singularities of the fibration, so-called *atoms*.

The calculation of the Fomenko–Zieschang invariant is not an algorithmic procedure and for some integrable systems it turns out to be quite a non-trivial task. However, in some cases it can be substantially simplified. One of such cases is a class of *algebraically solvable systems*. These systems admit the separation of variables in the sense that the Hamiltonian equations on each fiber can be written in the form

$$\dot{u}_i = \sqrt{P(u_i)} \quad \text{or} \quad \dot{u}_i = \frac{\sqrt{P(u_i)}}{u_1 - u_2}, \quad i = 1, 2 \quad (P \text{ is a polynomial}).$$

Moreover, all the phase coordinates are expressed as rational functions on the radicals of the form $\sqrt{u_i - \alpha_j}$ where u_1, u_2 are the variables of separation, $\alpha_j \in \mathbb{R}$ (the constants α_j and the polynomial P depend on the values of the first integrals).

The first systematic approach to the study of the topology of algebraically solvable systems was made by M. P. Kharlamov [2]. He noticed that the expressions of the phase coordinates via radicals permit to understand the topological properties of the projection of each fiber onto the plane $\mathbb{R}^2(u_1, u_2)$. We follow this idea and suggest (under some natural conditions) an algorithm for calculating the Fomenko–Zieschang invariant for algebraically solvable systems. As a corollary, we can list the types of all elementary (i.e. corresponding to the coincidence of a unique pair of constants α_k, α_l) bifurcations that occur in such systems.

Theorem 1. Any elementary compact non-degenerate 3-dimensional singularity of an algebraically solvable system has the type of one of the following atoms:
$A, B, C_2, P_4, D_1, A^*, A^{**}$.

Remark. The exact definition of the atoms (=bifurcation types) mentioned in the theorem can be found in [1]. For instance, the atom A denotes the disappearance of a torus via the critical circle, the atom B stands for the transformation of a torus into two tori via the direct product of the figure “eight” and the cirle.

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STRUCTURALLY STABLE CORANK 1 SINGULARITIES OF INTEGRABLE SYSTEMS WITH THREE DEGREES OF FREEDOM

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Let us fix $s \in \mathbb{N}$ and consider the action of the group $\Gamma = \mathbb{Z}_s$ on the plane $\mathbb{R}^2$ of the form $z \longrightarrow e^{2\pi i/s} z$, where $z = x + iy \in \mathbb{C} \approx \mathbb{R}^2$. Consider the Morse functions $F_{s,0} = F_{s,0}^{\pm,\pm}(z) = \pm|z|^2 = \pm(x^2 + y^2)$ for $s \geq 1$, and $F_{s,0} = F_{s,0}^{+,-}(z) = x^2 - y^2$ for $s = 1, 2$. Consider two families of $\mathbb{Z}_s-$invariant germs $F_{s,k} = F_{s,k}(z, \lambda, a)$, $k = 1, 2$, at zero:

$$F_{s,1} = F_{s,1}(z, a, \lambda) = \begin{cases} \pm x^2 + y^3 + \lambda y, & s = 1, \\ \pm x^2 \pm y^4 + \lambda y^2, & s = 2, \\ Re(z^3) + \lambda |z|^2, & s = 3, \\ Re(z^s) \pm a|z|^4 + \lambda |z|^2, & s \geq 4,\ a^2 \neq 1 \text{ with } s = 4,\ a > 0 \\ & \text{with } s \geq 5, \end{cases}$$

$$F_{s,2} = F_{s,2}(z, a, \lambda) = \begin{cases} \pm x^2 \pm y^4 - \lambda_2 y^2 + \lambda_1 y, & s = 1, \\ \pm x^2 \pm y^6 + \lambda_2 y^4 + \lambda_1 y^2, & s = 2, \\ Re(z^4) \pm (1 + \lambda_2)|z|^4 \pm a|z|^6 + \lambda_1 |z|^2, & s = 4,\ a > 0, \\ Re(z^5) \pm a|z|^6 + \lambda_2 |z|^4 + \lambda_1 |z|^2, & s = 5,\ a > 0, \\ Re(z^6) + a_1|z|^6 \pm a_2|z|^8 + \lambda_2|z|^4 + \lambda_1|z|^2, & s = 6,\ a_1^2 \neq 1, \\ & a_1 a_2 \neq 0. \end{cases}$$

Here $\lambda = (\lambda_i) \in \mathbb{R}^k$ is a small parameter, $a = (a_i) \in \mathbb{R}^m$ is *moduli*, $m \in \{0, 1, 2\}$ is the *modality*.

Theorem 1. *Consider the* right *$\Gamma-$equivalence classes of the $\Gamma-$invariant function germs $F_{s,k}(z, 0, \hat{a})$ in two variables at zero, $k = 0, 1, 2$. These singularities have $\Gamma-$codimension k, $\Gamma-$Milnor number $k+m+1$, $\Gamma-$versal deformation $F_{s,k}(z, \lambda, a) + \lambda_0$, and form a complete list of $\Gamma-$invariant singularities with $\Gamma-$codimensions $k \leq 2$. The complement to their union in the space $\mathfrak{n}_\Gamma^2$ of $\Gamma-$invariant germs at zero having a critical point 0 with a critical value 0, has codimension > 2 in $\mathfrak{n}_\Gamma^2$.*

An *integrable Hamiltonian system* (IHS) on a $2n$-dimensional symplectic manifold (M, ω) is given by a smooth map $\mathcal{F} = (f_1, \ldots, f_n) : M \longrightarrow \mathbb{R}^n$, where $\{f_i, f_j\} = 0$. A *singular Lagrangian foliation* arises on M, whose fibers are the connected components of the sets $\mathcal{F}^{-1}(c)$, $c \in \mathbb{R}^n$. Suppose M is compact, the vector fields $X_{f_j} = \operatorname{sgrad} f_j$ are tangent to ∂M, and $\mathcal{F}$ has a “good” behavior near ∂M. The mapping $\mathcal{F}$ generates a Hamiltonian $\mathbb{R}^n-$action on M.

Consider the free action of the group $\Gamma = \mathbb{Z}_s$ on the solid torus $V := D^2 \times S^1 \subset \mathbb{R}^2 \times S^1$ of the form $(z, \varphi_1) \longrightarrow (e^{2\pi i \frac{\ell}{s}} z, \varphi_1 + \frac{2\pi}{s})$, where $\varphi_1 \in S^1$, $0 \le \ell < s$, $(\ell, s) = 1$. Consider the “cylinder” $W := D^{n-1} \times (S^1)^{n-2}$ with coordinates $(\lambda, \varphi') = (\lambda_1, \dots, \lambda_{n-1}, \varphi_2, \dots, \varphi_{n-1})$. It turns out that, for a typical integrable system, *local singularities* (i.e. $\mathbb{R}^n-$orbits) of rank $n-1$ have neighborhoods C^∞-smoothly fibrewise equivalent to a *standard model* of the form

$$M_{\frac{\ell}{s}} = (V/\Gamma) \times W, \quad \omega_{\frac{\ell}{s}} = \mathrm{d}x \wedge \mathrm{d}y + \sum_{j=1}^{n-1} \mathrm{d}\lambda_j \wedge \mathrm{d}\varphi_j,$$

$$\mathcal{F}_{\frac{\ell}{s},k,a} : (V/\Gamma) \times W \to \mathbb{R}^n, \quad \mathcal{F}_{\frac{\ell}{s},k,a}(z, \varphi_1, \lambda, \varphi') = (\lambda, F_{s,k}(z, a(\lambda), \lambda')),$$

where $0 \le k < n$, $\lambda' = (\lambda_1, \dots, \lambda_k)$, $a(\lambda)$ and $a_1(\lambda)$ are smooth functions for $(s,k) \in \{(4,1),(6,2)\}$, $a(\lambda) \equiv 1$ for other pairs (s,k); $a(\lambda) = (a_1(\lambda), 1)$ for $(s,k) = (6,2)$.

Theorem 2 (E. A. Kudryavtseva, M. V. Onufrienko)**.** *Consider the space of integrable Hamiltonian systems (IHS) on M for which the functions $f_1, \dots, f_{n-1}$ generate a locally free Hamiltonian action of the $(n-1)-$dimensional torus on M. If $n = \frac{1}{2}\dim M \in \{2,3\}$ then the $\mathbb{R}^n-$orbits whose neighborhoods are C^∞-smoothly fibrewise equivalent to a standard model, have the following “genericity” properties:*

(i) *they are structurally stable (with respect to small integrable perturbations in the indicated IHS space), and even smoothly structurally stable in the case $(s,k) \notin \{(4,1),\ (6,2)\}$;*

(ii) *the set of IHSs having only such local singularities has a full measure in the indicated IHS space (i.e., the complement to this set has measure 0, and a positive codimension).*

In particular, any structurally stable $\mathbb{R}^n-$orbit has a neighborhood topologically fibrewise equivalent to a standard model model $(M_{\ell/s}, \omega_{\ell/s}, \mathcal{F}_{\ell/s,k,a})$.

We conjecture that the standard models with different pairs $(\pm\ell/s \ mod\ \mathbb{Z}, k)$ are not topologically fibrewise equivalent. We also conjecture that "typical" corank 1 singularities always admit a C^∞-smooth locally free Hamiltonian action of the $(n-1)-$dimensional torus on M (in the case $(n,s,k)=(2,1,1)$ of parabolic orbits, this was proved in [3]).

Theorem 2 describes *parabolic orbits with resonances* [2] that correspond to $(n,k)=(2,1)$, and their typical bifurcations that correspond to $(n,k)=(3,2)$. Thus, by Theorem 2, these local singularities are structurally stable and "generic" for integrable systems with 3 degrees of freedom.

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CONTRAST STRUCTURES IN A PROBLEM WITH A WEAK REACTION DISCONTINUITY

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A boundary value problem is investigated for a singularly perturbed reaction-diffusion-advection equation in a two-dimensional region in the case of discontinuous weak reaction coefficient, the break of which occurs on a previously known curve lying in this region:

$$\begin{aligned}&\varepsilon^2\Delta u=\varepsilon(\mathbf{a}(u,x),\nabla u)+f(u,x,\varepsilon)+\varepsilon f_1(u,x,\varepsilon),\ x=(x_1,x_2)\in D,\\&\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial D}=0,\quad x\in\partial D.\end{aligned}\tag{1}$$

Here D is a simply connected region on the plane (x_1, x_2) with a smooth boundary ∂D, ε is a small parameter lying in the interval $[0;\varepsilon_0)$, $\varepsilon_0 > 0$, $\mathbf{a}(u,x) = (a_1(u,x), a_2(u,x))$ is a two-dimensional vector function, $\mathbf{n}$ - outer normal to the curve ∂D. Let C_0 be a simple smooth closed curve lying entirely in the domain D which divides the domain into two parts: $D^{(-)}$ bounded by the curve C_0 and $D^{(+)}$, bounded by curves C_0 and ∂D.

Function $f_1(u,x,\varepsilon)$ represented as

$$f_1(u,x,\varepsilon) = \begin{cases} f_1^{(-)}(u,x,\varepsilon), & u \in I_u, \quad x \in \bar{D}^{(-)}, \ \varepsilon \in [0,\varepsilon_0); \\ f_1^{(+)}(u,x,\varepsilon), & u \in I_u, \quad x \in \bar{D}^{(+)}, \ \varepsilon \in [0,\varepsilon_0); \end{cases}$$

where $f_1^{(+)}(u,x,\varepsilon) \neq f_1^{(-)}(u,x,\varepsilon), \ x \in C_0$, I_u the interval of variation of the function u, $f_1^{(\pm)}(u,x,\varepsilon)$ sufficiently smooth functions in their respective domains of definition.

Function $f(u,x,\varepsilon)$, $\mathbf{a}(u,x)$ are sufficiently smooth in $u \in I_u$, $x \in \bar{D}$, $\varepsilon \in [0,\varepsilon_0)$ and $u \in I_u$, $x \in \bar{D}$ respectively.

For this problem conditions are obtained under which a solution with an inner transition layer appearing near the curve C_0 exists. An asymptotic approximation for such a solution is constructed using the Vasil'eva method. An existence theorem is proved using the asymptotic method of differential inequalities. An example is given to illustrate the proposed algorithms.

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BKP AND NEW MATRIX MODELS

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I will talk about new family of matrix models which are solvable in the sense that the related partitions function and correlation functions of each model are equal to certain (2-component) BKP tau functions and and can be written as explicit series over Young diagrams of the projective Schur functions and can be presented as a Pfaffian. Let us call it the target 2-BKP tau function to make difference with the 2-BKP tau functions which can be considered as the part of the integration measure. The two sets of the target 2-BKP higher times play the role of coupling constants of the model. Each model is an integral over a pair of matrices., say X and Y, The model is defined by the choice

1. of the type of matrices (both X and Y are Hermitian, both X and Y are unitary, both are complex and Hermitian conjugated $Y = X^{\dagger}$). The choice defines measures dX and dY;

2. the choice of the deformed measures $\tau_1^B(t, X)dX$ and $\tau_2^B(Y, t)dY$ where $\tau_{1,2}^B$ is a pair of 2-BKP tau functions and t, t play the role of two sets of the coupling constants (deformation parameters of the measure);

3. the choice of the interaction term $\tau^{KP}(X^2Y^2)$ which is the choice of the KP tau function. (This is the choice of a diagonal matrix which defined the tau function of hypergeometric type)

   Symbolically it can be written as

$$\int \tau_1^B(t, X)\tau^{KP}(X^2Y^2)\tau_2^B(Y, t)dXdY = \tau_3^B(t, t).$$

The choice of these three tau functions inside the integral should define a convergent integral. Examples are presented, the simplest are described by the vacuum tau functions which are exponentials.

It is conjectured that each matrix model of the family can be presented as the action of a certain abelian subgroup of B_∞ where the group times plays the role of the coupling constants.

ON NON-COMPACT BIFURCATION IN THE RESTRICTED PROBLEM OF THREE MAGNETIC VORTICES

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Since the time of Helmholtz [1], the model of N point vortices in an ideal fluid with constant intensities Γ_α ($\alpha = 1 \dots N$) is well known. The model of N interacting *magnetic vortices* in ferromagnets [2, 3] is a more general case than the hydrodynamic model: in addition to vorticities Γ_α, there are polarities λ_α which take values ± 1 depending on magnetization directed up or down.

In this talk, we consider a *restricted problem* of three magnetic vortices, where in the system of three magnetic vortices at positions $r_\alpha = (x_\alpha, y_\alpha)$, $\alpha \in \{0, 1, 2\}$, the vortex with vorticity Γ_0 is fixed at point $\mathcal{O}(0,0)$. Equations of motion in our generalized model have the following complex form:

$$i\lambda_\alpha \dot{z}_\alpha = \frac{1}{\bar{z}_\alpha} + \frac{\Gamma_\beta}{\lambda_\beta}\frac{1}{\bar{z}_\alpha - \bar{z}_\beta}, \quad \alpha \neq \beta \in \{1, 2\},$$

where $z_\alpha = x_\alpha + iy_\alpha$ is a complex coordinate specifying the position of vortex with vorticity Γ_α. The system can be written in Hamiltonian form:

$$\begin{gathered} \Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \qquad \alpha = 1, 2, \\ H = \frac{\Gamma_1}{\lambda_1}\ln \ell_1 + \frac{\Gamma_2}{\lambda_2}\ln \ell_2 + \frac{\Gamma_1}{\lambda_1}\frac{\Gamma_2}{\lambda_2}\ln \ell_{12}, \end{gathered} \tag{1}$$

where $\ell_\alpha = |r_\alpha|$ and $\ell_{\alpha\beta} = |r_\alpha - r_\beta|$. In addition, this system has an integral of the *angular momentum of vorticity* $F = \Gamma_1 \ell_1^2 + \Gamma_2 \ell_2^2$, so it is completely Liouville integrable with two degrees of freedom.

The main role in the study of such dynamical systems is played by the *bifurcation diagram* Σ of the momentum map $\mathcal{F}(\boldsymbol{z}) = (F(\boldsymbol{z}), H(\boldsymbol{z}))$. In the restricted system of three magnetic vortices in the case of a "vortex pair" (encountered in hydrodynamics [4, 5]), a bifurcation diagram contains *non-compact bifurcations* of the $\big(\mathbb{T}^2 + \mathrm{Cyl}\big) \to \mathrm{Cyl}$ type. Here

$\mathbb{T}^2$ denotes the presence of a two-dimensional Liouville torus, and Cyl denotes a two-dimensional cylinder.

For the system (1) an explicit reduction to a Hamiltonian system with one degree of freedom was performed. Level lines of the reduced Hamiltonian help to better understand the non-compact bifurcations present in the model.

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NONLOCAL KINETIC EQUATION, DESCRIBING A SOLITON GAS

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In this talk we consider a new class of hydrodynamic type systems, which are NOT diagonalisable, but integrable by Tsarev's generalised hodograph method.

We consider El's nonlocal kinetic equation and its linearly-degenerate reductions.

These reductions are non-diagonalisable hydrodynamic type systems. Nevertheless, they are integrable by Tsarev's generalised hodograph method.

We construct a general solution for these hydrodynamic type systems.

NEGATIVE NUMBERS OF TIMES OF INTEGRABLE HIERARCHIES

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Time evolutions of the dressing operators of the integrable hierarchies, like Kadomtsev–Petviashvili or Davey–Stewartson, are given by linear differential operators. In the standard situation these operators result from dressing of positive powers of a ∂_x. It is natural to call these semiinfinite hierarchies of integrable (2+1)-dimensional equations as hierarchies with positive numbers of times. Here we develop hierarchies directed to negative numbers of times. Derivation of such systems, as well as of the corresponding hierarchies, is based on the commutator identities. This approach enables introduction of linear differential

equations that admit lift up to nonlinear integrable ones by means of the special dressing procedure. Thus one can construct not only nonlinear equations but corresponding Lax pairs, as well. Lax operator of such evolutions coincide with the Lax operator of the "positive" hierarchy. We also derive (1+1)-dimensional reductions of equations of such hierarchies.

TRAFFIC FLOW MODEL

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This paper presents a construction of the new traffic flow mathematical model, which describes the movement of $N \in \mathbb{N}$ vehicles. The position of the vehicle is denoted by x, and $\dot{x}$ and $\ddot{x}$ denote speed and acceleration, respectively. All vehicles are considered as material points, thus their internal structure and external dimensions are not taken into account.

The main result is construction of traffic flow mathematical model, which is represented by system of delay-differential equations:

$$\begin{cases} \ddot{x}_n(t) = \\ \quad = R(\Delta x_n(t,\tau))\left[a\left(\dfrac{v_{max}-\dot{x}_{n-1}(t-\tau)}{1+e^{p(x_n(t)-x_{n-1}(t-\tau)+S)}}+\dot{x}_{n-1}(t-\tau)-\dot{x}_n(t)\right)\right]+ \\ \quad\quad +(1-R(\Delta x_n(t,\tau)))\left[q\left(\dfrac{\dot{x}_n(t)\left[\dot{x}_{n-1}(t-\tau)-\dot{x}_n(t)\right]}{x_{n-1}(t-\tau)-x_n(t)-l+k}\right)\right], \\ x_n(t)=\lambda_0-(n-1)\lambda, \quad \dot{x}_n(t)=v_n, \quad \text{for } t\in[-\tau,0] \text{ and } n\geq 2. \end{cases} \tag{1}$$

where τ — driver's response time, $\Delta x_n(t,\tau) = x_{n-1}(t-\tau) - x_n(t)$ — distance between adjacent vehicles, $a > 0$ — acceleration factor, $q > 0$ — deceleration factor, $l > 0$ — safely distance between vehicles, $k > 0$ — correction factor, $p > 0$ — driver's behavior parameter, $S > 0$ — distance of the front vehicle influence, λ_0 — first vehicle's position at the initial time, λ — distance between adjacent vehicles at the initial time, v_n — vehicles speed at the initial time and $R(\Delta x_n(t,\tau))$ — and relay function as follows:

$$R(\Delta x_n(t,\tau)) = \begin{cases} 1, & \text{if } \Delta x_n(t,\tau) > \dfrac{\dot{x}_n^2(t)}{2\mu g}+l, \\ 0, & \text{if } \Delta x_n(t,\tau) \leq \dfrac{\dot{x}_n^2(t)}{2\mu g}+l. \end{cases}$$

where μ — coefficient of friction, and g — acceleration of gravity. Function $R(\Delta x_n(t,\tau))$ describes the "acceleration-deceleration" switch.

The resulting model (1) describes all the vehicles in the flow. To describe the first vehicle ($n = 1$) the model is extended with the values of $x_0(t)$ and $\dot{x}_0(t)$. Value of $x_0(t)$ is the distance that the first vehicle have to drive and it is denoted by $x_0 = L$.

Value of $\dot{x}_0(t)$ in the first term is the maximum desired speed $\dot{x}_0 = v_{max}$, and value of $\dot{x}_0(t)$ in the second term is the minimum desired speed, namely the speed to which the current speed should be reduced $\dot{x}_0 = v_{min}$.

It may also be noted, that in this paper the main dynamic properties of the (1) model were investigated, and the values and units of measurement for the parameters were determined. In addition, an analysis of the regime stability was performed, in which n vehicles move at a speed of v_{max} at a distance of $\Delta c_n = c_n - c_{n-1}$, where c_n is the position of nth vehicle.

CONNECTIONS BETWEEN THE RING OF COAJOINT INVARIANTS AND THE JORDAN-KRONECKER INVARIANTS OF THE 7-DIMENSIONAL NILPOTENT LIE ALGEBRAS

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Let us consider a Lie algebra $\mathfrak{g}$. On its dual space $\mathfrak{g}^*$ the Lie-Poisson bracket $\mathcal{A}_x$ and the Poisson bracket with constant coefficients $\mathcal{A}_a$ are naturally defined. The Jordan-Kronecker decomposition theorem gives the classification of pairs of skew-symmetric bilinear forms by reducing them simultaneously to a canonical block-diagonal form. For any two points $x, a \in \mathfrak{g}^*$ there exists a pencil of bilinear forms $\mathcal{A}_x - \lambda\mathcal{A}_a$ at the point x. The sizes of blocks in the Jordan-Kronecker decomposition of $\mathcal{A}_x$ and $\mathcal{A}_a$ are called the algebraic type of a pencil. For almost all pencils their algebraic types are the same. So we can call the algebraic type of a common pencil the Jordan-Kronecker invariants of a Lie algebra. A.Yu. Groznova calculated the Jordan-Kronecker invariants for nilpotent Lie algebras with dimension seven in her diploma work [1].

On the other hand, the article [2] written by Alfons I. Ooms is concerned with studying the properties of the rings of coajoint invariants for the same Lie algebras. Groznova's calculations led to the conclusion that the existence of Kronecker pencils of different algebraic types but of the same rank can be a characteristic property for the non-free generatedness of the ring of coajoint invariants for Kronecker nilpotent Lie algebras. So the problem was to find such pencils for every 7-dimensional algebra with Kronecker type and non-free generated ring of coajoint invariants.

Thus the algorithm for finding these pencils or proving that they cannot exist will be presented. Also we will discuss the full list of the results of this algorithm work.

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MONODROMY OF THE FOCUS-FOCUS POINT OF THE CIRCULAR BILLIARD IN THE POTENTIAL FIELD

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Mathematical billiard is a dynamical system that describes the motion of a material point in closed bounded domain with a reflection law on the boundary. Consider a closed circular domain with Hooke repulsing potential of parameter $k < 0$ at the center. A material point (particle) moves in the domain under the action of the potential and reflects from the boundary absolutely elastic. This system appears to be integrable on the phase space M^4 with the Hamiltonian

$$H = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{k}{2}(x^2 + y^2)$$

and additional first integral

$$F = x\dot{y} - y\dot{x}.$$

Now glue n numerated copies of previous simple billiard along their common border. Reflection law is the same but particle changes their billiard leaf according the permutation $(12...n)$. This type of billiard is called a billiard book, that was invented by V. V. Vedyushkina and I. S. Kharcheva in [1]. Our billiard book is integrable with the same first integrals H and F. The following theorem tell us about topology of isoenergy manifold.

Theorem 1. *Consider the isoenergy space $Q^3 = \{H = const\}$. If $H < 0$, manifold Q^3 is homeomorphic to direct product $S^2 \times S^1$. If $H > 0$, manifold Q^3 is homeomorphic to lens space $L(n; 1)$.*

The momentum map $\mathcal{F} : M^4 \to \mathbb{R}^2$, $\mathcal{F}(x,y,\dot{x},\dot{y}) = (H(x,y,\dot{x},\dot{y}), F(x,y,\dot{x},\dot{y}))$, has only singular value at the point $H = F = 0$ corresponding to focus-focus case. We consider this singularity in terms of Fomenko-Zieschang invariants which are described in [2]. In particular let us calculate a loop molecule for this singularity which is described by monodromy matrix.

Theorem 2. *Consider an n-leaf circular billiard book with repulsing Hooke potential. Then the monodromy matrix for the singular value corresponding to focus-focus case has the form* $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

As a corollary we have that billiards can model any focus-focus singularity up to Liouville equivalence.

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ON THE INTERACTION OF BINARY ASTEROID WITH RESONANCE

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As it is known, about 15 persents of asteroids in main belt are binary or has a satellite [1]. On the other side, the numerous mean motion resonances have crossed the main belt. As we note in our previous paper, many of unbounded asteroid pairs moved in vicinity of resonances and strongly perturbed by them [2]. By this reason, the modelling of the process of resonance crossing by asteroid with satellite is an actual problem.

The main condition of encounter asteroid with resonance is the semimajor axis change. It can be possible by different reason, in particular by thermal Yarkovsky effect.

In this paper we have considered this problem in the pendulum model of resonance. As a result, we have obtained the estimation of the difference in the accelerations of primary and secondary in pair:

$$\delta \ddot{a} = -2.5 * 10^{-3} n n_s a_s F$$

where n - mean motion of the primary, n_s, a_s - mean motion and semimajor axis of the secondary, maximal value of F is about unit.

After that we compare obtained acceleration with solar tidal ones and with mutual attraction. In result we made the conclusion that binary asteroid at the specific conditions can be destroyed during the resonance transfer.

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ATLAS OF BIFURCATION DIAGRAMS FOR THE ONE MODEL OF A LAGRANGE TOP WITH A VIBRATING SUSPENSION POINT

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We consider a completely integrable Hamiltonian system with two degrees of freedom that describes the dynamics of a Lagrange top with a vibrating suspension point. For a dynamically symmetric rigid body with the center of mass lying on the axis of dynamic symmetry, the corresponding system of differential equations has the form of generalized Kirchhoff equations

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}} \tag{1}$$

with Hamiltonian function

$$H = \frac{1}{2}\left(M_1^2 + M_2^2 + cM_3^2\right) + a\gamma_3 - \frac{1}{2}b\gamma_3^2.$$

Here, $\boldsymbol{M} = \{M_1, M_2, M_3\}$ and $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$ denote the angular momentum vector and the vertical unit vector in the system of principal axes attached to the rigid body and passing through the point of restraint. The parameters a, b and c can be physically interpreted according to [1], [2]. Specifically, a is related to the location of the center of mass on the vertical axis; in what follows, for a particular study, the sign of a is assumed to be fixed. The parameter b characterizes the difference between the averaged squared projections of the suspension point's velocity onto the OX and OZ axes in the frame $OXYZ$ with the origin at the suspension point; the values of b can be positive or negative. The parameter c is a positive one characterizing the ratio of the principal inertia-tensor components for the dynamically symmetric rigid body.

The phase space $\mathcal{P}$ is specified as the tangent bundle $T\mathbb{S}^2$ of the two-dimensional sphere $\mathbb{S}^2$:

$$\mathcal{P} = \{(\boldsymbol{M}, \boldsymbol{\gamma}) : (\boldsymbol{M}, \boldsymbol{\gamma}) = \ell,\ |\boldsymbol{\gamma}|^2 = 1\}.$$

System (1) has one additional first integral, namely, the Lagrange integral

$$F = M_3$$

The function F and the Hamiltonian H form a complete involute set of integrals of system (1) on $\mathcal{P}$. According to the Liouville-Arnold theorem, a regular level surface of first integrals of a completely integrable Hamiltonian system is a nonconnected union of tori filled with conditionally periodic trajectories. The integral mapping $\mathcal{F} : \mathcal{P} \to \mathbb{R}^2$ is defined by setting $(f, h) = \mathcal{F}(\boldsymbol{x}) = (F(\boldsymbol{x}), H(\boldsymbol{x}))$. Let $\mathcal{C}$ denote the set of all critical points of the integral mapping, i.e., points at which $\operatorname{rank} d\mathcal{F}(\boldsymbol{x}) < 2$. The set of critical values $\Sigma = \mathcal{F}(\mathcal{C} \cap \mathcal{P})$ is called the bifurcation diagram. The type of rank-zero singularities of the integral mapping, which are associated with equilibrium positions, are determined in the paper [3]. In contrast to the classical approach used for stability analysis of the upper equilibrium in [2], an analysis of the type of singularities of the integral mapping revealed relations under which the lower equilibrium position becomes unstable. Additionally, a unique phenomenon is observed in the considered mechanical system, namely, the appearance of a double pinched torus.

The goal of this report is to determine and analyze the bifurcation diagram Σ of the integral mapping $\mathcal{F}$.

Theorem 1. *The bifurcation diagram Σ is part of the discriminant set of the polynomial $R(x)$, where*

$$R(x) = -bx^4+2ax^3+[b+(c-1)f^2-2h]x^2+2(f\ell-a)x+2h-cf^2-\ell^2.$$

The polynomial $R(x)$, as expected, is the base for the explicit integration of the phase variables in terms of Jacobi elliptic functions.

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CHAOTIC BEHAVIOR OF DISORDERED NONLINEAR LATTICES

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We extend the findings of [1,2] and study, in two spatial dimensions, the chaotic behavior of two typical disordered nonlinear lattices, the Klein-Gordon (DKG) model and discrete nonlinear Schrodinger equation (DDNLS), for different initial excitations. We compute the most commonly used chaos indicator, i.e. the maximum Lyapunov characteristic exponent (mLCE) and classify the different dynamical behaviors of the models according to the time evolution of the mLCE and the corresponding deviation vector distribution.

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THE WADA PROPERTY AND HOMOGENEOUS SYSTEMS REMARKS

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I am sometimes asked on Birkhoff curves qualitative theory and Wada property with respect to ordinary differential equations systems. How to guess the structure of such a system?

In order to demonstrate the Wada property direct inheritance by systems of differential equations, let us consider the simplest situation. Suppose an ordinary differential equations system

$$\dot{x} = X(x, y, v), \quad \dot{y} = Y(x, y, v), \quad \dot{v} = V(x, y, v) \tag{1}$$

be a homogeneous in the sense that differentiable functions $X, Y, V\colon \mathbb{R}^3 \to \mathbb{R}$ belong to same homogeneity class, or that too the vector field $(X, Y, V)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a homogeneous with respect to the point $(0, 0, 0)$ being the projection centre. Then eigen direction one can to define as follows

$$x \stackrel{def}{=} \lambda v, \quad y \stackrel{def}{=} \eta v \quad \text{with} \quad \lambda, \eta \stackrel{def}{=} const.$$

Therefore all eigen directions has been defined by two equations system

$$\lambda \stackrel{def}{=} P(\lambda, \eta) = \frac{X(\lambda, \eta, 1)}{V(\lambda, \eta, 1)}, \quad \eta \stackrel{def}{=} Q(\lambda, \eta) = \frac{Y(\lambda, \eta, 1)}{V(\lambda, \eta, 1)},$$

or that too the points set $\{(\lambda, \eta)\}$ be a fixed points set for a characteristic map $\psi_1\colon \mathbb{E}^2 \to \mathbb{E}^2$ defined by the formula

$$P(\lambda, \eta) + iQ(\lambda, \eta) \mapsto \lambda + i\eta \tag{2}$$

acting when iterations.

Recently author has published prime examples for dissipative dynamic systems acting on the plane possessing the Wada property [1]. The doubling variables universal scheme in combination with the construction of a homogeneous differential system has been applied, so that the condition that the stationary points qualitative properties with respect to the diffeomorphism has been preserved. Thereby the add-in problem has been solved for a generic diffeomorphism acting on the plane.

Theorem 1 (Main). *If characteristic map* (2) *possesses Wada property then system* (1) *defines topologically the only system*

$$\begin{aligned} \dot{\lambda}_1 &= P(\lambda_1, \eta_1) - \lambda_2,\ \dot{\eta}_1 = Q(\lambda_1, \eta_1) - \eta_2, \\ \dot{\lambda}_2 &= \lambda_1 - P(\lambda_2, \eta_2),\ \dot{\eta}_2 = \eta_1 - Q(\lambda_2, \eta_2). \end{aligned} \tag{3}$$

such that some stroboscopic cross section for system (3) *possesses Wada property.*

The specific vector fields prime examples construction have been discussed.

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OPERATIONS ON UNIVERSAL ENVELOPING ALGEBRA AND THE "ARGUMENT SHIFT" METHOD

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If a vector field X on a Poisson manifold (M, π) is given such that the square of the Lie derivative in the X direction "kills" the Poisson bivector π, then there is a well-known simple method of "shifting the argument" (along X) to construct a commutative subalgebra (with respect to the Poisson bracket) inside the algebra of functions on M. In a particular case, this method can be applied to the Poisson-Lie bracket on the symmetric algebra of an arbitrary Lie algebra and gives (according to a well-known result, the proven Mishchenko-Fomenko conjecture) maximal commutative subalgebras in the symmetric algebra. However, lifting of these algebras to commutative subalgebras in the universal enveloping algebra, although possible, is based on rather nontrivial results from the theory of infinite-dimensional Lie algebras. In my talk, I will describe results that allow one to construct on the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$ the operators of "quasidifferentiation" and with their help construct a commutative subalgebra in $U\mathfrak{gl}_n$ in a manner similar to the procedure of "shifting the argument", when n is small enough. I will also describe how, in the general case, this question can reduced to the combinatorial question of commuting a certain set of operators in tensor powers of $\mathbb{R}^n$. The talk is based on collaborations with Dimitry Gurevich, Pavel Saponov and Ikeda Yasushi.

THE SMALLER (SALI) AND THE GENERALIZED (GALI) ALIGNMENT INDEX METHODS OF CHAOS DETECTION

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Determining the chaotic or regular nature of orbits of dynamical systems is a fundamental problem of nonlinear dynamics, having applications to various scientific fields. The most commonly employed method for distinguishing between regular and chaotic behavior is the evaluation of the maximum Lyapunov exponent (MLE) σ_1, because if $\sigma_1 > 0$ the orbit is chaotic [1 and references therein]. The main problem of using this chaos indicator is that its numerical evaluation may take a long -and not known a priori- amount of time to provide a reliable estimation of the MLE's actual value. Over the years, many techniques have been developed, which try to overcome this problem (review presentations of some of them can be found in [2]).

In this talk we will focus our attention on two very efficient methods of chaos detection: the Smaller (SALI) and the Generalized (GALI)

Alignment Index techniques [3-9]. We will first recall the definitions of the SALI and the GALI and will briefly discuss the behavior of these indices for conservative Hamiltonian systems, emphasizing that these methods are based on the evolution of more than one deviation vectors from the studied orbit, in contrast to the computation of the MLE where only one deviation vector is needed.

Then, we will explain how one can use these methods to investigate the dynamics of time-dependent dynamical systems by considering a barred galaxy model whose parameters evolve in time [10]. We will show that the SALI/GALI technique is as a reliable criterion to estimate the relative fraction of chaotic versus regular orbits in such time-dependent systems, which proves to be much more efficient than the computation of the MLE. In particular, we will demonstrate that these indices are able to capture subtle changes in the dynamical nature of an orbit (or ensemble of orbits) even for relatively small time intervals; a property which makes them ideal for detecting dynamical transitions in time-dependent systems.

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NON-ABELIAN GENERALIZATIONS OF INTEGRABLE PDES AND ODES

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A general procedure for non-abelinization of given integrable polynomial differential equation is described. The main idea is: we generalize simultaneously both the equation and its series of first integrals of conserved densities. As an example, we consider the NLS type equations [1]. We also find non-abelinating Euler's top [2]. Using the matrix Painlevé–Kovalevskaya test, we find non-abelian Painlevé-2 [3] and Painlevé-4 [4] equations.

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TETRAHEDRON EQUATION, WEIGHTED GRAPHS AND LOOP QUANTUM GRAVITY

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The Zamolodchikov tetrahedron equation plays a key role in three-dimensional statistical models, exactly-solvable quantum mechanical models on two-dimensional lattices, the theory of cluster algebras and topological invariants of 2-knots. It turns out that this algebraic structure, which is a 3-dimensional analog of the Yang-Baxter equation, also allows to efficiently perform evaluations on spin foams, which are one of the approaches to the loop quantum gravity. I will talk about several joint results with I. Korepanov, G. Sharygin, V. Gorbunov, B. Bychkov and A. Kazakov.

THE LOCAL DYNAMICS OF DIFFERENTIAL EQUATIONS WITH DELAY AND PERIODIC COEFFICIENTS

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Consider delay differential equation

$$\dot{x} + x = ax(t-T) + f(x).$$

The local (in the vicinity of the zero equilibrium state) dynamics of a differential equation with delay is investigated in the case when the parameter T characterizing the delay is sufficiently large, i.e.

$$T \gg 1,\ \varepsilon = \frac{1}{T} \ll 1$$

and the parameter

$$a = \pm(a_0 + \varepsilon^2 a_1(t))$$

is not a constant.

For $|a_0| < 1$ zero solution is stable, for $|a_0| > 1$ there are no stable solutions in the vicinity of the zero equilibrium state. If $|a_0|$ is close to 1, it is necessary to investigate further. In these cases close to critical, the theory of quasinormal forms [1] was applied for the analysis.

Using asymptotic methods, analogs of normal - quasinormal - forms are constructed in critical cases. We found that in the case under consideration with a large delay, the normal form becomes non-autonomous.

For a_0 close to 1 we have normal form – parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{dr^2} + a_{1*}(r) \cdot u(\tau, r) + f_2 u^2 \tag{1}$$

with periodic boundary conditions

$$u(\tau, r) = u(\tau, r + 1), \tag{2}$$

where $a_{1*}(r)$ is a function containing such terms $a_1(r)$ whose frequencies are multiples of 2π.

For a_0 close to -1 we have normal form – parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} - a_{1*}(r) \cdot u(\tau, r) + (f_2^2 - f_3)u^3 \tag{3}$$

with antiperiodic boundary conditions

$$u(\tau, r) = -u(\tau, r + 1). \tag{4}$$

where $a_{1*}(r)$ is a function containing such terms $a_1(r)$ whose frequencies are multiples of 2π.

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FUNCTIONAL DIFFERENTIAL EQUATION WITH STRETCH AND TWIST

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The paper considers a boundary value problem in a bounded plane domain for a second-order functional differential equation containing a combination of dilations and rotations of the highest derivatives of the required function. Necessary and sufficient conditions are found in the algebraic form for the fulfillment of a Gording-type inequality that ensures the unique (Fredholm) solvability, discreteness, and sectorial structure of the spectrum of the Dirichlet problem. In the literature, the term strongly elliptic equation is adopted in this situation. The derivation of the above conditions, expressed directly in terms of the coefficients of the equation, is based on a combination of the Fourier and Gel'fand transforms of the elements of the commutative B-algebra generated by the expansion and rotation operators. The main point here is to clarify the structure of the space of maximal ideals of this algebra. It was proved that the space of maximal ideals is homeomorphic to the direct product of the spectra of the dilatation operator (circle) and the rotation operator (the whole circle in the case when the angle of rotation α is incommensurable with π, and a finite set of points on the circle when $O\pm$ is commensurable with π). This difference between the two cases for α leads to the fact that, depending on α, the conditions for the unique solvability of the boundary value problem may have significantly different forms and, for example, for α comparable to π, may depend not only on the absolute value, but also on the sign of the coefficient at term with rotation.

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PARTIALLY HYPERBOLIC SYMPLECTIC AUTOMORPHISMS OF 4-TORUS

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We study topological properties of automorphisms of 4-dimensional torus generated by an integer matrices being symplectic either with respect to the standard symplectic structure in $\mathbf{R}^4$ or a nonstandard symplectic structure generated by an integer non-degenerate skew-symmetric unimodular matrix. Such symplectic matrix generates a partially hyperbolic automorphism of the torus, if its eigenvalues consist of a pair of real numbers outside the unit circle and a pair of complex conjugate numbers on the unit circle. The main classifying tool is the structure of a foliation generated by unstable (stable) leaves of the automorphism and the automorphism action on the center manifold.

We study the one-dimensional foliations on the torus generated by the projection on the torus of unstable and stable eigen-lines. We prove that the related foliation on the torus T^4 can be either transitive or decomposable into 2-tori. Each case requires a special investigation for its classification. Automorphisms realizing all possible cases are provided (details and proofs see in [1]).

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TENSOR FIELDS WITH SIMULTANEOUSLY NON-ZERO NIJENHUIS AND HAANTJES TENSORS

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In 1951-55 Nijenhuis and Haantjes proposed construction of the integrable distributions as the complements to eigenvector fields X_i of a (1,1) tensor field A on some n-dimensional manifold. This construction is applicable if

- all n eigenvalues of A are real, distinct, and functionally independent;
- either Nijenhuis tensor N_A is equal to zero or Haantjes tensor H_A is equal to zero.

Now Nijenhuis and Haantjes tensors can be found in various parts of mathematics, mathematical physics, and classical mechanics, but the overwhelming majority of applications is related to the vanishing of one of these tensors

$$N_A(u,v) = 0 \qquad \text{or} \qquad H_A(u,v) = 0 \tag{1}$$

for any vector fields u and v, which guarantees integrability of all the distributions spanned by the $n-1$ eigenvector fields X_i, i.e. with geometry of the so-called Nijenhuis and Haantjes manifolds.

We want to discuss (1,1) tensor fields A with the simultaneously non-zero Nijenhuis and Haantjes tensors

$$N_A(u,v) \neq 0\,, \qquad \text{and} \qquad H_A(u,v) \neq 0 \tag{2}$$

and the corresponding integrable systems outside Nijenhuis and Haantjes geometry.

DIRICHLET-TO-NEUMANN MAP FOR EVOLUTION PDES ON THE HALF-LINE WITH TIME-PERIODIC BOUNDARY CONDITIONS

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This talk is about a new approach to the asymptotic analysis for large times of the generalised "Dirichlet-to-Neumann map", i.e., the problem of determining the unknown boundary values in terms of the given initial and boundary conditions, for evolution PDEs on the half-line with time-periodic boundary conditions [1]. Our analysis includes linear and integrable nonlinear PDEs; these types of equations share the property that they possess a Lax pair [2,3]. We show that the time-dependent part of the Lax pair yields the large t asymptotics of the periodic unknown boundary values in terms of the given periodic boundary data via an elegant algebraic calculation. Using the time-dependent part of the Lax pair we construct the "Q-equation" on which the explicit determination of the unknown boundary values is based. The Q-equation was first introduced by Lenells and Fokas for the Nonlinear Schrödinger (NLS) equation [4], but we simplify the techniques of this paper and we show the usefulness of the Q-equation approach also for linear equations. The latter is of particular interest, since it illustrates the power of Lax pairs even in the case of linear problems. Lax pairs were a crucially important ingredient of the Fokas Method [5] and it is noteworthy that in this problem their significance is re-confirmed. Our analysis of the generalised Dirichlet-to-Neumann map via an appropriately constructed Q-equation focuses mainly (though not exclusively) on linear equations including the heat equation, the convection-diffusion equation, the linearised KdV equation and the linearised version of the NLS equation. With regards to the applicability of the method to nonlinear equations, we show that the Q-equation provides a straightforward way for deriving the remarkable results of Boutet de Monvel, Kotlyarov and Shepelsky for the focusing NLS equation [6].

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WDVV EQUATIONS AND INVARIANT BI-HAMILTONIAN FORMALISM

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The WDVV equations are central in Topological Field Theory and Integrable Systems. We prove that in low dimensions the WDVV equations are bi-Hamiltonian. The invariance of the bi-Hamiltonian formalism is proved for $N = 3$. More examples in higher dimensions show that the result might hold in general. The invariance group of the bi-Hamiltonian pairs is the group of projective reciprocal transformations. The significance of projective invariance of WDVV equations is discussed.

Based on a joint work with J. Vasicek, https://arxiv.org/abs/2104.13206

ASYMPTOTIC SOLUTION OF THE BOUNDARY CONTROL PROBLEM FOR THE INTERIOR LAYER BURGERS-TYPE EQUATION WITH MODULAR ADVECTION

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Recent results of using asymptotic analysis for asymptotic solving of some classes inverse problems for reaction-diffusion-advection equations are presented. This approach was applied to a new class of time-periodic reaction-diffusion-advection problems with internal transition layers. Particularly, for Burgers-type equation, which has a time-periodic solution of moving front type, asymptotic analysis was applied to solve the boundary control problem by known information about the observed solution of the direct problem at a given time interval (period).

These results will be illustrated by the following problem

$$\begin{aligned}
&\varepsilon\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}+\frac{\partial |u|}{\partial x}-K\cdot|u|=0,\\
&(x,t)\in D:=\{x\in(-1,1);\quad t\in\mathbb{R}\},\\
&u(-1,t;\varepsilon)=u^{(-)}(t),\qquad u(1,t;\varepsilon)=u^{(+)}(t),\qquad t\in\mathbb{R},\\
&u(x,0;\varepsilon)=u(x,t+T;\varepsilon),\qquad x\in[-1,1],\qquad t\in\mathbb{R},
\end{aligned}\tag{1}$$

where ε – small parameter ($0<\varepsilon\ll 1$), functions $u^{(-)}(t)$ and $u^{(+)}(t)$ are sufficiently smooth and T-periodic in t, $K>0$ – given constant. Some application are in [1,2].

The inverse problem is to restore one of the boundary conditions, at which the front will move according to a given time law, or this boundary function must be determined by observing the moving front location. The second boundary condition is considered known in this case.

Main idea of our approach is based on the fact that asymptotic analysis allows to reduce the original problem to a much simpler problems which connect with given accuracy some parameters (boundary conditions) of the original model to be restored with the observed data of the mowing front location. The concept of an asymptotic solution of coefficient inverse problems is introduced. The accuracy of the solution is estimated.

The proposed approach can be applied to some other classes of inverse problems for reaction-diffusion-advection equations with boundary and internal layers [3,4].

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QUASI-DIFFERENTIATION OF CENTRAL ELEMENTS OF THE UNIVERSAL ENVELOPING ALGEBRA $U\mathfrak{gl}_n$

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We describe the explicit formula for the first-order quasi-differentiation of an arbitrary central element of the universal enveloping algebra of a general linear Lie algebra. We apply it to show that derivations of any two central elements of the universal enveloping algebra commute. This contributes to the Vinberg problem - finding commutative subalgebras in universal enveloping algebras, so that the underlying Poisson algebras are determined by the argument shift method.

QUADRATICALLY INTEGRABLE GEODESIC FLOWS ON NON-ORIENTABLE TWO-DIMENSIONAL SURFACES AND BILLIARDS WITH SLIPPING

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In the work [3] by A.T. Fomenko was introduced a new class of billiards with slipping. Let's consider F — the isometry of the boundary of a flat ellipse, which translates the point x into the diametrically opposite point y. Let the material point move rectilinearly through the region and fall into the point x on the ellipse at some angle α. Then it continues its movement from a diametrically opposite point, leaving with the same angle α, ‘slipping ” along the border. On this basis, the billiard is called a ”billiard with slipping”.

This system has the same integral as the billiard in an ellipse — the parameter of the confocal quadric, which is the caustic of the trajectory. This allows us to raise the question of the topology of the Liouville foliation and the calculation of the Fomenko-Zieschang invariants that characterize the closures of the solutions of the system [1].

As it turned out, billiards with slipping implement some such flows on **non-orientable** surfaces. Thus, we can prove that any geodesic flow of quadratically integrable flows on a two-dimensional undirectable manifold (Klein bottle or projective plane) is Liouville equivalent to a suitable billiard with slipping. In this case, the quadratic integrals of geodesic flows are reduced to one canonical quadratic integrals on the billiard.

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SPATIOTEMPORAL CONDENSATION OF WALK-OFF MULTIMODE SOLITONS

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Multimode fibers (MMFs) have recently attracted considerable interest, motivated by their potential for increasing the transmission capacity of long-distance optical links via mode-division-multiplexing, which exploits the multiple transverse modes of the fiber as information carriers. Moreover, the capacity of MMF to carry high-energy beams is important for scaling the power delivery capabilities of fiber laser sources. Graded-index (GRIN) MMFs support the propagation of MM solitons over long distances [1].

Theoretical treatments of spatiotemporal soliton propagation in MMFs mostly rely on the simple variational approach (VA) [2]-[3]. The validity of the VA is based on the assumption that the initial beam shape is maintained unchanged upon propagation, except for a limited set of slowly-varying parameters (e.g., the beam amplitude and width).

In this work, we reveal that the generation of femtosecond MM soliton beams in GRIN fibers exhibits previously unexpected and rich nonlinear dynamics, which make them very different from their well-known single mode counterparts. In single mode fibers, a soliton forms when chromatic dispersion pulse broadening is compensated for by self-phase modulation. This means that a single mode soliton can have an arbitrary temporal duration, provided that its peak power is properly adjusted. In contrast, the soliton condition in a MMF also requires the compensation of modal dispersion. This leads to the (previously un-noticed) property that MM solitons composed of non-degenerate modes with axial symmetry have a fixed pulse width and energy at each wavelength, independently of the input pulse duration.

Moreover, we observed that, for long propagation distances, the beam content of MM solitons is irreversibly attracted toward the fundamental mode of the MMF, which provides a condensate. This is due to the action of intermodal four-wave mixing (FWM) and stimulated Raman scattering (SRS) [4]. Our results are of high importance for different applications of nonlinear MMFs, such as high-power mode-locked laser sources, and soliton-based transmissions.

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