Assignment 3

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Download all python codes from

https://github.com/VIB2020/AI1103/blob/main/Assignment%203/code/Assignment%203.py

and latex-tikz codes from

https://github.com/VIB2020/AI1103/blob/main/Assignment%203/Assignment%203.pdf

1 GATE 2015 (EE paper 01 new 2), Q 27 (Electrical Engg section)

Two players A, and B alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is:

(A) $\frac{5}{11}$

(B) $\frac{1}{2}$

(C) $\frac{7}{13}$

(D) $\frac{6}{11}$

2 Solution

- Given the die is fair.
- So, for any given throw by A or B:
  The probability of getting 6 = $\frac{1}{6}$ = p (say)
  The probability of NOT getting 6 = $\frac{5}{6}$ = q (say)

State	Corresponding description
1	Player who plays first. Here: PLAYER A
2	Player who plays second. Here: PLAYER B
3	WINNER

Table 1: State description table

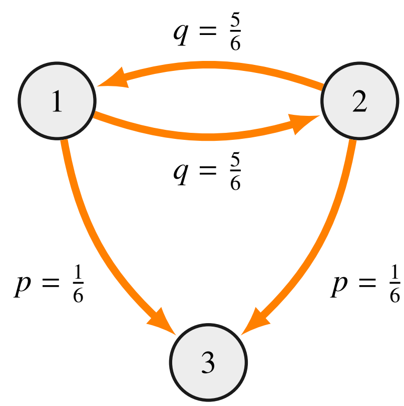

Figure 4: Markov chain for the given problem

Definition 2.1. $\implies$ The chain is non-*ergodic* and irregular.
State 3 is an ABSORBING state.
Transition/Stochastic matrix corresponding to single throw of dice: X

$$\begin{array}{cc} & \text{(To)} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{(From)} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & q & p \\ q & 0 & p \\ 0 & 0 & 1 \end{bmatrix} = X \end{array}$$

$$X^n \text{ corresponds to n throws of dice}$$

$$\Pr(\text{A wins in 'i' throws}) = p_{13}(i) = (X^i)_{13} \tag{1}$$

Definition 2.2. Canonical form of X^2 is:

$$\left(\begin{array}{cc|c} q & 0 & p \\ 0 & q & p \\ \hline 0 & 0 & 1 \end{array}\right)$$

Every transition matrix in it's canonical form can be partitioned as $\left(\begin{array}{c|c} Q & R \\ \hline O & J \end{array}\right)$
where Q and R arise from transition probabilities between non-absorbing states
O = Null matrix, J = Identity matrix
X^k approaches $\overline{X}$ as k increases. $\overline{X}$ is the limiting matrix

$$Q = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, R = \begin{pmatrix} p \\ p \end{pmatrix}, O = \begin{pmatrix} 0 & 0 \end{pmatrix}, J = [1] \tag{2}$$

$$\overline{X} = \begin{pmatrix} O & NR \\ O & J \end{pmatrix} \tag{3}$$

- Given constraint: A wins
- From the above Markov chain [??]:
  - We can go To state 3 From state 1 only in ODD no.of steps.
  - We can come back to our initial state only in EVEN no.of steps.
  - A wins in $2n + 1^{th}$ step $\implies$ Both A and B should not get 6 in any of the previous 2n throws.
- From Def 2.2: Matrix Q corresponds to the transtition between non-absorbing states in 1 step.
  $\implies$ We need to come back to state 1 each time we go to state 2.
  $\implies$ We need to increment by two steps.
  i.e we need to consider only Q^{2n} $(n \geq 0, n \in \mathbb{Z})$

The modified fundamental matrix is:

$$N = I + Q^2 + Q^4 + ... \tag{4}$$

$$= (I - Q^2)^{-1} = \begin{pmatrix} 1 - q^2 & 0 \\ 0 & 1 - q^2 \end{pmatrix}^{-1} \tag{5}$$

$$= \begin{pmatrix} \frac{1}{1 - q^2} & 0 \\ 0 & \frac{1}{1 - q^2} \end{pmatrix} \tag{6}$$

Remark.
The matrix D = NR gives the probability of ending up in the absorbing state (state 3) when the chain starts from a non-absorbent state (states 1 and 2).

$$D = \begin{pmatrix} \frac{1}{1 - q^2} & 0 \\ 0 & \frac{1}{1 - q^2} \end{pmatrix} \begin{pmatrix} p \\ p \end{pmatrix} = \begin{pmatrix} \frac{p}{1 - q^2} \\ \frac{p}{1 - q^2} \end{pmatrix} \tag{7}$$

Given A starts the game.

$\implies$ Markov chain starts with A (from State 1)

$$\therefore \Pr(\text{A wins}) = \frac{p}{1 - q^2} = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11} \tag{8}$$

OPTION D is correct