EE 3900 - Assignment 1

VIBHAVASU

Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer . Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```python
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

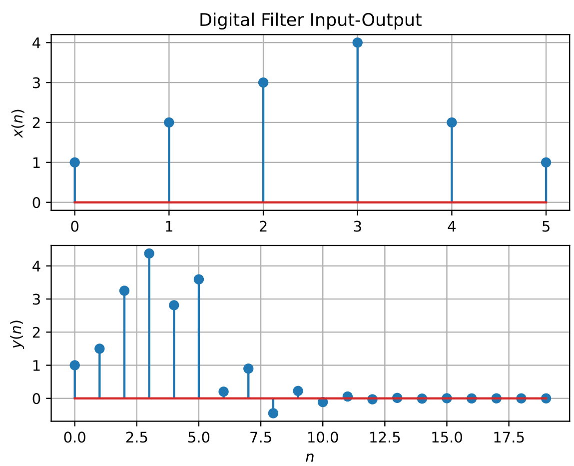

Fig. 3.2

3.3 Repeat the above exercise using a C code.

Solution: : C Code

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e3-3.c
```

Solution: : Python Code

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e3-3.py
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.7}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.8}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.9}$$

4.4 a) Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

b) and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.12}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.13}$$

and from (4.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.14}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

using the fomula for the sum of an infinite geometric progression.

(i) **Solution:**

$$\Delta(z) = \mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = 1 \tag{4.16}$$

(ii) **Solution:**

$$U(z) = \mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} u[n] z^{-n} \tag{4.17}$$

$$= 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} \tag{4.18}$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution:

$$a^n u[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{4.20}$$

$$U'(z) = \mathcal{Z}\{a^n u[n]\} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \tag{4.21}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots \tag{4.22}$$

Given: $|z| > |a|$

$$U'(z) = \frac{1}{1 - az^{-1}} \tag{4.23}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.24}$$

Plot $|H(e^{\jmath\omega})|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/dtft.
    py
```

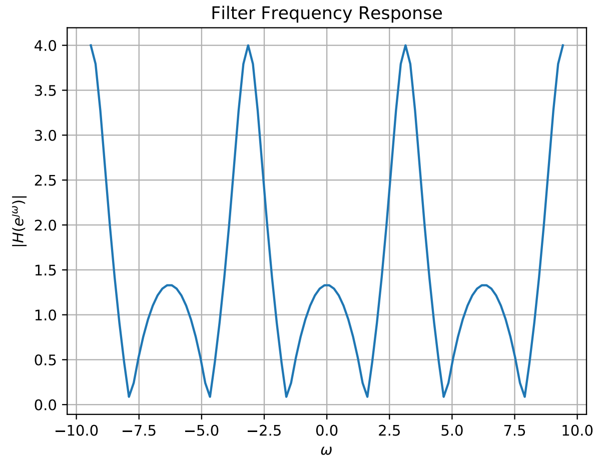

Fig. 4.6: $|H(e^{\jmath\omega})|$

Solution: From (4.9) we get

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.25}$$

$$= \frac{e^{\jmath\omega} + e^{-\jmath\omega}}{e^{\jmath\omega} + \frac{1}{2}} = \frac{2\cos(\omega)}{e^{\jmath\omega} + \frac{1}{2}} \tag{4.26}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| \tag{4.27}$$

$$= \frac{2\cos\omega}{\sqrt{\left(\cos\omega + \frac{1}{2}\right)^2 + \sin^2\omega}} \tag{4.28}$$

$$= \frac{2\cos\omega}{\sqrt{\cos^2\omega + \sin^2\omega + \cos\omega + \frac{1}{4}}} \tag{4.29}$$

$$= \frac{2\cos\omega}{\sqrt{\frac{5}{4} + \cos\omega}} \tag{4.30}$$

For a periodic function of period T,

$$f(x) = f(x + T), T \neq 0 \tag{4.31}$$

Checking if π is a period,

$$\frac{2\cos(\omega + \pi)}{\sqrt{\frac{5}{4} + \cos(\omega + \pi)}} \tag{4.32}$$

$$= \frac{-2\cos\omega}{\sqrt{\frac{5}{4} - \cos\omega}} \tag{4.33}$$

$$\implies H\left(e^{\jmath(\omega+\pi)}\right) \neq H\left(e^{\jmath\omega}\right) \tag{4.34}$$

Checking if 2π is a period

$$\frac{2\cos(\omega+2\pi)}{\sqrt{\frac{5}{4}+\cos(\omega+2\pi)}} = \frac{2\cos\omega}{\sqrt{\frac{5}{4}+\cos\omega}} \tag{4.35}$$

$$\implies H\left(e^{\jmath(\omega+2\pi)}\right) = H\left(e^{\jmath\omega}\right) \tag{4.36}$$

$\therefore$ Period of $H(e^{\jmath\omega})$ is 2π

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$

Solution:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n')e^{-j\omega n'} \tag{4.37}$$

$$\implies \int_{\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega \tag{4.38}$$

$$= \sum_{n'=-\infty}^{\infty} \int_{-\pi}^{\pi} h(n')e^{-j\omega n'}e^{j\omega n}d\omega \tag{4.39}$$

$$= \sum_{n'=-\infty}^{\infty} h(n')2\pi\delta(n'-n) = 2\pi h(n) \tag{4.40}$$

$$\therefore h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega}d\omega \tag{4.41}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.9).
Solution: from (4.9)

$$H(z) = (1+z^{-2})\left(1+\frac{1}{2}z^{-1}\right)^{-1} \tag{5.2}$$

$$\text{For ROC:} \quad \left|\frac{1}{2}z^{-1}\right| < 1 \tag{5.3}$$

$$\implies |z| > \frac{1}{2} \text{ is the ROC for this case} \tag{5.4}$$

$$\begin{array}{rl}
 & 1-\frac{1}{2}z^{-1}+\frac{5}{4}z^{-2}-\frac{5}{8}z^{-3}+\frac{5}{16}z^{-4}\ldots\ldots \\ \hline
1+\frac{1}{2}z^{-1}\Big) & 1+z^{-2} \\
 & \underline{1+\frac{1}{2}z^{-1}} \\
 & -\frac{z^{-1}}{2}+z^{-2} \\
 & \underline{-\frac{z^{-1}}{2}-\frac{1}{4}z^{-2}} \\
 & \underline{\frac{5}{4}z^{-2}} \\
 & \underline{\frac{5}{4}z^{-2}+\frac{5}{8}z^{-3}} \\
 & \underline{-\frac{5}{8}z^{-3}} \\
 & \underline{-\frac{5}{8}z^{-3}-\frac{5}{16}z^{-4}} \\
 & \underline{\frac{5}{16}z^{-4}} \\
 & \frac{5}{16}z^{-4}+\frac{5}{32}z^{-5} \\
 & \vdots
\end{array}$$

$$\implies H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} = 1-\tfrac{1}{2}z^{-1}+\sum_{n=2}^{\infty}\frac{5}{4}z^{-n} \tag{5.5}$$

$$\text{We know that } H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \tag{5.6}$$

comparing coefficients: in the ROC $|z| > \frac{1}{2}$

$$h(n) = \begin{cases} 0 & \text{if} \quad n < 0 \\ 1, & \text{if} \quad n = 0 \\ -\frac{1}{2}, & \text{if} \quad n = 1 \\ 5\left(-\frac{1}{2}\right)^n, & \text{if} \quad n \geq 2 \end{cases} \tag{5.7}$$

$$h(0) = 1, \quad h(1) = -\tfrac{1}{2}, \quad h(2) = \tfrac{5}{4} \tag{5.8}$$

$$h(3) = -\tfrac{5}{8}, \quad h(4) = \tfrac{5}{16} \tag{5.9}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.10}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.9),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.11}$$

$$\therefore h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.12}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hn.py
```

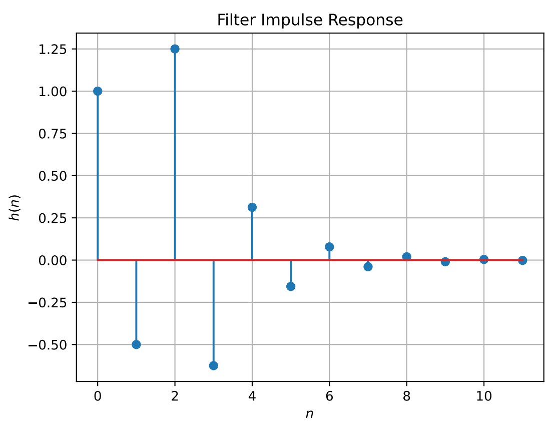

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 Convergent? Justify using the ratio test.
Solution:

$$h(n) = \begin{cases} 0 \quad \text{if} \quad n < 0 \\ 1, \quad \text{if} \quad n = 0 \\ -\frac{1}{2}, \quad \text{if} \quad n = 1 \\ 5\left(-\frac{1}{2}\right)^n, \quad \text{if} \quad n \geq 2 \end{cases} \tag{5.13}$$

Using ratio test:

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \left|\frac{5\left(-\frac{1}{2}\right)^{n+1}}{5\left(-\frac{1}{2}\right)^{n}}\right| = \frac{1}{2} < \infty \tag{5.14}$$

$$\implies h(n) \text{ is convergent}$$

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.15}$$

Is the system defined by (3.2) stable for the impulse response in (5.10)
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = 0 + 1 - \frac{1}{2} + 5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n \tag{5.16}$$

$$= \frac{1}{2} + 5\left(1 - \frac{1}{2} - \left(\frac{1}{1+\frac{1}{2}}\right)\right) \tag{5.17}$$

$$= \frac{1}{2} + \frac{5}{6} = \frac{8}{6} = 1.333 < \infty \tag{5.18}$$

$\therefore h(n)$ is Stable

5.6 Verify the above result using a python code.
Solution: The Following code computes and proves the aboves result

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e5−6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.19}$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hndef
    .py
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.20}$$

Comment. The operation in (5.20) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.py
```

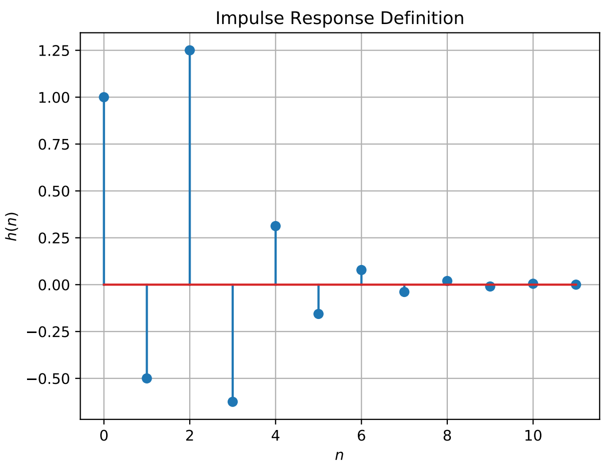

Fig. 5.7: $h(n)$ from the definition

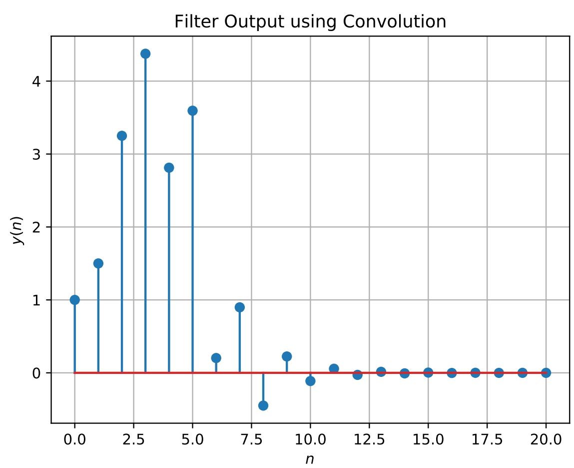

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.
Solution: From (3.1) $x(n) = \{1, 2, 3, 4, 2, 1\}$
From (5.20) $y(n) = x(n) * h(n)$

$$\begin{pmatrix} x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h(1) \\ h(2) \\ h(3) \\ h(4) \\ h(5) \\ h(6) \\ h(7) \\ h(8) \\ h(9) \\ h(10) \\ h(11) \\ h(12) \end{pmatrix} \quad (5.21)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1. \\ -0.5 \\ 1.25 \\ -0.625 \\ 0.3125 \\ -0.15625 \\ 0.078125 \\ -0.0390625 \\ 0.01953125 \\ -0.00976562 \\ 0.00390625 \\ -0.00195312 \end{pmatrix} \quad (5.22)$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.8984375 \\ -0.44921875 \\ 0.224609375 \\ -0.112304688 \\ 0.0561523438 \\ -0.0280761719 \\ 0.0140380859 \\ -7.01904297\times10^{-3} \\ 3.50952148\times10^{-3} \\ -1.75476074\times10^{-3} \\ 8.77380371\times10^{-4} \\ -4.38690186\times10^{-4} \\ 0 \end{pmatrix} \quad (5.23)$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.24)$$

Solution:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (5.25)$$

$$H(z) = \mathcal{Z}\{h(m)\} = \sum_{m=-\infty}^{\infty} h(m)z^{-m} \quad (5.26)$$

$$Y(z) = \mathcal{Z}\{h(m)\} = \sum_{k=-\infty}^{\infty} y(m)z^{-k} \quad (5.27)$$

$$X(z)H(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \sum_{m=-\infty}^{\infty} h(m)z^{-m} \quad (5.28)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n]h[m]z^{-(n+m)} \quad (5.29)$$

Let $m = k - n$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x[n]h[n-k] \right) z^{-k} \tag{5.30}$$

$$= \sum_{k=-\infty}^{\infty} y[n]z^{-k} = Y(\mathrm{z}) \tag{5.31}$$

$$\implies Y(z) = X(z) \cdot H(z) \tag{5.32}$$

now put $n + m = k \quad n = -\infty$

$$\Rightarrow Y(z) = \sum_{k=-\infty}^{\infty} x(m-k) \sum_{m=-\infty}^{\infty} h(m)z^{-k} \tag{5.33}$$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m-k]h[k] \right) z^{-k} \tag{5.34}$$

$$\text{but } Y(z) = \sum_{k=-\infty}^{\infty} y(m)z^{-k} \tag{5.35}$$

$$\Rightarrow y(m) = \sum_{m=-\infty}^{\infty} x[m-k]h(k) \tag{5.36}$$

$$\implies y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.37}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution:

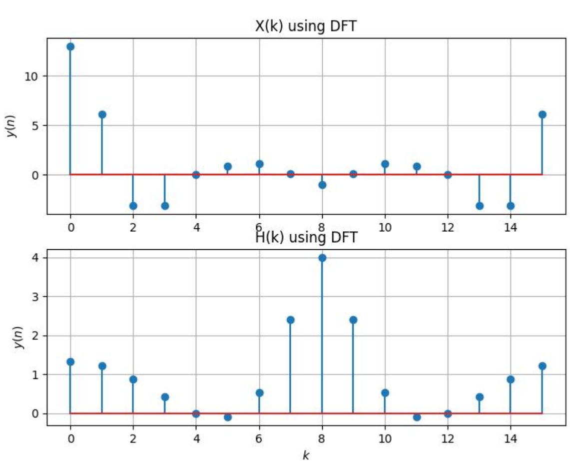

Fig. 6.1: $X(k)$ and $H(k)$ using DFT

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution:

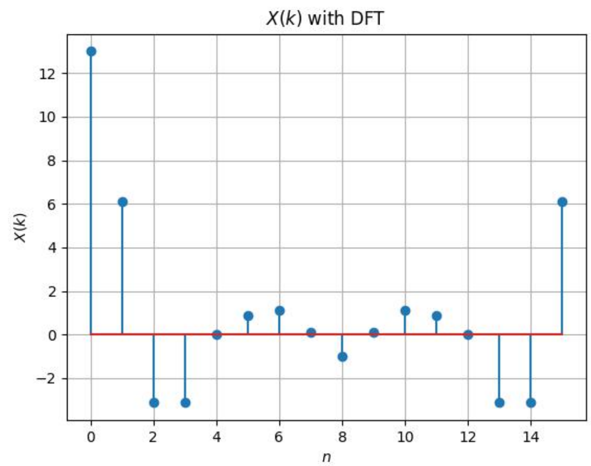

Fig. 6.2: $Y(k)$ using DFT

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

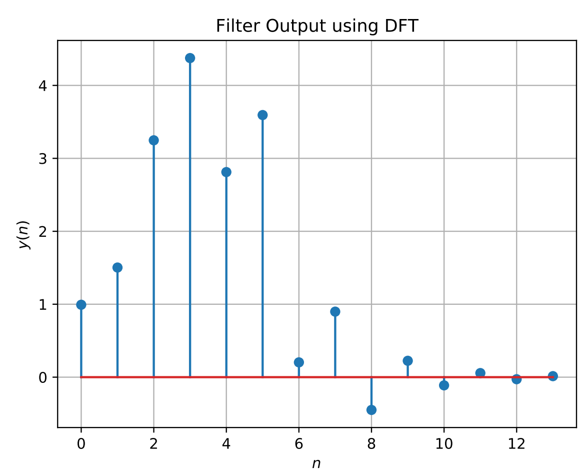

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The following code plots Fig. 6.4, and **??**.

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e6.4.py
```

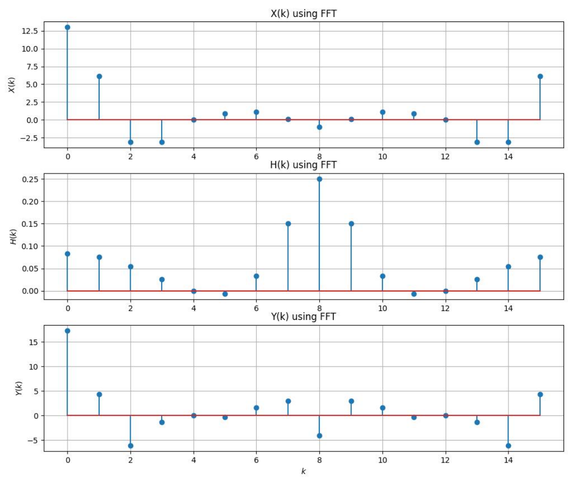

Fig. 6.4: $X(k)$, $Y(k)$ and $H(k)$ using FFT

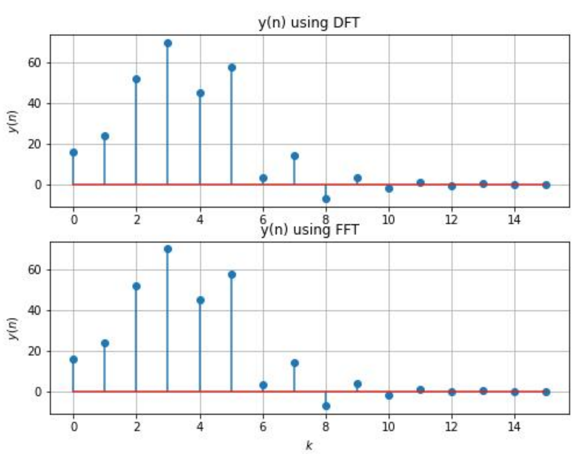

Fig. 6.4: Comparision of $y(n)$ obtained from DFT (above) and IFFT (below)

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_4^0 & W_4^1 & W_4^2 & W_4^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N = e^{-j2\pi/N} \tag{7.8}$$

$$\implies W_{N/2} = e^{-j2\pi/(N/2)} \tag{7.9}$$

$$\therefore W_N^2 = e^{2(-j2\pi/N)} = e^{-j2\pi/(N/2)} = W_{N/2} \tag{7.10}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.11}$$

Solution:

$$\mathbf{F}_2 = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.12}$$

$$\mathbf{D}_{4/2} = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \tag{7.13}$$

$$\text{R.H.S} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_{4/2}\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_{4/2}\mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.14}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{bmatrix} \mathbf{P}_4 \tag{7.15}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{7.16}$$

$$= \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \mathbf{F}_4 \tag{7.17}$$

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.18}$$

Solution:

$$\mathbf{F}_N = \begin{bmatrix} W_N^{00} & \cdots & W_N^{0(N-1)} \\ \vdots & & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.19}$$

Take Even - Odd permutation. i.e, multiply with $\mathbf{P}_N$

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix} W_N^{00} & W_N^{02} & \cdots & \cdots & W_N^{01} & W_N^{03} & \cdots \\ \vdots & & & & \vdots & & \\ W_N^{(N-1)0} & W_N^{(N-1)2} & \cdots & \cdots & W_N^{(N-1)1} & W_N^{(N-1)3} & \cdots \end{bmatrix} \tag{7.20}$$

$$= \begin{bmatrix} \left(W_N^0\right)^0 & \left(W_N^0\right)^2 & \cdots & W_N^1\left(W_N^0\right)^0 & W_N^1\left(W_N^0\right)^2 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_N^{N-1}\right)^0 & \left(W_N^{N-1}\right)^2 & \cdots & W_N^{N-1}\left(W_N^{N-1}\right)^0 & W_{N/2}^{N-1}\left(W_N^{N-1}\right)^2 & \cdots \end{bmatrix} \tag{7.21}$$

$$= \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & W_N^1\left(W_{N/2}^0\right)^0 & W_{N/2+1}^1\left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_N^{N-1}\right)^0 & \left(W_N^{N-1}\right)^2 & \cdots & W_N^{N-1}\left(W_{N/2}^{N-1}\right)^0 & W_N^{N-1}\left(W_{N/2}^{N-1}\right)^1 & \cdots \end{bmatrix} \tag{7.22}$$

$$\begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & W_{N/2}^0\left(W_{N/2}^0\right)^0 & W_{N/2}^0\left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots & W_N^{N/2-1}\left(W_{N/2}^{N/2+1}\right)^0 & W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^1 & \cdots \\ \left(W_{N/2}^{N/2+0}\right)^0 & \left(W_{N/2}^{N/2+0}\right)^1 & \cdots & W_N^{N/2+0}\left(W_{N/2}^{N/2+0}\right)^0 & W_N^{N/2}\left(W_{N/2}^{N/2+0}\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_N^{N-1}\right)^0 & \left(W_N^{N-1}\right)^2 & \cdots & W_N^{N/2+(N/2-1)}\left(W_{N/2}^{N-1}\right)^0 & W_N^{N/2+(N/2-1)}\left(W_{N/2}^{N-1}\right)^1 & \cdots \end{bmatrix} \tag{7.23}$$

$$W_N^{N/2+k} = -W_N^k \tag{7.24}$$

$$W_{N/2}^{N/2+k} = W_{N/2}^k \tag{7.25}$$

$$\begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & W_{N/2}^0\left(W_{N/2}^0\right)^0 & W_{N/2}^0\left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots & W_N^{N/2-1}\left(W_{N/2}^{N/2+1}\right)^0 & W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^1 & \cdots \\ \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & -W_N^0\left(W_{N/2}^0\right)^0 & -W_N^0\left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_N^{N-1}\right)^0 & \left(W_N^{N-1}\right)^2 & \cdots & -W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^0 & -W_{(N/2-1)}^{N-1}\left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \tag{7.26}$$

$$\begin{bmatrix} \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} & \begin{bmatrix} W_{N/2}^0\left(W_{N/2}^0\right)^0 & W_{N/2}^0\left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^0 & W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \\ \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} & -\begin{bmatrix} W_{N/2}^0\left(W_{N/2}^0\right)^0 & W_{N/2}^0\left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^0 & W_N^{N/2-1}\left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \end{bmatrix} \tag{7.27}$$

$$= \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \tag{7.28}$$

Where

$$\mathbf{D}_{N/2} = diag\left(W_N^0 \quad \cdots \quad W_N^{N/2-1}\right) \tag{7.29}$$

$$\mathbf{F}_{N/2} = \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \tag{7.30}$$

$$\implies \mathbf{F}_N\mathbf{P}_N = \tag{7.31}$$

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \tag{7.32}$$

Multiply on both sides by $\mathbf{P}_N$ $\quad (\mathbf{P}_N)^2 = \mathbf{I}$

$$\therefore \mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.33}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.34}$$

Solution:

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.35}$$

$$\mathbf{x}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \tag{7.36}$$

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix} 1 & 3 & 2 & 4 \end{pmatrix} \tag{7.37}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.38}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution:

$$(\mathbf{F}_N\mathbf{x})_k = \sum_{m=0}^{N-1} W_N^{mk} x(m) \tag{7.39}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi km/N} = X(k) = \mathbf{X}_k \tag{7.40}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0)\\ X_1(1)\\ X_1(2)\\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0)\\ X_2(1)\\ X_2(2)\\ X_2(3) \end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix} X(4)\\ X(5)\\ X(6)\\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0)\\ X_1(1)\\ X_1(2)\\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0)\\ X_2(1)\\ X_2(2)\\ X_2(3) \end{bmatrix} \tag{7.42}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0)\\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0)\\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0\\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0)\\ X_4(1) \end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} X_1(2)\\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0)\\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0\\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0)\\ X_4(1) \end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix} X_2(0)\\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0)\\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0\\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0)\\ X_6(1) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_2(2)\\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0)\\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0\\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0)\\ X_6(1) \end{bmatrix} \tag{7.46}$$

$$P_8 \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\\ x(4)\\ x(5)\\ x(6)\\ x(7) \end{bmatrix} = \begin{bmatrix} x(0)\\ x(2)\\ x(4)\\ x(6)\\ x(1)\\ x(3)\\ x(5)\\ x(7) \end{bmatrix} \tag{7.47}$$

$$P_4 \begin{bmatrix} x(0)\\ x(2)\\ x(4)\\ x(6) \end{bmatrix} = \begin{bmatrix} x(0)\\ x(4)\\ x(2)\\ x(6) \end{bmatrix} \tag{7.48}$$

$$P_4 \begin{bmatrix} x(1)\\ x(3)\\ x(5)\\ x(7) \end{bmatrix} = \begin{bmatrix} x(1)\\ x(5)\\ x(3)\\ x(7) \end{bmatrix} \tag{7.49}$$

Therefore,

$$\begin{bmatrix} X_3(0)\\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0)\\ x(4) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X_4(0)\\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2)\\ x(6) \end{bmatrix} \tag{7.51}$$

$$\begin{bmatrix} X_5(0)\\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1)\\ x(5) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_6(0)\\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3)\\ x(7) \end{bmatrix} \tag{7.53}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1\\ 2\\ 3\\ 4\\ 2\\ 1 \end{pmatrix} \tag{7.54}$$

compte the DFT using (7.38)
Solution:

$$\mathbf{X} = \mathbf{F}_6\mathbf{x} \tag{7.55}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 1 & \left(e^{\frac{-j2\pi}{6}}\right) & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^5\\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{10}\\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^9 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{15}\\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{16} & \left(e^{\frac{-j2\pi}{6}}\right)^{20}\\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^5 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} & \left(e^{\frac{-j2\pi}{6}}\right)^{25} \end{bmatrix} \begin{pmatrix} 1\\ 2\\ 3\\ 4\\ 2\\ 1 \end{pmatrix} \tag{7.56}$$

$$= \begin{pmatrix} 13\\ -4-\sqrt{3}j\\ 1\\ -1\\ 1\\ -4+\sqrt{3}j \end{pmatrix} \tag{7.57}$$

12. Repeat the above exercise using the FFT after zero padding **x**.

Solution:

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e7.12.py
```

From the above code we get this output:
$$\begin{bmatrix} 13 \\ -3.1213-6.5355j \\ j \\ 1.1213-0.5355j \\ -1 \\ 1.1213+0.5355j \\ -j \\ -3.1213+6.5355j \end{bmatrix}$$

13. Write a C program to compute the 8-point FFT.

Solution:

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e7.13.c
```

From the above code we get this output:
$$\begin{bmatrix} 13 \\ -3.1327-j6.5545 \\ j \\ 1.1327-j0.5545 \\ -1 \\ 1.1327+j0.5545 \\ -j \\ -3.1327+j6.5545 \end{bmatrix}$$

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution:

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/Assignment%201/e8.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.