Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

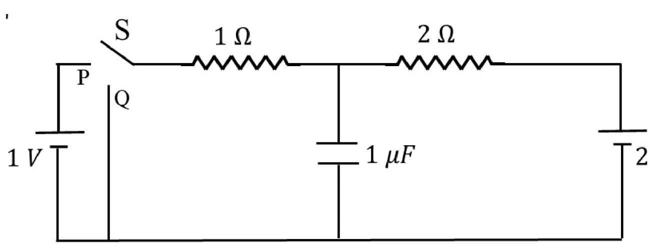

Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:

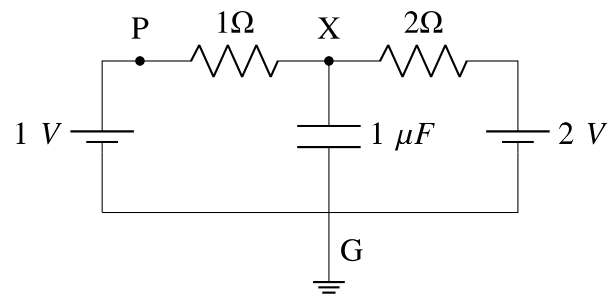

Fig. 2.2

3. Find q_1.
Solution:
After infinite time with switch at P
The capacitor is charged
Applying KCL at X:

$$\frac{V_x - 1}{1} = \frac{2 - V_x}{2} \tag{2.1}$$

$$\implies V_x = \frac{4}{3}\ \text{V} \tag{2.2}$$

$$q_1 = CV = 1\ \mu\text{C} \tag{2.3}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:

$$\mathcal{U}(s) = \int_{0}^{\infty} u(t)e^{-st}dt \tag{2.4}$$

$$= \int_{0}^{0} \frac{1}{2}e^{-st}dt + \int_{0}^{\infty} e^{-st}dt = \frac{1}{s} \tag{2.5}$$

$$\text{R.O.C: } Re(s) > 0 \tag{2.6}$$

5. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \tag{2.7}$$

and find the ROC. **Solution:**

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty u(t)e^{-(s+a)t}dt \tag{2.8}$$

$$= \frac{1}{s+a} \tag{2.9}$$

$$\text{R.O.C: } Re(s) > -a \tag{2.10}$$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

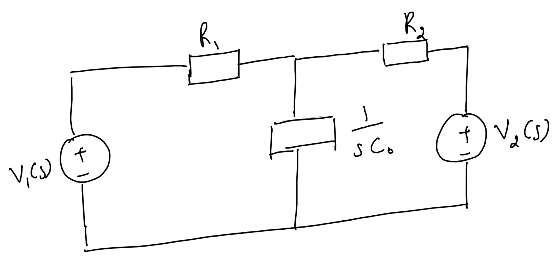

Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.11}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.12}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: Applying KCL at X:

$$\frac{V_x - \frac{1}{s}}{R_1} + s(VC_0) = \frac{\frac{2}{s} - V_x}{R_2} \tag{2.13}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.14}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.15}$$

7. Find $v_{C_0}(t)$. Plot using python.

$$v_{C_0}(t) = \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.16}$$

$$v_{C_0}(t) = \frac{4}{3}\left(1 - e^{-(1.5\times10^6)t}\right)u(t) \tag{2.17}$$

8. Verify your result using ngspice.
Solution:

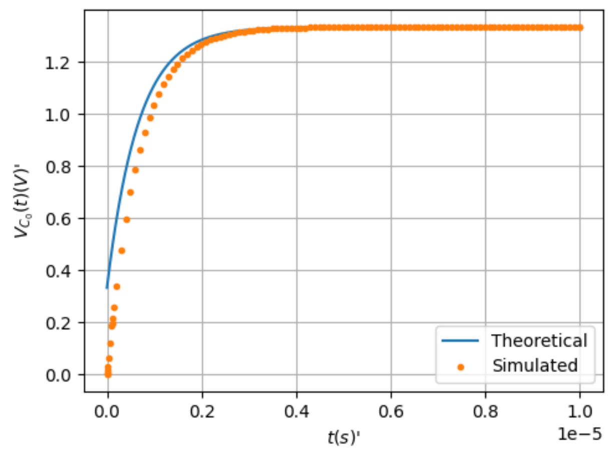

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

9. Obtain Fig. 2.3 using the equivalent differential equation.

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The circuit at steady state when the switch is at Q:
At steady state: Capacitor is charged

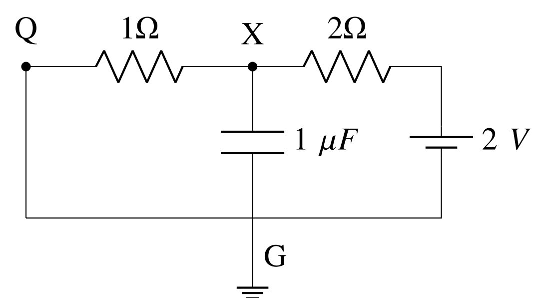

Fig. 3.1

Applying KCL at X.

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \tag{3.1}$$

$$\implies V = \frac{2}{3}\text{V}\, q_2 = \frac{2}{3}\,\mu\text{C} \tag{3.2}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:

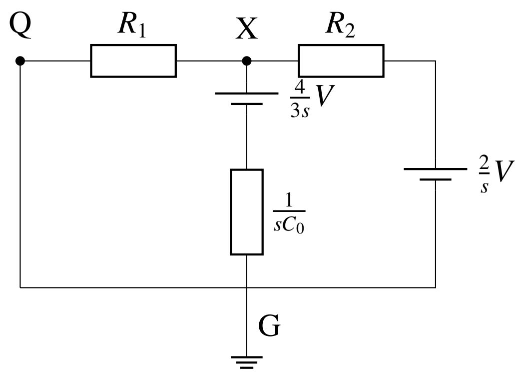

Fig. 3.2

3. $V_{C_0}(s) = ?$ **Solution:**
   Applying KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.3}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.4}$$

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.5}$$

4. $v_{C_0}(t) = ?$ Plot using python. Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.6}$$

   Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.7}$$

   The python code used to plot Fig. 3.3

   https://github.com/DarkWake9/EE3900/blob/main/cktsig/codes/e3.4.py

5. Verify your result using ngspice.
   Solution: The following ngspice script simulates the given circuit

   https://github.com/DarkWake9/EE3900/blob/main/cktsig/codes/e3.cir

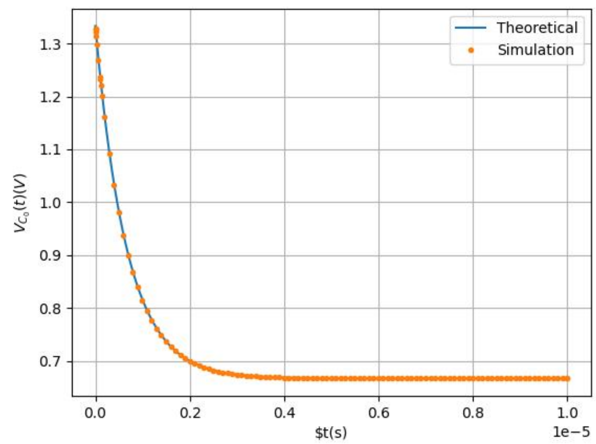

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
   Solution:

$$v_{C_0}(0-) = \lim_{t\to0-} v_{C_0}(t) = \frac{q_1}{C} = \frac{4}{3}\text{ V} \tag{3.8}$$

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}\text{ V} \tag{3.9}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}\text{ V} \tag{3.10}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
   Solution: The equivalent circuit in the t-domain is shown below.

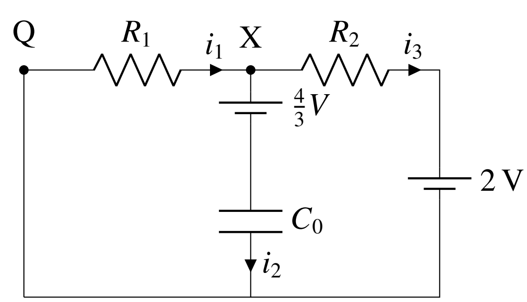

Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.11}$$

$$i_1R_1 + \frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt = 0 \tag{3.12}$$

$$\frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt - i_3R_2 - 2 = 0 \tag{3.13}$$

$$\tag{3.14}$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$\text{Let } i(t) \xleftrightarrow{\mathcal{L}} I(s) \tag{3.15}$$

$$\implies I_1 = I_2 + I_3 \tag{3.16}$$

$$I_1R_1 + \frac{4}{3} + \frac{1}{sC_0}I_2 = 0 \tag{3.17}$$

$$\frac{4}{3} + \frac{1}{sC_0}I_2 - I_3R_2 - 2 = 0 \tag{3.18}$$

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

   Solution: The equivalent circuit in the t-domain is shown below.

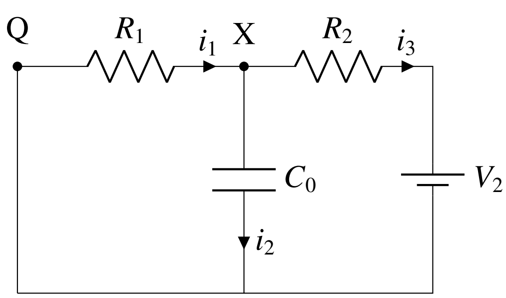

Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \tag{4.1}$$

$$i_1R_1 + \frac{1}{C_0}\int_0^t i_2\, dt = 0 \tag{4.2}$$

$$i_3R_2 + 2 - \frac{1}{C_0}\int_0^t i_2\, dt = 0 \tag{4.3}$$

Differentiating the above equations,

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{\mathrm{d}i_3}{\mathrm{d}t} \tag{4.4}$$

$$R_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + \frac{i_2}{C_0} = 0 \tag{4.5}$$

$$R_2\frac{\mathrm{d}i_3}{\mathrm{d}t} - \frac{i_2}{C_0} = 0 \tag{4.6}$$

Using (4.4) and (4.6) in (4.5),

$$R_1\left(\frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{\mathrm{d}i_3}{\mathrm{d}t}\right) + \frac{i_2}{C_0} = 0 \tag{4.7}$$

$$R_1\frac{\mathrm{d}i_2}{\mathrm{d}t} + \left(1 + \frac{R_1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.8}$$

$$\frac{\mathrm{d}i_2}{\mathrm{d}t} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.9}$$

$$\frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{i_2}{\tau} = 0 \tag{4.10}$$

where $\tau = \frac{C_0R_1R_2}{R_1+R_2}$ is the RC time constant of the circuit. Note that $i_2(0) = \frac{V_2}{R_2}$ A and $i_2 = C_0\frac{\mathrm{d}V}{\mathrm{d}t}$, where V is the voltage of the capacitor. Hence, integrating (4.10),

$$C_0\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{V_2}{R_2} + \frac{C_0V}{\tau} = 0 \tag{4.11}$$

$$\implies \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{\tau} = \frac{V_2}{C_0R_2} \tag{4.12}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
   Solution:

$$H(s) = \frac{V_{C_0}(s)}{V_2(s)} \tag{4.13}$$

In the s domain:

$$\frac{V_{C_0}}{R_1} + \frac{V_{C_0}}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2}{R_2} = 0 \tag{4.14}$$

$$H(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{R_2} \tag{4.15}$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} = \frac{1}{3 + sC_0} \tag{4.16}$$

3. Plot $H(s)$. What kind of filter is it?

It is a Low-Pass filter

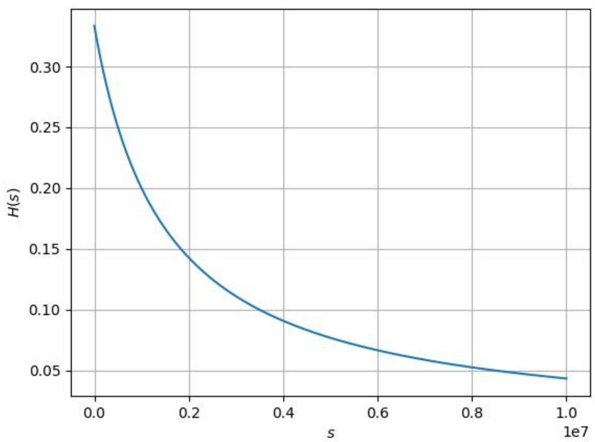

Fig. 4.2: Plot of $H(s)$.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.17}$$

Solution:

5. Find $H(z)$.
6. How can you obtain $H(z)$ from $H(s)$? **Solution:** Apply a Bilinear Transform

$$s \to \frac{2}{T}\frac{z-1}{z+1} \tag{4.18}$$