Fourier Series

VIBHAVASU - EP20BTECH11015

Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \tag{1.1}$$

1.1 Plot $x(t)$. **Solution:**

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e1.1.py

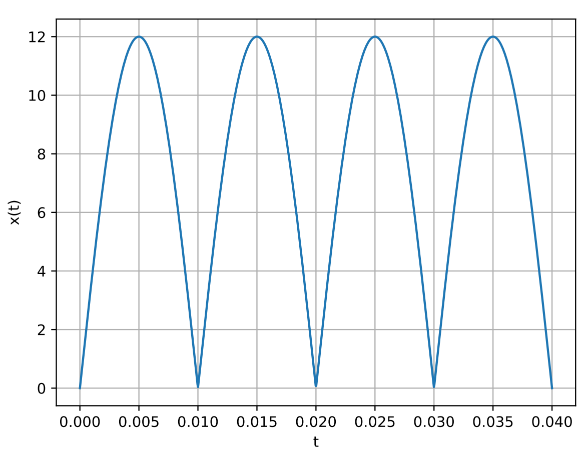

Fig. 1.1

1.2 Show that $x(t)$ is periodic and find its period. **Solution:** From Fig. (**??**), we see that $x(t)$ is periodic. Further,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right)\right| \tag{1.2}$$

$$= A_0 |\sin(2\pi f_0 t + \pi)| \tag{1.3}$$

$$= A_0 |\sin(2\pi f_0 t)| \tag{1.4}$$

$\therefore$ period of $x(t)$ is $\dfrac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:
Multiplying both sides by $e^{-\jmath 2\pi k f_0 t}$ Integrating w.r.t t from $-\frac{1}{2f_0}$ to $\frac{1}{2f_0}$:

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t} dt \tag{2.3}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \left(\sum_{n=-\infty}^{\infty} c_n e^{\jmath 2\pi n f_0 t}\right) e^{-\jmath 2\pi k f_0 t} dt \tag{2.4}$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (n-k) f_0 t} dt \tag{2.5}$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{\delta(n-k)}{f_0} = \frac{c_k}{f_0} \tag{2.6}$$

$$\therefore c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.7}$$

2.2 Find c_k for (1.1)
Solution:

$$c_k = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| e^{-\jmath 2\pi k f_0 t} dt \tag{2.8}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| \cos(2\pi n f_0 t)\ dt$$
$$+ \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| \sin(2\pi n f_0 t)\ dt \tag{2.9}$$

$$= 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t)\ dt \tag{2.10}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin(2\pi(n+1) f_0 t)\right)\ dt$$
$$- f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin(2\pi(n-1) f_0 t)\right)\ dt \tag{2.11}$$

$$= A_0 \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.12}$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.13}$$

2.3 Verify (1.1) using python.

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e2.3.py

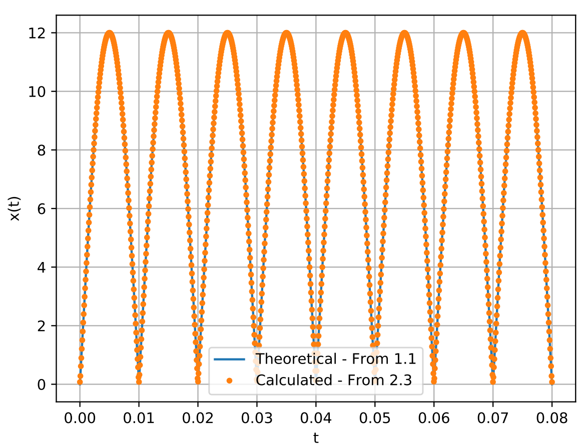

Fig. 2.3

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right) \tag{2.14}$$

and obtain the formulae for a_k and b_k.
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.16}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos(2\pi k f_0 t)$$
$$+ \sum_{k=0}^{\infty} \left(c_k - c_{-k}\right) \sin(2\pi k f_0 t) \tag{2.17}$$

$$\implies a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.18}$$

$$b_k = c_k - c_{-k} \tag{2.19}$$

2.5 Find a_k and b_k for (1.1)
Solution:
From (1.1):

$$x(-t) = x(t) \tag{2.20}$$

$$\implies \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

$$\implies c_k = c_{-k} \tag{2.22}$$

$$a_k = \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \ = 0 \\ \frac{4A_0}{\pi(1-n^2)} & n \ \text{ is even} \\ 0 \quad \text{otherwise} \end{cases} \tag{2.23}$$

$$b_k = 0 \tag{2.24}$$

2.6 Verify (2.14) using python.

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e2.6.py

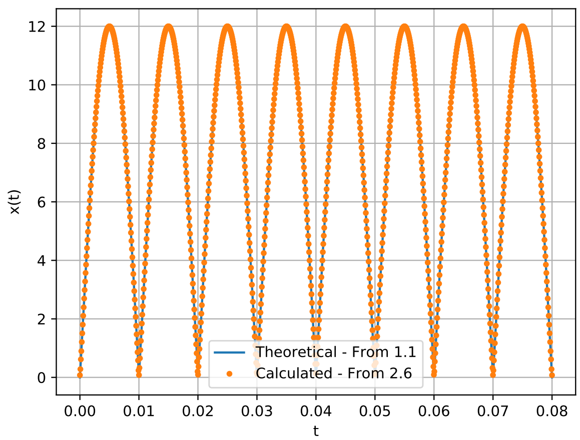

Fig. 2.6: Fourier Expansion of x(t)

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

Solution:
Put $t - t_0 = t'$,

$$g(t - t_0) = g(t') \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t') e^{-\jmath 2\pi f t}\, dt \tag{3.5}$$

$$\int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f (t' + t_0)}\, dt' \tag{3.6}$$

$$= G(f) e^{-j2\pi f t_0} \tag{3.7}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.8}$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{\jmath 2\pi f t}\, df = \int_{-\infty}^{\infty} G(f') e^{\jmath 2\pi f' t}\, df' \tag{3.9}$$

Replace t with $-f$ and f' with t and df' with dt,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-\jmath 2\pi f t}\, dt \tag{3.10}$$

$$\implies G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.11}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-\jmath 2\pi f t}\, dt \tag{3.12}$$

$$= e^{\jmath 2\pi (0) t} = 1 \tag{3.13}$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t} dt \tag{3.14}$$

$$\int_{-\infty}^{\infty} e^{-j2\pi (f + f_0) t} dt = \delta(f + f_0) \tag{3.15}$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$
Solution:
Using the linearity of the Fourier Transform and (3.15),

$$\begin{aligned} \cos(2\pi f_0 t) &= \frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right) \\ &\xleftrightarrow{\mathcal{F}} \frac{1}{2}\left(\delta(f + f_0) + \delta(f - f_0)\right) \end{aligned} \tag{3.16}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.
Solution: From (2.1) and (2.13)

$$x(t) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} c_k \delta(f + k f_0) \tag{3.17}$$

$$X(f) = \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f + 2k f_0)}{1 - 4k^2} \tag{3.18}$$

Python code used to verify (3.18):

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e3.8.py

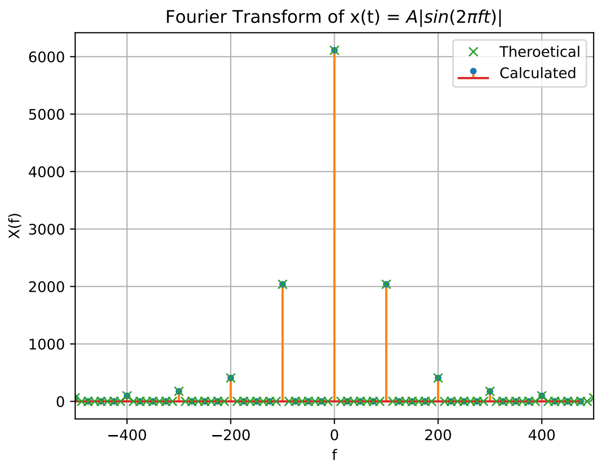

Fig. 3.8

3.9 Show that

$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}(t) \tag{3.19}$$

Verify using python.
Solution:

$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \text{rect}(t)\, e^{-\jmath 2\pi f t}\, dt \tag{3.20}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t}\, dt \tag{3.21}$$

$$= \frac{e^{\jmath\pi f} - e^{-\jmath\pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \text{sinc}(f) \tag{3.22}$$

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e3.9.py

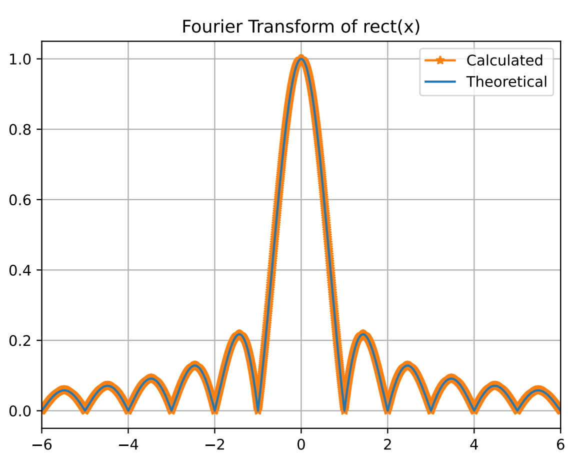

Fig. 3.9

3.10 $\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow}$?. Verify using python.
Solution: From (3.11)

$$\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(-f) = \text{rect}(f) \tag{3.23}$$

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e3.10.py

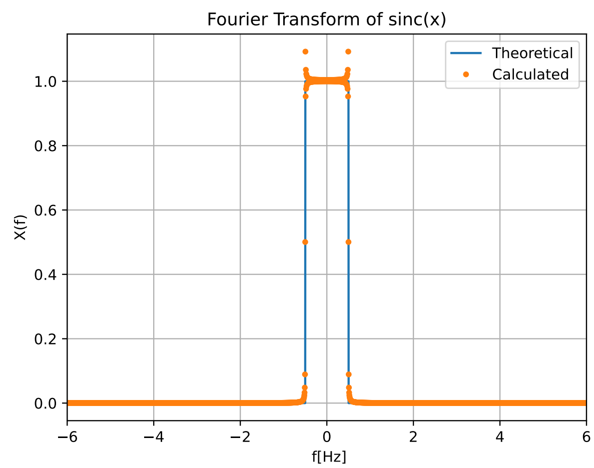

Fig. 3.10

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution:
H(f) is a *Low-pass* filter which allows only the zeroth harmonic and filters the rest.

If f_0 is the cut-off frequency then an ideal Low-pass filter is described by:

$$H(f) = \text{rect}\left(\frac{f}{2f_0}\right) = \begin{cases} 1 & \text{if } |f| < f_0 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0} \text{rect}\left(\frac{f}{2f_0}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.
Solution:

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{\jmath 2\pi f t} df \tag{4.3}$$
$$= \frac{\pi V_0}{2A_0} \int_{-\infty}^{\infty} \text{rect}\left(\frac{f}{2f_0}\right) e^{\jmath 2\pi f t} df \tag{4.4}$$
$$= \frac{\pi V_0}{2A_0} \int_{-f_0}^{f_0} e^{\jmath 2\pi f t} df \tag{4.5}$$
$$= \frac{\pi V_0}{2A_0} \frac{e^{\jmath 2\pi f_0 t} - e^{-\jmath 2\pi f_0 t}}{\jmath 2\pi t} \tag{4.6}$$
$$= \frac{\pi V_0}{A_0} \frac{\jmath \sin(2\pi f_0 t)}{\jmath 2\pi t} \tag{4.7}$$
$$= \frac{\pi V_0}{A_0} f_0 \text{sinc}\,(2 f_0 t) \tag{4.8}$$

4.3 Verify your result using through convolution. **Solution:**

https://github.com/DarkWake9/EE3900/blob/main/charger/codes/e4.3.py

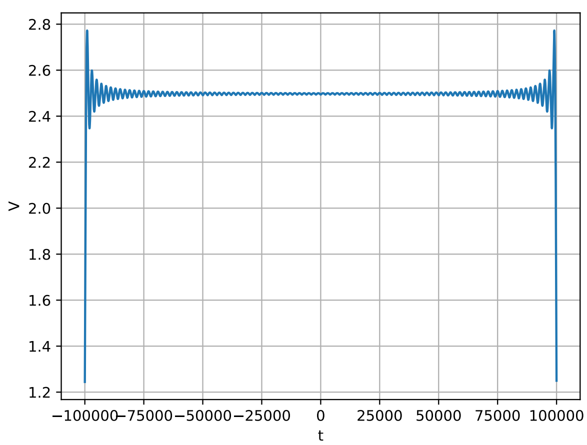

Fig. 4.3

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$. **Solution:**
For an n^{th} order Butterworth filter:
The Gain response $G(f)$ is given by

$$G^2(f) = |H_n(f)|^2 = \frac{1}{1 + \left(\frac{f}{f_0}\right)^{2n}} \tag{5.1}$$

Where f_0 is the cutoff frequency.

5.2 Design a Chebyschev filter for $H(f)$. **Solution:**
For an n^{th} order Chebyschev filter:
The Gain response $G(f)$ is given by

$$G(f) = H_n(f) = \frac{1}{\sqrt{1 + \epsilon^2 T_n^2 \frac{f}{f_0}}} \tag{5.2}$$

Where :

a) f_0 is the cutoff frequency.

b) T_n is a Chebyshev polynomial of the n^{th} order.

$$T_n = \begin{cases} \cos\left(n \cos^{-1} x\right) & |x| < 1 \\ \cosh\left(n \cosh^{-1} x\right) & |x| \geq 1 \end{cases} \tag{5.3}$$

c) ϵ is the ripple

5.3 Design a circuit for your Butterworth filter.
5.4 Design a circuit for your Chebyschev filter.