Pingala Series

VIBHAVASU - EP20BTECH11015

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***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution:

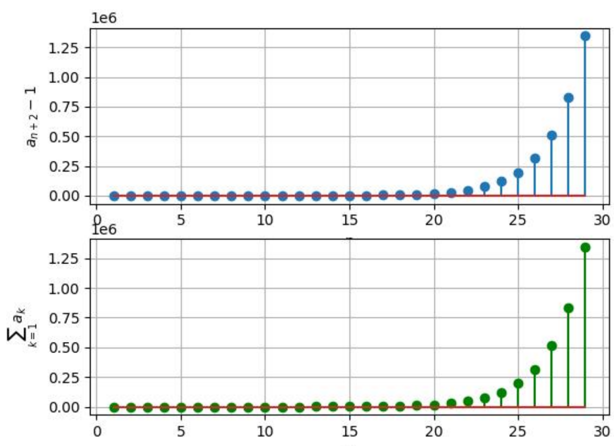

Fig. 1.1

From 1.1 above equation is **True**

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution:
Proposed value of above summation = $\frac{10}{89}$ = 0.11235955056179775
Calculated value from the code = 0.11235955056179774
$\therefore$ The above equation is **True**

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution:

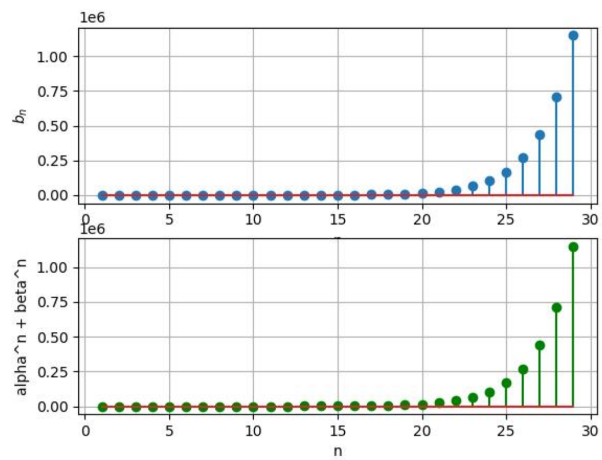

Fig. 1.3

From 2.2 above equation is **True**

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution:
Proposed value of above summation = $\frac{8}{89}$ = 0.0898876404494382
Calculated value from the code = 0.1348314606741573
$\therefore$ The above relation is **False**

Solution:
The solution to all above questions can be found at:

```
wget https://github.com/DarkWake9/EE3900/
    blob/main/pingala/codes/e1.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution:

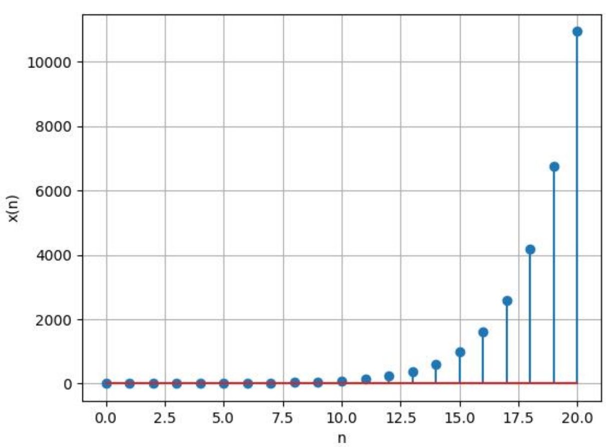

Fig. 2.2

2.3 Find $X^+(z)$.
Solution: From (2.2)

$$\mathcal{Z}^+(x(n+2)) = \tag{2.3}$$
$$\mathcal{Z}^+(x(n+1)) + \mathcal{Z}^+(x(n)) \tag{2.4}$$
$$\implies z^2X^+(z) - z^2x(0) - zx(1) = \tag{2.5}$$
$$zX^+(z) - zx(0) + X^+(z) \tag{2.6}$$
$$\left(z^2 - z - 1\right)X^+(z) = z^2 \tag{2.7}$$
$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.8}$$

Let, α and β be solutions to eqn $z^2 - z - 1 = 0$, $\implies$ they are solutions of $z^{-2} + z^{-1} - 1 = 0$

$$X^+(z) = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)} \quad |z| > \alpha \tag{2.9}$$

2.4 Find $x(n)$.
Solution: Apply $\mathcal{Z}^+$ transform on both sides

$$X^+(z) = \frac{1}{1 - z^{-1} - 2^{-2}} \tag{2.10}$$
$$= \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)} \tag{2.11}$$
$$= \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}}\right) \tag{2.12}$$
$$= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)} u(n) z^{-n} \tag{2.13}$$
$$\implies x_n = a_{n+1}u(n) \tag{2.14}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.15}$$

Solution:

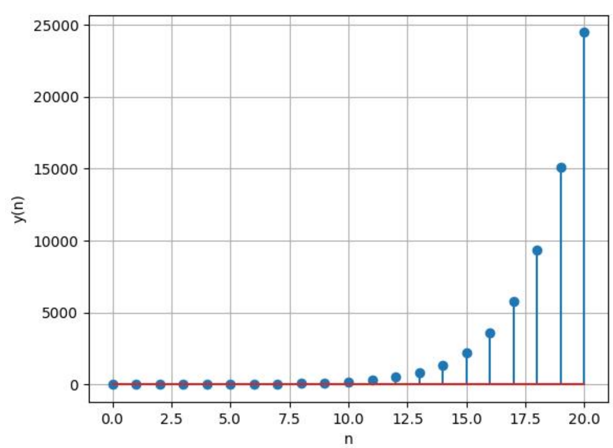

Fig. 2.5

2.6 Find $Y^+(z)$.
Solution:

$$Y^+(z) = \mathcal{Z}^+\left(x(n-1) + x(n+1)\right) \tag{2.16}$$
$$= z^{-1}(X^+(z)) + zX^+(z) - zx(0) \tag{2.17}$$
$$= (z + z^{-1})X^+(z) - z \tag{2.18}$$
$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.19}$$
$$= \frac{2z^{-1} + 1}{1 - z^{-1} - z^{-2}} \tag{2.20}$$

2.7 Find $y(n)$.
Solution: From eq (2.15)

$$y(n) = (x(n-1) + x(n+1))u(n) \tag{2.21}$$
$$= (a_n + a_{n+2})u(n) \tag{2.22}$$
$$\therefore y(n) = b_{n+1}u(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \tag{2.23}$$

3 Power of the Z transform

3.1 Show that

$$u(n)\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{n-1} x(k) \tag{3.2}$$
$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.3}$$
$$= x(n) * u(n-1) \tag{3.4}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.5}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.6}$$

Solution:

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \tag{3.7}$$
$$u(n) = 1 \quad n > 0 \tag{3.8}$$
$$\implies a_{n+2} - 1 = [x(n+1) - 1]\, u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10}X^+(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10}X^+(10) \tag{3.12}$$
$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.13}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.14}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.15}$$

and find $W(z)$.
Solution:

$$\alpha^n + \beta^n \quad n \geq 1 \tag{3.16}$$
$$\equiv (\alpha^{n+1} + \beta^{n+1}) \quad n \geq 0 \tag{3.17}$$
$$= (\alpha^{n+1} + \beta^{n+1})u(n) \tag{3.18}$$

From eq (2.23)

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \tag{3.19}$$
$$\therefore W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.20}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10}Y^+(10) \tag{3.21}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.22}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.23}$$
$$= \frac{1}{10}Y^+(z) \tag{3.24}$$
$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.25}$$

3.6 Solve the JEE 2019 problem.
Solution:

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.26}$$

also as

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \tag{3.27}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{3.28}$$

$$= z\left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}}\right] \tag{3.29}$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z\sum_{n=0}^{\infty}\left(x(n) - 1\right)z^{-n} \tag{3.30}$$

$$= \sum_{n=0}^{\infty}\left(x(n) - 1\right)z^{-n+1} \tag{3.31}$$

$$= \sum_{n=0}^{\infty}\left(x(n+1) - 1\right)z^{-n} \tag{3.32}$$

And from (3.9), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.33}$$

From (3.33), (3.10), (3.21):

Options a, b, and c are correct

Option d is incorrect