Assignment-1

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1 PROBLEM

1.1. Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \quad (1.1.1)$$

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \quad (1.1.2)$$

1.2. Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

$$(1.2.1)$$

and $H(k)$ using h(n).

1.3. Compute

$$Y(k) = X(k)H(k) \quad (1.3.1)$$

2 SOLUTION

2.1. Impulse response h(n) can be found from given difference equation as follows

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \quad (2.1.1)$$

2.2. Let $W_N = e^{-j2\pi/N}$

We can express X as Matrix Multiplication of DFT Matrix and x.

$$X = \left[W_N^{ij}\right]_{N\times N} x, \quad i, j = 0, 1, \ldots, N-1 \quad (2.2.1)$$

2.3. In the given problem, we have N = 6

$$\implies W_6 = e^{-j2\pi/6} = \frac{1}{2} - \frac{\sqrt{3}}{2}j \quad (2.3.1)$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

$$(2.3.2)$$

$$\implies \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \quad (2.3.3)$$

2.4. Similarly, we have

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.625 \\ 0.3125 \\ -0.15625 \end{bmatrix}$$

$$(2.4.1)$$

$$\implies \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} 1.28125 \\ 0.51625 - 0.5142j \\ -0.07813 + 1.1096j \\ 3.84375 \\ -0.07183 - 1.1096j \\ 0.51625 + 0.5142j \end{bmatrix} \quad (2.4.2)$$

2.5. We can find Y using,

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} \times \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} \quad (2.5.1)$$

$$\implies \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} 16.65625 \\ -2.95312 + 1.16372j \\ -0.07813 + 1.1096j \\ -3.84375 \\ -0.07813 - 1.1096j \\ -2.95312 - 1.16372j \end{bmatrix} \tag{2.5.2}$$

2.6. Consider the following property of Complex Exponentials

$$W_N^2 = W_{N/2} \tag{2.6.1}$$

2.7. Let F_N be the N-point DFT Matrix.
Using the property of Complex Exponentials we can express F_N in terms of $F_{N/2}$

$$F_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix} P_N \tag{2.7.1}$$

For N = 6

$$\implies F_6 = \begin{bmatrix} I_3 & D_3 \\ I_3 & -D_3 \end{bmatrix} \begin{bmatrix} F_3 & 0 \\ 0 & F_3 \end{bmatrix} P_6 \tag{2.7.2}$$

where I_3 is the 3x3 identity matrix

$$D_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & W_3^1 & 0 \\ 0 & 0 & W_3^2 \end{bmatrix} \tag{2.7.3}$$

$$P_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7.4}$$

$$\implies P_6 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.7.5}$$

Let

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \end{bmatrix} = F_{N/2} \begin{bmatrix} x(0) \\ x(2) \\ x(4) \end{bmatrix} \tag{2.7.6}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} = F_{N/2} \begin{bmatrix} x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.7.7}$$

be the N/2 point DFTs.

2.8. By replacing the above results in the equation

$X = F_N x$, we get

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & W_6^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^2 \\ 1 & 0 & 0 & -W_6^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -W_6^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -W_6^2 \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.8.1}$$

2.9. Using the above method we have broken down an N-point DFT into 2 N/2-point DFTs

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \end{bmatrix} + \begin{bmatrix} W_6^0 & 0 & 0 \\ 0 & W_6^1 & 0 \\ 0 & 0 & W_6^2 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.9.1}$$

$$\begin{bmatrix} X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \end{bmatrix} - \begin{bmatrix} W_6^0 & 0 & 0 \\ 0 & W_6^1 & 0 \\ 0 & 0 & W_6^2 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.9.2}$$

By recursively breaking down N-point DFT into 2 N/2-point DFTs we can reduce our time complexity from $O(N^2)$ to $O(NlogN)$

2.10. The following code computes Y and generates magnitude and phase plots of X, H, Y

https://github.com/jpln135/EE3025/tree/main/Assignment_1/codes/EE18BTECH11021.py

2.11. The following plots are obtained

2.12. Let $x(n) = \left\{ 1, \underset{\uparrow}{2}, 3, 4, 2, 1, 4, 3 \right\}$

$N = 8 = 2^3$

Recursively breaking down,

$$F_8 = \begin{bmatrix} I_4 & D_4 \\ I_4 & -D_4 \end{bmatrix} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} P_8 \tag{2.12.1}$$
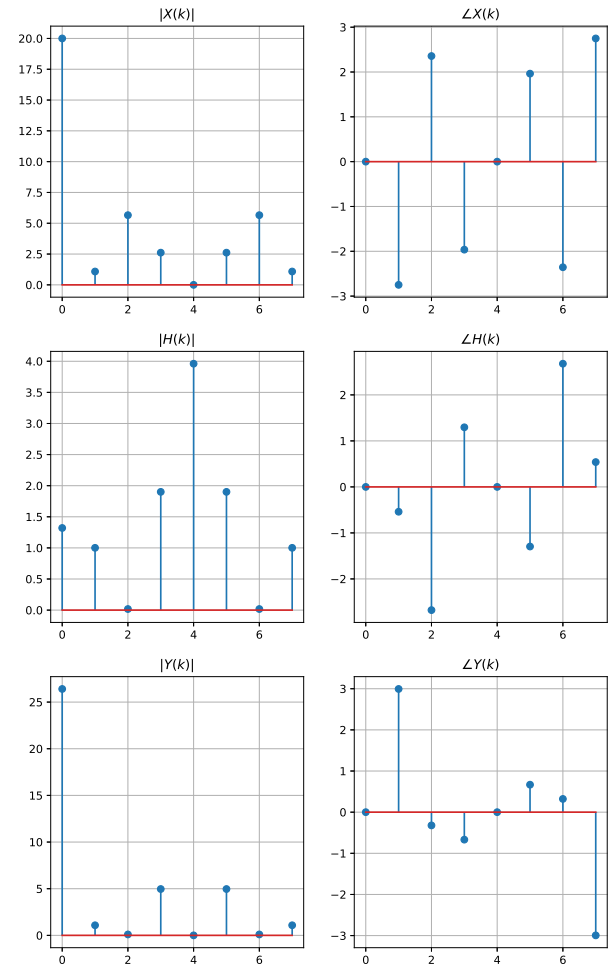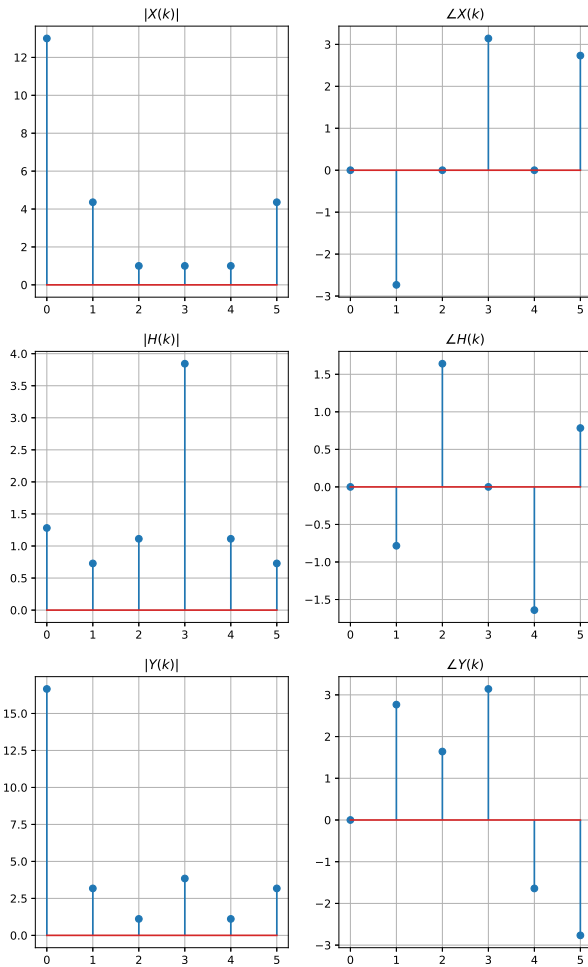
$$F_4 = \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} P_4 \tag{2.12.2}$$

2-point FFT is a base case

$$F_2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + b \\ a - b \end{bmatrix} \tag{2.12.3}$$

2.13. The following code computes Y and generates magnitude and phase plots of X, H, Y

https://github.com/jpln135/EE3025/tree/main/Assignment_1/codes/EE18BTECH11021_2.py

2.14. The following plots are obtained

2.15. The following code compares computation times for N-point DFTs and N-point FFTs of the form N = 2^n for N = 1 (2^0) to 2048 (2^{11})
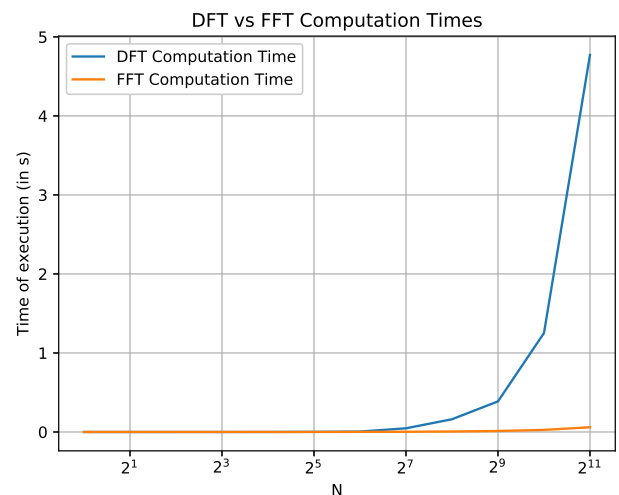
> https://github.com/jpln135/EE3025/tree/main/
> Assignment_1/codes/
> EE18BTECH11021_3.py

2.16. The following plot is obtained
We can see that the computation time for DFT computation rises exponentially as we increase N in powers of 2. But FFT is much faster.

2.17. Explanation:
In DFT - Matrix multiplication of NxN matrix with Nx1 vector. Hence $O(N^2)$ time complex-

ity.

In FFT - N-point FFT is broken down recursively into 2 N/2-point FFTs recursively. Additionally O(N) operation of Vector multiplication is performed on the N/2 point FFTs.

$$T(n) = 2T(n/2) + O(n) \qquad (2.17.1)$$

Solving this recurrence gives $O(NlogN)$ time complexity.

2.18. Step-by-step visualisation 8-point FFTs into 4-point FFTs

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \qquad (2.18.1)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \qquad (2.18.2)$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \qquad (2.18.3)$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \qquad (2.18.4)$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \qquad (2.18.5)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \qquad (2.18.6)$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \qquad (2.18.7)$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \qquad (2.18.8)$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \qquad (2.18.9)$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \qquad (2.18.10)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \qquad (2.18.11)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \qquad (2.18.12)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \qquad (2.18.13)$$

X_3, X_4 combined to give X_1 from Eq.(2.18.3), (2.18.4)

X_5, X_6 combined to give X_2 from Eq. (2.18.5), (2.18.6)

X_1, X_2 combined to give X from Eq.(2.18.1), (2.18.2)

2.19. The following is the C program to compute and print the FFT (N-point where N is of the form 2^n)

https://github.com/jpln135/EE3025/tree/main/
   Assignment_1/codes/
   EE18BTECH11021_FFT.c