Discrete Mathematics

Homework 3
Deadline : 5/27

Exercise 4.1

18. Consider the following four equations:

1) $$1 = 1$$
2) $$2 + 3 + 4 = 1 + 8$$
3) $$5 + 6 + 7 + 8 + 9 = 8 + 27$$
4) $$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$$

Conjecture the general formula suggested by these four equations, and prove your conjecture.

Assume general formula:

$$a_n = \sum_{i=1}^{(2n+1)} (n^2 + i) = = n^3 + (n+1)^3$$

Proof:

$$a_n = \sum_{i=1}^{(2n+1)} (n^2 + i) = \sum_{i=1}^{(2n+1)} (n^2) + \sum_{i=1}^{(2n+1)} (i) = n^2(2n+1) + \sum_{i=1}^{(2n+1)} i = 2n^3 + n^2 + \frac{(2n+1)(2n+1+1)}{2} = 2n^3 + n^2 + 2n^2 + 3n + 1$$

$$= n^3 + (n^3 + 3n^2 + 3n + 1) = n^3 + (n+1)^3$$

24. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 1 \qquad a_2 = 2 \qquad a_n = a_{n-1} + a_{n-2},\ n \geq 3.$$

a) Determine the values of a_3, a_4, a_5, a_6, and a_7.

b) Prove that for all $n \geq 1$, $a_n < (7/4)^n$.

$a) a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, a_7 = 21$

b) For n=1, $a_1 = 1 < (\frac{7}{4})^1$ **is true.**

Assume for all $1 \leq \text{n} \leq \text{k}$ **, k>=1,** $a_k = a_{k-1} + a_{k-2} < (\frac{7}{4})^k$

is true.

For n=k+1,

$a_{k+1} = a_k + a_{k-1} < (\frac{7}{4})^k + (\frac{7}{4})^{k-1} = (\frac{7}{4})^{k-1} * \left(\frac{7}{4} + 1\right) = \left(\frac{7}{4}\right)^{k-1} * \frac{11}{4} = \left(\frac{7}{4}\right)^{k-1} * \left(\frac{44}{16}\right) < \left(\frac{7}{4}\right)^{k-1} * \frac{49}{16} = \left(\frac{7}{4}\right)^{k-1} * \left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{k+1}$

Exercise 4.2

12. For $n \geq 0$ let F_n denote the nth Fibonacci number. Prove that

$$F_0 + F_1 + F_2 + \cdots + F_n = \sum_{i=0}^{n} F_i = F_{n+2} - 1.$$

Basis step: when $n = 0$, $\sum_{i=0}^{0} F_i = F_0 = 0 = F_2 - 1$ is true.

Induction step: when $n = k$, $\sum_{i=0}^{k} F_i = F_{k+2} - 1$ is true for $k \geq 0$.

Then $\sum_{i=0}^{k+1} F_i = F_{k+1} + \sum_{i=0}^{k} F_i = F_{k+1} + (F_{k+2} - 1) = F_{k+3} - 1$.

So when $n = k + 1$, $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$ is true for all $n \geq 0$.

16. Give a recursive definition for the set of all

a) positive even integers

b) nonnegative even integers

(a) Define the set X recursively by

(1) $2 \in X$

(2) for each $a \in X$, $a+2 \in X$

(b) Define the set Y recursively by

(1) $0 \in Y$

(2) for each $a \in Y$, $a+2 \in Y$

Exercise 4.3

10. If $n \in \mathbf{Z}^+$, and n is odd, prove that $8 | (n^2 - 1)$.

Because n is odd, $n + 1$ and n - 1 is even.

Assume n - 1 is divisible by 4, then $8|(n^2 - 1)$ holds.
Assume n - 1 is not divisible by 4.
Let $n - 1 = 4k + r$, $0 \le r < 4$, $k, r \in Z$.
Because $4 \nmid (n - 1) \Rightarrow r \neq 0$
Because $2|(n - 1) \Rightarrow 2|r \Rightarrow r = 2$
Thus, $(n + 1) = 4k + r + 2 = 4(k + 1)$
$\Rightarrow 4|(n + 1) \Rightarrow 8|(n - 1)(n + 1) \Rightarrow 8|(n^2 - 1)$ holds.

12. Determine the quotient q and remainder r for each of the following, where a is the dividend and b is the divisor.

a) $a = 23, \quad b = 7$ **b)** $a = -115, \quad b = 12$

c) $a = 0, \quad b = 42$ **d)** $a = 434, \quad b = 31$

(a) $q = 3, r = 2$
(b) $q = -10, r = 5$ ($q = -9, r = -7$)
(c) $q = 0, r = 0$
(d) $q = 14, r = 0$

18. For what base do we find that $251 + 445 = 1026$?

Let b be the base
$\Rightarrow (2b^2 + 5b + 1) + (4b^2 + 4b + 5) = b^3 + 2b + 6$
$\Rightarrow 6b^2 + 9b + 6 = b^3 + 2b + 6$
$\Rightarrow b^3 - 6b^2 - 7b = 0$
$\Rightarrow b(b - 7)(b + 1) = 0$
$\Rightarrow b = 0, 7$, or -1, but $b \epsilon Z^+$, so $b = 7$.

Exercise 4.4

12. Let $a, b \in \mathbf{Z}^+$ where $a \ge b$. Prove that $\gcd(a, b) = \gcd(a - b, b)$.

1. We want to show $\gcd(a, b) \ge \gcd(a - b, b)$.
Let $c = \gcd(a, b)$, that is, there are x, y such that $ax + by = c$
Then $c = ax - bx + by + bx = (a - b)x + b(x + y)$ where $x, x + y \in \mathbb{Z}^+$
Hence, $\gcd(a - b, b) \le c = \gcd(a, b)$

2. We show that $\gcd(a, b) \le \gcd(a - b, b)$.
Let $d = \gcd(a - b, b)$, that is, there are i, j such that $(a - b)i +$

$bj = d$
Then $d = ai - bi + bj = ai + b(j - i)$
Hence, $\gcd(a, b) \le d = \gcd(a - b, b)$

By 1 and 2, we can get $\gcd(a, b) = \gcd(a - b, b)$.

16. Let $a, b \in \mathbf{Z}^+$. Prove that there exist $c, d \in \mathbf{Z}^+$ such that $cd = a$ and $\gcd(c, d) = b$ if and only if $b^2 | a$.

要證兩邊，只證一邊的扣 5 分

"=>"
$gcd((c, d)) = b \Rightarrow b \mid c$ and $b \mid d$
$\Rightarrow b^2 \mid cd$
$\Rightarrow b^2 \mid a$
"<="
$b^2 | a \Rightarrow a = kb^2$ where $k \in \mathbb{Z}^+$.
Choose $c = kb, d = b$ satisfy that $cd = a$ and $\gcd(kb, b) = b$

Exercise 4.5

8. a) How many positive divisors are there for

$$n = 2^{14}3^9 5^8 7^{10} 11^3 13^5 37^{10}?$$

b) For the divisors in part (a), how many are

i) divisible by $2^3 3^4 5^7 11^2 37^2$?
ii) divisible by 1,166,400,000?
iii) perfect squares?
iv) perfect squares that are divisible by $2^2 3^4 5^2 11^2$?
v) perfect cubes?
vi) perfect cubes that are multiples of $2^{10} 3^9 5^2 7^5 11^2 13^2 37^2$?
vii) perfect squares and perfect cubes?

錯一個扣兩分，五個以上 -10

(a) (14+1)(9+1)(8+1)(10+1)(3+1)(5+1)(10+1) = **3920400**

(b)

(i). (14-3+1)(9-4+1)(8-7+1)(10+1)(3-2+1)(5+1)(10-2+1) = 171072

(ii). $1166400000 = 2^9 3^6 5^5$

(14-9+1)(9-6+1)(8-5+1)(10+1)(3+1)(5+1)(10+1) = **278784**

(iii). $(\lfloor 14/2 \rfloor + 1)(\lfloor 9/2 \rfloor + 1)(\lfloor 8/2 \rfloor + 1)(\lfloor 10/2 \rfloor + 1)(\lfloor 3/2 \rfloor + 1)(\lfloor 5/2 \rfloor + 1)(\lfloor 10/2 \rfloor + 1) = \mathbf{43200}$

(iv). $\left(\left\lfloor \frac{14-2}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{9-4}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{8-2}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{10}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{3-2}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{5}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{10}{2} \right\rfloor + 1\right) = \mathbf{9072}$

(v). $(\lfloor 14/3 \rfloor + 1)(\lfloor 9/3 \rfloor + 1)(\lfloor 8/3 \rfloor + 1)(\lfloor 10/3 \rfloor + 1)(\lfloor 3/3 \rfloor + 1)(\lfloor 5/3 \rfloor + 1)(\lfloor 10/3 \rfloor + 1) = \mathbf{3840}$

(vi). $1 \times 1 \times 2 \times 2 \times 1 \times 1 \times 3 = \mathbf{12}$

一一列舉出可能性(2 只有 12 可以選擇，3 只有 9 可以選擇，5 有 3 或 6 可以選，以此類推)

(vii). $(\lfloor 14/6 \rfloor + 1)(\lfloor 9/6 \rfloor + 1)(\lfloor 8/6 \rfloor + 1)(\lfloor 10/6 \rfloor + 1)(\lfloor 3/6 \rfloor + 1)(\lfloor 5/6 \rfloor + 1)(\lfloor 10/6 \rfloor + 1) = \mathbf{48}$