Algorithm 2021 Spring HW1: solution

(Chapter 1 to Chapter 4)

Give asymptotic tight bound(θ) for question 1-4 by using master theory.

1. (2.5pts) $T(n) = 7T\left(\frac{n}{3}\right) + n^2$

   (a) $\theta(1)$
   (b) $\theta(n)$
   (c) $\theta(n^2)$
   (d) $\theta(n^3)$

Ans: (c)

$n^{\log_3 7} < n^2$

Master theorem case 3

2. (2.5pts) $T(n) = 7T\left(\frac{n}{49}\right) + \sqrt{n}$

   (a) $\theta(n)$
   (b) $\theta(\sqrt{n}\log(n))$
   (c) $\theta(n^{3/2})$
   (d) $\theta(n^2)$

Ans: (b)

$n^{\log_{49} 7} = \sqrt{n}$

Master theorem case 2

3. (2.5pts) $T(n) = T\left(\frac{9}{10}n\right) + 100$

   (a) $\theta(1)$
   (b) $\theta(n\log(n))$
   (c) $\theta(\log(n))$
   (d) $\theta(\log^2(n))$

Ans: (c)

$n^{\log_{\frac{9}{10}} 1} = n^0 = O(1)$

Master theorem case 2

4. (2.5pts) $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}\log(n)$

   (a) $\theta(n^2)$
   (b) $\theta(\sqrt{n}\log(n))$
   (c) $\theta(\sqrt{n}\log^2(n))$
   (d) $\theta(n\log(n))$

Ans: (c)

$n^{\log_4 2} = \sqrt{n}$

$\frac{f(n)}{n^{\log_b a}} = (\log n)^k$ ，$k > -1$

Extended Master Theorem case 3

Give asymptotic tight bound(θ) for question 5-6. (Assume that T(n) is a constant for sufficiently small n.)

5. (15pts) $T(n) = T(n-2) + n\log\frac{n}{2}$ (Assume n is even.)

Ans: $\theta(n^2\log n)$

We take two parts. One for O, and another for Ω.

(1) Proof O:

$$T(n) = T(n-2) + n\log\frac{n}{2}$$

$$= T(n-4) + (n-2)\log\frac{n-2}{2} + n\log\frac{n}{2}$$

$$= c + 2\log\frac{2}{2} + 4\log\frac{4}{2} + \cdots + (n-2)\log\frac{n-2}{2} + n\log\frac{n}{2}$$

$$\leq c + n\log\frac{n}{2} + n\log\frac{n}{2} + \cdots + n\log\frac{n}{2} + n\log\frac{n}{2}$$

$$= c + \frac{n}{2} * n\log\frac{n}{2} = c + \frac{n^2}{2}\log\frac{n}{2}$$

$Therefore, T(n) = O(n^2\log n)$

(2) Proof Ω:

$$T(n) = T(n-2) + n\log\frac{n}{2}$$

$$= T(n-4) + (n-2)\log\frac{n-2}{2} + n\log\frac{n}{2}$$

$$= c + 2\log\frac{2}{2} + \cdots + (\frac{n}{2} - 2)\log\frac{(\frac{n}{2} - 2)}{2} + \frac{n}{2}\log\frac{n}{4} + \cdots + n\log\frac{n}{2}$$

$\geq c + 0 + \cdots + 0 + \frac{n}{2}\log\frac{n}{4} + \cdots + n\log\frac{n}{2}$ (half of number)

$$= c + \frac{n}{4} * \frac{n}{2}\log\frac{n}{4} = c + \frac{n^2}{8}\log\frac{n}{4}$$

$Therefore, T(n) = \Omega(n^2 \log n)$

By (1)(2), $T(n) = \theta(n^2 \log n)$

6. (15pts) $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{log^2 n}$

Ans: $\theta(n)$

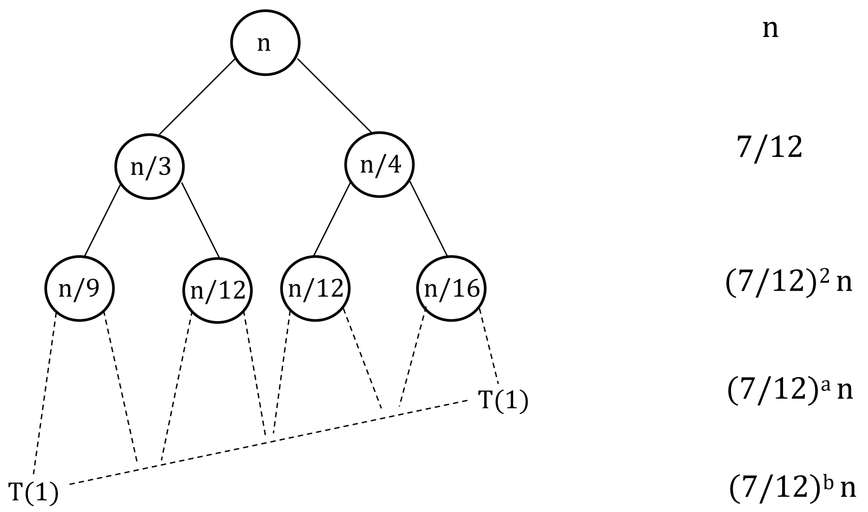
Use extended master theorem,

$f(n) = \frac{n}{log^2 n} = O\left(n^{\log_3 3}(\log_3 n)^{-2}\right) = O(\frac{n}{log^2 n})$

Then, $T(n) = \theta\left(n^{\log_3 3}\right) = \theta(n)$

7. (15pts) $T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{n}{4}\right) + n$

Ans: $\Theta(n)$



$4^{\mathrm{a}} = \mathrm{n}, \mathrm{a} = \log_4 \mathrm{n}$ $\rightarrow$ rightmost branch peters out after $\log_4 \mathrm{n}$ levels

$3^{\mathrm{b}} = \mathrm{n}, \mathrm{b} = \log_3 \mathrm{n}$ $\rightarrow$ lefttmost branch peters out after $\log_3 \mathrm{n}$ levels

Lower bound:

$Guess: T(n) \geq dn \ for \ some \ positive \ constant \ d$

Substitution:

$$T(n) \geq T\left(\frac{n}{3}\right) + T\left(\frac{n}{4}\right) + cn$$

$$\geq d\left(\frac{n}{3}\right) + d\left(\frac{n}{4}\right) + cn$$

$$= d\left(\frac{7n}{12}\right) + cn$$

$$\geq dn$$

$$if -\frac{5d}{12} + c \geq 0, d \leq \frac{12}{5c}$$

$Therefore, T(n) = \Omega(n)$

Upper bound:

$Guess: T(n) \leq dn \ for \ some \ positive \ constant \ d$

Substitution:

$$T(n) \leq T\left(\frac{n}{3}\right) + T\left(\frac{n}{4}\right) + cn$$

$$\leq d\left(\frac{n}{3}\right) + d\left(\frac{n}{4}\right) + cn$$

$$= d\left(\frac{7n}{12}\right) + cn$$

$$\leq dn$$

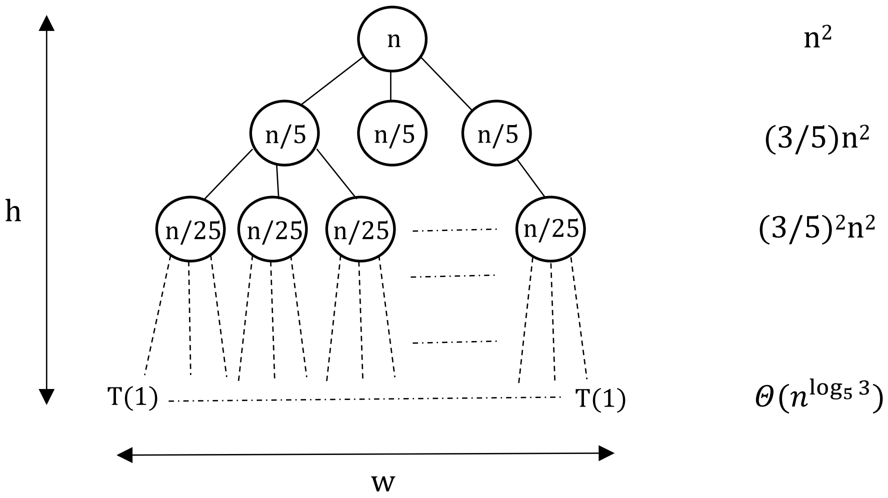
$$if -\frac{5d}{12} + c \leq 0, d \geq \frac{12}{5c}$$

$Therefore, T(n) = O(n)$

Since $T(n) = \mathrm{O}(n)$ and $T(n) = \Omega(n)$, we conclude that $T(n) = \Theta(n)$

8. (15pts) $T(n) = 3T\left(\frac{n}{5}\right) + n^2$

Ans: $\Theta(n^2)$



$$5^h = n, h = \log_5 n$$
$$w = 3^h = 3^{log_5 n} = n^{log_5 3}$$

$$T(n) = cn^2 + \left(\frac{3}{5}\right) cn^2 + \left(\frac{3}{5}\right)^2 cn^2 + \cdots + \left(\frac{3}{5}\right)^{\log_5 n - 1} cn^2 + \Theta(n^{log_5 3})$$

$$= \sum_{i=0}^{log_5 n-1} \left(\frac{3}{5}\right)^i cn^2 + \Theta(n^{log_5 3})$$

$$= \frac{1 - (\frac{3}{5})^{log_5 3}}{1 - \frac{3}{5}} cn^2 + \Theta(n^{log_5 3})$$

$$= \left(\frac{5}{2} - \frac{5}{2} \times 3^{\log_5 \frac{3}{5}}\right) cn^2 + \Theta(n^{log_5 3})$$

$$= \Theta(n^2)$$

9. (15pts) Compare the following time complexity and arrange them in increasing order.
   (1) $4^{\lg n}$
   (2) $\log(n!)$
   (3) 2^n
   (4) $n!$
   (5) $n^{\frac{1}{\log n}}$
   (6) $(\log n)^{\log n}$

   (a) 125634
   (b) 521634
   (c) 521364
   (d) 123465

Ans: (b)
   (1) $n^{\log_2 4}$ → n^2
   (2) $\text{n}\log n$
   (3) 2^n
   (4) n!
   (5) $n^{\frac{\log 10}{\log n}}$ → $n^{\log_n 10}$ → $10^{\log_n n}$ → 10 → O(1)
   (6) $n^{\log\log n}$

10. (15pts) True or False
   (a) $\text{n}\log(2n) = O\,(\text{n}\log n)$
   (b) $5n^e + 24\,\text{n}\log n = O(n^{2.8})$ (e = natural logarithm)
   (c) $2^{2n} = O(2^n)$

Ans: True, True, False
   (a) $\text{n}\log 2 + \text{n}\log n$ → $\text{n} + \text{n}\log n$ → $\text{O}(\text{n}\log n)$
   (b) $n^e + \text{n}\log n$ → $\text{O}(n^e)$ → $\text{O}(n^{2.8})$
   (c) 4^n → $\text{O}(4^n)$