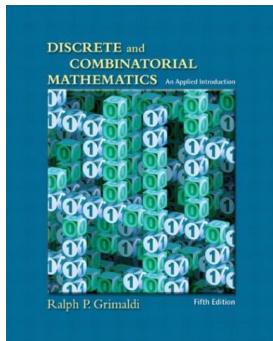
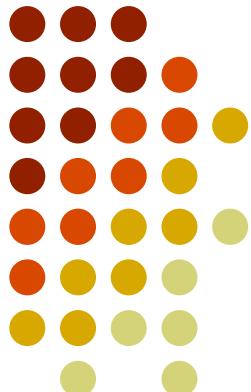


Discrete Mathematics

-- *Chapter 9: Generating Function*



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Outline

- Calculational Techniques
- Partitions of Integers
- The Exponential Generating Function
- The Summation Operator



Enumeration again

- Chapter 1: $c_1+c_2+c_3+c_4=25$, where $c_i \geq 0$
- Chapter 8: $c_1+c_2+c_3+c_4=25$, where $10 > c_i \geq 0$
- In chapter 9, c_2 to be even and c_3 to be a multiple of 3
- the coefficient xy^2 in $(x+y)^3$
- the coefficient x^4 in $(x+x^2)(x^2+x^3+x^4)(1+x+2x^2)$



9.1 Introductory Examples

- **Ex 9.1 :**

- One mother buys 12 oranges for three children, Grace, Mary, and Frank.
- Grace gets at least four, and Mary and Frank gets at least two, but Frank gets no more than five.

- **Solution**

- $c_1 + c_2 + c_3 = 12$, where $4 \leq c_1, 2 \leq c_2$, and $2 \leq c_3 \leq 5$

- Generating function:

$$f(x) = (x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5)$$

product $x^i x^j x^k \rightarrow$ every triple (i, j, k)

- The coefficient of x^{12} in $f(x)$ yields the solution.

Table 9.1

G	M	F	G	M	F
4	3	5	6	2	4
4	4	4	6	3	3
4	5	3	6	4	2
4	6	2	7	2	3
5	2	5	7	3	2
5	3	4	8	2	2
5	4	3			
5	5	2			



Introductory Examples

- Ex 9.2 : 寫成 x 的多項式，並求得其系數
 - There is an unlimited number of red, green, white, and black jelly beans.
 - In how many ways can we select 24 jelly beans so that we have an even number of white beans and at least six black ones?
 - **Solution**
 - red (green): $1 + x^1 + x^2 + \dots + x^{23} + x^{24}$
 - white: $1 + x^2 + x^4 + \dots + x^{22} + x^{24}$
 - black: $x^6 + x^7 + \dots + x^{23} + x^{24}$
 - Generating function:
$$f(x) = (1 + x^1 + x^2 + \dots + x^{23} + x^{24})^2 (1 + x^2 + x^4 + \dots + x^{22} + x^{24}) \\ (x^6 + x^7 + \dots + x^{23} + x^{24})$$
 - The coefficient of x^{24} in $f(x)$ is the answer.



Introductory Examples

- Ex 9.3 : How many nonnegative integer solutions are there for $c_1+c_2+c_3+c_4=25$?
 - **Solution**
 - Alternatively, in how many ways 25 pennies can be distributed among four children?
 - Generating function:
 $f(x) = (1 + x^1 + x^2 + \dots + x^{24} + x^{25})^4$ (polynomial)
 - The coefficient of x^{25} is the solution.

Note:

- $g(x) = (1 + x^1 + x^2 + \dots + x^{24} + x^{25} + x^{26} + \dots)^4$ (**power series**) can also generate the answer

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

- Easier to compute with a power series than with a polynomial



9.2 Definition and Examples: Calculational Techniques

- Definition 9.1: 使用這個函式產生序列
Let a_0, a_1, a_2, \dots be a sequence of real numbers. The function

$$f(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{i=0}^{\infty} a_i x^i$$

- is called the *generating function* for the given sequence. x 的係數
- Ex 9.4 : $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$
so, $(1+x)^n$ is the generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}, \underline{0}, \underline{0}, \underline{0}, \dots \text{ 無窮多項}$$



Definition and Examples: Calculational Techniques

- **Ex 9.5 :**

- a) $(1 - x^{n+1})/(1 - x)$ is the generating function for the sequence 1, 1, ..., 1, 0, 0, 0, ..., where the first $n+1$ terms are 1.

$$\therefore (1 - x^{n+1}) = (1 - x)(1 + x + x^2 + \cdots + x^n).$$

無窮等比級數

- b) $1/(1-x)$ is the generating function for the sequence 1, 1, 1, 1, ... \because while $|x| < 1, \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

- c) $1/(1-x)^2$ is the generating function for the sequence 1, 2, 3, 4, ...

$$\begin{aligned}\therefore \frac{d}{dx} \frac{1}{1-x} &= \frac{d}{dx} (1-x)^{-1} = (-1)(1-x)^{-2}(-1) \\ &= \frac{1}{(1-x)^2} = \frac{d}{dx} (1 + x + x^2 + x^3 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + \cdots\end{aligned}$$

- d) $x/(1-x)^2$ is the generating function for the sequence 0, 1, 2, 3, ...

$$\therefore \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \cdots$$



Definition and Examples: Calculational Techniques

- **Ex 9.5 :**

e) $(x+1)/(1-x)^3$ is the generating function for the sequence

$$\underline{1^2, 2^2, 3^2, 4^2, \dots}$$

$$\therefore \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{d}{dx} (0 + x + 2x^2 + 3x^3 + \dots)$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \dots$$

$$\begin{aligned}\therefore \frac{d}{dx} \frac{x}{(1-x)^2} \\ &= \frac{d}{dx} x(1-x)^{-2} \\ &= (1-x)^{-2} + x(-2)(1-x)^{-3}(-1) \\ &= \frac{(1-x)+2x}{(1-x)^3} = \frac{x+1}{(1-x)^3}\end{aligned}$$

f) $x(x+1)/(1-x)^3$ is the generating function for the sequence

$$\underline{0^2, 1^2, 2^2, 3^2, 4^2, \dots}$$

$$\therefore \frac{x(x+1)}{(1-x)^3} = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \dots$$



Definition and Examples: Calculational Techniques

- **Ex 9.5 :**

g) Further extensions:

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\begin{aligned} f_1(x) &= x \frac{d}{dx} f_0(x) = \frac{x}{(1-x)^2} \\ &= 0 + x + 2x^2 + 3x^3 + \dots \end{aligned}$$

$$\begin{aligned} f_2(x) &= x \frac{d}{dx} f_1(x) = \frac{x^2 + x}{(1-x)^3} \\ &= 0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \dots \end{aligned}$$

$$\begin{aligned} f_3(x) &= x \frac{d}{dx} f_2(x) = \frac{x^3 + 4x^2 + x}{(1-x)^4} \\ &= 0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \dots \end{aligned}$$

$$\begin{aligned} f_4(x) &= x \frac{d}{dx} f_3(x) = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-x)^5} \\ &= 0^4 + 1^4 x + 2^4 x^2 + 3^4 x^3 + \dots \end{aligned}$$



Definition and Examples: Calculational Techniques

- **Ex 9.6 :**

- a) $1/(1 - ax)$ is the generating function for the sequence $\underline{a^0}, \underline{a^1}, \underline{a^2}, \underline{a^3}, \dots$
- b) $f(x) = 1/(1 - x)$ is the generating function for the sequence $\underline{1}, \underline{1}, \underline{1}, \dots$ Then
 - $g(x) = f(x) - x^2$ is the generating function for the sequence $\underline{1}, \underline{1}, \underline{0}, \underline{1}, \underline{1}, \underline{1}, \dots$
 - $h(x) = f(x) + 2x^3$ is the generating function for the sequence $\underline{1}, \underline{1}, \underline{1}, \underline{3}, \underline{1}, \underline{1}, \dots$
- c) Can we find a generating function for the sequence 0, 2, 6, 12, 20, 30, 42, ...?



Definition and Examples: Calculational Techniques

- **Ex 9.6 :**

c) Observe $0, 2, 6, 12, 20, \dots$

$$a_0 = 0 = 0^2 + 0, \quad a_1 = 2 = 1^2 + 1,$$

$$a_2 = 6 = 2^2 + 2, \quad a_3 = 12 = 3^2 + 3,$$

$$a_4 = 20 = 4^2 + 4, \dots$$

$$\therefore a_n = n^2 + n$$

$$\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(x+1) + x(1-x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

is the generating function.



Extension of Binomial Theorem

- Binomial theorem:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

- When $n \in \mathbb{Z}^+$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

- If $n \in \mathbb{R}$, we define

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

- If $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{r!} \\ &= \frac{(-1)^r(n)(n+1)\dots(n+r-1)}{r!} \\ &= \frac{(-1)^r(n+r-1)!}{(n-1)!r!} = (-1)^r \binom{n+r-1}{r}\end{aligned}$$



Extension of Binomial Theorem

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

- Ex 9.7 :

For $n \in \mathbb{Z}^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is given by

$$\begin{aligned}(1+x)^{-n} &= 1 + (-n)x + (-n)(-n-1)x^2/2! \\&\quad + (-n)(-n-1)(-n-2)x^3/3! + \dots \\&= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} x^r \\&= \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r.\end{aligned}$$

$(1-x)^{-n} ?$

Hence $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \dots = \sum_{r=0}^{\infty} \binom{-n}{r} x^r$. This generalizes the binomial theorem of Chapter 1 and shows us that $(1+x)^{-n}$ is the generating function for the sequence $\binom{-n}{0}, \binom{-n}{1}, \binom{-n}{2}, \binom{-n}{3}, \dots$



Extension of Binomial Theorem

- **Ex 9.8** : Find the coefficient of x^5 in $(1-2x)^{-7}$.
 - **Solution**

$$(1-2x)^{-7} = \sum_{r=0}^{\infty} \binom{-7}{r} (-2x)^r$$

The coefficient of x^5 :

$$\binom{-7}{5} (-2)^5 = (-1)^5 \binom{7+5-1}{5} (-32) = (32) \binom{11}{5}$$

- **Ex 9.9** : Find the coefficient of all x^i in $(1+3x)^{-1/3}$

$$\begin{aligned}(1+3x)^{-1/3} &= 1 + \sum_{r=1}^{\infty} \frac{(-1/3)(-4/3)(-7/3) \cdots ((-3r+2)/3)}{r!} (3x)^r \\&= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7) \cdots (-3r+2)}{r!} x^r,\end{aligned}$$

↑
 $1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!} x^r$



Definition and Examples: Calculational Techniques

- **Ex 9.10** : Determine the coefficient of x^{15} in $f(x) = (x^2+x^3+x^4+\dots)^4$.
 - **Solution**
 - $(x^2+x^3+x^4+\dots) = x^2(1+x+x^2+\dots) = x^2/(1-x)$
 - $f(x)=(x^2/(1-x))^4 = x^8/(1-x)^4$
 - Hence the solution is the coefficient of x^7 in $(1-x)^{-4}$:
 $C(-4, 7)(-1)^7 = (-1)^7 C(4+7-1, 7)(-1)^7 = C(10, 7) = 120.$
 - 四個變數相加為15，每個變數>1的正整數解

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 15 \quad x_1, x_2, x_3, x_4 > 1 \\ \Rightarrow x_1 + x_2 + x_3 + x_4 &= 7 \quad x_1, x_2, x_3, x_4 \geq 0 \\ C(4+7-1, 7) &= C(10, 7) = 120 \end{aligned}$$



Table 9.2

For all $m, n \in \mathbf{Z}^+$, $a \in \mathbf{R}$,

1) $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

2) $(1 + ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^n x^n$

3) $(1 + x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$

4) $(1 - x^{n+1})/(1 - x) = 1 + x + x^2 + \cdots + x^n$

5) $1/(1 - x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$

6) $1/(1 - ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$

$$= \sum_{i=0}^{\infty} (ax)^i = \sum_{i=0}^{\infty} a^i x^i$$

$$= 1 + ax + a^2 x^2 + a^3 x^3 + \cdots$$



7) $1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \dots$
 $= \sum_{i=0}^{\infty} \binom{-n}{i}x^i$
 $= 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \dots$
 $= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i}x^i$

8) $1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \dots$
 $= \sum_{i=0}^{\infty} \binom{-n}{i}(-x)^i$
 $= 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \dots$
 $= \sum_{i=0}^{\infty} \binom{n+i-1}{i}x^i$

check $n=1$

If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, and $h(x) = f(x)g(x)$, then
 $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where for all $k \geq 0$,

$$c_k = a_0 b_k + a_1 b_{k-1} + \dots + a_{k-1} b_1 + a_k b_0 = \sum_{j=0}^k a_j b_{k-j}.$$



Definition and Examples: Calculational Techniques

- **Ex 9.11** : In how many ways can we select, with repetition allowed, r objects from n distinct objects?
 - **Solution**
 - For each object (with repetitions), $1+x+x^2+\dots$ represents the possible choices for that object (namely none, one, two,...)
 - Consider all of the n distinct objects, the generating function is $f(x) = (1+x+x^2+\dots)^n$
$$(1 + x + x^2 + \cdots)^n = \left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$
 - The answer is the coefficient of x^r in $f(x)$, $\binom{n+r-1}{r}$.



Definition and Examples: Calculational Techniques

- **Ex 9.12** : Counting the compositions of a positive integer n .
 - **Solution**
 - E.g., $n = 4$
 - One-summand: $(x^1 + x^2 + x^3 + x^4 + \dots) = [x/(1-x)]$, coefficient of $x^4 = 1$
 - Two-summand: $(x^1 + x^2 + x^3 + x^4 + \dots)^2 = [x/(1-x)]^2$, coefficient of $x^4 = 3$
 - Three-summand: $(x^1 + x^2 + x^3 + x^4 + \dots)^3 = [x/(1-x)]^3$, coefficient of $x^4 = 3$
 - Four-summand: $(x^1 + x^2 + x^3 + x^4 + \dots)^4 = [x/(1-x)]^4$, coefficient of $x^4 = 1$
 - The number of compositions of 4: coefficient of x^4 in $\sum_{i=1}^4 [x/(1-x)]^i$
Total= $1+3+3+1=8=2^3$

How about $n=5$?



Definition and Examples: Calculational Techniques

- **Ex 9.12** : Counting the compositions of a positive integer n .
 - The number of ways to form an integer n is the coefficient of x^n in the following generating function.

$$\sum_{i=1}^{\infty} (x^1 + x^2 + x^3 + \dots)^i = \sum_{i=1}^{\infty} [x/(1-x)]^i$$

$f(x) = \sum_{i=1}^{\infty} [x/(1-x)]^i$. But if we set $y = x/(1-x)$, it then follows that

$$\begin{aligned} f(x) &= \sum_{i=1}^{\infty} y^i = y \sum_{i=0}^{\infty} y^i = y \left(\frac{1}{1-y} \right) = \left(\frac{x}{1-x} \right) \left[\frac{1}{1 - \left(\frac{x}{1-x} \right)} \right] = \left(\frac{x}{1-x} \right) \left[\frac{1}{\frac{1-x-x}{1-x}} \right] \\ &= x/(1-2x) = x[1 + (2x) + (2x)^2 + (2x)^3 + \dots] \\ &= 2^0 x + 2^1 x^2 + 2^2 x^3 + 2^3 x^4 + \dots \end{aligned}$$

So the number of compositions of a positive integer n is the coefficient of x^n in $f(x)$ — and this is 2^{n-1} (as we found earlier in Examples 1.37, 3.11, and 4.12.)



Definition and Examples: Calculational Techniques

- **Ex 9.14 :** In how many ways can a police captain distribute 24 rifle shells to four police officers, so that each officer gets at least three shells but not more than eight.

- **Solution**

- $f(x) = (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$

$$= x^{12}(1+x+x^2+x^3+x^4+x^5)^4$$

$$= x^{12}[(1-x^6)/(1-x)]^4$$

$$x_1+x_2+x_3+x_4=24, 8 \geq x_i \geq 3$$

- The answer is the coefficient of x^{12} in $(1-x^6)^4(1-x)^{-4}$

$$= [1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + x^{24}] [\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \dots]$$

$$[\binom{-4}{12}(-1)^{12} - \binom{4}{1}\binom{-4}{6}(-1)^6 + \binom{4}{2}\binom{-4}{0}] = [\binom{15}{12} - \binom{4}{1}\binom{9}{6} + \binom{4}{2}] = 125$$

Can you explain the last term by the principle of inclusion and exclusion?



Definition and Examples: Calculational Techniques

- **Ex 9.16** : Determine the coefficient of x^8 in $\frac{1}{(x-3)(x-2)^2}$.
 - **Solution**

Since $\frac{1}{x-a} = (-1/a)(1/(1-(x/a))) = (-1/a)[1 + (x/a) + (x/a)^2 + \dots]$ for any $a \neq 0$, we could solve this problem by finding the coefficient of x^8 in $\frac{1}{[(x-3)(x-2)^2]}$ expressed as $(-1/3)[1 + (x/3) + (x/3)^2 + \dots](1/4)[\binom{-2}{0} + \binom{-2}{1}(-x/2) + \binom{-2}{2}(-x/2)^2 + \dots]$.

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

$$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3),$$

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= \left(\frac{-1}{3}\right) \frac{1}{1-(x/3)} + \left(\frac{1}{2}\right) \frac{1}{1-(x/2)} + \left(\frac{-1}{4}\right) \frac{1}{(1-(x/2))^2}$$

$$= \left(\frac{-1}{3}\right) \sum_{i=0}^{\infty} \left(\frac{x}{3}\right)^i + \left(\frac{1}{2}\right) \sum_{i=0}^{\infty} \left(\frac{x}{2}\right)^i$$

$$+ \left(\frac{-1}{4}\right) \left[\binom{-2}{0} + \binom{-2}{1} \left(\frac{-x}{2}\right) + \binom{-2}{2} \left(\frac{-x}{2}\right)^2 + \dots \right].$$

The coefficient of x^8 is $(-1/3)(1/3)^8 + (1/2)(1/2)^8 + (-1/4)\binom{-2}{8}(-1/2)^8 = -[(1/3)^9 + 7(1/2)^{10}]$.



Definition and Examples: Calculational Techniques

- **Ex 9.17** : How many four-element subsets of $S = \{1, 2, \dots, 15\}$ contains no consecutive integers?

Solution 每個difference相加 = difference sum

- E.g., one subset $\{1, 3, 7, 10\}$, $1 \leq 1 < 3 < 7 < 10 \leq 15$, difference 0, 2, 4, 3, 5, difference sum = 14.
- These suggest the integer solutions to $c_1 + c_2 + c_3 + c_4 + c_5 = 14$ where $0 \leq c_1, c_5$ and $2 \leq c_2, c_3, c_4$. 因為要不連續
- The answer is the coefficient of x^{14} in
$$f(x) = (1+x+x^2+x^3+\dots)(x^2+x^3+x^4+\dots)^3(1+x+x^2+x^3+\dots)$$
$$= x^6(1-x)^{-5}$$
- The coefficient of x^8 in $(1-x)^{-5}$.

$$\binom{-5}{8}(-1)^8 = \binom{5+8-1}{8} = \binom{12}{8} = 495$$



Convolution of Sequences

- Ex 9.19 : Let
 - $f(x) = x/(1-x)^2 = 0+1x+2x^2+3x^3+\dots$, for the sequence $a_k = k$
 - $g(x) = x(x+1)/(1-x)^3 = 0+1^2x+2^2x^2+3^2x^3+\dots$, for the sequence $b_k = k^2$
 - $h(x) = f(x)g(x)$
 $= a_0b_0 + (a_0b_1+a_1b_0)x + (a_0b_2+a_1b_1+a_2b_0)x^2 + \dots$, for the sequence $c_k = a_0b_k + a_1b_{k-1} + a_2b_{k-2} + \dots + a_{k-2}b_2 + a_{k-1}b_1 + a_kb_0$
 - $$c_k = \sum_{i=0}^k i(k-i)^2.$$

$c_0 = 0 \times 0^2$ $c_1 = 0 \times 1^2 + 1 \times 0^2 = 0$ $c_2 = 0 \times 2^2 + 1 \times 1^2 + 2 \times 0^2 = 1$ $c_3 = 6$
- The sequence c_0, c_1, c_2, \dots is the convolution of the sequences a_0, a_1, a_2, \dots and b_0, b_1, b_2, \dots



Convolution of Sequences

- Ex 9.20 : Let
 - $f(x) = 1/(1-x) = 1+x+x^2+x^3+\dots$
 - $g(x) = 1/(1+x) = 1-x+x^2-x^3+\dots$
 - $h(x) = f(x)g(x)$
 $= 1/[(1-x)(1+x)] = 1/(1-x^2) = 1+x^2+x^4+x^6+\dots$
- The sequence 1, 0, 1, 0, ... is the convolution of the sequences 1, 1, 1, 1, ... and 1, -1, 1, -1, ...



9.3 Partition of Integers 無順序性

- $p(n)$: the number of partitioning a positive integer n

$$p(1) = 1: \quad 1$$

$$p(2) = 2: \quad 2 = 1 + 1$$

$$p(3) = 3: \quad 3 = 2 + 1 = 1 + 1 + 1$$

$$p(4) = 5: \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

$$p(5) = 7: \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1$$

$$= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$$

- The number of 1's is 0 or 1 or 2 or 3.... The power series is $1+x+x^2+x^3+x^4+\dots$
- The number of 2's can be kept tracked by the power series $1+x^2+x^4+x^6+x^8+\dots$
- For n , the number of 3's can be kept tracked by the power series $1+x^3+x^6+x^9+x^{12}+\dots$



Partition of Integers

- Determine $p(10)$

- The coefficient of x^{10} in $f(x)$

$$= (1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)(1+x^3+x^6+x^9+\dots)\dots \\ (1+x^{10}+x^{20}+\dots)$$

$$f(x) = \frac{1}{(1-x)} \frac{1}{(1-x^2)} \frac{1}{(1-x^3)} \cdots \frac{1}{(1-x^{10})} = \prod_{i=1}^{10} \frac{1}{(1-x^i)}$$

- By the coefficient of x^n in $P(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)}$, we get the sequence $p(0), p(1), p(2), p(3), \dots$



Partition of Integers

- **Ex 9.21** : Find the generating function for the number of ways an advertising agent can purchase n minutes of air time if the time slots come in blocks of 30, 60, or 120 seconds.

Solution

- Let 30 seconds represent one time unit.
- Find integer solutions to $a+2b+4c = 2n$
- Generating function:

$$\begin{aligned}f(x) &= (1+x+x^2+x^3+x^4+\dots)(1+x^2+x^4+x^6+x^8+\dots)(1+x^4+x^8+x^{12}+\dots) \\&= \frac{1}{(1-x)} \frac{1}{(1-x^2)} \frac{1}{(1-x^4)}.\end{aligned}$$

- Answer: the coefficient of x^{2n} is the number of partitions of $2n$ into 1's, 2's, and 4's.



Partition of Integers

- Ex 9.22 : Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into **distinct** summands.
 - One time of occurrence per summand
 - $P_d(x) = (1+x)(1+x^2)(1+x^3)\dots$
- Ex 9.23 : Find the generating function for $p_o(n)$, the number of partitions of a positive integer n into **odd** summands.
 - $P_o(x) = (1+x+x^2+x^3+\dots)(1+x^3+x^6+\dots)(1+x^5+x^{10}+\dots)\dots$
 - $= 1/(1-x) \times 1/(1-x^3) \times 1/(1-x^5) \times 1/(1-x^7) \times \dots$
 - $P_d(x) = P_o(x) ?$

$$\begin{aligned}6 &= 1+5 \\6 &= 1+2+3 \\6 &= 2+4\end{aligned}$$

Now because

$$1+x = \frac{1-x^2}{1-x}, \quad 1+x^2 = \frac{1-x^4}{1-x^2}, \quad 1+x^3 = \frac{1-x^6}{1-x^3}, \quad \dots,$$

we have

$$\begin{aligned}P_d(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)\dots \\&= \frac{1-x^2}{1-x} \frac{1-x^4}{1-x^2} \frac{1-x^6}{1-x^3} \frac{1-x^8}{1-x^4} \dots = \frac{1}{1-x} \frac{1}{1-x^3} \dots = P_o(x).\end{aligned}$$

$$\begin{aligned}6 &= 1+1+1+3 \\6 &= 1+5 \\6 &= 3+3\end{aligned}$$



Partition of Integers

- **Ex 9.24** : Find the generating function for the number of partitions of a positive integer n into odd summands and occurring an odd number of times.

Solution

$$f(x) = (1+x+x^3+x^5+\dots)(1+x^3+x^9+x^{15}+\dots) \\ (1+x^5+x^{15}+x^{25}+\dots)\dots$$

$$\begin{aligned} 6 &= 1+1+1+3 \\ 6 &= 1+5 \\ 6 &= 3+3 \end{aligned}$$

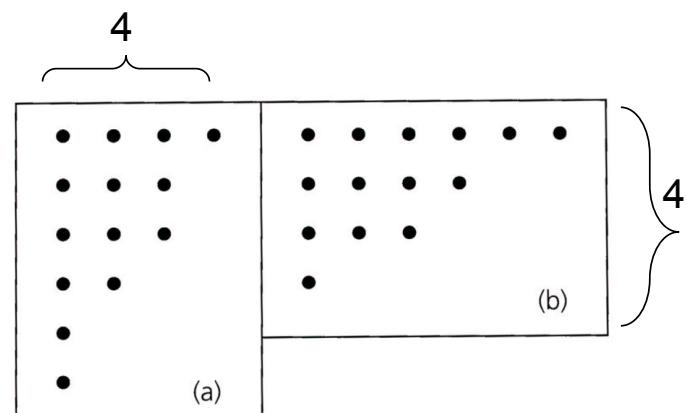
$$= \prod_{k=0}^{\infty} \left(1 + \sum_{i=0}^{\infty} x^{(2k+1)(2i+1)} \right).$$



Partition of Integers

- Ferrers graph uses rows of dots to represent a partition of an integer
- In fig. 9.2, two Ferrers graphs are transposed each other for the partitions of 14.
 - (a) $14 = 4+3+3+2+1+1$
 - (b) $14 = 6+4+3+1$

The number of partitions of an integer n into m summands is equal to the number of partitions of n into summands where m is the largest summand.



<http://mathworld.wolfram.com/FerrersDiagram.html>



9.4 The Exponential Generating Function

Now for all $0 \leq r \leq n$,

$$C(n, r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n, r),$$

where $P(n, r)$ denotes the number of permutations of n objects taken r at a time. So

$$\begin{aligned}(1+x)^n &= C(n, 0) + C(n, 1)x + C(n, 2)x^2 + C(n, 3)x^3 + \cdots + C(n, n)x^n \\&= P(n, 0) + P(n, 1)x + P(n, 2)\frac{x^2}{2!} + P(n, 3)\frac{x^3}{3!} + \cdots + P(n, n)\frac{x^n}{n!}.\end{aligned}$$

For a sequence $a_0, a_1, a_2, a_3, \dots$ of real numbers,

$$f(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the *exponential generating function* for the given sequence.



The Exponential Generating Function

- Ex 9.25 : Examining the Maclaurin series expansion for e^x , we find

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

so e^x is the exponential generating function for the sequence 1, 1, 1,....

What 's the sequence for the exponential generating function $1/(1-x)$?

Ans: 0!, 1!, 2!, 3!,



The Exponential Generating Function

- **Ex 9.26** : In how many ways can four of the letters in ENGINE be arranged?

Solution

Table 9.4

E E N N	4!/(2! 2!)	E G N N	4!/2!
E E G N	4!/2!	E I N N	4!/2!
E E I N	4!/2!	G I N N	4!/2!
E E G I	4!/2!	E I G N	4!

- Using exponential generating function: $f(x) = [1+x+(x^2/2!)]^2[1+x]^2$
 - E, N: $[1+x+(x^2/2!)]$ 不挑、挑一個、挑2個
 - G, I: $[1+x]$
- The answer is the coefficient of $x^4/4!$.

In the complete expansion of $f(x)$, the term involving x^4 [and, consequently, $x^4/4!$] is

$$\begin{aligned} & \left(\frac{x^4}{2! 2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + x^4 \right) \\ &= \left[\left(\frac{4!}{2! 2!} \right) + \left(\frac{4!}{2!} \right) + 4! \right] \left(\frac{x^4}{4!} \right). \end{aligned}$$



The Exponential Generating Function

- Ex 9.27 : Consider the Maclaurin series expansion of e^x and e^{-x} .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$



The Exponential Generating Function

- Ex 9.28 : A ship carries 48 flags, 12 each of the colors red, white, blue and black. Twelve flags are placed on a vertical pole to communicate signal to other ships.
 - How many of these signals use an even number of blue flags and an odd number of black flags?

$$\begin{aligned}f(x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \\f(x) &= (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right) (e^{2x})(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{4x} - 1) \\&= \frac{1}{4} \left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1 \right) = \left(\frac{1}{4}\right) \sum_{i=1}^{\infty} \frac{(4x)^i}{i!},\end{aligned}$$

the coefficient of $x^{12}/12!$ in $f(x)$ yields $(1/4)(4^{12}) = 4^{11}$ signals made up of 12 flags with an even number of blue flags and an odd number of black flags.



The Exponential Generating Function

- how many of these use at least three white flags or no white flag at all?

$$\begin{aligned}g(x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \underbrace{\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)}_{e^{3x}} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^2 \\&= e^x \left(e^x - x - \frac{x^2}{2!}\right) (e^x)^2 = e^{3x} \left(e^x - x - \frac{x^2}{2!}\right) = e^{4x} - xe^{3x} - \left(\frac{1}{2}\right) x^2 e^{3x} \\&= \sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - x \sum_{i=0}^{\infty} \frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right) \left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right).\end{aligned}$$



The Exponential Generating Function

- i) $\sum_{i=0}^{\infty} \frac{(4x)^i}{i!}$ — Here we have the term $\frac{(4x)^{12}}{12!} = 4^{12} \left(\frac{x^{12}}{12!}\right)$, so the coefficient of $x^{12}/12!$ is 4^{12} .
- ii) $x \left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!} \right)$ — Now we see that in order to get $x^{12}/12!$ we need to consider the term $x[(3x)^{11}/11!] = 3^{11}(x^{12}/11!) = (12)(3^{11})(x^{12}/12!)$, and here the coefficient of $x^{12}/12!$ is $(12)(3^{11})$; and
- iii) $(x^2/2) \left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!} \right)$ — For this last summand we observe that $(x^2/2)[(3x)^{10}/10!] = (1/2)(3^{10})(x^{12}/10!) = (1/2)(12)(11)(3^{10})(x^{12}/12!)$, where this time the coefficient of $x^{12}/12!$ is $(1/2)(12)(11)(3^{10})$.

Consequently, the number of 12 flag signals with at least three white flags, or none at all, is

$$4^{12} - 12(3^{11}) - (1/2)(12)(11)(3^{10}) = 10,754,218.$$

1個白色 2個白色



The Exponential Generating Function

- **Ex 9.29 :** A company hires 11 new employees, and they will be assigned to four **different** subdivisions. Each subdivision has at least one new employee. In how many ways can these assignments be made?
 - **Solution**

$$f(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right)^4 = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1.$$

The answer then is the coefficient of $x^{11}/11!$ in $f(x)$:

$$4^{11} - 4(3^{11}) + 6(2^{11}) - 4(1^{11}) = \sum_{i=0}^4 (-1)^i \binom{4}{i} (4-i)^{11}.$$



9.5 The Summation Operator

- Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$. Then $f(x)/(1-x)$ generate the sequence of $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \dots$. So we refer to $1/(1-x)$ as the summation operator.

$$\begin{aligned}\frac{f(x)}{1-x} &= f(x) \cdot \frac{1}{1-x} = [a_0 + a_1x + a_2x^2 + a_3x^3 + \dots][1 + x + x^2 + x^3 + \dots] \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \dots,\end{aligned}$$



The Summation Operator

Summation
operator

- Ex 9.30 :
 - $1/(1-x)$ is the generating function for the sequence 1, 1, 1, 1, 1, ...

 - $[1/(1-x)] \times [1/(1-x)]$ is the generating function for the sequence $1, 1+1, 1+1+1, \dots \Rightarrow 1, 2, 3, \dots$
 - $x+x^2$ is the generating function for the sequence 0, 1, 1, 0, 0, 0, ...
 - $(x+x^2) \times [1/(1-x)]$ is the generating function for the sequence 0, 1, 2, 2, 2, 2, ...
 - $(x+x^2)/(1-x)^2$ is the generating function for the sequence 0, 1, 3, 5, 7, 9, 11, ...
 - $(x+x^2)/(1-x)^3$ is the generating function for the sequence 0, 1, 4, 9, 16, 25, 36, ...



The Summation Operator

- **Ex 9.31** : Find a formula to express $0^2+1^2+2^2+\dots+n^2$ as a function of n .

Solution

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\begin{aligned} f_1(x) &= x \frac{d}{dx} f_0(x) = \frac{x}{(1-x)^2} \\ &= 0 + x + 2x^2 + 3x^3 + \dots \end{aligned}$$

$$\begin{aligned} f_2(x) &= x \frac{d}{dx} f_1(x) = \frac{x^2 + x}{(1-x)^3} \\ &= 0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \dots \end{aligned}$$

so $x(1+x)/(1-x)^3$ generates $0^2, 1^2, 2^2, 3^2, \dots$. As a consequence of our earlier observations about the summation operator, we find that

$$\frac{x(1+x)}{(1-x)^3} \frac{1}{(1-x)} = \frac{x(1+x)}{(1-x)^4}$$

is the generating function for $0^2, 0^2 + 1^2, 0^2 + 1^2 + 2^2, 0^2 + 1^2 + 2^2 + 3^2, \dots$



The Summation Operator

Hence the coefficient of x^n in $[x(1+x)]/(1-x)^4$ is $\sum_{i=0}^n i^2$. But the coefficient of x^n in $[x(1+x)]/(1-x)^4$ can also be calculated as follows:

$$\frac{x(1+x)}{(1-x)^4} = (x+x^2)(1-x)^{-4} = (x+x^2) \left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \dots \right],$$

so the coefficient of x^n is

$$\begin{aligned} & \binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2} \\ &= (-1)^{n-1} \binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2} \binom{4+(n-2)-1}{n-2}(-1)^{n-2} \\ &= \binom{n+2}{n-1} + \binom{n+1}{n-2} = \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} \\ &= \frac{1}{6}[(n+2)(n+1)(n) + (n+1)(n)(n-1)] \\ &= \frac{1}{6}(n)(n+1)[(n+2) + (n-1)] = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$