HW7

Handwrite

7.12

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

since X_1, X_2 are linearly independent

$f(x_1, x_2) = f(x_1) f(x_2) = e^{-(x_1+x_2)}$ $\quad x_1 > 0,\ x_2 > 0$

$Y_1 = X_1 + X_2$ $\quad X_1 = Y_1 Y_2$

$Y_2 = X_1/(X_1+X_2) = X_1/Y_1$ $\quad X_2 = Y_1(1-Y_2)$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 \\ (1-Y_2) & -Y_1 \end{vmatrix} = |-Y_1|$$

$Y_1 > 0 \quad 1 > Y_2 > 0$

Find $g(y_1, y_2)$ and if $g(y_1, y_2) = g(y_1) g(y_2)$, Y_1 and Y_2 is linearly independent

$g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2)) |J|$

$= e^{-(y_1y_2 + y_1(1-y_2))} \cdot y_1 = e^{-(y_1)} \cdot y_1$

$g(y_1) = \int_0^1 e^{-y_1} \cdot y_1 \, dy_2 = e^{-y_1} \cdot y_1$

$g(y_2) = \int_0^\infty e^{-y_1} \cdot y_1 \, dy_1 = \Gamma(2) = 1$ $\quad \Gamma(2) = (2-1)! = 1$

$g(y_1) g(y_2) = e^{-y_1} \cdot y_1 = g(y_1, y_2)$

so Y_1, Y_2 is linear independent

7.14

$$f(x) = \begin{cases} \frac{1+x}{2} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

let $x_1 = w_1(y) = -\sqrt{y}$, $J_1 = \frac{d(-\sqrt{y})}{dy} = \frac{-1}{2\sqrt{y}} = -|J_1|$

$x_2 = w_2(y) = \sqrt{y}$ $\quad |J_2| = \left|\frac{d(\sqrt{y})}{dy}\right| = \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}}$

$P(-1 < x < 1) = P(-1 < x < 0) + P(0 < x < 1)$

$P(-1<x<1) = P(-1<x<0) + P(0<x<1)$

$= \int_{-1}^{0} f(-\sqrt{y}) \cdot |\sqrt{y}| \, dy + \int_{0}^{1} f(\sqrt{y}) \frac{1}{2\sqrt{y}} \, dy$

$= -\int_{0}^{1} f(-\sqrt{y}) \frac{-1}{2\sqrt{y}} \, dy + \int_{0}^{1} f(\sqrt{y}) \frac{1}{2\sqrt{y}} \, dy$

$= \int_{0}^{1} (f(-\sqrt{y}) + f(\sqrt{y})) \cdot \frac{1}{2\sqrt{y}} \, dy$

$= \int_{0}^{1} \left(\frac{(1-\sqrt{y}) + (1+\sqrt{y})}{2} \cdot \frac{1}{2\sqrt{y}} \right) dy = \int_{0}^{1} \frac{1}{2\sqrt{y}} \, dy$

$$g(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

7.18

$g(x; p) = p q^{x-1} \quad x = 1, 2, 3 \ldots$

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x) & X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) \, dx \end{cases}$$

$M_X(t) = \sum_{x=1}^{\infty} p q^{x-1} e^{tx} = \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$

$= \frac{p}{q} qe^t (1 + (qe^t)^2 + (qe^t)^3 + \cdots)$

$= \frac{p}{q} qe^t \frac{1}{1-(qe^t)}$ and $|qe^t| < 1$

$= \frac{pe^t}{1-qe^t} \qquad qe^t < 1 \quad e^t < \frac{1}{q}$

$\qquad t < -\ln q$

since $q<1$, $\ln q < -\ln q$，所以當 $t < \ln q$

時等式仍成立

$$\mu = \frac{dM_X(t)}{dt}\bigg|_{t=0} = \frac{d\,\frac{pe^t}{1-qe^t}}{dt}\bigg|_{t=0} = \frac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2}\bigg|_{t=0}$$

$$= \frac{pe^t}{(1-qe^t)^2}\bigg|_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$\sigma^2 = \mu_2' - \mu^2$$

$$\mu_2 = \frac{d^2M_X(t)}{dt}\bigg|_{t=0} = \frac{d\,\frac{pe^t}{(1-qe^t)^2}}{dt}\bigg|_{t=0}$$

$$= \frac{pe^t(1-qe^t)^2 - pe^t\,2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\bigg|_{t=0} = \frac{2-p}{p^2}$$

$$\sigma^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

7.22 $M_X(t) = (1-2t)^{-\nu/2}$

$$\mu = M_X'(t)\big|_{t=0} = -\frac{\nu}{2}\cdot -2\,(1-2t)^{-\frac{\nu}{2}-1}\bigg|_{t=0} = \nu$$

$$\sigma^2 = \mu_2 - (\mu)^2$$

$$\mu_2 = M_X''(t)\big|_{t=0} = \nu\cdot\left(-\frac{1}{2}\nu-1\right)\cdot -2(1-2t)^{-\frac{\nu}{2}-2}\bigg|_{t=0}$$

$$= \nu(\nu+2)$$

$$\sigma^2 = \nu(\nu+2) - \nu^2 = 2\nu$$

Matlab

1(a)

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function[X]=HW7_1_a(n)
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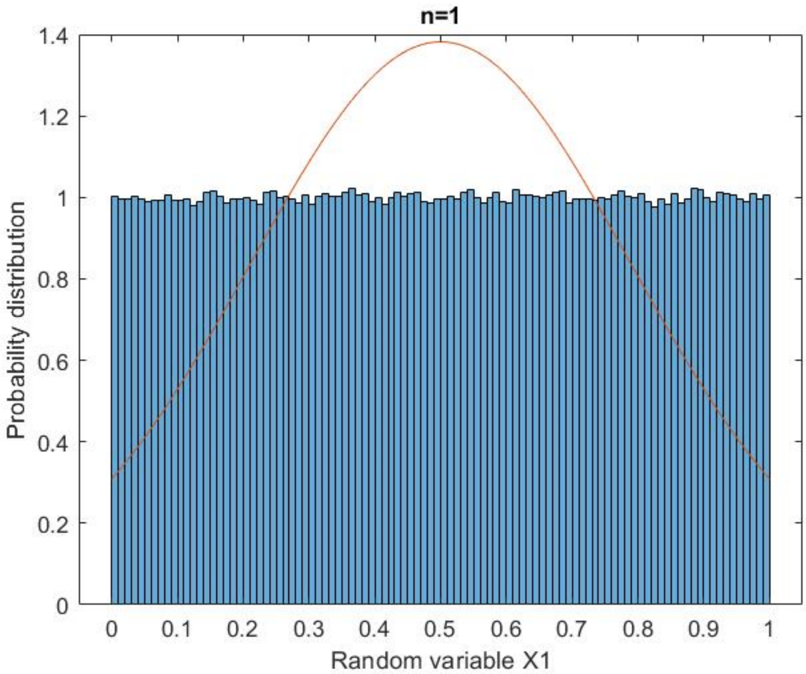
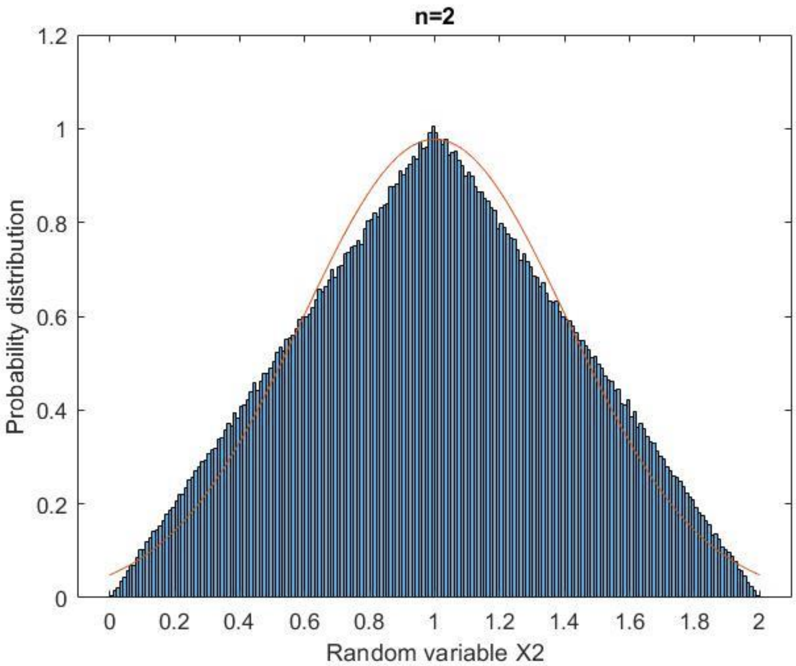
傳出來的 X 為一(1,1000000)矩陣，傳入值 n 為 Xn 之 n

1(b)

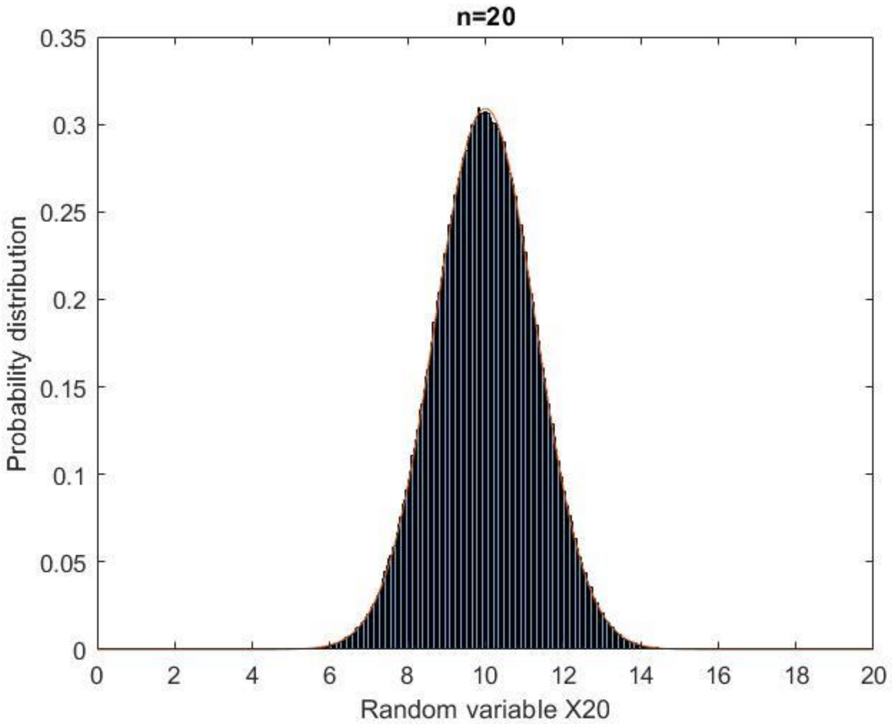
當 n=1


n=1
Probability distribution
Random variable X1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
1.2
1.4

n=2


n=2
Probability distribution
Random variable X2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

n=20



n=1 時，Irwin-Hall distribution 和 normal distribution 沒有任何關聯；然而當 n=2 時，儘管 Irwin-Hall distribution 和 normal distribution 並不吻合，但分布的趨勢均是先上升後下降；而當 n=20，也就是 n 是較大的數字時，Irwin-Hall distribution 和 normaldistribution 就非常接近，幾乎是貼在一起了。