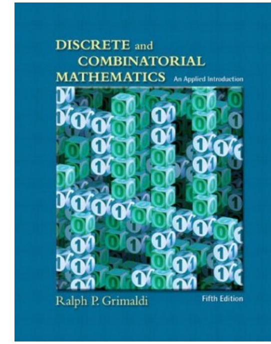
Discrete Mathematics
-- *Chapter 7: Relations: The Second Time Round*



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Outline

- Relations Revisited: Properties of Relations
- Computer Recognition: Zero-One Matrices and Directed Graphs
- **Partial Orders**: Hasse Diagrams
- **Equivalence Relations** and Partitions
- Finite State Machine: The Minimization Process
  - Application of equivalence relation
  - Minimization process: find a machine with the same function but fewer internal states

7.1 Relations Revisited: Properties of Relations

- Definition 7.1: For sets A, B, any subset of A × B is called a (binary) relation from A to B. Any subset of A × A is called a (binary) relation on A .

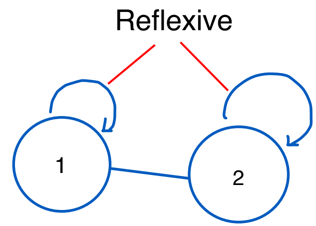
- **Ex 7.1**
  - Define the relation $\Re$ on the set Z by $a\Re b$, if $a \leq b$.
  - For x, $y \in Z$ and $n \in Z^+$, the modulo *n* relation $\Re$ is defined by $x \Re y$ if x - y is a multiple of *n*, e.g., with n=7, $9\Re 2$, $-3\Re 11$, but $3 \not\Re 7$

- **Ex 7.2** : Language $A \subseteq \Sigma^*$. For x, $y \in A$, define $x\Re y$ if x is a prefix of y.

Relations Revisited: Properties of Relations

- Finite state machine M = (S, I, O, v, w)
  - Reachability
    - $s_1 \Re s_2$ if $v(s_1, x) = s_2$, $x \in I$. $\Re$ denotes the first level of reachability.
    - $s_1 \Re s_2$ if $v(s_1, x_1 x_2) = s_2$, $x_1 x_2 \in I^2$. $\Re$ denotes the second level of reachability.
  - Equivalence
    - 1-equivalence relation: $s_1 E_1 s_2$ if $w(s_1, x) = w(s_2, x)$ for $x \in I$.
    - k-equivalence relation: $s_1 E_k s_2$ if $w(s_1, y) = w(s_2, y)$ for $y \in I^k$.
    - If two states are k-equivalent for **all $k \in \mathbf{Z}^+$**, they are called equivalent.

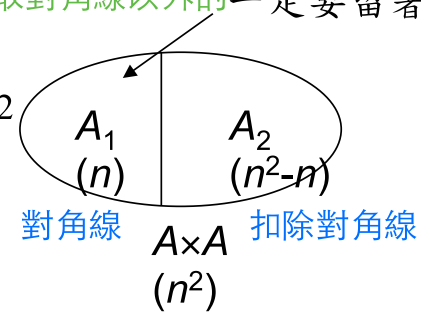
Reflexive



- Definition 7.2: A relation $\Re$ on a set A is called **reflexive** if $(x, x)\in\Re$, for **all** $x\in A$. 自己指向自己

- **Ex 7.4** : For $A = \{1, 2, 3, 4\}$, a relation $\Re \subseteq A\times A$ will be reflexive if and only if $\Re \subseteq \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. But $\Re_1 = \{(1, 1), (2, 2), (3, 3)\}$ is not reflexive, $\Re_2 = \{(x, y) | x \leq y, x, y\in A\}$ is reflexive.

- **Ex 7.5** : Given a finite set A with $|A| = n$, we have $|A\times A| = n^2$, so there are 2^{n^2} relations on A. Among them $2^{(n^2-n)}$ are reflexive.
  對角線的留著，取或不取對角線以外的一定要留著
  - $A = \{a_1, a_2, \ldots, a_n\}$
  - $A\times A = \{(a_i, a_j) | 1 \leq i, j \leq \mathrm{n}\} = A_1 \cup A_2$
  - $A_1 = \{(a_i, a_i) | 1 \leq i \leq \mathrm{n}\}$
  - $A_2 = \{(a_i, a_j) | i \neq j, 1 \leq i, j \leq \mathrm{n}\}$



Symmetric

- Definition 7.3: A relation $\Re$ on a set A is called **symmetric** if for **all x, $y \in A$**, **if** $(x, y) \in \Re \Rightarrow (y, x) \in \Re$.

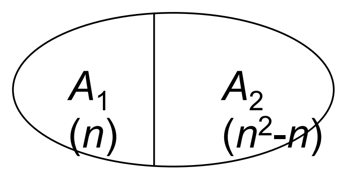
- **Ex 7.6** : $A = \{1, 2, 3\}$
  - $\Re_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, symmetric, but not reflexive.
  - $\Re_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$, reflexive, but not symmetric.
  - $\Re_3 = \{(1, 1), (2, 2), (3, 3)\}$ and $\Re_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, both reflexive and symmetric.
  - $\Re_5 = \{(1, 1), (2, 3), (3, 3)\}$, neither reflexive nor symmetric.

Symmetric

- To count the symmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$.
  - $A \times A = A_1 \cup A_2$, $A_1 = \{(a_i, a_i) | 1 \le i \le \mathrm{n}\}$, $A_2 = \{(a_i, a_j) | i \ne j, 1 \le i, j \le n\}$
  - A_1 contains n pairs, and A_2 contains n^2-n pairs.
  - A_2 contains $(n^2$-$n)/2$ subsets $S_{i,j}$ of the form $\{(a_i, a_j), (a_j, a_i) \mid i < j\}$.
  - So, we have totally $\underbrace{2^n}_{\text{對角線}} \times \underbrace{2^{(1/2)(n^2-n)}}_{\text{上三角}}$ symmetric relations on A.

- If the relations are both reflexive and symmetric, we have $2^{(1/2)(n^2-n)}$ choices.

1



$A \times A$
(n^2)

Transitive

- Definition 7.4: A relation $\Re$ on a set A is called **transitive if** $(x, y), (y, z) \in \Re \Rightarrow (x, z) \in \Re$ for all $x, y, z \in A$.

- **Ex 7.8** : Define the relation $\Re$ on the set Z^+ by $a \Re b$ if a divides b. This is a transitive and reflexive relation but not symmetric.
- **Ex 7.9** : Define the relation $\Re$ on the set Z by $a \Re b$ if $a \times b \geq 0$. What properties do they have?
  - Reflexive, symmetric
  - Not transitive, e.g., (3,0),(0,-7) $\in \Re$, but (3,-7) not

Antisymmetric

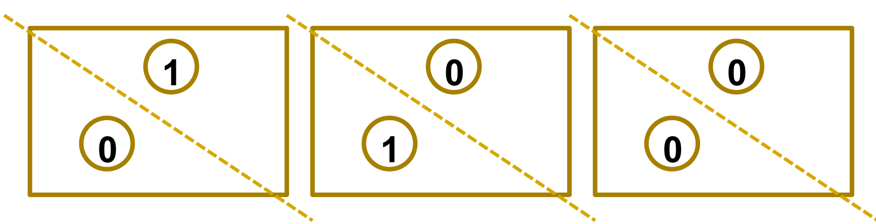
- Definition 7.5: A relation $\Re$ on a set A is called **antisymmetric** if $(x, y)\in\Re$ and $(y, x)\in\Re \Rightarrow x = y$ for all x, $y\in A$.
  - Both **a** related to **b** and **b** related to **a,** if **a** and **b** are one and the same element from A

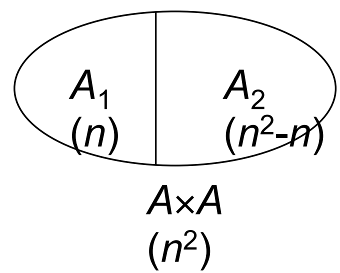
- **Ex 7.11** : Define the relation $(A, B)\in\Re$ if $A\subseteq B$. Then it is an anti-symmetric relation.

- Note that "*not symmetric*" is different from anti-symmetric.
- **Ex 7.12** : A = {1, 2, 3}, what properties do the following relations on A have?
  - $\Re=\{(1, 2), (2, 1), (2, 3)\}$ (not symmetric, not antisymmetric)
  - $\Re=\{(1, 1), (2, 2)\}$ (symmetric and antisymmetric)

Antisymmetric

- To count the antisymmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$.
  - $A \times A = A_1 \cup A_2$, $A_1 = \{(a_i, a_i) | 1 \le i \le \mathrm{n}\}$, $A_2 = \{(a_i, a_j) | i \ne j,\ 1 \le i, j \le n\}$
  - A_1 contains n pairs, and A_2 contains n^2-n pairs.
  - A_2 contains $(n^2-n)/2$ subsets $S_{i,j}$ of the form $\{(a_i, a_j), (a_j, a_i) \mid i < j\}$.
  - Each element in A_1 can be selected or not.
  - Each element in $S_{i,j}$ can be selected **in three alternatives**: ***either*** (a_i, a_j)***, or*** (a_j, a_i)***, or none*****.**
  - So, we have totally $2^n \times 3^{(1/2)(n^2-n)}$ anti-symmetric relations on A.





Antisymmetric

- **Ex 7.13** : Define the relation $\Re$ on the functions by $f \Re g$ if f is dominated by g (or $f \in O(g)$). What are their properties?
  - Reflexive
  - Transitive
  - not symmetric (e.g., g=n, f =n^{2,} g=O(f), but f $\neq$O(g))
  - not antisymmetric (e.g., g(n)= n, $f(n)$ = n+5, $f \Re g$ and $g \Re f$, *but* $f \neq g$)

Example

- A={a1,a2,…an}, students in class discrete math.
- a relation R on a set A, if (ai, aj)∈$\Re$, the midterm score (ai) $\geqq$ midterm score (aj)
  - No equal score
- R is reflexive
- R is transitive
- R is not symmetric
- R is antisymmetric

Partial Order

- Definition 7.6: A relation $\mathfrak{R}$ is called a partial order (partial ordering relation), if $\mathfrak{R}$ is *reflexive, anti-symmetric and transitive*.

- (A,R) is a **p**artially **o**rdered **set** / poset if R is a partial ordering on A. Typical notation: (A,≤);
  - "**no loops**"

- If a≤b or b≤a, the elements a and b are comparable.
- If all pairs are comparable, ≤ is a total ordering or chain.

Partial Order

- <u>**Ex 7.15**</u> : Let A be the set of positive integers divisors of n, the relation $\Re$ on A by $a\Re b$ if a divides b, it defines a *partial order*. How many ordered pairs does it occur in $\Re$.
  - E.g. n=12, $A = \{1, 2, 3, 4, 6, 12\}$, $\Re = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12)\}$
  - If $(a, b) \in \Re$, then $a = 2^m \cdot 3^n$ and $b = 2^p \cdot 3^q$ with **$0 \le m \le p \le 2, 0 \le n \le q \le 1$**.
  - Selection of size 2 from a set of size 3, with **repetition**.

$$\binom{3+2-1}{2} = \binom{4}{2} = 6 \text{ for } m, p; \ \binom{2+2-1}{2} = \binom{3}{2} = 3 \text{ for } n, q$$

$$\therefore \text{total} = 6 \cdot 3 = 18 \text{ ordered pairs}$$

  - For $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \Rightarrow$ the number of ordered pairs $= \prod_{i=1}^{k} \binom{(e_i+1)+2-1}{2} = \prod_{i=1}^{k} \binom{e_i+2}{2}$

Maximal element

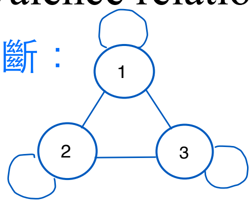
Equivalence relation

- Definition 7.7. A relation $\Re$ is called an **equivalence relation**, if $\Re$ is *reflexive, symmetric and transitive*.

- Given an equivalence relation R on A, for each a∈A the equivalence class [a] is defined by {x | (x,a)∈R }.
  - E.g., Modulo 3 equivalences on Z, such that [0] = {…,–6,–3,0,3,6,…} and [1] = {…,–5,–2,1,4,7,…}

- **Ex 7.16** (b): If $A = \{1, 2, 3\}$, the following are all equivalence relations
  - $\Re_1 = \{(1, 1), (2, 2), (3, 3)\}$
  - $\Re_2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3,2)\}$
  - $\Re_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$
  - $\Re_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

畫點判斷：



Examples

- Ex 7.16 (c): For a finite set A, $A \times A$ is the largest equivalence relation on A. If $A = \{a_1, a_2, \ldots, a_n\}$, then the equality relation $\mathfrak{R} = \{(a_i, a_i) | 1 \le i \le n\}$ is the smallest equivalence relation on A.

- Ex 7.16 (d): Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{x, y, z\}$, and $f: A \rightarrow B$ be the onto function. $f = \{(1, x), (2, z), (3, x), (4, y), (5, z), (6, y), (7, x)\}$. Define the relation $\mathfrak{R}$ on A by $a\mathfrak{R}b$ if $f(a) = f(b)$.
  $\mathfrak{R}$ is reflexive, symmetric, and transitive, so it is an equivalence relation. (e.g., f(a)=f(b), f(b)=f(c)=> f(a)=f(c))
  使用equivalence relation 分群：f1 = {(1,x) (3,x) (7,x)} , f2 = {(4,y) (6,y)} , f3 = {(2,z) (5,z)}

- Ex 7.16 (e): If $\mathfrak{R}$ is a relation on A, then $\mathfrak{R}$ is both an equivalence relation and a partial order relation iff $\mathfrak{R}$ is the equality relation on A.
  - equality relation $\{(a_i, a_i) | a_i \in A\}$ 對角線

  滿足reflexive, symmetric, transitive, antisymmetric

7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

- Definition 7.8: Let relations $\Re_1 \subseteq A \times B$ and $\Re_2 \subseteq B \times C$. The composite relation $\Re_1 \circ \Re_2$ is a relation defined by $\Re_1 \circ \Re_2 = \{(x, z) \mid \exists y \in B \text{ such that } (x, y) \in \Re_1 \text{ and } (y, z) \in \Re_2\}$. 有順序性

  (Note the different ordering with function composition.)

  $f : A \to B, \quad g : B \to C, \quad g \circ f : A \to C$

- **Ex 7.17** : Consider $\Re_1 = \{(1, x), (2, x), (3, y), (3, z)\}$ and $\Re_2 = \{(w, 5), (x, 6)\}$, and $\Re_3 = \{(w, 5), (w, 6)\}$. $\Re_1 \circ \Re_2 = \{(1, 6), (2, 6)\}$, and $\Re_1 \circ \Re_3 = \emptyset$?
- **Ex 7.18** : Let A be the set of employees {L. Alldredge,…} at a computer center, while B denotes a set of programming language {C++, Java,…}, and C is a set of projects $\{p_1, p_2, \ldots\}$, consider $\Re_1 \subseteq A \times B$, $\Re_2 \subseteq B \times C$. What is the means of $\Re_1 \circ \Re_2$?

join operation in database

Composite Relation

- Theorem 7.1: $\Re_1 \subseteq A \times B$, $\Re_2 \subseteq B \times C$, and $\Re_3 \subseteq C \times D \Rightarrow \Re_1 \circ (\Re_2 \circ \Re_3) = (\Re_1 \circ \Re_2) \circ \Re_3$
- Definition 7.9. We define the powers of relation $\Re$ by (a) $\Re^1 = \Re$; (b) $\Re^{n+1} = \Re \circ \Re^n$.

- **Ex 7.19** : If $\Re = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, then $\Re^2 = \{(1, 4), (1, 2), (3, 4)\}$, $\Re^3 = ?$ and $\Re^4 = ?$

$\Re^3 = \{(1,4)\}$
and for $n \geq 4$, $\Re^n = \emptyset$

$\aleph^2 = \aleph \circ \aleph$
$\aleph^3 = \aleph^2 \circ \aleph$

Relation Matrix

- Definition 7.10: An $m \times n$ zero-one matrix $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in m rows and n columns, where each e_{ij} denotes the entry in the ith row and jth column of E, and each such entry is 0 or 1.
- **Relation matrix**: A relation can be represented by an $m \times n$ zero-one matrix.

  1：存在relation 0：無relation

- **Ex 7.21** : Consider $\Re_1 = \{(1, x), (2, x), (3, y), (3, z)\}$, $\Re_2 = \{(w, 5), (x, 6)\}$, and $\Re_1 \circ \Re_2$ to be represented by relation matrices?

$$M(\mathcal{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{cccc} (w) & (x) & (y) & (z) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad M(\mathcal{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = M(\mathcal{R}_1 \circ \mathcal{R}_2).$$

Boolean addition' with 1+1=1

OR

Relation Matrix

- **Ex 7.22**: If $\Re$={(1, 2), (1, 3), (2, 4), (3, 2)}, then what are the relation matrices of $\Re^2$, $\Re^3$ and $\Re^4$?

$$M(\mathscr{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M(\mathscr{R}))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M(\mathscr{R}))^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrix

- Let A be a set with $|A| = n$ and $\Re$ be a relation on A. If $M(\Re)$ is the relation matrix for $\Re$, then
  - $M(\Re) = \mathbf{0}$ if and only if $\Re = \phi$.
    element皆為0
  - $M(\Re) = \mathbf{1}$ if and only if $\Re = A \times A$.
    element皆為1
  - $M(\Re^m) = [M(\Re)]^m$
- Definition 7.11: Let $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$ be two mxn ero-one matrices. We say that E precedes, or is less than, F, written as $E \le F$, if $e_{ij} \le f_{ij}$ for all i, j.

- Ex 7.23 : $E \le F$. How many zero-one matrices G do have the results of $E \le G$?

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$2^3=8$

Relation Matrix

- Definition 7.12: $I_n = (\delta_{ij})_{n\times n}$ is the $n\times n$ zero-one matrix, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

- Definition 7.13: $A = (a_{ij})_{m\times n}$ is a zero-one matrix, the transpose of A, written A^{tr}, is the matrix $(a^*_{ji})_{n\times m}$ where $a^*_{ji} = a_{ij}$

- **Ex 7.24** :

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad A^{\text{tr}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- **Theorem 7.2**: If M denote the relation matrix for $\mathfrak{R}$ on A, then
  - (A) $\mathfrak{R}$ is reflexive if and only if $I_n \leq M$.
  - (B) $\mathfrak{R}$ is symmetric if and only if $M = M^{\text{tr}}$.
  - (C) $\mathfrak{R}$ is transitive if and only if $M^2 \leq M$.
  - (D) $\mathfrak{R}$ is anti-symmetric if and only if $M \cap M^{\text{tr}} \leq I_n$.

Directed Graph

relation具有方向性

- Definition 7.14. A directed graph can be denoted as $G = (V, E)$, where V is the vertex set and E is the edge set.
  - (a, b): if $a, b \in V$ $(a, b) \in E$, then there is a edge from a to b. Vertex a is called source (origin) of the edge, and b is terminating vertex.
  - (a, a): is called a loop.
- $V = \{1, 2, 3, 4, 5\}$, $E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$
  - Isolated vertex: vertex 5 in Fig. 7.1.
- Single undirected edge $\{a, b\} = \{b, a\}$ in Fig. 7.2 (b) is used to represent the two directed edges shown in Fig. 7.2 (a).

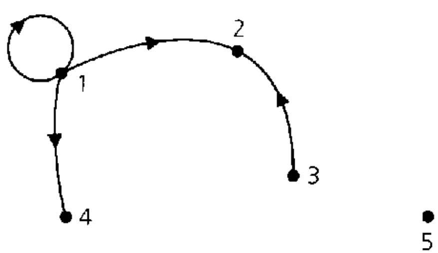


Figure 7.1

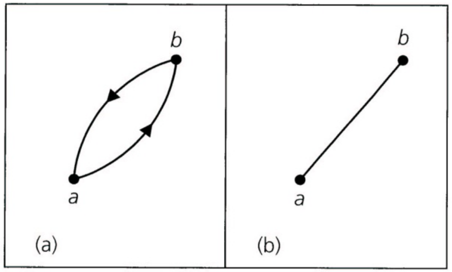


Figure 7.2

Directed Graph

- <u>Ex 7.26</u> precedence graph

先後順序關係

(s_1) $b := 3$
(s_2) $c := b + 2$
(s_3) $a := 1$
(s_4) $d := a * b + 5$
(s_5) $e := d - 1$
(s_6) $f := 7$
(s_7) $e := c + d$
(s_8) $g := b * f$

(a)

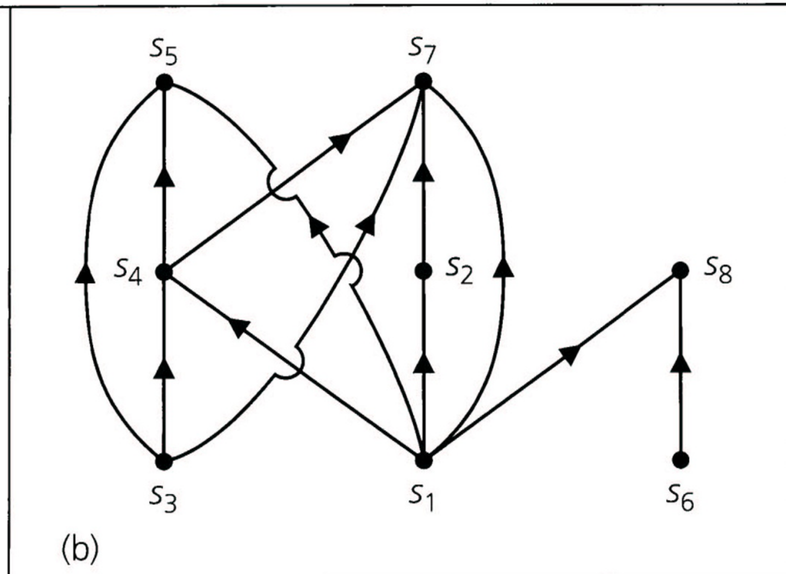


Figure 7.3

Directed Graph

- Ex 7.27 : R = {(1,1),(1,2),(2,3),(3,2),(3,3),(3,4),(4,2)}
  - directed graph in Fig. 7.4 (a)
  - (associated) undirected graph in Fig. 7.4 (b)
  - path: In the connected graph, any two vertices x, y, with $x \neq y$, there is a path starting at x and ending at y.
  - cycle: a closed path starts and terminates at the same vertex, containing at least three edges.
    - E.g.: {3, 4}, {4, 2}, and {2, 3}

No repeated vertex

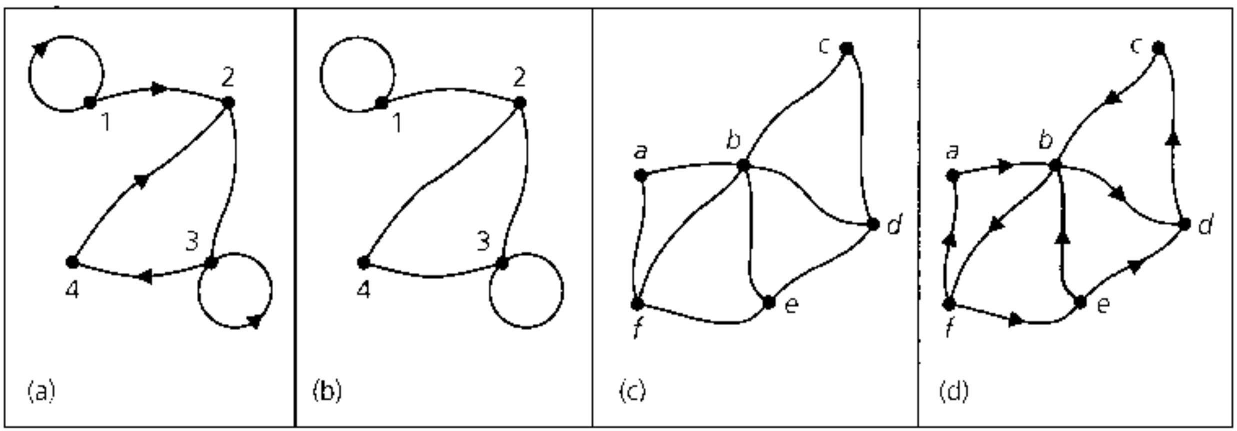


Figure 7.4

Directed Graph

- Definition 7.15: A directed graph G on V is called **strongly connected**, if for all $x, y \in V$, where $x \neq y$, there is a path (in G) of directed edges from x to y.
  - e.g., Fig. 7.5
- **Disconnected graph**: is the union of two connected pieces called the components of the graph.
  - e.g., Fig. 7.6

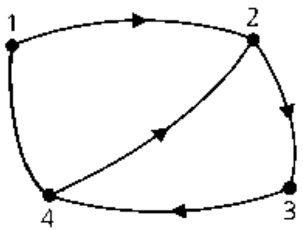


Figure 7.5

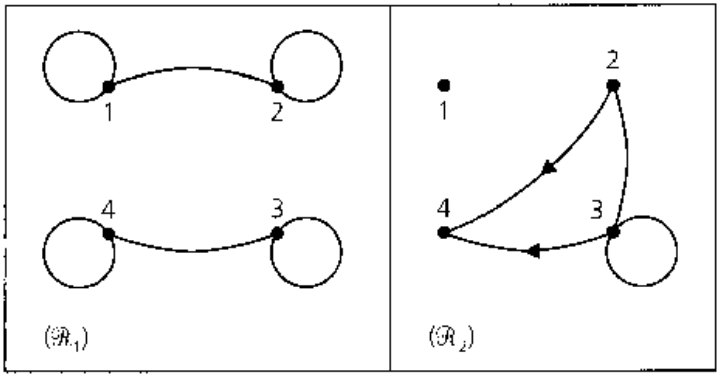


Figure 7.6

Directed Graph

- **Complete graph**: the graphs of undirected graphs that are loop-free and have an edge for every pair of distinct vertices, which are denoted by K_n.
  - e.g., Fig. 7.7

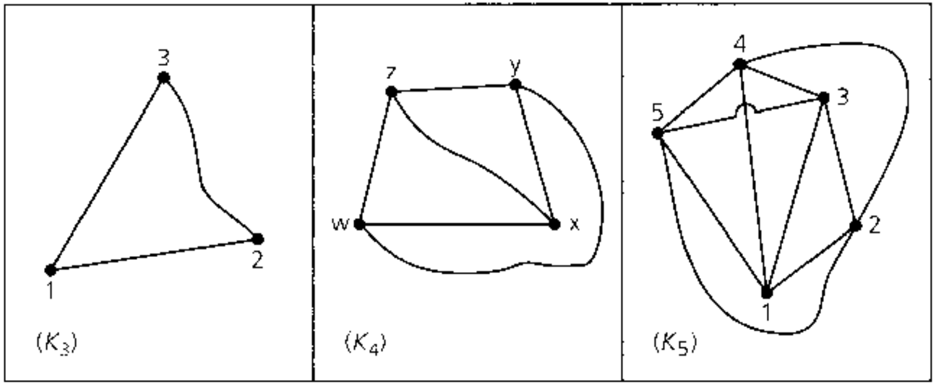
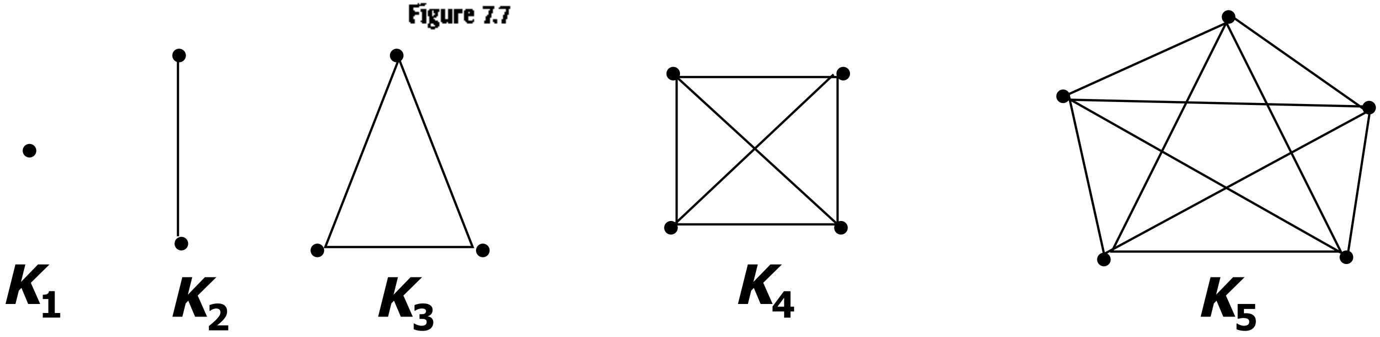


Figure 7.7



Directed Graph

- **Ex 7.30** : $\mathfrak{R}$ is reflexive if and only if its directed graph contains a loop at each vertex. 每點都有一個loop指向自己
  - e.g., Fig 7.8, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1,1), (1, 2), (2, 2), (3, 3), (3, 1)\}$
- **Ex 7.31** : $\mathfrak{R}$ is symmetric if and only if its directed graph may be drawn only by loops and undirected edges. 不會有direct edge
  - e.g., Fig 7.9, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1,1), (1, 2), (2, 1), (2, 3), (3, 2)\}$
- **Ex 7.32** : $\mathfrak{R}$ is anti-symmetric if and only if for any $x \neq y$ the graph contains at most one of the edges (x, y) or (y, x)
  - e.g., Fig 7.10, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1,1), (1, 2), (2, 3), (1, 3)\}$

任兩點間最多只有1個direct edge（可無edge）

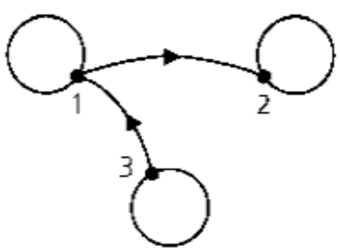


Figure 7.8

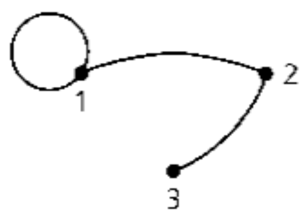


Figure 7.9

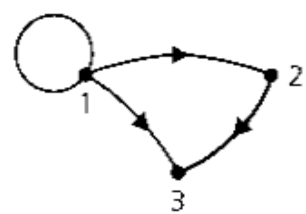


Figure 7.10

Directed Graph

- **Ex 7.32** : $\mathfrak{R}$ is transitive if and only if for all $x, y \in A$, if there is a **path** from x to y in the associated graph, then there is an **edge** (x, y) also.
  - e.g., Fig 7.10, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$
- **Ex 7.33** : Fig 7.11, a relation is an equivalence relation if and only if its graph is one **complete graph** augmented by loops at every vertex or consists of disjoint union of complete graphs augmented by loops at each vertex.
  - e.g., Fig 7.11, $A = \{1, 2, 3, 4, 5\}$ and $\mathfrak{R}_1 = \{(1,1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$, $\mathfrak{R}_2 = \{(1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4) (5, 5)\}$.

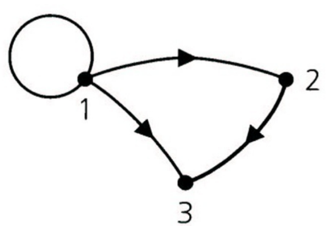


Figure 7.10

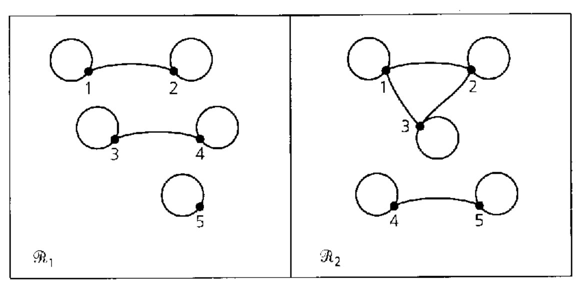


Figure 7.11

Directed Graph

reflexive: *loop on each vertex*

symmetric: *undirected edge + loops*

transitive: *one path* → *one edge*

equivalence: *disjoint union of complete graphs + loops at every vertex*

directed graphs ⟷ relations

⇩ ⇩

adjacency matrices ⟷ relation matrices

7.3 Partial Orders: Hasse Diagrams

- Definition: Let A be a set with $\Re$ a relation on A. The pair (A, $\Re$) is called a **partially ordered set**, or poset, if relation $\Re$ on A is partially ordered.
  - If A is called a poset, we understand that there is a partially order $\Re$ on A that makes A into this set.

natural counting: **N**

x+5=2 : **Z**

2x+3=4 : **Q**

x^2-2=0 : **R**

x^2+1=0 : **C**

Something was lost when we went from **R** to **C**. We have lost **the ability to "order"** the elements in **C**.

$2+i < 1+2i$?

7.3 Partial Orders: Hasse Diagrams

- **Ex 7.34** : Let A be the set of courses offered at a college. Define the relation $\Re$ on A by $x\Re y$ if x, y are the same course or if x is a prerequisite for y.

- **Ex 7.35** : Define $\Re$ on $A = \{1, 2, 3, 4\}$ by $x\Re y$ if x divide y. Then $\Re = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$ is a partial order, and $(A, \Re)$ is a poset.

- **Ex 7.36** : PERT (Program Evaluation and Review Technique) network is first used by U.S. Navy in 1950.
  - E.g., Let A be the set of tasks that must be performed to build a house. Define the relation $\Re$ on A by $x\Re y$ if x, y are the same task or if x must be performed before y.

Partial Orders: Hasse Diagrams

- **Ex 7.37** : Figure 7.17 (b) illustrates a simpler diagram for (a), called the **Hasse diagram**. The directions are assumed to go from the bottom to the top.

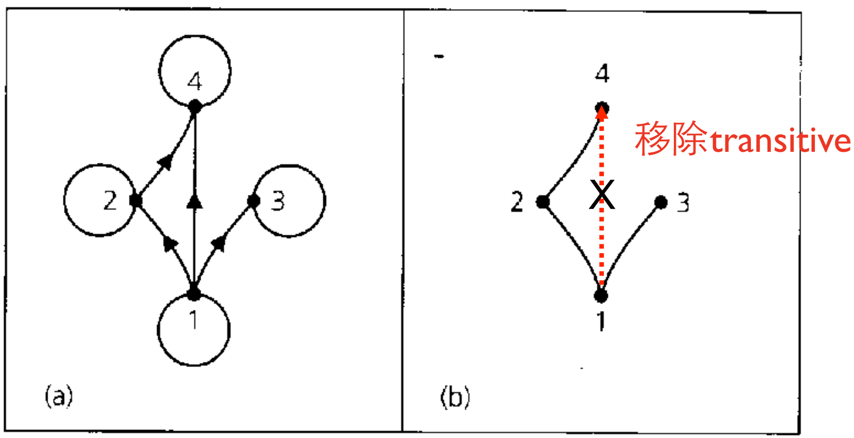
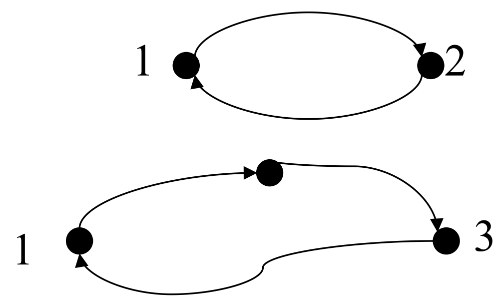


Figure 7.17

not partial order



Hasse Diagram

- If $(A, \Re)$ is a poset, we construct a Hasse diagram for $\Re$ on A by drawing a line segment from x up to y, if
  - $x\Re y$
  - there is no other z such that $x\Re z$ and $z\Re y$. *(in between x and y)*
- **Ex 7.38** : In Fig. 7.18 we have the Hasse diagrams for the following four posets.
  - (a) $\Re$ is the subset relation on A is the power set of $\mathcal{U}$ with $\mathcal{U} = \{1, 2, 3\}$
  - (b), (c), and (d) are the divide relations.

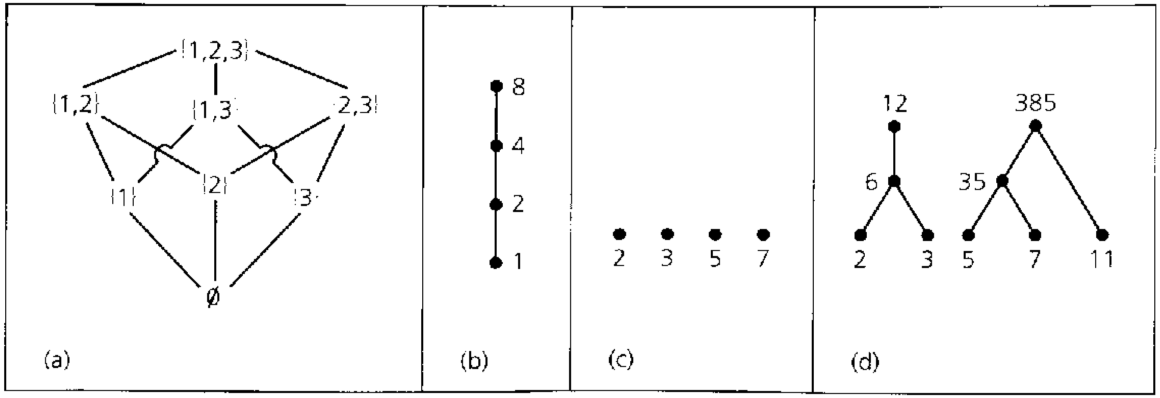


Figure 7.18

Totally Ordered

- Definition 7.16. If $(A, \Re)$ is a poset, we say that A is **totally ordered** (linearly ordered) if for all $x, y \in A$ either $x \Re y$ or $y \Re x$. In this case, $\Re$ is called a total order.
- **Ex 7.40**
  a) On the set **N**, the relation $\Re$ defined by $x \Re y$ if $x \leq y$ is a total order.
  b) The subset relation is a partial order but not total order, e.g., $\{1, 2\}, \{1, 3\} \in A$, but $\{1, 2\} \not\subseteq \{1, 3\}$ or $\{1, 3\} \not\subseteq \{1, 2\}$.
  c) Fig 7.19 is a total order.

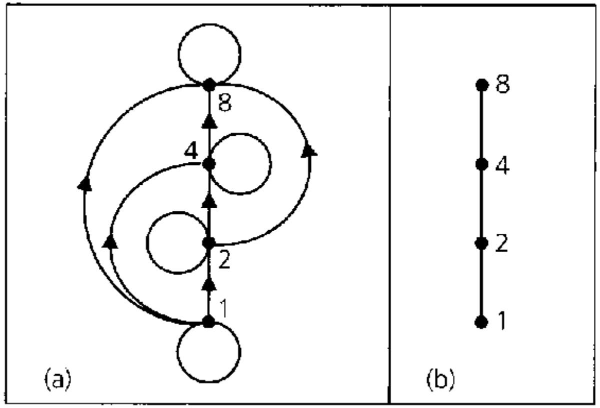


Figure 7.19

Topological Sorting

- Given a Hasse diagram for a partial order relation $\Re$, how to find a total order $\Im$ for which $\Re \subseteq \Im$.

$3! \times 2! = 12$種 total order

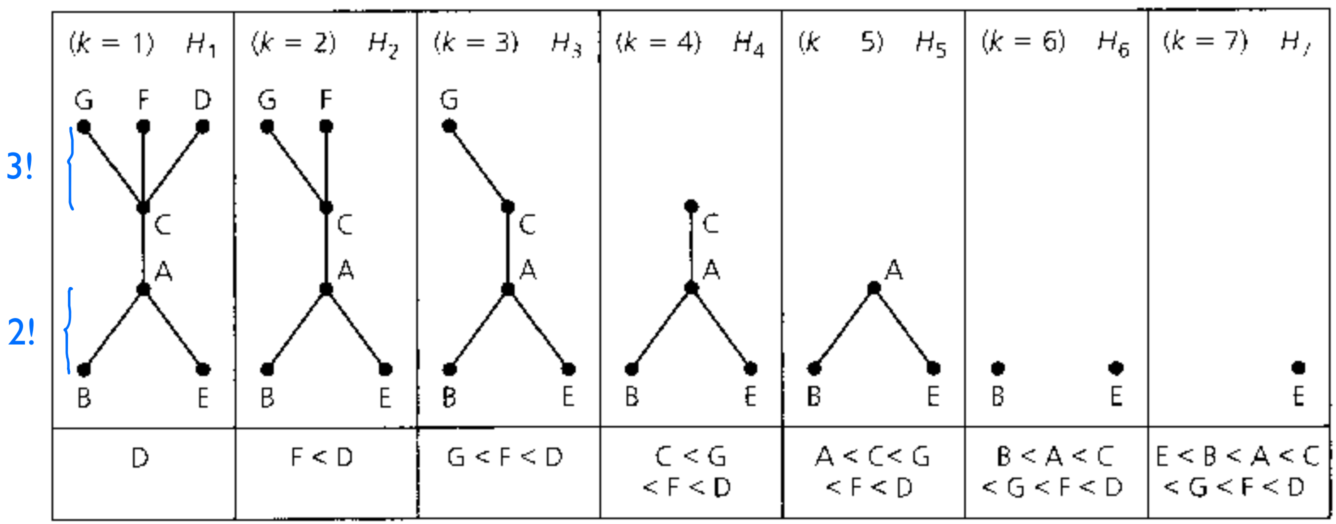


Figure 7.21

Not unique, 12 answers

Topological Sorting

- For a partial order $\Re$ on a set A with $|A| = n$
  - Step 1: Set $k = 1$. Let H_1 be the Hasse diagram of the partial order.
  - Step 2: Select a vertex v_k in H_k such that no edge in H_k starts at v_k.
  - Step 3: If $k = n$, the process is completed and we have a total order

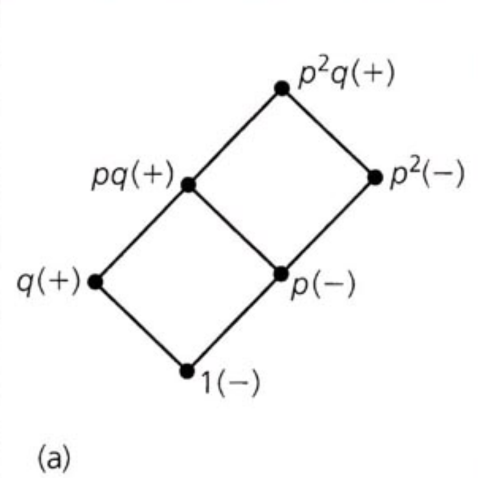
$$\Im : v_n < v_{n-1} < \cdots < v_1$$

    that contains $\Re$.
  - If $k < n$, then remove from H_k the vertex v_k and all edges of H_k that terminate at v_k. Call the result H_{k+1}. Increase k by 1 and return to step (2).

Topological Sorting

- **Ex 7.41**

任何一點［+］一定會比［-］號多



$\mathcal{T}_1$: $p^2q > pq > q > p^2 > p > 1$
+, +, +, −, −, −

$\mathcal{T}_2$: $p^2q > pq > p^2 > p > q > 1$
+, +, −, −, +, −

$\mathcal{T}_3$: $p^2q > p^2 > pq > q > p > 1$
+, −, +, +, −, −

$\mathcal{T}_4$: $p^2q > pq > p^2 > q > p > 1$
+, +, −, +, −, −

$\mathcal{T}_5$: $p^2q > p^2 > pq > p > q > 1$
+, −, +, −, +, −

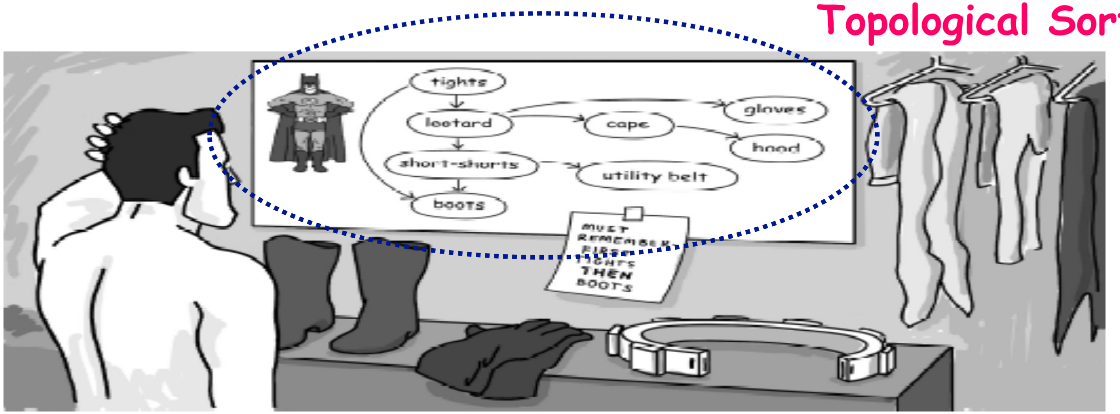
(b)

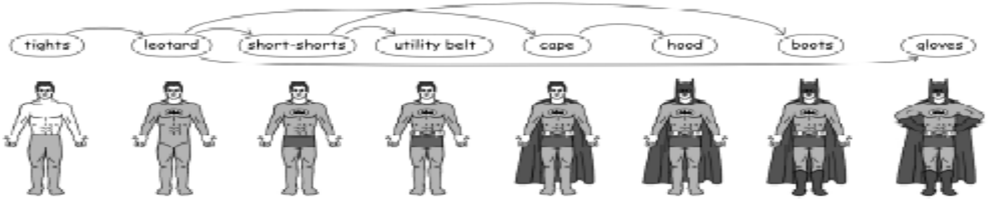
Figure 7.22

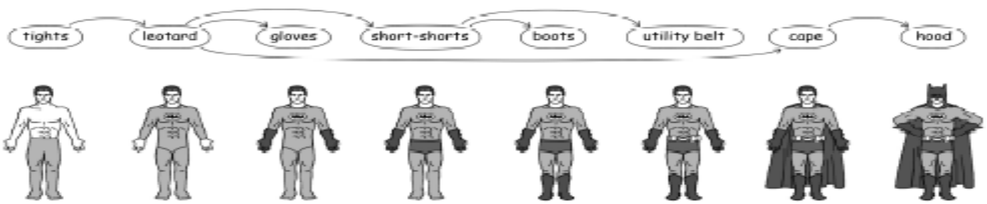
More practices, exercise7.27

EX: Dressing in the morning

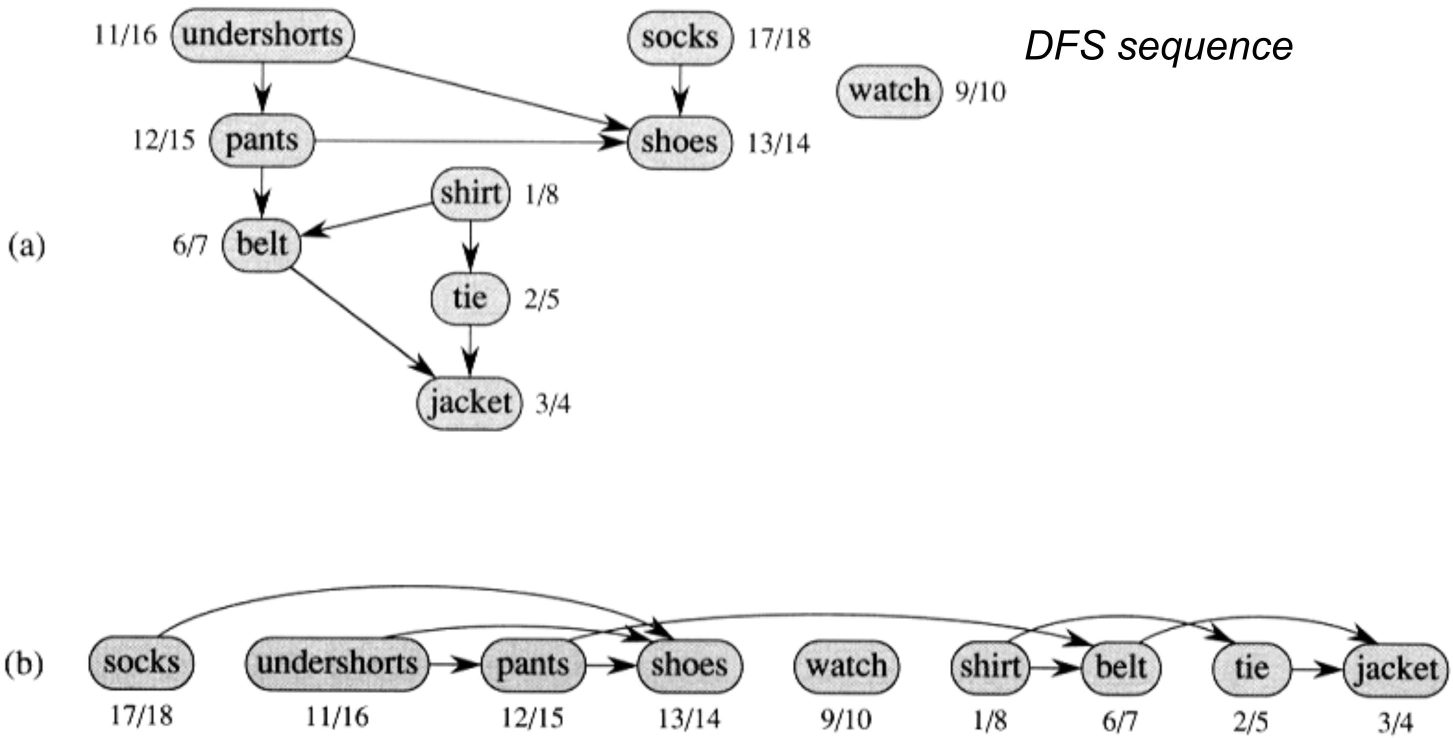
Topological Sort





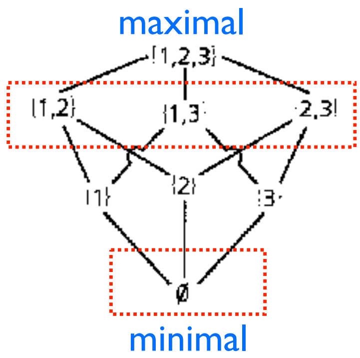


DFS sequence



Maximal and Minimal

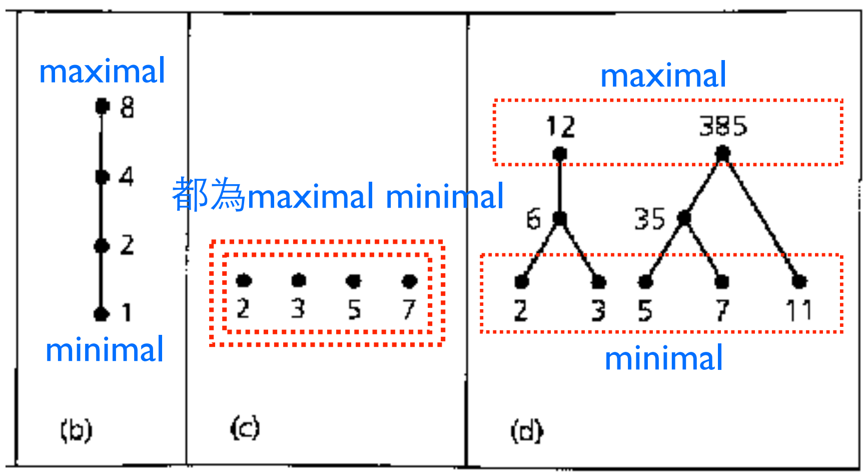
- Definition 7.17: If $(A, \mathfrak{R})$ is a poset, then x is a maximal element of A if for all $a \in A$, $a \neq x \Rightarrow x \not\mathfrak{R}\, a$. Similarly, y is a minimal element of A if for all $b \in A$, $b \neq y \Rightarrow b \not\mathfrak{R}\, y$. Proper subset：subset不能和原本的set相同
- **Ex 7.42** : $\mathcal{U} = \{1, 2, 3\}$, $A = P(\mathcal{U})$.
  - For the poset $(A, \subseteq)$, $\mathcal{U}$ is the maximal and ϕ is the minimal.
  - Let B be the *proper subsets* of $\{1, 2, 3\}$. Then we have multiple maximal elements $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ for the poset $(B, \subseteq)$, and ϕ is still the only minimal element.



- **Ex 7.43** : For the poset $(\mathbf{Z}, \leq)$, we have neither a maximal nor a minimal element. The poset $(\mathbf{N}, \leq)$, has no maximal element but a minimal element 0.

Maximal and Minimal

- Ex 7.44 : How about the poset in (b), (c), and (d) of Fig. 7.18? Do they have maximal or minimal elements?
- Theorem 7.3: If $(A, \mathfrak{R})$ is a poset and <u>A is finite</u>, then A has both a maximal and a minimal element.



Least and Greatest

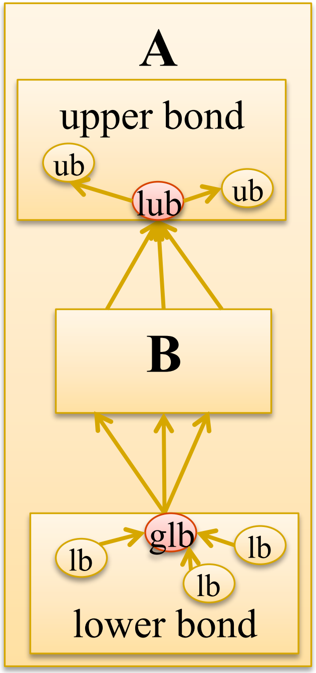
- Definition 7.18: If $(A, \Re)$ is a poset, then x is a **least** element of A if for all $a \in A$, $x \Re a$. Similarly, y is a **greatest** element of A if for all $a \in A$, $a \Re y$.
- <u>**Ex 7.45**</u> : $\mathcal{U} = \{1, 2, 3\}$, $A = P(\mathcal{U})$.
  - For the poset $(A, \subseteq)$, $\mathcal{U}$ is the greatest and ϕ is the least.
  - Let B be the <u>nonempty subsets</u> of $\mathcal{U}$. Then we have $\mathcal{U}$ as the greatest element and three minimal elements for the poset $(B, \subseteq)$, but no least element.
- Theorem 7.4: If poset $(A, \Re)$ has a greatest **or** a least element, then that element is unique.
  - Proof: Assume x and y are both greatest elements.
    Since x is a greatest element, $y \Re x$. Likewise, $x \Re y$ while y is a greatest element. As $\Re$ is antisymmetric, it follows x = y.

Lower and Upper Bound

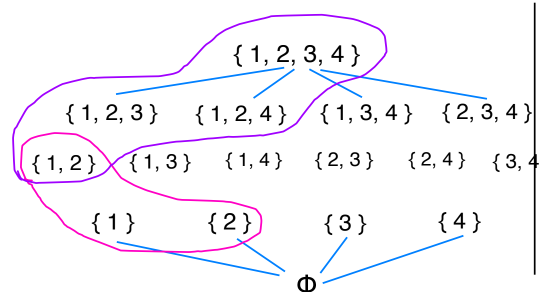
- Definition 7.19: If $(A, \Re)$ is a poset with $B \subseteq A$, then
  - $x \in A$ is called a **lower bound** of B if $x \Re b$ **for all** $\boldsymbol{b \in B}$
  - $y \in A$ is called an **upper bound** of B if $b \Re y$ **for all** $\boldsymbol{b \in B}$

- An element $x' \in A$ is called a *greatest lower bound* (**glb**) of B if for all other lower bounds x'' of B we have $x'' \Re x'$. Similarly, an element $x' \in A$ is called a *least upper bound* (**lub**) of B if for all other upper bounds x'' of B we have $x' \Re x''$.

- Theorem 7.5: If $(A, \Re)$ is a poset and $B \subseteq A$, then B has **at most one** lub (glb).



Lower and Upper Bound



- **Ex 7.47** : Let U $=\{1, 2, 3, 4\}$ with $A = P(U)$ and let $\Re$ be the subset relation on B. If $B = \{\{1\}, \{2\}, \{1, 2\}\}$, then what are the upper bounds of B, lower bounds of B, the greatest lower bound and the least upper bound?
  - Upper bounds: $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 2, 3, 4\}$
  - lub: $\{1, 2\}$
  - glb = ϕ

{2, 3, 4} is not.

- **Ex 7.48** : Let $\Re$ be the "≤" relation on A. What are the results for the following cases?
  - A = **R** and $B = [0, 1]$ => lub:1, glb:0
  - A = **R** and $B = \{q \in \mathbf{Q} \mid q^2 < 2\}$ => $\text{lub}: \sqrt{2}, \ \text{glb}: -\sqrt{2}$
  - A = **Q** and $B = \{q \in \mathbf{Q} \mid q^2 < 2\}$ => ? *No lub and glb*

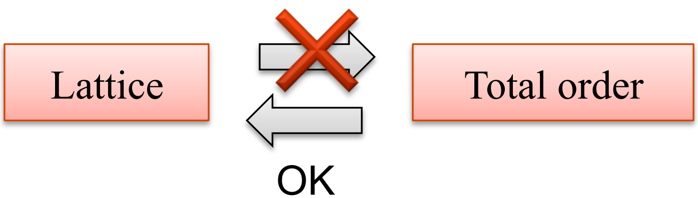
Lattice

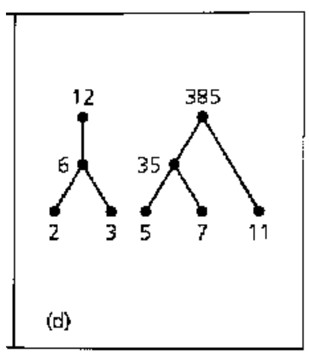
- Definition 7.20. The poset $(A, \mathfrak{R})$ is called a **lattice** if for *all* $x, y \in A$ the elements lub$\{x, y\}$ and glb$\{x, y\}$ both exist in A.

- <u>**Ex 7.49**</u> : For $A = \mathbf{N}$ and $x, y \in \mathbf{N}$, define $x \mathfrak{R} y$ by $x \le y$. Then lub$\{x, y\}$ = max$\{x, y\}$, glb$\{x, y\}$ = min$\{x, y\}$, and $(\mathbf{N}, \le)$ is a lattice.

- <u>**Ex 7.50**</u> : For the poset $(P(\mathcal{U}), \subseteq)$, if $S, T \subseteq \mathcal{U}$, we have lub$\{S, T\} = S \cup T$ and glb$\{S, T\} = S \cap T$ and it is a lattice.

- <u>**Ex 7.51**</u>: consider the poset in Example 7.38(d). Here we find that lub$\{2, 3\} = 6$ exists, but there is no glb for the elements 2 and 3.
  - This partial order is not a lattice.





7.4 Equivalence Relations and Partitions

- Equivalence relation: reflexive, symmetric, and transitive
- Examples:
  - For any set $A \neq \phi$, the relation of equality is an equivalence relation on A.
    - "sameness" among the elements of A
  - Let the relation on **Z** defined by $x \Re y$ if x-y is a multiple of 2, then $\Re$ is an equivalence relation on **Z**, where all even integers are related, as are all odd integers.
    - The above relation **splits Z** into two subsets: $\{\ldots, -3, -1, 1, 3, \ldots\} \cup \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

Partition

- Definition 7.21. Given a set A and index set I, let $\phi \neq A_i \subseteq A$ for $i \in I$.
  - Then $\{A_i\}_{i\in I}$ is a partition of A if (a) $A = \cup_{i\in I} A_i$ and (b) $A_i \cap A_j = \phi$ for $i \neq j$.
  - Each subset A_i is called a cell (block) of the partition.

- **Ex 7.52** : $A = \{1, 2,\ldots,10\}$
  - $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$.
  - $A_i = \{i, i+5\}$, $1 \leq i \leq 5$.
- **Ex 7.53** : Let $A = \mathbf{R}$, for each $i \in \mathbf{Z}$,
  let $A_i = [i, i+1)$. Then $\{A_i\}_{i\in Z}$ is a partition of $\mathbf{R}$.

Equivalence Class

- Definition 7.22: Let $\Re$ be an equivalence relation on a set A. For each $x \in A$, the **equivalence class** of x, denoted $[x]$, is defined by $[x] = \{y \in A \mid y \Re x\}$

(1) $x \Re y$?

(2) [x]=[y]?

- **Ex 7.54** : Define the relation $\Re$ on **Z** by $x \Re y$ if $4 \mid (x-y)$.
  - $[0] = \{\ldots, -8, -4, 0, 4, \ldots\} = \{4k \mid k \in \mathbf{Z}\}$
  - $[1] = \{\ldots, -7, -3, 1, 5, \ldots\} = \{4k+1 \mid k \in \mathbf{Z}\}$
  - $[2] = \{\ldots, -6, -2, 2, 6, \ldots\} = \{4k+2 \mid k \in \mathbf{Z}\}$
  - $[3] = \{\ldots, -5, -1, 3, 7, \ldots\} = \{4k+3 \mid k \in \mathbf{Z}\}$
- **Ex 7.55** : Define the relation $\Re$ on **Z** by $a \Re b$ if $a^2=b^2$, $\Re$ is an equivalence relation.
  - $[n] = [-n] = \{-n, n\}$

$$Z = \{0\} \cup \left(\bigcup_{n \in Z^+} \{-n, n\} \right)$$

Equivalence Class

- **Theorem 7.6**: If $\Re$ is an equivalence relation on a set A and $x, y \in A$, then
  - (a) $x \in [x]$
  - (b) $x\Re y$ if and only if $[x] = [y]$
  - (c) $[x] = [y]$ or $[x] \cap [y] = \phi$. *(identical or disjoint)*

- **Ex 7.56** :
  - Let $A = \{1, 2, 3, 4, 5\}$, $\Re = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$. $[1] = \{1\}$, $[2] = \{2, 3\} = [3]$, $[4] = \{4, 5\} = [5]$. Then, we have $A = [1] \cup [2] \cup [4]$.
  - Consider an onto function $f: A \rightarrow B$. $f(\{1, 3, 7\}) = x$; $f(\{4, 6\}) = y$; $f(\{2, 5\}) = z$. The relation $\Re$ defined on A by $a\Re b$ if $f(a) = f(b)$. equivalence class
    - $A = [1] \cup [4] \cup [2] = f^{-1}(x) \cup f^{-1}(y) \cup f^{-1}(z)$.

- **Ex 7.58** : If an equivalence relation $\Re$ on $A = \{1, 2, 3, 4, 5, 6, 7\}$ induces the partition $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$, what is $\Re$? [2007台大資工]
  - $[1] = \{1, 2\} = [2] = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$
  - $\Re = (\{1,2\} \mathrm{x} \{1,2\}) \cup (\{3\} \mathrm{x} \{3\}\} \cup (\{4, 5, 7\} \mathrm{x} \{4, 5, 7\}) \cup (\{6\} \mathrm{x} \{6\})$
  - $|\Re| = 2^2 + 1^2 + 3^2 + 1^2 = 15$

{(1,1), (2,2)} v.s. {1,2}x{1,2}

Equivalence and Partition

- **Theorem 7.7**: If A is a set, then
  - (a) any equivalence relation $\mathfrak{R}$ on A induces a partition of A; and
  - (b) any partition of A gives rise to an equivalence relation $\mathfrak{R}$ on A.
- **Theorem 7.8**: For any set A, there is one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.

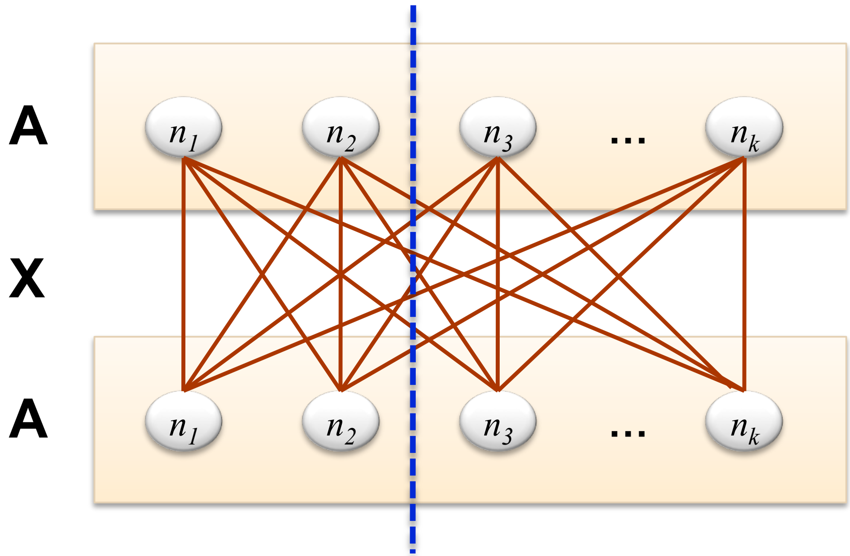
equivalence relation會將index分群
Partition也會

- Ex 7.59 :
  - (a) If A= {1, 2, 3, 4, 5, 6}, how many relations on A are equivalence relations? (identical containers)
    - *a partition of A: a distribution of the (distinct) elements of A into identical containers with no container left empty* $\sum_{i=1}^{6} S(6,i) = 203$
  - (b) How many of the equivalence relations in part (a) satisfy 1, 2 $\in$ [4]?

$$\sum_{i=1}^{4} S(4,i) = 15$$

{ 1, 2, 4, 3, 5, 6 }

Equivalence and Partition



k=10

k^2 *100*

$2^2+(k-2)^2$ *4+64=68*

r partitions?

7.5 Finite State Machines: The Minimization Process

- Two finite state machines of the same function may have different number of internal states.
  - Some of these states are redundant.
- A process of transforming a given machine into one that has *no redundant internal states* is called the minimization process.
  - Rely on the concepts of *equivalence relation* and *partition*.

Finite State Machines: The Minimization Process

- 1-Equivlence: Given the finite state machine $M = \{S, I, O, v, w\}$, we define the ***relation*** $\boldsymbol{E_1}$ on S by $s_1 E_1 s_2$ if $w(s_1, x) = w(s_2, x)$ for all $x \in I$.
- The relation E_1 is an equivalence relation on S, and it partitions S into subsets such that two states are in the same subset if they produce the same output for each $x \in I$. 單位長度為1
- Here s_1 and s_2 are called 1-equivlent.

給定長度為1的訊號，可得到相同的output

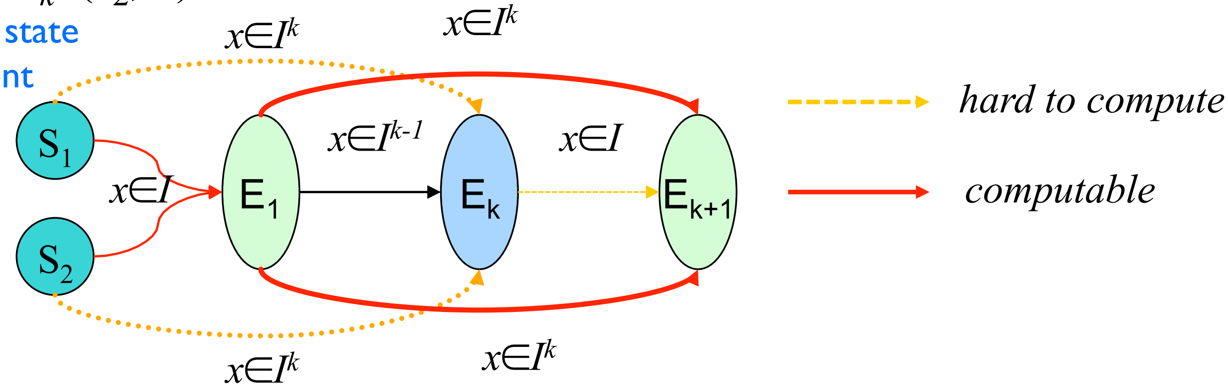
Finite State Machines: The Minimization Process

- For the states S, we define the *k-equivalence* relation E_k on S by $s_1 E_k s_2$ if $w(s_1, x) = w(s_2, x)$ for all $x \in I^k$. 單位長度為k
- The relation E_k is an equivalence relation on S, and it partitions S into subsets such that two states are in the same subset if they produce the same output for each $x \in I^k$. k越大partition 越多（partition會越分越細）
- We call two states s_1 and s_2 equivalent if they are k-equivalent for all $k \geq 1$.

Finite State Machines: The Minimization Process

- Goal: *Determine the partition of S induced by E and select one state for each equivalent class.*
- Observations:
  - If two states are not 2-equivalent, they can not be 3-equivalent.
  - For $s_1, s_2 \in S$, where $s_1 E_k s_2$, we find that $s_1 E_{k+1} s_2$ if and only if $v(s_1, x) E_k v(s_2, x)$ for all $\boldsymbol{x \in I}$.

判斷s1 s2的next state
是否為k-equivalent



An Algorithm for the Minimization of a Finite State Machine

原來被切開的，不需再做測試，只需測試未被分開的部分

1. Set $k = 1$. $s_1 \mathrm{E}_1 s_2$ when s_1 and s_2 have the same output rows. (P_i be the partitions of S induced by E_i)
2. Having determined P_k, we want to obtain P_{k+1}. Determine the states that are (k+1)-equivalent. Note that if $s_1 \mathrm{E}_k s_2$, then $s_1 \mathrm{E}_{k+1} s_2$ if and only if $v(s_1, x) \mathrm{E}_k v(s_2, x)$ for all $x \in I$.
3. If $P_{k+1} = P_k$, the process is completed.
   If $P_{k+1} \neq P_k$, $k = k+1$, goto step 2.

從next state去推k-equivalent （得到最終partition結果）=》化簡state

Minimization of a Finite State Machine

- <u>**Ex 7.60**</u> : M is given by the state table shown in Table 7.1.
  - Looking at the output rows: which states are 1-equivalent?
  - P_1 partitions S as P_1: $\{s_1\}$, $\{s_2, s_5, s_6\}$, $\{s_3, s_4\}$
  - P_2: $\{s_1\}$, $\{s_2, s_5\}$, $\{s_6\}$, $\{s_3, s_4\}$ (Table 7.2), $P_2 \neq P_1$, continue
  - P_3: $\{s_1\}$, $\{s_2, s_5\}$, $\{s_6\}$, $\{s_3, s_4\}$, $P_3 = P_2$, stop

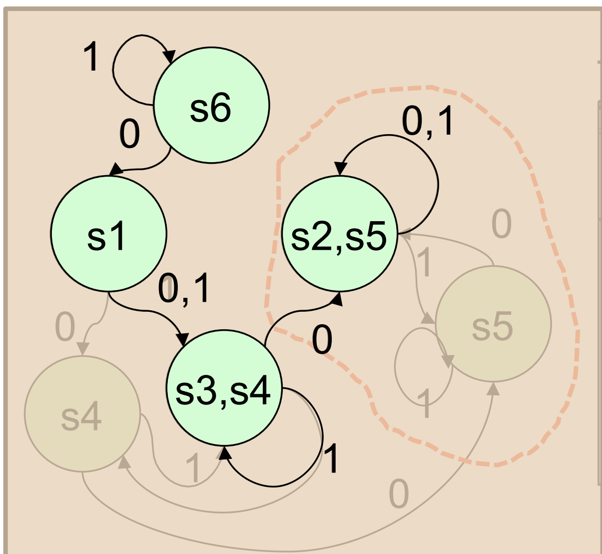


Table 7.1

	ν		ω	
	0	1	0	1
s_1	s_4	s_3	0	1
s_2	s_5	s_2	1	0
s_3	s_2	s_4	0	0
s_4	s_5	s_3	0	0
s_5	s_2	s_5	1	0
s_6	s_1	s_6	1	0

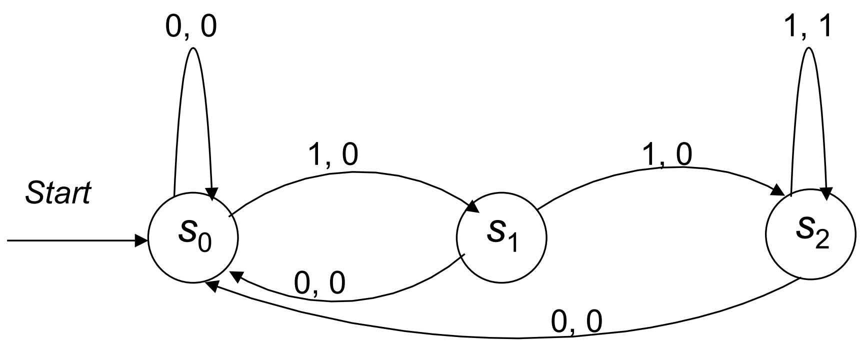
Table 7.2

	ν		ω	
	0	1	0	1
s_1	s_3	s_3	0	1
s_2	s_2	s_2	1	0
s_3	s_2	s_3	0	0
s_6	s_1	s_6	1	0

See also 15.2 Karnaugh Maps, a similar idea

More Minimization example

- In Ex 6.20 : Construct a machine that recognizes each occurrence of the sequence 111.



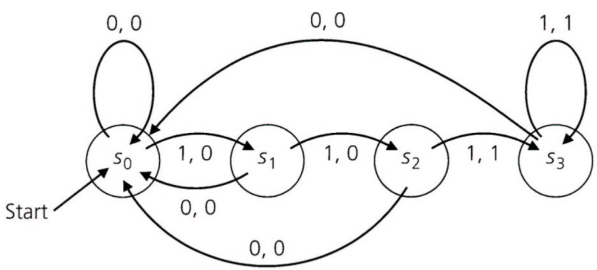


Figure 6.10

	v		w	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_0	s_2	0	0
s_2	s_0	s_3	0	1
s_3	s_0	s_3	0	1

P_1: $\{s_0, s_1\}, \{s_2, s_3\}$

P_2: $\{s_0\}, \{s_1\}, \{s_2, s_3\}$

P_3: $\{s_0\}, \{s_1\}, \{s_2, s_3\}$

Refinement

- Definition 7.23: If P_1 and P_2 are partitions of set A, then P_2 is called a refinement of P_1, denoted as $P_2 \leq P_1$, if every cell of P_2 is contained in a cell of P_1.
- When $P_2 \leq P_1$ and $P_2 \neq P_1$, we write $P_2 < P_1$.
  - In Example 7.60, $P_3 = P_2 < P_1$

- Theorem 7.9: In the minimization process, if $P_{k+1} = P_k$, then $P_{r+1} = P_r$ for all $r \geq k+1$.

Distinguishing String

- If $s_1 E_k s_2$ but $s_1 \not{E}_{k+1} s_2$, then we have a string $x = x_1 x_2 \ldots x_k x_{k+1} \in I^{k+1}$ such that $w(s_1, x) \neq w(s_2, x)$ but $w(s_1, x_1 x_2 \ldots x_k) = w(s_2, x_1 x_2 \ldots x_k)$. We call this string x as **distinguishing string**. 不為一
- $s_1 \not{E}_{k+1} s_2 \Rightarrow \exists x_1 \in I, [v(s_1, x_1) \not{E}_k \, v(s_2, x_1)]$

x讓s1 s2不為k-equivalent

Distinguishing String

- **Ex 7.61** : From Example 7.60, $s_2 E_1 s_6$ but $s_2 \not{E}_2 s_6$, so we seek a distinguishing string of length 2.
- $x = 00$ is the minimal distinguishing string for s_2 and s_6
  - $w(s_2, 00) = 11 \neq 10 = w(s_6, 00)$

next state 不為equivalent

P_2: $\{s_1\}, \{s_2, s_5\}, \{s_3, s_4\}, \{s_6\}$

0, 1   0, 1

P_1: $\{s_1\}, \{s_2, s_5, s_6\}, \{s_3, s_4\}$

0, 0   0, 1

Table 7.1

	v		ω	
	0	1	0	1
s_1	s_4	s_3	0	1
s_2	s_5	s_2	1	0
s_3	s_2	s_4	0	0
s_4	s_5	s_3	0	0
s_5	s_2	s_5	1	0
s_6	s_1	s_6	1	0

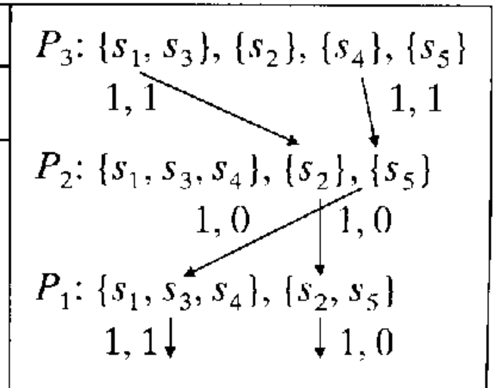
Distinguishing String

- <u>Ex 7.62</u> : s_1 and s_4 are 2-equivalent but are not 3-equivalent.
- $x = 111$ is the minimal distinguishing string for s_1 and s_4
  - $w(s_1, 111) = 100 \neq 101 = w(s_4, 111)$

Table 7.3

	ν		ω	
	0	1	0	1
s_1	s_4	s_2	0	1
s_2	s_5	s_2	0	0
s_3	s_4	s_2	0	1
s_4	s_3	s_5	0	1
s_5	s_2	s_3	0	0

(a)



(b)

can't choose '0'

X=11 to distinguish s2 and s5

typo

2) Then $\nu(s_1, 1) \not{E}_2\ \nu(s_4, 1) \Rightarrow \exists x_2 \in \mathcal{J}$ (here $x_2 = 1$) with v $(\nu(s_1, 1), 1) \not{E}_1$ v $\nu(\nu(s_4, 1), 1)$, or $\nu(s_1, 11) \not{E}_1\ \nu(s_4, 11)$. We used the partitions P_2 and P_1 to obtain $x_2 = 1$.

3) Now we use the partition P_1 where we find that for $x_3 = 1 \in \mathcal{J}$.