Discrete Mathematics Midterm Exam

Date: 2021/04/22

Caution: Please write down your answer **clearly**.

Scope: Ch.1~3

Note: Total 3 pages, full mark would be 100 points.

1. (a) How many quadrilateral are determined by the vertices of a regular polygon of n sides? ($n \geq 4$) (4 points)

   (b) How many if no side of the polygon is to be a side of any quadrilateral? ($n \geq 8$) (6 points)

   Note: quadrilateral (四邊形)

   (a) C_4^n

   (b) $C_4^n - n - n(n-5) - \frac{n(n-5)}{2} - \frac{n((n-4)(n-5)-2(n-5))}{2}$

   $= C_4^n - \dfrac{n^3 - 8n^2 + 17n}{2}$

2. We have $A \triangle B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\} = (A \cup B) - (A \cap B) = (A \cup B) \cap \overline{(A \cap B)}$. Please **simplify** the expression $\overline{A \triangle B} = \overline{(A \cup B) \cap \overline{(A \cap B)}}$ and **explain the reasons**. (10 points)

$\overline{A \triangle B} = \overline{(A \cup B) \cap \overline{(A \cap B)}}$	**Reasons**
$= \overline{(A \cup B)} \cup \overline{\overline{(A \cap B)}}$	**DeMorgan's Law**
$= \overline{(A \cup B)} \cup (A \cap B)$	**Law of Double Complement**
$= (A \cap B) \cup \overline{(A \cup B)}$	**Commutative Law of $\cup$**
$= (A \cap B) \cup (\overline{A} \cap \overline{B})$	**DeMorgan's Law**
$= [(A \cap B) \cup \overline{A}] \cap [(A \cap B) \cup \overline{B}]$	**Distributive Law of $\cup$ over $\cap$**

$= [(A \cup \overline{A}) \cap (B \cup \overline{A})] \cap [(A \cup \overline{B}) \cap (B \cup \overline{B})]$

Distributive Law of ∪ over ∩

$= [\mathcal{U} \cap (B \cup \overline{A})] \cap [(A \cup \overline{B}) \cap \mathcal{U}]$ **Inverse Law**

$= (B \cup \overline{A}) \cap (A \cup \overline{B})$ **Identity Law**

$= (\overline{A} \cup B) \cap (A \cup \overline{B})$ **Commutative Law of ∪**

$= (\overline{A} \cup B) \cap \overline{(\overline{A} \cap B)}$ **DeMorgan's Law**

$= \overline{A} \triangle B$

$= (A \cup \overline{B}) \cap (\overline{A} \cup B)$ **Commutative Law of ∩**

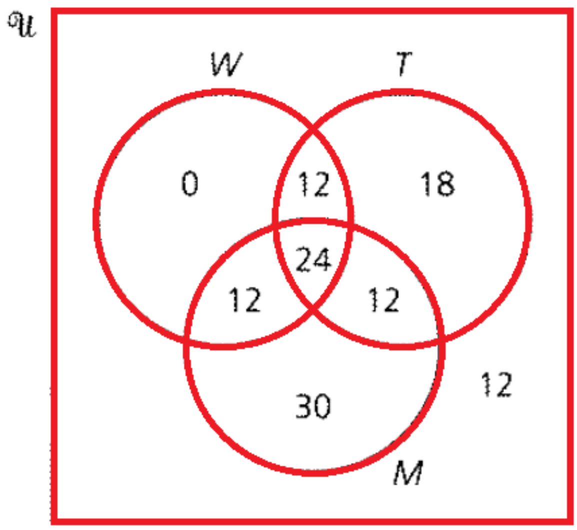
$= (A \cup \overline{B}) \cap \left(\overline{A \cap \overline{B}}\right)$ **DeMorgan's Law**

$= A \triangle \overline{B}$

3. In a survey of 120 passengers, an airline found that 48 enjoyed wine with their meals, 78 enjoyed mixed drinks, and 66 enjoyed iced tea. In addition, 36 enjoyed any given pair of these beverages and 24 passengers enjoyed them all. If two passengers are selected at random from the survey sample of 120, what is the probability that **they both enjoy exactly two of the three beverage offerings**? (5 points) **Please draw the Venn diagram**. (2 points)

(a) $\frac{C_2^{36}}{C_2^{120}} = \frac{3}{34}$

(b)



4. **Negate** each of the following and **simplify** the resulting statement.
   (a) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$ (3 points)

   $$= \neg\left(\boldsymbol{p} \vee \left((\boldsymbol{q} \vee \neg \boldsymbol{p}) \wedge (\boldsymbol{q} \wedge \boldsymbol{r})\right)\right)$$

   $$= \neg\left((\boldsymbol{p} \vee \boldsymbol{q}) \vee \left((\neg \boldsymbol{p} \vee \boldsymbol{p}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})\right)\right)$$

   $$= \neg(\boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r})$$

   (b) $p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)$ (5 points)

   $$= \neg\ \left(\boldsymbol{p} \wedge (\boldsymbol{r} \vee (\boldsymbol{q} \wedge (\neg \boldsymbol{q} \vee \neg \boldsymbol{p})))\right)$$

   $$= \neg\left(\boldsymbol{p} \wedge \left(\boldsymbol{r} \vee (\boldsymbol{q} \wedge \neg \boldsymbol{p})\right)\right)$$

   $$= \neg((\boldsymbol{p} \wedge \boldsymbol{r}) \vee (\boldsymbol{p} \wedge \neg \boldsymbol{p} \wedge \boldsymbol{q}))$$

   $$= \neg(\boldsymbol{p} \wedge \boldsymbol{r})$$

5. Write the dual for (a) (2 points) $p \rightarrow (q \wedge r)$ (b) (2 points) $p \underline{\vee} q$, where p, q, and r are primitive statements.
   (a)

   $p \rightarrow (q \wedge r) \Leftrightarrow \neg p \vee (q \wedge r)$
   dual of $p \rightarrow (q \wedge r)$ is $\neg p \wedge (q \vee r)$

(b)

$p \veebar q \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$
dual of $p \veebar q$ is $(p \vee \neg q) \wedge (\neg p \vee q)$

6. Given primitive statement p, q, r, show that the implication $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a **tautology**. (8 points)

注意：不能用特例證明全部

Method1: By truth table.(要寫結論)

Method2: By inference.

	Steps	Reasons
(1)	$\neg p \vee r$	premise
(2)	$p \rightarrow r$	By $\neg p \vee r \Leftrightarrow p \rightarrow r$
(3)	$p \vee q$	premise
(4)	$q \vee p$	By Commutative Laws
(5)	$\neg q \rightarrow p$	By $q \vee p \Leftrightarrow \neg q \rightarrow p$
(6)	$\neg q \rightarrow r$	By step (2), (5) and Law of the Syllogism
(7)	$q \vee r$	By $q \vee r \Leftrightarrow \neg q \rightarrow r$

So $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology.

7. For x a real number and n a positive integer, show that
(a) $1 = (2+x)^n - C_1^n(x+1)(2+x)^{n-1} + C_2^n(x+1)^2(2+x)^{n-2} - \cdots + (-1)^n C_n^n (x+1)^n$. (5 points)

注意：不能用特例證明全部

By binomial theorem, RHS $= \sum_{k=0}^{n} \binom{n}{k} (2+x)^{n-k}(-(x+1))^k$

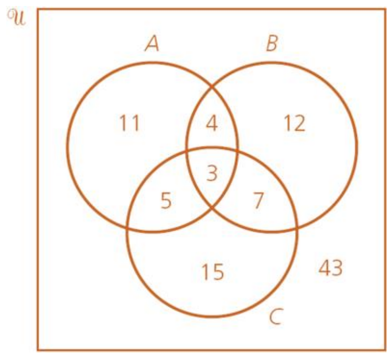
$$= \big((2+x)-(x+1)\big)^n = 1^n = 1$$

(b) $2^n = (2+x)^n - C_1^n x(2+x)^{n-1} + C_2^n x^2 (2+x)^{n-2} - \cdots + (-1)^n C_n^n x^n$. (5 points)

By binomial theorem, RHS $= \sum_{k=0}^{n} \binom{n}{k} (2+x)^{n-k}(-x)^k =$

$((2+x)-x)^n = 2^n$

8. An AND gate in an ASIC (Application Specific Integrated Circuit) has two inputs: I_1, I_2, and one output: O. Such an AND gate can have any or all of the following defects:
   D_1: The input I_1 is stuck at 0.
   D_2: The input I_2 is stuck at 0.
   D_3: The output O is stuck at 1.
   For a sample of 100 such gates we let A, B, and C be the subsets (of these 100 gates) having defects D_1, D_2, and D_3, respectively. With $|A| = 23$, $|B| = 26$, $|C| = 30$, $|A \cap B| = 7$, $|A \cap C| = 8$, $|B \cap C| = 10$, and $|A \cap B \cap C| = 3$, how many gates in the sample have at least one of the defects D_1, D_2, D_3? (8 points)
   Please draw the Venn diagram. (2 points)



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 23 + 26 + 30 - 7 - 8 - 10 + 3$$
$$= 57$$

9. **Negate** and **simplify** each of following
   (a) $\exists x[p(x) \vee q(x)]$ (2 points)
   (b) $\forall x[p(x) \wedge \neg q(x)]$ (2 points)
   (c) $\forall x[p(x) \rightarrow q(x)]$ (2 points)
   (d) $\exists x[(p(x) \vee q(x)) \rightarrow r(x)]$ (2 points)

a) $\forall x\ [\neg p(x) \wedge \neg q(x)]$

b) $\exists x[\neg p(x) \vee q(x)]$

c) $\exists x[p(x) \vee \neg q(x)]$

d) $\forall x\ [(p(x) \vee q(x)) \wedge \neg r(x)]$

10. How many bit string of length 20 contain at most Nineteen 1s? Please expressed as $A^B + C$

(a) A =______ (1 point)

(b) B =______ (2 points)

(c) C =______ (2 points)

A=2

B=20

C=-1

11. Derive

A =_____ (5 points)

B =_____ (5 points)

in $\binom{2n}{n+1} + \binom{2n}{n} = \frac{\binom{A}{B}}{2}$

$$\frac{2n!\,n(n+1)}{(n+1)!\,(n+1)!} + \frac{2n!\,(n+1)(n+1)}{(n+1)!\,(n+1)!} =$$

$$= \frac{2n!(n+1)(2n+1)}{(n+1)!(n+1)!}$$

$$= \frac{(2n+1)!(n+1)}{(n+1)!(n+1)!}$$

$$= \left(\frac{(2n+1)!(2n+2)}{(n+1)!(n+1)!}\right) * 1/2$$

$$= \frac{\binom{2n+2}{n+1}}{2}$$

A= 2n+2

B= n+1

12. A={1,2,3,4} 、B={x,y} 、C={}

consider a = |Power Set(A)|

b = |A×B| (× is cross product)

c = |Power Set(C)|

Please calculate $2^c + b + a$. (10 points)

A= 16
B= 8
C= 1