Population Games and Evolutionary Dynamics

(MIT Press, 200x; draft posted on my website)

1. Population games
2. Revision protocols and evolutionary dynamics
3. Potential games and their applications
4. Survival of dominated strategies

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a hypnodisk

Population Games and Evolutionary Dynamics

Population games model strategic interactions in which:

1. The number of agents is large.
2. Individual agents are small.
3. The number of roles is finite.
4. Agents interact anonymously.
5. Payoffs are continuous.

Examples:

economics: externalities, macroeconomic spillovers, centralized markets

biology: animal conflict, genetic natural selection

transportation science: highway congestion, mode choice

computer science: selfish routing

We consider games played by a single unit-mass population of agents.

$S = \{1, \dots, n\}$ strategies

$X = \{x \in \mathbf{R}^n_+ : \sum_{i\in S} x_i = 1\}$ population states/mixed strategies

$F_i : X \to \mathbf{R}$ payoffs to strategy i (Lipschitz continuous)

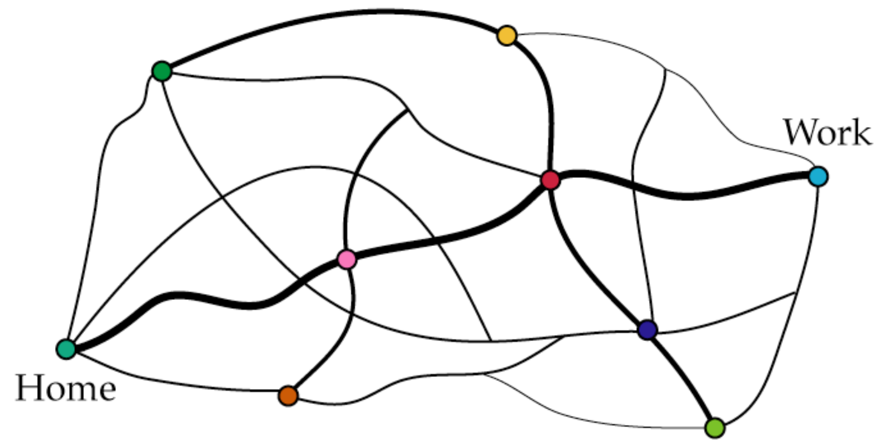
$F : X \to \mathbf{R}^n$ payoffs to all strategies = the payoff vector field

(geometry: see Sandholm, Dokumaci, and Lahkar (2006))

State $x \in X$ is a Nash equilibrium ($x \in NE(F)$) if

$$x_i > 0 \quad \Rightarrow \quad F_i(x) \geq F_j(x) \text{ for all } j \in S$$

Example: Congestion games
(Beckmann, McGuire, and Winsten (1956))



Home and Work are connected by paths $i \in S$ consisting of links $\phi \in \Phi$.

The payoff to choosing path i is
–(the delay on path i) = –(the sum of the delays on the links on path i)

formally:	$F_i(x) = -\sum_{\phi \in \Phi_i} c_\phi(u_\phi(x))$	payoff to path i
	x_i	mass of players choosing path i
	$u_\phi(x) = \sum_{i:\, \phi \in \Phi_i} x_i$	utilization of link ϕ
	$c_\phi(u_\phi)$	cost of delay on link ϕ

Examples: highway congestion $\Rightarrow$ c_ϕ increasing

positive externalities $\Rightarrow$ c_ϕ decreasing

Revision Protocols and Evolutionary Dynamics

Traditionally, predictions in game theory are based on some notion of equilibrium.

These concepts in turn rely on the assumption of equilibrium knowledge.

In contexts with large numbers of agents, this assumption seems untenable.

As an alternative, we consider an explicitly dynamic model of individual choice.

Microfoundations for evolutionary dynamics

The choice procedure individual agents follow is called a revision protocol.

ρ: $\mathbf{R}^n \times X \to \mathbf{R}_+^{n\times n}$ revision protocol

$\rho_{ij}(F(x), x)$ conditional switch rate from strategy i to strategy j

An all-purpose interpretation:

Each agent is equipped with a rate R Poisson alarm clock ($R = n \max_{i,j,x} \rho_{ij}(F(x), x)$).

When an i player's clock rings, he receives a revision opportunity.

He switches to strategy j with probability $\rho_{ij}(F(x), x)/R$.

Often, simpler interpretations work for specific revision protocols.

The mean dynamic

$$\left.\begin{array}{r}\text{population game } F \\ \text{revision protocol } \rho \\ \text{population size } N\end{array}\right\} \quad \Rightarrow \quad \text{Markov process } \{X_t^N\}$$

Over finite time spans, as N grows large, the process $\{X_t^N\}$ is well-approximated by solutions to an ODE. (Kurtz (1970), Benaim and Weibull (2003), Sandholm (2003))

This ODE, the mean dynamic, is defined by the expected increments of $\{X_t^N\}$.

(D)
$$\begin{aligned}\dot{x}_i = V_i^F(x) &= \sum_{j\in S} x_j \rho_{ji}(F(x),x) - x_i \sum_{j\in S} \rho_{ij}(F(x),x) \\ &= \text{inflow into } i \; - \; \text{outflow from } i\end{aligned}$$

Example 1: Proportional imitation (Schlag (1998))

$$\rho_{ij} = x_j\,[F_j - F_i]_+$$

The x_j term suggests a different interpretation for ρ:
pick an *opponent* at random; then decide whether to imitate him.

$$\Rightarrow \quad \dot{x}_i = x_i(F_i(x) - \bar{F}(x)), \text{ where } \bar{F}(x) = \sum_{j\in S} x_j F_j(x)$$

= the replicator dynamic (Taylor and Jonker (*Mathematical Biosciences* 1978))

Example 2: Proportional switching with direct selection of alternatives

$$\rho_{ij} = [F_j - F_i]_+$$

$$\Rightarrow \quad \dot{x}_i = \sum_{j\in S} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j\in S} [F_j(x) - F_i(x)]_+$$

= the pairwise difference dynamic (Smith (*Transportation Science* 1984))

Classes of population games

Can we find broad classes of population games that allow

- simple characterizations of Nash equilibrium sets;
- global convergence results for general classes of evolutionary dynamics?

Three important classes of examples:

1.	Potential games	Monderer and Shapley (1996), Sandholm (2001)
2.	Supermodular games	Topkis (1979), Hofbauer and Sandholm (2005)
3.	Stable games	Hofbauer and Sandholm (2006a) (cf Maynard Smith and Price (1973)) (cf Minty (1967), Smith (1979), Dafermos (1980)))

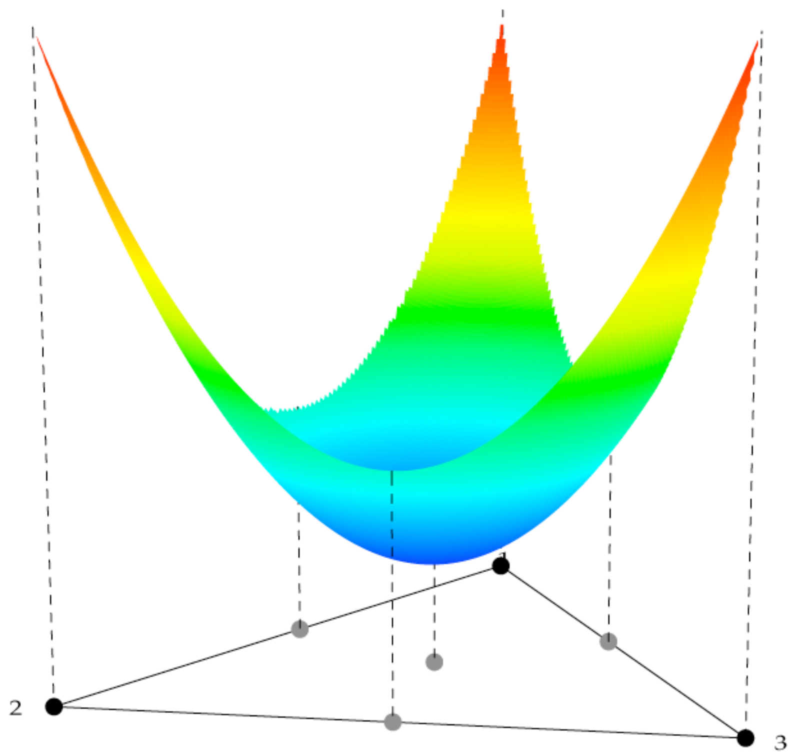
1. Potential Games

F is a potential game if there exists an f: $X \to \mathbf{R}$ s.t. $\nabla f(x) = F(x)$ for all $x \in X$

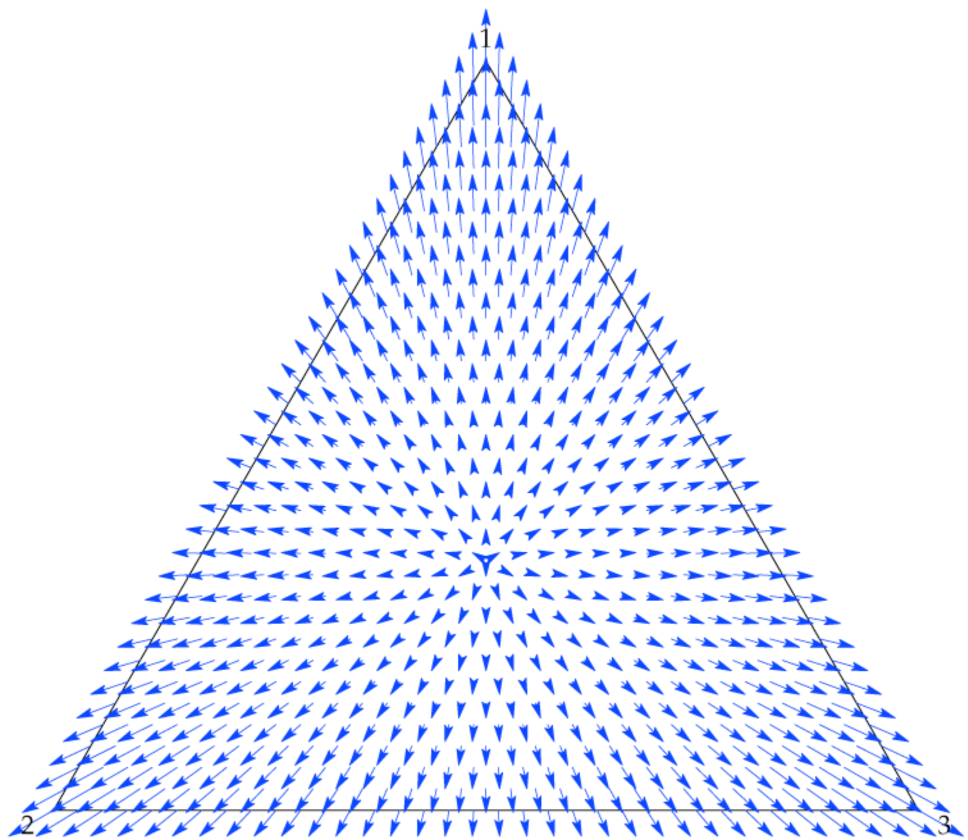
$\Leftrightarrow$ externality symmetry: $\dfrac{\partial F_i}{\partial x_j}(x) = \dfrac{\partial F_j}{\partial x_i}(x)$ for all $i, j \in X$ and $x \in X$.

Example 1: Pure coordination

$$F(x) = x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



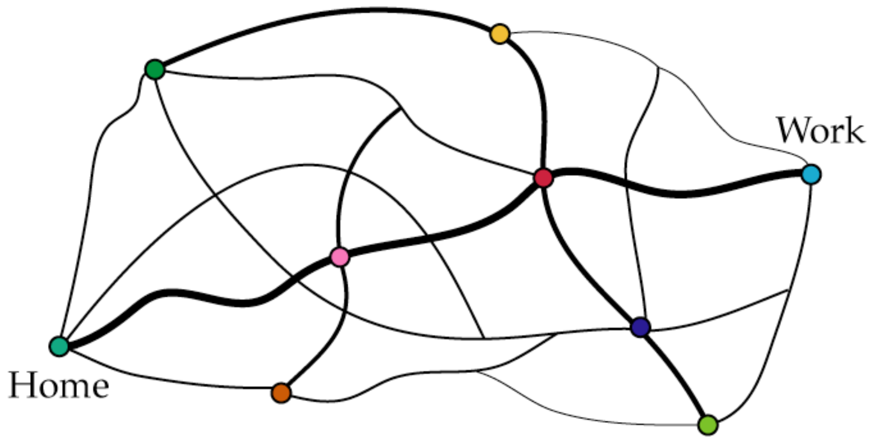
$$f(x) = \tfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$



$$F(x) = x$$

(projected onto the tangent space of X)

Example 2: Congestion games (Beckmann, McGuire, and Winsten (1956))



$$F_i(x) = -\sum_{\phi \in \Phi_i} c_\phi(u_\phi(x)) \qquad \Rightarrow \qquad f(x) = -\sum_{\phi \in \Phi_i} \int_0^{u_\phi(x)} c_\phi(z)\,dz$$

$$\neq -\sum_{\phi \in \Phi_i} u_\phi(x) c_\phi(x) = \bar{F}(x)$$

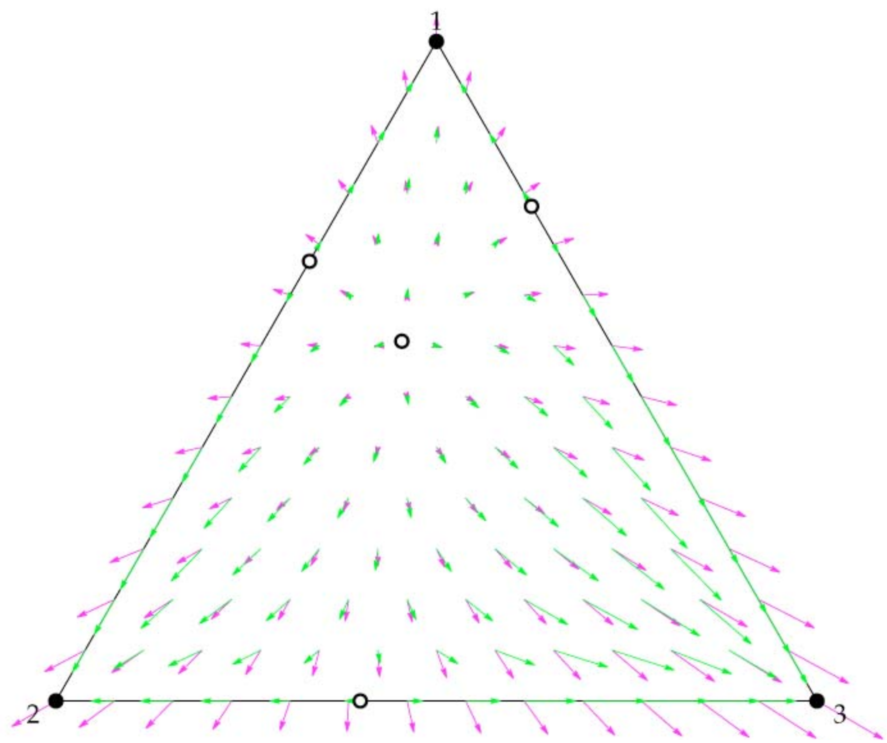
Note: if the c_ϕ are strictly increasing, then f is strictly concave,
and so has a unique maximizer (= the unique NE (see below)).

Global convergence of evolutionary dynamics

(D) $$\dot{x}_i = V_i^F(x) = \sum_{j\in S} x_j \rho_{ji}(F(x),x) - x_i \sum_{j\in S} \rho_{ij}(F(x),x)$$

We say that the dynamic (D) satisfies positive correlation if

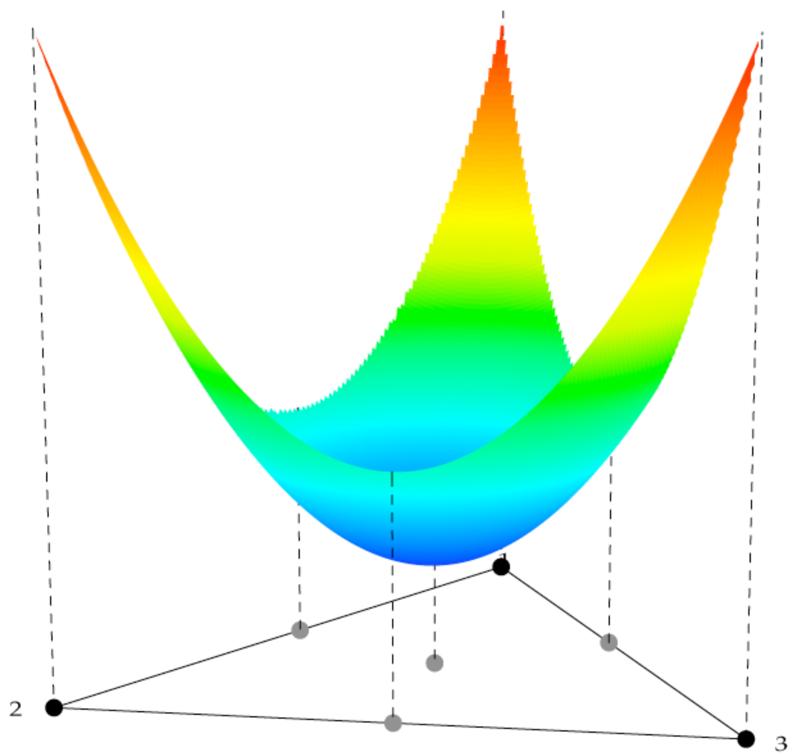
(PC) $V^F(x)' F(x) = n \operatorname{Cov}(V^F(x), F(x)) \geq 0$, with equality if and only if $x \in NE(F)$.



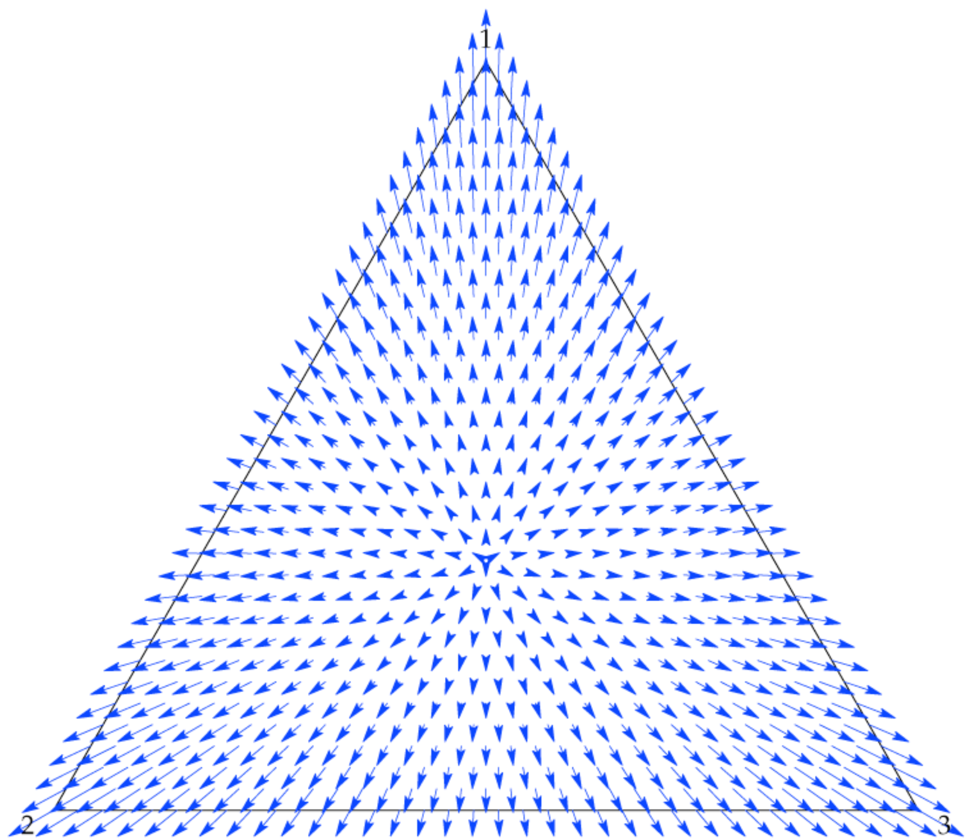
(PC) illustrated: The replicator dynamic in 123 Coordination

Theorem: If F is a potential game, and the dynamic V^F satisfies (PC), then nonstationary solutions of V^F ascend the potential function f. Therefore, every solution trajectory converges to a set of NE.

Proof 1: $\frac{d}{dt} f(x_t) = \nabla f(x_t)' \dot{x}_t = F(x_t)' V^F(x_t) \geq 0.$ Proof 2: below



$f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$



$F(x) = x$

Example 3: Games generated by variable externality pricing schemes

Let F be an arbitrary game.

Let $E_i(x) = \sum_{j \in S} x_j \frac{\partial F_j}{\partial x_i}(x)$ = the total (marginal) externalities created by an i player

Let $\hat{F} = F + E$ be the population game obtained by imposing pricing scheme E on F.

Theorem: $\hat{F}$ is a potential game whose potential function is $\bar{F}$, the average payoff function of the original game.

Proof: $$\frac{\partial}{\partial x_i}\bar{F}(x) = \frac{\partial}{\partial x_i}\sum_{j \in S} x_j F_j(x) = F_j(x) + \sum_{j \in S} x_j \frac{\partial F_j}{\partial x_i}(x) = F_j(x) + E_j(x).$$

Application: Evolutionary Mechanism Design

ex. A planner would like to ensure efficient behavior in a highway network.

He knows the cost functions c_ϕ, but he does not know any agent's valuation for driving to work.

He therefore cannot compute the efficient state.

Theorem (Sandholm (2002)):
By imposing the pricing scheme *E*, the planner can ensure that regardless of the agents' valuations, efficient behavior is

(i) the unique Nash equilibrium, and

(ii) globally stable under any dynamic satisfying (PC).

(That the original game is a congestion game is irrelevant: see Sandholm (2005, 2006).)

Survival of Dominated Strategies under Evolutionary Dynamics

In potential games, a wide class of evolutionary dynamics converge to Nash equilibrium.

Can evolutionary dynamics provide foundations for rationality-based solution concepts more generally?

For Nash equilibrium, the answer is no: there are games in which most solutions of any reasonable evolutionary dynamic fail to converge to Nash equilibrium (Hofbauer and Swinkels (1996), Hart and Mas-Colell (2003)).

We therefore consider a more modest goal: elimination of strictly dominated strategies.

(Strategy $i \in S$ is strictly dominated by strategy $j \in S$ if $F_j(x) > F_i(x)$ for all $x \in X$.)

Dominance is the mildest requirement employed in standard game-theoretic analysis, so it is natural to expect evolutionary dynamics to accord with it.

But:

There is no *a priori* reason to expect dominated strategies
to be eliminated by evolutionary dynamics.

Evolutionary dynamics are based on the idea that agents switch to strategies
whose current payoffs are reasonably good.

Even if a strategy is dominated, it can have reasonably good payoffs in many states.

Put differently: decision making in evolutionary models is "local"; domination is "global".

The result

(PC) Positive correlation: $V^F(x)'F(x) \geq 0$, with equality if and only if $x \in NE(F)$.

(IN) Innovation: If $x \notin NE(F)$, $x_i = 0$, and $i \in BR^F(x)$, then $V_i^F(x) > 0$.

Theorem (Hofbauer and Sandholm (2006b)):

Any continuous dynamic that satisfies (PC) and (IN) does not eliminate strictly dominated strategies in all games.

Proof of special cases (BNN dynamic, PD dynamic) (cf Berger and Hofbauer (2006))

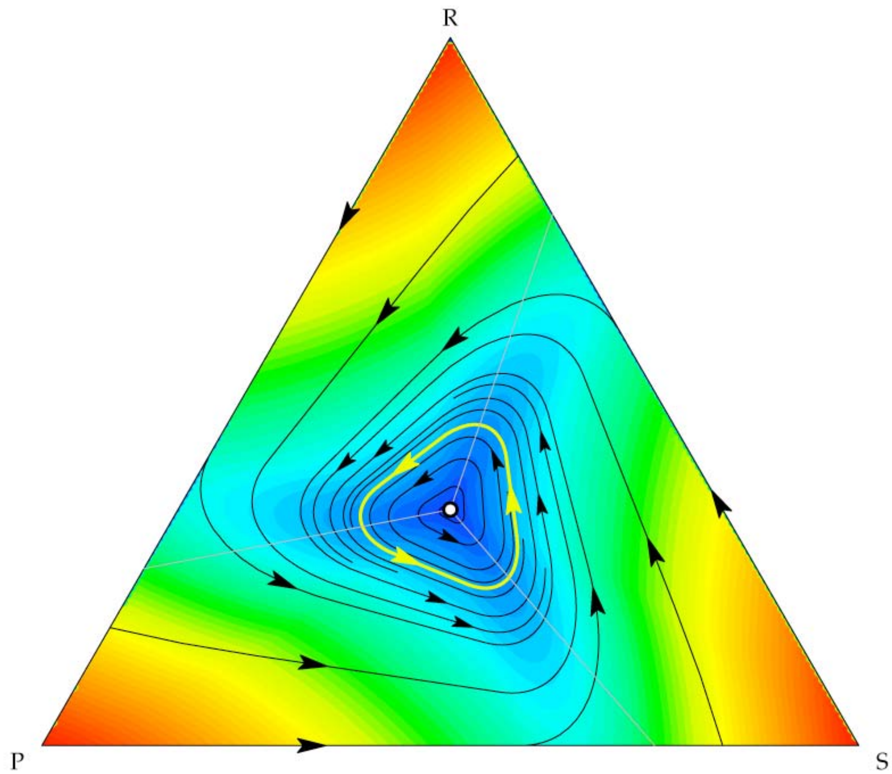
Step 1: Cycling in bad RPS

Suppose agents are randomly matched to play bad Rock-Paper-Scissors.

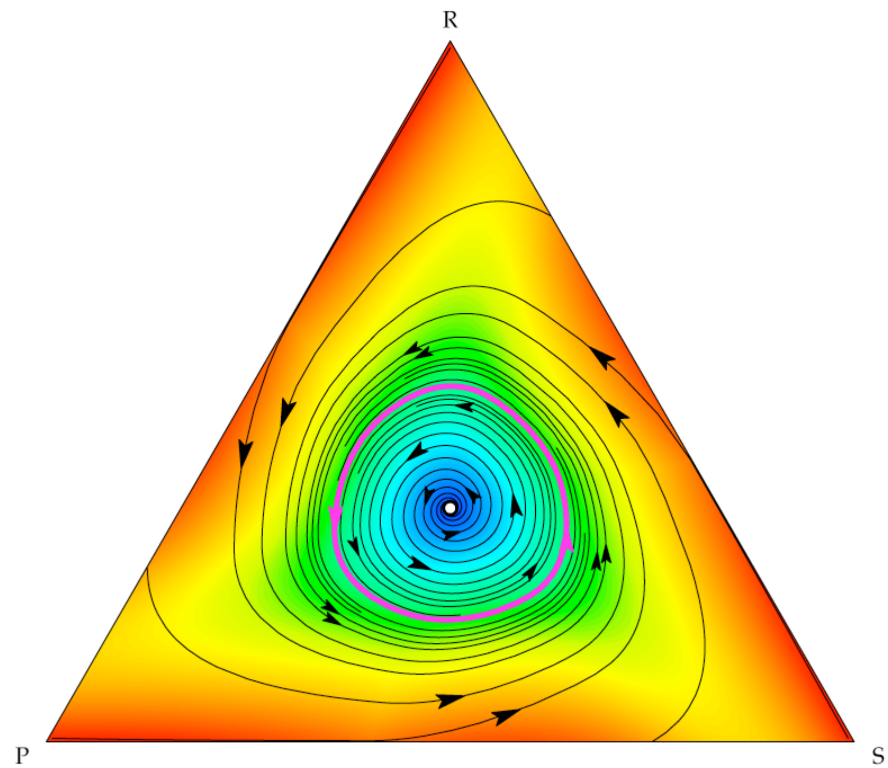
$$A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -2 \\ -2 & 1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad F(x) = \begin{pmatrix} F_R(x) \\ F_P(x) \\ F_S(x) \end{pmatrix} = \begin{pmatrix} x_S - 2x_P \\ x_R - 2x_S \\ x_P - 2x_R \end{pmatrix}$$

Unless the initial state is the unique NE $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, play approaches a limit cycle.

Proof: Use a Lyapunov function. (The function depends on the dynamic at issue).



the BNN dynamic in bad RPS



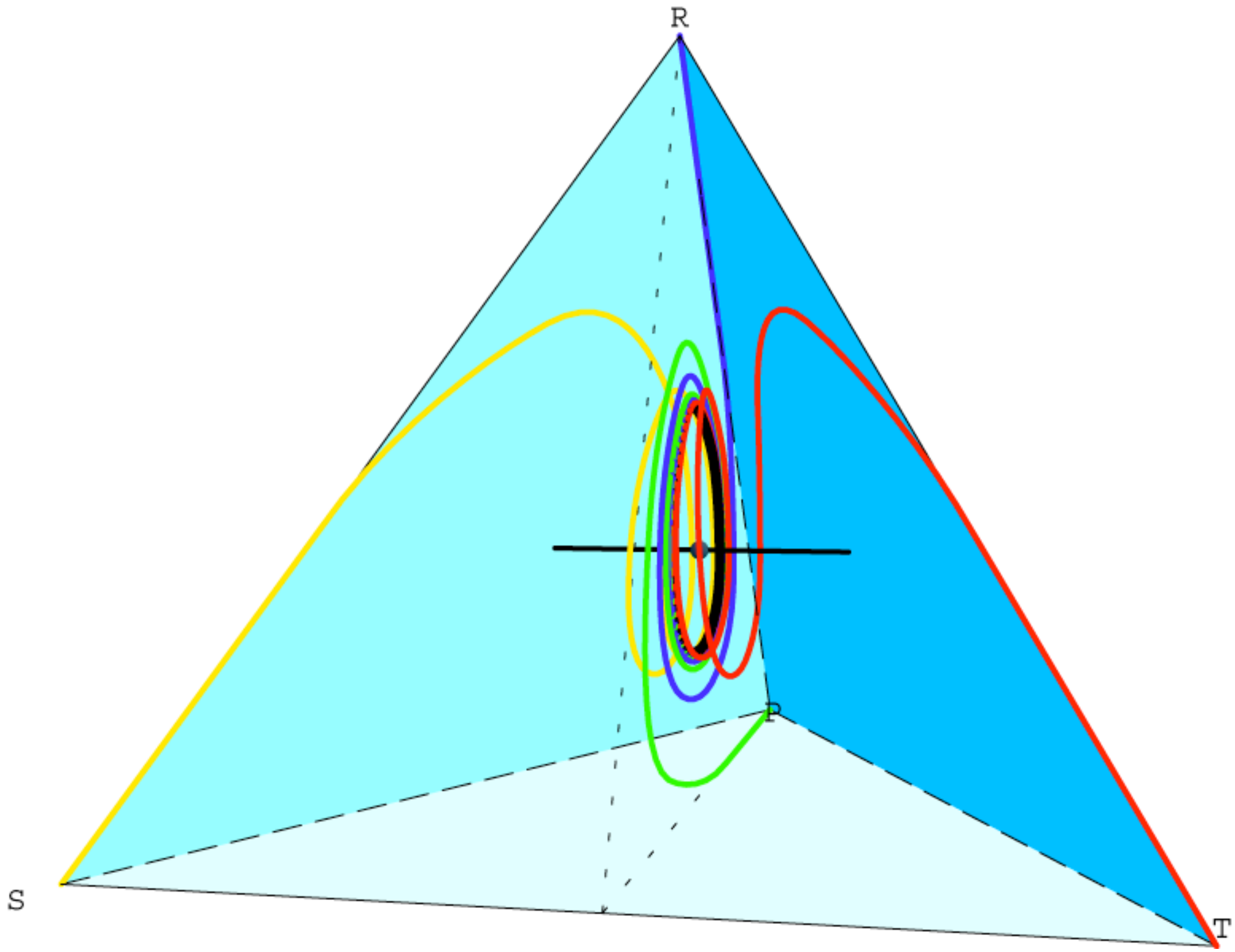
the PD dynamic in bad RPS

Step 2: Add a twin to bad RPS

Add a fourth strategy to bad RPS that duplicates Scissors.

$$A = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$

Except from the line of NE $\{x \in X\colon\ x = ((\frac{1}{3}, \frac{1}{3}, \frac{1}{6} - c, \frac{1}{6} + c)\}$,
play converges to a cycle on the plane where $x_S = x_T$.



the PD dynamic in “bad RPS with a twin”

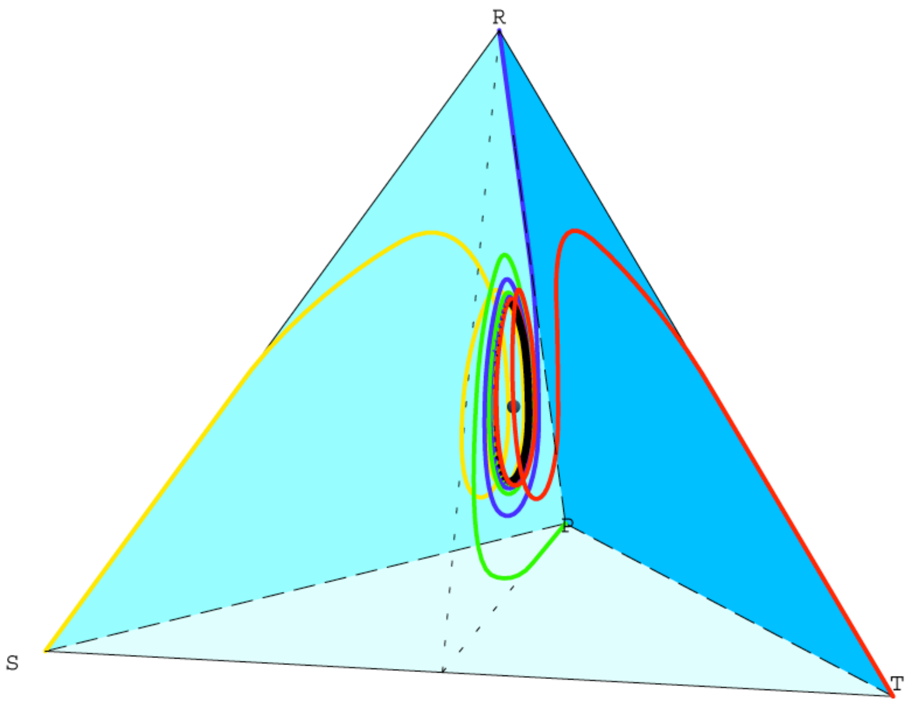
Step 3: A feeble twin

Uniformly reduce the payoff of the twin by ε:

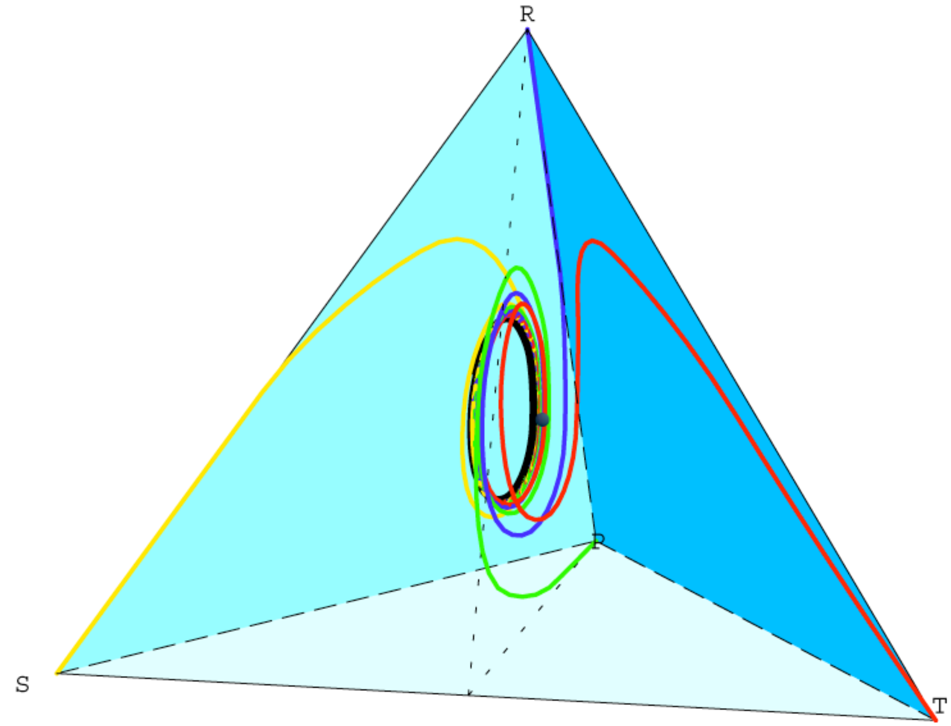
$$A = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2-\varepsilon & 1-\varepsilon & -\varepsilon & -\varepsilon \end{pmatrix}$$

Since the dynamic depends continuously on the game's payoffs,
the attracting interior limit cycle persists when ε is slightly greater than zero.

$\therefore$ The feeble twin, a strictly dominated strategy, survives.



the PD dynamic in "bad RPS + twin"



the PD dynamic in "bad RPS + evil twin"

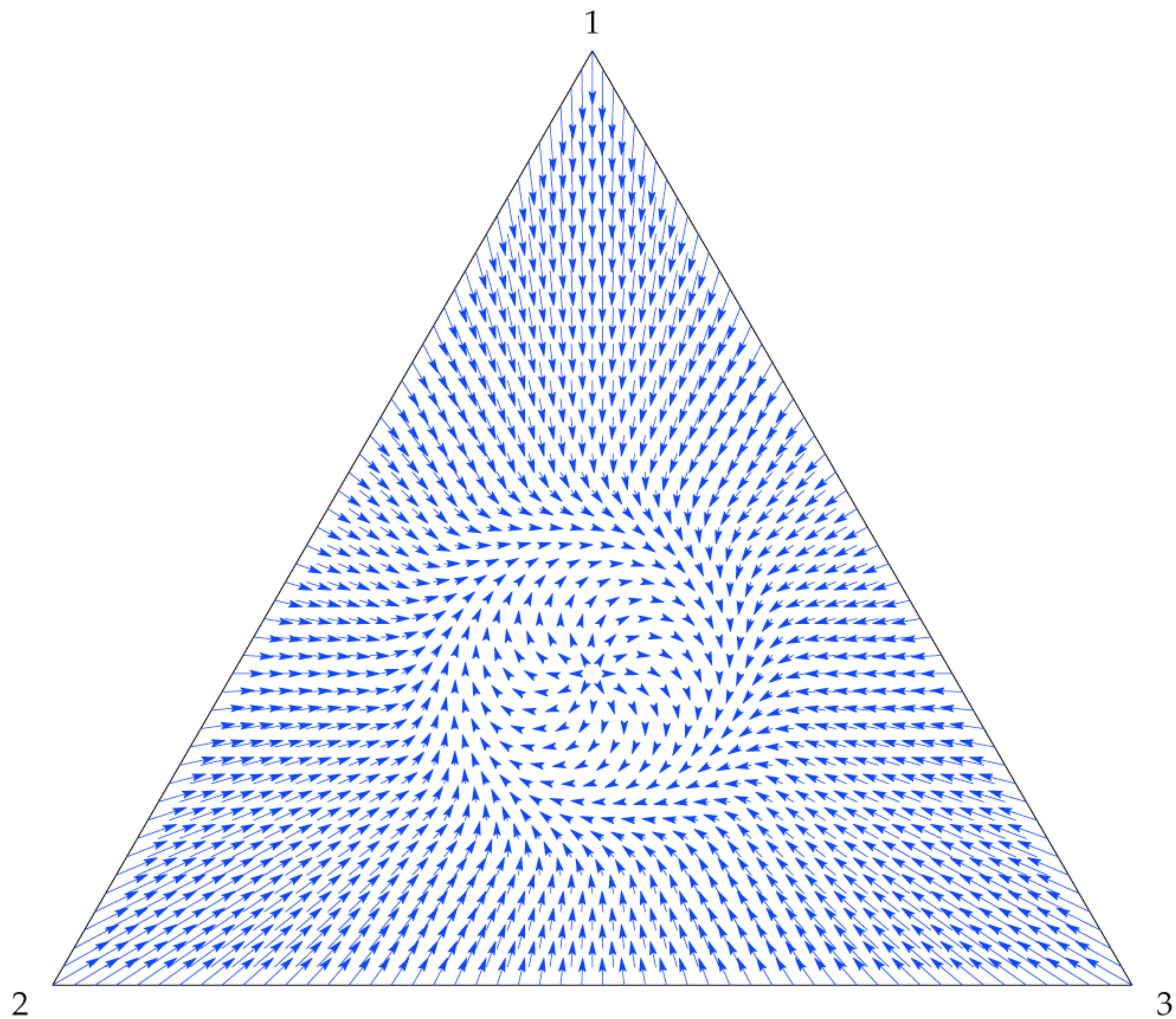
Where does this argument fail for arbitrary dynamics that satisfy (PC)?

1. In a game with identical twins, (PC) does not imply convergence to the plane where the twins receive equal weight.

2. In Bad RPS, (PC) is not enough to ensure cycling.

Item 1 is handled using tools from dynamical systems theory.

For item 2, we need a game that is “worse” than bad RPS.



The hypnodisk game

Population Games and Evolutionary Dynamics: Summary

1. Population games
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3. Potential games and their applications
4. Survival of dominated strategies under evolutionary dynamics

Some open topics

New classes of dynamics, perhaps based on psychologically motivated models of choice

Prevalence of convergence/of cycling/of survival of dominated strategies

Applications of all kinds