Solutions of Sep. 13th, 2022 Test

1 PROPOSITIONAL LOGIC

$\varphi_1 : (p \to q)$

$\varphi_2 : (q \to (s \wedge t))$

$\varphi_3 : (r \to (s \wedge t))$

$\varphi_4 : (p \vee r)$

Is $(s \wedge t)$ a logical consequence of $\{\varphi_1, \dots, \varphi_4\}$?

$\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \overset{?}{\models} (s \wedge t)$

R We reduce the problem to deciding whether the formula

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \neg (s \wedge t)$$

is satisfiable.

If it is NOT satisfiable then $s \wedge t$ is logical consequence of $\{\varphi_1, \dots, \varphi_4\}$; otherwise it is not.

$\varphi_1 : p \to q$

$\varphi_2 : (q \to (s \wedge t))$

$\varphi_3 : (r \to (s \wedge t))$

$\varphi_4 : (p \vee r)$

$\varphi_1 : \boxed{\neg p \vee q}$ ✓

$\varphi_2 : \neg q \vee (s \wedge t) \iff \boxed{(\neg q \vee s) \wedge (\neg q \vee t)}$ ✓

$\varphi_3 : \neg r \vee (s \wedge t) \iff \boxed{(\neg r \vee s) \wedge (\neg r \vee t)}$ ✓

$\varphi_4 : \boxed{p \vee r}$ ✓

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \neg(s \wedge t) \iff \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge (\neg s \vee \neg t)$$

$\boxed{\neg p \vee q}$ φ_1

$\boxed{\neg q \vee s \quad \neg q \vee t}$ φ_2

$\boxed{\neg r \vee s \quad \neg r \vee t}$ φ_3

$\boxed{p \vee r}$ φ_4

$\neg s \vee \neg t$

(DPLL)

NO UNIT CLAUSE

NO PURE LITS

⇓

SPLIT!

$q := T$

s, t, $\neg r \vee s$, $\neg r \vee t$, $p \vee r$, $\neg s \vee \neg t$

$\xrightarrow{s := T,\ t := T}$ $p \vee r$, $\{\}$ ← empty clause! ✗

$q := \bot$

$\neg p$, $\neg r \vee s$, $\neg r \vee t$, $p \vee r$, $\neg s \vee \neg t$

$q := \bot$

$\xrightarrow{p := \bot}$ $\neg r \vee s$, $\neg r \vee t$, r, $\neg s \vee \neg t$

$\xrightarrow{r := T}$ s, t, $\neg s \vee \neg t$

$\xrightarrow{s := T,\ t := T}$ $\{\}$ ✗

We proved that $\varphi_1 \land \varphi_2 \land \varphi_3 \land \varphi_4 \land \neg(s \land t)$ is NOT satisfiable. Therefore $\varphi_1 \land \varphi_2 \land \varphi_3 \land \varphi_4 \models s \land t$.

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② FIRST ORDER LOGIC

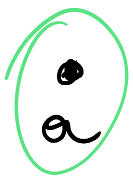
$\varphi_1 : \forall x. R(x,x)$

$\varphi_2 : \forall x. \forall y. (R(x,y) \to R(y,x))$

$\varphi_3 : \forall x. \forall y. \forall z. ((R(x,y) \land R(y,z)) \to R(x,z))$

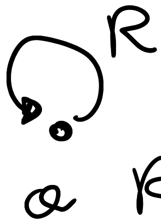
INTUITIVELY:

R is symmetric, reflexive and transitive



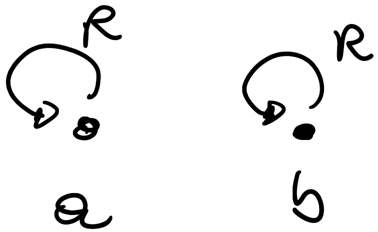
2nd constant

THIS IS NOT A MODEL OF $\varphi_1 \land \varphi_2 \land \varphi_3$ because $\rho(R) = \{\}$

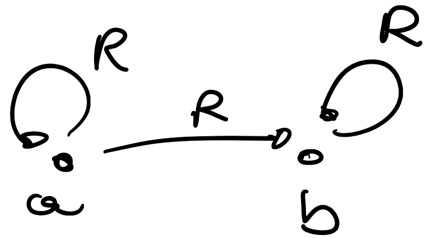


$R(a,a)$

THIS IS A MODEL OF $\varphi_1 \land \varphi_2 \land \varphi_3$



THIS IS A MODEL OF $\varphi_1 \land \varphi_2 \land \varphi_3$

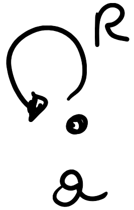


THIS IS NOT A MODEL OF $\varphi_1 \land \varphi_2 \land \varphi_3$

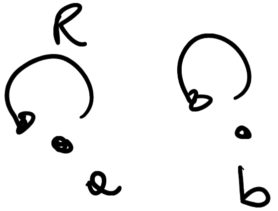
- The theory $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is CONSISTENT (at least one model exists for the theory), therefore it is NOT the case that any formula will be a logical consequence of the theory.

$$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \overset{?}{\models} \overbrace{\forall x \forall y \forall z \left(\left(R(x,y) \wedge \neg R(y,z)\right) \to \neg R(x,z)\right)}^{\psi}$$

- For this to be true, all the models of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ need to be models of ψ.

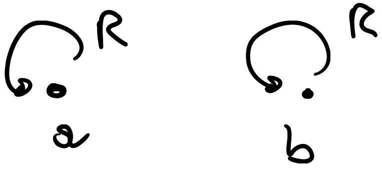


$\psi \rightsquigarrow \underbrace{R(a,a) \wedge \neg R(a,a)}_{\text{FALSE}} \to \neg R(a,a)$ $\quad$ ψ is trivially true in this model



$\psi \rightsquigarrow R(a,b) \wedge \neg R(b,a) \to \neg R(a,a)$ $\quad$ ψ is trivially true in this model

$x := a \quad y := b \quad z := a$



$$R(x,y) \wedge \neg R(y,z) \to \neg R(x,z)$$

$$x := a \quad y := a \quad z := b$$

$$\underbrace{R(a,a) \wedge \neg R(a,b)}_{T} \to \underbrace{\neg R(a,b)}_{T}$$

In order to prove $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \models \psi$ we can check whether there exists an interpretation I such that $\models_I \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and $\not\models_I \psi$ (iff $\models_I \neg\psi$)

$$\psi := \forall x \forall y \forall z \left(R(x,y) \wedge \neg R(y,z) \to \neg R(x,z)\right)$$

$$\neg\psi := \neg \forall x \forall y \forall z \left(R(x,y) \wedge \neg R(y,z) \to \neg R(x,z)\right) \Longleftrightarrow$$

$$\exists x \exists y \exists z \neg \left(R(x,y) \wedge \neg R(y,z) \to \neg R(x,z)\right) \Longleftrightarrow$$

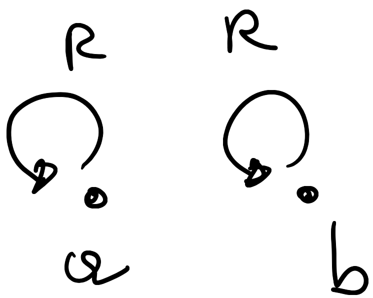
$$\exists x \exists y \exists z \left(R(x,y) \wedge \neg R(y,z) \wedge R(x,z)\right)$$

$$\neg\psi := \exists x\, \exists y\, \exists z\ \big(R(x,y) \wedge \neg R(y,z) \wedge R(x,z)\big)$$

$$\underbrace{R(a,a) \wedge \neg R(a,a)}_{\text{False already}} \wedge \ldots$$

Cannot satisfy $\neg\psi$ over a domain with just one constant

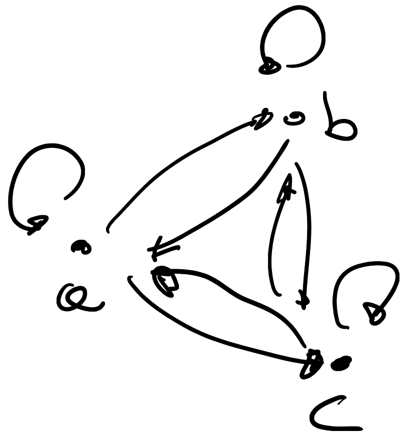
Model of theory $\{\varphi_1, \varphi_2, \varphi_3\}$



$$R(a,a) \wedge \underbrace{\neg R(a,b) \wedge R(a,b)}_{\text{false}}$$

Model of theory $\{\varphi_1, \varphi_2, \varphi_3\}$

$$\underbrace{R(a,b) \wedge \cancel{\neg R(b,c)} \wedge R(a,c)}_{\text{false}}$$



- It is not possible to build an interpretation that satisfies $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and also $\neg \psi$
  therefore $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \models \psi$

ALTERNATIVE Convert $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \neg \psi$ to clauses (prenex form, CNF, skolemize)
and show that the formula is not satisfiable on the corresponding Herbrand domain.

Let's convert the theory to prenex normal form

$$\underbrace{\forall x . R(x,x)}_{①} \wedge \underbrace{\forall x \forall y (R(x,y) \to R(y,x))}_{②} \wedge \underbrace{\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(x,z))}_{③}$$

$$\forall x . R(x,x) \wedge \forall s \forall t (\neg R(s,t) \vee R(t,s)) \wedge \forall w \forall y \forall z (\neg R(w,y) \vee \neg R(y,z) \vee R(w,z))$$

$$\forall x . \forall s . \forall t . \forall w . \forall y . \forall z \underbrace{\left(R(x,x) \wedge (\neg R(s,t) \vee R(t,s)) \wedge (\neg R(w,y) \vee \neg R(y,z) \vee R(w,z))\right)}_{\tau(x,s,t,w,y,z)}$$

Theory in Prenex CNF

Must check satisfiability of

$$\forall x . \forall s . \forall t . \forall w . \forall y . \forall z . \tau(x,s,t,w,y,z) \wedge \neg \underbrace{(\forall x \forall y \forall z (R(x,y) \wedge \neg R(y,z) \to \neg R(x,z)))}_{(*)}$$

(*) becomes $\exists x . \exists y . \exists z \neg ((R(x,y) \wedge \neg R(y,z)) \to \neg R(x,z))$ $\quad \neg(\varphi_1 \to \varphi_2) \equiv \varphi_1 \wedge \neg \varphi_2$

$\exists x . \exists y . \exists z \ (R(x,y) \wedge \neg R(y,z) \wedge \neg\neg R(x,z))$

$\exists x . \exists y . \exists z \ (R(x,y) \wedge \neg R(y,z) \wedge R(x,z))$

So we have

$$\overbrace{\forall x. \forall s. \forall t. \forall w. \forall y. \forall z\ \tau(x,s,t,w,y,z) \wedge \exists x \exists y \exists z \left(R(x,y) \wedge \neg R(y,z) \wedge R(x,z)\right)}^{(*)}$$

NOT THE SAME VARIABLE!

Remember

$$\underbrace{\forall x. \varphi(x) \wedge \exists y. \psi(y)}_{①} \to \forall x \exists y \left(\varphi(x) \wedge \psi(y)\right) ②$$

$$\forall x. \varphi(x) \wedge \exists y. \psi(y) \to \exists y \forall x \left(\varphi(x) \wedge \psi(y)\right) ③$$

①, ②, ③ are all equivalent

② and ③ are prenex, but ③ is better for skolemization (avoids skolem function $f_y(x)$)

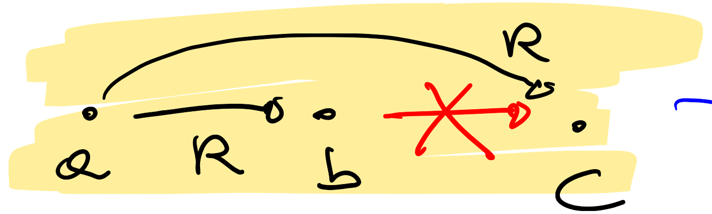
Therefore (*) in prenex form and skolemized is

$$\forall x \forall s \forall t \forall w \forall z.\ \tau(x,s,t,w,y,z) \wedge R(a,b) \wedge \neg R(b,c) \wedge R(a,c)$$

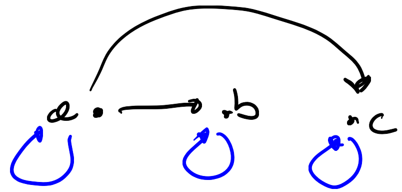
We can check the satisfiability of this formula!

1. $R(x,x)$
2. $\neg R(s,t) \vee R(t,s)$
3. $\neg R(w,y) \vee \neg R(y,z) \vee R(w,z)$
4. $R(a,b)$
5. $\neg R(b,c)$
6. $R(a,c)$

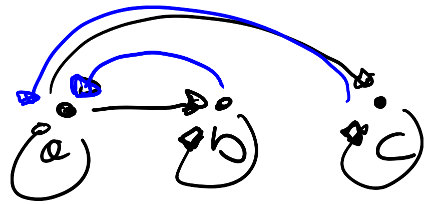
- To satisfy 4, 5, 6 our interpretation must have the following elements **at least**



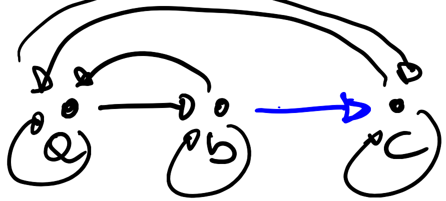
- To satisfy 1 we must also have



- To satisfy 2 we must also have



- To satisfy 3 we must also have



$w := c \quad y := a \quad z := b$

BUT now 5 is not satisfied any more!