Giving physical meaning to $\mathcal{W}^k(\mathfrak{g}, x, f)$

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September 2024

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Quantum field theories

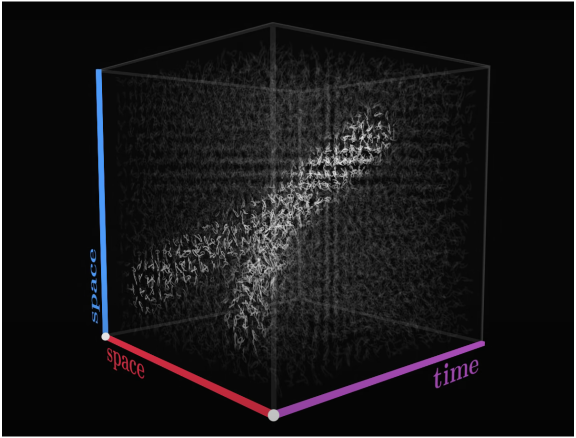

Figure 1: Illustration of a field

Correlation functions

In the operator formalism

$$\langle 0|\mathcal{T}\left(\varphi(x)\phi(y)\dots\right)|0\rangle \tag{1}$$

In the path-integral formalism

$$\frac{1}{Z}\int \mathcal{D}\psi\ \varphi(x)\phi(y)\dots e^{i\mathcal{S}(\psi)} \tag{2}$$

with $Z = \int \mathcal{D}\psi\ e^{i\mathcal{S}(\psi)}$

Conformal field theories

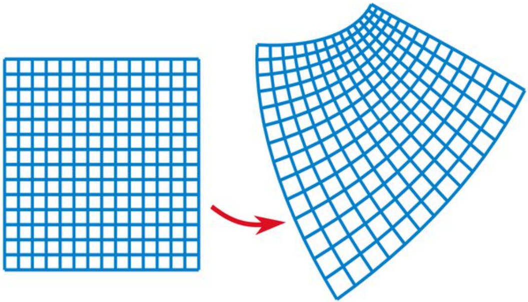

Figure 2: Conformal Transformation

$$\eta^{\mu\nu} \to \lambda(x^\alpha)\eta^{\mu\nu} \tag{3}$$

Conformal generators

$$< M_{\mu\nu}, P_\mu, K_\mu, D > \tag{4}$$

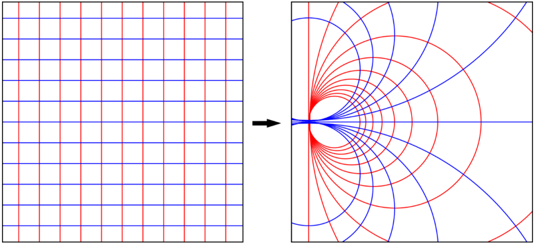

Figure 3: Special Conformal Transformation

2D CFTs and chirality

Chiral coordinates

$$z = x + t \qquad \bar{z} = x - t \tag{5}$$

The energy-momentum tensor decouples

$$\begin{gathered} T^{z\bar{z}} = T^{\bar{z}z} = 0 \\ T^{zz} = T(z) \qquad T^{\bar{z}\bar{z}} = \bar{T}(\bar{z}) \end{gathered} \tag{6}$$

Virasoro algebra

$$T(z) = \sum_n L_n z^{-n-2} \tag{7}$$

The Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{8}$$

Partition of the Hilbert space

$$\mathcal{H} = \sum_{i,j} V(c, h_i) \otimes \overline{V}(c, \bar{h}_j) \tag{9}$$

RCFTs

$$\mathcal{H} = \sum_{i,j} V_{\mathcal{A}}(h_i) \otimes \overline{V}_{\bar{\mathcal{A}}}(\bar{h}_j) \tag{10}$$

OPE and algebra

$$A(z)B(w) \sim \sum_i \frac{1}{(z-w)^i} C_i(w) \tag{11}$$

The structure is called a vertex algebra

$\mathcal{W}$-algebra

A $\mathcal{W}$-algebra is a chiral algebra of the form $< T, \varphi^{h_1}, \varphi^{h_2}, \cdots >$, written $\mathcal{W}(2, h_1, h_2, \dots)$

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The Sigma model

Defined from Σ to X by the action

$$\mathcal{S}(\varphi) = \frac{1}{2}\int_{\Sigma} g^{ij}(\varphi)\partial^{\mu}\varphi_i\partial_{\mu}\varphi_j \tag{12}$$

- $\mathcal{M} \to \mathbb{C}$: non-interacting, massless quantum mechanics
- $\mathbb{R} \to \mathcal{M}$: relativistic particle
- $\mathbb{R}^2 \to \mathcal{M}$: string on a manifold

Sigma model on Lie algebras

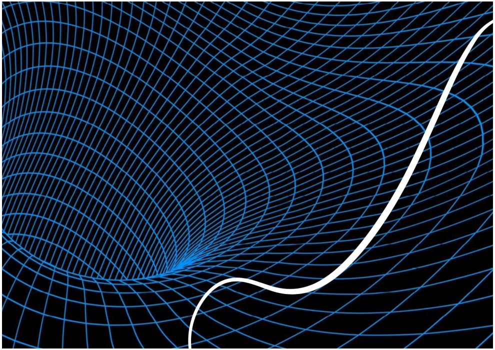

Figure 4: Illustration of a string on a Lie group

$$\mathcal{L} = 2\mathrm{Tr}(g^{-1}\partial g \; g^{-1}\overline{\partial} g) \tag{13}$$

Two levels of conformality

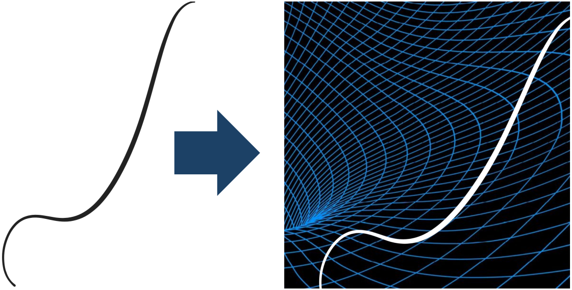

Figure 5: Time slice of the base manifold and target manifold

$G \times G$ symmetry

$$J^\mu = g^{-1}\partial^\mu g \qquad \tilde{J}^\mu = \partial^\mu g \, g^{-1} \tag{14}$$

Geometrical obstruction

Equation of conservation:

$$\partial J^{\bar{z}} + \bar{\partial} J^z = 0 \tag{15}$$

But

$$\begin{aligned} \partial J^{\bar{z}} = \bar{\partial} J^z = 0 \\ \Rightarrow [J^\mu, J^\nu] = 0 \end{aligned} \tag{16}$$

The topological term

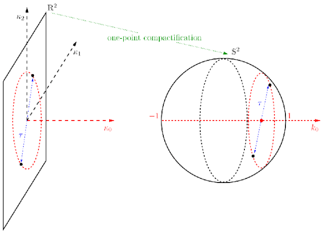

Figure 6: Compactification of $\mathbb{R}^2$ into $\mathbb{S}^2$

$$\mathcal{S}_{\mathrm{W}} = \int_{\mathbb{B}^3} \mathrm{Tr}(\wedge^3 g^{-1} \mathrm{d}g) \tag{17}$$

The WZW model

We consider $\frac{k}{2\pi}\mathcal{S}_{\mathrm{W}}$, for $k \in \mathbb{Z}$ the level of the theory. The equation of conservation becomes

$$\left(1 + \frac{k}{2\pi\lambda}\right)\partial J^{\bar{z}} + \left(1 - \frac{k}{2\pi\lambda}\right)\overline{\partial} J^{z} = 0 \tag{18}$$

So we consider

$$\mathcal{S}_{\mathrm{WZW}} = \frac{k}{\pi}\int_{\mathbb{S}^2} \mathrm{Tr}(g^{-1}\partial g\; g^{-1}\overline{\partial} g) - \frac{k}{2\pi}\int_{\mathbb{B}^3} \mathrm{Tr}(\wedge^3 g^{-1} \mathrm{d} g) \tag{19}$$

With conserved currents

$$J(z) = \tilde{J}^{z} \qquad \bar{J}(\bar{z}) = J^{\bar{z}} \tag{20}$$

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Toda field theory

Study of $(\mathcal{A}, \overline{\mathcal{A}})$ such that

$$\mathcal{A} = g\partial g^{-1} \qquad \overline{\mathcal{A}} = g\overline{\partial} g^{-1} \tag{21}$$

$$\partial\overline{\partial} g = \overline{\partial}\partial g \tag{22}$$

$$\mathcal{A} \in \mathfrak{g}_+ \qquad \overline{\mathcal{A}} \in \mathfrak{g}_- \tag{23}$$

We get the Casimir algebras

$$A_n \Rightarrow \mathcal{W}(2, 3, \ldots n) \tag{24}$$

Generalizing conditions

We want to fix $J(z)$ along Γ

$$J(z) = M + j(z) \qquad j(z) \in \Gamma^{\perp} \tag{25}$$

$$\phi_\gamma(z) = \langle \gamma, J(z) \rangle - \langle \gamma, M \rangle = 0 \quad \forall \gamma \in \Gamma \tag{26}$$

First-classness of the conditions

$$
\begin{aligned}
\{\phi_\alpha, \phi_\beta\} = 0 & \\
& \Rightarrow [\Gamma, \Gamma^\perp] \\
& \Rightarrow [M, \Gamma] \subset \Gamma^\perp \\
& \Rightarrow \Gamma \subset \Gamma^\perp
\end{aligned}
\tag{27}
$$

which in particular implies

$$
M \in [\Gamma, \Gamma]^\perp / \Gamma^\perp \tag{28}
$$

Conformality of the reduction

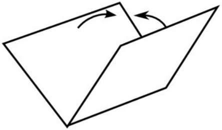

Figure 7: Illustration of a valley fold

Gravity potential

$$E_p = \langle m\vec{g}, \vec{z} \rangle \tag{29}$$

Our potential

$$E_p = \langle J'(z), H \rangle \tag{30}$$

Conditions on H

$$
\begin{aligned}
&H \in \Gamma^{\perp} \\
&[H, \Gamma^{\perp}] \subset \Gamma^{\perp} \\
&([H, M] + M) \in \Gamma^{\perp}
\end{aligned}
\tag{31}
$$

This makes us want to have

$$[H, M] = -M \tag{32}$$

The Drinfield-Sokolov gauge

To compute the algebra, we should fix a gauge

$$J_{\text{red}}(z) = M + j_{\text{red}} \qquad j_{\text{red}} \in \mathcal{V} \tag{33}$$

with

$$\Gamma^{\perp} = [M, \Gamma] + \mathcal{V} \tag{34}$$

we choose

$$\mathcal{V} = \Gamma^{\perp_\omega} \cap \Gamma^{\perp} \qquad \omega_M(\alpha, \beta) = \langle M, [\alpha, \beta] \rangle \tag{35}$$

Last conditions

For $\Gamma^{\perp_\omega}$ to exist

$$\Gamma \cap \mathrm{Ker}(\mathrm{ad}_M) \tag{36}$$

For positive conformal dimensions

$$\Gamma^{\perp} \subset \mathfrak{g}_{>-1} \tag{37}$$

Good gradings

Good Grading

(H, M) such that H is diagonalizable, $M \in \mathfrak{g}_{-1}$ and

$$\mathfrak{g}^M \subset \mathfrak{g}_{\leq} \tag{38}$$

We write $\mathcal{W}^k(\mathfrak{g}, x, f)$

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Good gradings

A pair (x, e) such that x induces a grading and

- ad_e is injective starting from $\mathfrak{g}_<$
- ad_e is surjective arriving in $\mathfrak{g}_>$

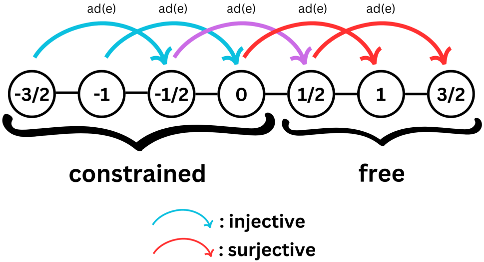

Figure 8: Illustration of a good grading

From shifting constant to blocks

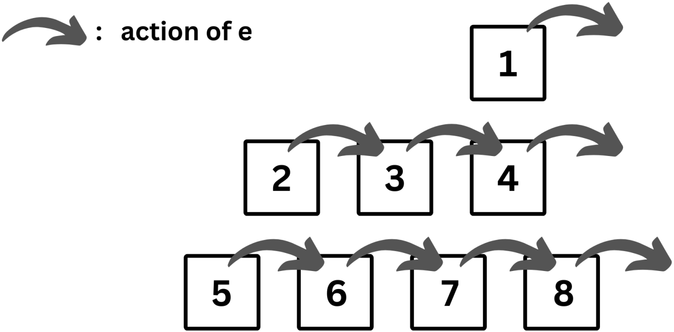

Figure 9: Representation of the nilpotent element e in a pyramid

From Hamiltonian term to pyramids

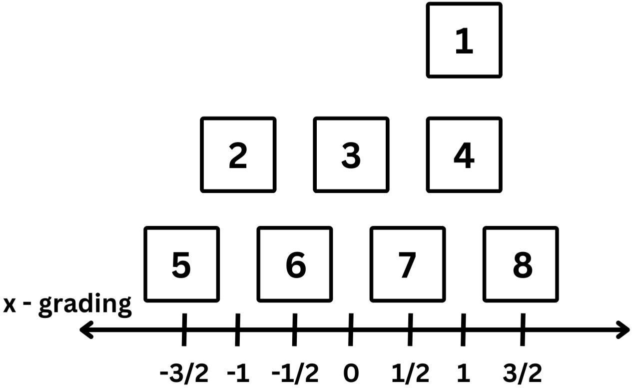

Figure 10: Representation of the diagonalizable element x in a pyramid

Reading pyramids

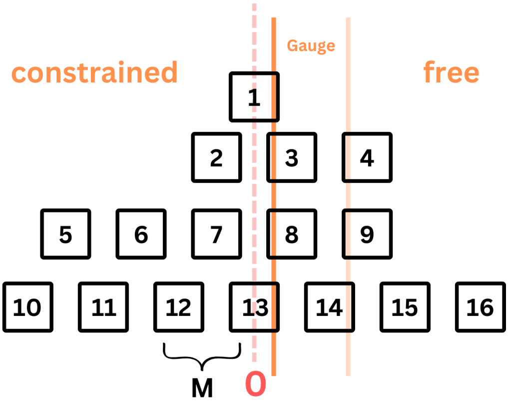

Figure 11: Diagram of the informations in a pyramid