

# Giving physical meaning to $\mathcal{W}^k(\mathfrak{g}, x, f)$

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# Table of Contents

- ① Introduction: CFTs and RCFTs
- ② The WZW model
- ③ Reductions of the model
- ④ Classifying through pyramids

# Table of Contents

① Introduction: CFTs and RCFTs

② The WZW model

③ Reductions of the model

④ Classifying through pyramids

# Quantum field theories

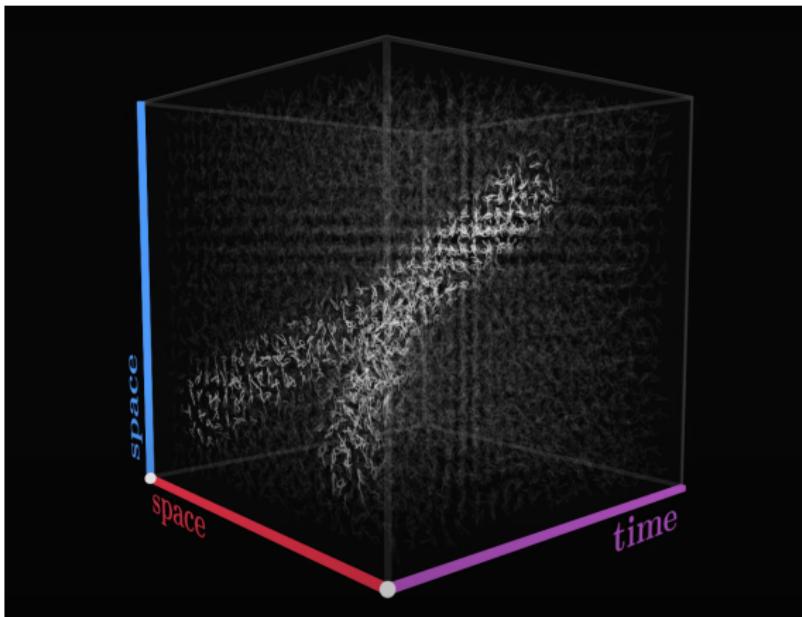


Figure 1: Illustration of a field

# Correlation functions

In the operator formalism

$$\langle 0 | \mathcal{T}(\varphi(x)\phi(y) \dots) | 0 \rangle \quad (1)$$

In the path-integral formalism

$$\frac{1}{Z} \int \mathcal{D}\psi \varphi(x)\phi(y) \dots e^{i\mathcal{S}(\psi)} \quad (2)$$

with  $Z = \int \mathcal{D}\psi e^{i\mathcal{S}(\psi)}$

# Conformal field theories

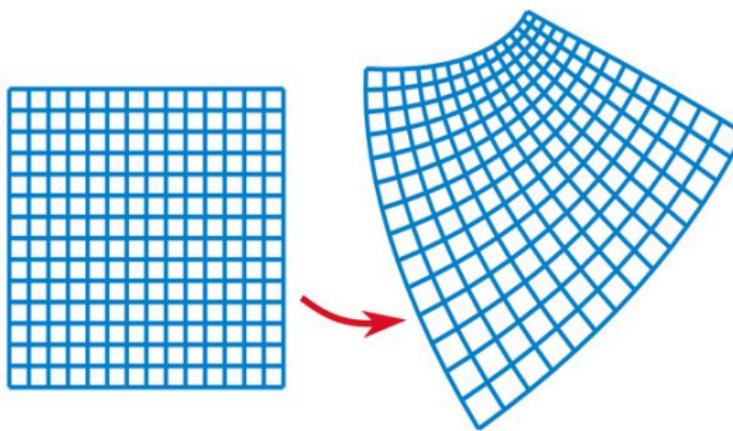


Figure 2: Conformal Transformation

$$\eta^{\mu\nu} \rightarrow \lambda(x^\alpha) \eta^{\mu\nu} \quad (3)$$

# Conformal generators

$$\langle M_{\mu\nu}, P_\mu, K_\mu, D \rangle \quad (4)$$

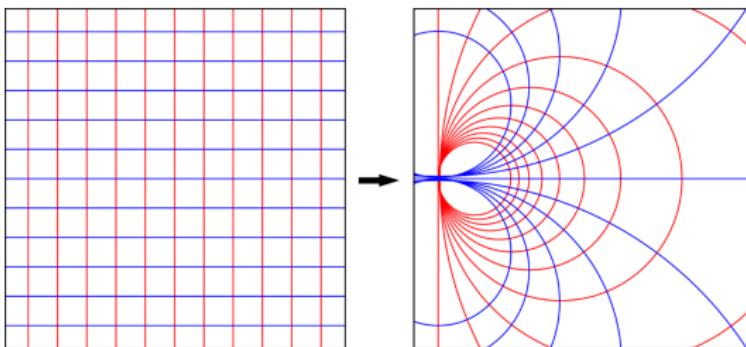


Figure 3: Special Conformal Transformation

## 2D CFTs and chirality

### Chiral coordinates

$$z = x + t \quad \bar{z} = x - t \quad (5)$$

The energy-momentum tensor decouples

$$\begin{aligned} T^{z\bar{z}} &= T^{\bar{z}z} = 0 \\ T^{zz} &= T(z) \quad T^{\bar{z}\bar{z}} = \bar{T}(\bar{z}) \end{aligned} \quad (6)$$

## Virasoro algebra

$$T(z) = \sum_n L_n z^{-n-2} \quad (7)$$

### The Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (8)$$

### Partition of the Hilbert space

$$\mathcal{H} = \sum_{i,j} V(c, h_i) \otimes \overline{V}(c, \bar{h}_j) \quad (9)$$

## RCFTs

$$\mathcal{H} = \sum_{i,j} V_A(h_i) \otimes \overline{V}_{\bar{A}}(\bar{h}_j) \quad (10)$$

# OPE and algebra

$$A(z)B(w) \sim \sum_i \frac{1}{(z-w)^i} C_i(w) \quad (11)$$

The structure is called a **vertex algebra**

## $\mathcal{W}$ -algebra

A  $\mathcal{W}$ -algebra is a chiral algebra of the form  $\langle T, \varphi^{h_1}, \varphi^{h_2}, \dots \rangle$ ,  
written  $\mathcal{W}(2, h_1, h_2, \dots)$

# Table of Contents

① Introduction: CFTs and RCFTs

② The WZW model

③ Reductions of the model

④ Classifying through pyramids

# The Sigma model

Defined from  $\Sigma$  to  $X$  by the action

$$\mathcal{S}(\varphi) = \frac{1}{2} \int_{\Sigma} g^{ij}(\varphi) \partial^{\mu} \varphi_i \partial_{\mu} \varphi_j \quad (12)$$

- $\mathcal{M} \rightarrow \mathbb{C}$  : non-interacting, massless quantum mechanics
- $\mathbb{R} \rightarrow \mathcal{M}$  : relativistic particle
- $\mathbb{R}^2 \rightarrow \mathcal{M}$  : string on a manifold

## Sigma model on Lie algebras

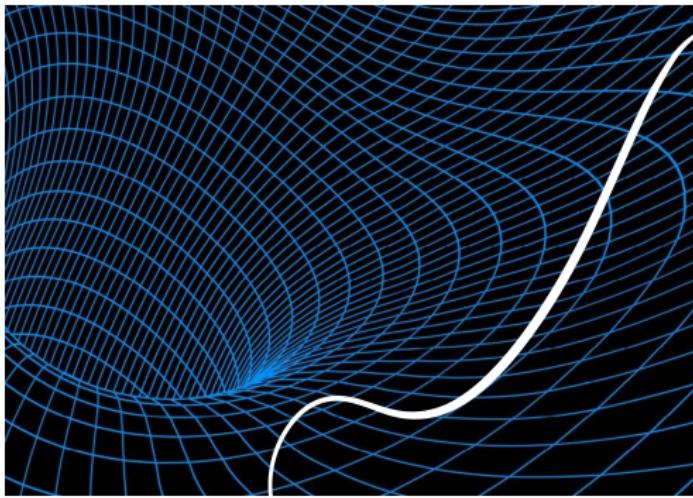


Figure 4: Illustration of a string on a Lie group

$$\mathcal{L} = 2\text{Tr}(g^{-1}\partial g \ g^{-1}\bar{\partial}g) \quad (13)$$

## Two levels of conformality

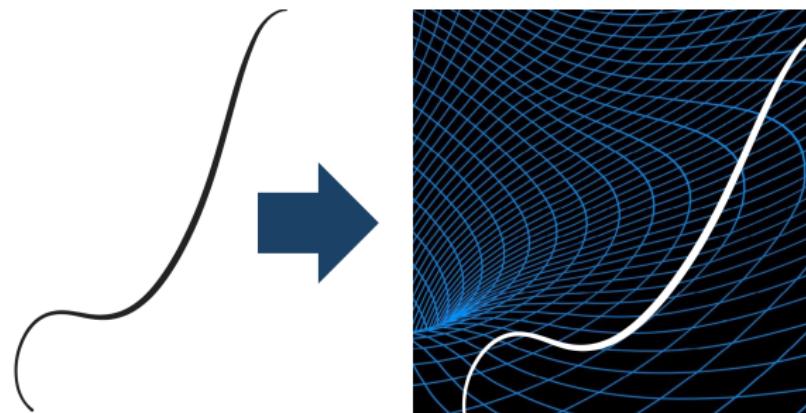


Figure 5: Time slice of the base manifold and target manifold

$G \times G$  symmetry

$$J^\mu = g^{-1} \partial^\mu g \quad \tilde{J}^\mu = \partial^\mu g \ g^{-1} \quad (14)$$

## Geometrical obstruction

Equation of conservation:

$$\partial J^{\bar{z}} + \bar{\partial} J^z = 0 \quad (15)$$

But

$$\begin{aligned} \partial J^{\bar{z}} &= \bar{\partial} J^z = 0 \\ \Rightarrow [J^\mu, J^\nu] &= 0 \end{aligned} \quad (16)$$

# The topological term

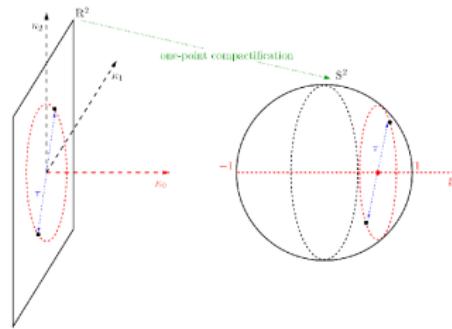


Figure 6: Compactification of  $\mathbb{R}^2$  into  $S^2$

$$\mathcal{S}_W = \int_{\mathbb{B}^3} \text{Tr}(\wedge^3 g^{-1} dg) \quad (17)$$

# The WZW model

We consider  $\frac{k}{2\pi}\mathcal{S}_W$ , for  $k \in \mathbb{Z}$  the level of the theory. The equation of conservation becomes

$$\left(1 + \frac{k}{2\pi\lambda}\right)\partial J^{\bar{z}} + \left(1 - \frac{k}{2\pi\lambda}\right)\bar{\partial}J^z = 0 \quad (18)$$

So we consider

$$\mathcal{S}_{WZW} = \frac{k}{\pi} \int_{\mathbb{S}^2} \text{Tr}(g^{-1}\partial g \ g^{-1}\bar{\partial}g) - \frac{k}{2\pi} \int_{\mathbb{B}^3} \text{Tr}(\wedge^3 g^{-1}dg) \quad (19)$$

With conserved currents

$$J(z) = \tilde{J}^z \qquad \bar{J}(\bar{z}) = J^{\bar{z}} \quad (20)$$

# Table of Contents

① Introduction: CFTs and RCFTs

② The WZW model

③ Reductions of the model

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## Toda field theory

Study of  $(\mathcal{A}, \overline{\mathcal{A}})$  such that

$$\mathcal{A} = g \partial g^{-1} \quad \overline{\mathcal{A}} = g \overline{\partial} g^{-1} \quad (21)$$

$$\partial \overline{\partial} g = \overline{\partial} \partial g \quad (22)$$

$$\mathcal{A} \in \mathfrak{g}_+ \quad \overline{\mathcal{A}} \in \mathfrak{g}_- \quad (23)$$

We get the Casimir algebras

$$A_n \Rightarrow \mathcal{W}(2, 3, \dots, n) \quad (24)$$

## Generalizing conditions

We want to fix  $J(z)$  along  $\Gamma$

$$J(z) = M + j(z) \quad j(z) \in \Gamma^\perp \quad (25)$$

$$\phi_\gamma(z) = \langle \gamma, J(z) \rangle - \langle \gamma, M \rangle = 0 \quad \forall \gamma \in \Gamma \quad (26)$$

## First-classness of the conditions

$$\begin{aligned}\{\phi_\alpha, \phi_\beta\} &= 0 \\ \Rightarrow [\Gamma, \Gamma^\perp] \\ \Rightarrow [M, \Gamma] &\subset \Gamma^\perp \\ \Rightarrow \Gamma &\subset \Gamma^\perp\end{aligned}\tag{27}$$

which in particular implies

$$M \in [\Gamma, \Gamma]^\perp / \Gamma^\perp\tag{28}$$

## Conformality of the reduction

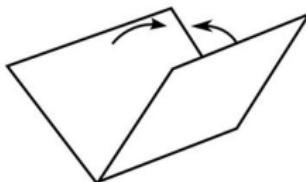


Figure 7: Illustration of a valley fold

## Gravity potential

$$E_p = \langle m\vec{g}, \vec{z} \rangle \quad (29)$$

## Our potential

$$E_p = \langle J'(z), H \rangle \quad (30)$$

## Conditions on $H$

$$H \in \Gamma^\perp$$

$$[H, \Gamma^\perp] \subset \Gamma^\perp \quad (31)$$

$$([H, M] + M) \in \Gamma^\perp$$

This makes us want to have

$$[H, M] = -M \quad (32)$$

## The Drinfeld-Sokolov gauge

To compute the algebra, we should fix a gauge

$$J_{\text{red}}(z) = M + j_{\text{red}} \quad j_{\text{red}} \in \mathcal{V} \quad (33)$$

with

$$\Gamma^\perp = [M, \Gamma] + \mathcal{V} \quad (34)$$

we choose

$$\mathcal{V} = \Gamma^{\perp_\omega} \cap \Gamma^\perp \quad \omega_M(\alpha, \beta) = \langle M, [\alpha, \beta] \rangle \quad (35)$$

## Last conditions

For  $\Gamma^{\perp_\omega}$  to exist

$$\Gamma \cap \text{Ker}(\text{ad}_M) \quad (36)$$

For positive conformal dimensions

$$\Gamma^\perp \subset \mathfrak{g}_{>-1} \quad (37)$$

## Good gradings

### Good Grading

$(H, M)$  such that  $H$  is diagonalizable,  
 $M \in \mathfrak{g}_{-1}$  and

$$\mathfrak{g}^M \subset \mathfrak{g}_\leq \tag{38}$$

We write  $\mathcal{W}(\mathfrak{g}, x, f)$

# Table of Contents

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## Good gradings

A pair  $(x, e)$  such that  $x$  induces a grading and

- $\text{ad}_e$  is injective starting from  $\mathfrak{g}_{<}$
- $\text{ad}_e$  is surjective arriving in  $\mathfrak{g}_{>}$

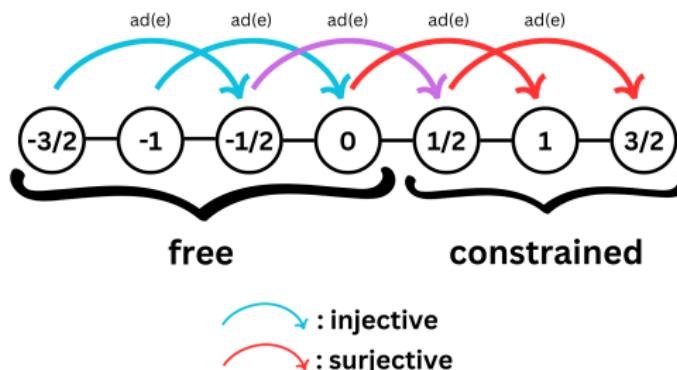


Figure 8: Illustration of a good grading

## From shifting constant to blocks

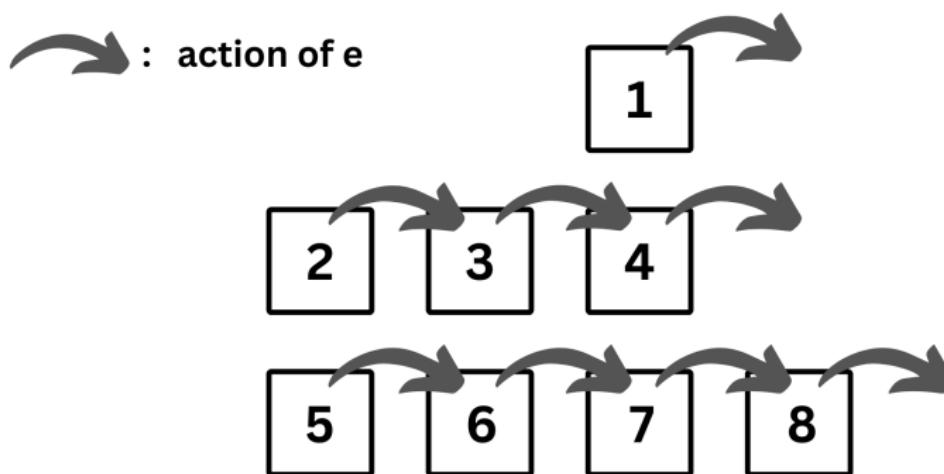


Figure 9: Representation of the nilpotent element  $e$  in a pyramid

## From Hamiltonian term to pyramids

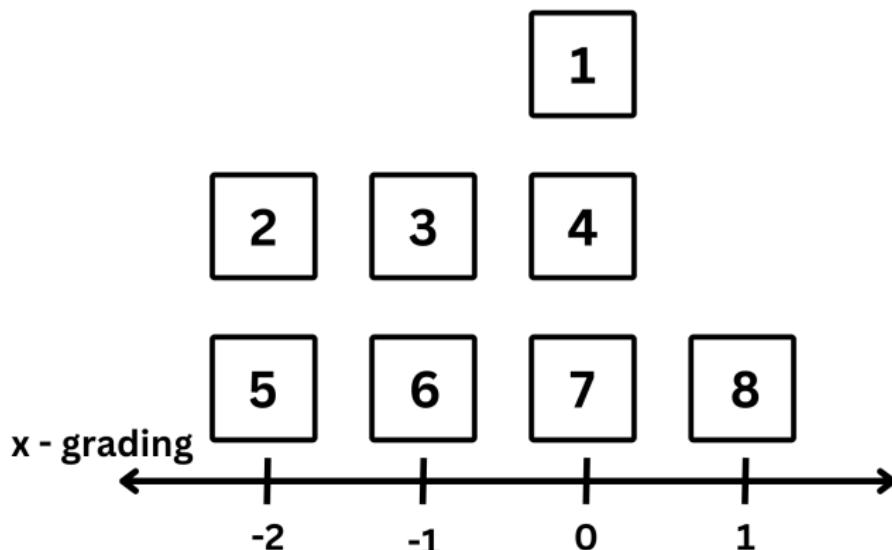


Figure 10: Representation of the diagonalizable element  $x$  in a pyramid

## Reading a pyramid

Exemple of  $W(A_n, \text{princip}) = W(2, 3, 4, \dots)$

## Hypothesis: explaining infinite W algebras

Hypothesis about infinite W algebra, linked to infinite pyramid, nice meaning