Random geometry and non-unitary quantum field theory

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1 Duality and exact critical points of lattice models

The goal of this section is to introduce the concept of duality transformations for lattice models. We shall consider in turn the Q-state Potts model and the site percolation problem. In both cases the duality transformation exists for arbitrary two-dimensional lattices (i.e., planar graphs). It relates a high-density expansion on the given (or *primal*) lattice to a low-density expansion on a related (or *dual*) lattice. For a finite lattice we consider these expansions to all orders, so the duality transformation is an exact identity between partition functions.

For the Potts model we shall consider planar lattices only, i.e., lattices that can be embedded in the sphere. But we shall consider site percolation on the torus, in order to show that non-trivial boundary conditions can be accounted for as well.

When the primal and dual lattices coincide, the lattice is said to be *self-dual*. In that case, the duality transformation fixes the critical temperature, under the mild assumption that there is a unique critical point. As a side result, we shall show that bond percolation on the square lattice and site percolation on the triangular lattice both have the percolation threshold $p_{\rm c} = \frac{1}{2}$.

Series 1: Preliminaries

Exercise 1-1: Let $G = (V, E, F)$ be a connected planar graph with vertices V, edges E and faces F. The *dual* graph $G^* = (V^*, E^*, F^*)$ is constructed by placing a dual vertex inside each face of G (including the infinite, external face), and connecting a pair of dual vertices by an edge if and only if the corresponding faces are adjacent in G. By construction, G^* is also connected and planar. Show that there are bijections $V^* \sim F$, $E^* \sim E$, and $F^* \sim V$.

Exercise 1-2: Show that duality is an involution, i.e., that $(G^*)^* = G$.

Exercise 1-3: Let $G_A = (V, A)$ be the *induced* graph, obtained from G by drawing only a subset $A \subseteq E$ of the edges. The corresponding dual is $G^*_A = (V^*, B)$ with $B = (E \setminus A)^*$, so that an edge is present in A if and only if the corresponding dual edge is absent from B. Let $c(A)$ be the number of independent cycles in G_A, and $k(B)$ be the number of connected components in G^*_A. Show that

$$c(A) = k(B) - 1 \,. \tag{1}$$

Exercise 1-4: Show the Euler-like relation:

$$k(A) = |V| - |A| + c(A) \,, \tag{2}$$

where $|\cdot|$ means the number of elements in the given set. You may want to use induction in $|A|$.

Exercise 1-5: By taking $A = E$, recover the familiar form of Euler's relation.

Series 2: Potts model

Exercise 1-1: Recall the random-cluster representation of the partition function of the Q-state Potts model:

$$Z_G(Q, v) = \sum_{A \subseteq E} v^{|A|} Q^{k(A)} \,, \tag{3}$$

where $v = e^{\beta J} - 1$ is the temperature parameter. Write the corresponding expression $Z_{G^*}(Q, v^*)$ for the partition function of the dual model at the (yet undetermined) temperature v^*.

Exercise 1-2: Show that one may chose v^* so that

$$Z_G(Q, v) = K Z_{G^*}(Q, v^*) \,, \tag{4}$$

where K is independent of $|A|$, i.e., it depends only on the parameters (Q, v) via $|E|$ and $|V|$.

Exercise 1-3: Show that the square lattice is self-dual, $G^* = G$, up to boundary effects. Assuming the uniqueness of the phase transition, show that the critical point is given by $v_c = \sqrt{Q}$.

Exercise 1-4: Determine the bond-percolation threshold on the square lattice.

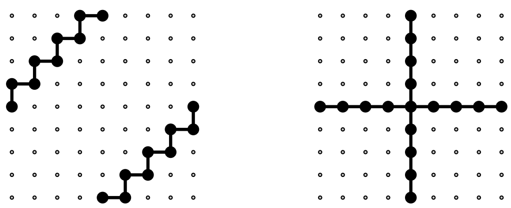
Series 3: Site percolation

Exercise 1-1: Consider site percolation on a planar lattice $G = (V, E)$ that we call primal. The corresponding *matching* lattice $\hat{G} = (V, \hat{E})$ is obtained from G by adding further edges inside each face, so that all vertices around the face become pairwise connected. Colour vertices black (resp. white) with probability p (resp. $1 - p$). We define black (resp. white) vertices to be connected through the edges E (resp. $\hat{E}$). Any configuration induces a black subgraph $\mathscr{C}$ of G, and a white subgraph $\hat{\mathscr{C}}$ of $\hat{G}$. Obviously, each black connected component is surrounded by white vertices, and each white connected component is surrounded by black vertices. Show that all white vertices that surround a black component are connected in $\hat{\mathscr{C}}$, and that all black vertices that surround a white component are connected in $\mathscr{C}$.

Exercise 1-2: On a planar graph, the number of connected components (or *clusters*) N of a black subgraph $\mathscr{C}$ with vertices V, edges E and faces F would be given by Euler's relation,

$$N = |V| - |E| + |F| \,, \tag{5}$$

where here we do *not* count the unbounded region outside the graph as a face. But we wish now to consider the model on the torus. Clusters may wrap around the torus, which modifies (5). We call clusters with the following topologies *single-wrapping* and *cross-wrapping*:



Show how (5) is modified for configurations with k single-wrapping clusters, or with a cross-wrapping cluster.

Exercise 1-3: Some faces of $\mathscr{C}$ are elementary in the sense that they correspond to faces of the underlying primal lattice G. Other faces are larger and enclose some empty vertices of the primal lattice. Let $|F_0|$ denote the number of elementary faces of $\mathscr{C}$. Let $\hat{N}$ denote the number of connected components of white vertices. Show that

$$N - \hat{N} - (|V| - |E| + |F_0|) = \begin{cases} 1 & \mathscr{C} \text{ cross-wrapping,} \\ -1 & \hat{\mathscr{C}} \text{ cross-wrapping,} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Exercise 1-4: Consider now G being a torus with $L \times L$ vertices. Define the function

$$\chi(p) = L^{-2} \left\langle |V| - |E| + |F_0| \right\rangle \,, \tag{7}$$

where $\langle \cdot \rangle$ denotes mean value. Show that $\chi(p)$ is a polynomial in p (it is called the *matching polynomial*). Compute it for the square lattice.

Exercise 1-5: Define the matching function

$$M(p) := N(p) - \hat{N}(1 - p) - L^2 \chi(p) \,. \tag{8}$$

Show that $\lim_{L \to \infty} L^{-2} \langle M(p) \rangle = 0$.

Exercise 1-6: Suppose that site percolation on G has a unique phase transition at $p = p_c$. Show that site percolation on the matching lattice has a transition at $q = 1 - p_c$.

Exercise 1-7: Show that site percolation on *any* triangulation (i.e., a lattice G all of whose faces are triangles) has $p_c = \frac{1}{2}$.

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2 Conformal transformations

The goal of this section is to acquire some familiarity with conformal transformations. We shall focus on some of their simple properties which are repeatedly used in SLE arguments. In the first part we shall examine the linear fractional transformations (LFT)

$$w(z) = \frac{az+b}{cz+d}\,, \quad \text{with } ad-bc \neq 0 \tag{9}$$

In the second part we shall investigate a more elaborate conformal transformation. Consider a polygon P in the complex plane. By the Riemann mapping theorem there exists a conformal transformation $f(z)$ from the upper half plane to the interior of P. We shall admit without proof that this is given by the Schwarz-Christoffel (SC) mapping:

$$f(z) = \int^z \frac{K}{(w-a)^{1-(\alpha/\pi)}(w-b)^{1-(\beta/\pi)}(w-c)^{1-(\gamma/\pi)}\ldots}\,\mathrm{d}w\,, \tag{10}$$

wheree K is a constant, $a < b < c < \ldots$ are the points along the real axis in the z-plane that map to the vertices of P, and $\alpha, \beta, \gamma, \ldots$ are the interior angles of P.

Series 1: Linear fractional transformations

Exercise 2-1: Verify that the composition of two LFT corresponds to the multiplication of the matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of their parameters. Conclude that the LFT form a group isomorphic to $\mathrm{SL}(2,\mathbb{C})$.

Exercise 2-2: Show that the LFT

$$w(z) = \frac{z-i}{z+i} \tag{11}$$

maps $\mathbb{R}$ to the unit circle, and the upper half-plane to the unit disc.

Exercise 2-3: Show that any LFT can be written as a composition of affine transformation ($z \mapsto az+b$) and inversions ($z \mapsto 1/z$).

Exercise 2-4: Show that an LFT maps the union of straight lines and circles into itself.

Exercise 2-5: Let $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ be two sets of three distinct points. Show that there exists an LFT that maps $z_1, z_2, z_3 \mapsto w_1, w_2, w_3$.

Series 2: Schwarz-Christoffel mapping

Exercise 2-1: Consider the semi-infinite strip

$$S := \{w = x+iy \,|\, x > 0, 0 < y < \pi\} \tag{12}$$

Make explicit the SC mapping from the upper half-plane to S.

Exercise 2-2: Write the conformal mapping from the upper half-plane to a generic triangle with interior angles $\alpha = \pi\bar{a}$, $\beta = \pi\bar{b}$ and $\gamma = \pi(1-\bar{a}-\bar{b})$. Then specialise to an equilateral triangle. Compare with the probability of having a percolation cluster connecting two segments of the real axis, discussed in the lectures. Conclusion?

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3 Temperley-Lieb algebra

The goal of this section is to complete some of the arguments given in the course on the Temperley-Lieb algebra and to acquire some familiarity with algebraic methods. We shall also see that the TL algebra underlies other famous models in statistical physics and adjacent areas.

Series 1: Dimension of the diagram algebra

According to the lectures, the number $d_{n,p}$ of link states on n points with p links equals the number of increasing paths from $(0, 0)$ to $(n - p, p)$.

Exercise 3-1: Show that an increasing path ending at $(n-p, p)$ must pass through either $(n-p-1, p)$ or $(n - p, p - 1)$, but not both. Deduce a recursion relation on $d_{n,p}$.

Exercise 3-2: Argue that the boundary values are $d_{n,0} = 1$ and $d_{2p-1,p} = 0$.

Exercise 3-3: Show that the solution of the problem is

$$d_{n,p} = \binom{n}{p} - \binom{n}{p-1}, \tag{13}$$

where we understand, as usual, that $\binom{n}{-1} = 0$.

Exercise 3-4: Give then the dimension $d_{2n,n}$ of the n-diagram algebra.

Exercise 3-5: We wish to rederive this result by the technique of generating functions. Let

$$f(x) = \sum_{n=0}^{\infty} d_{2n,n} x^n \,. \tag{14}$$

Derive a functional equation satisfied by $f(x)$ by reasoning on the link emanating from the first point.

Exercise 3-6: Find the solution of this equation which is regular for $x \to 0$. Show that it reproduces the previous result for $d_{2n,n}$.

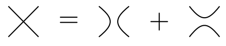
Exercise 3-7: Show that $d_{2n,n}$ can always be written as a sum of squares.

Series 2: Six-vertex model and the XXZ spin chain

We define a loop model on the square lattice with partition function

$$Z = \sum_{\mathscr{S}} \beta^{\ell(\mathscr{S})} \,, \tag{15}$$

where the sum is over all splittings $\mathscr{S}$ of the square-lattice vertices according to



(16)

and $\ell(\mathscr{S})$ is the resulting number of loops in $\mathscr{S}$. The boundary conditions are non-periodic (the loops bounce back at the boundaries of the system). We can think of (16) as the linear operator $\check{R}$ that propagates through the vertex in a transfer-matrix formalism, with time flowing upwards. If we label the left and right sites as $m, m + 1$ we have then

$$\check{R}_m = \mathbf{1} + u_m \,, \tag{17}$$

with u_m the Temperley-Lieb generator.

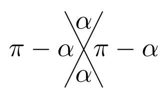
Exercise 3-1: We define a refined model in which each loop also has an overall orientation, which can be positive (trigonometric = anticlockwise) or negative (antitrigonometric = clockwise). We write

$$\beta = q + q^{-1} \tag{18}$$

and give the weight $q^{\pm 1}$ to a loop with orientation $\pm$. Show that Z results from giving a local weight $q^{\alpha/(2\pi)}$ to each place where an oriented loop turns through an angle α, followed by a sum over all orientations.

Exercise 3-2: If instead we keep the orientations but sum over the splittings, show that there are six possible arangements of oriented edges around each vertex. The resulting model is called the *six-vertex model*.

Exercise 3-3: Specify the local weights of the six-vertex model. It is useful to deform the square-lattice vertex as follows



(19)

and state the result in terms of the deformation angle α.

Exercise 3-4: In the limit $\alpha \to 0$, rewrite this as $\check{R} = \mathbf{1} + u_i$, where the objects on the right-hand side are 4×4 matrices in the basis of two spins $1/2$.

Exercise 3-5: Verify that the matrix representation of u_i obeys the relations of $\mathsf{TL}_n(\beta)$.

Exercise 3-6: Express u_i in terms of Pauli matrices. You should find that

$$\begin{aligned} u_m &= \tfrac{1}{2}\left[\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y \right. \\ &- \tfrac{1}{2}(q+q^{-1})(\sigma_m^z \sigma_{m+1}^z - I) \\ &- \left. \tfrac{1}{2}(q-q^{-1})(\sigma_m^z - \sigma_{m+1}^z)\right] . \end{aligned} \tag{20}$$

Exercise 3-7: The *XXZ spin chain*, a popular model in statistical and condensed-matter physics, is defined by the Hamiltonian

$$H = \tfrac{1}{2}\sum_m \left[\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \sigma_m^z \sigma_{m+1}^z\right] , \tag{21}$$

where Δ is called the anisotropy parameter. Comment on this in view of the above findings.

Series 3: Temperley-Lieb algebra and knot theory

Define, for any $i = 1, 2, \ldots, n-1$, the so-called braid operators (*opérateurs de tresses*)

$$\begin{aligned} g_i &= iq^{1/2}(\mathbf{1} - q^{-1}u_i) , \\ g_i^{-1} &= -iq^{-1/2}(\mathbf{1} - qu_i) , \end{aligned} \tag{22}$$

where we the loop weight is parameterised as

$$\beta = q + q^{-1} . \tag{23}$$

Exercise 3-1: Show that g_i and g_i^{-1} are indeed each other's inverses in $\mathsf{TL}_n(\beta)$:

$$g_i \; g_i^{-1} = g_i^{-1} \; g_i = \mathbf{1} . \tag{24}$$

Exercise 3-2: Prove the so-called braid relations:

$$g_i \; g_{i\pm1} \; g_i = g_{i\pm1} \; g_i \; g_{i\pm1} . \tag{25}$$

Exercise 3-3: One introduces the diagrammatic representation $g_i =$  (called "over-crossing"), and $g_i^{-1} =$  (called "under-crossing"). In both cases, the strands shown are those labelled i and $i+1$. Draw the diagrams corresponding to (24), (25) and the inverse of (25). Interpret these diagrams in words and try to justify the use of the word "braid".

Exercise 3-4: Prove the following relations

$$\begin{aligned} u_i g_{i+1}^{\pm1} g_i^{\pm1} &= g_{i+1}^{\pm1} g_i^{\pm1} u_{i+1} = u_i u_{i+1} , \\ u_{i+1} g_i^{\pm1} g_{i+1}^{\pm1} &= g_i^{\pm1} g_{i+1}^{\pm1} u_i = u_{i+1} u_i , \\ g_i^{\pm1} u_i &= u_i g_i^{\pm1} = \mp i q^{\mp3/2} u_i , \end{aligned} \tag{26}$$

and write the result of each line as an expression depending only on the TL generators u_i. Interpret these results diagrammatically. Do they confirm the intuition gained in the preceeding question?

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4 Indecomposability

The course has provided an example of indecomposability in percolation ($Q \to 1$), built on the mixing of two Potts observables acting on two spins. In the exercises we shall study a simpler example in which the mixing operators act on only one spin. It stems from the $Q \to 0$ limit of the Potts model that will be shown to describe spanning trees on a lattice. In this case the correlation function containing the logarithm can be computed exactly in the lattice model. It is related to the problem of conductance in a network of electrical resistors.

Series 1: Spanning trees and the $Q \to 0$ state Potts model

Exercise 4-1: Consider the loop model that corresponds to the square-lattice Potts model at its critical point. By factoring out a suitable power of Q, show that in the limit $Q \to 0$ the partition function is proportional to the number of *spanning trees* on the lattice. (A spanning tree is a connected subgraph with no cycles.)

Exercise 4-2: Define the new expectation values

$$\langle \cdots \rangle_0 := \frac{Z}{Q^{|V|/2}} \langle \cdots \rangle \,, \tag{27}$$

where $|V|$ is the number of vertices in the lattice. Interpret $\langle 1 \rangle_0$.

Exercise 4-3: Define the renormalised operator acting on one spin

$$\phi_a = \sqrt{Q} t_a^{[Q-1,1]} \,, \tag{28}$$

which reads explicitly

$$\phi_a(\sigma) = \sqrt{Q}\delta_{\sigma,a} - \tfrac{1}{\sqrt{Q}} \,. \tag{29}$$

Express its two-point function $\langle \phi_a(\sigma_i)\phi_b(\sigma_j)\rangle$ in terms of the probability $\mathbb{P}_1$ that the two points, i and j, belong to the same FK cluster.

Exercise 4-4: Show that 1 and ϕ_a must mix in the $Q \to 0$ limit. Define the corresponding physical operator φ_a and give a geometrical interpretation of the two-point function $\langle \varphi_a(\sigma_i)\varphi_b(\sigma_j)\rangle_0$ for $a \neq b$.

Exercise 4-5: We now consider the continuum limit, where spins σ_i, σ_j reside at the points r_i, r_j. Assume that $\langle 1 \rangle_0 = \sqrt{Q}A(Q)$, where $A(Q)$ is a regular function of Q with a finite limit as $Q \to 0$. Write the continuum-limit correlation function $\langle \phi_a(r_i)\phi_b(r_j)\rangle_0$ under the assumption that $\mathbb{P}_1$ decays with the critical exponent $\Delta_\sigma(Q)$ (usually called the 'magnetic' exponent).

Exercise 4-6: Compute then $\langle \varphi_a(r_i)\varphi_b(r_j)\rangle_0$ and $\langle \varphi_a(r_i)\rangle_0$. Argue that their ratio gives a logarithmic correlation function with a universal prefactor. Recall the geometric interpretation of the ratio.

Exercise 4-7: It can be shown that $\Delta_\sigma(Q) = 2h_{1/2,0}$, where the so-called Kac formula from CFT takes the form

$$h_{r,s} = \frac{\big(r(x+1) - sx\big)^2 - 1}{4x(x+1)} \tag{30}$$

and we have parameterised $\sqrt{Q} = 2\cos\left(\frac{\pi}{x+1}\right)$. From this information, compute the universal prefactor.

Series 2: Electrical resistor network and the discrete Laplacian

Exercise 4-1: Let $\mathscr{L}$ be a resistor network of $\mathscr{N}$ nodes numbered $i = 1, 2, \ldots, \mathscr{N}$. The resistor connecting nodes i and j has resistance $r_{ij} = r_{ji}$ and conductance $c_{ij} = r_{ij}^{-1}$. Denote the potential at node i by V_i. Let I_i be the net current flowing into i from any external source, subject to the constraint

$$\sum_{i=1}^{\mathscr{N}} I_i = 0 \,. \tag{31}$$

Recall Kirchhoff's law and restate it in matrix form

$$\sum_{j=1}^{\mathscr{N}} L_{ij} V_j = I_i \,. \tag{32}$$

Exercise 4-2: In this setup, how can we compute the equivalent resistance $R_{\alpha\beta}$ between two nodes α and β?

Exercise 4-3: Let Ψ_i and λ_i be the eigenvectors and eigenvalues of L, and denote by $\psi_{i\alpha}$, $\alpha = 1, 2, \ldots, \mathscr{N}$ the components of Ψ_i. Argue that Ψ_i can be taken to be orthonormal. Show that there is a zero eigenvalue, $\lambda_1 = 0$, and give the corresponding normalised eigenvector.

Exercise 4-4: Show that

$$R_{\alpha\beta} = \sum_{i=2}^{\mathscr{N}} \frac{|\psi_{i\alpha} - \psi_{i\beta}|^2}{\lambda_i}\,, \tag{33}$$

where the sum is over non-zero eigenvalues. *Hint*: To invert (32), it is useful to replace L_{ij} by $L_{ij}(\epsilon) = L_{ij} + \epsilon\delta_{ij}$ and set $\epsilon = 0$ in the end.

Exercise 4-5: As a warmup, we first investigate the case of a one-dimensional periodic lattice. We take $r_{ij} = r$ if and only if i and j are nearest neighbour nodes. Make explicit L_{ij} and give its eigenvalues λ_n and eigenfunctions ψ_{nx} for $n, x = 0, 1, \ldots, N-1$.

Exercise 4-6: Show that the equivalent resistance between nodes x_1 and x_2 is $R(x_1, x_2) = rG_N(x_1 - x_2)$, where

$$G_N(\ell) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos(2\ell\phi_n)}{1 - \cos(2\phi_n)}\,, \tag{34}$$

where $\phi_n = \frac{n\pi}{N}$.

Exercise 4-7: Derive the recursion relation

$$G_N(\ell) - G_N(\ell - 1) = 1 - \tfrac{1}{N}(2\ell - 1)\,. \tag{35}$$

Exercise 4-8: Conclude that

$$G_N(\ell) = |\ell| - \tfrac{\ell^2}{N}\,. \tag{36}$$

Exercise 4-9: We now attack the problem of an $M \times N$ square lattice. We take the nearest-neighbour resistance to be r in the x-direction, and s in the y-direction. The boundary conditions are periodic in both directions. Write the Laplacian in tensor-product notation, using the constituents from the one-dimensional case. Give its eigenfunctions and eigenvalues.

Exercise 4-10: Then write the equivalent resistance between nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$. Utilise the 1D result to separate from the double sum the terms where the wave number in one of the directions is zero.

Exercise 4-11: Take the thermodynamic limit $M, N \to \infty$ with $|\mathbf{r}_1 - \mathbf{r}_2|$ finite. Setting $x = |x_1 - x_2|$ and $y = |y_1 - y_2|$, show that

$$R(\mathbf{r}_1, \mathbf{r}_2) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{1 - \cos[x\theta + y\phi]}{\frac{1}{r}(1 - \cos\theta) + \frac{1}{s}(1 - \cos\phi)}\,. \tag{37}$$

Exercise 4-12: For the homogeneous problem $(s = r := R_0)$, perform the asymptotic expansion of the integral, in the limit of large $|\mathbf{r}_1 - \mathbf{r}_2|$.

Exercise 4-13: Compare the result with the preceeding problem on spanning trees. You may use a classical result by Kirchhoff[1] which states that the equivalent resistance is

$$R(\mathbf{r}_1, \mathbf{r}_2) = R_0 \frac{\mathscr{T}_2(\mathbf{r}_1, \mathbf{r}_2)}{\mathscr{T}}\,, \tag{38}$$

where $\mathscr{T}$ is the number of spanning trees on the network $\mathscr{L}$, while $\mathscr{T}_2(\mathbf{r}_1, \mathbf{r}_2)$ is the number of two-component spanning trees with the points $\mathbf{r}_1$ and $\mathbf{r}_2$ residing on different trees.

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[1] G. Kirchhoff, *Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird*, Annalen der Physik und Chemie **72**, 497 (1847)