Quantum Field Theory 2

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Chapter 1

Non-abelian gauge symmetry

1.1 Introducing the gauge field

Let's consider a quantum field ϕ with the standard symmetry, transforming in a finite dimensional unitary representation $\mathcal{U}$ of a compact Lie group G.

$$\phi \to \mathcal{U}(g)\phi \qquad \text{for } g \in G \tag{1.1}$$

Since it is finite dimensional, we can give the representation a finite index $i \in \mathbb{N}_d$

$$\phi_i \to \mathcal{U}(g)_{ij}\phi_j \qquad \text{for } g \in G \tag{1.2}$$

A potential term as $f(\phi^\dagger\phi)$ is invariant, as is a kinetic term $\partial_\mu\phi^\dagger\partial^\mu\phi$.

What if g varies over space-time (what if the symmetry acts locally instead of globally)?

$$\phi(x) \to \mathcal{U}(g(x))\phi(x) \qquad \text{for } g \in G \tag{1.3}$$

The potential term is still invariant, but something happens to the kinetic term due to the derivative.

$$\partial_\mu\phi(x) \to \mathcal{U}(g(x))\partial_\mu\phi(x) + \left[\partial_\mu\mathcal{U}(g(x))\right]\phi(x) \tag{1.4}$$

We would like to promote the symmetry to a global one, but the default derivative doesn't seem to make it work in general. We would like to modify the derivative ∂_μ to $\mathcal{D}_\mu$ such that

$$\mathcal{D}_\mu\phi(x) \to \mathcal{U}(g(x))\mathcal{D}_\mu\phi(x) \tag{1.5}$$

That is we want to take into account the symmetry in the geometry, or modify the symplectic structure to take into account the gauge degrees of freedom, or take secretly into account the coupling to the degrees of freedom in the kinetic term. To do so, we introduce a new field (the gauge field) A_μ and write[1]

$$\mathcal{D}_\mu = \partial_\mu - A_\mu \tag{1.6}$$

We have

$$\mathcal{D}'_\mu\phi'(x) = (\partial_\mu - A'_\mu)\mathcal{U}(g)\phi = \mathcal{U}(g)\partial_\mu\phi + \left[\partial_\mu\mathcal{U}(g)\right]\phi - A'_\mu\mathcal{U}(g)\phi \tag{1.7}$$

[1] the conventions will change

But we want it to be equal to

$$\mathcal{U}(g)\mathcal{D}_\mu\phi = \mathcal{U}(g)(\partial_\mu - A_\mu)\phi \tag{1.8}$$

Such that

$$\mathcal{U}(g)A_\mu = A'_\mu\mathcal{U}(g) - \partial_\mu\mathcal{U}(g) \tag{1.9}$$

In other terms

$$A'_\mu = \mathcal{U}(g)A_\mu\mathcal{U}^{-1}(g) + [\partial_\mu\mathcal{U}(g)]\,\mathcal{U}^{-1}(g) \tag{1.10}$$

What kind of object is A_μ? Using a matrix realization of the Lie algebra, considering

$$\exp : \mathfrak{g} \to G \tag{1.11}$$

We write (at least close to the identity)

$$\mathcal{U}(g) = e^{i\alpha^i(g)\tau_i} \tag{1.12}$$

with τ^i a basis of $\mathfrak{g}$ in appropriate representation.

$$[\partial_\mu\mathcal{U}(g)]\,\mathcal{U}^{-1}(g) = \left[\partial_\mu\alpha^i(g)\right]\tau_i \tag{1.13}$$

So A_μ is Lie algebra valued, $A_\mu = A^i_\mu\tau_i$. Following physics conventions, we write the generators of the Lie algebra as $i\tau_i$ (since $\mathcal{U}$ is unitary, the τ_i are then hermitians, which is likeable). So the convention changes for $\mathcal{D}_\mu$

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu \quad \text{with } A_\mu = A^i_\mu\tau_i \tag{1.14}$$

Substituting using the new convention, we have

$$A'_\mu = \mathcal{U}(g)A_\mu\mathcal{U}^{-1}(g) - i\,[\partial_\mu\mathcal{U}(g)]\,\mathcal{U}^{-1}(g) \tag{1.15}$$

Note that

$$\tau \to \mathcal{U}\tau\mathcal{U}^{-1} \tag{1.16}$$

is the adjoint action of the Lie group on the Lie algebra.

Now, let's rewrite things infinitesimally, for $\alpha = \alpha^i\tau_i$.

$$\mathcal{U}(g)A_\mu\mathcal{U}^{-1}(g) = A_\mu + i[\alpha, A_\mu] + O(\alpha^2) \tag{1.17}$$

and

$$-i\,[\partial_\mu\mathcal{U}(g)]\,\mathcal{U}^{-1}(g) = \partial_\mu\alpha^i(g)\tau_i \tag{1.18}$$

Such that

$$A'_\mu = A_\mu + \partial_\mu\alpha + i[\alpha, A_\mu] \tag{1.19}$$

In compontents[2],

$$\begin{aligned} A^k_\mu\tau_k &\to A^k_\mu\tau_k + \partial_\mu\alpha^k\tau_k + i\alpha^i A^j_\mu[\tau_i, \tau_j] \\ &\to A^k_\mu + \partial_\mu\alpha^k - \alpha^i A^j_\mu C^k_{ij} \end{aligned} \tag{1.20}$$

Notice how the last term vanishes when the Lie algebra is abelian.

A^K_μ is called the gauge field, $\mathcal{D}_\mu$ the gauge covariant derivative. An action (without A_μ) with global symmetry group G (ie the fields transform in a unitary representation of G) becomes invariant under a local symmetry with gauge group G upon replacing $\partial_\mu \to \mathcal{D}_\mu$. This introduces the gauge field into the action.

[2]There is no meaning to the height of the indices. They can be raised or lowered at will

1.2 The kinetic term of the gauge field

Looking for gauge invariant 2^{nd} order in derivatives quadratic term in A_μ. Consider $\mathcal{D}_\mu \mathcal{D}_\nu \phi$ for any field ϕ transforming in some representation of G.

- Includes a derivative of A^μ
- transforms nicely

How do we get rid of ϕ in this term?

$$\mathcal{D}_\mu \mathcal{D}_\nu \phi = \partial_\mu \partial_\nu \phi - i(\partial_\mu A^k_\nu)\tau_k \phi - iA^k_\nu \tau_k \partial_\mu \phi - iA^i_\mu \tau_i \partial_\nu \phi - A^i_\mu A^k_\nu \tau_i \tau_k \phi \quad (1.21)$$

We can try to consider the commutator of the covariant derivatives

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi = \left(-i(\partial_\mu A^j_\nu - \partial_\nu A^j_\mu) - iA^i_\mu A^k_\nu C^j_{ik}\right)\tau_j \phi \quad (1.22)$$

Hence $[\mathcal{D}_\mu, \mathcal{D}_\nu]$ is a matrix operator, in contrast to a derivative operator. We define

$$F_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu] \quad (1.23)$$

the field strength, for $\mathcal{D}_\mu$ in some representation.

$$F_{\mu\nu} = F^k_{\mu\nu}\tau_k = \left(-i(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu) - iA^i_\mu A^j_\nu C^k_{ij}\right)\tau_k \quad (1.24)$$

How does $F_{\mu\nu}$ transforms?

$$\mathcal{D}'_\mu \mathcal{D}'_\nu \phi' = \mathcal{U}(g)\mathcal{D}_\mu \mathcal{D}_\nu \phi \quad (1.25)$$

So

$$F'_{\mu\nu} = \mathcal{U}(g) F_{\mu\nu} \mathcal{U}^{-1}(g) \quad (1.26)$$

Hence $\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})$ is gauge invariant.

Remark. To define this product, we can either work in the universal envelopping algebra or in any representation.

Now, there are two Lorentz-invariant contractions.

1. $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$: it is the kinetic term
2. $\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})\varepsilon^{\mu\nu\rho\sigma}$: it will play a role later. Notice that it is a total derivative

1.3 Assorted facts about Lie algebras

1.3.1 The trace bilinear

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = F_{\mu\nu}{}^k F^{\mu\nu,l}\mathrm{Tr}(\tau_k \tau_l) \quad (1.27)$$

To have a well defined kinetic term, we need $\mathrm{Tr}(\tau_k\tau_l)$ to be non-degenerate and positive or negative definite. In the case of the Standard model, we can fix a specific representation of $su(2), su(3), \ldots$ and check the relations. Towards negative definites statement of $\mathrm{Tr}(\tau_i\tau_j)$

Definition 1.3.1. Given a represent. (π, V) of $\mathfrak{g}$, the bilinear form $B_V(.,.) = \mathrm{Tr}(\pi(.)\pi(.))$ is the trace bilinear form with respect to V.

With respect to the adjoint representation, it is the Killing form.

Theorem 1.3.1. *The killing form is non-degenerate iff $\mathfrak{g}$ is non-degenerate.*

Theorem 1.3.2. *Given a semisimple $\mathfrak{g}$, its Killing form is negative definite iff its Lie group is compact*

1.3.2 Normalisation of the trace bilinear

Definition 1.3.2. Let (π, V) be a rep of $\mathfrak{g}$. A bilinear form B on V is invariant wrt π if for $x \in \mathfrak{g}$, for $v, w \in V$,

$$B(\pi(x)v, w) + B(v, \pi(x)w) = 0 \tag{1.28}$$

Proposition: the trace bilinear is invariant with respect to the adjoint rep. Proposition: Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. Then there exist, up to a scalar, at most one invariant bilinear form.

Since the adjoint action always give a invariant bilinear form in any representation, we end up with the fact that our trace bilinear must be the Killing form up to a scalar. The Dynkin index $T(\pi)$ is the scalar in front of the bilinear (or a rescaled version of it).

1.3.3 Relating the Dynkin index to the Casimir of the representation

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\{x_i\}$ be a basis of $\mathfrak{g}$, $\{y_i\}$ be a dual basis of $\mathfrak{g}$ with respect to the Killing form. The Dynkin index $T(\pi)$ is defined as the scalar relating the trace bilinear in the representation π to the Killing form.

Prop: $J = \sum x_i y_i$ commutes with all elements of $\mathfrak{g}$. It is independant of the choice of basis. J is called the quadratic Casimir element of $\mathfrak{g}$. Let (π, V) be an irreducible representation of $\mathfrak{g}$. $\pi(J)$ commutes with all elements of $\pi(\mathfrak{g})$. By Schur's lemma, $\pi(J) = C(\pi)\mathrm{Id}$, with $C(\pi) \in \mathbb{C}$. $C(\pi)$ is called the Casimir of the representation. By a quick computation, we see that

$$C(\pi) \times \frac{d}{n} = T(\pi) \tag{1.29}$$

with d the dimension of the representation and n the dimension of the Lie algebra.

1.3.4 Exemple $SU(N)$

Lie algebras defined as matrix subalgebras of $\mathfrak{gl}(n)$ come with a canonical representation on $\mathbb{C}^n$, which is the defining representation.
NB: Physics convention: replace Killing form in definition of the Dynkin index by

$$2\mathrm{Tr}_{\pi_0} \tag{1.30}$$

where (π_0, V_0) is the defining representation. Hence the Dynkin index of the defining representation is $\frac{1}{2}$ in these conventions. In these conventions we also define the Casimir element using dual basis with respects to the inner product $2\mathrm{Tr}_{\pi_0}$.

For explicit computation, we can introduce an orthonormal basis $\{T_i\}$ of the defining representation of $\mathfrak{su}(\mathfrak{N})$ with regards to 2Tr.

$$\begin{aligned} \mathrm{Tr}(T_iT_j) &= \frac{1}{2}\delta_{ij} \\ \mathrm{Tr}((\pi(T_i)\pi(T_j)) &= T(\pi)\delta_i j \end{aligned} \tag{1.31}$$

With respect to $2\mathrm{Tr}_{\pi_0}$, $\{T_i\}$ is a self-dual basis.

$$\dim \pi_0 = N \tag{1.32}$$

On the other hand,

$$\dim(\mathfrak{su}(N)) = \#\{\text{basis of traceless hermitian matrices}\} = N^2 - 1 \tag{1.33}$$

Such that

$$C(\pi_0) = \frac{N^2-1}{2N} \tag{1.34}$$

Note: the structure constants in such a basis are completely antisymmetric.

$$[T_i, T_j] = iC_{ij}^k T_k \tag{1.35}$$

with $f_{ijk} = C_{ij}^k$, f is completely antisymmetric

1.4 Quantizing gauge theories

By construction of the gauge symmetry, the second order differential operator defininf the kinetic term of the gauge field is not invertible. We need to gauge fix to define the propagator. But what is left of the gauge fixing, that can help us fix the renormalization counterterms?

1.4.1 The Fadeev-Popov procedure

Recall

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.36}$$

Then $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ gives kinetic terms plus interaction terms. In an orthogonal basis, $\mathrm{Tr}(T_iT_j) \propto \delta_{ij}$ so the gauge kinetic term is $\dim \mathfrak{g}$ copies of the photon kinetic term

$$A_i^\mu(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A_i^\nu \tag{1.37}$$

Indeed, for A_i^μ an eigenvector of the momentum operator, we get $(g_{\mu\nu}k^2 - k_\mu k_\nu)$ which has a non trivial kernel. However, we could eliminate this kernel by shifting the coefficient of $\partial_\mu\partial_\nu$ away from -1:

$$\begin{aligned} &\mathcal{L}[A,\phi_i] + \frac{1}{2\xi}A_i^\mu\partial_\mu\partial_\nu A_i^\nu \\ &= \mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu) + \text{ total derivatives} \end{aligned} \tag{1.38}$$

How can we introduce a term $e^{-\frac{i}{2\xi}\int \mathrm{d}^4x(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu)}$ into the action? The strategy is to multiply the partition function Z action by field independant terms which do not change the amplitude.

$$e^{-\frac{i}{2\xi}\int \mathrm{d}^4x(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu)} = \int \mathcal{D}f e^{-\frac{i}{2\xi}\int \mathrm{d}^4x ff}\delta(\partial_\nu A_i^\nu - f) \tag{1.39}$$

We write $B_\xi[f] = e^{-\frac{i}{2\xi}\int \mathrm{d}^4x ff}$, and $G[A] = \delta(\partial_\nu A_i^\nu - f)$. We also write A^g the field A transformed by the action of g. Integrating over the gauge group, we trivially have

$$1 = \int \mathcal{D}g\delta(G[A^g]) \det \frac{\delta G[A^g]}{\delta g} \tag{1.40}$$

Now

$$\int \mathcal{D}g \int \mathcal{D}f B_\xi[f]\delta(G[A^g]) \det \frac{\delta G[A^g]}{\delta g} = C(\xi) \tag{1.41}$$

does not depend on A. Let's rescale

$$Z = \int \mathcal{D}A\mathcal{D}\phi e^{iS[A,\phi]} \tag{1.42}$$

into $Z' = C(\xi)Z$

$$Z' = \int \mathcal{D}g \int \mathcal{D}f \int \mathcal{D}A\mathcal{D}\phi\ e^{iS[A,\phi]}B_\xi[f]\delta(G[A^g]) \det \frac{\delta G[A^g]}{\delta g} \tag{1.43}$$

We want to use gauge invariance to replace dependance on A^g by dependance on A. First note

$$\left.\frac{\delta G[A^{g\tilde{g}}]}{\delta \tilde{g}}\right|_{\tilde{g}=e} = \left.\frac{\delta G[A^g]}{\delta g}\frac{\delta \tilde{g}g}{\delta \tilde{g}}\right|_{\tilde{g}=e} \tag{1.44}$$

Such that

$$\det \frac{\delta G[A^g]}{\delta g} = \det \left.\frac{\delta G[A^{g\tilde{g}}]}{\delta \tilde{g}}\right|_{\tilde{g}=e} \left[\det \left.\frac{\delta \tilde{g}g}{\delta \tilde{g}}\right|_{\tilde{g}=e}\right]^{-1} \tag{1.45}$$

We can write $\mu(g) = \left[\det \left.\frac{\delta \tilde{g}g}{\delta \tilde{g}}\right|_{\tilde{g}=e}\right]^{-1}$.

$$Z' = \int \mathcal{D}g\mu(g)\mathcal{D}f\mathcal{D}A\mathcal{D}\phi\ e^{iS[A,\phi]}B_\xi[f]\delta(G[A^g]) \det \left.\frac{\delta G[A^{g\tilde{g}}]}{\delta \tilde{g}}\right|_{\tilde{g}=e} \tag{1.46}$$

But if the gauge symmetry is preserved by the quantum theory, the measure should be invariant under a gauge transform. So we can integrate over $A^{g^{-1}}$ instead of over A, and (necessarily) similarly for ϕ. So

$$Z' = \int \mathcal{D}g\mu(g)\mathcal{D}f\mathcal{D}A^{g^{-1}}\mathcal{D}\phi^{g^{-1}}e^{iS[A^{g^{-1}},\phi^{g^{-1}}]}B_\xi[f]\delta(G[A]) \det \left.\frac{\delta G[A^{\tilde{g}}]}{\delta \tilde{g}}\right|_{\tilde{g}=e} \tag{1.47}$$

The g dependance has factored out into a coefficient

$$\int \mathcal{D}g\mu(g) = \int \mathcal{D}g \overset{-1}{\det} \left.\frac{\delta \tilde{g}g}{\delta \tilde{g}}\right|_{\tilde{g}=e} = V_G \tag{1.48}$$

which can be easily interpreted as the volume of the Gauge group. Indeed, integrating over all of the possible fields before gauge fixing should yield a factor

V_G. Since the gauge group is unphysical, we should have divided at some point the whole partition function by V_G to take into account the gauge redundancy. We are happy to see that V_G naturally decouples from Z'. We also notice that V_G naturally generalizes the Haar measure of the Lie group to a gauge group. We can define

$$Z = \frac{\int \mathcal{D}A\mathcal{D}\phi \; e^{iS}}{V_G} \tag{1.49}$$

We would now like to reformulate the other not nice term, to get it in a nice shape and understand its reason for appearing.

$$\det \frac{\delta G[A^g]}{\delta g}\bigg|_{g=e} = \det \frac{\delta[\partial_\mu A^{g,\mu} - f]}{\delta g}\bigg|_{g=e} \tag{1.50}$$

To evaluate this derivative around $g = e$, we can consider infinitesimal gauge transformations

$$A_\mu^{e+\pi} = A_\mu + \mathcal{D}_\mu \pi = A_\mu + \partial_\mu \pi - i[A_\mu, \pi] \tag{1.51}$$

Such that

$$\frac{\delta G[A^{e+\pi}]}{\delta \pi}\bigg|_{\pi=0} = \partial_\mu \mathcal{D}_\mu \tag{1.52}$$

Identity from fermionic path integral:

$$\det \mathcal{O} = \int \mathrm{d}\bar{\psi}\mathrm{d}\bar{\psi} \; e^{-i\int \mathrm{d}^4 x \bar{\psi}\mathcal{O}\psi} \tag{1.53}$$

Such that

$$\det \partial_\mu \mathcal{D}_\mu = \int \mathcal{D}\bar{c}\mathcal{D}c \; e^{i\int \mathrm{d}^4 x \bar{c}(-\partial_\mu \mathcal{D}_\mu)c} \tag{1.54}$$

Such that we end up with new fermionic fields

$$Z' = \int \mathcal{D}A\mathcal{D}\phi\mathcal{D}\bar{c}\mathcal{D}c e^{i\int \mathrm{d}^4 x\left[\mathcal{L}[A,\phi] - \frac{1}{2\xi}(\partial_\mu A_i^\mu)^2 - \bar{c}_i \partial_\mu D^\mu c_i\right]} \tag{1.55}$$

Note the fields $c_i, \bar{c}_i$:

- Are fermionic, to obtain the correct determinant
- Lorentz scalars in the adjoint representation, as must have same quantum number as π
- They break the spin statistics theorem, and hence cannot appear in physical states. They cannot enter in/out states.
- This justify their names, ghosts and anti ghosts

We will study the Feynman rules that govern such theories presently. Consider again

$$Z'/V_G == \int \mathcal{D}f\mathcal{D}A\mathcal{D}\phi e^{iS[A,\phi]} B[f]\delta(G[A]) \det \frac{\delta G[A^g]}{\delta g}\bigg|_{g=e} \tag{1.56}$$

Note: Z and Z' are related by a field independant constant for any choice of $B[f]$, $G[A]$. It reflects the freedom to choose any gauge we want. The choice

$$B_\xi[f] = e^{-\frac{i}{2\xi}\int \mathrm{d}^4 x f_i f^i} \qquad G[A] = \delta(f - \partial_\nu A_i^\nu) \tag{1.57}$$

are called a generalized ξ-gauges. How is it related to gauge fixing? A generalization thereof, but it reduces to standard notion in the limit $\xi \to 0$ limit. In this case, we can evaluate the $\mathcal{D}f$ integral via stationary phase ($B[f]$ oscillates rapidly except at f =0), where $\delta(f - \partial_\nu A_i^\nu)$ becomes $\delta(\partial_\nu A_i^\nu)$, imposing the Lorenz gauge.

1.4.2 BRST symmetry

The Fadeev Popov Lagrangian is BRST invariant

Symmetries constrain the form of counterterms required to renormalize the theory. What about gauge fixed gauge symmetries?

$$L_{FP} = L[A,\phi] - \frac{1}{2\xi}(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu) + \partial_\mu \bar{c}_i \mathcal{D}^\mu c_i \tag{1.58}$$

The usual Lagrangian is invariant under gauge transformations, whilst $L_{GF} = -\frac{1}{2\xi}(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu) + \partial_\mu \bar{c}_i \mathcal{D}^\mu c_i$ corresponds to the gauge fixing Lagrangian. We would like to recover some derivative. What if we do a gauge transformation proportional to c? Actually, since c is fermionic, we can't do it directly. But that is the general idea of BRST symmetry. Let's consider a gauge transformation of L_{FP} with gauge parameter $\alpha(x) = \theta c(x)$, with θ a grassmanian variable and $c(x)$ a ghost which is Lie algebra valued, ie $c(x) = c^i(x)\tau_i$

$$\begin{aligned} A_i^\mu &\to A_i^\mu + \theta D^\mu c_i \\ -\frac{1}{2\xi}(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu) &\to -\frac{1}{2\xi}(\partial_\mu A_i^\mu)(\partial_\nu A_i^\nu) - 2\frac{\theta}{2\xi}(\partial_\mu A_i^\mu)(\partial_\nu D^\nu c_i) \end{aligned} \tag{1.59}$$

How should $\bar{c}_i, c_i$ transform to make L_{FP} invariant?

$$\bar{c}_i \to \bar{c}_i - \frac{1}{\xi}\theta \partial_\mu A_i^\mu \tag{1.60}$$

is almost good enough, except that the gauge covariant derivative in front of c_i also transforms under the transformation due to its dependance in A_μ, messing things up by a little bit.

$$\begin{aligned} c &\to c + \theta\delta c \\ A^\mu &\to A^\mu + \theta D^\mu c \\ \mathcal{D}^\mu c = \partial^\mu c - i[A^\mu, c] &\to D^\mu c + \theta\partial^\mu \delta c - i[A^\mu, \theta\delta c] - i[D^\mu \theta c, c] \end{aligned} \tag{1.61}$$

Where

$$D^\mu \theta c = \theta \partial^\mu c + i[\theta c, A^\mu] \tag{1.62}$$

We need

$$\theta(\partial^\mu \delta c - i[A^\mu, \delta c]) - i[\theta\partial^\mu c, c] + [[\theta c, A^\mu], c] = 0 \tag{1.63}$$

We just have to solve this for δc! We find

$$\delta c = \frac{i}{2} c_i c_j [\tau^i, \tau^j] \tag{1.64}$$

which guarantees that

$$\mathcal{D}_\mu c \to \mathcal{D}_\mu c \tag{1.65}$$

Hence, we found a (fermionic) symmetry of the Lagrangian, the BRST symmetry.

$$\begin{aligned}
\delta_{\mathrm{BRST}} A^\mu &= \mathcal{D}^\mu \theta c \\
\delta_{\mathrm{BRST}} \phi &= \delta^{\text{gauge}}_{\theta c}\phi = i\theta\pi(c)\phi \\
\delta_{\mathrm{BRST}} \bar{c} &= -\frac{1}{\xi}\theta\partial_\mu A^\mu \\
\delta_{\mathrm{BRST}} c &= \theta\frac{i}{2}c_i c_j[\tau^i,\tau^j] = \theta\frac{i}{2}\{c_i,c_j\}
\end{aligned} \tag{1.66}$$

for some representation π. Hence, the only counterterms required to renormalize this Lagrangian are BRST invariant. In particular, terms not involving ghosts are gauge invariants.

1.4.3 BRST without FP

We will argue

1. There exist an operator Q_{BRST} such that for ϕ, ϕ_i physical fields (ie not ghosts?gauge?x?)
$$\langle [Q_{\mathrm{BRST}},\phi]\pi\phi_i\rangle = 0 \tag{1.67}$$

2. The gauge fixing part of the Lagrangian is of the form $[Q_{\mathrm{BRST}},\phi]$.

Consider the transformation δ_{BRST} introduced above, but introduce an additional field $N = N_i\tau^i$ and replace $\delta_{\mathrm{BRST}}\bar{c} = \theta N$, $\delta_{\mathrm{BRST}}N = 0$. We also introduce a multiplicative[3] grading called the ghost number: c has a ghost number 1, $\bar{c}$ has a ghost number -1, and all other fields 0. We write

$$\delta_{\mathrm{BRST}}\psi = \theta\triangle\psi \tag{1.68}$$

Where $\triangle$ is the Slavnov operator. This operator increases the ghost number by one (except when it annihilates).

Remark. $\triangle$ is a graded differential, ie

$$\triangle(\phi\psi) = (\triangle\phi)\psi + (-1)^{\text{ghost number of }\phi}\phi\triangle\psi \tag{1.69}$$

$\triangle^2 = 0$, ie $\triangle$ is nilpotent (this is why we had to introduce N). Hence it acts as a differential with respect to the grading by the ghost number. It is also closed. Hence suppose we modify the action before gauge fixing by a BRST exact term

$$\mathcal{L} \to \mathcal{L} + \triangle\psi \tag{1.70}$$

By gauge invariance of $\mathcal{L}$ and nilpotency of $\triangle$, this modified Lagrangian density is BRST invariant. Let's introduce a charge Q_{BRST} that implements the BRST transformation.

$$[Q_{\mathrm{BRST}},\psi]_{-s} = \triangle\psi \tag{1.71}$$

[3] the ghost number of the product of terms is the sum of the ghost numbers

where $[.,.]_{-1}$ is the anticommutator and $[.,.]_{+1}$ is the commutator, with $s = (-1)^{\text{ghost number of } \psi}$.

Assume a BRST invariant vacuum exists. Then

$$\langle 0|[Q_{\mathrm{BRST}}, T\pi\phi_i]_{-s=\pi s_i}|0\rangle \tag{1.72}$$

Also assuming that ϕ_j with $j = 2, \ldots n$ are BRST closed, ie $[Q_{\mathrm{BRST}}, \phi_j]_{-s_j} = 0$, then

$$\langle 0|T(\phi \prod_{k\neq 1} \phi_k)|0\rangle = 0 \tag{1.73}$$

correlators of BRST closed operator with a BRST exact operator vanish. Hence. the modification

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} \tag{1.74}$$

with $\delta\mathcal{L} = [Q_{\mathrm{BRST}}, \psi]_+ = \triangle\psi$ for any operator ψ of ghost number -1 leave correlators of BRST invariant operators invariant. In particular, BRST invariant means gauge invariant for operators not involving ghosts.

We pick $\psi = \bar{c}_i(F_i + \frac{1}{2}\xi N_i)$, with F_i a functional of ordinary fields (fields with ghost number 0 except N). Then

$$\delta\mathcal{L} = \triangle\psi = N_i F_i - \int \mathrm{d}^4y\ \bar{c}_i(x)\frac{\delta F_i}{\delta\phi_A(y)}[Q_{\mathrm{BRST}}, \phi_A(y)]_- + \frac{1}{2}\xi N_i^2 \tag{1.75}$$

where the index A runs over all ordinary fields. We can integrate out N by completing the spinor/sphere/??

$$\frac{1}{2}\xi N_i^2 + N_i F_i = \frac{1}{2}\xi(N_i + \frac{1}{\xi}F_i)^2 - \frac{1}{2}\frac{1}{\xi}F_i^2 \tag{1.76}$$

such that

$$\delta\mathcal{L} = -\int \mathrm{d}^4y\ \bar{c}(x)\frac{\delta F_i(x)}{\delta\phi_A(y)}[Q_{\mathrm{BRST}}, \phi_A(y)] - \frac{1}{2\xi}F_i^2 \tag{1.77}$$

Notice that $[Q_{\mathrm{BRST}}, \phi_A(y)] \propto c$. The first term is the kinetic term for ghosts.

For exemple, for $F_i = \partial_\mu A_i^\mu$:

$$\delta\mathcal{L} = \partial_\mu\bar{c}(x)\mathcal{D}^\mu c(x) - \frac{1}{2\xi}(\partial_\mu A_i^\mu)^2 = \mathcal{L}_{GF} - \frac{1}{2\xi}(\partial_\mu A_i^\mu)^2 \tag{1.78}$$

We recall that gauge invariant correlators are not modified by such a change in the Lagrangian. Note that we arrived at this result without invoking the FP procedure.

1.5 Perturbation theory of Yang-Mills theory

1.5.1 Feynman rules

Introducing the gauge coupling

To introduce gauge coupling, replace

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu \to \mathcal{D}_\mu = \partial_\mu - igA_\mu \tag{1.79}$$

which can be seen as just renormalizing A_μ. This change changes the gauge transformation. Before, we had

$$A'_\mu = \mathcal{U}(h) A_\mu \mathcal{U}^{-1}(h) - i\partial_\mu \mathcal{U}(h)\, \mathcal{U}(h)^{-1} \tag{1.80}$$

But now,

$$gA'_\mu = g\mathcal{U}(h) A_\mu \mathcal{U}^{-1}(h) - i\partial_\mu \mathcal{U}(h)\, \mathcal{U}(h)^{-1} \tag{1.81}$$

Which means

$$A'_\mu = \mathcal{U}(h) A_\mu \mathcal{U}^{-1}(h) - \frac{i}{g}\partial_\mu \mathcal{U}(h)\, \mathcal{U}(h)^{-1} \tag{1.82}$$

Infinitesimally,

$$A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \alpha + i[\alpha, A_\mu] \tag{1.83}$$

To remove the g dependance of derivative term in $F^{\mu\nu}$, redefine

$$F_{\mu\nu} = \frac{i}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{1.84}$$

The Lagrangian

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}\sum_a (F_a)_{\mu\nu}(F_a)^{\mu\nu} - \frac{1}{2\xi}\sum_a (\partial_\mu A_a{}^\mu)^2 + \partial_\mu \bar{c}_a(\delta_{ac}\partial^\mu + g f_{abc} A_b{}^\mu) c_c \\ & + \bar{\psi}_i(\delta_{ij} i\partial\!\!\!/ + g A\!\!\!/_a T^a_{ij} - m\delta_{ij})\psi_j \\ & + [(\delta_{ki}\partial_\mu - ig(A_a)_\mu T^a_{ki})\phi_i]^*[(\delta_{kj}\partial^\mu - igA^\mu_a T^a_{kj})\phi_j] - M^2\phi_i^*\phi_i\end{aligned} \tag{1.85}$$

The indices in greek letters are Lorentz indices, the beginning of the alphabet indices are Lie indices, and the middle of the alphabet indices are field indices. The T are the representation of the Lie algebra in our theory. We can divide this Lagrangian into several parts. The kinetic term is

$$\begin{aligned}\mathcal{L}_{\text{kin}} = & -\frac{1}{4}(\partial_\mu(A_a)_\nu - \partial_\nu(A_a)_\mu)(\partial^\mu(A_a)^\nu - \partial^\nu(A_a)^\mu) \\ & -\frac{1}{2\xi}\sum_a(\partial_\mu A_a{}^\mu)^2 + \bar{\psi}_i(i\partial\!\!\!/ - m)\psi_i \\ & -\phi_i^*(\Box + M^2)\phi_i - \bar{c}_a\Box c_a\end{aligned} \tag{1.86}$$

Propagators

The Gauge boson. In momentum space,

$$\frac{1}{4}(p_\mu A_\nu - p_\nu A_\mu)^2 + \frac{1}{2\xi}(p_\mu A^\mu)^2 = -\frac{1}{2}A^a_\mu(-p^2 g^{\mu\nu} + (1-\frac{1}{\xi})p^{\mu\nu})\delta_{ab}A^b_\nu \tag{1.87}$$

To find the propagator, we make the ansatz

$$\Pi^A_{\mu\nu} = A(p^2)p_\mu p_\nu + B(p^2)g_{\mu\nu} \tag{1.88}$$

Writing $\Pi^A_{\mu\nu}(-p^2 g^{\mu\nu} + (1-\frac{1}{\xi})p^{\mu\nu}) = g^\rho_\mu$ gives

$$A = \frac{-(1-\frac{1}{\xi})\frac{1}{p^2}}{\frac{1}{\xi}p^2} \qquad B = -\frac{1}{p^2} \tag{1.89}$$

Which results in

$$i(\Pi^A_{ab})_{\mu\nu} = -i\frac{g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}}{p^2 + i\epsilon}\delta_{ab} \tag{1.90}$$

We draw the propagator as a bubbly line? as a cloud line going in spirals, labelled by a Lie index and a Lorentz index. For ghosts, we have

$$i\Pi^{\bar{c}c}_{ab} = \frac{i\delta_{ab}}{p^2 + i\epsilon} \tag{1.91}$$

We draw the propagator with a dotted line, oriented by an arrow from the bar ghost to the ghost. For the charged fermion, we draw a solid line oriented by the arrow and indexed by the field i, j and the spin α, β. The propagator is

$$i(\Pi^{\psi\bar{\psi}}_{ij})_{\alpha\beta} = \left(\frac{i\delta}{\not{p} - m + i\epsilon}\right) \tag{1.92}$$

Finally, the charged scalar is represented by a dashed line with long dashes, oriented by an arrow and indexed by the fields.

$$i\Pi^{\phi\phi^*}_{ij} = \frac{i\delta_{ij}}{p^2 - M^2 + i\epsilon} \tag{1.93}$$

Vertices

Gauge self interaction

$$\begin{aligned}\mathcal{L}^{A^3}_{\text{int}} &= -\frac{1}{4}(F_a)_{\mu\nu}(F_a)^{\mu\nu}\big|_{A^3} \\ &= -gf_{abc}\partial_\mu(A_a)_\nu(A_b)^\mu(A_c)^\nu\end{aligned} \tag{1.94}$$

It is a cubic vertex where each line carries a Lorentz index and a Lie algebra index. Assume all momentums are inflowing, with k, p, q, and recall that a ∂_μ gives in momentum space a $-ik_\mu$. To give a value to the vertex, we should add a i in front (as a result of expanding e^{iS}) and suppose we are contracting with 3 other fields. There are 6 (3!) possible contractions, and we get

$$\begin{aligned}&-gf_{abc}(k_\nu\Pi^{\mu\kappa}_{ad}(k)\Pi^{\nu\lambda}_{be}(p)\Pi^{\rho\sigma}_{cf}(q) + \dots) \\ &\quad = -g\Pi^{\mu\kappa}_{ad}(k)\Pi^{\nu\lambda}_{be}(p)\Pi^{\rho\sigma}_{cf}(q)(f_{abc}k_\nu g_{\mu\rho} + f_{abc}k_\rho g_{\mu\nu} + \dots)\end{aligned} \tag{1.95}$$

such that when the dust settles, we get for the vertex

$$-gf_{abc}(g_{\mu\rho}(k_\nu - q_\nu) + g_{\mu\nu}(p_\rho - k_\rho) + g_{\nu\rho}(q_\mu - p_\mu)) \tag{1.96}$$

Now, let's look at the quartic term

$$\mathcal{L}^{A^3}_{\text{int}} = -\frac{1}{4}g^2 f_{eab}f_{ecd}(A_a)_\mu(A_b)_\nu(A_c)^\mu(A_d)^\nu \tag{1.97}$$

We get for the vertex

$$\begin{aligned}-ig^2[&f_{eab}f_{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &+ f_{eac}f_{ebd}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &+ f_{ead}f_{ebc}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})]\end{aligned} \tag{1.98}$$

Now, let's turn to the ghost-gauge boson interaction

$$\mathcal{L}_{\text{int}}^{ccA} = g f_{abc} \partial_\mu \bar{c}_a A_b^\mu c_c \tag{1.99}$$

which gives a factor

$$g f_{abc} p_\mu \tag{1.100}$$

with p the momentum of the outgoing ghost. For the fermion-gauge boson interaction, we find

$$\mathcal{L}_{\text{int}}^{\bar{\psi} A \psi} = g \bar{\psi}_i \not{A}_a \psi_j T_{ij}^a \tag{1.101}$$

which gives

$$i g \gamma_{\alpha\beta}^\mu T_{ij}^a \tag{1.102}$$

For the scalar field - gauge boson interaction,

$$\mathcal{L}[\text{int}]^{\phi^2 A} = -i g \partial_\mu \phi_k^* A_a^\mu \phi_j T_{kj}^a + i g A_a^\mu T_{ik}^a \phi_i^* \partial_\mu \phi_k = i g T_{ij}^a (q+k)_\mu \tag{1.103}$$

where k and q are the momentum of the scalar fields flowing in the direction of their $U(1)$ current. Finally, for the last interaction, we have

$$\mathcal{L}_{\text{int}}^{\phi^2 A^2} = g^2 (A_a)_\mu A_b^\mu T_{ik}^a T_{kj}^b \phi_i^* \phi_j \tag{1.104}$$

such that we get

$$i g^2 g^{\mu\nu} \{T^a, T^b\}_{ij} \tag{1.105}$$

1.5.2 Taking the Feynman rules for a spin

Fermion 2-point functions

We have obviously a straight line, but also a diagram with a gauge loop, and a counter term. In the Feynman gauge ($\xi = 1$), we have for the diagram with a gauge loop

$$\begin{aligned} &\int \frac{\mathrm{d}^4 k}{(2\pi)^4} i g T_{jk}^a \frac{i}{\not{k} - m + i\epsilon} i g \gamma^\mu T_{ki}^b i \frac{-g_{\mu\nu}\delta^{ab}}{(p-k)^2 + i\epsilon} \\ &= -g^2 \sum_a (T^a T^a)_{ji} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma_\mu (\not{k} + m) \gamma^\mu}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - i\epsilon} \end{aligned} \tag{1.106}$$

We notice that $\sum_a (T^a T^a)_{ji}$ is the casimir element, and is equal to $C(\pi)\delta_{ij}$. For exemple, in the defining representation of $\mathfrak{su}(n)$, $C_D = \frac{n^2-1}{2n}$.

Remark. It coincides with the QED fermion self-energy up to a group theory factor $C(\pi)\delta_{ij}$.

We can try to evaluate it in dimensional regularization. We first introduce the scale μ to keep g dimensionless. We replace g by $g\mu^{\frac{4-d}{2}}$ thanks to dimensional analysis. We have

$$-g^2 \mu^{4-d} C(\pi) \delta_{ij} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma_\mu (\not{k} + m) \gamma^\mu}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - i\epsilon} \tag{1.107}$$

Then we can do a bit of gamma algebra: $\gamma_\mu \gamma^\mu = g_\mu^\mu = d$. So

$$\gamma_\mu \not{k} \gamma^\mu = k_\rho \gamma_\mu \gamma^\rho \gamma^\mu = k_\rho (2 g^{\mu\rho} \gamma_\mu - \gamma^\rho \gamma^\mu \gamma_\mu) = (2-d)\not{k} \tag{1.108}$$

Hence

$$\gamma_\mu(\not{k}+m)\gamma^\mu = (2-d)\not{k}+dm \tag{1.109}$$

Then we introduce Feyman parameters. Recall that we have general formulas for integrals of the form

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^{2a}}{(k^2-\triangle)^b} \tag{1.110}$$

Using

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[A+(B-A)x]^2} \tag{1.111}$$

Writing $A = k^2 - m^2$, $B = (p-k)^2$,

$$A + (B-A)x = (k-xp)^2 - m^2(1-x) \tag{1.112}$$

Next shift the integration variable $k \to k + xp$, such that our expression becomes

$$-g^2\mu^{4-d}C(\pi)\delta_{ij}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(2-d)(\not{k}+x\not{p})+dm}{(k^2-\triangle+i\epsilon)^2} \tag{1.113}$$

with $\triangle = (m^2 - p^2 x)(1-x)$. We can then evaluate the whole expression in dimensional regularization,

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\triangle+i\epsilon)^2} = \frac{2}{(4\pi)^{d/2}}\frac{1}{\triangle^{2-d/2}}\Gamma(\frac{4-d}{2}) \equiv (*) \tag{1.114}$$

such that for the 1-loop term, we get

$$-g^2\mu^{4-d}C(\pi)\delta_{ij}\int_0^1 \mathrm{d}x((2-d)\not{p}x+dm)(*) \tag{1.115}$$

Now, we want to extract the divergences by setting $d = 4-\epsilon$ and by expanding around $\epsilon = 0^+$. We get

$$ig^2\mu^\epsilon C(\pi)\delta_{ij}\int_0^1 \mathrm{d}x((2-\epsilon)\not{p}x-(4-\epsilon)m)\frac{1}{(4\pi)^2}(4\pi)^{\epsilon/2}\triangle^{-\epsilon/2}(\frac{2}{\epsilon}-\gamma_E+\dots) \tag{1.116}$$

which, expanded, gives

$$ig^2C(\pi)\delta_{ij}\frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}x \tag{1.117}$$