Statistical field theory and applications

Buisine Léo
Ecole Normale Superieure of Paris

October 15, 2024

Contents

Chapter 1

Introduction

Email adress: adam.nahum@phys.ens.fr
TDman: xiangyu.cao@phys.ens.fr (he also has a website)

Books

1. Cardy, but condensed

Statistical field theory: just applying statistical mechanics idea to systems

1.1 Renormalization group (preview)

Very general tool from 20th century physics to understand complex systems (spatially extended and many degrees of freedom). Explains why QFT is so powerful in high energy physics and statistical physics

Example. Classical lattice models, modelizes magnetism for example

$$Z = \sum_{\{S\} e-\beta E(s)} \qquad \beta = \frac{1}{k_B T} \tag{1.1}$$

Example. Statistical field theory

$$Z = \int \mathcal{D}\varphi(x) e^{-\mathcal{H}(\varphi)} \tag{1.2}$$

Example. Qauntum lattice model, chain with spin up or down for example

$$Z = \mathrm{Tr}(e^{-\beta H}) \tag{1.3}$$

Example. Quantum field theories

$$Z = \int \mathcal{D}\varphi(x,t) e^{-\mathcal{S}(\varphi)} \tag{1.4}$$

with

$$\mathcal{S}(\varphi) = \int \mathrm{d}^d x \int_0^\beta \mathrm{d}t[(\partial_t \varphi)^2 + (\nabla \varphi)^2 \dots] \tag{1.5}$$

Renormalization group:
Key idea: The useful theoretical description is different at different scales.
Aim: understand how description changes as we "zoom out".
Objective of finding a universal description and forgetting useless information

A simple SFT and complicated microscopic model may be the same at large scales: might be easier to study the SFT in this case.

Consider a theory space (heuristic). We can imagine any kind of space of theories. We start with a point $\mathcal{H}$ in the space, which is a microscopic theory we know well. The objective is to then zoom out and try to describe the new zoomed out theory by a new theory in the theory space. By doing this and this and this again, we get a flow which eventually accumulates somewhere $\mathcal{H}_*$, which would be the most macroscopic description of the model. For example, the description of a molecule of water would end up to the navier-stokes equations. Then, maybe starting with another point (another theory), we can arrive at the same point (different fluids are all described by navier stokes equation). $\mathcal{H}_*$ has the property that when zooming out, it is fixed. Invariant under zooming out. Eventually, we find a lot of theories flowing to $\mathcal{H}_*$. It is called the Basin of attraction of $\mathcal{H}_*$. It leads to the notion of universality of $\mathcal{H}_*$.

If we go far enough, we might escape the basin of attraction of $\mathcal{H}_*$. We then may find another fixed point, $\mathcal{H}'_*$. Between two basins of attraction there is a limit, at which occurs a phase transition. Different phases just have different fixed points.

Different kind of phases:
Disordered vs Ordered

Example. Liquid vs Solid

Example. Paramagnet vs Ferromagnet

Phase transition theories also correspond to fixed points, but they are unstable fixed points.

The key notion is Universality. If we understand the fixed points, we understand a large class of models at once, wether the models are field theories, lattice models, magnets, gases, whatever

Remark. Status of RG: It is not a set of formula, but more of a paradigm very vague and heuristic. It is a way of thinking, that has to be tailored for each theory. Only in special cases we can follow the standard algorithmic procedure. There are some very special cases (ex: weakly interacting field theories)

1.2 The central limit theorem: toy exemple of RG

Let N identical independant random variables $X_1, \dots, X_N \in \mathbb{R}$. We can think of them as the degrees of freedom of the system. For exemple, we can imagine a chain where each point has a magnetisation described by X_i, and where each cell is non-interacting.

The "microscopic model" is the probability distribution of each X. We assume

$$< x >= 0 \qquad < x^2 >= \sigma^2 \tag{1.6}$$

There are a vast amount of such laws, from discrete to continous to non-compactly supported laws.

The "coarse-grained" variables are

$$X = \frac{1}{N}\sigma_{i=0}^{N} x_i \tag{1.7}$$

We zoom out of the chain. This variables will have distributions, $p_N(x)$.

The central limit theorem says that under some weak assumptions ($\sigma < \infty$), as $N \to \infty$, $p_N(x)$ converges pointwise to a gaussian law of standard deviation $\sigma/\sqrt{N}$.

Aside: Cumulants
CLT proved by computing cumulants

$$< e^{\mu x} >= 1 + \mu < x > + \frac{\mu^2}{2!} < x^2 > + \dots \tag{1.8}$$

The moments of x are the $< x^n >$, appearing in the developpement above.

$$< e^{\mu x} >= e\hat{}[\mu < x > + \frac{\mu^2}{2!}(< x^2 > - < x >^2) + \dots] \tag{1.9}$$

The terms in x are the cumulants.

In term of RG, in the space of distributions, a lot of distribution laws converge to a gaussian. However, if we go sufficiently far, we can avoid ending up in the gaussian limit. To do so, we can create interactions between the variables (but they must be sufficiently strong/long range). We can also have non identical distributions, and dominate everything with only a few of the variables. We can have a variance going to infinity (rare in physics, but it exists, ex: Levy-stable distributions).

1.3 Ising model and $O(N)$ model

Family of models important in the history of statistical field theory, still studied. It is rich enough to feature the key ideas of phase transitions and RG flows. It is classical stat mech (no plank constant).

$$Z = \int_{\text{configs}} e^{-\beta E} \tag{1.10}$$

The degrees of freedom in the $O(N)$ model are made by the spins vector $\vec{S}$:

$$\begin{aligned} &\vec{S} = (S^1, S^2, \dots, S^n) \\ &\vec{S}^2 = 1 \end{aligned} \tag{1.11}$$

We place such a spin at each point i of an hypercubic lattice in d dimensions.

$$-\beta E = J \sum_{<i,j>} \vec{S}_i \vec{S}_j \tag{1.12}$$

β is absorbed in J, J is dimensionless, $J > 0$ ferromagnetic, $< i, j >$ goes through the links of the lattice.

Low T implies high β implies large J. In this case, the energy is the lowest when the spins are aligned. The sign of J doesn't matter if the lattice is bipartite.

The model is called $O(N)$ because there is a global $O(N)$ symmetry, the global rotation of the spins. E is invariant under this symmetry. $O(N)$ is defined by $R^T R = 1 \Rightarrow$, there are two kinds of such matrices: rotations and reflections (times rotations).

To solve it, we can study

1. Mean field approximation
2. Real space RG
3. "$4 - \epsilon$" expansion
4. "$2 + \epsilon$" expansion
5. Large n approximation
6. Transfer matrix
7. Duality
8. ...

1.4 First look at the Ising model: order and disorder

We take $d = 2, n = 1$. We have $s \pm 1$
There is only a $\mathbb{Z}_2$ symmetry, $S_i \to -S_i$

$$Z_{\text{ising}} = \sum_{\{S\}} e^{J \sum_{<ij>} S_i S_j} \tag{1.13}$$

The simplest model having a transtion between ordered and disordered phase. If $J \to \infty$, all spins will align, either up or down. If $J = 0$, each spin will be independantly up or down. These are the two extreme limits of two different phases, separated by a critical point J_c.

How can we distinguish the phases?

- correlation function $< S_i S_j >$, for i and j far away.

For $J < J_c$:
It is the disordered phase

$$< S_i S_j > \sim e^{-r/\xi} \tag{1.14}$$

where r is the distance between i and j.
The exponential decay is characteristic of a weakly interacting system.
We can understand by perturbation in J.
ξ is the *correlation length*.

For $J > J_c$:
It is the ordered phase

$$\lim_{r\to\infty} < S_i S_j >= M^2 \tag{1.15}$$

with $M > 0$. It is the magnetisation.

$$\bar{S} = \frac{1}{L^2} \sum_i S_i \tag{1.16}$$

In the disordered phase, $\bar{S}$ follows a gaussian centered on 0, it is the CLT. In the ordered phase, $\bar{S}$ accumulates as gaussians on M or $-M$.

In the ordered phase, we consider the fluctuations around $\bar{S}$, which are exponentially decaying.

$$< S_i S_j >= M^2 + Ae^{-r/\xi} \tag{1.17}$$

We define $\delta S_i = S_i - \bar{S}$, with $\bar{S} = \pm M$. Then

$$< \delta S_i \ \delta S_j >= Ae^{-r/\xi} \tag{1.18}$$

At $J = J_c$

$$< S_i S_j > \simeq \frac{B}{r^{2x}} \qquad x = \frac{1}{8} \tag{1.19}$$

We can't understand this using the perturbation theory. We need to use RG.

We zoom out: We turn squares of 3 by 3 (9 cells) into 1 single square, with as spin the majority spin in the square of 9 spins. This is the majority rule. We suppose for simplicity that $L = \infty$. With this rescaling, we can heuristically expect that the correlation length gets reduced by 3.

If we do this from the disordered phase, we will approach the configurations of $J = 0$. Similarly, if we do this from the ordered phase, we will approach the configurations of $J \sim \infty$. If we start from the critical point, neither of these things happen. Actually, we approach a scale invariant configuration. There are non-trivial correlations that don't change under coarse-graining.

Chapter 2

Lattice techniques

1. high T
2. low T
3. Duality (preview)
4. transfer matrix (1D)
5. Preview of quantum-classical correspondance

2.1 High T

The High T expansion: two key ideas

1. This is a perturbative expansion from $T = \infty$, so $(J = 0)$
2. high T expansion is a geometrical way of thinking about corrolations of spins

2.1.1 Free energy

We consider a 2D square lattice without periodic boundary condition. We write $N = L^2$ the number of spins, and $N_b = 2L^2$ the number of bounds. The partition function takes the formula

$$\begin{aligned} Z &= e^{-\beta F} \\ &= \sum_{\{S\}} e^{J \sum_{<i,j>} S_i S_j} \\ &= \sum_{\{S\}} \prod_{<i,j>} e^{J S_i S_j} \end{aligned} \tag{2.1}$$

observe that

$$e^{J S_i S_j} = \cosh(J)[1 + \tanh(J S_i S_j)] \qquad \forall S_i, S_j = \pm 1 \tag{2.2}$$

Plugging this in,

$$Z = \sum_{\{S\}} \prod_{<i,j>} (1 + \mu S_i S_j) \times (\cosh(J))^{N_b} \tag{2.3}$$

where

$$\mu \equiv \tanh(J) \sim J + O(J^2) \tag{2.4}$$

We want to expand $\prod_{<i,j>}(1+\mu S_i S_j)$ geometrically. We say a bond is occupied if we have $\mu S_i S_j$ on it. We call the collection of occupied bonds a graph, written G.

$$Z = (\cosh(J))^{N_b} \sum_{\{S\}} \sum_{G} \prod_{<i,j>} \mu S_i S_j \tag{2.5}$$

Then we sum over all spin configurations. We notice by some symmetry arguments that a graph leads a term 0 unless all sites are adjacent to an even number of occupied bonds. So G survives iff G is "closed". In general,

$$Z = (\cosh(J))^{N_b} 2^N \sum_{G \text{ closed}} \mu^{|G|} \tag{2.6}$$

where $|G|$ is the number of closed bounds in G. This is exact on finite lattices, with

$$u = \tanh(J) \qquad J = \frac{J_{\text{phys}}}{T} \tag{2.7}$$

Now, we want to expand f.

$$Z = e^{-\beta F} \qquad F = -\frac{1}{\beta} \ln(Z) \tag{2.8}$$

$$f = \lim_{L \to +\infty} \frac{F}{L^2} \tag{2.9}$$

Using what we found above,

$$-\beta F = \ln(Z) = N \ln(2\cosh(J)) + \ln(\sum_{G \text{ closed}} \mu^{|G|}) \tag{2.10}$$

Thinking about the graphs possible, we get

$$\sum_{G} \mu^{|G|} = 1 + N\mu^4 + 2N\mu^6 + O(\mu^8) \tag{2.11}$$

So

$$\ln(\sum_{G \text{ closed}} \mu^{|G|}) = N(\mu^4 + 2\mu^6) + O(\mu^8) \tag{2.12}$$

So

$$-\beta f = \ln(2\cosh(J)) + \mu^4 + 2\mu^6 + O(\mu^8) \tag{2.13}$$

If we tried to expand μ more, we would get disconnected graphs, and as such we would have a factor in N^2, which is not extensive. f would explode when going to $N \to 0$. As such, there must be some cancellation of these powers when taking the log.

At the order of μ^8, there should be a term in $\frac{1}{2}N^2\mu^8$. How does it cancels out? Idk, see lecture notes

Remark. 1. $f \to^{\text{derivatives}} E, C_v$

2. high order high-T expansion is exact and allows to probe the model at $J \sim< J_c$
3. high-T expansion is general (eg 3D) but in 2D the sum over the closed graphs can be done exactly.
4. For $O(n)$ model etc with continuous degrees of freedom the high-T expansion can still be done (see tutorial)
5. Useful for Monte-Carlo

2.1.2 high-T expansion of correlation

$$\begin{aligned} < S_i S_j > &= \frac{\sum_S e^{-H} S_i S_j}{\sum_S e^{-H}} \\ &= \frac{\sum_S e^{-H} S_i S_j}{2^N (\cosh(J))^{N_b} \sum_G \mu^{|G|}} \end{aligned} \tag{2.14}$$

we focus on

$$\begin{aligned} \sum_S e^{-H} S_i S_j &= (\cosh(J))^{N_b} \sum_G \sum_{\{S\}} S_i S_j \prod_{<k,l> \in G} u S_k S_l \\ &= 2^N (\cosh(J))^{N_b} \sum_{G(i \leftrightarrow j)} \mu^{|G|} \end{aligned} \tag{2.15}$$

So now survive the same graphs as above, but where the sites i and j have an odd number of occupied bounds instead of even number.

So we have

$$< S_i S_j >= \frac{\sum_{G(i \leftrightarrow j)} \mu^{|G|}}{\sum_G \mu^{|G|}} \tag{2.16}$$

For $\mu \to 0$,

$$< S_i S_j >= \binom{x+y}{y} u^{x+y} + O(u^{x+y+2}) \tag{2.17}$$

where $\binom{x+y}{y}$ corresponds to the number of possible shortest paths from i to j. Using Stirling's formula, we have

$$< S_i S_j >= e^{r/\xi(\theta)} \frac{A(\theta)}{\sqrt{r}} \tag{2.18}$$

with

$$\frac{1}{\xi(\theta)} = (c+s)\ln(\frac{1}{\mu}) + c\ln(c) + s\ln(s) - (c+s)\ln(c+s) \tag{2.19}$$

where θ is the angle formed by (x, y), and where

$$c = \cos(\theta) \qquad s = \sin(\theta) \tag{2.20}$$

There is no rotation symmetry!

2.2 Low T expansion

$$Z = \sum_{\{S\}} e^{J \sum_{<i,j>} S_i S_j} \tag{2.21}$$

We want to describe the things in term of domain walls, which are graphs on the dual lattice. The domain walls are the walls between 2 different spins

$$\begin{aligned} Z &= e^{JN_b} \sum_S \prod_{<i,j>} e^{S_i S_j - 1} J \\ &= e^{JN_b} \sum_{\{S\}} (e^{-J})^{|D.W|} \end{aligned} \tag{2.22}$$

where $|D.W|$ is the length of the domain wall.

$$Z = 2e^{JN_b} \sum_{\{G\}} \tilde{\mu}^{|G|} \tag{2.23}$$

with $\tilde{\mu} = e^{-J}$. This is basically the low T expansion.

If $T \to 0, J \to +\infty, \tilde{\mu} \to 0$, we have

$$Z = 2e^{JN_b}(1 + N\tilde{\mu}^4 + O(\tilde{\mu}^6)) \tag{2.24}$$

2.3 Glimpse of Kramers-Wannier duality

1. High T: $Z = (2\cosh^2(J)^{N_b})^N \sum_{G \text{ closed}} u^{|G|}$
2. Low T: $Z = 2e^{2JN} \sum_{G \text{ closed (on dual lattice)}} \tilde{u}^{|G|}$

So they are essentially the same, except for a factor in front. From this, we can find the critical point (will be done in TD)

2.4 Transfer Matrix

- Simple method with profound consequences
- underlies connection between classical stat mech in d spatial dimensions and quantum stat mech in $(d-1)$ spatial dimensions
- 1D solves any problem with finite number of states per site

We consider the Ising model on a ring of size L

$$\begin{aligned} Z &= \sum_{S_1,\dots,S_L} e^{JS_1S_2} \dots e^{JS_LS_1} \\ &= \sum_{S_1,\dots,S_L} M_{S_1S_2} \dots M_{S_LS_1} \end{aligned} \tag{2.25}$$

with

$$M = \begin{pmatrix} M_{1,1} & M_{1,-1} \\ M_{-1,1} & M_{-1,-1} \end{pmatrix} = \begin{pmatrix} e^J & e^{-J} \\ e^{-J} & e^J \end{pmatrix} \tag{2.26}$$

$$Z = \sum_{S_i} (M^L)_{S_i S_i} \tag{2.27}$$

$$Z_{PBC} = \mathrm{tr}(M^L) \tag{2.28}$$

M is real symmetric zith eigenvalues $\lambda_1 \geq \lambda_2$

$$\begin{aligned} M &= |1\rangle\lambda_1\langle 1| + |2\rangle\lambda_2\langle 2| \\ M &= |1\rangle\lambda_1^L\langle 1| + |2\rangle\lambda_2^L\langle 2| \end{aligned} \tag{2.29}$$

$$Z_{PBC} = \lambda_1^L + \lambda_2^L \sim \lambda_1^L \tag{2.30}$$

free energy density f:

$$Z \sim e^{-\beta L f} \tag{2.31}$$

$$f = -\frac{1}{\beta}\ln(\lambda_1) = -\frac{1}{\beta}\ln(2\cosh(J)) \tag{2.32}$$

2.4.1 Correlation functions (PBC)

We want $< S_x S_{x+r} >$

$$< S_x S_{x+r} >= \frac{1}{Z}\sum_{\{S\}} e^{JS_1S_2}\ldots e^{JS_LS_1} S_x S_{x+r} \tag{2.33}$$

We can make it as a matrix

$$< S_x S_{x+r} >= \frac{1}{Z}\sum_{\{S\}}\sum_{S'_x} e^{JS_1S_2}\ldots e^{S_{x-1}S_x} S_x \delta_{S_x,S'_x} e^{S_xS_{x+1}}\ldots e^{JS_LS_1} S_{x+r} \tag{2.34}$$

$$\mathcal{S}_{S_x,S'_x} = S_x\delta_{S_x,S'_x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.35}$$

which gives

$$< S_x S_{x+r} >= \frac{1}{Z}\mathrm{tr}(\mathcal{S}M^r\mathcal{S}M^{L-r}) \tag{2.36}$$

$\underline{L \to \infty}$:

$$\begin{aligned} M^L &\sim |1\rangle\lambda_1^L\langle 1| \\ M^r &= |2\rangle\lambda_2^r\langle 2| + |1\rangle\lambda_1^r\langle 1| \end{aligned} \tag{2.37}$$

such that

$$< S_x S_{x+r} >= \left(\frac{\lambda_2}{\lambda_1}\right)^r |\langle 1|\mathcal{S}|2\rangle|^2 \tag{2.38}$$

So

$$\lim_{L\to\infty} < S_1 S_{1+r} >= Ae^{-r/\xi} \tag{2.39}$$

with

$$\xi = \frac{1}{\ln(\lambda_1/\lambda_2)} \tag{2.40}$$

The ratio of the leading and subleading eigenvalues given the correlation length.

In a more general way, if M is a $d \times d$ real symmetric matrix, assuming

$$\lambda_1 > \lambda_2 > \lambda_3 > \ldots \tag{2.41}$$

given an observable O_x which maps to a matrix $\mathcal{C}$, we have

$$\lim_{L\to\infty} < O_x >= \langle 1|\mathcal{C}|1\rangle \tag{2.42}$$

$$\lim_{L\to\infty} [< O_x O_{x+r} > - < O_x >< O_{x+r} >] = \sum_{\mu=2}^{d} \left(\frac{\lambda_\mu}{\lambda_1}\right)^r \dots \tag{2.43}$$

2.5 Preview of quantum-classical correspondance

Reinterpret space in D ising model as "imaginary time" (PBC).

$$Z_{1D,\text{ classical}} = \mathrm{Tr} M^L \tag{2.44}$$

$$M = \begin{pmatrix} e^J & e^{-J} \\ e^{-J} & e^J \end{pmatrix} \tag{2.45}$$

Take $J >> 1$.

$$M = e^J \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \tag{2.46}$$

with $\varepsilon = e^{-2J}$

$$Z_{1D,\text{ classical}} = e^{JL} \hat{Z}_{qm} \tag{2.47}$$

with

$$\hat{Z}_{qm} = \mathrm{Tr}(e^{\varepsilon L \sigma^x}) \tag{2.48}$$

We define

$$\beta_{qm} = \varepsilon L \tag{2.49}$$

and

$$\hat{H}_{qm} = e^{\sigma^x} \tag{2.50}$$

$$Z_{1D,\text{classical}}(J, L) = e^{JL} \hat{Z}_{qm,0D}(\beta_{qm}) \tag{2.51}$$

with

$$\hat{Z}_{qm,0D} = \mathrm{Tr}(e^{-\beta_{qm}\hat{H}}) \tag{2.52}$$

The Cardy Book discusses this in more details.

Chapter 3

RG Formalism via real space RG

Yeomans Cardy chapter 3

3.1 Block spins

We descale the space, and associate blocks of spins to a single spin on a new grid, which corresponds to the descaled grid. For exemple, on a square lattice, we replace 9 spins with 1. S_I is a function of S_i for $i \in I$. We also rescale the metric such that the lattice spacing is always 1. Here, we divide all metrics by 3.

3.2 Linear RG near fixed points

Take $1 >> \delta K = K - K_*$. We suppose there is a single parameter in the space of theories.

$$\begin{aligned} K' &= R(K) = R(K_* + \delta K) \\ &= K_* + R'(K_*)\delta K + O(K^2) \\ &= K_* + \delta K' \end{aligned} \tag{3.1}$$

with $\delta K' = R'(K_*)\delta K$.

Definition 3.2.1. "RG eigenvalue": y where

$$R'(K_*) = b^y \tag{3.2}$$

$$y = \frac{\ln(R'(K_*))}{\ln(b)} \tag{3.3}$$

Then

$$\delta K' = b^y \delta K + O(\delta K^2) \tag{3.4}$$

if $\mathrm{Re}(y) > 0$ then δK goes up under RG. y is a universal quantity which only depends on fixed point theory. It does not depend on b, not on RG scheme, not on model we start with.

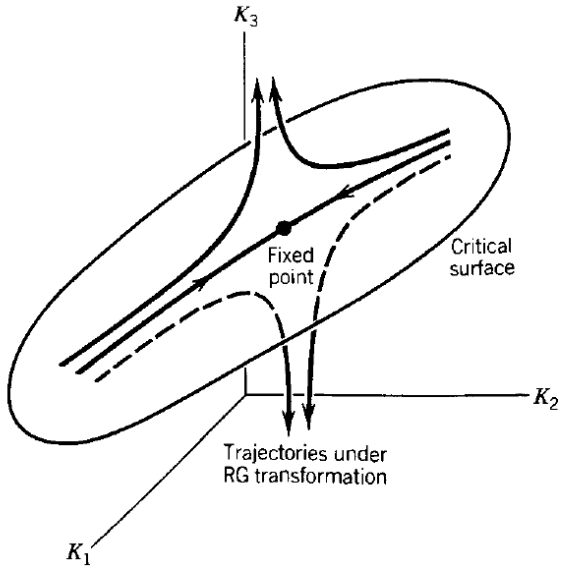

Figure 3.1: Critical surface

3.3 Correlation length

Define $\nu = 1/y$ the correlation length exponent. We can find

$$\xi \sim |\delta K|^{-\nu} = \frac{c_{\pm}}{|\delta K|^{\nu}} \tag{3.5}$$

3.4 Adding a magnetic field

(no new calculation required in our approximation). Consider $h << 1$.

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{3.6}$$

$$e^{-\mathcal{H}'[S']} = \sum_{S \to S'} e^{-\mathcal{H}_0[S]} e^{-\mathcal{H}_i[S] + h\sum_i S_i} \tag{3.7}$$

So

$$\mathcal{H}'[S'] = \frac{N}{3} \ln Z_{\Delta} + K \sum_{<ij>} < S_i >_{S'_I} < S_j >_{S'_J} + h \sum_i < S_i >_{S'_I} \tag{3.8}$$

We easily read

$$h' = 3\gamma_K h + O(h^3) \tag{3.9}$$

where we remanining is $O(h^3)$ and not $O(h^2)$ due to the $\mathbb{Z}_2$ symmetry in h. We find

$$y_h > y_t > 0 \tag{3.10}$$

which means that the system is more sensible to the magnetic field than to the temperature.

If the critical surface has "codimension" k:

1. There are k relevant perturbations
2. Experimentalists must tune k parameters to acoss critical surface

3.5 Extensions

We can experiment with modifications

- Square lattice
- have the possibility of "vacancies", $S = \pm 1, 0$ (we had a term in S_i^2 to the Hamiltonian)
- models with more states (ex Potts model, percolation)
- have boundaries, boundary conditions

3.6 Symmetries and RG flows

Recall Ising $\mathbb{Z}_2$ symmetry.

1. $\mathbb{Z}_2$-even couplings: $K \sum_{<ij>} S_i S_j$, etc
2. $\mathbb{Z}_2$-odd couplings: $K \sum_i S_i$, etc

Linearized RG equation

$$\delta K' = T \delta K + O(\delta K^2) \tag{3.11}$$

Previously $\underline{\delta K} = (\delta K, h)$

$$T = \begin{pmatrix} b^{y_t} & 0 \\ 0 & b^{y_h} \end{pmatrix} \tag{3.12}$$

More generally (more couplings), we have two blocks

$$T = \begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} \tag{3.13}$$

associated to the even and odd sectors. There is a hierarchy of eigenvalues in each sector. There is a theorem which states there is only one relevant parameter per sector?

3.6.1 Emergent symmetries

RG preserves symmetries of $\mathcal{H}$ (should not lose a symmetry by zooming out). However, RG might generate new symmetries asymptotically (at fixed points). For exemple, conformal symmetry is emergent in critical ising model.

3.6.2 $\mathbb{Z}_2$ as an emergent symmetry

Recall: $\mathbb{Z}_2$ symmetry means $\mathcal{H}[S] = \mathcal{H}[-S]$.

In 3D, the Ising model has only 2 relevant perturbations, δT and h.

Interesting consequence for models that do not have microscopic $\mathbb{Z}_2$ but which have 2 experimental tuning parameters. Even though we cannot reach the critical point directly (we cannot have the $\mathbb{Z}_2$ symmetry microscopically), we can fix ourselves on the critical surface, which will flow to the critical point (we will have the $\mathbb{Z}_2$ symmetry macroscopically).

3.6.3 Scaling operators

Remember the idea of scaling variables. We start with

$$\begin{aligned} \mathcal{H} &= K_1 \sum SS + K_2 \sum SSS + \dots \\ &= H_*[S] + \sum_i K_i \sum \dots \end{aligned} \tag{3.14}$$

And linearizing the action of the RG on δK_i, we have

$$\delta K' = T \delta K \tag{3.15}$$

We can change the coordinates $\delta K_i \to u_\alpha = M^{-1}_{\alpha i} \delta K_i$ to diagonalize the transformation, ie

$$u'_\alpha = b^{y_\alpha} u_\alpha \tag{3.16}$$

We then have

$$\mathcal{H} = H_*[S] + \sum_\alpha u_\alpha \sum_r O_\alpha(r) \tag{3.17}$$

where r is the sum over positions, and O_α is a somewhat local operator in the spins near r.

The O_α are called the scaling operators. They have simple behaviour under the RG, and simple correlation functions. Good basis of operators at fixed points.

Consider a coarse graining by a factor b. $R = r/b$. $O'_\alpha(R)$ is some function but of S' instead of S.

Theorem 3.6.1.

$$O_\alpha(r) \sim b^{-x_\alpha} O'_\alpha(R) \tag{3.18}$$

- *x_α is the scaling dimension of O_α*
- *$\sim$: equivalent for correlation fnctions at distances » lattice spacing*
- $x_\alpha = d - y_\alpha$

Example. Previous Ising calculation . Within our approx, h was a scaling variable and the corresponding scaling operator was just S_r. We found that

$$< S_r >_{S'_R} = \gamma S'_R \tag{3.19}$$

In correlation function, $S_r \sim \gamma S'_R$. Thus at large distance,

$$< S_r S_{r'} > \sim \gamma^2 < S_R S_{R'} > \tag{3.20}$$

3.6.4 Relation between RG eigenvalues y_α and scaling dimensions x_α

$$O_\alpha(r) = b^{-x_\alpha} O'_\alpha(R) \tag{3.21}$$

is consistent if and only if $x_\alpha = d - y_\alpha$.

Example.

$$O(r) = S_r \qquad u = h \tag{3.22}$$

Previously we found $b^y = 3\gamma$, but $3 = b^d$ and $\gamma = b^{-x}$

3.6.5 off-cricitcal correlators, "scaling forms"

$$G^{\text{lattice}}(r, T) \simeq G(r, u_t) \tag{3.23}$$

we ignore irrelevant couplings, $u_t \propto T - T_c$

$$G(r, u_t) = b^{-2x_\alpha} G(r/b, b^{y_\alpha} u_t) \tag{3.24}$$

We can see better the solution by setting $b = r$. Claim: it is a function of $r/\xi(u_t)$, with

$$\xi(u_t) = u_t^{-\frac{1}{y_\alpha}} \tag{3.25}$$

$$u_t = \xi^{-y_\alpha} \tag{3.26}$$

So

$$G_\alpha(r, T) = r^{-2x} H(T/\xi(T)) \tag{3.27}$$

with

$$\frac{c}{|T - T_c|^\nu} \tag{3.28}$$