General Relativity

Buisine Léo
Ecole Normale Superieure of Paris

September 25, 2024

Contents

Chapter 1

Introduction

Danièle Steer: daniel.steer@phys.ens.fr
work on theories of gravity, gravitational waves, primordial cosmology

We should do the TD before the TD.

This will be a course more physical than mathematical. No differential geometry or Cartan geometry. The TD will be more formal. The objective is to understand current experiments and current measures. As such, we will mostly stay outside of black holes horizons. We will also use cevtors in coordinate basis. We should remember that physics is independant of the choice of coordinates. The objective is to be able to do accurate computations.

Aims:

1. Handle GR and its tools
2. Be able to do accurate calculations
3. Have a feeling for different works in GR today

The current GR fails at cosmological scale. It doesn't explain the Hubble constant, Dark energy nor Dark matter. Many theories try to modify GR, but many of these modified gravities are constantly ruled out by new experiments and new data. For exemple, a whole lot of theories were ruled out in 2017 by an experiment showing that

$$\left|\frac{c_{GW} - c}{c}\right| < 10^{-15} \tag{1.1}$$

The Newtonian gravity potential on the surface of a body of mass M and radius R is given by

$$\Phi_N = \frac{GM}{R} \tag{1.2}$$

The Newtonian gravity is valid in the limit $\Phi_N << 1$.

1. On earth, we have $\Phi_N \sim 10^-9$
2. On the sun, we have $\Phi_N \sim 10^-6$

3. On a neutron star, we have $\Phi_N \sim 10^-1$
4. In a black hole, there is no upper limit

Semi classical gravity is doing QFT in curved spacetime. The idea is to start with a classical solution of Einstein's equation. Then we quantise the field in this metric. Then we can loop, compute the change in the metric due to the field, then re quantise the field, etc... The result of this computation gives the famous Hawking temperature of black holes

$$T_H = \frac{hbarc^3}{8\pi G M k_B} \tag{1.3}$$

For a BH of mass of the sun M_Θ, $T_H \sim 10^{-8}$K. This causes the decay of black holes such as primordial black holes.

GR is a very unique theory. Therem of lovelock (1972): Einstein's equations are the unique 2nd order local equation of motion for a metric $g_{\mu\nu}$ derivable from an action in 4D.

So if we want to go beyond GR, we can either work in more than 4D, add extra fields (ex scalar-tensor theories of gravity), we can try higher order equations of motion, or we can remove the locality.

Chapter 2

Overview

2.1 Special Relativity

For our Minkowski metric, we will take the signature $(-,+,+,+)$. SR describes all forces apart from gravity, and they live on a fixed Minkowski spacetime.

In SR, there is a class of globally defined non-accelerating inertial observers, with associated pseudo-cartesian coords

$$x^\mu = (ct, x, y, z) \qquad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \tag{2.1}$$

IN GR, gravitational force is a manifestation of curvature $\eta_{\mu\nu} \to g_{\mu\nu}(x)$, given by Einstein's equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.2}$$

We have $\frac{8\pi G}{c^4} \sim 10^{43}$. $G_{\mu\nu}$ is intrisicly related to the curvature of spacetime, by the RIemann tensor for exemple. It has dimension inverse length square? $T_{\mu\nu}$ is the energy-momentum tensor.

We have 2nd order non-linear PDE for $g_{\mu\nu}(x)$. To find the solutions, we can impose symmetries

1. Cylinder symmetry: string-like BH
2. Time independant spherical symmetry: Schwarchild BH
3. Planar symmetry: Domorin-Wall cosmological solution in 5D, it is a modified gravity theory

Another idea is to do perturbative physics, and to find perturbative solutions.

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \tag{2.3}$$

$\bar{g}_{\mu\nu}$ is a known solution, but can be a lot of things. Either Minkowski spacetime, either Schwarchild blackhole, either FRWL, ...

Another possibility to solve the equations is when $\Phi_N << 1$, where we have

$$\nabla^2 \Phi = 4\pi G \rho \tag{2.4}$$

with ρ the mass density.

Geodesic equation for test masses, subject only to gravity.

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \tag{2.5}$$

equation for $x^\mu(\lambda)$, the world line of the test mass ($m > 0$), where λ is the time perceived by the body following the geodesic, an (often, but not always: see 1st TD) affine parameter of the proper time.

The Christoffel symbol is given by

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \tag{2.6}$$

In the limit $\Phi_N << 1$, it reduces to $\vec{a} = -\vec{\nabla}\Phi_N$. This is independant of the composition of the particle (universal equation). It is a consequence of $m_i = m_g$. THe current experiments say that

$$\left| \frac{m_i - m_g}{m_g} \right| \sim 10^{-15}$$

Weak equivalence principle: $m_i = m_g$.

In GR, a consequence of the WEP is that locally, gravity and acceleration are indistinguishable. Locally, the effect of gravity can be removed by going to a locally freely falling frame (LFFF).

Einstein's EP: in the LFFF, the laws of physics are those of SR. Locally, $g_{\mu\nu} \sim \eta_{\mu\nu}$.

2.2 Rindler's space

A few work on SR and Rindler Space

- $x^\mu = (t, x, y, z)$ associate to globally defined non accelerating inertial observer
- $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +)$
- different inertial frames related by Lorentz transformations $x'^\mu = \Lambda^\mu_\nu x^\nu$
- Free massive particles satisfy $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0$ where τ is the proper time, $ds^2 = -d\tau^2$

Now let's work in general curvilinear coords

$$x^\mu \to y^\mu(x^\alpha)(\text{assumed invertible}) \tag{2.7}$$

Example. spherical polar coords $y^\alpha = (t, r, \theta, \phi)$

Example. Rindler: $y^\alpha = (\eta, \rho, y, z)$

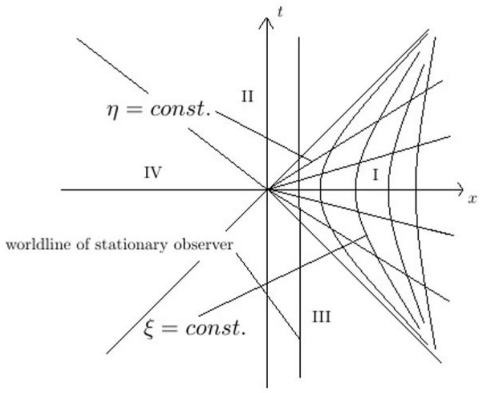

Figure 2.1: Rindler's observers

Metric:

$$ds^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \mathrm{d}y^\alpha \mathrm{d}y^\beta \tag{2.8}$$

with

$$\eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} = g_{\alpha\beta}(y) \tag{2.9}$$

The inverse is

$$g^{\alpha\beta} = \eta^{\mu\nu} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \tag{2.10}$$

Check

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \tag{2.11}$$

In the new coord system, $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0$. So

$$\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}\tau^2} + \tilde{\Gamma}^\mu_{\alpha\beta} \frac{\mathrm{d}y^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}y^\beta}{\mathrm{d}\tau} = 0 \tag{2.12}$$

where

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{\partial y^\mu}{\partial x^\omega} \frac{\partial^2 x^\omega}{\partial y^\alpha \partial y^\beta} = \frac{1}{2} g^{\mu\epsilon} (\partial_\alpha g_{\epsilon\beta} + \partial_\beta g_{\epsilon\alpha} - \partial_\epsilon g_{\alpha\beta}) \tag{2.13}$$

Geodesic equation, but note that we are in flat spacetime!

Now consider a particular class of non-inertial observers, namely one with constant eternal proper acceleration along x axis $1/r_0$ with $r_0 > 0$. (this is very unphysical).

$$A^\mu A_\mu = \text{const} = 1/r_0^2 \tag{2.14}$$

$$A^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}, \qquad u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = (\gamma, \gamma v, 0, 0) \tag{2.15}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \quad v = \frac{\mathrm{d}x}{\mathrm{d}t} \quad a = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{2.16}$$

We can send these Rindler observers in space, resulting in the trajectories

$$\begin{aligned} t &= r_0 \sinh(\eta) \\ x &= r_0 \cosh(\eta) \end{aligned} \tag{2.17}$$

with $\eta = \tau / r_0$

The Rindler space is the space occupied by the trajectory of the Rindler observers, corresponding to the right wedge of the Minkowski spacetime.

This allows us to parametrize the right wedge of the Minkowski spacetime, as

$$\begin{aligned} t &= \rho \sinh(\eta) \\ x &= \rho \cosh(\eta) \end{aligned} \tag{2.18}$$

These give the Kindler coordinates (ρ, η, y, z) with $\rho > 0$ and $-\infty < \eta < +\infty$. In these coords,

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{2.19}$$

Sometimes, we write $\rho = e^{\chi}$ in which case

$$\mathrm{d}s^2 = e^{2\chi} \left(-\mathrm{d}\eta^2 + \mathrm{d}\chi^2\right) + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{2.20}$$

Now, let's look at the Schwarchild metric. For $r > r_s$

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 + \frac{1}{(1 - \frac{r_s}{r})} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \tag{2.21}$$

where $r_s = 2GM$ and

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta) \mathrm{d}\phi^2 \tag{2.22}$$

We want to consider the Schwarchild metric very near the BH horizon. We expand at first order $r = r_s + \varepsilon$

$$\mathrm{d}s^2 = -\frac{\varepsilon}{r_s} \mathrm{d}t^2 + \frac{r_s}{\varepsilon} \mathrm{d}\varepsilon^2 + r_s^2 \mathrm{d}\Omega^2 \tag{2.23}$$

Let

$$\begin{aligned} \rho &= 2\sqrt{r_s \varepsilon} \\ \mathrm{d}\rho &= \sqrt{\frac{r_s}{\varepsilon}} \mathrm{d}\varepsilon \end{aligned} \tag{2.24}$$

We have

$$\mathrm{d}s^2 = -\frac{\rho^2}{4r_s^2} \mathrm{d}t^2 + \mathrm{d}\rho^2 + r_s^2 \mathrm{d}\Omega^2 \tag{2.25}$$

So with

$$\eta = \frac{t}{2r_s} \tag{2.26}$$

We get

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + r_s^2 \mathrm{d}\Omega^2 \tag{2.27}$$

Which is, for the particular part, exactly the Rindler metric. So the Rindler space modelizes at first order the exterior near horizon of a black hole.

Chapter 3

Formalism

GR is based on 4 postulates

1. Spacetime is 4D Lorentzian manifold, equipped with a Levi-Civita connexion ($\Gamma^{\mu}_{\alpha\beta}$)
2. Free particles follow time-like or null geodesics ($m > 0, \quad m = 0$)
3. The energy, momentum and stresses of matter are described by a symmetric tensor $T_{\mu\nu}$, which is covariantly conserved $\nabla^{\mu} T_{\mu\nu} = 0$
4. The curvature of space time is related to $T_{\mu\nu}$ via the Einstein's equation $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

Differentiable manifold, scalars, vectors, tensors, forms

1. In GR, space time has a curved geometry; in general no particular symmetry and hence no preferred set of coords
2. Furthermore, a single set of coords is not always sufficient to describe all of spacetime (ex. Schwarchild)
3. Manifold structure gives us the framework for smoothly meshing these coordinates together; and that any point p on $\mathcal{M}$ can be thought in term of curvilinear coordinates on $\mathbb{R}^n$ (if $\mathcal{M}$ has dimension n)

$$\begin{aligned} x(p) &= (x^1(p), x^2(p), \ldots, x^n(p)) \\ x'(p) &= (x^{'1}(p), x^{'2}(p), \ldots, x^{'n}(p)) \end{aligned} \tag{3.1}$$

   and the map between x and x' is smooth and invertible

4. Scalars are invariant under coord transformations
5. vectors are abstractly ($\bar{x} = x^{\mu}\partial_{\mu}$), but we will think of them as (x^{μ}). They are defined on the tangent space T_p at point p, for which a natural coordinate basis is $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$. $\bar{x}$ is invariant under coordinate transformations. Thus, under the change $x \to x'$, we have

$$x^{'\mu}(x) = \frac{\partial x^{'\mu}}{\partial x^{\alpha}} x^{\alpha}(x) \tag{3.2}$$

6. covectors have as natural coord basis $\mathrm{d}x^\mu$; $\omega = \omega_\mu \mathrm{d}x^\mu$. We have under the transformation of coordinate

$$\omega'_\mu(x) = \frac{\partial x^\alpha}{\partial x'^\mu}\omega_\alpha(x) \tag{3.3}$$

7. (k, l) tensors transform as k vectors and l covectors.

8. Principle of covariance: physical laws must be the same for all observers whatever their coord system. They must preserve their form under general coord transformations. For exemple, the equation for geodesics is invariant under coordinate systems

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \tag{3.4}$$

Writing it out gives us the transformation law of Γ. We find

$$\Gamma'\nu_{\alpha\beta} = \frac{\partial x'^\nu}{\partial x^\omega}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\lambda}{\partial x'^\beta}\Gamma'\omega_{\mu\lambda} + \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\lambda}{\partial x'^\beta}\frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\lambda} \tag{3.5}$$

In particular, Γ is not a tensor.

3.1 (Anti-) symmetrisation of tensors

.
Given some tensor $B_{\mu\nu}$, we can symmetrize it by taking

$$S_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) \tag{3.6}$$

we can antisymmetrize it by taking

$$A_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}) \tag{3.7}$$

For a tensor with n indices, we consider permutations, and we have

$$T_{(\mu_1,\ldots,\mu_n)} = \frac{1}{n!}\sum_{\text{perm }\sigma} T_{\sigma(\mu_1),\ldots,\sigma(\mu_n)} \qquad T_{[\mu_1,\ldots,\mu_n]} = \frac{1}{n!}\sum_{\text{perm }\sigma} \mathrm{sign}(\sigma) T_{\sigma(\mu_1),\ldots,\sigma(\mu_n)} \tag{3.8}$$

where we write a tensor with indices in () for a symmetrized tensor, and in [] for an antisymmetrized tensor.

By convention, we write

$$T^{(\mu_1\mu_2)}{}_{[\mu_3,\mu_4](\mu_5|\mu_6|\mu_7)} \tag{3.9}$$

a tensor symmetric in $\mu_1 \leftrightarrow \mu_2$ and in $\mu_5 \leftrightarrow \mu_7$, and antisymmetric in $\mu_3 \leftrightarrow \mu_4$

3.2 Derivatives

Since we haven't introduced a metric, we only have ∂_μ.

1. $\partial_\mu f$ transforms like a vector
2. $\partial_\mu A^\nu$ does not transform as a $(1,1)$ tensor

there is a class of tensors for which ∂_μ does take tensors to $(k, l+1)$ tensors. These are totally antisymmetric $(0,p)$ tensors $\omega_{\mu_1,\dots\mu_p} = \omega_{[\mu_1,\dots\mu_p]}$ known as p-forms. In dimension n, there are $\binom{n}{p}$ independant functions in a p-forms.

The exterior derivative d takes a p-form to a $(p+1)$-form

$$(\mathrm{d}A)_{\mu_1,\dots,\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2,\dots,\mu_{p+1}]} \tag{3.10}$$

$\mathrm{d}A$ is a tensor. Also, $\mathrm{d}^2 = 0$. GR cannot be only formulated in term of forms.

3.3 Metric and tensor densities

1. Metric gives a notion of distance on the manifold
2. $(0,2)$ symmetric tensor $g_{\mu\nu} = g_{\nu\mu}$
3. non degenerate $g \equiv \det(g_{\mu\nu}) \neq 0$ (the det is not a scalar, it changes under coord transforms)
4. $g_{\mu\nu}$ can be diagonalized, diag elements are non-zero. We can rescale basis vectors, so the values are ± 1. Lorentzian manifolds have signature $(-++\dots)$
5. Inverse metric $g^{\mu\nu}$ such that $g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha$
6. Metric and inverse, raise and lower indices
7. $A^\mu = g^{\mu\nu}A_\nu$
8. The distance between two space time points is invariant, distance between x^μ and $x^\mu + \mathrm{d}x^\mu$ is $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$
9. How does $g = \det(g_{\mu\nu})$ change under coord transforms?

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} g_{\alpha\beta} \frac{\partial x^\beta}{\partial x'^\nu} \tag{3.11}$$

Writing $J^\alpha_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}$, we have $g' = J^\dagger g J$, such that

$$\det(g') = \det(J)^2 \det(g) \tag{3.12}$$

So

$$\sqrt{-g'} = \sqrt{-g}\left|\frac{\partial x^\alpha}{\partial x'^\beta}\right| \tag{3.13}$$

Since physics is coordinate invariant, any action S must be a scalar, and thus must be of the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(\phi, \nabla\phi) \tag{3.14}$$

with $\mathcal{L}$ a scalar.

Notation: Given a tensor T, $\sqrt{-g}T$ is known as a tensor density. Later we will be varying the actions with $g_{\mu\nu}$ (which will define the stress energy tensor). We thus need to calculate

$$\begin{aligned}\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} \\ &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\end{aligned} \tag{3.15}$$

3.4 Geodesics and curves of extremal proper time

Which curve $x^\mu(\lambda)$ extremises the proper time between two fixed points p ($\lambda = 0$) and q ($\lambda = 1$)?

$$\tau = \int_p^q \mathrm{d}\tau = \int_0^1 \frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\mathrm{d}\lambda = \int_0^1 \sqrt{I}\mathrm{d}\lambda \tag{3.16}$$

where

$$I = -g_{\mu\nu}(x(\lambda))\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \tag{3.17}$$

since

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = g_{\mu\nu}(x(\lambda))\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda^2 \tag{3.18}$$

Euleur Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\sqrt{I}}{\partial\dot{x}^\mu}\right) = \frac{\partial\sqrt{I}}{\partial x^\mu} \tag{3.19}$$

with $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial I}{\partial\dot{x}^\mu}\right) - \left(\frac{1}{I}\frac{\mathrm{d}I}{\mathrm{d}\lambda}\right)\frac{\partial I}{\partial\dot{x}^\mu} = \frac{\partial I}{\partial x^\mu} \tag{3.20}$$

If we choose I to be constant along the path, the term in $\frac{1}{I}\frac{\mathrm{d}I}{\mathrm{d}\lambda}$ diseappears, the proper time τ becomes an affine parameter of λ? somthing like that; and we have a simplified equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial I}{\partial\dot{x}^\mu}\right) = \frac{\partial I}{\partial x^\mu} \tag{3.21}$$

Substituting somehting above gives

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \tag{3.22}$$

It is very important to notice that we can easily compute the Christoffel symbol this way, by identification using the Euler-Lagrange equation. The Euler-Lagrange equation is easy to find. We also have

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \tag{3.23}$$

Notice that we obtain the same equation by considering the following action instead

$$S = \int \mathrm{d}\lambda \; I \tag{3.24}$$

We recall

$$I = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{3.25}$$

It is common for time-like geodesics to set $I = -1$ $(m > 0)$. Massless particles satisfy the same geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \tag{3.26}$$

but λ cannot be proper time since $\mathrm{d}s^2 = 0$. We have $I = 0$ on null geodesics. Space-like geodesics, same equations, we can fix $I = 1$.

Example. In Minkoswki space, consider a curve $x^\mu(\lambda)$

$$x^\mu(\lambda) = (R\lambda, R\cos(\lambda), R\sin(\lambda), 0) \tag{3.27}$$

Is it a light-like curve which is not a geodesic.

3.5 Covariant derivatives ∇_μ

1. acts from tensors (k, l) to tensors $(k, l+1)$
2. reduces to ∂_μ on flat space
3. linear
4. Leibnitz derivation
5. On scalars f, $\nabla_\mu f = \partial_\mu f$
6. When acting on a vector v^α, these properties imply that
$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + (\gamma_\mu)^\alpha{}_\beta v^\beta \tag{3.28}$$
7. moreover, $(\gamma_\mu)^\alpha{}_\beta$ must transform like the Christoffel symbol
8. Since $v^\beta v_\beta$ is a scalr, we have $\nabla_\mu(v^\alpha v_\alpha) = \partial_\mu(v^\alpha v_\alpha)$, we have
$$\nabla_\mu v_\alpha = \partial_\mu v_\alpha - (\gamma_\mu)^\alpha{}_\beta v_\beta \tag{3.29}$$
9. we can also find the transformation rule of $T^{\mu\nu}$ by contracting $T^{\mu\nu}v_\mu v_\nu$.
10. The torsion tensor is $T^\lambda{}_{\alpha\beta} = (\gamma_\alpha)^\lambda{}_\beta - (\gamma_\beta)^\lambda{}_\alpha$.

In special relativitym we choose to have no torsion (it is assumed, some theories with torsion exist but they are not GR), aka $T^\lambda{}_{\alpha\beta} = 0$. We also choose $\nabla_\alpha g_{\mu\nu} = 0$, which is obvious in the rest frame (where we locally have a Minkoski spacetime), and which is true in any frame due to the equivalence principle. These two conditions result in

$$\gamma^\lambda{}_{\alpha\beta} = \Gamma^\lambda{}_{\alpha\beta} \tag{3.30}$$

the Christoffel symbol.

Check that $[\nabla_\mu, \nabla_\nu]f = 0$ for f a scalar, amd that $\nabla_\mu v^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu)$.

In this language, the geodesic equation rewrites

$$U^\mu \nabla_\alpha U^\alpha = 0 \tag{3.31}$$

where $U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$

3.6 Directional covariant derivative

if a tensor is only defined on some curve $x^\mu(\lambda)$ (eg. momentum P^μ of a particle), then the directional covariant derivative $\frac{D}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\nabla_\mu$

$\frac{DT^{\mu\dots}_{\nu\dots}}{\mathrm{d}\lambda}$ transforms as a tensor. Parrallel transport: T is parrallel transported along $x^\mu(\lambda)$ if $\frac{DT}{\mathrm{d}\lambda} = 0$

3.7 Geodesics deviation equation and the Riemann tensor

In flat spacetime, two intially parrallel straight lines (geodesics) remain parrallel. In curved spacetime, they converge / diverge. The objective is to find by how much they deviate. The idea is to consider two initially parrallel lines, $x^\mu(\lambda)$ and $\bar{x}^\mu(\lambda)$, where $\bar{x}^\mu(\lambda) = x^\mu(\lambda) + \xi^\mu(\lambda)$, $\xi^\mu << 1$. We then expand the geodesics equations at first order.

On rewriting $\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}\lambda^2}$ in terms of $\frac{D^2\xi^\mu}{\mathrm{d}\lambda^2}$, we find

$$\frac{D^2\xi^\rho}{\mathrm{d}\lambda} + R^\rho_{\sigma\mu\nu}\xi^\nu \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = 0 \tag{3.32}$$

where

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\rho\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \tag{3.33}$$

is the Riemann curvature tensor, of second order in term of the metric. A necessary and sufficient condition for $g_{\mu\nu}$ to describe globally flat space time is that $R^\mu{}_{\alpha\beta\gamma} = 0$. One can show that

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]v^\rho &= R^\rho_{\sigma\mu\nu}v^\sigma \\ [\nabla_\mu, \nabla_\nu]\omega_\sigma &= -R^\rho_{\sigma\mu\nu}\omega_\rho \end{aligned} \tag{3.34}$$

The Riemann tensor possesses multiple properties:

- The first two indices are symmetric with the two last

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \tag{3.35}$$

- It is antisymmetric in the two first indices and the two last indices

$$R_{\lambda\mu\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu} \tag{3.36}$$

- The last 3 indices obey a cyclicity condition

$$R_{\lambda[\mu\nu\kappa]} = 0 \tag{3.37}$$

To know the number of independant components of the Riemann tensor, we can count the number of constraints.

Bianchi identity:

$$\nabla_{[\gamma} R_{\mu\nu]\lambda\kappa} = 0 \tag{3.38}$$

There are multiple scalars or tensors that we can define

1. Ricci tensor: $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$, it is symmetric
2. Ricci scalar: $R = R^{a}lpha\ _{\alpha}$
3. Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. We have $\nabla^{\mu} G_{\mu\nu} = 0$
4. Kretschmann scalar: $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$

Remark. 1. $n = 1$: no intrisc curvature possible

2. $n = 2$: 1 independant component of Riemann tensor, which is entirely defined by the Ricci scalar: $R_{\mu\nu} = R g_{\mu\nu}$, $R_{\alpha\beta\gamma\delta} = R g_{\alpha[\gamma} g_{\delta]\beta}$
3. $n = 3$: Riemann has 6 independant components, just like the Ricci tensor. So Einstein's equation completely fix the geometry of the space: no matter = flat space.
4. $n = 4$: Riemann has 20 independant components, whilst Ricci has 10 independant components. A lot more possibilities.

Example. $n = 2$:

1. 2 sphere: $\mathrm{d}s^2 = a^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2)$. There is a constant positive curvature $R = 2/a^2$
2. flat space: $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, $R = 0$
3. 2D hyperbolic plane $\mathrm{d}s^2 = \frac{a^2}{what}(\mathrm{d}x^2 + \mathrm{d}y^2)$ for $y > 0$. There is a constant negative curvature $R = -2/a^2$

3.8 Lie Derivative

We don't need any supplementary structure (contrary to ∇^{μ} which required the connexion). Take a curve γ, tangent vector $U^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$, acceleration vector $A^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$. At a point P on the curve, we have the coordinates x^{μ}. A little bit later, we arrive at point Q, with coordinates $x'^{\mu} = x^{\mu} + U^{\mu}\mathrm{d}\lambda$. We interpret this as an infinitesimal coordinate transformation. Under this transformation,

$$A'^{\alpha}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}(x) = (\delta^{\alpha}_{\beta} + (\partial_{\beta} U^{\alpha})\mathrm{d}\lambda) A^{\beta}(x) \tag{3.39}$$

or in other words

$$A'^{\alpha}(Q) = A^{\alpha}(P) + (\partial_{\beta} U^{\alpha})\mathrm{d}\lambda A^{\beta}(P) \tag{3.40}$$

On the other hand,

$$A^\alpha(Q) = A^\alpha(P) + U^\beta(\partial_\beta A^\alpha)\mathrm{d}\lambda \tag{3.41}$$

In general, $A^{'\alpha}$ and A^α differ. Their difference defines the Lie derivative of A^α along the curve.

$$\mathcal{L}_U A^\alpha(P) = \lim_{\mathrm{d}\lambda \to 0} \frac{A^\alpha(Q) - A^{'\alpha}(Q)}{\mathrm{d}\lambda} = (\partial_\beta A^\alpha)U^\beta - (\partial_\beta U^\alpha)A^\beta \tag{3.42}$$

1. for a vector, $\mathcal{L}_U A^\alpha = (\nabla_\beta A^\alpha)U^\beta - (\nabla_\beta U^\alpha)A^\beta$
2. for a scalar, $\mathcal{L}_U \Phi = (\partial_\beta \Phi)U^\beta$
3. for a covector, $\mathcal{L}_U \omega_\alpha = (\nabla_\beta \omega_\alpha)U^\beta + (\nabla_\beta U^\alpha)\omega_\beta$

Holds true the Leibnitz rule for the Lie derivative

$$\mathcal{L}_u(p^\alpha q_\beta) = (\mathcal{L}_u p^\alpha)q_\beta + p^\alpha \mathcal{L}_u q_\beta \tag{3.43}$$

Lie transport: $\mathcal{L}_U T^{\alpha,\dots}{}_{\beta,\dots} = 0$. If this is satisfied, T is Lie transported along γ. What are the properties of Lie transported tensors?

We can change the coordinates such that the first coordinate correspond to the path, and the others are constant along γ. Then, if the tensor is Lie transported along this path, then its derivative along the first coordinate will be null, ie it will be conserved along the first coordinate.

Theorem 3.8.1. *1. for a (k,l) tensor T, if $L_U T = 0$, then a coordinate system can be constructed such that in this coordinate system, $U^\alpha = \delta_0^\alpha$ and $\partial_0 T = 0$.*

2. if in a given coordinate system, the coordinates of a tensor T don't depend on some coordinate x^0 say, then $\mathcal{L}_{\partial_0} T = 0$.

For a rank 2 tensor, we have

$$\mathcal{L}_u T_{\alpha\beta} = (\nabla_\mu T_{\alpha\beta})u^\mu + (\nabla_\alpha u^\mu)T_{\mu\beta} + (\nabla_\beta u^\mu)T_{\alpha\mu} \tag{3.44}$$

3.9 Killing vectors

(Lie derivative applied to the metric)

1. k^μ is a Killing vector if $\mathcal{L}_k g_{\mu\nu} = 0$, if $\nabla_\beta k_\alpha + \nabla_\alpha k_\beta = 0$ (killing equation)
2. if in a given coord system $g_{\mu\nu}$ doesn't depend on x^0, $\partial_0 g_{\mu\nu} = 0$, so $\mathcal{L}_{\partial_0} g_{\mu\nu} = 0$

As expected, for each Killing vector, there is a conserved quantity along a geodesic

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(k_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right) = 0 \tag{3.45}$$

If we have a single killing vector, we can choose the coord system such that the killing vector k_μ is equal to ∂_0. If we have more, it is not always possible to have a coordinate per killing vector.

If k and k' are killing vectors, then $[k, k']$ is still a killing vector.

Example.

$$n = 2: \quad \mathbb{R}^2: \qquad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.46}$$

For this example, ∂_x and ∂_y are killing vectors. But we can also take the polar coordinate system, for which the metric is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2, \quad g'_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{3.47}$$

In which ∂_θ is evidently a killing vector.

In the above example, we see that there is no coordinate system able to express nicely all 3 killing vectors that we just showed. This also shows that there can be more killing vectors than the dimension of the space. In fact, we will show in TD that there can be at most $\frac{n(n+1)}{2}$ linearly independant killing vectors.

If there are exactly $\frac{n(n+1)}{2}$ killing vectors, then the space time is maximally symmetric and it has a constant Ricci scalar. Indeed, in the above exemple, we have 3 killing vectors and we have $R = 0$.

Example.

$$n = 3: \quad \mathbb{R}^3: \qquad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{3.48}$$

This shows ∂_x, ∂_y and ∂_z as killing vectors. But switching the coordinates to

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2) \tag{3.49}$$

We find a 4th killing vector $\partial_\phi = -y\partial_x + x\partial_y$. The two lasts are $-x\partial_z + z\partial_x$ and $-z\partial_y + y\partial_z$.

Definition 3.9.1. A space-time is stationary if there is one time-like killing vector k^μ. ($k^\mu k_\mu < 0$)

In the coord system space time, $k = \partial_t$. In $x^\mu = (t, x^k)$, the metric is then of the form

$$\mathrm{d}s^2 = g_{00}(x^k)\mathrm{d}t^2 + g_{0i}(x^k)\mathrm{d}t\mathrm{d}x^i + g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j \tag{3.50}$$

With $g_{00}(x^k) < 0$ in our Lorentzian manifold. This describes rotating neutron stars, rotating black holes...

Definition 3.9.2. A static space time is a space-time with a time-like killing vector k^μ ($k^\mu k_\mu < 0$) and invariant under $t \to -t$

In $x^\mu = (t, x^k)$, the metric is then of the form

$$\mathrm{d}s^2 = g_{00}(x^k)\mathrm{d}t^2 + g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j \tag{3.51}$$

This describes Schwarchild black holes, non-rotating neutron stars, ...

Definition 3.9.3. A static spherical symmetric is a static space time with additional 3 space-like killing vectors (R, S, T)

In $x^\mu = (t, x^k)$, the metric is then of the form

$$\mathrm{d}s^2 = g_{00}(r)\mathrm{d}t^2 + g_{rr}(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2) \tag{3.52}$$

On an affinely parametrized time-like geodesic ($\lambda = \tau$), conserved quantity

$$k_\mu u^\mu = k^\nu u^\mu g_{\mu\nu} \tag{3.53}$$

Example. For $k = \partial_t$, we have $k^\mu = \delta^\mu_0$, and with $u^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$, the conserved quantity in a stationary spacetime is $E = -\dot{t}g_{00}$ the energy.

Example. For $R = \partial_\theta$, the corresponding conserved quantity is $L = R^\mu u^\nu g_{\mu\nu} = r^2 \sin^2(\theta)\dot{\phi}$ the momentum along θ.

Alternatively, we can also find conserved quantities through the equations of motion, by slightly offsetting the Lagrangian in the action.

Definition 3.9.4. Killing tensors are tensors $K_{\nu_1 \dots \nu_l}$ which are totally symmetric and satisfy

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_l)} = 0 \tag{3.54}$$

To each killing tensor, there exist an associated quantity conserved along geodesics,

$$C = K_{\nu_1 \dots \nu_l} \frac{\mathrm{d}x^{\nu_1}}{\mathrm{d}\lambda} \cdots \frac{\mathrm{d}x^{\nu_l}}{\mathrm{d}\lambda} \tag{3.55}$$

Proof. We want to prove that for a geodesic $x(\lambda)$, we have $\frac{\mathrm{d}C}{\mathrm{d}\lambda} = 0$

$$\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(K_{\nu_1 \dots \nu_l} \frac{\mathrm{d}x^{\nu_1}}{\mathrm{d}\lambda} \cdots \frac{\mathrm{d}x^{\nu_l}}{\mathrm{d}\lambda}\right) \\
&= \frac{\mathrm{d}K_{\nu_1 \dots \nu_l}}{\mathrm{d}\lambda} \dot{x}^{\nu_1} \dots \dot{x}^{\nu_l} + K_{\nu_1 \dots \nu_l} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\dot{x}^{\nu_1} \dots \dot{x}^{\nu_l}\right) \\
&= \dot{x}^\mu \nabla_\mu K_{\nu_1 \dots \nu_l} \dot{x}^{\nu_1} \dots \dot{x}^{\nu_l} + K_{\nu_1 \dots \nu_l} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\dot{x}^{\nu_1} \dots \dot{x}^{\nu_l}\right)
\end{aligned} \tag{3.56}$$

□

In a Kerr black hole for exemple, the Brandon Carter is constant (?), which will be used along with L and E to uniquely determine the geodesics.

For a killing vector k^ρ, we will show that

$$\nabla_\mu \nabla_\rho k^\rho = R^\rho_{\sigma\mu\nu} k^\nu \tag{3.57}$$

By contracting, we also find

$$k^\lambda \nabla_\lambda R = 0 \tag{3.58}$$

Meaning the directional derivative of the Ricci scalar along a killing vector vanishes.

Chapter 4

Physical laws in curved spacetime

- general covariance: physical laws should be the same for all observers, independant of their coordinates
- equiv principle: physical laws reduce to those of special relativity in a LFFF

To go from special relativity to general relativity, one procedure is to "covariantize".

1. replacing $\eta_{\mu\nu} \to g_{\mu\nu}$
2. replacing $\partial_\mu \to \nabla_\mu$

This prescription leaves some ambiguities.

Example. In Minkoswki spacetime

$$S'_\phi = \int \mathrm{d}x^4 \left(-\frac{1}{2}(\partial_\mu \phi)(\partial_\nu \phi)\eta^{\mu\nu} - V(\phi) \right) \tag{4.1}$$

We could "covariantize" this into

$$S'_\phi = \int \mathrm{d}x^4 \sqrt{-g} \left(-\frac{1}{2}(\nabla_\mu \phi)(\nabla_\nu \phi)\eta^{\mu\nu} - V(\phi) \right) \tag{4.2}$$

But we could also want to add a term in $R\phi^2$, or a contraction of the Riemann tensor with the field.

The minimal prescription doesn't tell us what to do in this case. There can be formal reasons (symmetries, keeping the 2nd order, ...) to avoid some terms, but ultimately experiments will have the final word. It is also this blur that allows many modified gravity theories.

We can vary with respect to ϕ, $\delta_\phi S = 0$. We get the equation

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}(\partial_\mu \phi)g^{\mu\nu}) = V' \tag{4.3}$$

Remark. Note that something similar can be done with vector fields instead of scalar fields

But we can also vary the action with respect to $g_{\mu\nu}$ with ϕ constant. This defines the stress energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \qquad T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{4.4}$$

$$T^{\phi}_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right) \tag{4.5}$$

Enforcing once again $\delta_g S = 0$, we get that the energy-momentum tensor satisfies $\nabla^\mu T^{\phi}_{\mu\nu} = 0$

If we don't know what is the action for our matter field, then we can describe $T_{\mu\nu}$ in term of macroscopic quantities

Example. For exemple, for a perfect fluid whose energy density is ρ, whose 4 velocity is u^μ and whose pressure is P, we have

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \tag{4.6}$$

Then enforcing $\nabla^\mu T_{\mu\nu} = 0$ enforces the Euler equations and the continuity equation. For non relativistic dust, $P = 0$ and for relativistic fluid, $P = \frac{1}{3}\rho$

Einstein equations are

$$E_{\mu\nu} = T_{\mu\nu} \tag{4.7}$$

where the Einstein's tensor is linked to the curvature of the space.

- $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow \nabla^\mu E_{\mu\nu} = 0$
- $E_{\mu\nu} = E_{\nu\mu}$
- Einstein equations are second order in the derivatives
- for weak, slowly varying fields and a source consisting of dust, we get Newtonian gravity

The action for Einstein equations is

$$\mathcal{S}_{EH} = \frac{c^4}{16\pi\mathcal{G}}\int \mathrm{d}^4x\sqrt{-g}R + \mathcal{S}_{\text{matter}} \tag{4.8}$$

The equation of motion for this action correspond to Einstein's equations.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{4.9}$$

$$\frac{8\pi G}{c^4}T = -R \tag{4.10}$$

which lead to (in 4D)

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{4.11}$$