Quantum field theory

Buisine Léo
Ecole Normale Superieure of Paris

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Chapter 1

Classical field theory

1.1 Lagrangian

We consider a system with N particles, described by a position and momentum $\vec{X}, \vec{P}$, such that

$$\dot{P}_A = -\frac{\partial V}{\partial X^A} \tag{1.1}$$

with $A = 1, \ldots, 3N$. We can define a Lagrangian

$$\mathcal{L}(\dot{X}^A, X^A) \equiv T(\dot{X}^A) - V(X^A) \tag{1.2}$$

where $V(X^A)$ is the potential, and $T(\dot{X}^A)$ the kinetic energy, usually

$$T(\dot{X}^A) = \sum_A \frac{1}{2} m_A (\dot{X}^A)^2 \tag{1.3}$$

We then define the action

$$\mathcal{S} = \int_{t_i}^{t_f} \mathrm{d}t \mathcal{L}(\dot{X}^A, X^A, t) \tag{1.4}$$

The principle of least action (extremum action) then says

$$\delta \mathcal{S} = 0 \tag{1.5}$$

where the extremum are fixed

$$X^A(t_{i,f}) = X^A_{i,f} \qquad \delta X^A(t_{i,f}) = 0 \tag{1.6}$$

To find the laws, we do a slight modification in the coordinates

$$X^A(t) = X^A(t) + \delta X^A(t) \tag{1.7}$$

Putting this in the action, commuting time derivatives and δ, and using integration by parts, we find

$$\frac{\partial \mathcal{L}}{\partial X^A} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{X}^A} = 0 \tag{1.8}$$

1.2 Equation of motion for fields

This is already a strong result, but people generalized it to fields, with infinity many degrees of freedom

$$X^A(t) \to \phi(t, vecx) \tag{1.9}$$

We define

$$\mathcal{L} = \int \mathrm{d}x^3 \mathcal{L}(\phi_a(t, \vec{x}), \partial_\mu \phi_a(t, \vec{x})) \tag{1.10}$$

With

$$\mathcal{S} = \int \mathrm{d}t \mathcal{L}(t) \tag{1.11}$$

We still want to enforce on shell

$$\delta \mathcal{S} = 0 \tag{1.12}$$

So we take

$$\phi_a \to \phi_a + \delta\phi_a \tag{1.13}$$

And we fix the extremum. We do the same procedure (commutation of μ derivative and δ, followed by integration by parts) and we get

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \tag{1.14}$$

1.3 Symmetries

A Classical (infinitesimal) "Symmetry" is a infinitesimal change $\delta\phi_a$ such that under the transformation

$$\phi_a \to \phi_a + \delta\phi_a \tag{1.15}$$

the Lagrangian changes as

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu F^\mu \\ &= \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\right)\delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_m u \phi_a}\delta\phi_a\right) \end{aligned} \tag{1.16}$$

This is called a symmetry because it leaves the action invariant: if ϕ_a is a solution of the equation of motion, then

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\delta\phi_a\right) \\ &= \partial - \mu F^\mu \end{aligned} \tag{1.17}$$

and we can define the conserved current

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\delta\phi_a - F^\mu \tag{1.18}$$

such that

$$\partial_\mu J^\mu = 0 \tag{1.19}$$

In this case, we can also define the conserved charge

$$Q \equiv \int_{\mathrm{space}} \mathrm{d}^3 x J^0 \tag{1.20}$$

And we have

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= \int \mathrm{d}^3 x \partial_t J^0 \\ &= \int \mathrm{d}x^3 (\vec{\nabla} \cdot \vec{J}) \\ &= 0 \end{aligned} \tag{1.21}$$

where the last equality comes from the assumption (almost always made) that $\phi_a \to 0$ at $\pm\infty$ Let's do an exemple. We consider a symmetry transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \tag{1.22}$$

with $|\omega| << 1$.

$$\begin{aligned} \phi'(x) &= \phi(\Lambda^{-1} x) \\ &= \phi(x^\mu - w^\mu_\nu x^\nu) \\ &\simeq \phi(x) - \omega^\mu_\nu x^\nu \partial_\mu \phi(x) \\ &= \phi(x) - \omega\rho\sigma x^\sigma \partial^\rho \phi(x) \end{aligned} \tag{1.23}$$

where we recognize $(\delta\phi_a)^{\rho\sigma} \simeq x^\sigma \partial^\rho \phi(x)$. We have

$$\partial \mathcal{L} = -\omega^\mu_\nu x^\nu \partial_\mu \mathcal{L} \tag{1.24}$$

Such that put in

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta\phi_a - F^\mu \tag{1.25}$$

we have

$$\begin{aligned} J^\mu &= (\partial^\mu \phi)[-\omega_{\rho\sigma} x^\sigma \partial^\rho \phi] - (-w^\rho_\sigma x^\sigma \mathcal{L} \\ &= -\omega_{\rho\sigma} \left[x^\sigma \partial^\mu \phi \partial^\rho \phi - \eta^{\sigma\mu} x^\rho \left(\frac{(\partial_\alpha \phi)^2}{2} - \frac{m^2}{2} \phi^2 \right) \right] \end{aligned} \tag{1.26}$$

$$\begin{aligned} (J^{\rho\sigma})^\mu &= x^\sigma \partial^\mu \phi \partial^\rho \phi - \eta^{\sigma\mu} x^\rho \mathcal{L} - (\sigma \leftrightarrow \rho) \\ &= x^\rho T^{\sigma\mu} - x^\sigma T^{\rho\mu} \end{aligned} \tag{1.27}$$

where $T^{\rho\mu}$ is the stress-energy / energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{1.28}$$

Chapter 2

TD

2.1 TD1

$$G(1+\omega,\epsilon) = \simeq 1 - \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu} - i\epsilon_{\mu}P^{\mu} \tag{2.1}$$