Aufgabe 1

a)

$$Ax = b$$

$$A = D - E - F$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$$

$$D = \begin{pmatrix} \diagdown \end{pmatrix}$$

$$-F = \begin{pmatrix} 0 & \blacktriangle \\ & 0 \end{pmatrix}$$

$$-E = \begin{pmatrix} 0 & \\ \triangle & 0 \end{pmatrix}$$

$$B_J = D^{-1}(E + F)$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right)$$

$$B_{GS} = (D - E)^{-1} \cdot F$$

$$B_J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$0 = \det \begin{pmatrix} -\lambda & 0 & -\frac{1}{2} \\ 0 & -\lambda & 0 \\ -\frac{1}{2} & 0 & -\lambda \end{pmatrix} = -\lambda^3 + \frac{1}{4}\lambda$$

$$\Rightarrow \lambda^2 = \frac{1}{4}$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}, \lambda_3 = 0$$

$$\rho(B) = \max(|\lambda|) = \frac{1}{2}$$

$$B_{GS} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \cdot F = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$0 = \det \begin{pmatrix} -\lambda + \frac{1}{4} & 0 & 0 \\ 0 & -\lambda & 0 \\ -\frac{1}{2} & 0 & -\lambda \end{pmatrix} = -\lambda^3 + \frac{1}{4}\lambda^2$$

$$\Rightarrow \lambda = \frac{1}{4}$$

$$\Rightarrow \lambda_1 = \frac{1}{4}, \lambda_2 = 0, \lambda_3 = 0$$

$$\rho(B) = \max(|\lambda|) = \frac{1}{4}$$

b)

$$b = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$$

Jacobi

$x_1^0 = 0$ $\quad$ $x_1^1 = \frac{1}{2} \cdot (4) = 2$ $\quad$ $x_1^2 = \frac{1}{2} \cdot (4 - 1 \cdot 2{,}5) = 0{,}75$

$x_2^0 = 0$ $\quad$ $x_2^1 = \frac{1}{1} \cdot (0) = 0$ $\quad$ $x_2^2 = \frac{1}{1} \cdot (0) = 0$

$x_3^0 = 0$ $\quad$ $x_3^1 = \frac{1}{2} \cdot (5) = 2{,}5$ $\quad$ $x_3^2 = \frac{1}{2} \cdot (5 - 1 \cdot 2) = 1{,}5$

$x_1^3 = \frac{1}{2} \cdot (4 - 1{,}5) = 1{,}25$ $\quad$ $x_1^4 = \frac{1}{2} \cdot (4 - 2{,}125) = 0{,}9125$

$x_2^3 = 1 \cdot 0 = 0$ $\quad$ $x_2^4 = 0$

$x_3^3 = \frac{1}{2} \cdot (5 - 0{,}75) = 2{,}125$ $\quad$ $x_3^4 = \frac{1}{2} \cdot (5 - 1{,}25) = 1{,}875$

Gauß-Seidel

$x_1^0 = 0$ $\quad$ $x_1^1 = \frac{1}{2} \cdot (4-0) = 2$ $\quad$ $x_1^2 = \frac{1}{2} \cdot (4 - \frac{3}{2}) = \frac{5}{4}$

$x_2^0 = 0$ $\quad$ $x_2^1 = 1 \cdot 0 = 0$ $\quad$ $x_2^2 = 0$

$x_3^0 = 0$ $\quad$ $x_3^1 = \frac{1}{2} \cdot (5 - 1 \cdot 2) = \frac{3}{2}$ $\quad$ $x_3^2 = \frac{1}{2} \cdot (5 - \frac{5}{4}) = \frac{15}{8}$

$x_1^3 = \frac{1}{2} \cdot (4 - \frac{15}{8}) = \frac{17}{16}$ $\quad$ $x_1^4 = \frac{1}{2} \cdot (4 - \frac{63}{32}) = \frac{65}{64}$

$x_2^3 = 0$ $\quad$ $x_2^4 = 0$

$x_3^3 = \frac{1}{2} (5 - \frac{17}{16}) = \frac{63}{32}$ $\quad$ $x_3^4 = \frac{1}{2} \cdot (5 - \frac{65}{64}) = \frac{255}{128}$