CHI-SQUARED TEST

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Introduction

- “Study the past if you would divine the future ."
  - **Confucius** Chinese philosopher & reformer (551 BC - 479 BC).

Chi-squared test

- Objectives:
  - To check if several groups share the same distribution.
  - To determine the distribution in the population.
  - To study the independence of two (or more) factors.

- Compare the observed frequencies in an empirical experience with theoretical frequencies, derived from some hypothetical distribution.

Chi-squared test

- Divide interval into k segments
  - with the same density, or equal distance.
- Calculate F^{-1} function we are evaluating.
- From this function calculate the intervals corresponding to each segment.
- Calculate X^2.

Measure the difference

- The discrepancy between the observed frequencies (O_i) and the expected values (E_i) is evaluated by:

$$X^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

- k is the number of classes.
- Can be brought on by a discrete variable.
- Or by discretizing a continuous variable.

Distribution of the Pearson statistic

- If the data come from a population described by the model given below,

$$X^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

It follows a Chi-Square law with k-1 degrees of freedom:

$$X^2 \sim \chi^2_{k-1}$$

Test of homogeneity

- We work with k classes on P populations, and we test if there is a common population.
- The data (observed values) are represented in a contingency table:

	C_1	...	C_i	...	C_k	
X_1	$O_{1,1}$		$O_{i,1}$			
...						
X_j	$O_{1,j}$		$O_{i,j}$			$n_{.,j}$
...				...		
X_P					$O_{k,P}$	
			$n_{i,.}$			n

Test of homogeneity

- If homogeneity exists on the populations, then a common probabilitiy exists for each class. Taking $E_{i,j} = n_{i,\cdot}\, n_{\cdot,j}/n$, we can see that:

$$X^2 = \sum_{j=1}^{P} \sum_{i=1}^{k} \frac{(O_{i,j} - E_{i,j})^2}{E_{i,j}}$$

- Follows a chi-square distribution with $(k\text{-}1)(P\text{-}1)$ degrees of freedom.

Example: Test of homogenity

- We have in a reparation service 4 priority levels for the works to be done:
  - Urgent
  - High
  - Mean
  - Low
- The probabilities for each reparation is urgent 10%, high 20%, mean 30% and low 40%.

Example: Test of homogenity

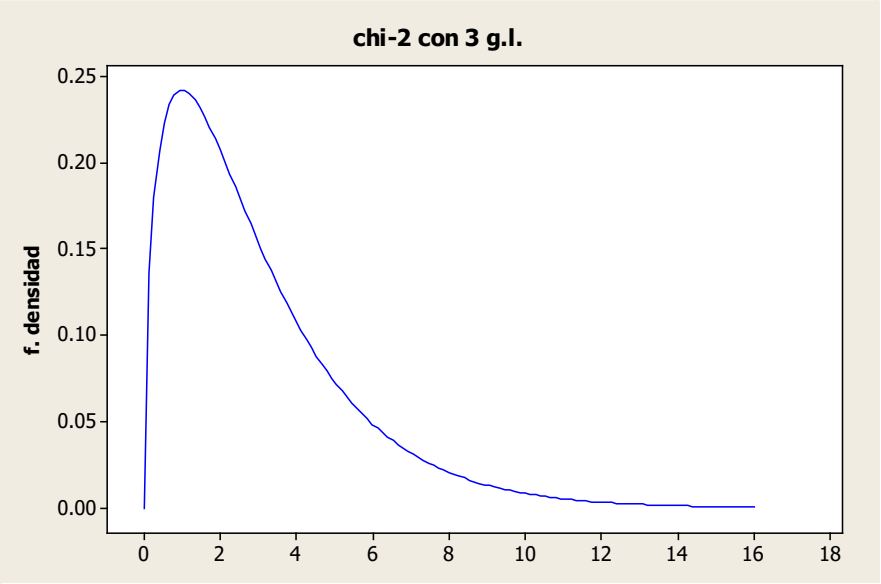
- From historical data we obtain the following table (533 observations in total).

	urgent	high	mean	low
Observed	78	107	145	223
Expected	55.3	110.6	165.9	221.2
Difference	22.7	-3.6	-20.9	1.8

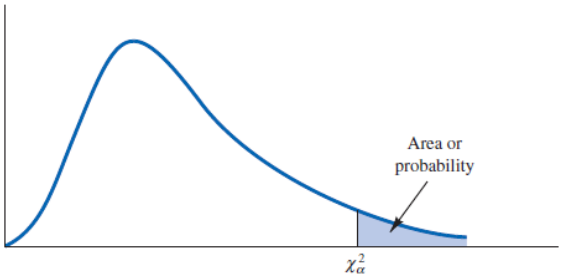
- «Are the differences between the observed values and our proposed probabilities for each category due to the random nature of the phenomenon?»

Example: Test of homogenity

- The model we are using is this.
  - 43% between 0 and 2
  - 31% between 2 and 4
  - 15% between 4 and 6
  - 7% between 6 and 8



Chi Square Table



Degrees of Freedom	Area in Upper Tail									
	.995	.99	.975	.95	.90	.10	.05	.025	.01	.005
1	.000	.000	.001	.004	.016	2.706	3.841	5.024	6.635	7.879
2	.010	.020	.051	.103	.211	4.605	5.991	7.378	9.210	10.597
3	.072	.115	.216	.352	.584	6.251	7.815	9.348	11.345	12.838
4	.207	.297	.484	.711	1.064	7.779	9.488	11.143	13.277	14.860
5	.412	.554	.831	1.145	1.610	9.236	11.070	12.832	15.086	16.750
6	.676	.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.647	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.041	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801

Example: Test of homogenity

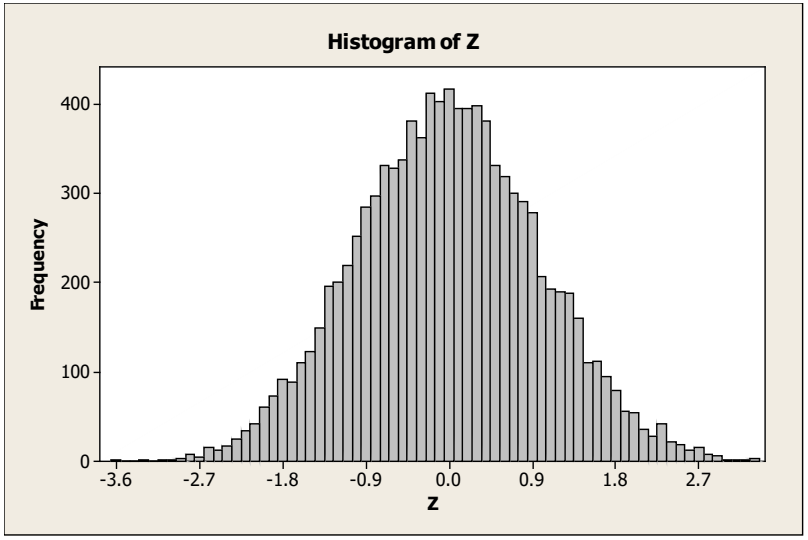
- Calculating the statistic:

Class	Prob (pi)	Expected Number (Ei)	Observed Number (Oi)	Squared Difference	Squared Standardizd Diff.
Urgente	0.10	(0.10)*553=55.3	78	515.29	9.31808318
High	0.20	(0.20)*553=110.6	107	12.96	0.11717902
Medium	0.30	(0.30)*553=165.9	145	436.81	2.63297167
Low	0.40	(0.40)*553=221.2	223	3.24	0.01464738
				SUM= 12.0829	

- In this case X^2 is 12.08, looking the table we detect that this value is so high. The proposed classification is not correct.

Testing a family of distributions for continuous variables

- We want to test the fitness of an algorithm that generates a normal distribution N(0,1).
- We generate 10000 values (empirically generated), it seems that the generation process is correct.
- To assure this we perform a Chi-square test.



Arbitrarily define a discretization of the variable in 10 classes to obtain a significant frequency.

Testing a family of distributions for continuous variables

Clas	Prob.	Expected number (over 10000)	Observed number	Standardized difference
< -3	0.001350	13.50	6	-2.04
$[-3, -2]$	0.021400	214.00	195	-1.30
$[-2, -1.5]$	0.044057	440.57	456	0.74
$[-1.5, -1]$	0.091848	918.48	934	0.51
$[-1, 0]$	0.341345	3413.45	3477	1.09
$[0, 1]$	0.341345	3413.45	3402	-0.20
$[1, 1.5]$	0.091848	918.48	886	-1.07
$[1.5, 2]$	0.044057	440.57	417	-1.12
$[2, 3]$	0.021400	214.00	220	0.41
> 3	0.001350	13.50	7	-1.77

Testing a family of distributions for continuous variables

$\chi^2_{9,0.05} = 16.919 > \chi^2_{obs} = 13.59$

Besides, the p value of the value 13.59 would be around 0.13. That is greater than $\alpha = 0.05$.

(You might check p-value calculator for chi-square table:https://www.socscistatistics.com/pvalues/chidistribution.aspx)

So, there is no evidences against that the distribution does not follow a Normal distribution.

Proposed exercises

Exercise 1: Epilachna varivestis

- Determine whether the following amount of insects (*Epilachna varivestis* by bean plants) can be represented by a Poisson distribution:

Y	fy (=Oi)	Pi	Ei	(Oi-Ei)^2/Ei
0	12		23.03	
1	56		36.01	
2	23		28.15	
3	10		14.67	
4	5		5.73	
5	4		2.39	
total	110		110	

df	0,05	0,025	0,01	0,005
4	9	11	13	14,8602
5	11	13	15	16,7496
6	13	14	17	18,5475

Exercise 1: Epilachna varivestis

- Determine whether the following amount of insects (*Epilachna varivestis* by bean plants) can be represented by a Poisson distribution:

Y	fy (=Oi)	Pi	Ei	(Oi-Ei)^2/Ei
0	12	0.21	23.03	5.28
1	56	0.33	36.01	11.09
2	23	0.26	28.15	0.942
3	10	0.13	14.67	1.487
4	5	0.05	5.73	0.093
5	4	0.02	2.39	1.084
total	110	1	110	**19.976**

df	0,05	0,025	0,01	0,005
4	9	11	13	14,8602
5	11	13	15	16,7496
6	13	14	17	18,5475

Exercice 3: Is the die biased?

X	1	2	3	4	5	6
Number of times	20	15	12	17	9	17

H_0 : The die is fair (The distribution of counts follows a Uniform distribution ($F = F_0$)).

H_1 :The die is not fair (The distribution of counts does not follow a Uniform distribution ($F \neq F_0$)).

The expected value is s

$$E[X] = np = 90 * \left(\frac{1}{6}\right) = 15.$$

$$\chi^2_{5,0.05} = 11.070$$

Is the die biased?

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i^2}$$

$$= \frac{(20-15)^2}{15} + \frac{(15-15)^2}{15} + \cdots + \frac{(17-15)^2}{15} = 5.2$$

$P(\chi^2 > 5.2) =$ between 0.1 and 0.9

or $\chi^2_{5,0.05} = 11.070 > \chi^2_{obs} = 5.2$

There is no evidence to reject the null hypothesis.