Homework 5

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5.1.4

a) $f_{XY}(xy) = \frac{\binom{30}{x}\binom{10}{y}\binom{60}{4-x-y}}{\binom{100}{4}}$, where $x + y \leq 4$

X	Y	Z	Moderate	High	Low	$f_{XY}(x,y)$
0	0	4	30	10	60	0.124357822
1	0	3	30	10	60	0.261805941
2	0	2	30	10	60	0.196354456
3	0	1	30	10	60	0.062123444
4	0	0	30	10	60	0.006988887
0	1	3	30	10	60	0.087268647
1	1	2	30	10	60	0.135416866
2	1	1	30	10	60	0.066560832
3	1	0	30	10	60	0.010353907
0	2	2	30	10	60	0.02031253
1	2	1	30	10	60	0.02065681
2	2	0	30	10	60	0.004992062
0	3	1	30	10	60	0.001836161
1	3	0	30	10	60	0.00091808
0	4	0	30	10	60	$5.35547E-05$

b)

x	$\mathrm{f}_X(x)$
0	0.233828714
1	0.418797697
2	0.267907350
3	0.072477351
4	0.006988887

c)

$$\begin{aligned} E[X] &= \sum_x x \cdot f_X(x) \\ &= (0 \cdot f_X(0)) + (1 \cdot f_X(1)) + (2 \cdot f_X(2)) + (3 \cdot f_X(3)) + (4 \cdot f_X(4)) \\ &\approx 0.418798 + 2(0.267907) + 3(0.072477) + 4(0.006989) \\ &= 1.2 \end{aligned}$$

d)

y	$f_{XY}(3,y)$	$f_{Y\|3}(y)$
0	0.062123444	0.857142857
1	0.010353907	0.142857143

e) $E[Y \mid X=3] = \sum_y y \cdot f_{Y|X=3}(y) = (0 \times f_{Y|X=3}(0)) + (1 \times f_{X|Y=3}(1)) = 1 \times 0.142857 = 0.142857$

f) $V[Y \mid X=3] = [\ \sum_y y^2 \cdot f_{Y|X=3}(y)\ ] - \mu^2 = (0^2 \times f_{Y|X=3}(0)) + (1^2 \times f_{X|Y=3}(1)) - 0.142857^2 = 0.142857 - 0.142857^2 \approx 0.122448$

g) They are not independent.
Take for example, $f_{XY}(0,0) = f_X(0) \cdot f_Y(0)$. Here, $f_{XY}(0,0) \approx 0.124358$, whereas $f_X(0) \cdot f_Y(0) \approx 0.2338287 \cdot 651630549 \approx 0.15237$. Since $0.124358 \neq 0.15237$, it means that X and Y are not independent.

5.1.9

We can find c by solving $\int_0^3 \int_0^3 cxy\ dxdy = 1$, for which we get c $= \frac{4}{81}$

a) $\int_0^3 \int_0^2 \frac{4xy}{81}\ dxdy = \frac{4}{9}$

b) We first find $f_X(x) = \int_0^3 \frac{4xy}{81} dy = \frac{2x}{9}$. Then we will find $P(X < 2.5)$:
$\int_0^{2.5} f_X(x)\ dx = \int_0^{2.5} \frac{2x}{9}\ dx = \frac{2}{9}[\frac{x^2}{2}\ |_0^{2.5}] = \frac{2}{9} \cdot \frac{5}{2}^2 \cdot \frac{1}{2} = \frac{25}{36}$

c) We first find $f_Y(y) = \int_0^3 \frac{4xy}{81} dx = \frac{2y}{9}$. Then we will find $P(1 < Y < 2.5)$:
$\int_1^{2.5} f_Y(y)\ dy = \int_0^{2.5} \frac{2y}{9}\ dy = \frac{2}{9}[\frac{y^2}{2}\ |_1^{2.5}] = \frac{2}{9}[\frac{5}{2}^2 \cdot \frac{1}{2} - \frac{1}{2}] = \frac{21}{36}$

d) $P(X > 1.8, 1 < Y < 2.5) = \int_1^{2.5} \int_{1.8}^3 \frac{4xy}{81}\ dxdy \approx 0.37333$

e) $E[X] = \int_0^3 x \cdot f_X(x)\ dx = \int_0^3 \frac{2x^2}{9}\ dx = 2$

f) Since the probability of \$X < 0 \$is equal to 0, $P(X < 0, Y < 4) = 0$. This is because PDF is constrained to $0 < X < 3$ and $0 < Y < 3$. We're basically solving for $\int_0^3 \int_0^0 \frac{4xy}{81}\ dxdy$, where the inner integral is equal to 0. Thus, the whole integral is $= 0$.

g) $f_X(x) = \int_0^3 \frac{4xy}{81} dy = \frac{2x}{9}$ for $0 < x < 3$.

5.4.2

To get value of c, we solve: $\sum_x \sum_y f_{XY}(x,y) = 1$ which is equal to $\sum_x \sum_y c(x+y) = 1$. From here we solve:

$$\begin{aligned}
1 &= \sum_x \sum_y c(x+y) \\
&= c[(1+1) + (1+2) + (1+3) + (2+1) + ... + (3+2) + (3+3)] \\
&= c[6(1) + 6(2) + 6(3)] \\
&= c(36) \\
\frac{1}{36} &= c
\end{aligned}$$

To find $Cov(X,Y)$ we will look for $E[XY] - \mu_x \mu_y$:

$$
\begin{aligned}
E[XY] &= \sum_y \sum_x cxy(x+y) \\
&= c[1(1+1) + 2(1+2) + 3(1+3) + 2(2+1) + 4(2+2) + 6(2+3) + 3(3+1) + 6(3+2) + 9(3+3)] \\
&= c[2 + 6 + 12 + 6 + 16 + 30 + 12 + 30 + 54] \\
&= \frac{1}{36}(168) \\
&\approx 4.6667
\end{aligned}
$$

$$
\begin{aligned}
f_X(x) &= \sum_y cy(x+y) \\
&= c[(x+1) + (x+2) + (x+3)] \\
&= \frac{1}{36}(3x+6) \\
&= \frac{x+2}{12}
\end{aligned}
$$

$$
\begin{aligned}
\mu_x &= \sum_x x \cdot f_X(x) \\
&= \sum_x x \cdot (\frac{x+2}{12}) \\
&= (\frac{1+2}{12}) + 2(\frac{2+2}{12}) + 3(\frac{3+2}{12}) \\
&= \frac{3}{12} + \frac{8}{12} + \frac{15}{12} \\
&= \frac{26}{12} \\
&= \frac{13}{6}
\end{aligned}
$$

μ_y is also $\frac{13}{6}$ since it covers the same range with the same relation in $f_{XY}(x, y)$.

Thus we get: $Cov(X, Y) = E(XY) - \mu_x \mu_y = \frac{168}{36} - (\frac{13}{6})^2 = \frac{168}{36} - \frac{169}{36} = \frac{-1}{36} \approx -0.0277778$.

To find correlation, we will solve $\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$. Since we already have $sigma_{XY}$, we will simply be finding σ_X and σ_Y:

$$
\begin{aligned}
\sigma_X &= Cov(X) \\
&= \sqrt{V(X)} \\
&= \sqrt{[\sum_x x^2 f_X(x)] - \mu_X^2} \\
&= \sqrt{[\sum_x x^2 \frac{x+2}{12}] - \mu_X^2} \\
&= \sqrt{\frac{3}{12} + 4(\frac{4}{12}) + 9(\frac{5}{12}) - \mu_X^2} \\
&= \sqrt{\frac{3}{12} + \frac{16}{12} + \frac{45}{12} - \frac{13}{6}^2} \\
&= \sqrt{\frac{64}{12} - \frac{169}{36}} \\
&= \sqrt{\frac{23}{36}} \\
&\approx 0.799305253885
\end{aligned}
$$

$\sigma_X = \sigma_Y$ since the two equations have the same bound and the equation $f_{XY}(x, y)$ is parallel.

We then get: $\rho_{XY} = \frac{\frac{-1}{36}}{\sqrt{\frac{23}{36}}^2} \approx -0.04347826$

In summary: $c = \frac{1}{36}$, $\sigma_{XY} = \frac{-1}{36}$ and $\rho_{XY} \approx -0.04347826$

5.4.6

To find E[X] and E[Y] we will need to find $f_X(x)$ and $f_Y(x)$:

$$
\begin{aligned}
f_X(x) &= \int_0^\infty e^{-x-y}\ dy \\
&= e^{-x} \int_0^\infty e^{-y} dy \\
&= e^{-x}[-e^{-y}\ |_0^\infty] \\
&= e^{-x}
\end{aligned}
$$

We can use the above equation to also show $f_Y(x) = e^{-y}$ by simply swapping the x's and y's. Thus we will get $f_{XY}(x, y) = e^{-x-y} = f_X(x) \cdot f_Y(y)$. This means that X and Y are independent random variables. This also means that $E[XY] = E[X] \cdot E[Y]$, meaning that $\sigma_{XY} = E[XY] - E[X]E[Y] = E[XY] - E[XY] = 0$.

Thus $\sigma_{XY} = Cov(X, Y) = 0$.

As for the correlation $\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$, we have already proven that $\sigma_{XY} = 0$. Hence, $\frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0$.

Thus, our correlation $\rho_{XY} = 0$.

In summary: $\sigma_{XY} = 0$ and $\rho_{XY} = 0$