Homework 4

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3.8.12

a) Let X be the number of visits in a day. $X \sim Poisson(\lambda T) = Poisson(1.8)$

$$
\begin{aligned}
P(X > 5) &= 1 - P(X \leq 5) \\
&= 1 - [P(X = 5) + P(X = 4) + P(X = 3) + P(X = 2) + P(X = 1) + P(X = 0)] \\
&= 0.01037804
\end{aligned}
$$

b) Let Y be the number of visits in a week. $Y \sim Poisson(\lambda T) = Possion(1.8 \times 7)$

$$
\begin{aligned}
P(Y < 5) &= P(4) + P(3) + P(2) + P(1) + P(0) \\
&= \frac{e^{-12.6}12.6^4}{4!} + \frac{e^{-12.6}12.6^3}{3!} + \frac{e^{-12.6}12.6^2}{2!} + \frac{e^{-12.6}12.6^1}{1!} + \frac{e^{-12.6}12.6^0}{0!} \\
&\approx 0.004979028
\end{aligned}
$$

c) We want to find how many days it would take for $P(Z \geq 1) \geq 0.99$ where Z is the number of visits in $x \in \mathbf{Z}$ days. We'll first solve for $P(Z \geq 1) = 0.99$:

$$
\begin{aligned}
P(Z \geq 1) &= 0.99 \\
1 - P(0) &= 0.99 \\
1 - \frac{e^{-\lambda \cdot x}(\lambda \cdot x)^0}{0!} &= 0.99 \\
e^{-\lambda \cdot x} &= -0.01 \\
e^{-\lambda \cdot x} &= 0.01 \\
\lambda \cdot x &= \ln(0.01) \\
x &= \frac{\ln(0.01)}{\lambda} \text{ , where } \lambda = 1.8 \\
x &\approx 2.558427
\end{aligned}
$$

Since Poisson takes discrete numbers / whole numbers, we'll round x up to 3 days.
Hence, it'll take atleast 3 days such that the probability of atleast one visit is 99%.

d) Let X be the number of visits per day, then $X \sim Pois(\lambda T)$. We want to find $P(X > 5) = 0.1$ which is equivalent to:
$1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)] = 0.01$

And we solve for the following, where $\lambda t = \mu$:
$1 - [\frac{e^{-\mu}\mu^0}{0!} + \frac{e^{-\mu}\mu^1}{1!} + \frac{e^{-\mu}\mu^2}{2!} + \frac{e^{-\mu}\mu^3}{3!} + \frac{e^{-\mu}\mu^4}{4!} + \frac{e^{-\mu}\mu^5}{5!}] = 0.1$

```r
1 - ppois(5, 1.8)
```

```
## [1] 0.01037804
```

```r
ppois(4, 12.6)
```

```
## [1] 0.004979028
```

```r
1 - ppois(0, 1.8 * 3)
```

```
## [1] 0.9954834
```

4.1.2

a) P(X<2) $= \int_1^2 \frac{2}{x^3}dx = \frac{2x^{-2}}{-2} \mid_1^2 = -[\frac{1}{x^2} \mid_1^2] = \frac{3}{4} = 0.75$

b) P(X>5) $= 1 - \int_1^5 \frac{2}{x^3}dx = 1 - [\frac{-1}{x^2} \mid_2^5] = 1 + [\frac{1}{25} - 1] = \frac{1}{25}$

c) P$(4 <$ X $< 8)$ $\int_4^8 \frac{2}{x^3}dx = \frac{-1}{x^2} \mid_4^8 = -[\frac{1}{64} - \frac{1}{16}] = \frac{3}{64}$

d) P(X < 4 or X > 8) $= 1 - (4 < X < 8) = 1 - \int_4^8 \frac{2}{x^3}dx = 1 - (-\frac{1}{x^2} \mid_4^8) = 1 + (\frac{1}{64} - \frac{1}{16}) = 1 - \frac{3}{64} = \frac{61}{64}$

e) Let x be the value such that F(X $<$ x) $= 0.95$

$$\begin{aligned}
\int_1^x \frac{2}{t^3}dt &= 0.95 \\
\frac{-1}{t^2} \mid_1^x &= 0.95 \\
\frac{-1}{x^2} + 1 &= 0.95 \\
\frac{1}{x^2} &= 0.05 \\
\frac{1}{x^2} &= \frac{5}{100} \\
\frac{1}{x^2} &= \frac{1}{20} \\
x &= \sqrt{20}
\end{aligned}$$

4.1.2 Extras

a) CDF for $x > 1 = P(X \leq x) = \int_1^x \frac{2}{t^3}dt = \frac{-1}{t^2} \mid_1^x = \frac{-1}{x^2} + 1$

b) $\mu = \int_1^\infty xf(x)dx = \int_1^\infty x \cdot \frac{2}{x^3}dx = \int_1^\infty \frac{2}{x^2}dx = \frac{-2}{x} \mid_1^\infty = \frac{-2}{\infty} - \frac{-2}{1} = 0 + 2 = 2$

c) $\sigma^2 = E[X^2] - E[X]^2 = \int_1^\infty x^2 f(x)dx - 2^2 = \int_1^\infty x^2 \cdot \frac{2}{x^3}dx - 4 = \int_1^\infty \frac{2}{x}dx - 4 = 2ln(x) \mid_1^\infty -4 = \infty$

4.4.2

a) CDF for $0.95 < x < 1.05 = \int_{0.95}^x \frac{1}{(1.05-0.95)}dt = \int_{0.95}^x 10dt = 10t \mid_{0.95}^x = 10x - 10(0.95)$

We then get:

$$F(x) = \begin{cases} 0, & \text{if } x < 0.95 \\ 10x - 10(0.95), & \text{if } 0.95 \leq x \leq 1.05 \\ 1, & \text{if } x > 1.05 \end{cases}$$

b) $P(X > 1.02) = \int_{1.02}^{1.05} \frac{1}{1.05-0.95}dx = 10x \mid_{1.02}^{1.05} = 10.5 - 10.2 = 0.3$

c) let a be the lower bound of 0.95, and b be the upper bound of 1.05. To find the thickness exceeded by 90% of flanks, we just need to find where P(X < x) = 1 - 0.9. Since our probability is uniform, we can simply get the values of the lower 10% range and add it to the lower bound.Thus we get:

0.95 + 0.1(1.05-0.95) = 1.02$(0.1 \cdot 0.1)$ = 0.95 + 0.01 = 0.96

d)
$\mu = \frac{b+a}{2} = \frac{1.05+0.95}{2} = \frac{2}{2} = 1$
$\sigma^2 = \frac{(b-a)^2}{12} = \frac{0.1^2}{12} \approx 0.000833333333333$

4.5.6

a)$X \sim N(260, 50^2)$

We want to find P(X>4*60) = P(X > 240). In order to do this, we'll first find Z, where $Z \sim N(0, 1)$:

Z = $\frac{X-\mu}{\sigma} = \frac{240-260}{50} = \frac{-20}{50} = \frac{-2}{5} = -0.4$
$P(X > 240) = P(Z > -0.4) = P(Z < 0.4) = 0.6554$

b) We can use the z-score of the first and third quartile:

The z-score for the third quartile (75%) is approximately 0.67, which we will then convert to the cumulative distribution of the problem:

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ 0.67 &= \frac{X-260}{50} \\ 33.5 &= X - 260 \\ 293.5 &= X \end{aligned}$$

We can also do the same to find our first quartile, who's z-score is just the negative of the third quartile. Hence, Z = -0.67:

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ -0.67 &= \frac{X-260}{50} \\ -33.5 &= X - 260 \\ 226.5 &= X \end{aligned}$$

Thus, the first quartile (25%) has a battery life of 226.5 minutes while the third quartile (75%) has a battery life of 293.5 minutes.

c) Looking at our z-table, we see that the z-score with 95% probability is approximately 1.64. We will take $Z = -1.64$ since we're looking for the battery life that is surpassed with 95% probability, while 1.64 gives us the value surpassed by 5%. We will convert -1.64 to our current distribution where $X \sim N(260, 50^2)$:

$$
\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
-1.64 &= \frac{X-260}{50} \\
-82 &= X-260 \\
178 &= X
\end{aligned}
$$

Thus, the value that is surpassed by 95% of battery lives is approximately 178 minutes.

```
1 - pnorm(240, 260, 50)
```

```
## [1] 0.6554217
```

```
pnorm(293.5, 260, 50)
```

```
## [1] 0.7485711
```

```
pnorm(226.5, 260, 50)
```

```
## [1] 0.2514289
```

```
pnorm(178, 260, 50)
```

```
## [1] 0.05050258
```

4.5.12

a) $X \sim N(310, 45^2)$ (units are in millions of gallons). We want to find when P(X > 350). Thus, we'll just find 1 - P(X < 350). But first, we'll convert our distribution to a standardized distribution $Z \sim N(0, 1)$:

$$
\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
Z &= \frac{350-310}{45} \\
Z &= \frac{40}{45} \\
Z &\approx 0.888889
\end{aligned}
$$

Then $P(Z < 0.89) = 0.81327$, and $1 - P(Z < 0.89) = 0.18673$. Thus, the probability that a day requires more water than what's stored in the city reservoir is 18.673%.

b) We will need to find where $P(X < x) = 0.99$. The z-score for 99% probability in a standard normal distribution is ≈ 2.33. Then we'll convert from standard $Z \sim N(0, 1)$ to $X \sim N(310, 45^2)$.

$$
\begin{aligned}
Z &= \frac{X-\mu}{\sigma} \\
2.33 &= \frac{X-310}{45} \\
104.85 &= X-310 \\
414.85 &= X
\end{aligned}
$$

Thus, the city will need a reservoir capacity of 414.85 million gallons in order for the probability of it being exceeded to be 1%.

c) We need to find where $P(Z < z) = 0.05$. The z-score for 0.95 is ≈ 1.64, thus we'll our z-score will need to convert $x = -1.65$ to the distribution where $X \sim N(310, 45^2)$:

$$\begin{aligned} Z &= \frac{X - \mu}{\sigma} \\ -1.65 &= \frac{X - 310}{45} \\ -74.25 &= X - 310 \\ 235.75 &= X \end{aligned}$$

Thus, the amount of water needed for water use to be exceeded with 95% probability is 235.75 million gallons.

d) We want to find the mean when $P(X > 350) = 0.01$ where $X \sim N(310, 45^2)$. This is equivalent to finding when $P(X < 350) = 0.99$. The z-score in the standard normal distribution for 0.99 is approximately 2.33:

$$\begin{aligned} Z &= \frac{X - \mu}{\sigma} \\ 2.33 &= \frac{350 - \mu}{45} \\ 104.85 &= 350 - \mu \\ 245.15 &= \mu \end{aligned}$$

Since we're looking for consumption per person, we'll get $\frac{245.15}{1.4} \approx 175.10714$. Thus, the mean daily consumption per person will need to be approximately 175.10714 gallons.

4.7.5

a) Let Y be the number of calls in 30 minutes, then $Y \sim Pois(2)$:
$P(Y = 0) = e^{-2} \approx 0.135335$

b) Let X be the number of calls in 10 minutes, then $X \sim Pois(\frac{2}{3})$. We want to find $P(X \leq 1)$:
$P(Z \leq 1) = 1 - P(Z > 1) = 1 - e^{\lambda t} = 1 - e^{\frac{2}{3}} \approx 0.48658$

c) $P(\frac{1}{3} \leq X \leq \frac{2}{3}) = F(\frac{2}{3}) - F(\frac{1}{3}) = (1 - e^{-2/3}) - (1 - e^{-1/3}) = 0.203114$

d) We're trying to find when F(t) = 0.9 since F(t) gives the probability of a call within a certain time interval:

$$\begin{aligned} F(t) &= 0.9 \\ P(X \leq t) &= 0.9 \\ 1 - e^{-\lambda t} &= 0.9 \\ -e^{-\lambda t} &= -0.1 \\ e^{-\lambda t} &= 0.1 \\ -\lambda t &= ln(0.1) \\ \lambda t &= -ln(0.1) \\ t &= \frac{-ln(0.1)}{\lambda} \\ t &= \frac{-ln(0.1)}{\frac{1}{15}}, \text{let } \lambda \text{ be } 1/15 \text{ since its the rate per minute} \\ t &\approx 34.538776 \end{aligned}$$

Thus, the time interval would have to be approx. 34.538776 minutes in order for the probability of at least one call in interval to be 0.90.

```r
dpois(0, 2)
```

```
## [1] 0.1353353
```

```r
1 - dpois(0, 2/3)
```

```
## [1] 0.4865829
```

```r
(1 - dpois(0, 2/3)) - (1 - dpois(0, 1/3))
```

```
## [1] 0.2031142
```

```r
qexp(0.9, 1/15)
```

```
## [1] 34.53878
```

4.7.11

a) Let Y be the number of calls in 30 minutes, then $Y \sim Pois(3)$. We want to find $P(X \leq 3)$. This will be equivalent to 1 - [P(0) + P(1) + P(2) + P(3)] since we're subtracting the probability of getting 3 or less calls in half an hour:

$P(0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-3}$
$P(1) = \frac{e^{-\lambda t}(\lambda t)^1}{1!} = 3e^{-3}$
$P(2) = \frac{e^{-\lambda t}(\lambda t)^2}{2!} = \frac{9e^{-3}}{2}$
$P(3) = \frac{e^{-\lambda t}(\lambda t)^3}{3!} = \frac{27e^{-3}}{6}$

$1 - [P(0) + P(1) + P(2) + P(3)] \approx 0.3527681$

Thus the probability of getting more than 3 calls per half hour is approximately 0.3528.

b) We're trying to find P(X > 1) = P(N=0):
$P(X > 1) = e^{-\lambda t} = e^{-3} \approx 0.0498$

c) We're solving for $P(X = 0) = 0.01$ where X is the number of calls in 1 hr, thus $X \sim Pois(6)$:

$$\begin{aligned} P(X = 0) &= 0.01 \\ e^{-\lambda x} &= 0.01 \\ -\lambda x &= ln(0.01) \\ x &= \frac{ln(0.01)}{-\lambda} \\ x &= \frac{ln(0.01)}{-6} \\ x &\approx 0.76753 \end{aligned}$$

Thus, the x hours required for the probability of no calls to be 0.01 is $x = 0.76753$ hours.

d) We're looking for P(X > 2) where t=2 and X is time between calls. This is equivalent to $P(N = 0)$, where N is the number of calls in 2 hours, and $N \sim Pois(12)$:
$P(X > 2) = P(N = 0) = e^{-12} \approx 0.0000061442123$

Thus, the probability that we get no call in 2 hours is 0.0000061442123.

```r
1 - (dpois(3, 3) + dpois(2, 3) + dpois(1, 3) + dpois(0, 3))
```

```
## [1] 0.3527681
```

```r
dpois(0, 3)
```

```
## [1] 0.04978707
```

```r
qexp(0.99, 2*3)
```

```
## [1] 0.7675284
```

```r
dpois(0, 6*2)
```

```
## [1] 6.144212e-06
```