# Control Systems
# Lecture 8: Stability Analysis for Linear Systems

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# Outline

1. Introduction
2. Routh's Stability Criterion
3. Stability in State Space

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition (chapter 5: page 212-217)

# Introduction

The most important problem in linear control systems is stability.

- Want the system to be stable
- Under what conditions will a system become unstable?
- if it is unstable, how should we stabilize the system?

- Determine the stability of the systems
  - transfer function
  - state-space form

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# Routh's Stability Criterion

consider the system represented in the transfer function below

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

where the $a$'s and $b$'s are constants and $m \leq n$.

- Stability of linear closed loop system determined by location of the poles.
    - Stable systems– all poles must lie on the left hand side of s plane.
    - Unstable systems – at least one pole lies on the right hand side
        - Output diverges for any input
    - Marginally stable systems – poles on the imaginary axis with all other poles on the left.
      (system stable for some input, unstable for others)

For each closed-loop transfer function $T(s)$, determine the stability system

- $T(s) = \frac{1}{(s+1)(s+3)}$, poles $(s = -1, s = -3) \Rightarrow$ Stable
- $T(s) = \frac{s}{(s^2+1)^2}$, poles $s = \pm - j \Rightarrow$ marginally stable

Recap: a control system is stable if and only if all closed-loop poles lie in the left-half s plane.

Consider the closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where the $a$'s and $b$'s are constant values and $m \leqq n$.

A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial.

# Routh's Stability Criterion

- Determines if any of roots lie outside the left half plane
- Doesn't give actual pole locations

1. Write the polynomial in $s$ in the following form:
$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$
where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.
2. Check:
   - Are any coefficient are equal to zero? Ex: $s^3 + 0s^2 + s + 3 = 0$
   (Not all roots lie on the left hand side)
   - Are any coefficients negative?
   ( At least one root is on the right hand side $\Rightarrow$ unstable)
3. Unstable system is possible with all positive coefficients $\Rightarrow$ use Routh's stability criterion

$$a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & \dots & \dots & \\
s^{n-3} & c_1 & c_2 & \dots & \dots & \\
s^{n-4} & d_1 & d_2 & \dots & \dots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_5 - a_4 a_1)$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_3)$$

$$c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_5)$$

$$d_1 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = -\frac{1}{c_1}(b_1 c_2 - c_1 b_2)$$

- Number of changes in sign in the first column = number of roots on right hand side
- No sign change $\Rightarrow$ Stable$\neq$ Sign changes $\Rightarrow$ unstable

Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the stability

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | 5 |
| $s^3$ | 2 | 4 | 0 |
| $s^2$ | 1 | 5 | |
| $s^1$ | -6 | | |
| $s^0$ | 5 | | |

Two poles are in the right half plane

$\Downarrow$

Unstable

Let us apply Routh's stability criterion to the following third order polynomial

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers.The array of coefficients becomes

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all roots have negative real parts is given as

$$a_1 a_2 > a_0 a_3$$

If the first-column term element in any row is zero, but the remaining terms are not zero or there is no remaining term, then

- Replace the first zero with a very small number $\epsilon$ (assume to be positive or negative)
- Calculations continue as normal. (some elements that follow will be function of $\epsilon$)
- Complete the array
- Count sign change in first column
  - Allow $\epsilon$ to be a very tiny number (almost zero)
  - Number of sign changes = number of roots on the right hand side

# Routh's Stability Criterion

## Special Case I: example

Consider the following equation

$$s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

The array of coefficient is

$$\begin{array}{llll} s^5 & 1 & 2 & 11 \\ s^3 & 2 & 4 & 10 \\ s^3 & 0 & 6 & \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array} \qquad \begin{array}{llll} s^5 & 1 & 2 & 11 \\ s^3 & 2 & 4 & 10 \\ s^3 & 0 \approx \epsilon & 6 & \\ s^2 & -12/\epsilon + 4 & 10 & \\ s^1 & 10\epsilon^2/12+6 & & \\ s^0 & 10 & & \end{array}$$

Assume $\epsilon$ to be positive, 2 sign changes $\Rightarrow$ unstable

If all the coefficients in any derived row are zero, it indicates that

- two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.
- The evaluation of the rest of the array can be continued by
  - forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row.
- Such roots with equal magnitudes and lying radially opposite in the s plane can be found by solving the auxi. polynomial, which is always even.
- For a 2n-degree auxi. polynomial, there are n pairs of equal and opposite roots

Consider the following equation:

$$s^3 + s^2 + 2s$$

The array of coefficient is

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & \color{red}{0} & \\ s^0 & & \end{array}$$

$$P(s) = s^2 + 2$$
$$\frac{dP(s)}{ds} = 2s$$

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & \color{blue}{2} & \\ s^0 & & \end{array}$$

NOte: auxi. polynomial is

$$1s^2 + 2s^0$$

$$s^3 + s^2 + 2s = (s+1)\color{blue}{(s^2+2)}$$

$$\Downarrow$$

Stable

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficient is

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 24 | -25 |
| $s^4$ | 2 | 48 | -50 |
| $s^3$ | 0 | 0 | |
| $s^2$ | | | |
| $s^1$ | | | |
| $s^0$ | | | |

Note that such a case occurs only in an odd numbered row. The auxi. polynomial is then formed from the coefficients of the $s^4$.

$$P(s) = 2s^4 + 48s^2 - 50$$
$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 24 | -25 |
| $s^4$ | 2 | 48 | -50 |
| $s^3$ | 8 | 96 | |
| $s^2$ | 24 | -50 | |
| $s^1$ | 338/3 | | |
| $s^0$ | -50 | | |

There is one sign change

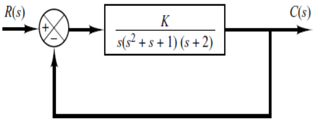
$\Rightarrow$ Unstable

We have the closed- loop transfer function as follows

$$T(s) = \frac{18}{s^5+s^4-7s^3-7s^2-18s-18}$$

Is the system stable?

Consider the closed loop system shown below




The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | K |
| $s^3$ | 3 | 2 | 0 |
| $s^2$ | 7/3 | K | |
| $s^1$ | $2\text{-}\frac{9}{7}K$ | | |
| $s^0$ | K | | |

For stability, $K$ must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

# Outline



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# Stability in State Space

- Location of poles determine stability
- How to find poles in state space?
    - Could convert state space to transfer function
    - Use eigenvalues

# Stability in State-Space

Recall that the number $\lambda$ is an eigenvalue if and only if there exists a non-zero vector $x$ such that

$$Ax = \lambda x$$

- All solutions will be null vector except when denominator

  $$det(\lambda I - A) = 0$$

  determine the stability.

# Example

Determine the stability. How many poles are on the left hand side, right hand side and $j\omega$ axis?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -4 \\ -1 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$det(sI - A) = s^3 - 9s^2 + 12s - 4$$

| | | |
|---|---|---|
| $s^3$ | 1 | 12 |
| $s^2$ | -9 | -4 |
| $s^1$ | 104/9 | |
| $s^0$ | -4 | |

3 sign changes $\Rightarrow$ 3 roots on the right hand side$\Rightarrow$ unstable.

recall

$$G(s) = C[Is - A]^{-1}B + D$$

In order to find the pole, we need to find

$$\det(Is - A) = 0$$

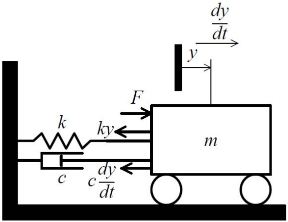
Which is equivalent to finding the eigenvalue of A.

$$det(\lambda I - A)$$

So $\lambda$ can be used to determine the stability the system.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

where $k$, $m$ and $c$ are constant values and
$x_1 = y, x_2 = \frac{dy}{dt}$




Compute the determinant

$$\det \begin{bmatrix} \lambda & -1 \\ k/m & s + c/m \end{bmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

The roots of the equation above are the poles of the system

# Reference

- Katsuhiko Ogata, Modern Control Engineering, Fifth Edition, Pearson,