

Control Systems

Lecture 9: Root Locus Techniques

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Outline

- 1 Introduction
- 2 Defining the Root Locus
- 3 Properties of the Root Locus
- 4 Root Locus Sketching
- 5 Plotting Root Loci with MATLAB
- 6 Root Locus for Positive-Feedback Systems

- Slide can be found at : <https://github.com/Daro12/Control-Systems>
- Reading:
 - Norman S. Nise, Control Systems Engineering, 7th edition, WILEY (chapter 8: page 382-448)
 - Katsuhiko Ogata, Modern Control Engineering, 5th edition, Pearson (chapter 6: page 290-301)



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Introduction

definition

Root locus is a graphical representation of the closed-loop poles as a system parameter is varied.



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Ex. Consider the closed-loop systems defined as

$$\frac{s^2+s+1}{s^3+4s^2+Ks+1}$$



Introduction

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Root locus is a graphical representation of the closed-loop poles as a system parameter is varied.

Ex. Consider the closed-loop systems defined as

$$\frac{s^2+s+1}{s^3+4s^2+Ks+1}$$

Recall: The stability of the system depends on the location the poles.

- What value of K should we choose to achieve the requirement?
- What is the effect of varying K on the system?



Introduction

Learning outcomes

- State the properties of root locus



Introduction

Learning outcomes

- State the properties of root locus
- Sketch the root locus



Introduction

Learning outcomes

- State the properties of root locus
- Sketch the root locus
- Use the computer program to plot the root locus



Introduction

Learning outcomes

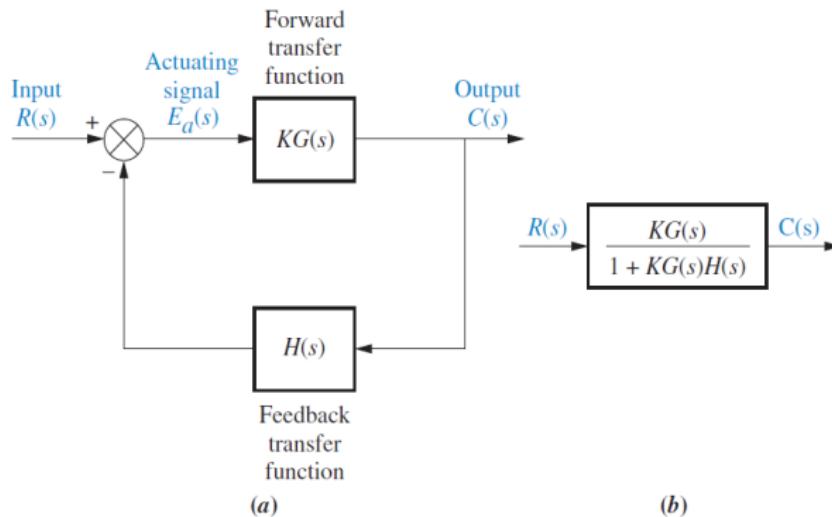
- State the properties of root locus
- Sketch the root locus
- Use the computer program to plot the root locus
- Sketch the root locus for positive feedback systems



Introduction

Control System Problem

Consider the typical closed-loop feedback control system below



From Norman S. Nise (figure 8.1)

The poles change with K



Introduction

Control System Problem

Let define

$$G(s) = \frac{N_G(s)}{D_G(s)} \text{ and } H(s) = \frac{N_H(s)}{D_H(s)}$$



Introduction

Control System Problem

Let define

$$G(s) = \frac{N_G(s)}{D_G(s)} \text{ and } H(s) = \frac{N_H(s)}{D_H(s)}$$

then,

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

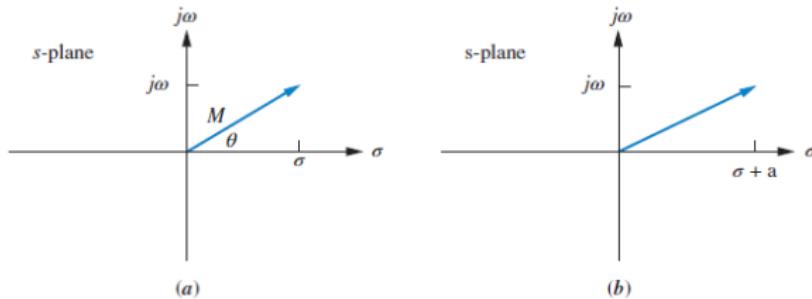
The system's **transient response** and **stability** are dependent upon the poles which is the function of **K**. We have no knowledge of the systems' performance unless we know the value of **K**.



Introduction

Vector Representation of Complex Numbers

We have the complex number $s = \sigma + j\omega$



From Norman S. Nise (Figure 8.2)

and consider the complex function

$$F(s) = (s + a),$$

then,

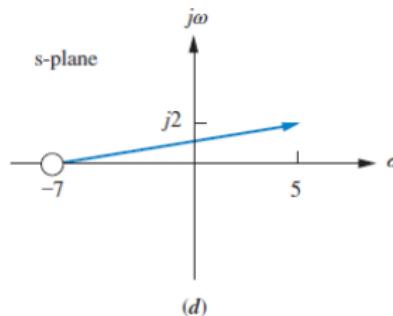
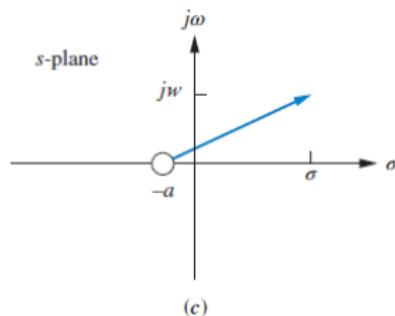
$$F(s) = \sigma + j\omega + a = (\sigma + a) + j\omega$$



Introduction

Vector Representation of Complex Numbers

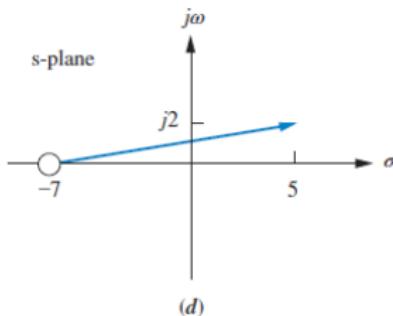
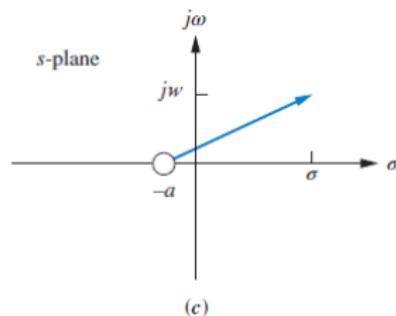
$F(s)$ has a zero at $-a$. If we translate the vector a units to the left, we have an alternative representation of a complex number originated at zeros of $F(s)$, and terminates on the point $s = \sigma + j\omega$



Introduction

Vector Representation of Complex Numbers

$F(s)$ has a zero at $-a$. If we translate the vector a units to the left, we have an alternative representation of a complex number originated at zeros of $F(s)$, and terminates on the point $s = \sigma + j\omega$



Conclusion: $(s + a)$ is a complex number and can be represented by a vector drawn from the zero of the function to the point s .

Ex. $(s + 7)|_{s \rightarrow 5+j2}$ is a complex number drawn from the zero -7 , to $s = 5 + j2$.



Introduction

Vector Representation of Complex Numbers

Let apply the concepts to a complicated function. Assume a function

$$F(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{j=1}^n (s + p_j)} \quad (1)$$

The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point s .



Introduction

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The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point s . The magnitude, M , of $F(s)$ at any point, s , is

$$M = \frac{\prod \text{(zeros lengths)}}{\prod \text{(pole lengths)}} = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{j=1}^n |(s + p_j)|} \quad (2)$$



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$$M = \frac{\prod \text{(zeros lengths)}}{\prod \text{(pole lengths)}} = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{j=1}^n |(s + p_j)|} \quad (2)$$

The angle, θ , of $F(s)$ at any point, s , is

$$\begin{aligned} \theta &= \sum \text{(zero angles)} - \sum \text{(pole angles)} \\ &= \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j) \end{aligned} \quad (3)$$



Introduction

Vector Representation of Complex Numbers

Ex. Given

$$F(s) = \frac{(s+1)}{s(s+2)}$$

Find $F(s)$ at the point $s = (-3 + j4)$



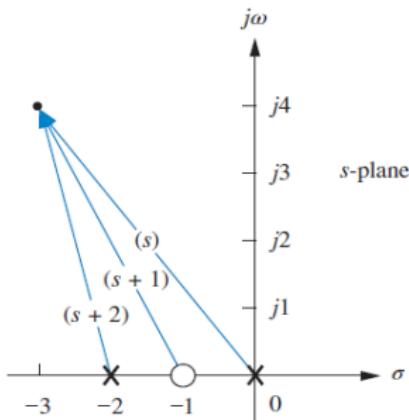
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From Norman S. Nise (Figure 8.3)



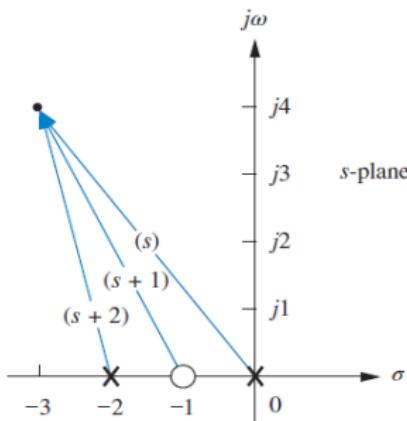
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From Norman S. Nise (Figure 8.3)

- At zero $-1: \sqrt{20} \angle 116.6^\circ$
- At pole 0: $5 \angle 126.9^\circ$
- At pole $-2: \sqrt{17} \angle 104^\circ$



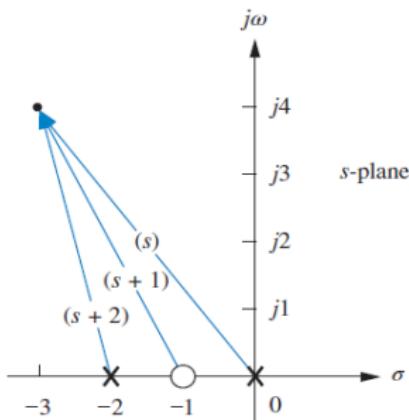
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- At zero $-1: \sqrt{20} \angle 116.6^\circ$
- At pole $0: 5 \angle 126.9^\circ$
- At pole $-2: \sqrt{17} \angle 104^\circ$

Thus,

$$\begin{aligned} M\angle\theta &= \frac{\sqrt{20}}{5\sqrt{15}} \angle (116^\circ - 126.9^\circ - 104.0^\circ) \\ &= 0.217 \angle (-114.3^\circ) \end{aligned}$$



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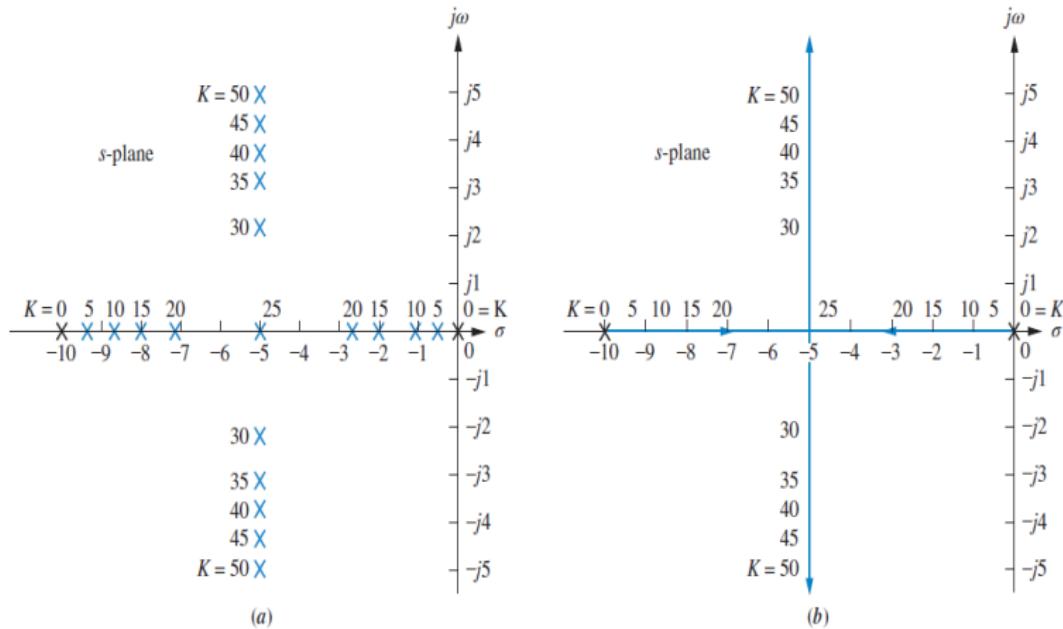
- Norman S. Nise, Control Systems Engineering, 7th edition, WILEY (chapter 8: page 382-448)
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Defining the Root Locus

consider the closed loop transfer function defined as

$$\frac{K}{s^2 + 10s + K}$$



From Norman S. Nise (Figure 8.5)



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Properties of the Root Locus

Consider the closed-loop transfer function for the system

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)}$$

Poles exist when the characteristic polynomial in the denominator become zero, or

$$KG(s)H(s) = -1 = 1\angle(2k + 1)180^\circ, k = 0, \pm 1, \pm 2, \dots$$



Properties of the Root Locus

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Since $KG(s)H(s)$ is the complex quantity, It can be written in the form of angle and magnitude.

Angle condition:

$$\angle KG(s)H(s) = (2k + 1)180^\circ, k = 0, \pm 1, \pm 2, \dots \quad (4)$$

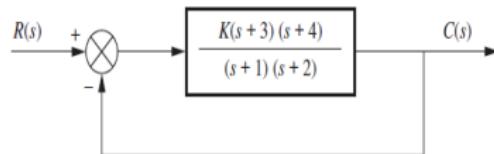
Magnitude condition:

$$|KG(s)H(s)| = 1$$



Properties of the Root Locus

Consider the system below

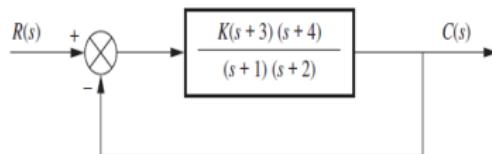


From Norman S. Nise (Figure 8.6)



Properties of the Root Locus

Consider the system below



From Norman S. Nise (Figure 8.6)

The closed-loop transfer function $T(s)$ is

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)}$$

If point s is a closed-loop system pole for some value of gain, K , then s must satisfy Eqs. (4),(5)



Properties of the Root Locus

Consider the point $-2 + j3$. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must equal an odd multiple of 180° .

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ$$



Properties of the Root Locus

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$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ$$

Thus, $-2 + j3$ is not a point on the root locus, or alternatively, $-2 + j3$ is not a closed-loop pole for any gain.



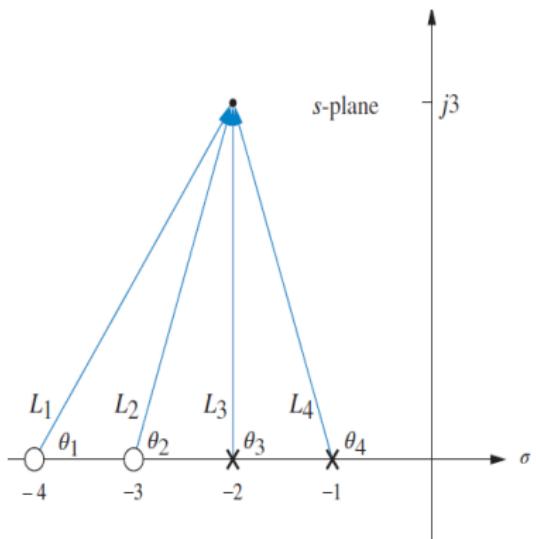
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$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ$$

Thus, $-2 + j3$ is not a point on the root locus, or alternatively, $-2 + j3$ is not a closed-loop pole for any gain.

If these calculations are repeated for the point $-2 + j(\sqrt{2}/2)$, the angles do add up to 180° . That is, $-2 + j(\sqrt{2}/2)$ is a point on the root locus for some value of gain. We now proceed to evaluate that value of gain.



From Norman S. Nise (Figure 8.7)



Properties of the Root Locus

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{pole length}}{\prod \text{zero lengths}} \quad (6)$$

Replace $-2 + j3$ by $-2 + j(\sqrt{2}/2)$, the gain K , is calculated as

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{(\sqrt{2}/2)(1.22)}{(2.12)(1.22)} = 0.33$$

Thus, the point $-2 + j(\sqrt{2}/2)$ is a point on the root locus for a gain of 0.33.



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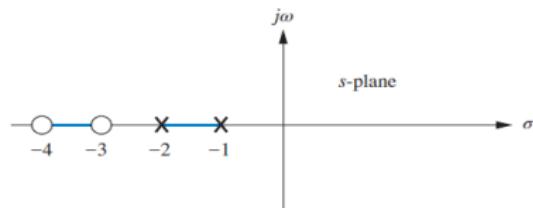
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Root Locus Sketching

Sketching Rule



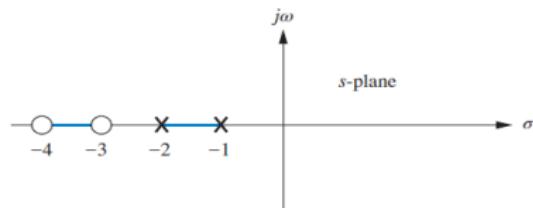
From Norman S. Nise (Figure 8.9)

- Rule 1: Number of branches

of branches of root locus = # of branches of closed-loop poles

Root Locus Sketching

Sketching Rule



From Norman S. Nise (Figure 8.9)

- **Rule 1: Number of branches**

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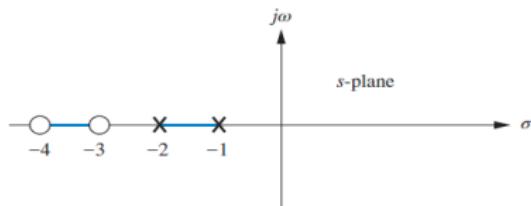
- **Rule 2: Symmetry**

The root locus is symmetrical about the real axis.



Root Locus Sketching

Sketching Rule



From Norman S. Nise (Figure 8.9)

- **Rule 1: Number of branches**

of branches of root locus = # of branches of closed-loop poles

- **Rule 2: Symmetry**

The root locus is symmetrical about the real axis.

- **Rule 3: Real-axis segments**

On the real axis, for $K > 0$, the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.



Root Locus Sketching

Sketching Rule

- Rule 4: Starting and ending points

The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

consider again the closed-loop transfer function

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$



Root Locus Sketching

Sketching Rule

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consider again the closed-loop transfer function

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

- As $K \rightarrow 0$ (small gain),

$$T(s) \approx \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + \epsilon} \quad (7)$$

We conclude that the root locus begin at the poles of $G(s)H(s)$.



Root Locus Sketching

Sketching Rule

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We conclude that the root locus begin at the poles of $G(s)H(s)$.

- As $K \rightarrow \infty$

$$T(s) \approx \frac{KN_G(s)D_H(s)}{\epsilon + KN_G(s)N_H(s)} \quad (8)$$

We conclude that the root locus ends at the zeros of $G(s)H(s)$



Root Locus Sketching

Sketching Rule

- Rule 5: Behavior at Infinity

Consider applying Rule 4 to the following open-loop transfer function:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

There are three finite poles, at $s = 0$, $s = -1$ and $s = -2$, and no finite zeros.



Root Locus Sketching

Sketching Rule

- Rule 5: Behavior at Infinity

Consider applying Rule 4 to the following open-loop transfer function:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

There are three finite poles, at $s = 0$, $s = -1$ and $s = -2$, and no finite zeros.

A function can also have infinite poles and zeros.

- If the function approaches infinity as s approaches infinity, then the function has a pole at infinity.
- If the function approaches zeros as s approaches infinity, then the function has a zero at infinity

Ex.

- $G(s) = s$ has the poles at ∞ since $G(s) \rightarrow \infty$ as $s \rightarrow \infty$.
- $G(s) = \frac{1}{s}$ has a zero at ∞ since $G(s) \rightarrow 0$ as $s \rightarrow \infty$



Root Locus Sketching

Sketching Rule

- **Rule 5:** (cont.)

Every function of s has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros.



Root Locus Sketching

Sketching Rule

- Rule 5: (cont.)

Every function of s has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros.

For illustration, consider

$$KG(s)H(s) \approx \frac{K}{s^3} = \frac{K}{s \cdot s \cdot s}$$

Each s in the denominator causes the open-loop function, $KG(s)H(s)$, to become 0 as that $s \rightarrow \infty$. It has three zeros at infinity



Root Locus Sketching

Sketching Rule

- Rule 5: (cont.)

Every function of s has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros.

For illustration, consider

$$KG(s)H(s) \approx \frac{K}{s^3} = \frac{K}{s \cdot s \cdot s}$$

Each s in the denominator causes the open-loop function, $KG(s)H(s)$, to become 0 as that $s \rightarrow \infty$. It has three zeros at infinity

So, the root locus begins at the finite poles of $KG(s)H(s)$ and ends at the infinite zeros.

- Where are the infinite zeros?



Root Locus Sketching

Sketching Rule

- Rule 5: (Cont.)

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:

$$\sigma_a = \frac{\sum_{\text{finite poles}} - \sum_{\text{finite zeros}}}{\#\text{finite poles} - \#\text{finite zeros}} \quad (9)$$

$$\theta_a = \frac{(2k + 1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \quad (10)$$

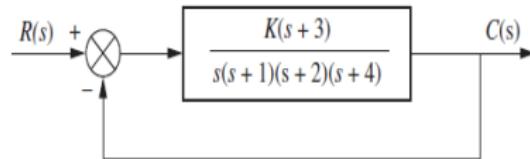
where $k = 0, \pm 1, \pm 2, \dots$ and the angle is given in radians with respect to the positive extension of the real axis.



Root Locus Sketching

Example: Sketching with Asymptotes

Consider the system below



From Nise Figure 8.11

The real-axis intercept is evaluated as

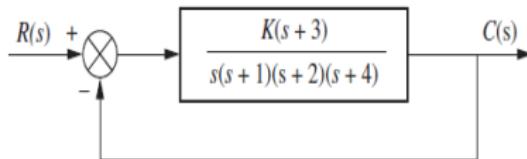
$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$



Root Locus Sketching

Example: Sketching with Asymptotes

Consider the system below



From Nise Figure 8.11

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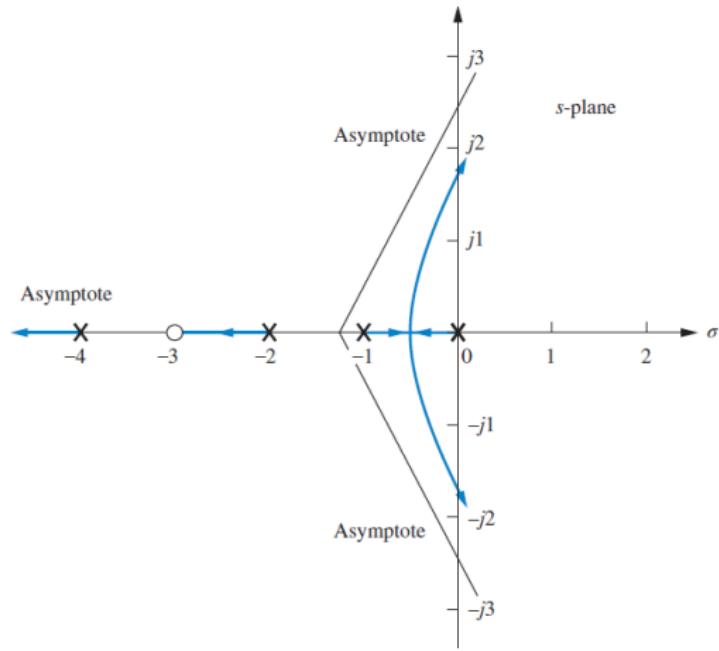
The angles of the lines that intersect at $-4/3$ is given as

$$\begin{aligned}\theta_a &= \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \\ &= \pi/3, \quad k = 0 \\ &= \pi, \quad k = 1 \\ &= 5\pi/3, \quad k = 2\end{aligned}$$



Root Locus Sketching

Example: Sketching with Asymptotes



From Nise Figure 8.12



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points

For each $s = \sigma$ on the real-axis segment of the root locus,

$$KG(\sigma)H(\sigma) = -1$$



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points

For each $s = \sigma$ on the real-axis segment of the root locus,

$$KG(\sigma)H(\sigma) = -1$$

or

$$K = -\frac{1}{G(\sigma)H(\sigma)} \quad (11)$$

Real-axis break-in and breakaway points are the real values of σ for which

$$\frac{dK(\sigma)}{d\sigma} = 0$$



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points

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Ex. consider

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points (cont.)
we have

$$KG(s)H(s) = -1$$



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points (cont.)
we have

$$KG(s)H(s) = -1$$

then,

$$K(\sigma) = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points (cont.)
we have

$$KG(s)H(s) = -1$$

then,

$$K(\sigma) = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$

Take the derivative with respect to σ

$$\frac{dK}{d\sigma} = -\frac{11\sigma^2 - 26\sigma - 61}{(\sigma^2 - 8\sigma + 15)^2}$$

Setting $dK/d\sigma = 0$, we find

$$\sigma_1 = -1.45 \text{ and } \sigma_2 = 3.82$$

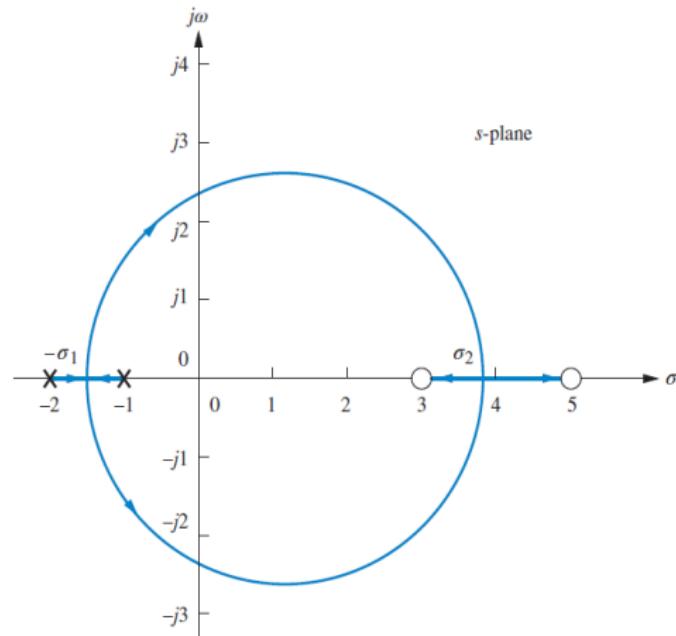
which are the breakaway and break-in points.



Root Locus Sketching

Sketching Rule

- Rule 6: Real axis break-in and breakaway points (cont.)



From Nise Figure 8.13



Root Locus Sketching

Sketching Rule

- Rule 7: The $j\omega$ -Axis Crossings

if $s = j\omega$ is a closed loop pole on the imaginary axis, then

$$KG(j\omega)H(j\omega) = -1 \quad (12)$$



Root Locus Sketching

Sketching Rule

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if $s = j\omega$ is a closed loop pole on the imaginary axis, then

$$KG(j\omega)H(j\omega) = -1 \quad (12)$$

Ex. Consider

$$KG(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$

substitute $s = j\omega$ then setting $KG(j\omega)H(j\omega) = -1$
we obtain

$$-\omega^4 + j7\omega^3 + 14\omega^2 - j(8 + K)\omega - 3K = 0$$

Separating real and imaginary parts,

$$\begin{aligned} -\omega^4 + 14\omega^2 - 3K &= 0 \\ 7\omega^3 - (8 + K)\omega &= 0 \end{aligned}$$



Root Locus Sketching

Sketching Rule

- Rule 7: The $j\omega$ -Axis Crossings: Form the second equation, discard the trivial solution, we obtain

$$\omega^2 = \frac{8+K}{7}$$

Substituting into the first equation. Then

$$K^2 + 65K - 720 = 0$$

Then,

$$K = -74.65 \text{ or } K = 9.65$$

Discard the negative one (negative feedback $K > 0$).

Thus,

$$K = 9.65 \text{ and } \omega = 1.59$$



Outline

- 1 Introduction
- 2 Defining the Root Locus
- 3 Properties of the Root Locus
- 4 Root Locus Sketching
- 5 Plotting Root Loci with MATLAB
- 6 Root Locus for Positive-Feedback Systems

Reading:

- Norman S. Nise, Control Systems Engineering, 7th edition, WILEY (chapter 8: page 382-448)
- Katsuhiko Ogata, Modern Control Engineering, 5th edition, Pearson (chapter 6: page 290-301)



Plotting Root Loci with MATLAB

In plotting root loci with MATLAB, we deal with the system equation given in the form

$$1 + K \frac{\text{num}}{\text{den}} = 0$$



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A MATLAB command commonly used for plotting root loci is

```
rlocus(num, den)
```

Using this command, the root-locus plot is drawn on the screen. The gain vector K is automatically determined.



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For the systems defined in state space, `rlocus(A, B, C, D)` plots the root locus of the system with the gain vector automatically determined.



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For the systems defined in state space, `rlocus(A, B, C, D)` plots the root locus of the system with the gain vector automatically determined.

Note that commands

```
rlocus(num, den, K) or rlocus(A, B, C, D, K)
```

use the user-supplied gain vector K .



Plotting Root Loci with MATLAB

If it is desired to plot the root loci with marks 'o' or 'x', it is necessary to use the following command:

```
r = rlocus(num,den)  
plot(r,'o') or plot(r,'x')
```

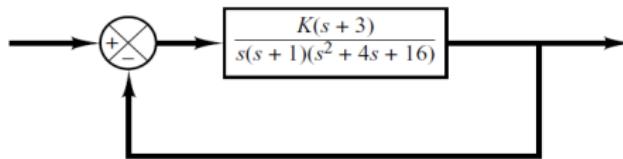
Plotting root loci using marks o or x is instructive, since each calculated closed-loop pole is graphically shown; in some portion of the root loci those marks are densely placed and in another portion of the root loci they are sparsely placed



Plotting Root Loci with MATLAB

Example 1

Consider the system shown below



From Ogata (Figure 6-15)

Plot root loci with a square aspect ratio so that a line with slope 1 is a true 45° line. Choose the region of root-locus plot to be

$$-6 \leq x \leq 6, -6 \leq y \leq 6$$

where x and y are the real-axis coordinate and imaginary-axis coordinate, respectively.



Plotting Root Loci with MATLAB

Example

To set the given plot region on the screen to be square, enter the command

```
v = [-6 6 -6 6]; axis (v); axis('square')
```

With this command, the region of the plot is as specified and a line with slope 1 is at a true 45° , not skewed by the irregular shape of the screen.

Define

$$a = s(s + 1); \quad a = [1 \ 1 \ 0]$$

$$b = s^2 + 4s + 16; \quad b = [1 \ 4 \ 16]$$



Plotting Root Loci with MATLAB

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Define

$$a = s(s + 1); \quad a = [1 \ 1 \ 0]$$

$$b = s^2 + 4s + 16; \quad b = [1 \ 4 \ 16]$$

Then, we use the following command:

```
c = conv(a,b)
```

The denominator polynomial is thus found to be

```
den = [1 5 20 16 0]
```



Plotting Root Loci with MATLAB

Example

To find the complex-conjugate open-loop poles (the roots of $s^2 + 4s + 16 = 0$), we may enter the roots command as follows:

```
r = roots(b)
```

Thus, the system has the following open-loop zero and open-loop poles:

Open-loop zero: $s = -3$

Open-loop poles: $s = 0, s = -1, s = -2 \pm j3.4641$



Plotting Root Loci with MATLAB

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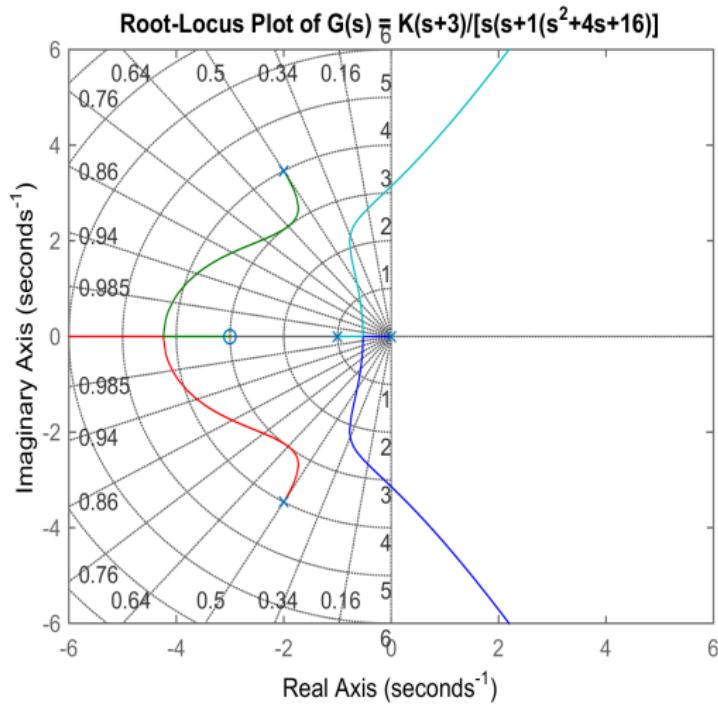
Open-loop poles: $s = 0, s = -1, s = -2 \pm j3.4641$

```
%-----Root-locus plot-----  
num = [1 3];  
den = [1 5 20 16 0];  
rlocus(num,den)  
v= [-6 6 -6 6];  
axis(v);axis('square')  
grid;  
title('Root-Locus Plot of G(s) = K(s+3)/[s(s+1)(s^2 + 4s + 16)]')
```



Plotting Root Loci with MATLAB

Example 1



Plotting Root Loci with MATLAB

Example 2

Consider the negative feedback system whose open-loop transfer function $G(s)H(s)$ is

$$G(s)H(s) = \frac{K}{s(s+0.5)(s^2+0.6s+10)}$$

There are no open-loop zeros. Open-loop poles are located at $s = -0.3 + j3.1480$, $s = -0.3 - j3.1480$, $s = -0.5$, and $s = 0$

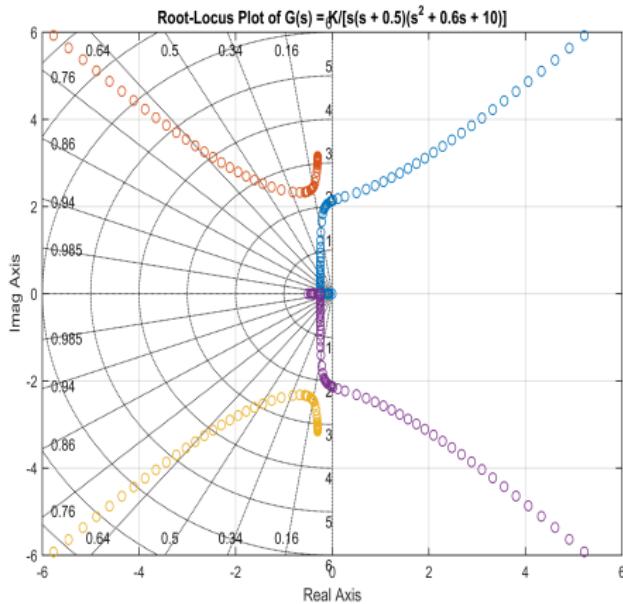
%-----Root-locus plot-----

```
num = [1];
den = [1 1.1 10.3 5 0];
rlocus(num,den)
plot(r,'o'); v= [-6 6 -6 6];
axis(v);axis('v'); grid
sgrid
title('Root-Locus Plot of G(s) = K/[s(s + 0.5)(s^2 + 0.6s + 10)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```



Plotting Root Loci with MATLAB

Example 2



we can also plot the root loci using smaller increments of K in the critical region.



Plotting Root Loci with MATLAB

Example 2

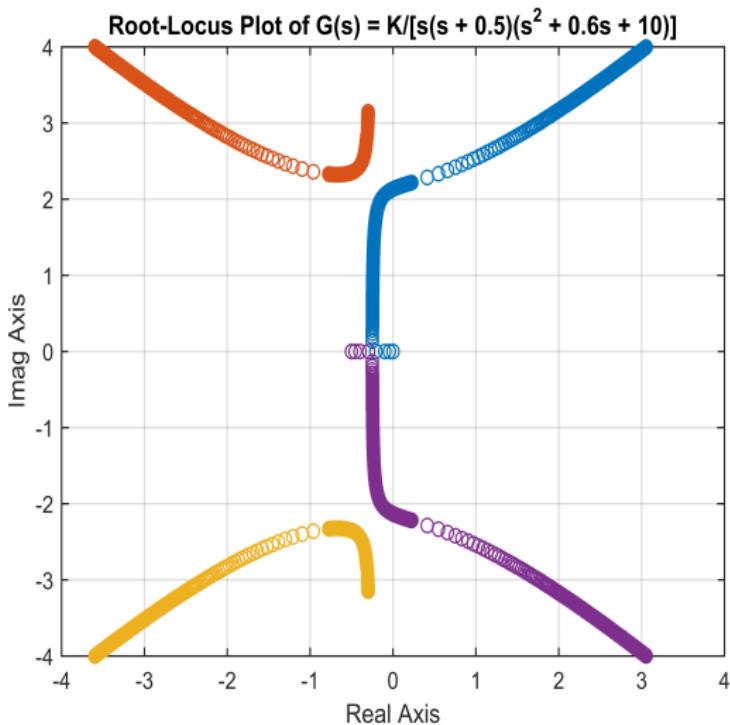
-----Root locus plot -----

```
num = [1]; den = [1 1.1 10.3 5 0];
K1 = 0:0.2:20; K2 = 20:0.1:30; K3 = 30:5:1000;
K = [K1 K2 K3];
r = rlocus(num,den,K);
plot(r, 'o')
v = [-4 4 -4 4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 0.5)(s^2 + 0.6s + 10)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```



Plotting Root Loci with MATLAB

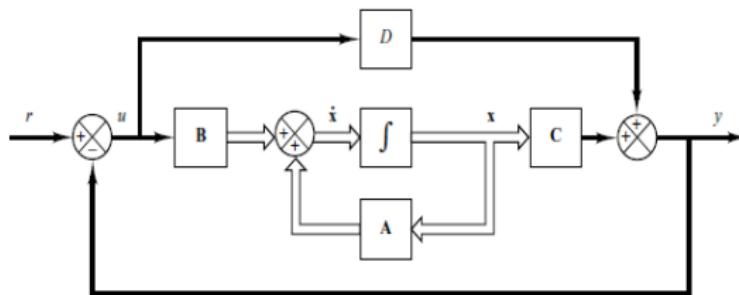
Example 2



Plotting Root Loci with MATLAB

Example 3

Consider the system shown below



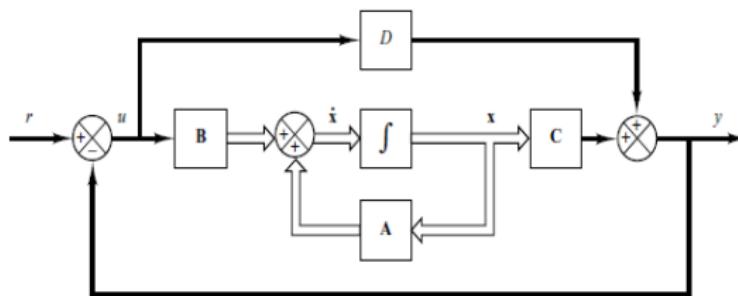
From Ogata (figure 6-19)



Plotting Root Loci with MATLAB

Example 3

Consider the system shown below



From Ogata (figure 6-19)

The system equations are

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$u = r - y$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix}$$
$$C = [1 \ 0 \ 0], D = [0]$$

In this example problem, we shall obtain the root-locus diagram of the system defined in state space.



Plotting Root Loci with MATLAB

Example 3

The root-locus plot for this system can be obtained with MATLAB by use of the following command:

```
rlocus(A,B,C,D)
```

This command will produce the same root-locus plot as can be obtained by use of the rlocus (num, den) command, where num and den are obtained from

```
[num,den]=ss2tf(A,B,C,D)
```



Plotting Root Loci with MATLAB

Example 3

The root-locus plot for this system can be obtained with MATLAB by use of the following command:

```
rlocus(A,B,C,D)
```

This command will produce the same root-locus plot as can be obtained by use of the rlocus (num, den) command, where num and den are obtained from

```
[num,den]=ss2tf(A,B,C,D)
```

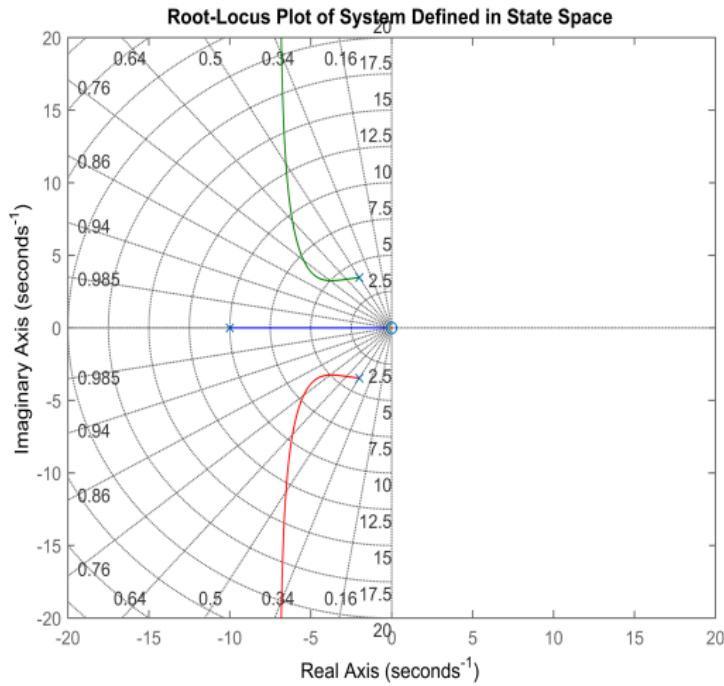
```
% -----Program-----
```

```
A = [0 1 0;0 0 1;-160 -56 -14]; B = [0;1;-14];
C = [1 0 0]; D = [0];K = 0:0.1:400;
rlocus(A,B,C,D,K);
v = [-20 20 -20 20]; axis(v)
grid
title('Root-Locus Plot of System Defined in State Space')
```



Plotting Root Loci with MATLAB

Example 3



Plotting Root Loci with MATLAB

Example 3: Constant ζ Loci and Constant ω_n Loci

The damping ratio ζ of a pair of complex-conjugate poles can be expressed in terms of the angle ϕ , which is measured from negative real axis with

$$\zeta = \cos \phi$$

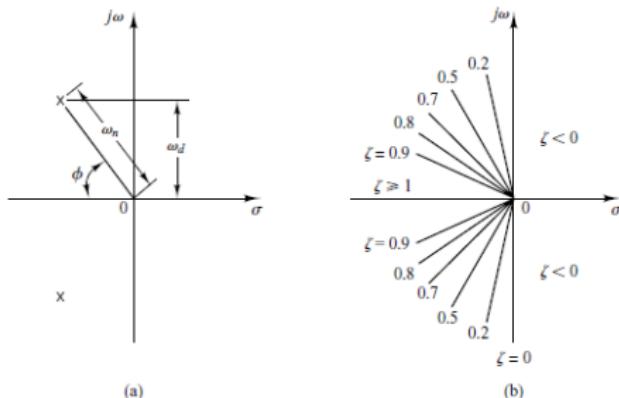


Plotting Root Loci with MATLAB

Example 3: Constant ζ Loci and Constant ω_n Loci

The damping ratio ζ of a pair of complex-conjugate poles can be expressed in terms of the angle ϕ , which is measured from negative real axis with

$$\zeta = \cos \phi$$



From Ogata (figure 6-21)

- The damping ratio determines the angular location of the poles.
- The distance of the pole from the origin is determined by the undamped natural frequency ω_n .



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram

The command

`sgrid`

overlays lines of constant damping ratio ($\zeta = 0$ to 1 with 0.1 increment) and circles of constant ω_n on the root-locus plot.



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram

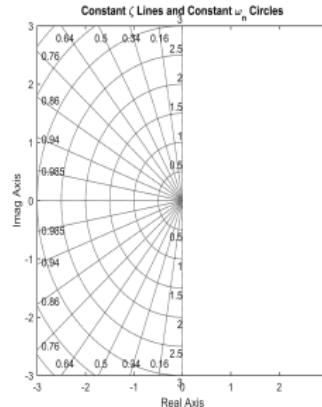
The command

`sgrid`

overlays lines of constant damping ratio ($\zeta = 0$ to 1 with 0.1 increment) and circles of constant ω_n on the root-locus plot.

-----Example 4 -----

```
sgrid  
v = [-3 3 -3 3];  
axis(v); axis('square')  
title('Constant  $\zeta$  Lines  
and Constant  $\omega_n$   
Circles')  
xlabel('Real Axis')  
ylabel('Imag Axis')
```



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram

If only particular constant ζ lines (such as the $\zeta = 0.5$ line and $\zeta = 0.707$ line) and particular constant ω_n circles (such as the $\omega_n = 0.5$ circle, $\omega_n = 1$ circle, and $\omega_n = 2$ circle) are desired, use the following command:

```
sgrid([0.5, 0.707], [0.5, 1, 2])
```



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram

If only particular constant ζ lines (such as the $\zeta = 0.5$ line and $\zeta = 0.707$ line) and particular constant ω_n circles (such as the $\omega_n = 0.5$ circle, $\omega_n = 1$ circle, and $\omega_n = 2$ circle) are desired, use the following command:

```
sgrid([0.5, 0.707], [0.5, 1, 2])
```

If we wish to overlay lines of constant ζ and circles of constant ω_n as given above to a root-locus plot of a negative feedback system with

```
num = [0 0 0 1]
den =[1 4 5 0]
```



Plotting Root Loci with MATLAB

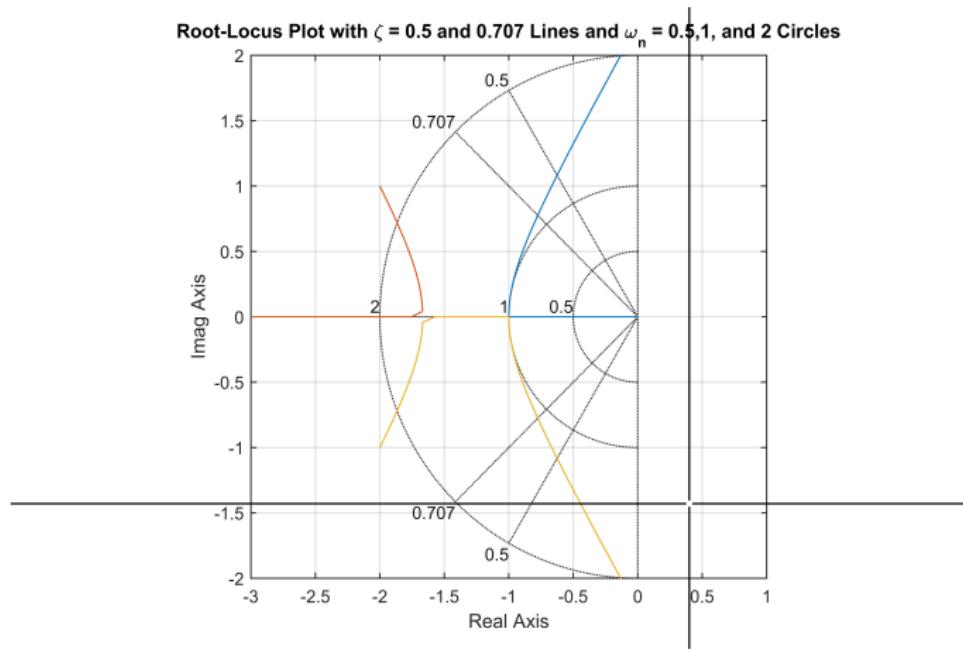
Plotting polar grid in the root-locus diagram

```
num = [1];
den = [1 4 5 0];
K = 0:0.01:1000;
r = rlocus(num,den,K);
plot(r,'-'); v = [-3 1 -2 2]; axis(v); axis('square')
sgrid([0.5,0.707], [0.5,1,2])
grid
title('Root-Locus Plot with \zeta = 0.5 and 0.707 Lines and
\omega_n = 0.5,1, and 2 Circles')
xlabel('Real Axis'); ylabel('Imag Axis')
gtext('\omega_n = 2')
gtext('\omega_n = 1') gtext('\omega_n = 0.5')
gtext('x')
gtext('x')
gtext('x')
```



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram

If we want to omit either the entire constant ζ lines or entire constant ω_n circles, we may use empty bracket [] in the arguments of the sgrid command.



Plotting Root Loci with MATLAB

Plotting polar grid in the root-locus diagram

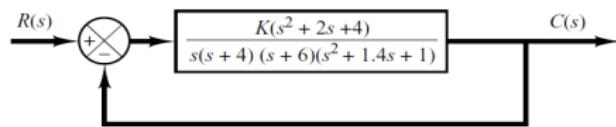
If we want to omit either the entire constant ζ lines or entire constant ω_n circles, we may use empty bracket [] in the arguments of the sgrid command. For example, if we want to overlay only the constant damping ratio line corresponding to $\zeta = 0.5$ and no constant ω_n circle on the root plot, the we may use this command

```
sgrid(0.5, [])
```



Plotting Root Loci with MATLAB

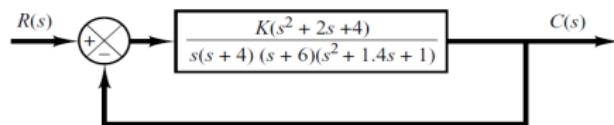
Conditionally stable systems



From Ogata (figure 6-24)

Plotting Root Loci with MATLAB

Conditionally stable systems



From Ogata (figure 6-24)

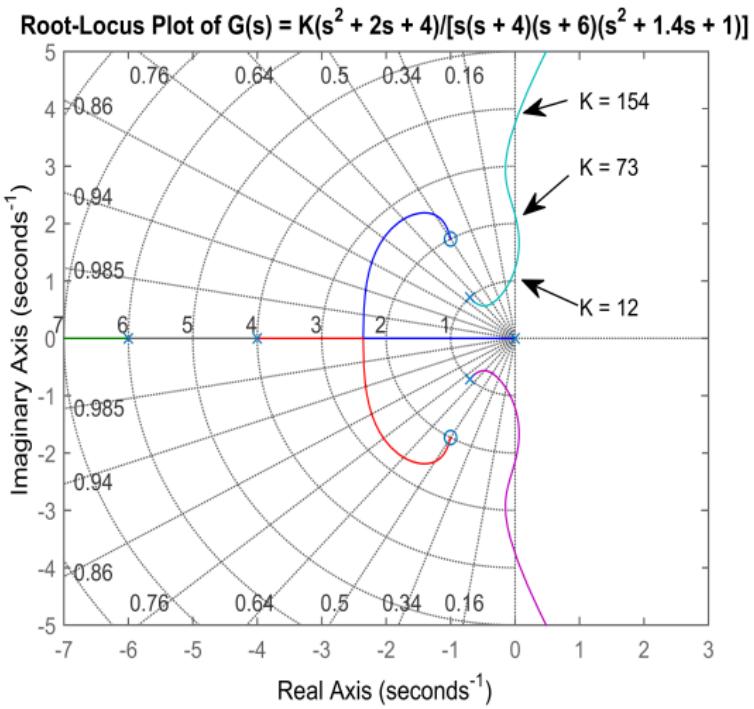
----- Example 6-----

```
num = [1 2 4];
den = conv(conv([1 4 0],[1 6]), [1 1.4 1]);
rlocus(num, den)
v = [-7 3 -5 5]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(s^2 + 2s + 4)/[s(s + 4)(s + 6)(s^2 + 1.4s + 1)]')
text(1.0, 0.55,'K = 12')
text(1.0,3.0,'K = 73')
text(1.0,4.15,'K = 154')
```



Plotting Root Loci with MATLAB

Conditionally stable systems



Plotting Root Loci with MATLAB

Conditionally stable systems

- This system is stable only for limited ranges of the value of K –that is,
 $0 \leq K \leq 12$ and $73 \leq K \leq 154$.



Plotting Root Loci with MATLAB

Conditionally stable systems

- This system is stable only for limited ranges of the value of K –that is, $0 \leq K \leq 12$ and $73 \leq K \leq 154$.
- The system becomes unstable for $12 \leq K \leq 73$ and $154 \leq K$.



Plotting Root Loci with MATLAB

Conditionally stable systems

- This system is stable only for limited ranges of the value of K –that is, $0 \leq K \leq 12$ and $73 \leq K \leq 154$.
- The system becomes unstable for $12 \leq K \leq 73$ and $154 \leq K$.
- If K assumes a value corresponding to unstable operation, the system may break down or may become nonlinear due to a saturation nonlinearity that may exist. Such a system is called conditionally stable.



Plotting Root Loci with MATLAB

Conditionally stable systems

In practice, conditionally stable systems are not desirable.



Plotting Root Loci with MATLAB

Conditionally stable systems

In practice, conditionally stable systems are not desirable.

Conditional stability is dangerous but does occur in certain systems—in particular, a system that has an unstable feedforward path. Such an unstable feedforward path may occur if the system has a minor loop.



Plotting Root Loci with MATLAB

Conditionally stable systems

In practice, conditionally stable systems are not desirable.

Conditional stability is dangerous but does occur in certain systems—in particular, a system that has an unstable feedforward path. Such an unstable feedforward path may occur if the system has a minor loop. It is

advisable to avoid such conditional stability since, if the gain drops beyond the critical value for any reason, the system becomes unstable.



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

- If all the poles and zeros of a system lie in the left half s plane, then the system is called **minimum phase**.



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

- If all the poles and zeros of a system lie in the left half s plane, then the system is called **minimum phase**.
- If a system has at least one pole or zero in the right-half s plane, then the system is called **nonminimum phase**.



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

- If all the poles and zeros of a system lie in the left half s plane, then the system is called **minimum phase**.
- If a system has at least one pole or zero in the right-half s plane, then the system is called **nonminimum phase**.

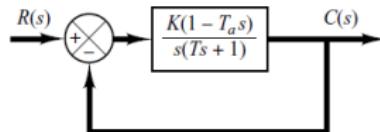
The term nonminimum phase comes from the phase-shift characteristics of such a system when subjected to sinusoidal inputs.



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

Consider the system shown



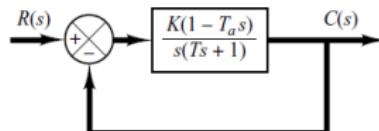
From Ogata (figure 6-26)



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

Consider the system shown



From Ogata (figure 6-26)

For this system,

$$G(s) = \frac{K(1-T_a s)}{s(Ts+1)}, (T_a \leq 0), H(s) = 1$$

This is a nonminimum-phase system, since there is one zero in the right-half s plane.



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

The angle condition becomes

$$\begin{aligned}\angle G(s) &= \angle -\frac{K(T_a s - 1)}{s(Ts + 1)} \\ &= \angle \frac{K(T_a s - 1)}{s(Ts + 1)} + 180^\circ \\ &= \pm 180^\circ (2k + 1), (k = 0, 1, 2, \dots)\end{aligned}$$



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

The angle condition becomes

$$\begin{aligned}\angle G(s) &= \angle -\frac{K(T_a s - 1)}{s(Ts + 1)} \\ &= \angle \frac{K(T_a s - 1)}{s(Ts + 1)} + 180^\circ \\ &= \pm 180^\circ (2k + 1), (k = 0, 1, 2, \dots)\end{aligned}$$

or

$$\angle \frac{K(T_a s - 1)}{s(Ts + 1)} = 0^\circ \quad (13)$$

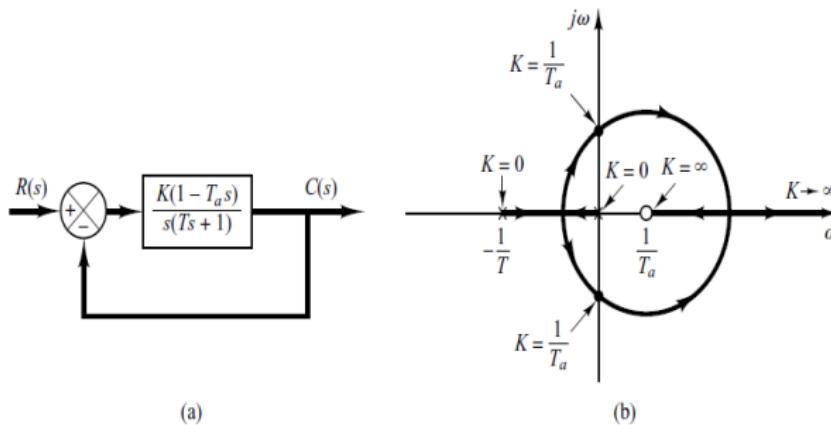
The root loci can be obtained from Equation (13).



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

Figure below shows a root-locus plot for this system. From the diagram, we see that the system is stable if the gain K is less than $1/T_a$.



From Ogata (figure 6-26)

Plotting Root Loci with MATLAB

Nonminimum-Phase Systems

For example, if $T = 1$ sec and $T_a = 0.5$ sec, The program is

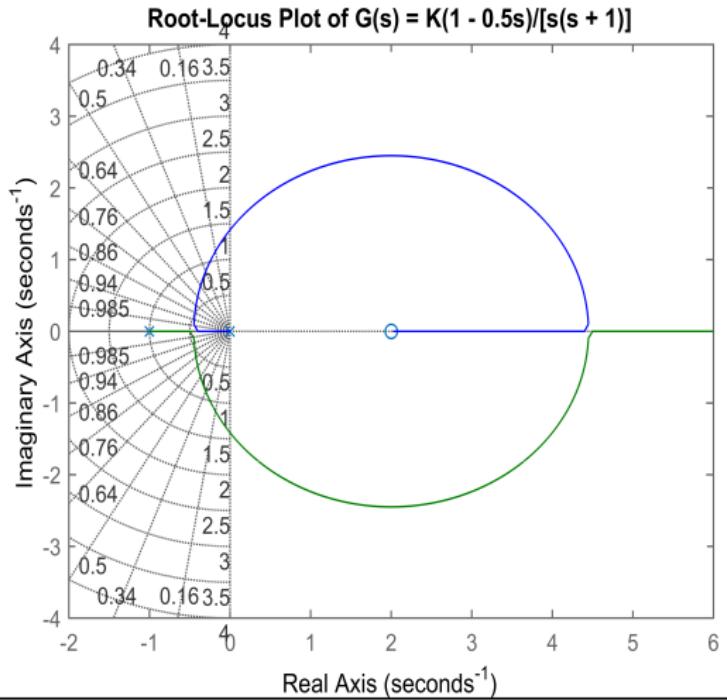
-----example 7 -----

```
num = [-0.5 1]; den = [1 1 0];
k1 = 0:0.01:30; k2 = 30:1:100;
K3 = 100:5:500; K = [k1 k2
k3];
rlocus(num,den,K)
v = [-2 6 -4 4]; axis(v);
axis('square')
grid
title('Root-Locus Plot of G(s)
= K(1 - 0.5s)/[s(s + 1)]')
gtext('x')
gtext('x')
gtext('o')
```



Plotting Root Loci with MATLAB

Nonminimum-Phase Systems



Plotting Root Loci with MATLAB

Finding the Gain value K at an Arbitrary Point on the Root Loci

It is frequently desired to find the gain value K at an arbitrary point on the root locus. This can be accomplished by using the following `rlocfind` command:

```
[K, r] = rlocfind(num, den)
```



Plotting Root Loci with MATLAB

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```
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```

- The `rlocfind` command, which must follow an `rlocus` command, overlays movable $x - y$ coordinates on the screen.



Plotting Root Loci with MATLAB

Finding the Gain value K at an Arbitrary Point on the Root Loci

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```
[K, r] = rlocfind(num, den)
```

- The `rlocfind` command, which must follow an `rlocus` command, overlays movable $x - y$ coordinates on the screen.
- Using the mouse, we position the origin of the $x - y$ coordinates over the desired point on the root locus and press the mouse button.



Plotting Root Loci with MATLAB

Finding the Gain value K at an Arbitrary Point on the Root Loci

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```
[K, r] = rlocfind(num, den)
```

- The `rlocfind` command, which must follow an `rlocus` command, overlays movable $x - y$ coordinates on the screen.
- Using the mouse, we position the origin of the $x - y$ coordinates over the desired point on the root locus and press the mouse button.
- Then MATLAB displays on the screen the coordinates of that point, the gain value at that point, and the closed-loop poles corresponding to this gain value.



Outline

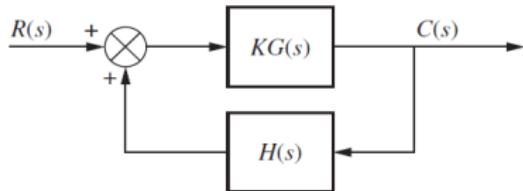
- 1 Introduction
- 2 Defining the Root Locus
- 3 Properties of the Root Locus
- 4 Root Locus Sketching
- 5 Plotting Root Loci with MATLAB
- 6 Root Locus for Positive-Feedback Systems

Reading:

- Norman S. Nise, Control Systems Engineering, 7th edition, WILEY (chapter 8: page 382-448)
- Katsuhiko Ogata, Modern Control Engineering, 5th edition, Pearson (chapter 6: page 290-301)



Root Locus for Positive-Feedback Systems

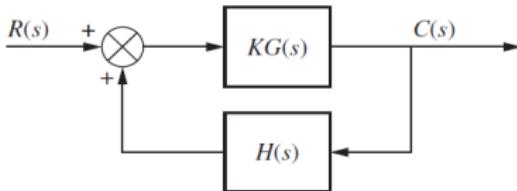


We find that the transfer function for the positive-feedback is given as

$$T(s) = \frac{KG(s)}{1 - KG(s)H(s)} \quad (14)$$



Root Locus for Positive-Feedback Systems



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From Eq. (14), obviously, a pole , s , exists when

$$KG(s)H(s) = 1 = 1\angle k360^\circ, k = 0, \pm 1, \pm 2, \pm 3, \dots \quad (15)$$



Root Locus for Positive-Feedback Systems

Sketching rule for positive feedback

- Rule 1: Number of branches

The same arguments as for negative feedback apply to this rule. There is no change.



Root Locus for Positive-Feedback Systems

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Root Locus for Positive-Feedback Systems

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The same arguments as for negative feedback apply to this rule. There is no change.

- Rule 3: Real-axis segments

Real-axis segments: On the real axis, the root locus for positive-feedback systems exists to the left of an even number of real-axis, finite open-loop poles and/or finite open-loop zeros.



Root Locus for Positive-Feedback Systems

Sketching rule for positive feedback

- Rule 4: Starting and ending points

Starting and ending points: The root locus for positive-feedback systems begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.



Root Locus for Positive-Feedback Systems

Sketching rule for positive feedback

- Rule 5: Behavior at infinity

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes for positive-feedback systems are given by the real-axis intercept, σ_a , and angle, θ_a , as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \quad (16)$$

$$\theta_a = \frac{2k\pi}{\#\text{finite poles} - \#\text{finite zeros}} \quad (17)$$



Root Locus for Positive-Feedback Systems

Sketching rule for positive feedback

- The imaginary axis crossing can be found using the root locus program.



Root Locus for Positive-Feedback Systems

Sketching rule for positive feedback

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- In a search of the $j\omega$ -axis, you are looking for the point where the angles add up to a multiple of 360° .



Root Locus for Positive-Feedback Systems

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Root Locus for Positive-Feedback Systems

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- In a search of the $j\omega$ -axis, you are looking for the point where the angles add up to a multiple of 360° .
- The breakaway points are found by looking for the maximum value of K .
- The break-in points are found by looking for the minimum value of K .



Reference

- Norman S. Nise, Control Systems Engineering, 7th edition, WILEY
- Katsuhiko Ogata, Modern Control Engineering, 5th edition, Pearson

