Control Systems
Lecture 8: Stability Analysis for Linear Systems

Daro VAN

Laboratory of Dynamics and Control
Department of Industrial and Mechanical Engineering
Institute of Technology of Cambodia

Phnom Penh, May 28th, 2020

Outline

Outline

1. Introduction
2. Routh's Stability Criterion
3. Stability in State Space

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition (chapter 5: page 212-217)

The most important problem in linear control systems concerns stability.

Introduction

The most important problem in linear control systems concerns stability.

- Want the system to be stable

The most important problem in linear control systems concerns stability.

- Want the system to be stable
- Under what conditions will a system become unstable?

Introduction

The most important problem in linear control systems concerns stability.

- Want the system to be stable
- Under what conditions will a system become unstable?
- if it is unstable, how should we stabilize the system?

- Determine the stability of the systems

- Determine the stability of the systems
    - transfer function

- Determine the stability of the systems
  - transfer function
  - state-space form

Outline

1 Introduction

2 Routh's Stability Criterion

3 Stability in State Space

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition (chapter 5: page 212-217)

Routh's Stability Criterion

consider the system represented in the transfer function below

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where the a's and b's are constants and $m \leq n$.

Routh's Stability Criterion

consider the system represented in the transfer function below

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where the a's and b's are constants and $m \leq n$.

- Stability of linear closed loop system determined by location of the poles.

Routh's Stability Criterion

consider the system represented in the transfer function below

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where the a's and b's are constants and $m \leq n$.

- Stability of linear closed loop system determined by location of the poles.
    - Stable systems– all poles must lie on the left hand side of s plane.

Routh's Stability Criterion

consider the system represented in the transfer function below

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

where the a's and b's are constants and $m \leqq n$.

- Stability of linear closed loop system determined by location of the poles.
  - Stable systems– all poles must lie on the left hand side of s plane.
  - Unstable systems – at least one pole lies on the right hand side
    - Output diverges for any input
  - Marginally stable systems – poles on the imaginary axis with all other poles on the left.
    (system stable for some input, unstable for others)

For each closed-loop transfer function $T(s)$, determine the stability system

- $T(s) = \frac{1}{(s+1)(s+3)}$, poles $(s = -1, s = -3) \Rightarrow$ Stable

For each closed-loop transfer function $T(s)$, determine the stability system

- $T(s) = \frac{1}{(s+1)(s+3)}$, poles $(s = -1, s = -3) \Rightarrow$ Stable
- $T(s) = \frac{s}{(s^2+1)^2}$, poles $s = \pm - j \Rightarrow$ marginally stable

Routh's Stability Criterion

Recap: a control system is stable if and only if all closed-loop poles lie in the left-half s plane.

Consider the closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where the a's and b's are constant values and $m \leq n$.

Recap: a control system is stable if and only if all closed-loop poles lie in the left-half s plane.

Consider the closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where the a's and b's are constant values and $m \leqq n$.

A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial.

Routh's Stability Criterion

- Determines if any of roots lie outside the left half plane

- Determines if any of roots lie outside the left half plane
- Doesn't give actual pole locations

1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.

1. Write the polynomial in s in the following form:
$$a_0s^n + a_1s^{n-1} + ... + a_{n-1}s + a_n = 0$$
where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.
2. Check:
   - Are any coefficient are equal to zero? Ex: $s^3 + 0s^2 + s + 3 = 0$
   (Not all roots lie on the left hand side)
   - Are any coefficients negative?
   (At least one root is on the right hand side $\Rightarrow$ unstable)

1. Write the polynomial in s in the following form:

$$a_0s^n + a_1s^{n-1} + ... + a_{n-1}s + a_n = 0$$

   where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.

2. Check:
   - Are any coefficient are equal to zero? Ex: $s^3 + 0s^2 + s + 3 = 0$
     (Not all roots lie on the left hand side)
   - Are any coefficients negative?
     (At least one root is on the right hand side $\Rightarrow$ unstable)

3. Unstable system is possible with all positive coefficients $\Rightarrow$ use Routh's stability criterion

$$a_0s^n + a_1s^{n-1} + ... + a_{n-1}s + a_n = 0$$

Routh's Criterion

Form the Routh Array

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

Routh's Criterion

Form the Routh Array

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2} & b_1 & b_2 & \ldots & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & \ldots & \\
s^{n-4} & d_1 & d_2 & \ldots & \ldots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2} & b_1 & b_2 & \ldots & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & \ldots & \\
s^{n-4} & d_1 & d_2 & \ldots & \ldots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_5 - a_4 a_1)$$

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2} & b_1 & b_2 & \ldots & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & \ldots & \\
s^{n-4} & d_1 & d_2 & \ldots & \ldots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_5 - a_4 a_1)$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_3)$$

Routh's Criterion

Form the Routh Array

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2} & b_1 & b_2 & \ldots & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & \ldots & \\
s^{n-4} & d_1 & d_2 & \ldots & \ldots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_5 - a_4 a_1)$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_3)$$

$$c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_5)$$

$$a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & \dots & \dots & \\
s^{n-3} & c_1 & c_2 & \dots & \dots & \\
s^{n-4} & d_1 & d_2 & \dots & \dots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_5 - a_4 a_1)$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_3)$$

$$c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_5)$$

$$d_1 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = -\frac{1}{c_1}(b_1 c_2 - c_1 b_2)$$

Routh's Criterion

Form the Routh Array

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2} & b_1 & b_2 & \ldots & \ldots & \\
s^{n-3} & c_1 & c_2 & \ldots & \ldots & \\
s^{n-4} & d_1 & d_2 & \ldots & \ldots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0a_3 - a_1a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0a_5 - a_4a_1)$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(a_1b_3 - b_1a_3)$$

$$c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{b_1}(a_1b_3 - b_1a_5)$$

$$d_1 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = -\frac{1}{c_1}(b_1c_2 - c_1b_2)$$

- Number of changes in sign in the first column = number of roots on right hand side

$$a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & \dots & \dots & \\
s^{n-3} & c_1 & c_2 & \dots & \dots & \\
s^{n-4} & d_1 & d_2 & \dots & \dots & \\
\vdots & \vdots & \vdots & & & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{a_1}(a_0 a_5 - a_4 a_1)$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_3)$$

$$c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{b_1}(a_1 b_3 - b_1 a_5)$$

$$d_1 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = -\frac{1}{c_1}(b_1 c_2 - c_1 b_2)$$

- Number of changes in sign in the first column = number of roots on right hand side
- No sign change $\Rightarrow$ Stable $\neq$ Sign changes $\Rightarrow$ unstable

Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the stability

Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the stability

$$\begin{array}{llll} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & \color{blue}{1} & 5 & \\ s^1 & \color{red}{-6} & & \\ s^0 & \color{blue}{5} & & \end{array}$$

Routh's Stability Criterion
Example

Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the stability

$$\begin{array}{llll} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & \color{blue}{1} & 5 & \\ s^1 & \color{red}{\text{-}6} & & \\ s^0 & \color{blue}{5} & & \end{array}$$

Two poles are in the right half plane

Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the stability

$$\begin{array}{llll} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & 1 & 5 & \\ s^1 & -6 & & \\ s^0 & 5 & & \end{array}$$

Two poles are in the right half plane

$\Downarrow$

Unstable

Let us apply Routh's stability criterion to the following third order polynomial

$$a_0s^3 + a_1s^2 + a_2s + a_3 = 0$$

where all the coefficients are positive numbers.The array of coefficients becomes

Let us apply Routh's stability criterion to the following third order polynomial

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers.The array of coefficients becomes

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

Let us apply Routh's stability criterion to the following third order polynomial

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers.The array of coefficients becomes

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all roots have negative real parts is given as

$$a_1 a_2 > a_0 a_3$$

If the first-column term element in any row is zero, but the remaining terms are not zero or there is no remaining term, then

If the first-column term element in any row is zero, but the remaining terms are not zero or there is no remaining term, then

- Replace the first zero with a very small number ϵ (assume to be positive or negative)

If the first-column term element in any row is zero, but the remaining terms are not zero or there is no remaining term, then

- Replace the first zero with a very small number ϵ (assume to be positive or negative)
- Calculations continue as normal. (some elements that follow will be function of ϵ)
- Complete the array

If the first-column term element in any row is zero, but the remaining terms are not zero or there is no remaining term, then

- Replace the first zero with a very small number ϵ (assume to be positive or negative)
- Calculations continue as normal. (some elements that follow will be function of ϵ)
- Complete the array
- Count sign change in first column
  - Allow ϵ to be a very tiny number (almost zero)
  - Number of sign changes = number of roots on the right hand side

Consider the following equation

$$s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

Routh's Stability Criterion

Special Case I: example

Consider the following equation

$$s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

The array of coefficient is

$$\begin{array}{llll} s^5 & 1 & 2 & 11 \\ s^3 & 2 & 4 & 10 \\ s^3 & 0 & 6 & \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array} \qquad \begin{array}{llll} s^5 & 1 & 2 & 11 \\ s^3 & 2 & 4 & 10 \\ s^3 & 0 \approx \epsilon & 6 & \\ s^2 & -12/\epsilon + 4 & 10 & \\ s^1 & 10\epsilon^2/12{+}6 & & \\ s^0 & 10 & & \end{array}$$

Assume ϵ to be positive, 2 sign changes $\Rightarrow$ unstable

If all the coefficients in any derived row are zero, it indicates that

If all the coefficients in any derived row are zero, it indicates that

- two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.

If all the coefficients in any derived row are zero, it indicates that

- two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.
- The evaluation of the rest of the array can be continued by
  - forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row.

If all the coefficients in any derived row are zero, it indicates that

- two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.
- The evaluation of the rest of the array can be continued by
  - forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row.
- Such roots with equal magnitudes and lying radially opposite in the s plane can be found by solving the auxi. polynomial, which is always even.

If all the coefficients in any derived row are zero, it indicates that

- two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.
- The evaluation of the rest of the array can be continued by
  - forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row.
- Such roots with equal magnitudes and lying radially opposite in the s plane can be found by solving the auxi. polynomial, which is always even.
- For a 2n-degree auxi. polynomial, there are n pairs of equal and opposite roots

Consider the following equation:

$$s^3 + s^2 + 2s$$

Consider the following equation:

$$s^3 + s^2 + 2s$$

The array of coefficient is

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

Note: auxi. polynomial is

$$1s^2 + 2s^0$$

Consider the following equation:

$$s^3 + s^2 + 2s$$

The array of coefficient is

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

Note: auxi. polynomial is

$$1s^2 + 2s^0$$

$$s^3 + s^2 + 2s = (s+1)(s^2+2)$$

Consider the following equation:

$$s^3 + s^2 + 2s$$

The array of coefficient is

$$P(s) = s^2 + 2$$

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

Note: auxi. polynomial is

$$1s^2 + 2s^0$$

$$s^3 + s^2 + 2s = (s+1)(s^2+2)$$

Consider the following equation:

$$s^3 + s^2 + 2s$$

The array of coefficient is

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & \color{red}{0} & \\ s^0 & & \end{array}$$

$$P(s) = s^2 + 2$$
$$\frac{dP(s)}{ds} = 2s$$

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & \color{blue}{2} & \\ s^0 & & \end{array}$$

Note: auxi. polynomial is

$$1s^2 + 2s^0$$

$$s^3 + s^2 + 2s = (s+1)\color{blue}{(s^2+2)}$$

Consider the following equation:

$$s^3 + s^2 + 2s$$

The array of coefficient is

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 0 & \\ s^0 & & \end{array}$$

$$P(s) = s^2 + 2$$
$$\frac{dP(s)}{ds} = 2s$$

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & 2 & \\ s^0 & & \end{array}$$

Note: auxi. polynomial is

$$1s^2 + 2s^0$$

$$s^3 + s^2 + 2s = (s+1)(s^2 + 2)$$

$$\Downarrow$$

Stable

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficient is

$$\begin{array}{llll}
s^5 & 1 & 24 & -25 \\
s^4 & 2 & 48 & -50 \\
s^3 & 0 & 0 & \\
s^2 & & & \\
s^1 & & & \\
s^0 & & &
\end{array}$$

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficient is

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	
s^2			
s^1			
s^0			

$$P(s) = 2s^4 + 48s^2 - 50$$
$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

Note that such a case occurs only in an odd numbered row. The auxi. polynomial is then formed from the coefficients of the s^4.

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficient is

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	
s^2			
s^1			
s^0			

$$P(s) = 2s^4 + 48s^2 - 50$$
$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	338/3		
s^0	-50		

Note that such a case occurs only in an odd numbered row. The auxi. polynomial is then formed from the coefficients of the s^4.

Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficient is

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	
s^2			
s^1			
s^0			

Note that such a case occurs only in an odd numbered row. The auxi. polynomial is then formed from the coefficients of the s^4.

$$P(s) = 2s^4 + 48s^2 - 50$$
$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	338/3		
s^0	-50		

There is one sign change

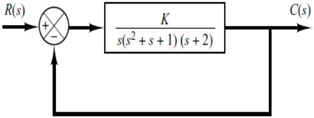
$\Rightarrow$ Unstable

We have the closed- loop transfer function as follows

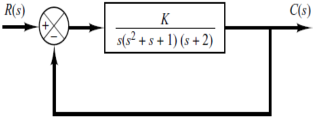
$$T(s) = \frac{18}{s^5+s^4-7s^3-7s^2-18s-18}$$

Is the system stable?

Consider the closed loop system shown below



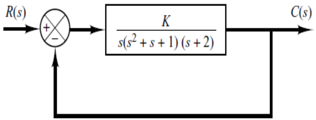
Consider the closed loop system shown below



The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Consider the closed loop system shown below

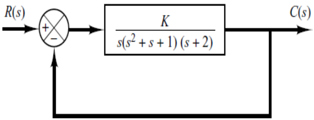


The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

s^4	1	3	K
s^3	3	2	0
s^2	7/3	K	
s^1	$2\text{-}\frac{9}{7}K$		
s^0	K		

Consider the closed loop system shown below



The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

s^4	1	3	K
s^3	3	2	0
s^2	7/3	K	
s^1	2-$\frac{9}{7}K$		
s^0	K		

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition (chapter 5: page 212-217)

Stability in State Space

- Location of poles determine stability
- How to find poles in state space?

Stability in State Space

- Location of poles determine stability
- How to find poles in state space?
    - Could convert state space to transfer function
    - Use eigenvalues

Stability in State-Space

Recall that the number λ is an eigenvalue if and only if there exists a non-zero vector x such that

$$Ax = \lambda x$$

Stability in State-Space

Recall that the number λ is an eigenvalue if and only if there exists a non-zero vector x such that

$$Ax = \lambda x$$

- All solutions will be null vector except when $(\lambda I - A)$ is singular matrix.

$$det(\lambda I - A) = 0$$

  determine the stability.

Example

Determine the stability. How many poles are on the left hand side, right hand side and $j\omega$ axis?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -4 \\ -1 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

Example

Determine the stability. How many poles are on the left hand side, right hand side and $j\omega$ axis?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -4 \\ -1 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$det(sI - A) = s^3 - 9s^2 + 12s - 4$$

Example

Determine the stability. How many poles are on the left hand side, right hand side and $j\omega$ axis?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -4 \\ -1 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$det(sI - A) = s^3 - 9s^2 + 12s - 4$$

s^3	1	12
s^2	-9	-4
s^1	104/9	
s^0	-4	

Example

Determine the stability. How many poles are on the left hand side, right hand side and $j\omega$ axis?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -4 \\ -1 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

$$det(sI - A) = s^3 - 9s^2 + 12s - 4$$

s^3	1	12
s^2	-9	-4
s^1	104/9	
s^0	-4	

3 sign changes $\Rightarrow$ 3 roots on the right hand side$\Rightarrow$ unstable.

Recall

$$G(s) = C[Is - A]^{-1}B + D$$

In order to find the pole, we need to find

$$\det(Is - A) = 0$$

Which is equivalent to finding the eigenvalue of A.

$$det(I\lambda - A)$$

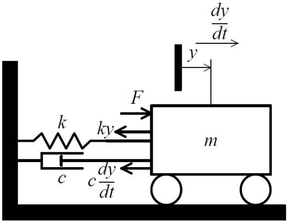
So λ can be used to determine the stability the system.

Stability in State Space

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

where k, m and c are constant values and

$x_1 = y, x_2 = \frac{dy}{dt}$

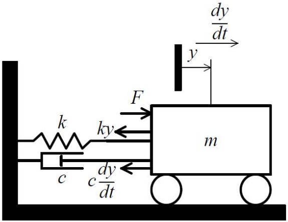


Stability in State Space

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

where k, m and c are constant values and

$x_1 = y, x_2 = \frac{dy}{dt}$



Compute the determinant

$$\det \begin{bmatrix} \lambda & -1 \\ k/m & s + c/m \end{bmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

The roots of the equation above are the poles of the system

Reference

- Katsuhiko Ogata, Modern Control Engineering, Fifth Edition, Pearson,