

Control Systems

L7: Transient and Steady State Response Analyses

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Phnom Penh, May 17th, 2020

Outline

- 1 Introduction
- 2 First Order Systems
- 3 Second Order Systems
- 4 Servo System with Velocity Feedback
- 5 Transient-Response Analysis with MATLAB

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition
(chapter 5)

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Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition
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Mathematical Models



Analysis of system performance
(various control systems)

Introduction

Typical Test Signals

The commonly used test input signals are:

- Step functions
- Ramp functions
- Impulse functions
- Acceleration functions
- Sinusoidal functions
- White noise

With these test signals, mathematical and experimental analyses of control systems can be carried out easily, since the signals are very simple functions of time.

Note: We will use the signals marked blue.

Introduction

Transient Response and Steady State Response

The time response of a control system consists of two parts:

- Transient response (from initial state to final state)
- Steady-state response (t approach infinity)

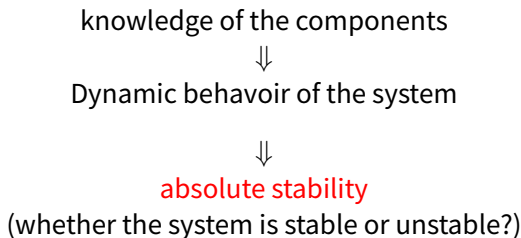
Thus, the system response $c(t)$ may be written as

$$c(t) = c_{tr}(t) + c_{ss}(t) \quad (1)$$

where the first term on the right-hand side of the equation is the transient response and the second term is the steady-state response.

Introduction

Absolute Stability, Relative Stability, and Steady-State Error



- A control system is in **equilibrium** if, in the **absence of any disturbance or input, the output stays in the same state**.
- A LTI system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition.
- A LTI is **critically stable** if oscillations of the output continue forever.
- It is **unstable** if the output **diverges** without bound from its equilibrium state when the system is subjected to an initial condition.

Introduction

Absolute Stability, Relative Stability and Steady State Error

Important system behavior (other than [absolute stability](#)) to which we must give careful consideration includes [relative stability](#) and [steady-state error](#).

Since a physical control system involves energy storage, the output of the system, when subjected to an input, cannot follow the input immediately but exhibits a transient response before a steady state can be reached.

If the output of a system at steady state does not exactly agree with the input, the system is said to have [steady-state error](#). This error is indicative of the accuracy of the system.

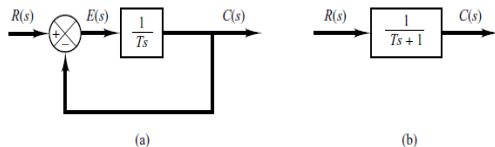
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First Order systems

Consider the first-order system shown below



The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1} \quad (2)$$

In the following, we shall analyze the system responses to such inputs as the **unit-step**, **unit-ramp**, and **unit-impulse functions**. The initial conditions are assumed to be zero.

First Order systems

Unit-Step Response of First Order Systems

Since the Laplace transform of the unit-step function is $1/s$, substituting $R(s) = 1/s$ into Equation (2), we obtain

$$C(s) = \frac{1}{Ts+1} \frac{1}{s}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s + (1/T)} \quad (3)$$

Taking the inverse Laplace transform, we obtain

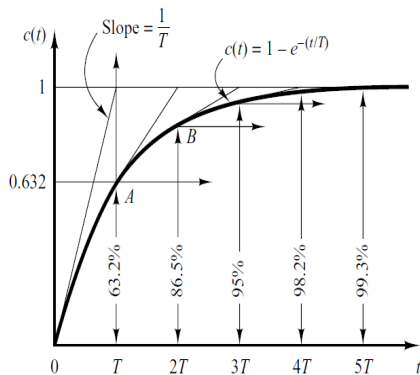
$$c(t) = 1 - e^{-t/T}, t \geq 0 \quad (4)$$

when $t = 0$ $c(t) = 0$
when $t = T$ $c(t) = 0.632$

The response $c(t)$ reaches 63.2% of its total change.

First-Order Systems

Unit-Step Response of First-Order Systems



- The smaller the time constant T , the faster the system response
- The slope of the response curve $c(t)$ decrease monotonically from $1/T$ at $t = 0$ to 0 at $t = \infty$.
- In practice, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value.

First Order systems

Unit-Ramp Response of First-Order Systems

Since the Laplace Transform of the unit-ramp function is $1/s^2$, we obtain the output of the system as

$$C(s) = \frac{1}{Ts + 1} \frac{1}{s^2} \quad (5)$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \quad (6)$$

Taking the inverse Laplace transform, we obtain

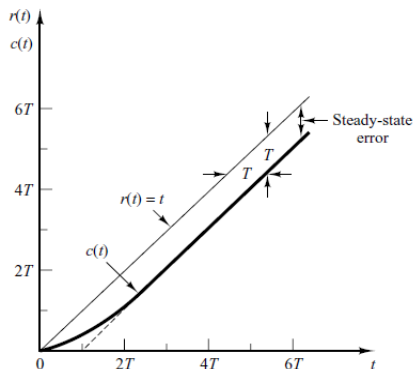
$$c(t) = t - T + Te^{-t/T}, t \geq 0 \quad (7)$$

The error signal $e(t)$ is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T(1 - e^{-t/T}) \end{aligned}$$

First Order Systems

Unit-Ramp Response of First-Order Systems



- As t approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches T or
$$e(\infty) = T$$
- The error in following the unit-ramp input is equal to T for sufficiently large t .

The smaller the time constant T , the smaller the steady-state error in following the ramp input.

First Order systems

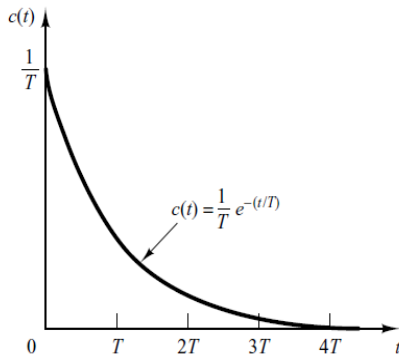
Unit-Impulse Response of First Order Systems

For the unit-impulse input, $R(s) = 1$ and the output of the system can be obtained as

$$C(s) = \frac{1}{Ts + 1} \quad (8)$$

The inverse Laplace Transform of Equation (8) gives

$$c(t) = \frac{1}{T} e^{-t/T}, t \geq 0 \quad (9)$$



First Order Systems

An Important Property of Linear Time Invariant Systems (LTI Systems)

It has been shown that the response of

- unit-ramp input is
$$c(t) = t - T + Te^{-t/T}, t \geq 0$$
- unit-step input is
$$c(t) = 1 - e^{-t/T}, t \geq 0$$
- unit-impulse input is
$$c(t) = \frac{1}{T}e^{-t/T}, t \geq 0$$

Note: This is a property of linear time-invariant systems. Linear time-varying systems and nonlinear systems do not possess these properties.

It can be seen that

- The response to the **derivative of an input signal** can be obtained by differentiating the response of the system to the original signal.
- The response to the **integral of the original signal** can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition.

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Second Order System

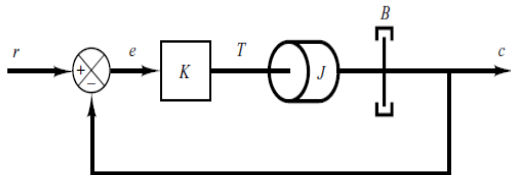
In this section, we shall obtain the response of a typical second-order control system to

- Step input
- Ramp input
- Impulse input

Second Order Systems

Servo System

The servo system consists of a proportional controllers and load elements (inertia and viscous-friction elements)



Suppose that we wish to control the output position c in accordance with the input position r . The equation for the load elements is

$$J\ddot{c} + B\dot{c} = T$$

where T is the torque produced by the proportional controller whose gain is K .

Second Order Systems

Servo System

By taking Laplace transforms of both sides of this last equation, assuming the zero initial conditions, we obtain

$$Js^2C(s) + BsC(s) = T(s)$$

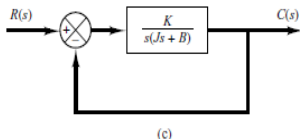
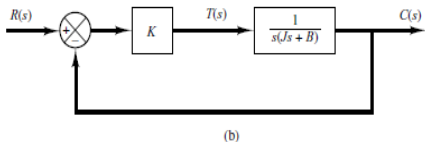
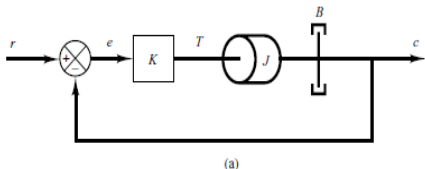
then,

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js+B)}$$

By using this transfer function, can you find $\frac{C(s)}{R(s)}$?

Second Order Systems

Servo System



The closed-loop transfer function is then obtained as

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{K}{Js^2 + Bs + K} \\ &= \frac{K/J}{s^2 + (B/J)s + (K/J)}\end{aligned}$$

Such a system where the closed-loop transfer function possesses **two poles** is called a **second-order system**.

Second Order Systems

Step Response of Second Order System

The closed-loop transfer function of the system shown in Figure (c) is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} \quad (10)$$

which can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{(s + \frac{B}{2J} + \sqrt{(\frac{B}{2J})^2 - \frac{K}{J}})(s + \frac{B}{2J} - \sqrt{(\frac{B}{2J})^2 - \frac{K}{J}})} \quad (11)$$

The closed-loop poles are complex conjugate if $B^2 - 4JK < 0$ and they are real if $B^2 - 4JK \geq 0$

Second Order Systems

Step Response of Second Order System

In the transient-response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2, \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where σ is called the attenuation, ω_n is called the undamped natural frequency and ζ is called damping ratio of the system.

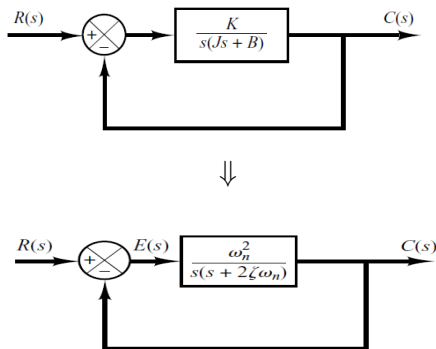
The damping ratio ζ is the ratio of the actual damping B to the critical damping $B_c = 2\sqrt{JK}$ or

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

Second Order Systems

Step Response of Second Order System

In term of ζ and ω_n , the system (c) can be modified to a generic form



Second Order Systems

Step Response of Second Order System

and the closed loop transfer function given in equation (10) can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (12)$$

This form is called **standard form** of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n .

Second-Order Systems

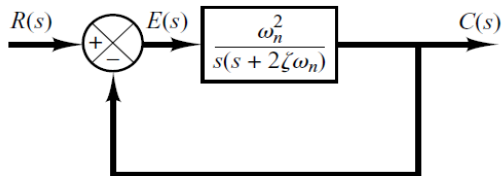
Step Response fo Second-Order System

- If $0 < \zeta < 1$, the closed-loop poles are complex conjugates and lie in the left-half s plane. The system is called **underdamped**, and the transient response is oscillatory.
- If $\zeta = 0$, the transient response does not die out.
- if $\zeta = 1$, the system is called **critically damped**.
- If $\zeta > 1$, the system is **overdamped**.

Second-Order Systems

Step Response of Second-Order System

We shall now solve for the response of the system shown below



Second-Order Systems

Step Response of Second-Order System

- **Underdamped case** ($0 < \zeta < 1$) : In the case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. ω_d is called **damped natural frequency**. For unit-step input, $C(s)$ can be written as

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} \quad (13)$$

The inverse Laplace transform can be obtained easily if $C(s)$ is written in the following form:

$$\begin{aligned} C(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \end{aligned}$$

Second-Order Systems

Step Response of Second-Order System

Referring to the Laplace transform Table, it can be shown that

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} \cos \omega_d t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} \sin \omega_d t$$

Hence, the inverse Laplace transform of $C(s)$ is obtained

$$\begin{aligned}\mathcal{L}^{-1}[C(s)] &= c(t) \\ &= 1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right) \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left(\omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right), t \geq 0\end{aligned}\tag{14}$$

Second-Order Systems

Step Response of Second-Order System

It can be seen that the **frequency of transient oscillation** is the damped natural frequency ω_d and thus varies with the damping ratio ζ . The error signal for this system is the difference between the input and output and is

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right), t \geq 0 \end{aligned}$$

This error signal exhibits a **damped sinusoidal oscillation**.

- At steady state, or at $t = \infty$, no error exists between the input and output.
- If the damping ratio ζ is equal to zero, the response becomes undamped oscillations continue indefinitely. The response $c(t)$ for zero damping can be found as

$$c(t) = 1 - \cos \omega_n t, t \geq 0 \quad (15)$$

Second-Order Systems

Step Response of Second-Order System

- **Critically damped case** ($\zeta = 1$) : If the two poles of $C(s)/R(s)$ are equal, the system is said to be critically damped one.

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written as

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s} \quad (16)$$

The inverse Laplace transform can be found as

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), t \geq 0 \quad (17)$$

Second-Order Systems

Step Response of Second-Order System

- **Overdamped case** ($\zeta > 1$): In this case, the two poles of $C(s)/R(s)$ are negative real and unequal. For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s} \quad (18)$$

The inverse Laplace transform of equation (18) is

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \\ - \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right), t \geq 0$$

where $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Second-Order Systems

Step Response of Second-Order System

- When ζ is appreciably greater than unity, one of the two decaying exponentials decreases much faster than the other, so the faster-decaying exponential term may be neglected. That is, if s_2 is located very much closer to the $j\omega$ axis than s_1 (which means $|s_2| \gg |s_1|$), then for an approximate solution we may neglect s_1 .
- This is permissible because the effect of s_1 on the response is much smaller than that of s_2 , since the term involving s_1 decays much faster than the term involving s_2 .
- Once the faster-decaying exponential term has disappeared, the response is similar to that of a first-order system, and $C(s)/R(s)$ may be approximated by

$$\frac{C(s)}{R(s)} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_1}$$

Second Order Systems

Step Response of Second Order Systems

With the appropriate transfer function $\frac{C(s)}{R(s)}$, the unit-step response can be obtained as

$$C(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

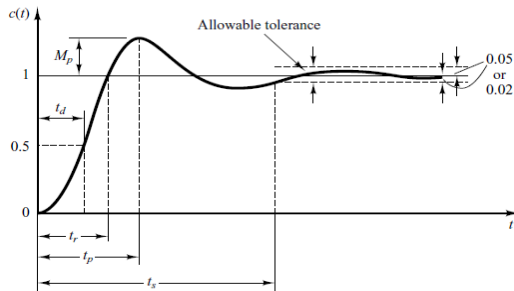
The time response $c(t)$ is then

$$c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \geq 0$$

This gives an approximate unit-step response when one of the poles of $C(s)/R(s)$ can be neglected.

Second Order Systems

Definition of Transient-Response Specifications



Unit-step response curve showing t_d, t_r, t_p, M_p and t_s

Second-Order Systems

Definition of Transient-Response Specifications

- **Delay time**, t_d : the time required for the response to reach half the final value the very first time.
- **Rise time**, t_r : the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped secondorder systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
- **Peak time**, t_p : the time required for the response to reach the first peak of the overshoot.

Definition of transient-Response Specification

- **Maximum overshoot**, M_p : the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- **Settling time**, t_s : the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).

Note: Not all these specifications necessarily apply to any given case. For example, for an overdamped system, the terms peak time and maximum overshoot do not apply.

Second-Order Systems

A Few Comments on Transient-Response Specifications

Except for certain applications where oscillations cannot be tolerated, it is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a desirable transient response of a second-order system, the damping ratio must be between 0.4 and 0.8.

- Small value of ζ ($\zeta < 0.4$) yield excessive overshoot in the transient response.
- Large value of ζ ($\zeta > 0.8$) responds sluggishly.

Note: The maximum overshoot and the rise time conflict with each other. Both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one of them is made smaller, the other necessarily becomes larger (you will see later)

Second Order Systems

Second-Order Systems and Transient-Response Specifications

- **Rise time** t_r : Referring to equation (14), we obtain the rise time by letting $c(t_r) = 1$.

$$c(t_r) = 1 = 1 - e^{\zeta\omega_n t_r} \left(\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right)$$

since $e^{\zeta\omega_n t_r} \neq 0$, we obtain the following equation:

$$\cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r = 0$$

Since $\omega_n \sqrt{1-\zeta^2} = \omega_d$ and $\zeta\omega_n = \sigma$, we have

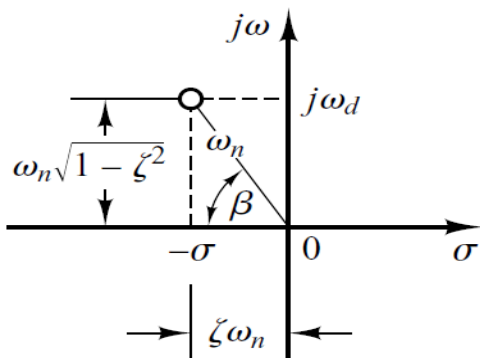
$$\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time t_r is

$$t_r = \frac{1}{\omega_d} \tan^{-1} \left(\frac{\omega_d}{-\sigma} \right) = \frac{\pi - \beta}{\omega_d}$$

Second Order Systems

Second Order System and Transient-Response Specifications



Clearly, for small value of t_r, ω_d , angle β must be large.

Second Order Systems

Second Order System and Transient-Response Specifications

- **Peak time** t_p : Referring to equation (14), we may obtain peak time by differentiating $c(t)$ with respect to time and letting this derivative equal to zero

$$\begin{aligned}\frac{dc}{dt} = & \zeta\omega_n e^{-\zeta\omega_n t} \left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t \right) \\ & + e^{-\zeta\omega_n t} \left(\omega_d \sin\omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} \cos\omega_d t \right)\end{aligned}$$

and the cosine terms in this last equation cancel each other, dc/dt , evaluated at $t = t_p$, can be simplified to

$$(\sin\omega_d t_p) \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin\omega_d t_p = 0 \text{ or } \omega_d t_p = 0, \pi, 2\pi, 3\pi, \dots$$

Since the peak time corresponds to the first peak overshoot, Hence

$$t_p = \frac{\pi}{\omega_d}$$

Second Order Systems

Second Order System and Transient-Response Specifications

- **Maximum overshoot** M_p : occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, M_p is obtained from Equation (14) as

$$\begin{aligned} M_p &= c(t_p) - 1 \\ &= -e^{-\zeta\omega_n(\pi/\omega_d)} \left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \\ &= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \end{aligned} \quad (19)$$

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

If the final value $c(\infty)$ of the output is not unity, then we need to use the following equation:

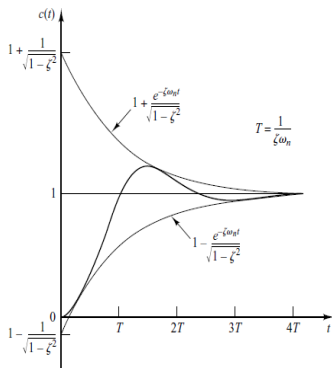
$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Second Order System

Second Order System and Transient-Response Specifications

- **Setting time** t_s : For an underdamped second-order system, the transient response is obtained from Equation (14) as

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}), t \geq 0$$



Second Order system

Second Order System and Transient Response Specification

For convenience in comparing with the responses of systems, we commonly define the settling time t_s to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \text{ (2\% criterion)}$$

or

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$$

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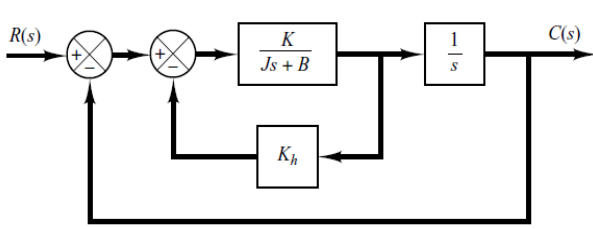
Servo System with Velocity Feedback

- The derivative of the output signal can be used to improve system performance. In obtaining the derivative of the output position signal, it is desirable to use a tachometer instead of physically differentiating the output signal.

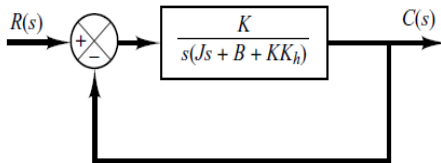
Note: The differentiation amplifies noise effects. In fact, if discontinuous noises are present, differentiation amplifies the discontinuous noises more than the useful signal.

Servo Systems with Velocity Feedback

In this system, the velocity signal, together with the position signal, is fed back to the input to produce the actuating error signal.



The block diagram can then be simplified as



Second-Order Systems

Servo Systems with Velocity Feedback

giving

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Comparing with equation (10), the damping ratio ζ becomes

$$\zeta = \frac{B + KK_h}{2\sqrt{KJ}} \quad (20)$$

The undamped natural frequency $\omega_n = \sqrt{K/J}$ is not affected by velocity feedback.

Note: the maximum overshoot for a unit-step input can be controlled by controlling the value of the damping ratio ζ , we can reduce the maximum overshoot by adjusting the velocity-feedback constant K_h so that ζ is between 0.4 and 0.7.

It is important to remember that velocity feedback has the effect of increasing the damping ratio without affecting the undamped natural frequency of the system.

Second-Order Systems

Impulse Response of Second-Order Systems

For a unit-impulse input $r(t)$, the corresponding Laplace transform is unity, or $R(s) = 1$. The unit-impulse response $C(s)$ of the second-order system is

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (21)$$

The inverse Laplace transform of this equation yields the time solution for the response $c(t)$ as follows:

For $0 \leq \zeta \leq 1$

$$c(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1 - \zeta^2} t, t \geq 0 \quad (22)$$

For $\zeta = 1$

$$c(t) = \omega_n^2 t e^{-\omega_n t}, t \geq 0 \quad (23)$$

For $\zeta > 1$,

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t}, t \geq 0 \quad (24)$$

Outline

- 1 Introduction
- 2 First Order Systems
- 3 Second Order Systems
- 4 Servo System with Velocity Feedback
- 5 Transient-Response Analysis with MATLAB**

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition
(chapter 5)

Transient-Response Analysis with MATLAB

MATLAB Representation of Linear Systems

The transfer function of a system is represented by two arrays of numbers. Consider the system

$$\frac{C(s)}{R(s)} = \frac{2s + 25}{s^2 + 4s + 25} \quad (25)$$

This system can be represented as two arrays, each containing the coefficients of the polynomials in decreasing powers of s as follows:

$$\begin{aligned} \text{num} &= [2, 25] \\ \text{den} &= [1, 4, 25] \end{aligned}$$

If `num` and `den` of the closed-loop transfer function are known, commands such as

$$\text{step}(\text{num}, \text{den}), \text{step}(\text{num}, \text{den}, t)$$

will generate plots of unit-step responses.

Transient-Response Analysis with MATLAB

MATLAB Representation of Linear Systems

For a control system defined in a state-space form, where state matrix A , control matrix B , output matrix C , and direct transmission matrix D of state-space equations are known, the commands

```
step(A,B,C,D), step(A,B,C,D,t)
```

will generate plots of unit-step responses.

Note: the command `step(sys)` may be used to obtain the unit-step response of a system. First, define the system by

```
sys = tf(num,den) or sys=ss(A,B,C,D)
```

Then, to obtain, for example, the unit-step response, enter

```
step(sys)
```

Transient-Response Analysis with MATLAB

MATLAB Representation of Linear Systems

Consider the following systems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Obtain the unit-step response curves.

After doing the Maths, we obtain, the transfer matrix $G(s)$ given as

$$G(s) = \frac{1}{s^2+s+6.5} \begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix}$$

and

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s^2+s+6.5} & \frac{s}{s^2+s+6.5} \\ \frac{s+7.5}{s^2+s+6.5} & \frac{6.5}{s^2+s+6.5} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Transient-Response Analysis with MATLAB

MATLAB Representation of Linear System

Since this system involves two inputs and two outputs, four transfer functions may be defined, depending on which signals are considered as input and output.

Note: when considering the signal u_1 as the input, we assume that signal u_2 is zero, and vice versa. The four transfer functions are

$$\begin{aligned}\frac{Y_1(s)}{U_1(s)} &= \frac{s-1}{s^2+s+6.5} & \frac{Y_1(s)}{U_2(s)} &= \frac{s}{s^2+s+6.5} \\ \frac{Y_2(s)}{U_1(s)} &= \frac{s+7.5}{s^2+s+6.5} & \frac{Y_2(s)}{U_2(s)} &= \frac{6.5}{s^2+s+6.5}\end{aligned}$$

In MATLAB, the plot can be obtained by using the following code

```
A = [1, -1; 6.5, 0];  
B = [1, 1; 1, 0];  
C = [1, 0; 0, 1];  
D = [0, 0; 0, 0];  
step(A, B, C, D)
```

Transient-Response Analysis with MATLAB

MATLAB Description of Standard Second-Order System

The second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (26)$$

is called the standard second-order system. Given ω_n and ζ , the command

`printsys(num,den)` or `printsys(num,den,s)`

Consider the case where $\omega_n = 5\text{rad/sec}$ and $\zeta = 0.4$, MATLAB program generate the standard second-order system.

MATLAB Program:

```
wn = 5;  
dampingratio = 0.4;  
[num0,den] = ord2(wn,dampingratio);  
num = 25 * num0;  
printsys(num,den's')
```

Transient-Response Analysis with MATLAB

Impulse Response

The unit-impulse response of a control system may be obtained by using any of the impulse commands such as

```
impulse(num,den)
```

```
impulse(A,B,C,D)
```

```
[y,x,t] = impulse(num,den)
```

```
[y,x,t]= impulse(num,den,t)
```

```
[y,x,t]= impulse(A,B,C,D)
```

```
[y,x,t] = impulse(A,B,C,D,iu)
```

```
[y,x,t] = impulse(A,B,C,D,iu,t)
```

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition
(page 183-211)

Transient-Response Analysis with MATLAB

Obtaining Response to Arbitrary Input

To obtain the response to an arbitrary input, the command `lsim` may be used. The commands like

```
lsim(num,den,r,t)
```

```
lsim(A,B,C,D,u,t)
```

```
y = lsim(num,den,r,t)
```

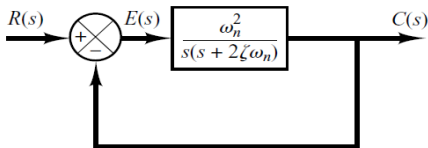
```
y = lsim(A,B,C,D,u,t)
```

will generate the response to input time function `r` or `u`.

Reading: Modern Control Engineering by Katsuhiko Ogata, 5th edition
(page 183-211)

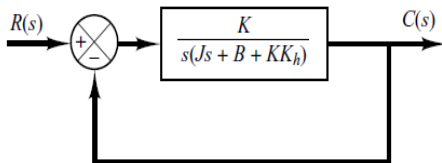
Homework

- Consider the system shown below, where $\zeta = 0.6$ and $\omega_n = 5\text{rad/sec}$, obtain the rise time t_r , peak time t_p , maximum overshoot M_p and setting time t_s . The system is subjected to the step input.



Homework

- Consider the system shown below, determine the value of gain K and velocity feedback constant K_h so that the maximum overshoot is the unit-step response is 0.2 and the peak time is 1sec. With these values of K and K_h , obtain the rise time and settling time. Assume that $J = 1\text{kg} - \text{m}^2$ and $B = 1\text{N} - \text{m}/\text{rad}/\text{sec}$.



- Katsuhiko Ogata, Modern Control Engineering, Fifth Edition, Pearson,

