Feedback Control Systems
Lecture 4 Output Feedback

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Content

- Introduction to State Estimation
- Observability
- Control Using Estimated State
- Kalman Filtering
- References

Recap

Model
State Space
approach

$$\dot{x} = f(x, u, d)$$
$$y = h(x, u, d)$$

Linearization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Analysis → **reachability**

Simulation

Prediction

Design → **Tuning**

Control Design
(State Feedback)
$u = -Kx + k_r r$

Recap

- We studied how to design a state feedback controller. The Design method based on the assumption that a state vector is available for measurements (This might not always the case)
- We will introduce **ways to get access to the state vector without actually measuring it**.

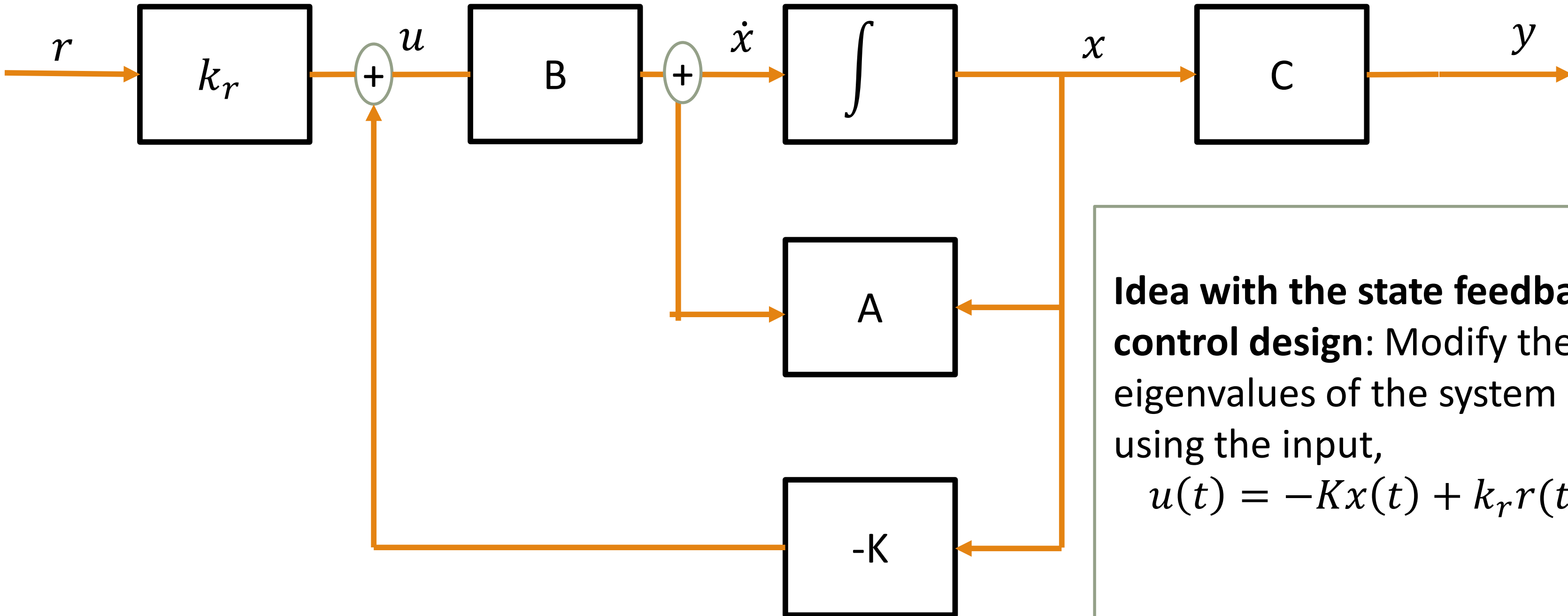
Introduction to State Estimation
Concepts

The **state feedback control design** requires that you have access to the complete state vector.

It is usually expensive to measure all states and sometimes it might be impossible to measure certain qualities (it might be physically impossible to fit a sensor or there might not be such a sensor available at all.

In automotive industry, the concept is called soft sensor. In other words, we use software to get access to the quantity that we would like to measure instead of hardware.

Introduction to State Estimation



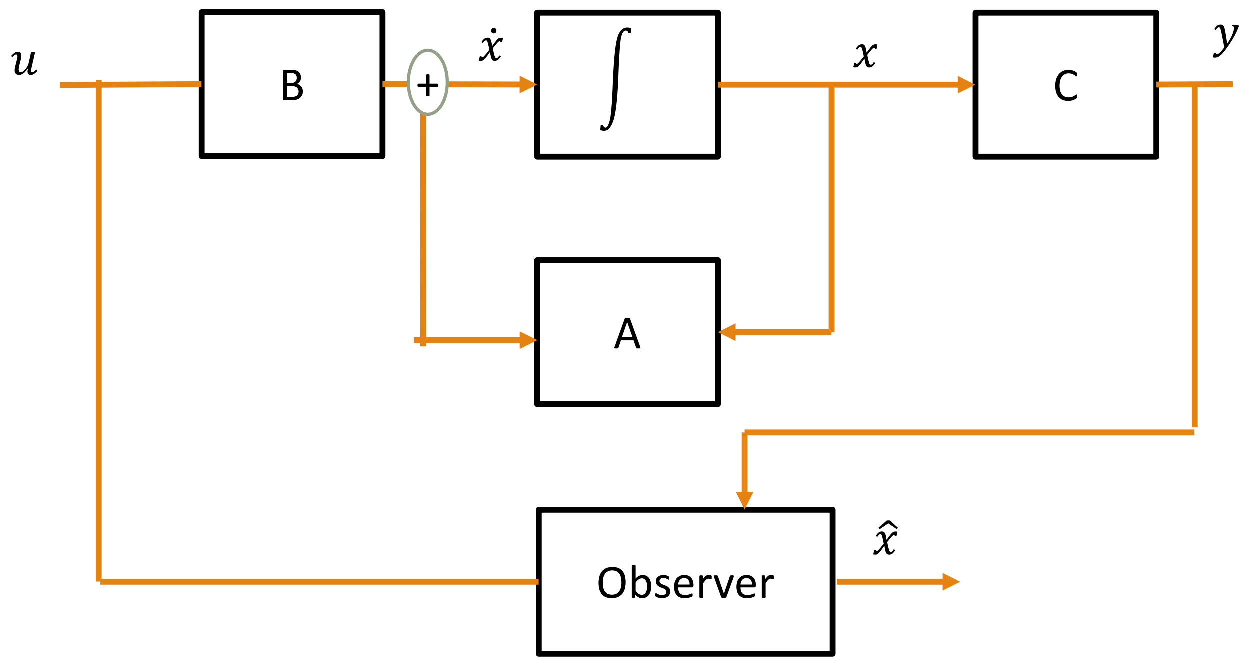
Idea with the state feedback control design: Modify the eigenvalues of the system by using the input,

$$u(t) = -Kx(t) + k_r r(t)$$

Problem: Requires full access to the state vector.

Introduction to State Estimation

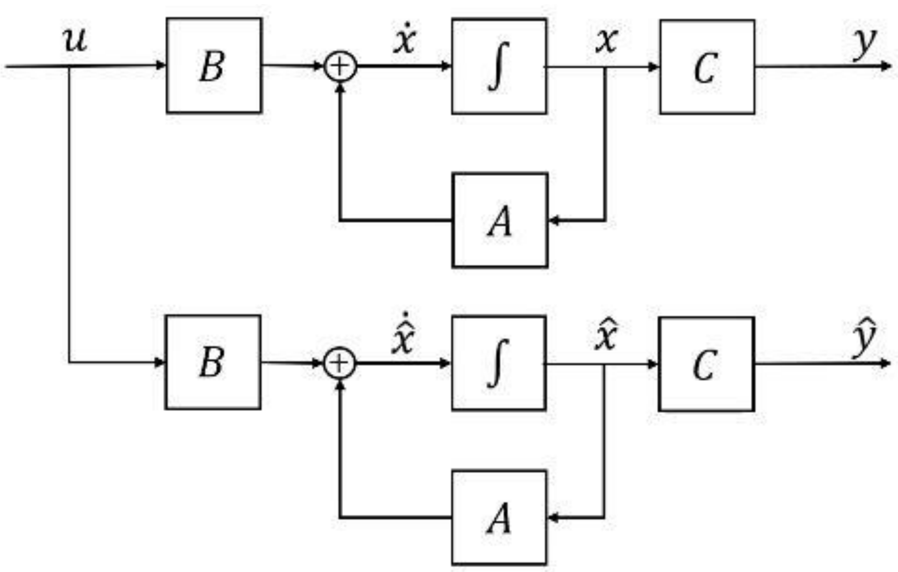
Idea of state estimation: Develop an observer of the dynamic systems that provides an estimate, $\hat{x}$, of the system's state.



Introduction to State Estimation

Open Loop Estimation

Goal of state estimation: $\tilde{x} = x - \hat{x} = 0$
Idea: Use a copy of the model description in the observer: $\dot{\hat{x}} = A\hat{x} + Bu$



Realistic?

Analyze the **error dynamics**:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu$$
$$= A(x - \hat{x}) = A\tilde{x}$$

Which has the solution:

$$\tilde{x}(t) = e^{At}\tilde{x}(0) = e^{At}\big(x(0) - \hat{x}(0)\big)$$

Problem: A needs to be stable, then $\tilde{x} \to 0 \; as \; t \to \infty$

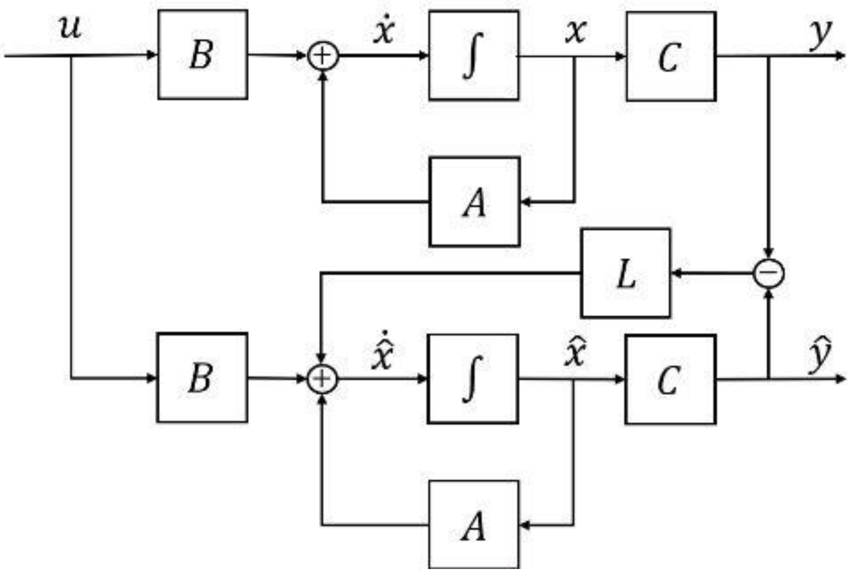
Open loop estimation does not seem to be a good idea!

State Estimation

Closed Loop Estimator

Goal of state estimation: $\tilde{x} = x - \hat{x} = 0$
Idea: Use feedback from y in the observer

to improve the estimates



Compare the estimated output with measured:
$\tilde{y} = y - \hat{y} = Cx - C\hat{x} = C\tilde{x}$
Feedback the error to the open loop estimation via feedback gain L:

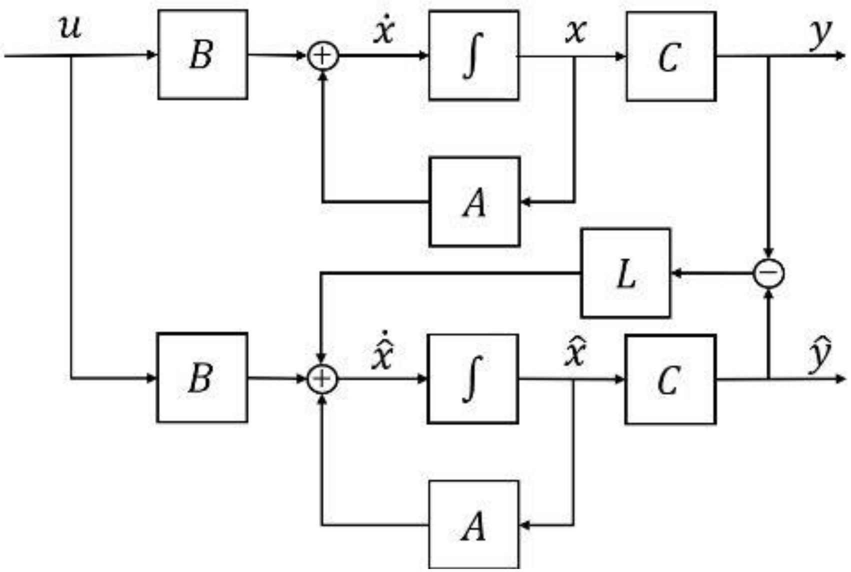
$$\dot{\hat{x}} = A\hat{x} + Bu + L\tilde{y}$$
$$\hat{y} = C\hat{x}$$

State Estimation

Closed Loop Estimator

Goal of state estimation: $\tilde{x} = x - \hat{x} = 0$
Idea: Use feedback from y in the observer

to improve the estimates



Analyze the **error dynamics**

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$
$$= Ax + Bu - A\hat{x} - Bu - L\tilde{y}$$
$$= A(x - \hat{x}) - L(y - \hat{y})$$
$$= A(x - \hat{x}) - L(Cx - C\hat{x})$$
$$= (A - LC)\tilde{x}$$

Which has the solution:

$$\tilde{x}(t) = e^{(A-LC)t}\tilde{x}(0)$$

L can be chosen such that the error dynamics converges, $\tilde{x} \to 0$ as $t \to \infty$, (**if observable**).

Note: This gives us the freedom to from the error dynamics using the gain L.

Introduction to State Estimation

Closed Loop Estimator

How do we choose the **estimated gain**?

The way to choose estimator gain is similar to that used for state feedback control design. Using the closed loop estimator, the error dynamics becomes:

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

Objective: Choose L such that the closed loop error dynamics $A - LC$ get desired properties, i.e **a suitable convergence with the specified time.**

The closed loop poles of the estimator are the roots to the characteristic polynomial:

$$\det(sI - A + LC) = 0$$

Use pole placement with the desired characteristic polynomial to choose the estimator gain, L.

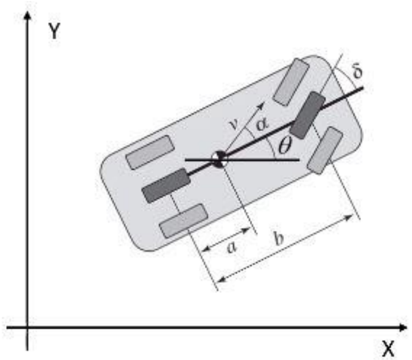
Introduction to State Estimation

Revisit Example – Vehicle Steering

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$



Where x_1 is the lateral position Y, x_2 is the heading orientation θ and u is the steering angle δ.

Design a state estimator to estimate the vehicle's states from the measurement of the lateral position.

Specification: Desired characteristic polynomial

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

Introduction to State Estimation

Revisit Example – Vehicle Steering

State estimator

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \left(\begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}\right) = \begin{bmatrix} -l_1 & 12 \\ -l_2 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda + l_1 & -12 \\ l_2 & \lambda \end{vmatrix} = \lambda^2 + l_1\lambda + 12l_2$$

Match with desired characteristic polynomial gives

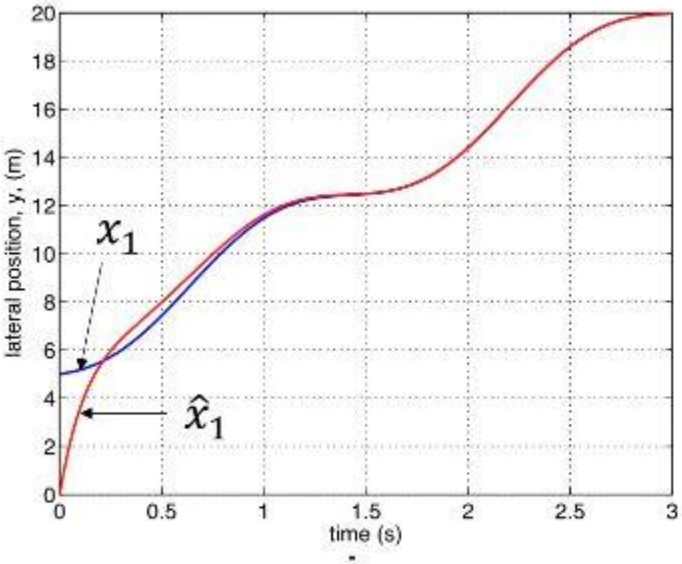
$$\lambda^2 + 10\lambda + 24 \equiv \lambda^2 + l_1\lambda + 12\lambda_2 \Longrightarrow l_1 = 10, l_2 = 2$$

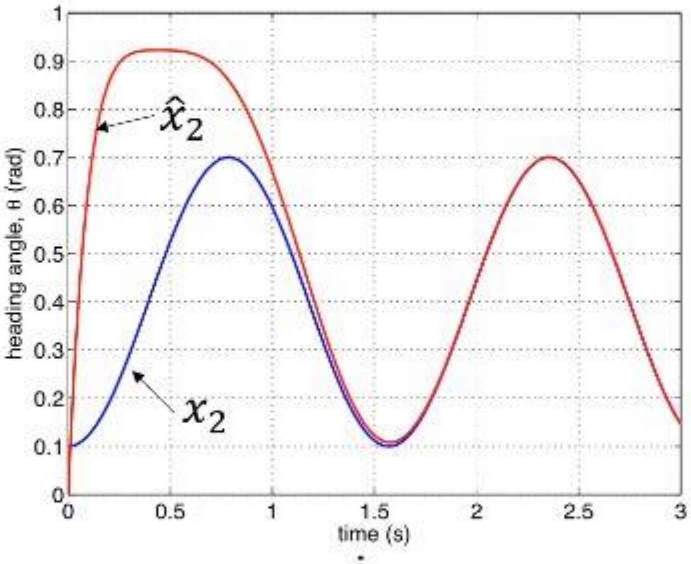
State estimator:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix}u + \begin{bmatrix} 10 \\ 2 \end{bmatrix}(y - \hat{x}_1)$$

Introduction to State Estimation
Revisit Example – Vehicle Steering

Simulations using a sinusoidal input with $x(0) = (5,0.1)$ and $\hat{x} = (0,0)$





Can you interpret the results?

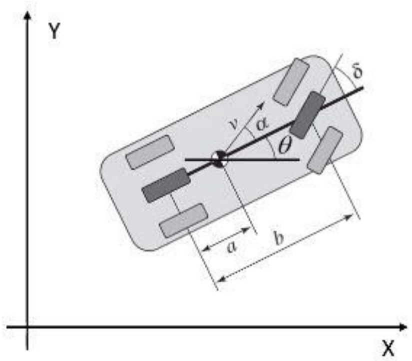
Introduction to State Estimation

Revisit Example – Vehicle Steering

Consider the same system with different measurement given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$



Where x_1 is the lateral position Y, x_2 is the heading orientation θ and u is the steering angle δ.

Design a state estimator to estimate the vehicle's states from the measurement of the heading angle.

Specification: Desired characteristic polynomial

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

Introduction to State Estimation

Revisit Example – Vehicle Steering

State estimator

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \begin{bmatrix} 0 & 12 - l_1 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The closed loop system has the characteristic polynomial

unobservable

$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda & -12 + l_1 \\ 0 & \lambda + l_2 \end{vmatrix} = \lambda(\lambda + l_2)$$

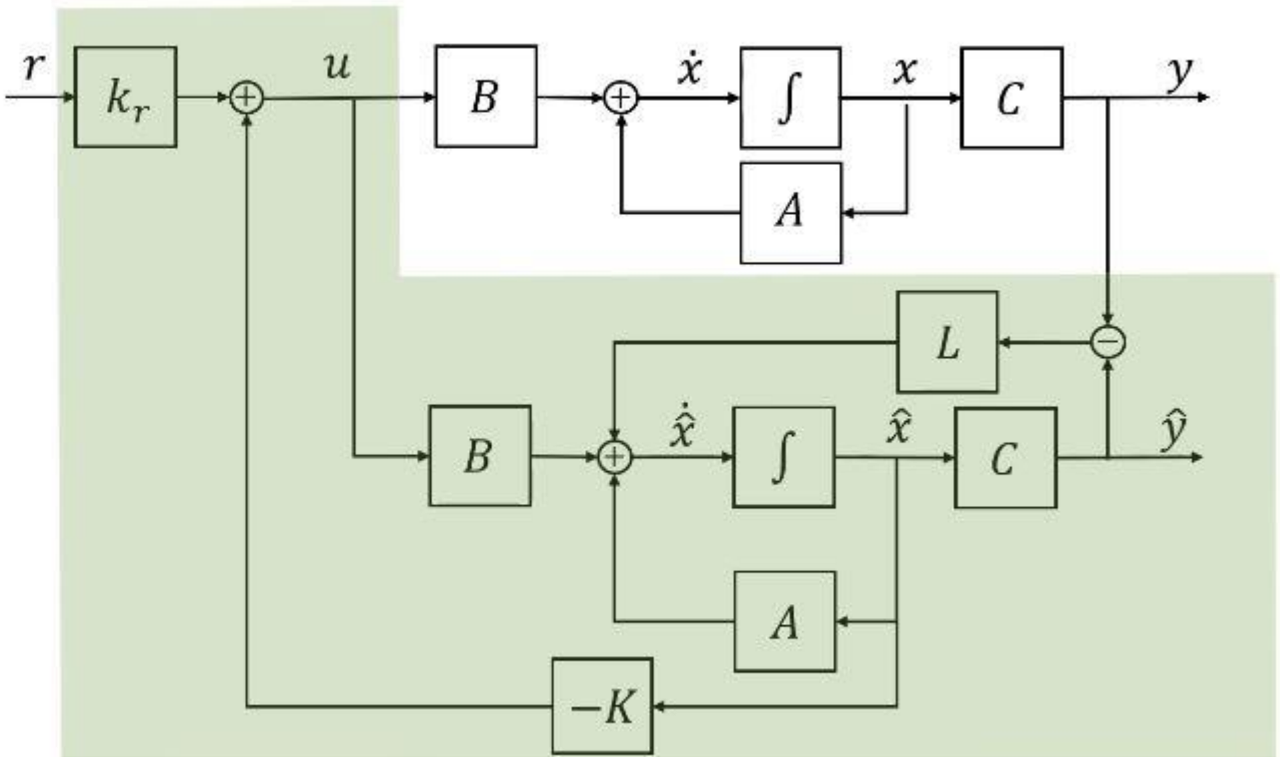
Match with desired characteristic polynomial gives

$$\lambda^2 + 10\lambda + 24 \equiv \lambda^2 + l_2\lambda$$

It is not possible to shape the error dynamics. We say that the system is not **observable.**

Control Using Estimated State

Use the estimated states for feedback, $u = -K\hat{x} + k_r r$



Control Using Estimated State

Given the system, $\dot{x} = Ax + Bu, y = Cx$, the controller, $u = -K\hat{x} + k_r r$, and the state estimator $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$, the closed system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - CL \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

The closed loop system has the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC)$$

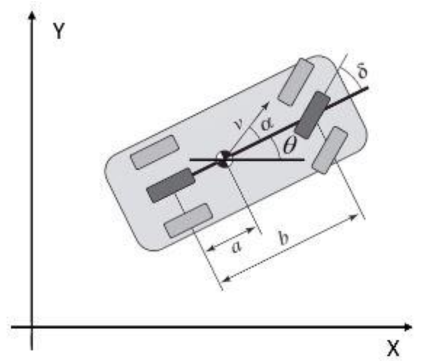
This polynomial can be assigned arbitrary roots if the system is **reachable** and **observable.**

Rule of thumb: Make the estimator poles 4-5 times faster then the feedback poles.
Note that we can design the state feedback controller based on the assumption that we have access to the complete state vector, and then we design the state estimator to get access to the state vector.

Control Using Estimated State

Revisit Example – Vehicle Steering

Consider the following system:



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]u$$

Where x_1 is the lateral position Y, x_2 is the heading orientation θ and u is the steering angle δ.

State feedback control (pole in -1 (double pole)):

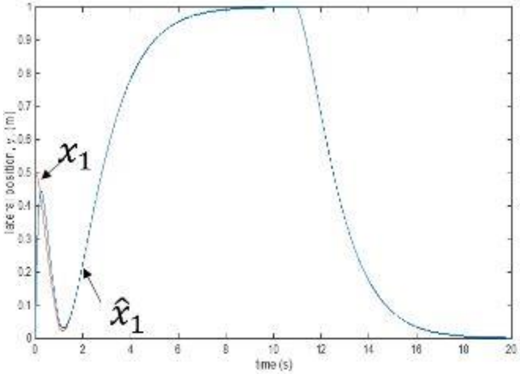
$$u = -0.0278x_1 - 0.6111x_2 + 0.0278r$$

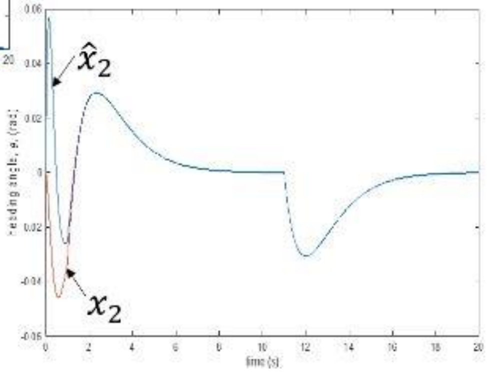
State estimator (poles in -4 and -6)

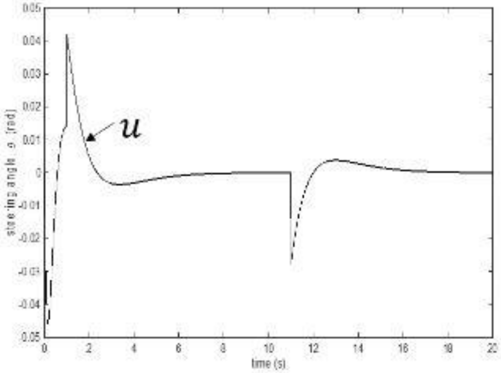
$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u + \begin{bmatrix}10\\ 2\end{bmatrix}(y - \hat{x}_1)$$

Control Using Estimated State

Revisit Example – Vehicle Steering





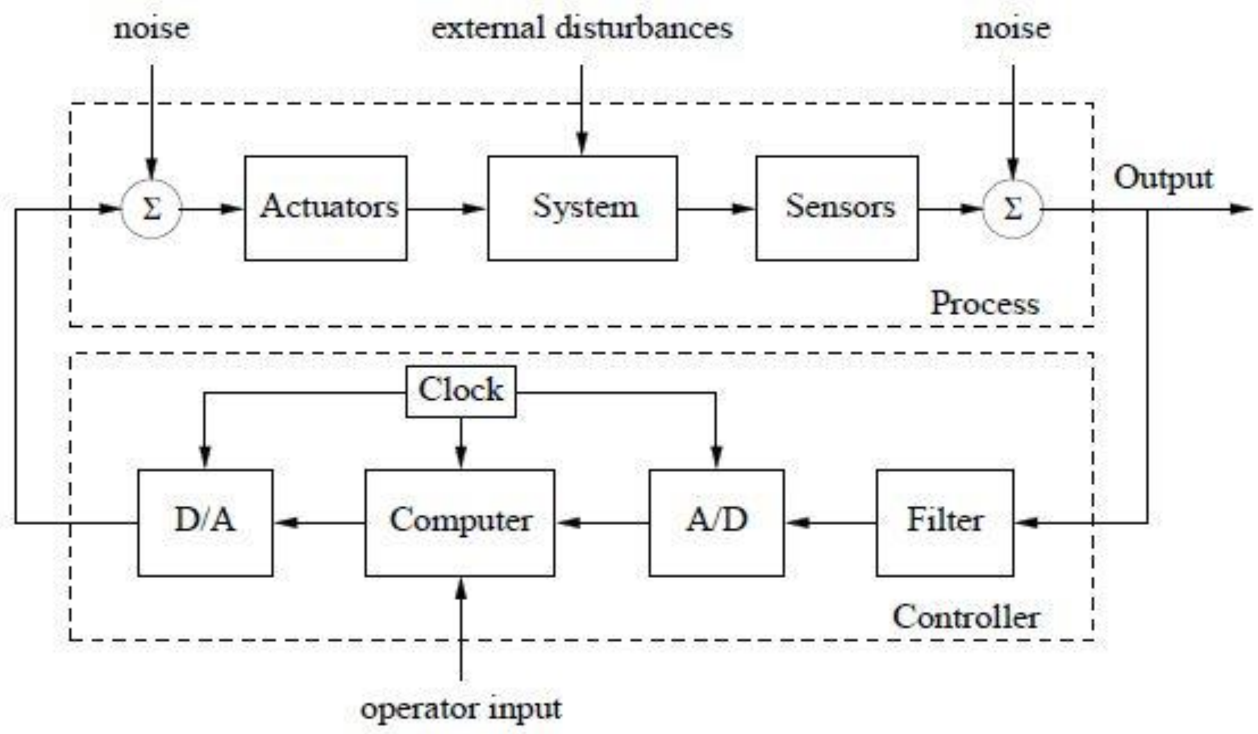


Control Using Estimated State

Implementation

Our controller consists of the state feedback controller, $u = -K\hat{x} + k_r r$,

And the state estimator, $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$.



Control Using Estimated State

Implementation

We need to discretize the controller to be able to implement it in the computer, by approximating the derivative by a difference.

$$\dot{\hat{x}} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{t_{k+1} - t_k} = A\hat{x}(t_k) + Bu(t_k) + L\big(y(t_k) - C\hat{x}(t_k)\big)$$

Rewrite it as a difference equation:

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + (t_{k+1} - t_k)(A\hat{x}(t_k) + Bu(t_k) + L\big(y(t_k) - C\hat{x}(t_k)\big),$$

h – sampling time

Control Using Estimated State

Model
State Space
approach

$$\dot{x} = f(x, u, d)$$
$$y = h(x, u, d)$$

Linearization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Analysis
Simulation
Prediction
Design

reachability

Control design

Estimation Design

State Feedback Control

$$u = -K\hat{x} + k_r r$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

Observability

Definition (**observability**): A linear system is **observable** if, for every $T > 0$, it is possible to determine the state of the system $x(T)$ through the measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

Recall the solution to the differential equation:

$$\mathrm{y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}\,Bu(\tau)d\tau + Du(t)}$$

Since we know $u(t)$, we only needs to consider zero input case.

$$\mathrm{y(t) = Ce^{At}}{\color{red}\mathrm{x(0)}}$$

So, if $x(0)$ can be determined, then we can reconstruct $x(t)$.

Observability

Definition: A **state** $x^* \neq 0$ is said to be **unobservable** if zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

So, if we can find a state x^* for which $Ce^{At}x(0) = 0$ for all $t \geq 0$, then the system is unobservable. For this to hold, all derivatives must be zero at $t = 0$.

$$\left.\begin{aligned}
&Ce^{At}x^*\Big|_{t=0} = 0 \implies Cx^* = 0 \\
&\frac{d}{dt}Ce^{At}x^*\Big|_{t=0} = 0 \implies CAe^{At}x^*\Big|_{t=0} = CAx^* = 0 \\
&\frac{d^2}{dt^2}Ce^{At}x^*\Big|_{t=0} = 0 \implies CA^2e^{At}x^*\Big|_{t=0} = CA^2x^* = 0 \\
&\qquad\qquad\vdots \\
&\frac{d^{n-1}}{dt^{n-1}}Ce^{At}x^*\Big|_{t=0} = 0 \implies CA^{n-1}e^{At}x^*\Big|_{t=0} = CA^{n-1}x^*
\end{aligned}\right\} \quad \overbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}^{W_0} x^* = 0$$

Observability

From linear algebra, we know that if the observability matrix, W_0, has full rank, then thee is only a solution to the equation. In other words, the system is observable.

Theorem (Observability rank condition): A linear system is observable if and only if the observability matrix

$$W_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full row rank.

Observability

Example: Vehicle Steering

Return to our vehicle steering example, with the lateral position as output signal

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

Observability matrix

$$W_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 12 \end{bmatrix}$$

So the system is observable.

Observability Example: Vehicle Steering

Return to our vehicle steering example, with the lateral position as output signal when the heading angle is the output.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

What do you think?

After doing the same procedure, we find that the system is not observable.

Observability

If the system is observable, then we can design the state estimator such that the poles of the error dynamics can be replaced arbitrarily. In order words, we can shape the state estimation error dynamics.

Observability can be used for determining which sensors to use in system design because if we measure the wrong quantity, then we might not be able to estimate the state, and as consequences, not be able to control the system.

Observability

Model
State Space
approach

$$\dot{x} = f(x, u, d)$$
$$y = h(x, u, d)$$

Linearization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Analysis
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Design

Reachability
Observability

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$$u = -K\hat{x} + k_r r$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

Kalman Filter

One of the principal uses of observers in practice is to estimate the state of a system in the presence of noisy measurements.

Kalman Filter named after Rudolf E. Kalman, one of the primary developer of its theory.

It has numerous of applications such as guidance, navigation, control of vehicle and many more.

Kalman Filter is an algorithm that uses a series of measurements observed over time, containing statistical noise, and produce an estimates of unknown variables that tend to be more accurate.

Kalman Filter

What is Filtering ?

Filtering is about recursively estimating parameters of interest based on measurement.

Notation

Let $\boldsymbol{x}_{\mathrm{k}}$ contains parameters of interest and $\boldsymbol{y}_k$ the measurement at time k.
(Time is usually discrete).

Objective

Compute $p(\boldsymbol{x}_{\boldsymbol{k}}|\boldsymbol{y}_{1:k})$ where $\boldsymbol{y}_{1:k} = [\boldsymbol{y}_1 \;\; \boldsymbol{y}_2 \quad \ldots . \; \boldsymbol{y}_k]$ contain all data up to time k.

Kalman Filter

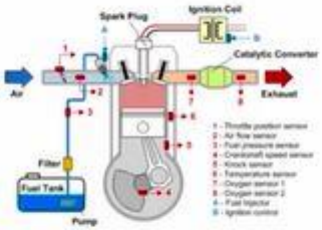
Applications

- Historically, positioning of airplanes and ships have been important examples

  $\boldsymbol{x}_k$: positions and velocities of planes

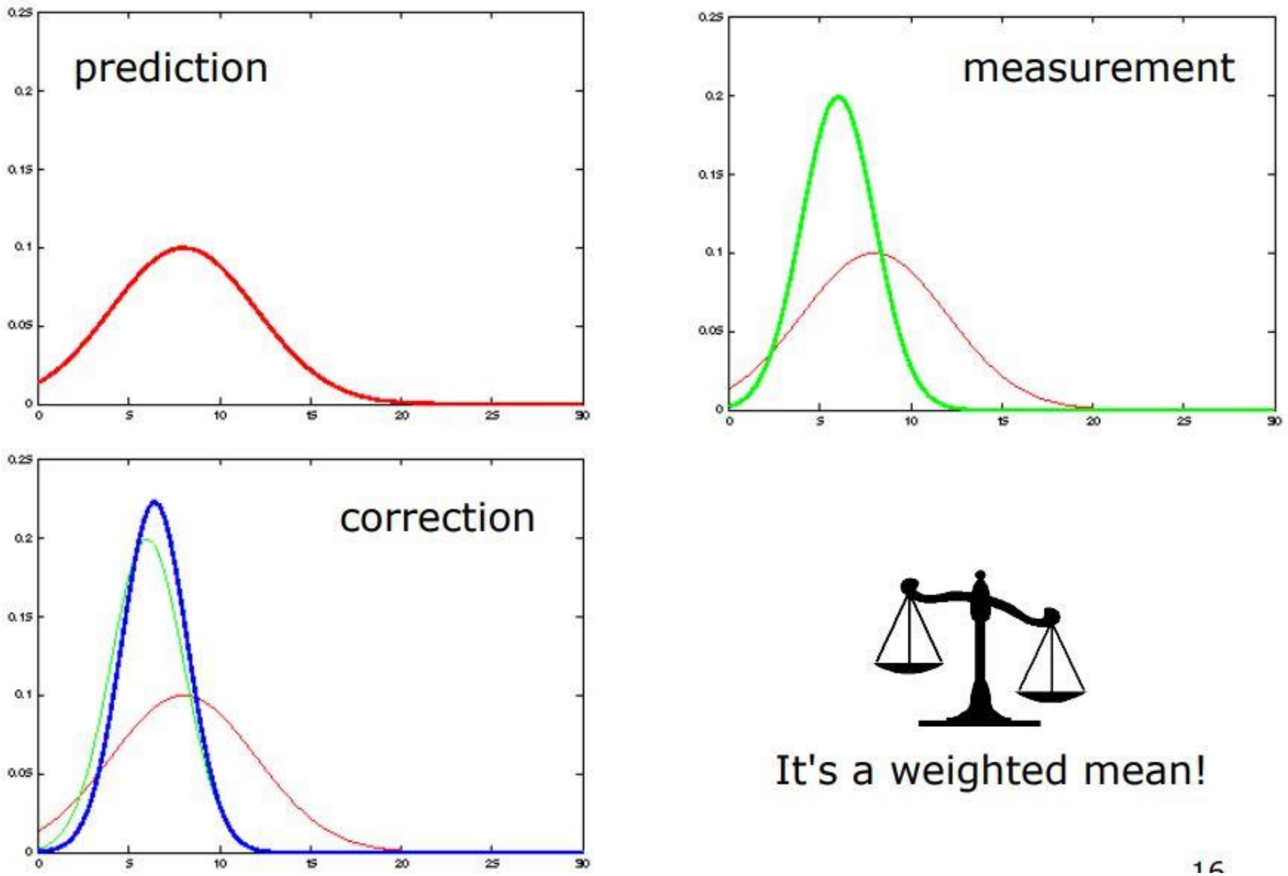
- Control of physical systems often require estimation of the interior state
  $\boldsymbol{x}_k$: angle of crankshaft, pressure, etc

- Often important to assesses the states in many other types of systems, e.g., biological or economical.
  $\boldsymbol{x}_k$: speed of disease, prices, consumption



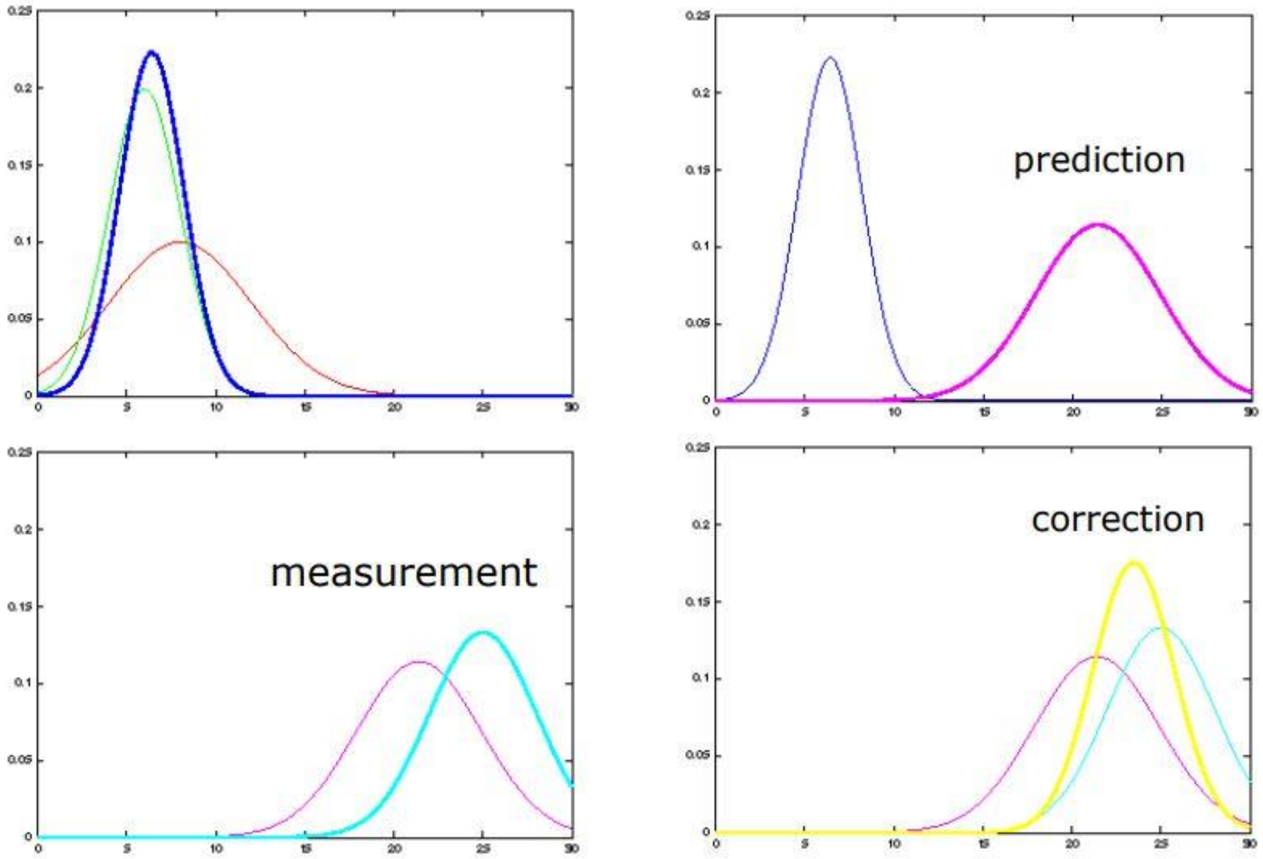
Kalman Filter

Concept

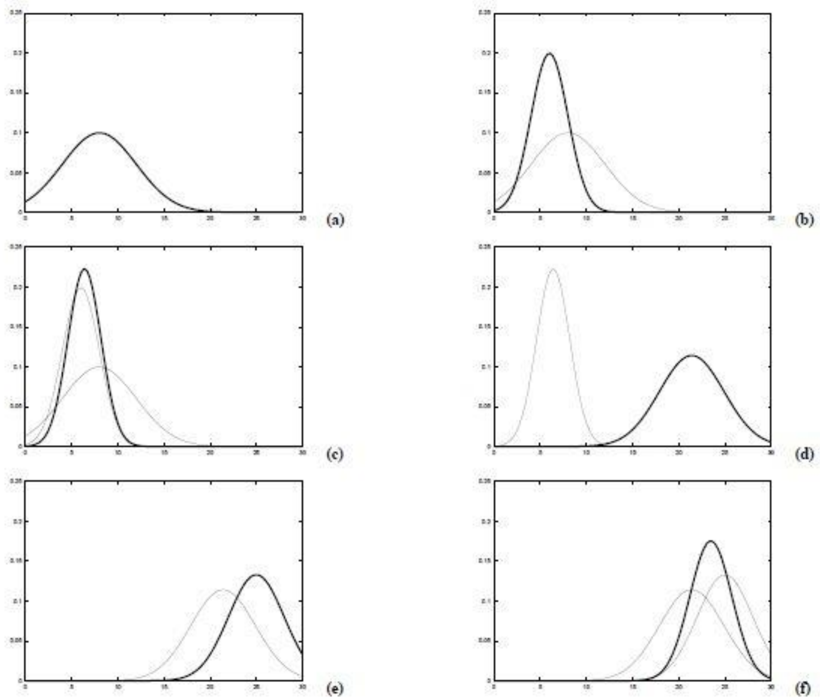


Kalman Filter

Concept



Kalman Filter
Concept



(a) Initial belief, (b) a measurement (in bold) with the associated uncertainty, (c) belief after integrating the measurement in to the belief using Kalman Filter, (d) belief of the motion to the right (which introduces uncertainties, (e) a new measurement with associated uncertainties, and (f) the result of the belief.

Kalman Filter

We start with the continuous time, then switch to the discrete time later.

Consider a linear time-invariant state-space model given by:

$$\dot{x} = Ax + Bu + v$$
$$y = Cx + w$$

Where x is the state vector, u is the control input and y is the output signal, v is the process disturbance and w is the measurement noise. The disturbance v and the noise w are zero mean and Gaussian.

$$\mathbb{E}\big(v(s)v^T(t)\big) = R_v\delta(t-s)$$
$$\mathbb{E}\big(w(s)w^T(t)\big) = R_w\delta(t-s)$$

Where δ is the unit impulse function (dirac function).

Kalman Filter

State Estimation

The state estimator (observer) is given as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The estimation error $\tilde{x} = x - \hat{x}$ can be computed as

$$\begin{aligned}\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} &= Ax + Bu + v - A\hat{x} - Bu - L(Cx + w - C\hat{x}) \\ &= (A - LC)\tilde{x} + v - Lw\end{aligned}$$

If $A - LC$ is stable, then the estimation error $\tilde{x}$ is a stationary stochastic process.

The covariance of the estimation error, $P_{\tilde{x}} = \mathbb{E}\left(\tilde{x}(t)\tilde{x}^T(t)\right)$, is given by the following equation

$$0 = (A - LC)P_{\tilde{x}} + P_{\tilde{x}}(A - LC)^T + R_v + LR_wL^T$$

The optimal observer minimizes $P_{\tilde{x}}$

Select L so that $P_{\tilde{x}}$ is minimized

Kalman Filter

The optimal observer gain, if the system is observable, is

$$L = P_{\tilde{x}} C^T R_w^{-1}$$

Where $P_{\tilde{x}} = P_{\tilde{x}}^T \geq 0$ is the solution to the Riccati equation

$$0 = AP_{\tilde{x}} + P_{\tilde{x}} A^T + R_v - P_{\tilde{x}} C^T R_w^{-1} C P_{\tilde{x}}$$

The observer is called the **Kalman-Bucy Filter**.

Similarities with LQR:

LQR ↔ Kalman

$A \leftrightarrow A^T, \quad \mathrm{B} \leftrightarrow C^T$

$S \leftrightarrow P, \quad K \leftrightarrow L^T$

$Q_x \leftrightarrow R_v, \quad Q_u \leftrightarrow R_w$

The Kalman-Bucy Filter is:

- Always stable
- The optimal linear filter for state estimation
- R_v and R_w are regarded as the design parameters. (how to select it might not be straight forward. The measurement noise covariance matrix can sometimes be found in the sensor specification. The disturbance matrix is more difficult (how much you should trust your model)

Kalman Filter

Revisit Example – Vehicle Steering

Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u + w$$

Where x_1 is the lateral position Y, x_2 is the heading orientation θ and u is the steering angle δ

The process disturbance and the measurement noise are zero mean with covariance

$$R_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_w = \rho$$

Design a Kalman filter estimate the vehicle's states form measurement of the lateral position.

Kalman Filter

Revisit Example – Vehicle Steering

State estimator

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The optimal observer gain, if the system is observable, is:

$$L = P_{\tilde{x}} C^T R_w^{-1}$$

Where $P_{\tilde{x}} = P_{\tilde{x}}^T \geq 0$ is the solution to the Riccati equation

$$0 = AP_{\tilde{x}} + P_{\tilde{x}} A^T + R_v - P_{\tilde{x}} C^T R_w^{-1} C P_{\tilde{x}} \qquad P_{\tilde{x}} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

The Riccati equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \rho^{-1} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

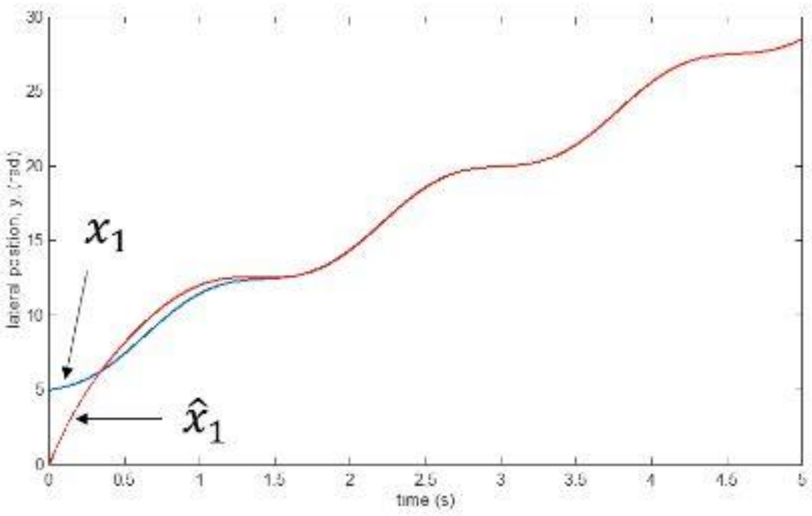
$$\rho = 1 \Rightarrow P_{\tilde{x}} = \begin{bmatrix} 5.0 & 1.0 \\ 1.0 & 0.4167 \end{bmatrix} \Longrightarrow L = \begin{bmatrix} 5.0 \\ 1.0 \end{bmatrix}$$

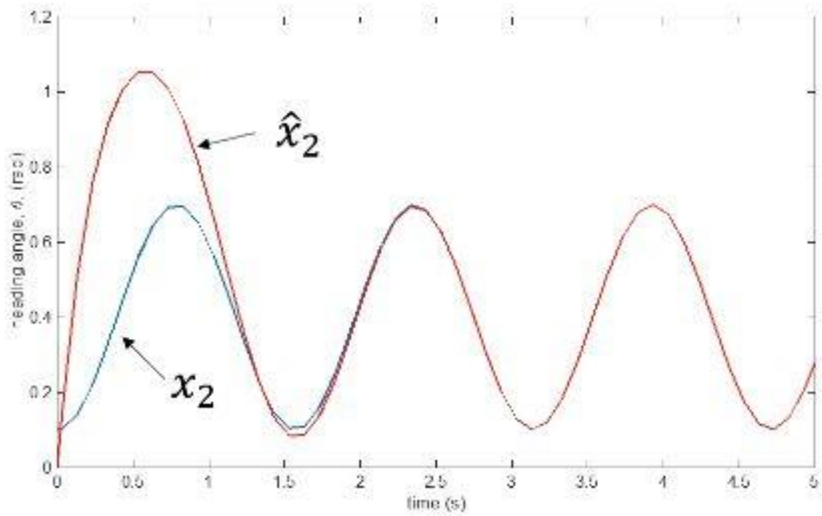
MATLAB:
[P,E,L]=care(A',C',[[1,0;[0,1]],1)

Kalman Filter

Revisit Example – Vehicle Steering

Simulations using a sinusoidal input, with $x(0) = (5,0.1)$, and $\hat{x} = (0,0)$

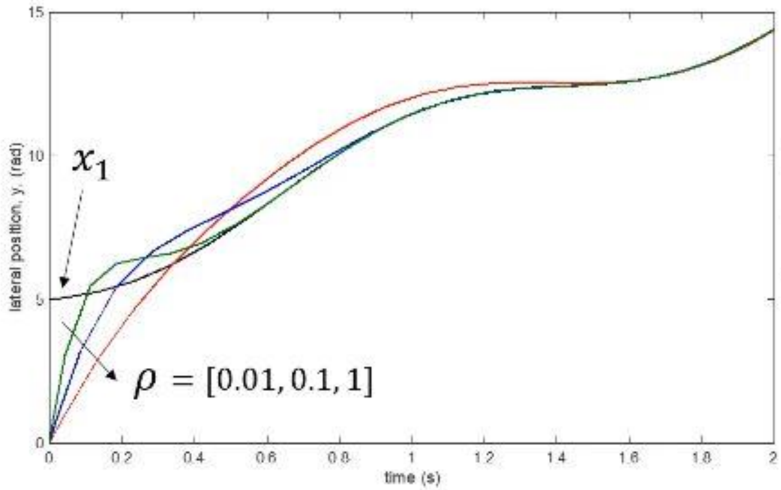


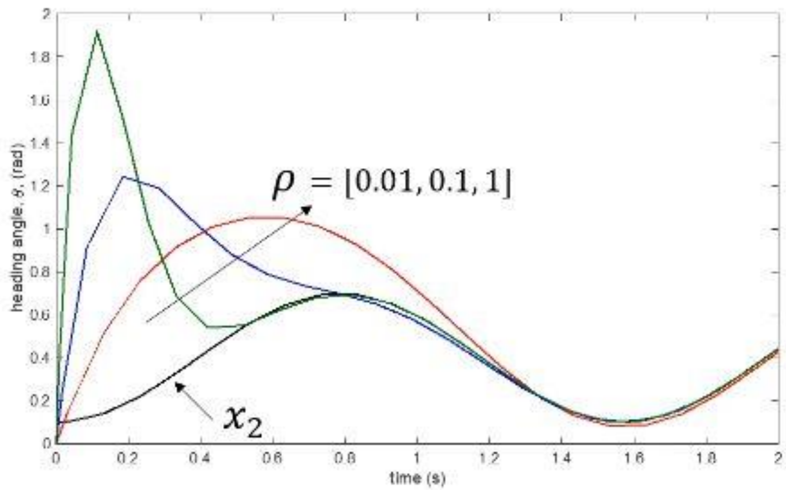


Kalman Filter

Revisit Example – Vehicle Steering

What will happen if we change the noise covariance?





The larger the covariance is, the slower the state converges. In other words, we trust the model more than the measurement.

Kalman Filter

State Estimation – Discrete time case

Consider an LTI state space model in discrete time given by

$$x[k+1] = Ax[k] + Bu[k] + v[k]$$
$$y[k] = Cx[k] + w[k]$$

Where x is the state vector, u is the control signal and y is the output signal, v is the process disturbance and w is measurement noise. The disturbance v and noise w are zero mean and Gaussian.

The state estimator (observer) is given as

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L[k](y[k] - C\hat{x}[k]$$

And the estimation error $\tilde{x}[k] = x[k] - \hat{x}[k]$ can be computed as

$$\tilde{x}[k+1] = (A - L[k]C)\tilde{x}[k] + v[k] - Lw[k]$$

Kalman Filter

State Estimation – Discrete time case

The covariance of the estimation error, $P_{\tilde{x}} = \mathbb{E}(\tilde{x}[k]\tilde{x}^T[k])$, is given by

$$\underline{P_{\tilde{x}}[k+1]} = (A - L[k]C)P_{\tilde{x}}[k](\mathrm{A} - \mathrm{L}[\mathrm{k}]\mathrm{C})^{\mathrm{T}} + \mathrm{R_v} + \mathrm{L}[\mathrm{k}]\mathrm{R_w}\mathrm{L^T}[\mathrm{k}]$$

The observer gain that minimizes $P_{\tilde{x}}[k]$ is given by

$$L[k] = AP_{\tilde{x}}[k]C^T(R_w + C\underline{P_{\tilde{x}}[k]}C^T)^{-1}$$

This is the **discrete time Kalman Filter**.

Note that the Kalman Filter is the **recursive filter** since its current covariance matrix depends on the previous covariance matrix. Moreover, if $P_{\tilde{x}}[k]$ converges, then L is constant.

Kalman Filter

Prediction

The state estimator (observer) is given as

$$\hat{x}[k+1|k] = A\hat{x}[k|k-1] + Bu[k] + L[k](y[k] - C\hat{x}[k|k-1]$$

is called the **one-step predictor**.

Using the Kalman Filter, **the optimal m-step predictor** can be derived as:

$$\hat{x}[k+m|k] = \boxed{A^m\hat{x}[k|k]} + \sum_{l=0}^{m-1} A^{m-1-l}Bu[k+l]$$

where

$$\hat{x}[k|k] = \hat{x}[k|k-1] + \tilde{L}[k](y[k] - C\hat{x}[k|k-1])$$

and

$$\tilde{L}[k] = P_{\tilde{x}}[k]C^T(R_w + CP_{\tilde{x}}[k]C^T)^{-1}$$

Discrete Kalman Filter Algorithm

Algorithm

Initialization at k = 1

$$\hat{x}[k-1|k-1], P_{\tilde{x}}[k-1|k-1]$$

Prediction step:

$$\hat{x}[k|k-1] = A[k-1]\hat{x}[k-1|k-1]$$
$$P[k|k-1] = A[k-1]P[k-1|k-1] + R_v$$

Update Step:

$$\tilde{L}[k] = P[k|k-1]C^T[k](R_w[k] + C[k]P[k|k-1]C^T[k])^{-1}$$
$$\hat{x}[k|k] = \hat{x}[k|k-1] + \tilde{L}[k](y[k] - C\hat{x}[k|k-1])$$
$$P[k|k] = P[k|k-1] - \tilde{L}[k]C[k]P[k|k-1]$$

Kalman Filtering

Prediction

Model
State Space
approach

$$\dot{x} = f(x, u, d)$$
$$y = h(x, u, d)$$

Linearization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Analysis
Simulation
Prediction
Design

Reachability
Observability

Control design

Estimation Design

State Feedback Control

$$u = -K\hat{x} + k_r r$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

Reference

Materials used in this lecture draws heavily from:

- The Lectures on Model-Based Automotive Systems Engineering taught professor Jonas Fredriksson, Chalmers University of Technology.
- The book, entitled Feedback Systems: An Introduction for Scientists and Engineering by Karl Johan Astrom and Richard M. Murray.

More about Kalman Filter can be found in Chapter 3 of the

- The book, entitled “Probabilistic Robotics” by Sebastian Thrun, Wolfram Burgard and Dieter Fox.