Feedback Control Systems
Lecture 2 Linear Systems

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Outline

1. State Space Representation
2. The Matrix Exponential
3. Input/Output Response
4. Linearization
5. Reference

State Space Representation

Many systems of interest are either linear, or correspond to the Linearization of a nonlinear system. The most complete theory of control applies to linear systems. The general form of a linear state space system is

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{Ax} + \boldsymbol{Bu} \\ \boldsymbol{y} &= \boldsymbol{Cx} + \boldsymbol{Du}, \end{aligned} \tag{1}$$

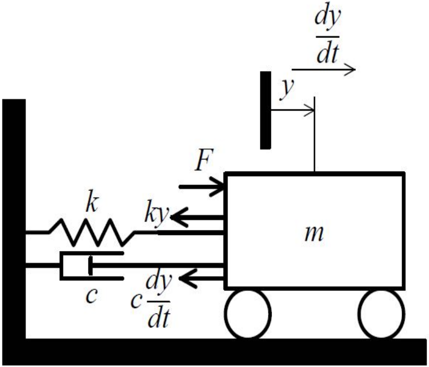
where, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}^{q\times p}$ and $\boldsymbol{D} \in \mathbb{R}^{q\times p}$. $\boldsymbol{A}$ is called state matrix. $\boldsymbol{B}$ is called the input matrix. $\boldsymbol{C}$ is called the output matrix. $\boldsymbol{D}$ is called the direct transmission matrix.

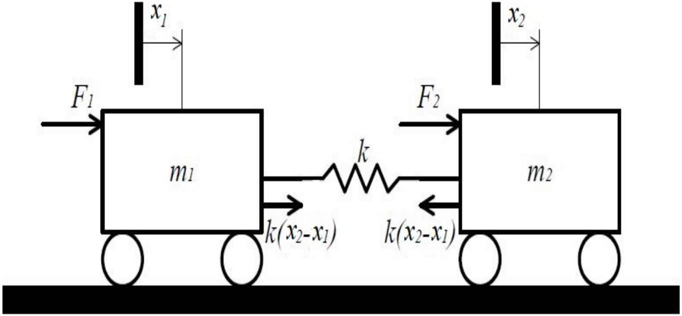
State Space Representation

- Step to get the state space representation
  - Find equations of motion
  - Choose state variables (position, velocity, ...)
  - Take the derivative of the state variables from step 2
  - Write in the state space form
  - Write the output equation

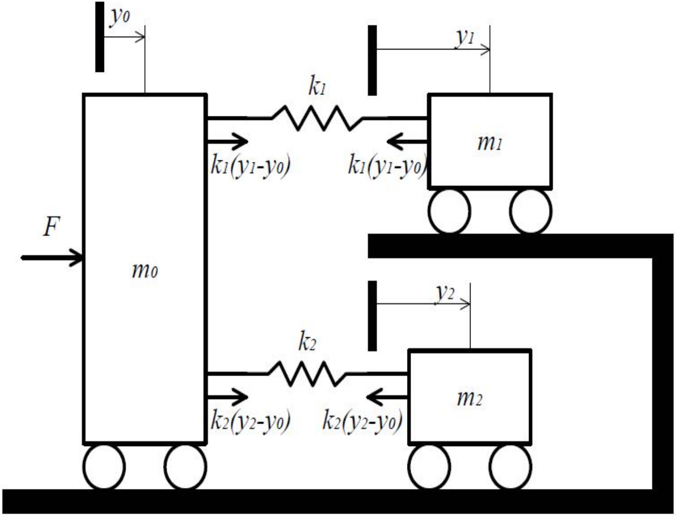
State Space Representation

Example 1: Mass-Spring-Damper System





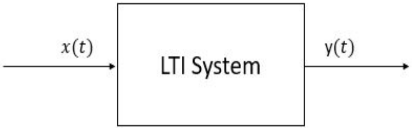
x1
x2
F1
F2
m1
m2
k
k(x2-x1)
k(x2-x1)



y_0
y_1
k_1
m_1
$k_1(y_1-y_0)$
$k_1(y_1-y_0)$
F
m_0
y_2
k_2
m_2
$k_2(y_2-y_0)$
$k_2(y_2-y_0)$

Basic Definition

Linear Time-Invariant Systems (LTI Systems)



LTI systems have to be

- Homogeneity
- Superposition (additivity)
- Time-Invariance

In practice, perhaps no real world systems can be satisfied the above properties. Then, why we study this?
Because, it is easy to solve, and most real world systems can be simplified.

Consider homogeneous equation given as

$$\frac{dx}{dt} = ax, x(0) = x_0$$

We can solve this equation easily

$$\frac{1}{x}dx = adt$$

Integrating both sides. The solution is given as

$$x(t) = e^{at}x(0)$$

Consider LTI scalar system give as

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ y(t) &= cx(t) + du(t), \end{aligned} \tag{2}$$

Useful properties

$$\tfrac{d}{dt}e^{at} = ae^{at}, \tfrac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}\dot{x}(t) - ae^{-at}x(t)$$

Multiply (2) by e^{-at} on both sides

$$e^{-at}\dot{x}(t) - ae^{-at}x(t) = e^{-at}bu(t) \Rightarrow \tfrac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}bu(t)$$

Solution

$$x(t) = e^{at}x(0) + \textstyle\int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$
$$y(t) = ce^{at}x(0) + \textstyle\int_0^t ce^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

Consider the homogeneous response corresponding to the system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} \tag{3}$$

For the scalar differential equation

$$\tfrac{dx}{dt} = ax, x \in \mathbb{R},$$

the solution is given by the exponential

$$x(t) = e^{at}x(0).$$

We wish to generalize this to the vector case, where $\boldsymbol{A}$ becomes a matrix. We define the matrix exponential as the infinite series, we obtain

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \frac{1}{3!}\boldsymbol{A}^3t^3 + \dots \tag{4}$$

Then the the derivative of $e^{\boldsymbol{A}t}$ with respect to t become

$$\frac{d}{dt}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}$$

Multiplying by $\boldsymbol{x}(0)$ from the right, then

$$\frac{d}{dt}e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

Definition

The Solution to the homogeneous system of differential equations is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}\mathbf{(0)} \tag{5}$$

Example

Double Integrator

A very simple linear system that is useful in understanding basic concepts in the second-order system given by

$$\ddot{q} = u, y = q$$

This system is called a double integrator because the input u is integrated twice to determine the output y.
In the state space form, we write $x = (q, \dot{q})$ and

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Tip: Use matrix exponential to find e^{At} Thus the homogeneous solution (u=0) for the double integrator is given by

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$
$$y(t) = x_1(0) + tx_2(0)$$

The Matrix Exponential

An important class of linear systems are those that can be converted into diagonal form. Suppose that we are given a system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}$$

Such that all the eigenvalues of $\boldsymbol{A}$ are distinct. It can be shown that we can find the invertible matrix $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$ is diagonal. If we choose a set of coordinates $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$, the in the new coordinates of dynamics become

$$\frac{d\boldsymbol{z}}{dt} = \boldsymbol{T}\frac{d\boldsymbol{x}}{dt} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}z.$$

Denote $\boldsymbol{D} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$ which is diagonal matrix , then,

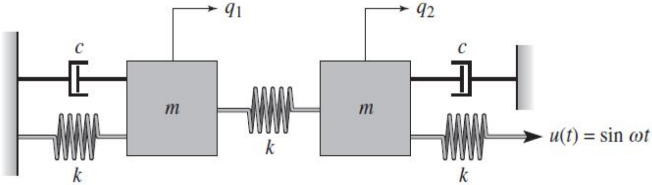
$$\boldsymbol{z}(t) = e^{\boldsymbol{D}t}\boldsymbol{z}(0)$$

Note that some matrices with equal eigenvalues cannot be transformed to diagonal form. They can ,however, be transformed to a closely related form, called Jordan form, in which the dynamics matrix have the eigenvalues along the diagonal.
We define a matrix to be a Jordan form if it can be written as

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & J_{k-1} & 0 \\ 0 & 0 & \dots & 0 & J_k \end{bmatrix}$$

Consider the spring-mass system shown below, but with the additional damper on each mass. The input to this system is the sinusoidal motion of the end of the rightmost spring, and the output is the position of each mass q_1 and q_2. The equations of motion of the system are

$$m\ddot{q}_1 = -2kq_1 - c\dot{q}_1 + kq_2 \text{ and } m\ddot{q}_2 = kq_1 - 2kq_2 - c\dot{q}_2 + ku$$



In state space form, we define the state to be $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$, and we can rewrite the equation as

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{bmatrix} u$$

We now define a transformation $z = Tx$ that puts this system into a simpler form. Let $z_1 = 1/2(q_1 + q_2), z_2 = \dot{z}_1, z_3 = 1/2(q_1 - q_2)$ and $z_4 = \dot{z}_3$

Then,

$$z = Tx = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

In the new coordinates, the dynamics becomes

$$\frac{dz}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3k}{m} & -\frac{c}{m} \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{k}{2m} \\ 0 \\ -\frac{k}{2m} \end{bmatrix} u$$

We can solve for the solutions by computing the solutions of two sets (z_1, z_2) and (z_3, z_4). Then we can compute for $x = T^{-1}z$.

For a linear, time-invariant system the transfer function, $G(s)$, is

$$G(s) = \frac{Y(s)}{X(s)}$$

where $X(s)$ and $Y(s)$ are the Laplace transforms of the input and output respectively, and $G(s)$ is the Transfer function. Then,

$$\mathrm{Y(s) = G(s)X(s)}$$

The inverse Laplace transform of $Y(s)$ is given by the following convolution integral:

$$y(t) = \int_0^t x(\tau)g(t-\tau)d\tau$$

where both $g(t)$ and $x(t)$ are 0 for $t < 0$.

Consider the system given as

$$\frac{dy}{dt} - ay = e^{ct}$$
$$sY(s) - aY(s) = \frac{1}{s-c}$$
$$Y(s) = \frac{1}{(s-a)(s-c)}$$

Apply convolution integral

$$y(t) = \int_0^t e^{a(t-\tau)} e^{c\tau} d\tau = \frac{e^{ct} - e^{at}}{c-a}$$

You can also use partial fraction for this. But here we want to review a little bit of convolution integral for later use.

The general form of a linear state space system is

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}, \tag{6}$$

where, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $\boldsymbol{C} \in \mathbb{R}^{q \times p}$ and $\boldsymbol{D} \in \mathbb{R}^{q \times p}$.

Theorem

The solution to the linear differential equation above is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \tag{7}$$

The input/output relation for the linear system is given by

Theorem

The solution to the linear differential equation above is given by

$$\boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t) \tag{8}$$

It is easy to see that from this equation that the output is jointly linear in both the initial conditions and input, which follows from the linearity of matrix/vector multiplication and integration.
Equation (8) is called convolution equation.

A common source of linear system models is through the approximation of a nonlinear system by a linear one. These approximations are aimed at the studying the local behavior of a system, where the the nonlinear affect are expected to be small.

Linearization Around a Fixed Point

Consider a simple nonlinear system given as

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

- Find some fixed point, $\bar{\mathbf{x}}$, such that $f(\bar{\mathbf{x}}) = 0$
- Linearize about $\bar{\mathbf{x}}$

$$\frac{Df}{D\mathbf{x}}\Big|_{\bar{\mathbf{x}}} = \left[\frac{\partial f_i}{\partial \mathbf{x}_j}\right]$$

Consider a simple nonlinear system of a pendulum given by

$$\ddot{\theta} = g/l \sin\theta - \delta\dot{\theta}$$

where δ is the damping coefficient. For simplicity, assume that $g/l = 1$. The state-space equation is written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 - \delta x_2 \end{bmatrix}$$

- Find fixed points

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

- Linearize around fixed points

$$\frac{Df}{D\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\delta \end{bmatrix}$$

Substitute the value of the fixed points and check the stability.

Choose small δ, then compute eigenvalue of value to check the stability.
Tip: in MATLAB, you can compute the eigenvalue of matrix A by $eig(A)$.

References

Materials used in this lecture draws heavily from the book entitled **Feedback Systems: An Introduction for Scientists and Engineers** by Karl Johan Astrom and Richard M. Murray.