Feedback Control Systems
Case Study: Inverted Pendulum on a Cart

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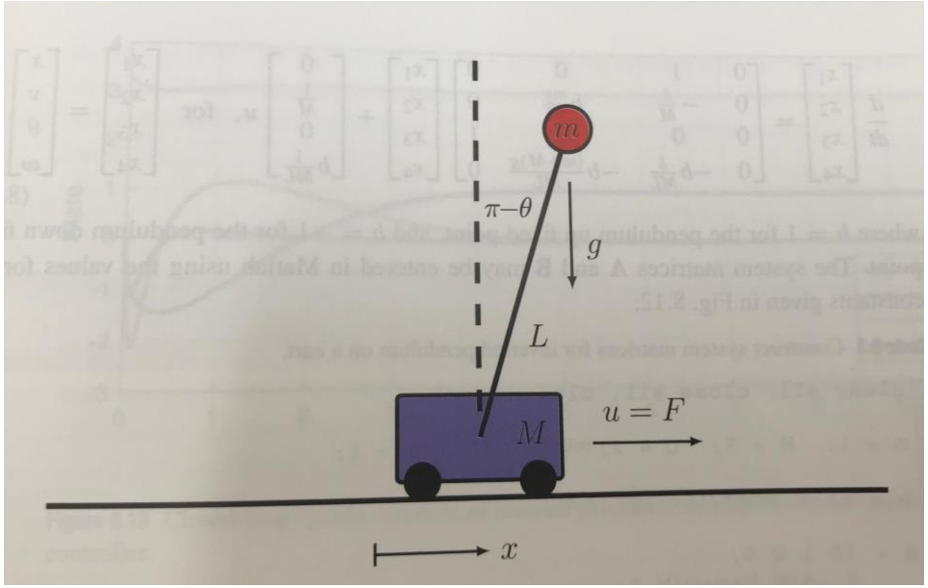
Case Study: Inverted Pendulum on a Cart

To consolidate the concepts of optimal control, we will implement a stabilizing controller for the inverted pendulum on a cart.



Schematic of inverted pendulum on a cart. The control forcing acts to accelerate or decelerate

Case Study: Inverted Pendulum on a Cart Full Nonlinear Dynamics

The full nonlinear dynamics are given by

$$\dot{x} = v$$

$$\dot{v} = \frac{-m^2L^2gcos(\theta)\sin(\theta) + mL^2(mL\omega^2\sin(\theta) - \delta v) + mL^2u}{mL^2\big(M + m(1 - \cos(\theta)^2)\big)}$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{(m + M)mgLsin(\theta) - mLcos(\theta)(mL\omega^2\sin(\theta) - \delta v) + mLcos(\theta)u}{mL^2\big(M + m(1 - \cos(\theta)^2)\big)}$$

Where x is the cart position, v is the velocity, θ is the pendulum angle, ω is the angular velocity, m is the pendulum mass, M is the cart mass, L is the pendulum arm, g is the gravitational acceleration, δ is a friction damping on the cart, and u is the control force applied on the cart.

Case Study: Inverted Pendulum on a Cart
Fixed Points

The **pencart** MATLAB function is used to simulate the full nonlinear systems.

There are two fixed pointed, corresponding to either the pendulum down ($\theta = 0$) or pendulum up ($\theta = \pi$) configuration. In both cases, $v = \omega = 0$ for the fixed point, and the cart position x is a free variable, as the equation do not depend explicitly on x. It is possible to linearize the nonlinear dynamics system.

Case Study: Inverted Pendulum on a Cart Linearization

The state space representation after linearization is as follows

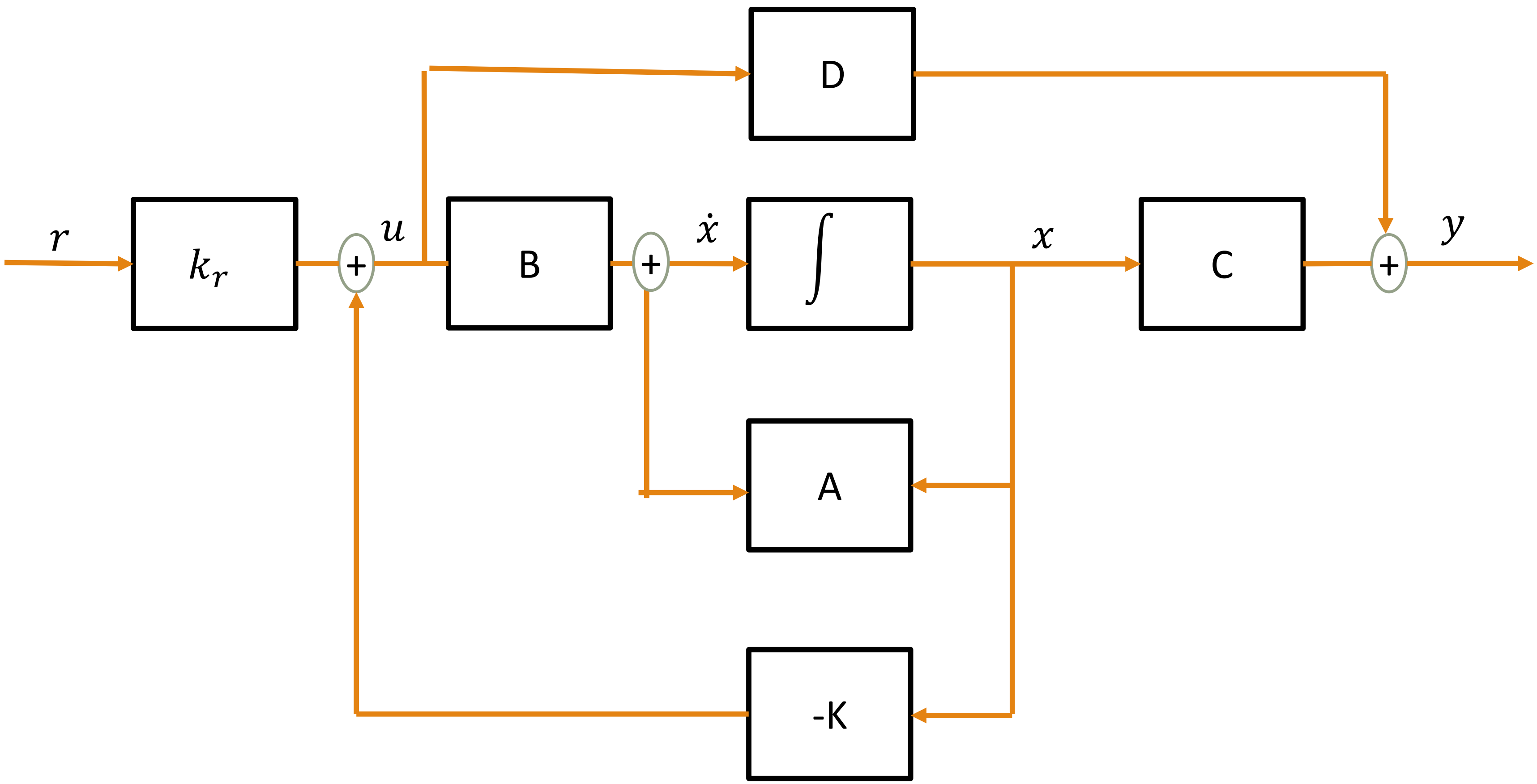
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\delta}{M} & \frac{bmg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{b\delta}{ML} & -\frac{b(m+M)g}{ML} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{b}{ML} \end{bmatrix} u,$$

Where $x = [x_1 \ \ x_2 \ \ x_3 \ \ x_4]^T = [x \ \ v \ \ \theta \ \ \omega]^T$

$b = 1$ for the pendulum up fixed point, and $b = -1$ for the pendulum down fixed point

Note: We may also confirm the stability of the open-loop system by checking eigenvalues of A.

Case Study: Inverted Pendulum on a Cart
Pole Placement



Case Study: Inverted Pendulum on a Cart
Pole Placement

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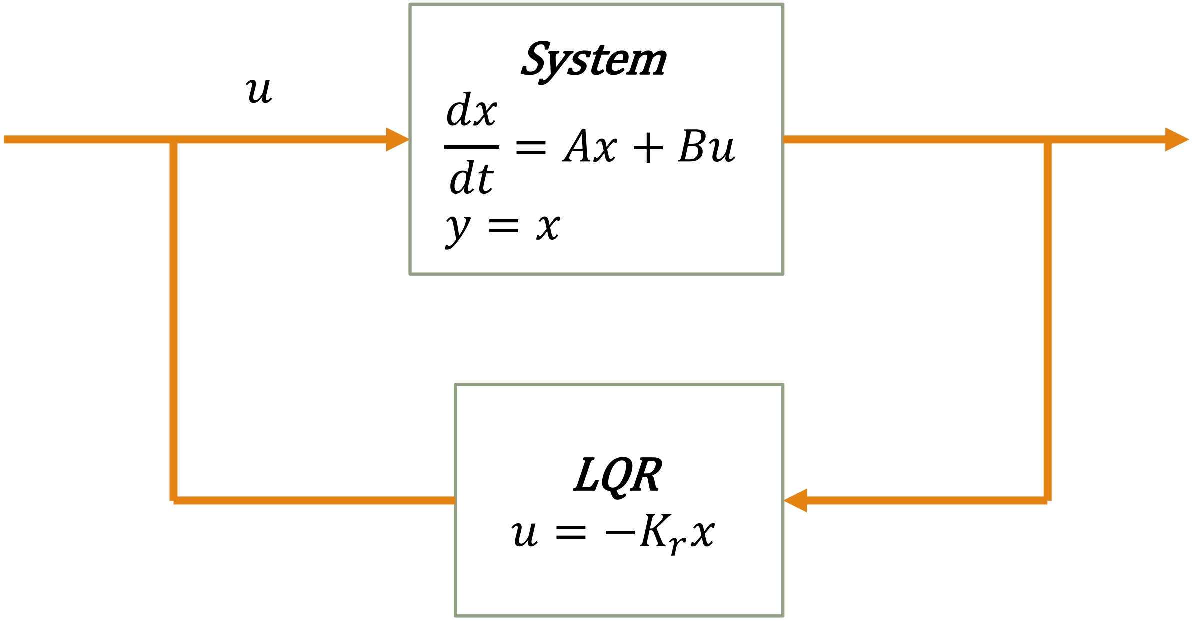
Case Study: Inverted Pendulum on a Cart

Optimal Full-State Control: Linear Quadratic Regulator



Schematic of LQR for optimal full-state feedback. The optimal controller for a linear system given measurement of the full state, $y = x$, is given by proportional control $u = -K_r x$ where K_r is a constant gain matrix obtained by solving an algebraic Riccati equation.

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Optimal Full-State Control: Linear Quadratic Regulator

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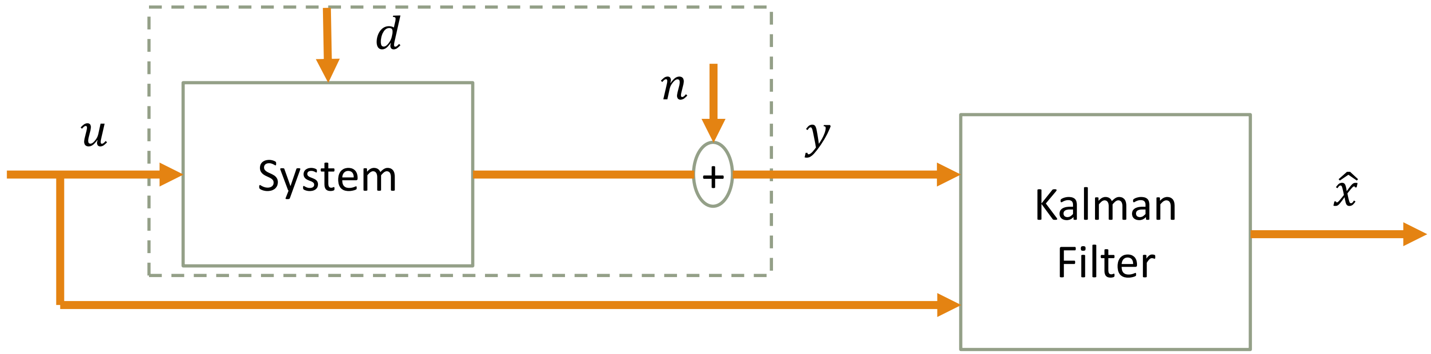
Case Study: Inverted Pendulum on a Cart
Kalman Filter

In this demonstration, we will do it in the down ward position. We have the system of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Where $C = [1\ 0\ 0\ 0]$



Then

$$\dot{x} = Ax + Bu + V_d d + 0n$$
$$y = Cx + 0u + 0d + V_n n$$

Case Study: Inverted Pendulum on a Cart
Kalman Filter

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Case Study: Inverted Pendulum on a Cart
Linear Quadratic Gaussian (LQG)

$$\dot{x} = Ax + Bu + w_d$$
$$y = Cx + w_n$$