Biol672/Biol470
Spring 2022
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Miniproject 2

Background

The following refers to the Lecture Notes. We use an epidemic model as a prototypical example of a systems approach to the study of biological systems. A number of chemical/biochemical reactions follow the same type of approach and formalism.

SIR model

Consider the SIR model, described by the following system of ODEs

$$\frac{dS}{dt} = -\frac{a}{N}\, I\, S, \tag{1}$$

$$\frac{dI}{dt} = \frac{a}{N}\, I\, S\, - c\, I, \tag{2}$$

$$\frac{dR}{dt} = c\, I. \tag{3}$$

In (1)-(3), the variables S, I and R represent the number of individuales in the *susceptible*, *infectious* and *removed* clases of the population, respectively, and the parameters a and c are the *transmission* and *recovery* rate constants, respectively. We consider arbitrary units (au) of time.

From eqs. (1)-(3), the *total population* $N = S + I + R$ satisfies

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0, \tag{4}$$

implying that N is constant and equal to the initial population.

Negative and positive feedback effects

The SIR model can be extended to include *negative* and *positive feedback* effects, controlled by the parameters λ and γ respectively,

$$\frac{dS}{dt} = -\frac{a}{N}\,(1+\gamma\, I)\, I\, S, +\lambda\, R \tag{5}$$

$$\frac{dI}{dt} = \frac{a}{N}\,(1+\gamma\, I)\, I\, S\, - c\, I, \tag{6}$$

$$\frac{dR}{dt} = c\, I - \lambda\, R. \tag{7}$$

Logistic dynamics in a reduced SIR model

The SIR model can also be reduced to the following system be eliminating the removed class

$$\frac{dS}{dt} = -\frac{a}{N}\, I\, S + c\, I, \tag{8}$$

$$\frac{dI}{dt} = \frac{a}{N}\, I\, S\, - c\, I. \tag{9}$$

By substituting $N = S + I$ in the eq. (9) and rearranging terms, one obtains the logistic equation

$$\frac{dI}{dt} = r\, I\, \left(1 - \frac{I}{K}\right) \tag{10}$$

where

$$r = a - c \qquad \text{and} \qquad K = \frac{(a-c)\, N}{a}. \tag{11}$$

Bistability in a reduced SIR model with saturation

A modified version of the model (8)-(9) that considers saturation reads

$$\frac{dS}{dt} = -\frac{a}{N}\, I\, S + c\, \frac{I}{1+1}, \tag{12}$$

$$\frac{dI}{dt} = \frac{a}{N}\, I\, S\, - c\, \frac{I}{1+1}. \tag{13}$$

By substituting $N = S + I$ in the eq. (12) and rearranging terms, one obtains the following equation, which becomes bistable for certain parameter set values,

$$\frac{dI}{dt} = \frac{a}{N}\, I\, (N - I) - c\, \frac{I}{N - I} \tag{14}$$

Codes

The codes *SIRmodel_AndExtensions.m*, *Population1D_Logistic.m*, and *Population1D_LogisticAndSaturation.m* can be used to simulate the models described above.

Projects

Graduate

1. Extend the SIR model with positive and negative feedback effects (5)-(7) to include
   (a) the death rate in the participating classes (e.g., for the SIR models the rate death should be included for the variables S, I and R)
   (b) vaccination effects
2. Extend the SIR model with positive and negative feedback effects (5)-(7) to include two interacting populations.
3. Design and discuss representative examples. Involve a group of undergraduate students in your discussions.

Undergraduate Option 1

1. Carry out simulations for the SIR model with positive and negative feedback effects (*SIRmodel_AndExtensions.m*, KSE=3) to reproduce the results of the data sets
   (a) SIRposneg_01.dat
   (b) SIRposneg_02.dat

   Use $N = 1000$. In these data sets, the columns correspond to t, S, I, R. Plot superimposed graphs of the results of your simulations and the data provided in the data sets. The distance between these graphs should be minimized for the parameter values you find: a, c, λ and γ.
2. Using the code *Population1D_LogisticAndSaturation.m* (parameter values: $a = 0.1$, $c = 2$ and $N = 100$), determine the equilibrium values for I ($I_{eq,1}$ and $I_{eq,2}$). Then, changing the initial condition ($I(1)$ in the code, since Matlab uses 1 as the first vector element), determine the threshold values for I, I_{thr}, such that for $I(1) < I_{thr}$, $\lim_{t\to\infty} I(t) = I_{eq,1}$ and for $I(1) > I_{thr}$, $\lim_{t\to\infty} I(t) = I_{eq,2}$.
3. Join one of the graduate projects for the design and discussion of the representative examples

Undergraduate Option 2

1. Build a parameter estimation code to estimate the value of the parameter a for the SIR model (*SIRmodel_AndExtensions.m*, KSE=3) to reproduce the results of the data set SIRposneg_03.dat.

   Use $N = 1000$, $c = 0.4$, $\lambda = \gamma = 0$. In these data sets, the columns correspond to t, S, I, R. Plot superimposed graphs of the results of your simulations and the data provided in the data sets. Suggestion: maximum likelihood estimation.

2. Adapt the code *Population1D_LogisticAndSaturation.m* (parameter values: $a = 0.1$, $c = 2$ and $N = 100$) to determine the equilibrium values for I ($I_{eq,1}$ and $I_{eq,2}$) and the threshold values for I, I_{thr}, such that for $I(1) < I_{thr}$, $\lim_{t\to\infty} I(t) = I_{eq,1}$ and for $I(1) > I_{thr}$, $\lim_{t\to\infty} I(t) = I_{eq,2}$.

3. Join one of the graduate projects for the design and discussion of the representative examples