(+44)7805025380
darraghferguson15@gmail.com

Darragh Ferguson

linkedin.com/in/darraghferguson

Intro to Portfolio Optimization

We will denote a universe of N assets by $\mathbf{p}_t \in R^N$, then linear returns for time index t (Any time period) are given by

$$\mathbf{r}_t = \frac{\mathbf{p}_t - \mathbf{p}_{t-1}}{\mathbf{p}_{t-1}} = \frac{\mathbf{p}_t}{\mathbf{p}_{t-1}} - 1$$

where operations are element-wise. It follows that log returns are given by $\mathbf{r}_t^{\log} = \log(\mathbf{p}_t) - \log(\mathbf{p}_{t-1})$. It is my goal to forecast the returns at time t (based on historical data up to time $t-1$), $\mathcal{F}_{t-1}$. (What are conditional first and second order moments? conditional mean vector and covariance matrix). We can define our forecasting in terms of conditional moments. That is the conditional mean vector and covariance matrix.

$$\mu_t^{\log} = E[\mathbf{r}_t^{\log} | \mathcal{F}_{t-1}]$$

$$\Sigma_t^{\log} = Cov[\mathbf{r}_t^{\log} | \mathcal{F}_{t-1}] = E[(\mathbf{r}_t^{\log} - \mu_t^{\log})(\mathbf{r}_t^{\log} - \mu_t^{\log})^T | \mathcal{F}_{t-1}]$$

This tells us that our forecasted mean at time t is a vector conditional on our historical forecasts $\mathcal{F}_{t-1}$. Similarly, our forecasted covariance matrix is conditional on $\mathcal{F}_{t-1}$.

A portfolio is an allocation of the available budget among N assets and unallocated cash. I define a portfolio at time t by a capital allocation $\mathbf{w}_t^{cap} \in R^N$, where $\mathbf{w}_{i,t}^{cap}$ is the capital allocated to the i-th asset. I could include cash as one extra riskless asset among the portfolio vector but this would produce a singular covariance matrix (why?), leading to issues with our optimization (why?). Similarly, a portfolio can be represented as the number of units of each asset held, $\mathbf{w}_t^{units} \in R^N$. If I fix the unit allocation over time so $\mathbf{w}_t^{units} = \mathbf{w}^{units}$, then the dollar amount of the portfolio will change as prices of the assets change, $\mathbf{w}_t^{cap} = \mathbf{w}_t^{units} \odot \mathbf{p}_t$. So the corresponding change in the portfolio is

$$\mathbf{w}_t^{cap} = \mathbf{w}_{t-1}^{cap} \odot (\mathbf{p}_t \oslash \mathbf{p}_{t-1})$$

where $\odot$ and $\oslash$ are element-wise product and division respectively.

My portfolio net asset value (NAV), is defined as the value of the portfolio at current market valuation and my cash balance:

$$NAV_t = 1^T \mathbf{w}_t^{cap} + c_t^{cap}$$

Using this formula and the earlier portfolio evolution then

$$NAV_t = 1^T (\mathbf{w}_{t-1}^{cap} \odot (1 + \mathbf{r}_t)) + c_{t-1}^{cap}$$

$$= NAV_{t-1} + (\mathbf{w}_{t-1}^{cap})^T \mathbf{r}_t$$

which simply says that net asset value in the present is equivalent to net asset value of the previous time step add the asset growth in the latest time step.

Portfolio return can then be described

$$R_t = \frac{NAV_t - NAV_{t-1}}{NAV_{t-1}} = \mathbf{w}_{t-1}^T \mathbf{r}_t$$

where $\mathbf{w}_t = \mathbf{w}_t^{cap} / NAV_t$ is the normalised portfolio with respect to the current NAV. Portfolio optimization seeks to determine the weights of $\mathbf{w}_{t-1}$ that will be held over period t, prior to realising returns. The next period normalised weights $\mathbf{w}_t$ result from the NAV change.