Brownian Motion

Brown (1827) Particle Motion on Water surface

Einstein (1905) Diffusion Equation $\Rightarrow$ Wiener 1930's Levy

Def.: $B(0)=0$; Independent Increment; $B(t)-B(s) \sim N(0, \sigma^2(t-s))$;

Continuous Sample Path $t \to B(w,t)$

Independent Increment.

Def. 2nd: $B(0)=0$; $B(t)$ Gaussian; $E[B(t)]=0$, $R_B(t,s)=\sigma^2 \min(t,s)$;

Continuous Sample Path $t \to B(w,t)$

$\forall n, \forall t_1, t_2, \cdots, t_n, \ (B(t_1), B(t_2), \cdots, B(t_n))^T \sim N$

$(B(t_1), B(t_2)-B(t_1), \cdots, B(t_n)-B(t_{n-1})) \sim ?$

$$\begin{bmatrix} B(t_1) \\ B(t_2)-B(t_1) \\ \vdots \\ B(t_n)-B(t_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \begin{bmatrix} B(t_1) \\ B(t_2) \\ \vdots \\ B(t_n) \end{bmatrix} \sim N$$

Independent / Stationary

Gaussian (Continuous Sample Path)

Poisson (Discrete Sample Path)

Def. 3rd: $B(0)=0$; Independent & Stationary Increments; $EB(t)=0$, $R_B(t,s)=\sigma^2\min(t,s)$;

Continuous Sample Path; (独立且平稳)

Ex. 1:

$B(t)$ is BM, $\sigma^2=1$, $B(3) \sim ?$ $\quad B(3)-B(0) \sim N(0, \sigma^2 \cdot 3) = N(0,3)$

Ex. 2:

$B(2)=4$, $E[B(4) \mid B(2)=4]$ $\quad B(4)-B(2) \sim N(0, 2\sigma^2)$ Martingale

$$E[B(4) \mid B(2)=4] = E[B(4)-B(2)+B(2) \mid B(2)=4] = E[B(4)-B(2) \mid B(2)=4] + E[B(2) \mid B(2)=4] = E[B(4)-B(2)] + 4 = 4$$

Ex. 3:

$B(2)=4$, $\quad P(B(4)>5 \mid B(2)=4) = P(B(4)-B(2)>1)$, $\quad B(4)-B(2) \sim N(0, 2\sigma^2)$

Conditional Gaussian: $(X_1, X_2)^T \sim N\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\right)$, $\quad X_2 \mid X_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1-\mu_1), \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

$$E[X_2 \mid X_1] = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1-\mu_1) \rightarrow \text{is a r.v.}$$

$$P(B(4)-B(2)>1)=\int_1^{\infty}\frac{1}{\sqrt{2\pi\cdot 2\sigma^2}}\exp\left(-\frac{t^2}{2\cdot 2\sigma^2}\right)dt$$

Ex. 4:

$B(t)$ is BM, $\frac{1}{c}B(c^2t)=Y(t)$ is BM?

1) $Y(0)=\frac{1}{c}B(0)=0$ 2) $\forall n,\ \forall t_1,\cdots,t_n,\ [Y(t_1),\cdots\cdots,Y(t_n)]=\frac{1}{c}[B(c^2t_1),\cdots\cdots,B(c^2t_n)]\sim N$

3) $E[Y(t)]=\frac{1}{c}E[B(c^2t)]=0$, $R_Y(t,s)=E[\frac{1}{c}B(c^2t)\cdot\frac{1}{c}B(c^2s)]=\frac{1}{c^2}\min(c^2t,c^2s)\cdot\sigma^2=\sigma^2\min(t,s)$

$\Rightarrow Y(t)$ is BM.

Ex. 5:

$Y(t)=tB(1/t)$ Suppose $Y(0)=0$, Prove $Y(t)$ is BM:

$(Y(t_1),\cdots\cdots,Y(t_n))=(t_1B(\frac{1}{t_1}),\ t_2B(\frac{1}{t_2}),\cdots\cdots,t_nB(\frac{1}{t_n}))$

Criteria for Gaussian Distribution: any linear combination of $\cdots$ is 1-Dim Gaussian

$\forall\alpha\in\mathbb{R}^n,\ \alpha^TX\sim N\Rightarrow X\sim N$

$\forall\lambda\in\mathbb{R}^n.\ \lambda^TY=\sum\lambda_it_iB(\frac{1}{t_i})\sim N\Rightarrow Y(t)$ Gaussian

$E[Y(t)]=0$

$E[Y(t)Y(s)]=E[tB(\frac{1}{t})\,sB(\frac{1}{s})]=ts\,\sigma^2\min(\frac{1}{t},\frac{1}{s})=\sigma^2\min(s,t)$

Interesting Properties of BM:

事件的等价性

① $M(t)=\max\limits_{0\le s\le t}B(s)\rightarrow$ model stock price. $P(M(t)>x)=P(T_x<t)$

Levy: $T_x=\min\{s:B(s)=x\}$ Hitting Time. $\{B(t)>x\}\Rightarrow B(T_x)=x\Rightarrow B(t)=B(t)-B(T_x)+\underbrace{B(T_x)}_{x}$

$|T_x<t)$

Goes up/down is equally possible $P(B(t)>x|T_x<t)=P(B(t)<x|T_x<t)=\frac{1}{2}$

$B(t)\,|\,B(t)>x)=\underbrace{B(t)-B(T_x)}_{\text{Expectation}=0}+x$ Reflection Trick

equivalent to: By the time t, $B(t)$ hasn't reached x yet.

$P\{B(t)>x\}=P(B(t)>x|T_x<t)P(T_x<t)+\underbrace{P(B(t)>x|T_x>t)}_{=0}\cdot P(T_x>t)$

$=\overbrace{P(B(t)>x|T_x<t)}^{=\frac{1}{2}}\cdot P(T_x<t)=\frac{1}{2}P(T_x<t)$

$\Rightarrow P(T_x<t)=2P(B(t)>x)$ $F_{T_x}(t)=2P(B(t)>x)=2\frac{1}{\sqrt{2\pi\sigma^2t}}\int_x^{\infty}\exp(-\frac{s^2}{2\sigma^2t})ds$

$\therefore P(M(t)>x)=P(T_x<t)=2P(B(t)>x)$

② Quadratic Variation : $B(t) \sim N(0, 6^2 t)$, $E[B^2(t)] = t$ $\quad 6^2=1$

$[0,t] \longrightarrow [0, \frac{t}{n}, \frac{2t}{n}, \cdots, \frac{n-1}{n}t, t] \to B(\frac{k}{n}t) - B(\frac{k-1}{n}t)$ $k=1\sim n$ writen as $\Delta B(\frac{k}{n})$

Prove: $\sum_{k=1}^{n} [\Delta B(\frac{k}{n})]^2 \xrightarrow[n\to\infty]{\text{mean square}} t$ In calculus, Bounded Variation, not Quadratic Variation

things are different in stochastic calculus.

i.e. $E[\sum_k (\Delta B(\frac{k}{n}))^2 - t]^2 \xrightarrow[n\to\infty]{} 0$ $\quad (B(\frac{k}{n}t) - B(\frac{k-1}{n}t) \sim N(0, 6^2\frac{t}{n})$ & $\Delta B(\frac{k}{n})$ indep.)

$= E[\sum_k ((\Delta B(\frac{k}{n}))^2 - \frac{t}{n})]^2 = E[\sum_k (\Delta B^2(\frac{k}{n}) - \frac{t}{n})^2 + \sum_{i\neq j} (\Delta B(\frac{i}{n})^2 - \frac{t}{n})(\Delta B(\frac{j}{n})^2 - \frac{t}{n})]$

$E[(\Delta B^2(\frac{i}{n}) - \frac{t}{n})(\Delta B^2(\frac{j}{n}) - \frac{t}{n})] = E[\Delta B^2(\frac{i}{n}) - \frac{t}{n}]\, E[\Delta B^2(\frac{j}{n}) - \frac{t}{n}] = 0 \Rightarrow$ 交叉项为0

$B(\frac{it}{n}) - B(\frac{i-1}{n}t)$ $\quad B(\frac{jt}{n}) - B(\frac{j-1}{n}t) \sim N(0, \frac{t}{n})$, $E[\Delta B^2(\frac{j}{n})] = \frac{t}{n}$ $\quad E[\Delta B^2(\frac{k}{n})] = \frac{t}{n}$

$\Rightarrow$ 原式 $= E[\sum_k (\Delta B(\frac{k}{n})^2 - \frac{t}{n})^2] = E[\sum_k (\Delta B^4(\frac{k}{n}) - 2\cdot\frac{t}{n}\Delta B^2(\frac{k}{n}) + \frac{t^2}{n^2})] = \sum_k (E[\Delta B^4(\frac{k}{n})] - 2\cdot\frac{t}{n}\cdot\frac{t}{n} + \frac{t^2}{n^2}) = \sum_k (E[\Delta B^4(\frac{k}{n})] - \frac{t^2}{n^2})$

Polynomial of Gaussian: $X \sim N(0, 6^2)$, $\phi_X(w) = \exp(jw\mu - \frac{w^2 6^2}{2}) = \exp(-\frac{w^2 6^2}{2})$

$$\phi_X(w) = E[\exp(jwX)],\quad EX = \frac{1}{j}\frac{\partial}{\partial w}\phi_X(w)\Big|_{w=0},\quad EX^2 = (\frac{\partial}{j\partial w})^2 \phi_X(w)$$

$$EX^4 = (\frac{\partial}{j\partial w})^4 \phi_X(w)\Big|_{w=0} = \frac{\partial^4}{\partial w^4}\exp(-\frac{w^2 6^2}{2})\Big|_{w=0} = \frac{\partial^3}{\partial w^3}(-6^2 w \exp(-\frac{6^2}{2}w^2))\Big|_{w=0} = \frac{\partial^2}{\partial w^2}((-6^2 + (-6^2 w)^2)\exp(-\frac{6^2}{2}w^2))\Big|_{w=0}$$

$$= \frac{\partial}{\partial w}(2w6^4 \cdot \exp(-\frac{6^2}{2}w^2) + (-6^2 w)\cdot(-6^2 + 6^4 w^2)\exp(-\frac{6^2}{2}w^2))\Big|_{w=0} = \frac{\partial}{\partial w}[(36^4 w - 6^6 w^3)\exp(-\frac{6^2}{2}w^2)]\Big|_{w=0}$$

$= 36^4$

$\therefore E[\Delta B(\frac{k}{n})^4] = 3\cdot(\frac{t}{n})^2$, $\quad \sum_k [3(\frac{t}{n})^2 - \frac{t^2}{n^2}] = \sum_k \frac{2t^2}{n^2} = \frac{2t^2}{n} \to 0$ $\quad \therefore E[\sum_k \Delta B^2(\frac{k}{n}) - t]^2 \longrightarrow 0$

How does Quadratic Variation $(\sum_n \Delta B^2(\frac{k}{n}) \xrightarrow{m.s} t)$ affect calculus? Stochastic

$dB \sim \sqrt{dt}$

$g(t, B(t))$: $dg(t, B(t)) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial B}dB + \frac{1}{2}\frac{\partial^2 g}{\partial B^2}\cdot \cancel{(dB)^2}\, dt$

mean square convergence

$\int_0^1 B(t)\,dB(t) = ?$ $\quad d(\frac{1}{2}B^2) = BdB + \frac{1}{2}dt \Rightarrow \int_0^1 BdB = \int_0^1 d\frac{1}{2}B^2 - \frac{1}{2}dt = \frac{1}{2}[B^2(1) - B^2(0)] - \frac{1}{2}$

Stock Price modelling

1900s: Bachelier $B(t)$ (may be less than 0) $\to$ 1930s Samuelson : $S(t) = \exp(\mu t + B(t))$

1970s: Black-Scholes-Mertons: Derivatives (Option. ...) $V(t, S(t))$ Risk Neutral

Portfolio : $S \to V$ (sell stock to buy option) — price ↓, sold stock ↑ / price ↑, option ↑ — Heritage

$$P(t) = V(t, S(t)) - \alpha S(t) \quad \left[S = \exp(\mu t + B(t)) \right)$$

$$dS = \mu dt \cdot S + dB \cdot S + \frac{1}{2} \cdot dt \cdot S = S(\mu dt + dB + \frac{1}{2} dt)$$

$$dP(t) = rP dt \quad (r : \text{interest rate})$$

$$dP(t) = d\left(V(t, S(t)) - \alpha S(t)\right) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 - \alpha \cdot dS = rP dt$$

$$\Downarrow$$

$$S^2 (dB)^2 = S^2 dt$$

$$rP dt = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 dt - \alpha dS$$

remove dS (variability): $\alpha = \frac{\partial V}{\partial S} \to$ Heritage

$$\frac{\partial V}{\partial t} dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 dt = rP dt = r(V - \alpha S) dt = r(V - \frac{\partial V}{\partial S} \cdot S) dt$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 + rS\frac{\partial V}{\partial S} - rV = 0$$

Black-Scholes Equation