Spectral Analysis (Power Spectrum Density)

$X(t)$ random function, $x(t)$ deterministic function

Fourier Transform:

1) Periodic Case : $x(t) = x(t+T)$, $\exists\, T$ $\Rightarrow$ $x(t) = \sum_k \alpha_k \exp(jk\frac{2\pi}{T}t)$ can be decomposed.

$$\alpha_k = \frac{1}{2\pi}\int_T x(t)\exp(-jk\frac{2\pi}{T}t)\,dt \Rightarrow \text{Fourier Series.}$$

i) $x(t) \sim \sum_k \alpha_k \exp(jk\frac{2\pi}{T}t)$ $\quad \lim_{n\to\infty}\int_T \left|x(t) - \sum_k \alpha_k \exp(jk\frac{2\pi}{T}t)\right|^2 dt = 0$ mean square sense

ii) inside a period $[-\frac{T}{2}, \frac{T}{2}]$ Non-periodic $\Rightarrow$ truncate in a range (周期延拓)

Fourier Series : $T \to \infty$

$$X(t) = \sum_k \alpha_k \exp(jk\frac{2\pi}{T}t) = \frac{1}{T}\sum_k \int_T x(\tau)\exp(-jk\frac{2\pi}{T}\tau)\,d\tau \cdot \exp(jk\frac{2\pi}{T}t)$$

$$= \frac{1}{2\pi}\sum_k \left(\int_T x(\tau)\exp(-jk\frac{2\pi}{T}\tau)\,d\tau\right)\cdot\frac{2\pi}{T}\exp(jk\frac{2\pi}{T}t)$$

Let $T \to \infty$, $k\frac{2\pi}{T} \to \omega$, $\sum \to \int$

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} x(\tau)\exp(-j\omega\tau)\,d\tau\right)d\omega\,\exp(j\omega t)$$

$$\therefore X(\omega) = \int_{\mathbb{R}} x(\tau)\exp(-j\omega\tau)\,d\tau$$

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\omega)\exp(j\omega t)\,d\omega$$

not convergent for random signals.

Fourier Transform $\Rightarrow$ adapt to random signals? (not deterministic)

$\int X(t)\exp(-j\omega t)\,dt$?

FT exists $\Leftarrow$ $x(t) \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} |x(t)|\,dt < \infty$

when Stochastic process is w.s.s., then the signal is not absolutely integrable

signal itself doesn't have a FT

How to adapt FT to random case? (2 approaches)

i) Perform FT on a short period of time: $\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)\,dt$ (short-time FT)

Little tweak on the equation : $\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)dt\right|^2$ (Signal → Power)

⇒ Expectation of it : $E\left[\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\exp(-j\omega t)dt\right|^2\right]$ (Remove randomness)

⇒ Normalize it & let $T\to\infty$: $\lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_T X(t)\exp(-j\omega t)dt\right|^2\right] = ?$ (Physics Perspective)

$$= \frac{1}{T}E\left[\int_T X(t)\exp(-j\omega t)dt\int_T X^*(\tau)\exp(j\omega\tau)d\tau\right] = \frac{1}{T}E\left[\iint X(t)X^*(\tau)\exp(-j\omega t+j\omega\tau)\,dt\,d\tau\right]$$

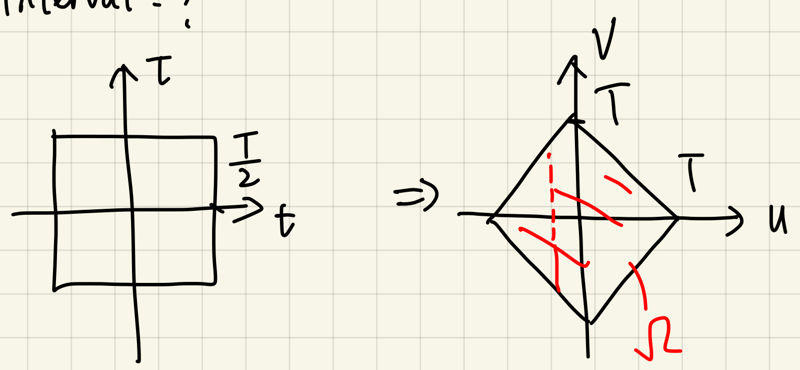
$$= \frac{1}{T}\iint E[X(t)X^*(\tau)]\exp(-j\omega t+j\omega\tau)\,dt\,d\tau \quad \text{(w.s.s.)}$$

$$= \frac{1}{T}\iint R_X(t-\tau)\exp(-j\omega(t-\tau))\,dt\,d\tau$$

?-

Let $\begin{cases} u=t-\tau \\ v=t+\tau \end{cases}$ $\quad \int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\,d\tau \quad$ Jacobian: $dt\,d\tau = \left|\det\left[\frac{\partial(t,\tau)}{\partial(u,v)}\right]\right|du\,dv \quad \det\left[\frac{\partial(t,\tau)}{\partial(u,v)}\right] = \left\{\det\left[\frac{\partial(u,v)}{\partial(t,\tau)}\right]\right\}^{-1} = \begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}^{-1} = \frac{1}{2}$

interval : ?



$$= \frac{1}{T}\iint_\Omega R_X(u)\exp(-j\omega u)\,du\,dv$$

$$\iint_\Omega = \int_{-T}^{0}du\left[\underbrace{\int_{-T-u}^{T+u}dv}_{\int_{-T+|u|}^{T-|u|}}\right] + \int_0^T du\left[\underbrace{\int_{-T+u}^{T-u}dv}_{\int_{-T+|u|}^{T-|u|}}\right]$$

$$\therefore = \left[\frac{1}{T}\int_{-T}^{0}\int_{-T-u}^{T+u}R_X(u)\exp(-j\omega u)\,dv\,du + \frac{1}{T}\int_0^T\int_{-T+u}^{T-u}R_X(u)\exp(-j\omega u)\,dv\,du\right]\frac{1}{2}$$

$$= \frac{1}{2T}\left[\int_{-T}^{0}R_X(u)\exp(-j\omega u)\cdot(2T+2u)\,du + \int_0^T R_X(u)\exp(-j\omega u)(2T-2u)\,du\right]$$

$$= \frac{1}{T}\left[\int_{-T}^{0}R_X(u)\exp(-j\omega u)(T-|u|)\,du + \int_0^T R_X(u)\exp(-j\omega u)(T-|u|)\,du\right]$$

$$= \int_{-T}^{T}R_X(u)\left[1-\frac{|u|}{T}\right]\exp(-j\omega u)\,du \quad \left(\lim_{T\to\infty}\left(1-\frac{|u|}{T}\right)=1\text{, even when } T\to\infty,\ u\to\infty \text{ as well}\right)$$

Real Analysis ⇒ Lebesgue Controlled Convergence

$$\xrightarrow{T\to\infty} \int_{-T}^{T}R_X(u)\exp(-j\omega u)\,du \quad (\text{if } R_X(u) \text{ is absolutely integrable!})$$

$$\Rightarrow \int_{\mathbb{R}}R_X(u)\exp(-j\omega u)\,du$$

Fourier Transform of Correlation Function

$$S_X(\omega) = \lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_{-\frac{T}{2}}^{\frac{T}{2}}X(t)\exp(-j\omega t)dt\right|^2\right] = \int_{\mathbb{R}}R_X(u)\exp(-j\omega u)\,du$$

Power Spectral Density (PSD) $\quad (W\cdot s^{-1} = J)$

Second-Order Information

$$S_{X_1+X_2}(w) \neq S_{X_1}(w) + S_{X_2}(w)$$

$$S_X(w) \geq 0$$ (Consistent with Bochner's Result (p.d. $\Leftrightarrow$ FT pos.))

$$S_X(w) = \int R_X(u) \exp(-jwu)\,du$$

Wiener-Khinchin Relation

$$S_X(w) = \int_{\mathbb{R}} R_X(\tau) \cos w\tau \, d\tau + j \int_{\mathbb{R}} \underbrace{R_X(\tau) \sin w\tau}_{\text{odd}} \, d\tau \rightarrow = 0$$

$$= \int_{\mathbb{R}} R_X(\tau) \cos w\tau \, d\tau$$

$$= S_X(-w)$$

Properties of PSD:

$S_X(w) \geq 0$, $S_X(w) = S_X(-w)$ ($S_X(w)$ real + $R_X(u)$ even $\Rightarrow$ $S_X(w)$ even)

Input-Output Relation of LTI Systems:

$X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t)$: $Y(t) = X(t) * h(t) = \int_{\mathbb{R}} X(\tau) h(t-\tau) d\tau$ Convolution

Impulse Response

$X(t)$ w.s.s. $\overset{?}{\Rightarrow}$ $Y(t)$ w.s.s. $\overset{?}{\Rightarrow}$ $S_Y(w) = ?$

$$E[Y(t) Y^*(s)] = E\left[\int_{\mathbb{R}} X(u) h(t-u) du \int_{\mathbb{R}} X^*(v) h^*(s-v) dv\right]$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} h(t-u) h^*(s-v) E[X(u) X^*(v)] du\, dv$$

$$= \iint_{\mathbb{R}^2} h(t-u) h^*(s-v) R_X(u-v) du\, dv$$ $\Rightarrow$ 2^{nd}-order convolution. still w.s.s. ($Y(t)$)

$$= \iint_{\mathbb{R}^2} h(t-u)\, h^*(-(v-s))\, R_X(u-v) du\, dv = [h \otimes h^*(-\cdot) \otimes R_X](t-s)$$

$$S_Y(w) = \int_{\mathbb{R}} R_Y(u) \exp(-jwu) du = \int_{\mathbb{R}} [h \otimes h^*(-\cdot) \otimes R_X](u) \exp(-jwu) du = H(w) \cdot H^*(w) \cdot S_X(w) = |H(w)|^2 S_X(w)$$

$S_X(w)$: 2nd-order property

$\Downarrow$

对 $R_X(\tau)$ 的正定性有认识

($\mathcal{F}\{R_X(\tau)\} \geq 0$)

$\Downarrow$

PSD

$$\frac{1}{2\pi} \sum_K \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} X(u) \exp(-jk\tfrac{2\pi}{T} u) du\right] \exp(jk\tfrac{2\pi}{T} t) \cdot \frac{2\pi}{T}$$

$\downarrow$ Assume

$$X(w) = \frac{1}{2\pi} \int X(t) \exp(jwt) dw$$

divergent

$$\rightarrow \frac{1}{2\pi} \int \exp(jwt)\, dF_X(w)$$

singularity corresponds to non-differetiable point

Spectral Representation

① $w \neq w' \Rightarrow E[dF_X(w)[dF_X(w')]^*] = 0$ Orthogonal Increment

② $w = w' \Rightarrow E[dF_X(w)(dF_X(w))^*] = E|dF_X(w)|^2 = S_X(w)dw$ ⊃ why?

$$R_X(t,s) = E[X(t)X^*(s)] = E\left[\int \exp(jwt)\, dF_X(w) \cdot \left[\int \exp(jw's)\, dF_X(w')\right]^*\right]$$

$$= \iint \exp(j(wt - w's))\, E(dF_X(w)[dF_X(w')]^*)$$

$$= \frac{1}{2\pi}\int \exp(jw(t-s))\, E|dF_X(w)|^2 \quad (w = w',\ \neq 0)$$

$$Y(t) = \int_{\mathbb{R}} h(t-u)X(u)\,du = \frac{1}{2\pi}\int h(t-u)\left(\int \exp(jwu)\,dF_X(w)\right)du = \frac{1}{2\pi}\int dF_X(w)\left(\int h(t-u)\exp(jwu)\,du\right) \quad (\tau = t-u,\ u = t-\tau)$$

$$= \frac{1}{2\pi}\int dF_X(w) \cdot \left(\int h(\tau)\exp(jw(t-\tau))\,d(t-\tau)\right)$$

$$= \frac{1}{2\pi}\int dF_X(w) \cdot \exp(jwt) \cdot H(w)$$

$$= \frac{1}{2\pi}\int \exp(jwt)\, H(w)\, dF_X(w)$$

$$dF_Y(w) = H(w)\,dF_X(w)$$

$$= \frac{1}{2\pi}\int \exp(jwt)\, dF_Y(w)$$

And $S_Y(w)dw = E|dF_Y(w)|^2$

⋮

$= |H(w)|^2 S_X(w)dw$

$$S_Y = |H|^2 \cdot S_X$$