Spectral Representation

$X(t)$ w.s.s. $\quad X(t) = \int \exp(j\omega t)\, dF_X(\omega) \cdot \frac{1}{2\pi}$

$F_X(\omega)$: spectral process (Random) $\rightarrow$ Orthogonal Increments,

$\forall\, \omega_1 \le \omega_2 \le \omega_3 \le \omega_4$, $\quad F_X(\omega_2) - F_X(\omega_1) \perp F_X(\omega_4) - F_X(\omega_3)$, $\quad E[(F_X(\omega_2) - F_X(\omega_1))(F_X(\omega_4) - F_X(\omega_3))] = 0 \quad$ Uncorrelated.

$E[dF_X(\omega_0)(dF_X(\omega_1))^*] = S_X(\omega_0)\, d\omega_0 \cdot \delta_{01}$

if $X(t) = X(t+T)$, then $X(t) = \sum_k \alpha_k \exp(jk\frac{2\pi}{T}t)$, $\qquad \alpha_k = \frac{1}{T}\int_T X(t)\exp(-jk\frac{2\pi}{T}t)\,dt$

Same expression

Spectral Representation: $X(t) = \int \exp(j\omega t)\, dF_X(\omega)$

Orthogonality between Fourier Series index : $E[\alpha_k \alpha_m^*] = 0 \quad k \ne m$. (Orthogonal Increment $F_X(\omega)$)

$\alpha_k = \frac{1}{T}\int_T X(t)\exp(-jk\frac{2\pi}{T}t)\,dt$

$E[\alpha_k \alpha_m^*] = \frac{1}{T^2}\int_T\int_T E[X(t)X^*(s)]\exp(-jk\frac{2\pi}{T}t)\exp(jm\frac{2\pi}{T}s)\,dt\,ds$

$= \frac{1}{T^2}\int_T\int_T \exp(-j\frac{2\pi}{T}(kt - ms))\, R_X(t-s)\,dt\,ds$

Why we use Fourier Series to prove orthogonality?

i) FS representation & Spectral representation are basically the same thing.

ii) Orthogonality of complex exponential functions & Periodicity are better visualized on a limited interval of integration.

$X(t)$ periodic : $E|X(t) - X(t+T)|^2 \Leftrightarrow R_X(t) = R_X(t+T)$

2nd Lecture.

Let $t' = t - s$. $\quad$ Jacobian $= \frac{\partial(t', s)}{\partial(t, s)} = 1$

$= \frac{1}{T^2}\int_T \int_{-\frac{T}{2}-s}^{\frac{T}{2}-s} R_X(t') \cdot \exp\left(-j\frac{2\pi}{T}(kt' - (m-k)s)\right) dt'\,ds$

$= \frac{1}{T^2}\int_T \exp(j\frac{2\pi}{T}(m-k)s)\left[\int_{-\frac{T}{2}-s}^{\frac{T}{2}-s} R_X(t')\exp(-j\frac{2\pi}{T}kt')\,dt'\right] ds$

$R_X(t)$ periodic : $\int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t') \cdot \exp(-j\frac{2\pi}{T}kt')\,dt$

$= \frac{1}{T}\int_T S_X(k\frac{2\pi}{T}) \cdot \exp(j\frac{2\pi}{T}(m-k)s)\,ds$

$= \frac{1}{T} S_X(k\frac{2\pi}{T}) \int_T \exp(j\frac{2\pi}{T}(m-k)s)\,ds$

when $k \ne m$, $E[\alpha_k \alpha_m^*] = 0$, $\quad \alpha_k$ is equivalent to $dF_X(\omega)$ in Spectral Representation.

$E[dF_X(\omega) \cdot dF_X^*(\omega')] = 0$

$\frac{1}{2\pi} E|dF_X(\omega)|^2 = S_X(\omega)\,d\omega$.

$R_X(\tau) = \frac{1}{2\pi}\int \exp(j\omega\tau)\, S_X(\omega)\,d\omega$.

$$R_X(\tau) = E[X(t)X^*(t-\tau)] = E\left[\frac{1}{4\pi^2}\int \exp(j\omega t)\, dF_X(\omega)\left(\int \exp(j\omega'(t-\tau))\, dF_X(\omega')\right)^*\right]$$

$$= \frac{1}{4\pi^2}\iint \exp(j\omega t - j\omega'(t-\tau))\, E[dF_X(\omega)\,(dF_X(\omega'))^*]$$

Orthogonal Increments $\Rightarrow \omega = \omega'$: $\frac{1}{4\pi^2}\int \exp(j\omega'\tau)\left[\int E[dF_X(\omega)\, dF_X^*(\omega')]\exp(j(\omega-\omega')t)\right]$

$$= \frac{1}{4\pi^2}\int \exp(j\omega'\tau)\cdot 2\pi S_X(\omega)\cdot d\omega'$$

$$= \frac{1}{2\pi}\int S_X(\omega)\exp(j\omega\tau)\,d\omega = \frac{1}{2\pi}\int \exp(j\omega\tau)\cdot E|dF_X(\omega)|^2$$

$X(t)$ W.S.S. $\quad X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{\exp(j\omega t)}_{\text{Orthogonality}}\, d\underbrace{F_X(\omega)}_{\text{Randomness.}}$

$X(t) \Longleftrightarrow \exp(j\omega t)$

Oscillating (W.S.S.)

$\underbrace{X(\omega, t)}_{\text{Sample Space}} \Longleftrightarrow \underbrace{\exp(j\omega t)}_{\text{Frequency.}}$

Define 2 Distances:

$$\|X(t) - X(s)\|_1 = \|\exp(j\omega t) - \exp(j\omega s)\|_2$$

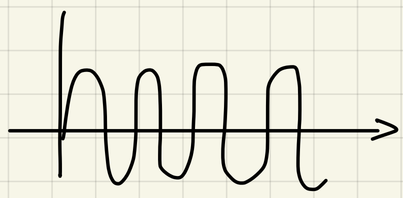
Isometry between Stochastic Processes & Complex Exponential Functions

$\|\cdots\|_1 = E\|\cdots\|^2$ (Stochastic)

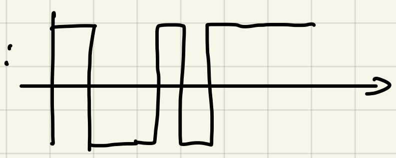
$\|\cdots\|_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\cdots|^2 \underbrace{S_X(\omega)}_{\text{PSD}}\,d\omega$ (Exponential)

W.S.S.:

Phase Modulation:



Random Telegram:



Oscillating (Exponential)

$$E\|X(t) - X(s)\|^2 = E\|X(t)\|^2 + E\|X(s)\|^2 - 2E[X(t)\cdot X^*(s)]$$

$$= 2R_X(0) - 2R_X(t-s)$$

$$\|\cdots\|_2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\exp(j\omega t) - \exp(j\omega s)|^2 S_X(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} (2 - 2\cdot\exp(j\omega(t-s)))\, S_X(\omega)\,d\omega$$

$$= 2R_X(0) - 2R_X(t-s)$$

Application of Oscillation in Sto. Pro.:

$\Rightarrow$ Sampling Theorem.

Shannon Sampling Theorem:

$X(t) \Longleftrightarrow \{X_k\}_{k=-\infty}^{+\infty} \Longleftrightarrow$ Expansion: $X(t) = \sum_k X_k \underbrace{\phi_k(\omega)}$ basis for expansion.

$X_k = X(k\Delta t) \qquad X(t) = \sum_k X(k\Delta t)\phi_k(t)$ Reproduce perfectly $\|\cdots\|_2 = 0$

$$\left\| X(t) - \sum X(k\Delta t)\phi_k(t) \right\|_1^2 = 0 \xleftrightarrow{\text{Isometry}} \left\| \exp(jwt) - \sum_k \exp(jwk\Delta t)\phi_k(t) \right\|_2^2 = 0$$

W.S.S. (Sampling Theorem's Requirement)

$$= \frac{1}{2\pi}\int_{\mathbb{R}} S_X(w)\left|\exp(jwt) - \sum \exp(jwk\Delta t)\phi_k(t)\right|^2 dw.$$

$X(t)$: Band-Limited ; $S_X(w) = 0,\ |w| \geq B.$

$$\Rightarrow \frac{1}{2\pi}\int_{-B}^{B} S_X(w)\left|\exp(jwt) - \sum \exp(jwk\Delta t)\phi_k(t)\right|^2 dw$$

$$\therefore \exp(jwt) = \sum \phi_k(t)\exp(jwk\Delta t), \quad w \in (-B, B)$$

$$\phi_k = \frac{1}{2B}\int_{-B}^{B} \exp(jwt)\exp(-jwk\Delta t)\, dw$$

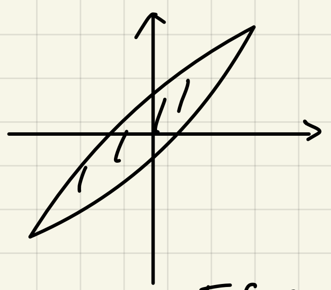
$$= \frac{1}{2B}\int_{-B}^{B} \exp(jw(t-k\Delta t))\, dw = \frac{1}{2B}\ \frac{\exp(jB(t-k\Delta t)) - \exp(-jB(t-k\Delta t))}{j(t-k\Delta t)}$$

$$= \frac{\sin(B(t-k\Delta t))}{B(t-k\Delta t)} = \operatorname{sinc}\left(B(t-k\Delta t)\right)$$

For a Band-Limited Signal, $X(t) = \sum_k X(k\Delta t)\cdot\frac{\sin B(t-k\Delta t)}{B(t-k\Delta t)}$ (Deterministic ✓)

Multi-Variate Correlation:

$X, Y \in \mathbb{R} \Rightarrow E(XY)$



Strongly Correlated:

$X_1, X_2, \cdots, X_n$, $n > 2$ $\quad E[X_iX_j] = R_{ij},\ i,j = 1, \cdots, n \quad r_{ij} = r_{ji}$ Organize them into a matrix.

$$\mathbb{R} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ \vdots & R_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix} = E[XX^T] \qquad X = [X_1, \cdots, X_n]^T$$

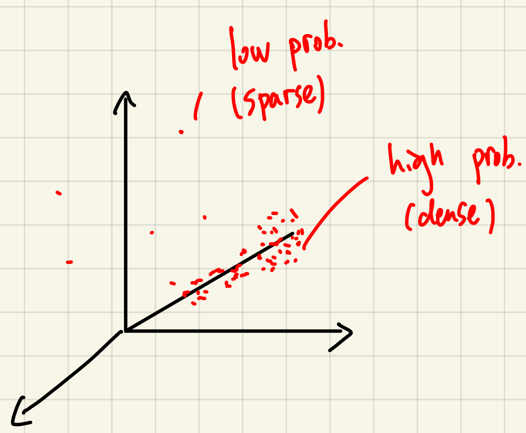
1. Decorrelation

$Y = AX$, $A \in \mathbb{R}^{n\times n}$, $E[YY^T] = \text{diag}.$

$$E[YY^T] = E[AX(AX)^T] = AE[XX^T]A^T = A\underline{R_X}A^T$$ positive definite.

$$R_X = U^T \text{diag}.(\lambda_1, \lambda_2, \cdots, \lambda_n)U = \sum_k \lambda_k U_k U_k^T \quad (U = [U_1 \cdots U_n])$$

2. PCA (Principal Component Analysis)



$X \in \mathbb{R}^n$ $\quad \text{Proj}_\alpha X \to$ Projection on an axis to test whether distribution revolves around it

as scattered as possible $E[\ ]^2$ to determine the range of projection

$E[\ ]^2 \uparrow \to$ Principal Component.

$E|\mathrm{Proj}_\alpha X|^2 = E\left|\frac{\langle\alpha, X\rangle}{\langle\alpha,\alpha\rangle} X\right|^2 \quad \|\alpha\|=1$

if the axis describes the distribution (prob.) most accurately, then the projection of sample pts onto the axis (more information) should be most scattered

$\max_\alpha E|\alpha^T X|^2, \ \text{s.t.} \ \|\alpha\|=1$

$\Rightarrow$ maximize variance of projection pts.

$$g(\alpha,\lambda) = E|\alpha^T X|^2 - \lambda(\|\alpha\|^2 - 1)$$
$$= E(\alpha^T X X^T \alpha) - \lambda(\alpha^T\alpha - 1)$$
$$= \alpha^T R_X \alpha - \lambda(\alpha^T\alpha - 1)$$

$\nabla_\alpha g(\alpha,\lambda) = (R_X + R_X^T)\alpha - \lambda(2\alpha) = 0 \Rightarrow R_X\alpha = \lambda\alpha$, Eigenvector Decomposition of Correlation Matrix

$\lambda \to$ max. eigenvalue of R_X

$\lambda = \alpha^T R_X \alpha \to$ optimized (maximized) subject

Energy-Wise

$\downarrow$ biggest eigenvalue of R_X

Energy $\neq$ Character.

$\Rightarrow 2^{nd}$ Principal Component (Perpendicular to 1^{st} Principal Component)

3. $Y = AX = A_1X_1 + A_2X_2 + \cdots + A_nX_n$ ($A_1 \cdots A_n$ orthogonal to each other)

$R_X = A^T \text{diag.} A \quad (AA^T = A^TA = 1)$

does $X_1 \cdots X_n$ possess orthogonality? No! $\Rightarrow X = A^TY = (A^T)_1 Y_1 + \cdots + (A^T)_n Y_n$

$Y_1 \cdots Y_n$ possess orthogonality $\to (E[Y_i Y_j] = 0, \ i \neq j)$

$(A^T)_1 \cdots (A^T)_n$ possess orthogonality

$\Rightarrow X = A^TY$: Bi-Orthogonal Expansion

$(A^T)_1 \cdots (A^T)_n \to$ geometrical characteristics (deterministic)

$Y_1, \cdots, Y_n \to$ stochastic characteristics (1-D)

while no constrain posed on X.

$\Rightarrow$ capturing energy characteristics as well

say $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, then

$(A^T)_1 Y_1 \to$ most essential direction

$(A^T)_2 Y_2 \to 2^{nd}$ essential direction

(Data Compression Algorithm)

$X = A^TY$ KL (Karhunen-Loève) Expansion

$A_k^T A_j = 0$

$X(t) = \sum_{-\infty}^{+\infty} \alpha_k \phi_k(t) \quad E(\alpha_k \alpha_m) = 0 \quad \int \phi_\ell \phi_j \, dt = 0$

Determine the basis $\{\phi_k\}$

2-06-03

$R_X A_k = \lambda_k A_k \Rightarrow \sum_{j=1}^{n} R_X(i,j) A_k(j) = \lambda_k A_k(i) \quad i = 1, \cdots, n$

continuous case $\int_I R_X(t,s)\phi_k(s)\,ds = \lambda_k \phi_k(t), \quad t \in I$

Solve the above integral in specific case:

1) Periodic Stochastic Processes: $X(t)=X(t+T)$ $R_X(\tau)=R_X(\tau+T)$ + W.S.S

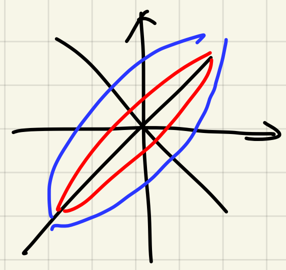
$$\int_I \underset{R_X(t-s)}{R_X(t,s)}\,\phi_k(s)\,ds=\lambda_k\phi_k(t) \quad (\phi_k \to \text{complex exponential composes a set of basis})$$

$$=\int_I R_X(\underbrace{t-s}_{s'=t-s})\exp(j\omega s)\,ds=\int_{I_s} R_X(s')\exp(j\omega(t-s'))\,d(t-s')=-\int_{\underset{\downarrow\,\text{periodicity}}{I}} R_X(s')\exp(-j\omega s')\,ds'\cdot\exp(j\omega t)=\lambda_k\exp(j\omega t)\ \left(\lambda_k=-\int_I R_X(s')\exp(-j\omega s')\,ds'\right)$$

KL Expansion

$$X(t)=\frac{1}{2\pi}\int_{\mathbb{R}}\exp(j\omega t)\,dF_X(\omega)$$

X_1, X_2, $EX_1=EX_2=0$, $EX_1^2=EX_2^2=1$ $E[X_1X_2]=\rho \Rightarrow$ 纺锤的胖瘦



$$R_X=\begin{bmatrix}1&\rho\\ \rho&1\end{bmatrix}\Rightarrow \det(\lambda I-R_X)=0,\quad \lambda_1=1+\rho,\quad \lambda_2=1-\rho$$

$$(R_X-\lambda_1 I)U_1=\begin{bmatrix}-\rho&\rho\\ \rho&-\rho\end{bmatrix}U_1=0\Rightarrow U_1=\begin{bmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{bmatrix}$$

Variance → direction of U_1, U_2

$$(R_X-\lambda_2 I)U_2=\begin{bmatrix}\rho&\rho\\ \rho&\rho\end{bmatrix}U_2=0\Rightarrow U_2=\begin{bmatrix}-\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{bmatrix}$$

5-th Multivariate Correlation

$X=(X_1,\cdots,X_n)^T$. $\underbrace{EX_iX_j}_{\downarrow}$ $(i\neq j)$: $\binom{n}{2}=\frac{n(n-1)}{2}$ Correlations

Distribution in $\mathbb{R}^n$? ⇐ knowledge from EX_iX_j → matrix

1) Decorrelation, $\exists A\in\mathbb{R}^{n\times n}$, $Y=AX\in\mathbb{R}^n$, $E[Y_iY_j]=0$, $i\neq j$

$$\underset{\Downarrow\ \text{Diagonal}}{R_Y}=E[YY^T]=E[AXX^TA^T]=AE[XX^T]A^T=AR_XA^T$$

$$R_X=U\,\mathrm{diag}(\lambda_1,\cdots,\lambda_n)\,U^T=\sum\lambda_kU_kU_k^T$$

$$U=(U_1,\cdots,U_n)$$

$$A=U^T$$

$$U^TR_XU=\mathrm{diag}(\lambda_1\cdots\lambda_n)$$