$$R_X(t,s) = E[X(t), X(s)]$$

Requirements on Correlation Function:

① $R_X(t,s) = R_X(s,t)$ $\Longrightarrow$ ① $R_X(\tau) = R_X(-\tau)$

② $|R_X(t,s)| \le |R_X(t,t)\, R_X(s,s)|^{1/2}$ w.s.s. ② $|R_X(\tau)| \le R_X(0)$

$\downarrow$ CDMA, Correlated RX

③ Positive Definite.

W.S.S. $\Longleftrightarrow$ $R_X(t,s) = R_X(\tau)$ $(\tau = t-s)$

$m(t) = E[X(t)] = m$

$f(t)$ is P.D. $\Longleftrightarrow$ $\forall n$, $\forall x_1, \cdots x_n$. $(f(x_i - x_j))_{ij} \geq 0$

$A \in \mathbb{R}^{n\times n} \geq 0 \Longleftrightarrow \forall \alpha \in \mathbb{R}^n,\ \alpha^T A \alpha \geq 0$

Correlation Function $R_X(\tau)$ is pos. def.

$R_X(\tau)$ is p.d. $\Rightarrow$ $R_X(0) \geq 0$, $n=1$, $\forall x_1$, $(\overset{R_X(0)}{R_X(x_1 - x_1)}) \geq 0$

$\Downarrow$

$|R_X(\tau)| \le R_X(0)$, $\forall \tau$. $n=2$ 时. let $(x_1, x_2) = (0, \tau)$ $\Rightarrow$ $\begin{bmatrix} R_X(0) & R_X(-\tau) \\ R_X(\tau) & R_X(0) \end{bmatrix}$ P.D.

P.D. Matrix $\Rightarrow$ Symmetric, $R_X(\tau) = R_X(-\tau)$

$\det(\text{Mat}) \geq 0$, $R_X^2(0) - R_X^2(\tau) \geq 0 \Rightarrow |R_X(\tau)| \le R_X(0)$.

$\forall n$, $\forall \tau_1, \cdots, \tau_n$, $\left[(R_X(\tau_i - \tau_j))_{ij}\right]$ $\forall \alpha \in \mathbb{R}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)^T$

$$\alpha^T \left[R_X(\tau_i - \tau_j)_{ij}\right] \alpha = \sum_{ij} \alpha_i R_X(\tau_i - \tau_j)\alpha_j = \sum_{ij} \alpha_i E\left[X(\tau_i)X(\tau_j)\right]\alpha_j = E\left[\sum_i\sum_j \alpha_i\alpha_j X(\tau_i)X(\tau_j)\right] = E\left[\left(\sum_i \alpha_i X(\tau_i)\right)\left(\sum_j \alpha_j X(\tau_j)\right)\right]$$

$$= E\left[\sum_i \alpha_i X(\tau_i)\right]^2 \geq 0$$

$X = [X(\tau_1), X(\tau_2), \cdots, X(\tau_n)]^T$ Correlation Matrix

$[R_X(\tau_i - \tau_j)]_{ij} = E[XX^T] = R$ $\quad \alpha^T R \alpha = \alpha^T E[XX^T]\alpha = E[\alpha^T X X^T \alpha] = E[\alpha^T X]^2$ $\Longrightarrow$ Characteristic Property

Unusual Properties of Correlation Functions

1) $R_X(T) = R_X(0)$ $(T \neq 0)$ $\Rightarrow$ periodic, $R_X(\tau) = R_X(\tau + T)$

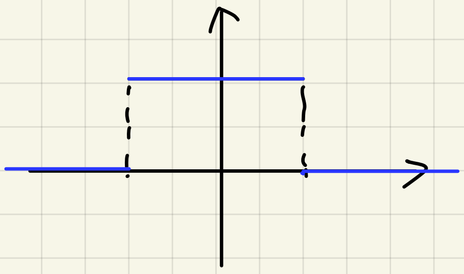
P.D. ? $\Downarrow$?

Prove: $E|X(\tau+T) - X(\tau)|^2 = 0$ mean square periodic, (Local $\Rightarrow$ General)

$$E|X(\tau+T) - X(\tau)|^2 = 2R_X(0) - 2R_X(T) = 0$$

$$|R_X(\tau+T) - R_X(\tau)| = |E(X(0)X(\tau+T)) - E(X(0)X(\tau))| = |E[X(0)(X(\tau+T) - X(\tau))]| \le \left[E(X^2(0))\, \underbrace{E(X(\tau+T) - X(\tau))^2}_{=0}\right]^{1/2} = 0$$

Can Rect. Windows be correlation functions?



Lemma: $R_X(t)$ is continuous at $0 \Rightarrow R_X(\tau)$ is continuous at τ, $\forall \tau$

"Local $\Rightarrow$ General"

Need a intermediate step (combining C.f. with mean square property)

Mean Square Continuous. $E|X(t+\Delta) - X(t)|^2 \to 0 \ (\Delta \to 0)$

衡量是否受过理科教育？

5 concepts：极限，连续，导数，微分，积分

Distance should satisfy 3 property:

1) Non-negativity

2) Symmetry

3) Triangular Inequality: ?

$$\left|\sqrt{E(X_1^2)} - \sqrt{EX_2^2}\right| \le \left|E(X_1 - X_2)^2\right|^{1/2}$$

square on each side:

LHS: $EX_1^2 + EX_2^2 - 2(EX_1^2)^{1/2}(EX_2^2)^{1/2}$

RHS: $EX_1^2 + EX_2^2 - 2EX_1EX_2$

plus: $EX_1EX_2 \le (EX_1^2)^{1/2}(EX_2^2)^{1/2}$ (Cauchy)

$\Rightarrow$ LHS $\le$ RHS

极限: $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall \Delta < \delta$. $|f(x+\Delta) - f(x)| \le \varepsilon \Rightarrow \lim\limits_{\Delta \to 0} f(x+\Delta) = f(x)$

Euclid.

Distance! (Approximating process $\to$ Metric!)

How do we define stochastic metric?

$\Rightarrow$ Mean Square Distance. $d(X, Y) = [E|X-Y|^2]^{1/2}$

Information Theory $\Rightarrow$ K-L Divergence?

$\underset{\text{r.v.}}{X_n} \xrightarrow{n \to \infty} \underset{\text{r.v.}}{X}$: $\forall \varepsilon > 0, \exists N > 0$, s.t. $\forall n > N$, $[E|X_n - X|^2]^{1/2} \le \varepsilon$

Diff. def. of distance $\Rightarrow$ Diff. limit $\Rightarrow$ Diff. calculus.

$\therefore$ Mean Square Distance is a Distance. (There are other def.s for stochastic distance.)

$$E|X(t+s) - X(t)|^2 = 2[R_X(0) - R_X(s)] \to 0 \ (s \to 0)$$ (Local $\to$ General)

$$R_X(\tau+s) - R_X(\tau) = E[X(0)X(\tau+s)] - E[X(0)X(\tau)] = E[X(0)(X(\tau+s) - X(\tau))] \le \cdots = 0 \ (s \to 0)$$

Same process

Random Variable (Sample Space $\to \mathbb{R}$. deterministic function)   3 elements (Sample Space, Probability Measure, Event Set?)

Statistical Experiment. $\Rightarrow$ Samples $\Rightarrow$ Sample Space Ω

Deterministic Set.

$\Rightarrow$ Probability Measure $P(A) \in [0, 1]$

(Deterministic, Prior)

Randomness vs. Chaos?

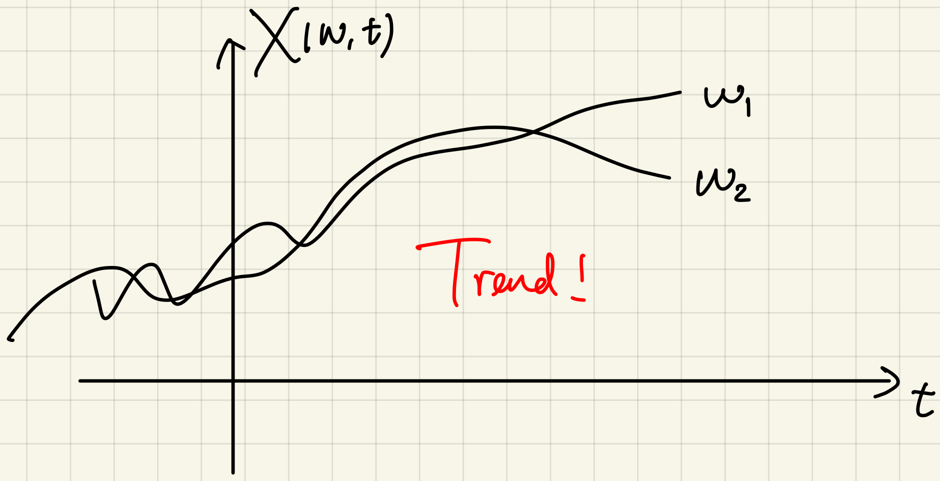
Initial-value sensitive

Accept uncertainty

$X: \Omega \to \mathbb{R}$ (Quantization)

Stochastic Processes

Randomness is expressed in Sample (Ω's element)



$X(t)$ t: index/time, t given $\rightarrow$ $X(w)$ r.v. (Ω's element)

$\downarrow$

$X(t,w)$ (index, Ω's sample)

$\rightarrow$ $X(t)$, deterministic : Sample Path

affected by correlations between r.v.s.

Correlation : $X, Y \Rightarrow E[XY] \rightarrow$ Inner Product (Algebraic) $\rightarrow$ Correlation Function

$\rightarrow$ Angle (Geometric)

$R_X(t,s) = E[X(t)X(s)]$

① $R_X(t,s) = R_X(s,t)$ ② $R_X(t,t) \geq 0$ ($E[X^2(t)] \geq 0$) ③ $|R_X(t,s)| \leq (R_X(t,t)R_X(s,s))^{1/2}$ (Cauchy-Schwarz)

$\Downarrow$

Uncertainty Implications (QM, Time × Freq. (Signal))

Cauchy-Schwarz $\Rightarrow$ Inner Product.

$|\sum x_k y_k| \leq [\sum x_k^2 \sum y_k^2]^{1/2}$ $\int fg\,dx \leq [\int f^2 dx \int g^2 dx]^{1/2}$

$|\langle x,y\rangle| \leq |\langle x,x\rangle\langle y,y\rangle|^{1/2}$

$\langle X,Y\rangle = E[XY] \rightarrow R_X(t_1,t_2)$ is pos. def.

$f(x,y)$ is pos. def. $\Rightarrow \forall n, \forall t_1 \sim t_n$, $F = (f(t_i,t_j))_{ij}$, F is p.d. matrix, $\forall \alpha \in \mathbb{R}^n$, $\alpha^T F \alpha \geq 0$

$R = [R_X(t_i,t_j)]_{ij}$, let $X = [X(t_1), \cdots, X(t_n)]^T$, $R = E[XX^T]$

$\forall \alpha \in \mathbb{R}^n$, $\alpha^T R \alpha = \underset{\text{deterministic}}{\alpha^T} \underset{\text{linear operation}}{E[XX^T]} \alpha = E[\alpha^T X X^T \alpha]$ let $n = \alpha^T X \rightarrow$ 1-D r.v., $\alpha^T R \alpha = E[n^2] \geq 0 \Rightarrow$ P.D.

$R_X(t_1,t_2)$ p.d. $\Rightarrow R_X(t,t) \geq 0$ $(n=1)$

characteristic property.

$\begin{bmatrix} R_X(0,0), & R_X(0,t) \\ R_X(t,0), & R_X(t,t) \end{bmatrix}$ P.D. $\Rightarrow$ $\begin{cases} \text{Symmetric}: R_X(0,t) = R_X(t,0) \\ \det \geq 0 : [R_X(0,0)R_X(t,t)]^{1/2} \geq R_X(0,t) \rightarrow \text{Cauchy-Schwarz.} \end{cases}$

Reduce the dimension of Correlation Function: (2D $\rightarrow$ 1D)

Stationary $\Leftrightarrow$ Invariance (Time-shift)

Diff. types of invariance $\Rightarrow$ Diff types of stationary (W.S.S., Wide-Sense Stationary)

W.S.S. enables local properties extend to general properties

1-Dim pos. def. : $\forall n$, $\forall t_1 \sim t_n$, $(f(t_i - t_j))_{ij}$ is p.d. (function is p.d.)

Bochner : $f(x)$ is p.d. $\iff \int f(x) \exp(-j\omega x) dx \geqslant 0$, $\forall x$

go to freq. domain to analyze