Non-Stationary Processes

1) Cyclostationary 2) Orthogonal Increment

$X(t)$ w.s.s. $\Leftrightarrow$ $E[X(t)] = m$, $E[X(t)X(s)] = R_X(t-s)$

$R_X(t+T, s+T) = R_X(t,s)$, $\forall T \in \mathbb{R}$ (w.s.s.) $\xrightarrow[\text{condition}]{\text{relax this}}$ $R_X(t+T, s+T) = R_X(t,s)$. $\exists T \in \mathbb{R}$ Cyclo-Stationary

$R_X(t+mT, s+mT) = R_X(t,s)$. $\forall m$

Relation between W.S.S. & cyclostationary:

Randomization $\Theta \sim U(0,T)$. Θ, $X(t)$ indep.

$Y(t) = X(t+\Theta)$: $X(t)$ Cyclo $\Rightarrow$ $Y(t)$ w.s.s.

$$R_Y(t,s) = E[Y(t)Y(s)] = E[X(t+\Theta) X(s+\Theta)] = E_\Theta\left[E_X[X(t+\theta)X(s+\theta) \mid \Theta = \theta]\right] = E_\Theta[R_X(t+\theta, s+\theta)]$$

$$\overset{\Theta \sim U(0,T)}{=\!=\!=} \frac{1}{T}\int_0^T R_X(t+\theta, s+\theta)\, d\theta$$

$$\forall T',\ R_Y(t+T', s+T') = \frac{1}{T}\int_0^T R_X(t+T'+\theta, s+T'+\theta)\, d\theta \overset{\theta' = T'+\theta}{=\!=\!=} \frac{1}{T}\int_{T'}^{T'+T} R_X(t+\theta', s+\theta')\, d\theta'$$

$$= \frac{1}{T}\int_0^T R_X(t+\theta', s+\theta')\, d\theta' = R_Y(t,s)$$

$\Rightarrow$ $Y(t)$ w.s.s.

Constellation (Phase + Amp. $\rightarrow$ Symbol / Bitstream)

PAM (Pulse (complex) Amplitude Modulation) BPSK, QPSK, QAM

30:00

$X(t) = \sum_{-\infty}^{+\infty} \alpha_k \phi(t-kT)$ α_k : r.v Information Symbol

$\phi(t)$: Baseband Waveform.

$$R_X(t,s) = E[X(t)X(s)] = E\left[\sum_k \sum_n \alpha_k \alpha_n \phi(t-kT)\phi(s-nT)\right] = \sum_k \sum_n E[\alpha_k \alpha_n]\, \phi(t-kT)\phi(s-nT)$$

$\{\alpha_n\}$ WSS. $\rightarrow$ $R_\alpha(k-n)$ $\Rightarrow \sum_k \sum_n R_\alpha(k-n) \cdot \phi(t-kT)\phi(s-nT)$

Generally : $R_X(t+T', s+T') \neq R_X(t,s)$

But when $T' = T$, $R_X(t+T, s+T) = \sum_k \sum_n R_\alpha(k,n)\, \phi(t+T-kT)\phi(s+T-nT) \overset{k'=k-1}{\underset{n'=n-1}{=\!=\!=}} \sum_{k',n'} R_\alpha(k'+1, n'+1)\, \phi(t-k'T)\, \phi(s-n'T)$

$R_\alpha(k'+1, n'+1) = R_\alpha(k'-n')$, and the sum $= R_X(t,s)$

$X(t)$: Cyclostationary Stochastic processes.

$$\tilde{X}(t) = X(t+\Theta), \quad \Theta \sim U(0,T)$$

$$R_{\tilde{X}}(t,s) = \frac{1}{T}\int_0^T R_X(t+\theta, s+\theta)\,d\theta = \frac{1}{T}\int_0^T \sum_k \sum_n R_\alpha(k-n)\phi(t-kT+\theta)\phi(s-nT+\theta)\,d\theta$$

$k' = k-n, \quad n' = n$

$$= \frac{1}{T}\int_0^T \sum_{k'}\sum_{n'} R_\alpha(k')\phi(t+\theta-(n'+k')T)\phi(s+\theta-n'T)\,d\theta$$

Let $\theta' = \theta - n'T$: $\frac{1}{T}\sum_{k'}\sum_{n'} R_\alpha(k')\int_{-n'T}^{-(n'-1)T}\phi(t+\theta'-k'T)\phi(s+\theta')\,d\theta'$

$X(t)$ deterministic

Correlation : $R_X(\tau) = \int_{\mathbb{R}} X(t+\tau)X(t)\,dt$

$$= \frac{1}{T}\sum_{k'} R_\alpha(k')\cdot\sum_{n'=-\infty}^{+\infty}\int_{-n'T}^{-(n'-1)T}\phi(t+\theta'-k'T)\,\phi(s+\theta')\,d\theta' = \frac{1}{T}\sum_{k'} R_\alpha(k')\int_{\mathbb{R}}\phi(t+\theta'-k'T)\phi(s+\theta')\,d\theta' \qquad \theta'' = s+\theta'$$

$$= \frac{1}{T}\sum_{k'} R_\alpha(k')\int_{\mathbb{R}}\phi(t-k'T-s+\theta'')\phi(\theta'')\,d\theta''$$

$$= \frac{1}{T}\sum_{k'} R_\alpha(k')\,R_\phi(t-s-k'T) = R_{\tilde{X}}(t-s)$$

$$S_{\tilde{X}}(w) = \int_{-\infty}^{+\infty} R_{\tilde{X}}(\tau)\exp(-jw\tau)\,d\tau = \frac{1}{T}\int_{-\infty}^{+\infty}\Big[\sum_k R_\alpha(k)R_\phi(\tau-kT)\Big]\exp(-jw\tau)\,d\tau = \frac{1}{T}\sum_k R_\alpha(k)\cdot\int_{\mathbb{R}} R_\phi(\tau-kT)\exp(-jw\tau)\,d\tau$$

$$= \frac{1}{T}\sum_k R_\alpha(k)\exp(-jwkT)\int_{\mathbb{R}} R_\phi(\tau')\exp(-jw\tau')\,d\tau'$$

$$= \frac{1}{T}\,S_\alpha(w)\cdot S_\phi(w) = \frac{1}{T}S_\alpha(w)\,|\Phi(w)|^2$$

$Z(t) = \sum_k \alpha_k\,\delta(t-kT) \qquad Z(t) \to \boxed{\phi(t)} \to\ ? \qquad Z(t) * \phi(t) = \sum \alpha_k \phi(t-kT)$

2) Orthogonal Increment

$X(t)$, $\forall\, t_1 < t_2 \le t_3 < t_4$, $X(t_4)-X(t_3) \perp X(t_2)-X(t_1)$: $E[(X(t_4)-X(t_3))(X(t_2)-X(t_1))] = 0$

$$R_X(t,s) = E[X(t)X(s)] \xrightarrow[X(0)=0]{s<t} E[(X(t)-X(s)+X(s)-X(0))(X(s)-X(0))] = E[X^2(s)]$$

$R_X(t,s) = E[X^2(\min(t,s))] = g(\min(t,s)) \rightarrow$ Characteristic Property of Orthogonal Increment.

i.e. if $R_X(t,s) = g(\min(t,s))$, $X(t)$ has Orthogonal Increment
$(\forall t,s)$

$$E[(X(t_4)-X(t_3))(X(t_2)-X(t_1))] = R_X(t_4,t_2) + R_X(t_3,t_1) - R_X(t_3,t_2) - R_X(t_4,t_1)$$

$t_4 > t_3 \ge t_2 > t_1$ — if $R_X(t,s) = g(\min(t,s))$, then

$$= g(t_2) + g(t_1) - g(t_2) - g(t_1) = 0$$

$\therefore$ Orthogonal Increment

Example of Orthogonal Increment : Brownian Motion def. of Brownian Motion:

① $B(0)=0$ ② Orthogonal Increment ③ $B(t)-B(s)\sim N(0,\sigma^2(t-s))$

任何事都是有道理的，区别是有没有必要知道它，和有没有能力去知道它.

$R_B(t,s)=E(B^2(s))=\sigma^2 s=\sigma^2\min(t,s)$

$B(s)-B(0)\sim N(0,\sigma^2(s-0))=N(0,\sigma^2 s)$
$(B(s))$

$E(B(s))=0$, $E(B^2(s))=var(B(s))=\sigma^2 S$

$Y(t)=\frac{d}{dt}B(t)$ (Not Rigorous)

$$R_Y(t,s)=E[Y(t)Y(s)]$$
$$=E\left[\frac{d}{dt}Y(t)\frac{d}{ds}Y(s)\right]$$
$$=\frac{\partial^2}{\partial t\partial s}E[Y(t)Y(s)]$$
$$=\frac{\partial^2}{\partial t\partial s}R_B(t,s)=\sigma^2\frac{\partial^2}{\partial t\partial s}\min(t,s)=\sigma^2\frac{\partial^2}{\partial t\partial s}\left[\frac{1}{2}(s+t-|s-t|)\right]$$
$$=-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial t\partial s}|t-s|$$

$$R_Y(t,s)=-\frac{\sigma^2}{2}\frac{\partial^2}{\partial t\partial s}|t-s| \quad \left(\frac{d}{dx}|x|=\mathrm{sgn}(x)\right)$$
$$=-\frac{\sigma^2}{2}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}|t-s|\right)$$
$$=-\frac{\sigma^2}{2}\frac{\partial}{\partial s}(\mathrm{sgn}(t-s))$$
$$=-\frac{\sigma^2}{2}[-2\delta(t-s)]$$
$$=\sigma^2\delta(t-s)$$

$\left(\frac{d}{dx}\mathrm{sgn}(x)=2\delta(x),\quad U(x)=\begin{cases}1, & x\geq 0\\ 0, & x<0\end{cases},\quad \frac{dU(x)}{dx}=\delta(x),\quad \mathrm{sgn}(x)=U(x)-U(-x)\right)$

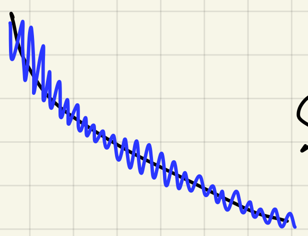
derivative is $\delta(\cdot)$ function $\Rightarrow$ White Noise

(Brownian Motion)

Orthogonal Increment $\xrightarrow{\text{differential}}$ W.S.S.

Isometry with Complex Exponentials (Oscillating)

$\therefore$ W.S.S. $\rightarrow$ High-Frequency



Stationary lies in
- ~~envelope~~ (Trend $\Rightarrow$ Non-Stationary)
- glitches * (Brownian Motion $\xrightarrow[\text{High-Pass Filter}]{\text{Derivative}}$ W.S.S.)