# Gaussian & Nonlinearity

## ① Square Law:

$X(t)$ Gaussian $\longrightarrow \boxed{x^2} \longrightarrow Y(t) = X^2(t) \geq 0$, not Gaussian.

$$E[Y(t)] = EX^2 = R_X(t,t) = R_X(0)$$

$R_Y(t,s) = E[Y(t)Y(s)] = E[X^2(t)X^2(s)]$ High-Order Moments (Gaussian) → did $E[X^n]$ in $9^{th}$ lecture

we want to make it more general

to calculate moments between different r.v.s

Example: $(X_1, X_2, X_3, X_4) \sim N$, $E[X_k] = 0$, $E[X_1X_2X_3X_4] = ?$

$$E[X_1X_2X_3X_4] = \int x_1x_2x_3x_4 f_{X_1X_2X_3X_4} \, dx_1dx_2dx_3dx_4$$

more ideal than $\Sigma^{-1}$ approach moments from $\phi_X(w)$ $(\Sigma)$ rather than distribution $(\Sigma^{-1})$ ???

$$\exp\left(-\frac{1}{2}(x_1,x_2,x_3,x_4)\Sigma^{-1}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\right) \Rightarrow \phi_X(w) = \exp\left(jw^T\mu - \frac{1}{2}w^T\Sigma w\right) \overset{E[X]=0}{=\!=\!=} \exp\left(-\frac{1}{2}w^T\Sigma w\right)$$

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$$X = (X_1, \cdots, X_n)^T \Rightarrow \phi_X(w) = E[\exp(j(w_1X_1 + \cdots + w_nX_n))]$$

$$EX_1 = \frac{1}{j}\frac{\partial}{\partial w_1}\phi_X(w_1\cdots w_n)\Big|_{w_1=\cdots=w_n=0}$$

$$\therefore EX_k^\alpha = \frac{1}{j^\alpha}\frac{\partial^\alpha}{\partial w_k^\alpha}\phi_X(w_1\cdots w_n)\Big|_{w_1=\cdots=w_n=0}$$

| Moment | |
|---|---|
| Distribution | Characteristic Function |
| ⇓ | ⇓ |
| Integral | Differential (Easier) |

$$E[X_1^{\alpha_1}\cdots X_n^{\alpha_n}] = \frac{1}{j^{\alpha_1+\cdots+\alpha_n}}\frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial w_1^{\alpha_1}\cdots\partial w_n^{\alpha_n}}\phi_X(w_1\cdots w_n)\Big|_{w_1=\cdots=w_n=0}$$

$$E[X_1X_2X_3X_4] = j^{-4}\frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\phi_X(w)\Big|_{w=0} = \cdots = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$$

$$\frac{\partial^4}{\partial x_1\partial x_2\partial x_3\partial x_4}E[\exp(jw^TX)] = \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\exp\left(jw^T\mu_X - \frac{1}{2}w^T\Sigma w\right) \underset{\text{normalized}}{\overset{\text{Suppose } \mu=0}{=\!=\!=}} \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\exp\left(-\frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4 w_i\Sigma_{ij}w_j\right)$$

$$= \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\exp\left(-\frac{1}{2}\sum_i w_i^2\Sigma_{ii} - \sum_{i<j}w_iw_j\Sigma_{ij}\right) = \frac{\partial^3}{\partial w_2\partial w_3\partial w_4}\left(-\sum_i w_i\Sigma_{i1}\right)\exp\left(-\frac{1}{2}\sum_i w_i^2\Sigma_{ii} - \sum_{i<j}w_iw_j\Sigma_{ij}\right)$$

$$= \frac{\partial^2}{\partial w_3\partial w_4}\left[-\Sigma_{21} + \left(-\sum_i w_i\Sigma_{i1}\right)\cdot\left(-\sum_j w_j\Sigma_{j2}\right)\right]\exp(\cdots) = \frac{\partial^2}{\partial w_3\partial w_4}\left[-\Sigma_{21} + \sum_i w_i\Sigma_{i1}\cdot\sum_j w_j\Sigma_{j2}\right]\exp(\cdots)$$

$$= \frac{\partial}{\partial w_4}\left[\Sigma_{31}\cdot\sum_j w_j\Sigma_{j2} + \Sigma_{32}\cdot\sum_i w_i\Sigma_{i1} + \left(-\Sigma_{21} + \sum_i w_i\Sigma_{i1}\cdot\sum_j w_j\Sigma_{j2}\right)\left(-\sum_k w_k\Sigma_{k3}\right)\right]\exp(\cdots)$$

Since we substitute $(w_1, w_2, w_3, w_4)$ with $(0,0,0,0)$, we only search $1^{st}$-order $w_4$ terms.

$$E[X_1X_2X_3X_4] = \Sigma_{31}\cdot\Sigma_{42} + \Sigma_{32}\Sigma_{41} + (-\Sigma_{21})\cdot(-\Sigma_{43}) = \Sigma_{12}\Sigma_{34} + \Sigma_{13}\Sigma_{24} + \Sigma_{14}\Sigma_{23}$$ Proved!

$$E[X^2(t)X^2(s)] = R_X(t,t)R_X(s,s) + 2\cdot R_X^2(t,s) \xrightarrow{\text{if } X \text{ is WSS}} R_X^2(0) + 2\cdot R_X^2(t-s), \text{ PSD}: \cdots$$

## ② Sign Function

$$\mathrm{sgn}(x)=\begin{cases}1, & x\ge 0\\ -1, & x<0\end{cases}$$

Gaussian

$$X(t)\to\boxed{\mathrm{sgn}(x)}\to Y(t)$$

$$E[X(t)]=0$$

$$EY=1\cdot P(X(t)\ge 0)+(-1)P(X(t)<0)=0\ (f_X:\text{ even function})$$

$$R_Y(t,s)=E[Y(t)Y(s)]=1\,P(Y(t)Y(s)=1)+(-1)\,P(Y(t)Y(s)=-1)=2P(Y(t)\cdot Y(s)=1)-1=2P(X(t)X(s)\ge 0)-1$$

$$P(X(t)X(s)\ge 0)=\left(\int_{-\infty}^{0}\int_{-\infty}^{0}+\int_{0}^{\infty}\int_{0}^{\infty}\right)\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1}{\sigma_1}\right)^2+\left(\frac{x_2}{\sigma_2}\right)^2-2\rho\frac{x_1x_2}{\sigma_1\sigma_2}\right)\right]dx_1dx_2$$

let $x_1'=\frac{x_1}{\sigma_1},\ x_2'=\frac{x_2}{\sigma_2}$

$$=\left(\int_{-\infty}^{0}\int_{-\infty}^{0}+\int_{0}^{\infty}\int_{0}^{\infty}\right)\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x_1'^2+x_2'^2-2\rho x_1'x_2')\right)dx_1'dx_2'$$

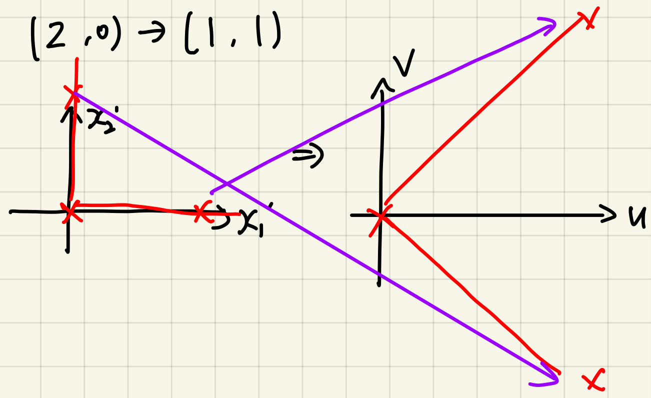
$$=2\cdot\int_{0}^{\infty}\int_{0}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x_1'^2+x_2'^2-2\rho x_1'x_2')\right)dx_1'dx_2'$$

看关键点

linear transform: $(0,0)\to(0,0)$

$(0,2)\to(1,-1)$

$(2,0)\to(1,1)$




let $\begin{cases}x_1'=u+v\\ x_2'=u-v\end{cases}$ $\quad\begin{bmatrix}x_1'\\x_2'\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$, $|J|=\left|\frac{\partial(x_1',x_2')}{\partial(u,v)}\right|=\left|\begin{matrix}1&1\\1&-1\end{matrix}\right|=2$

$$=2\cdot\int_{0}^{\infty}\int_{-u}^{u}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(2u^2+2v^2-2\rho(u^2-v^2)\right)\right)2\,du\,dv$$

$$=4\int_{0}^{\infty}\int_{-u}^{u}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{1-\rho^2}\left((1-\rho)u^2+(1+\rho)v^2\right)\right)du\,dv$$

$$=4\int_{0}^{\infty}\int_{-u}^{u}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{u^2}{1+\rho}-\frac{v^2}{1-\rho}\right)du\,dv$$

let $u'=\frac{u}{\sqrt{1+\rho}},\ v'=\frac{v}{\sqrt{1-\rho}}$, $\quad =4\int_{0}^{\infty}\int_{-\sqrt{\frac{1-\rho}{1+\rho}}u'}^{\sqrt{\frac{1-\rho}{1+\rho}}u'}\frac{1}{2\pi}\exp(-u'^2-v'^2)\,du'dv'$

$(1,1)\to\left(\frac{1}{\sqrt{1+\rho}},\frac{1}{\sqrt{1-\rho}}\right)$

$(1,-1)\to\left(\frac{1}{\sqrt{1+\rho}},\frac{-1}{\sqrt{1-\rho}}\right)$

polar coord. $\begin{cases}u'=r\cos\theta\\ v'=r\sin\theta\end{cases}$

$$=4\int_{0}^{\infty}\int_{-\phi}^{\phi}\frac{1}{2\pi}\exp(-r^2)\,r\,dr\,d\theta\quad\left(\phi=\tan^{-1}\sqrt{\frac{1+\rho}{1-\rho}}\right)$$

$$=4\cdot\frac{1}{2\pi}\cdot 2\phi\cdot\frac{1}{2}=\frac{2\phi}{\pi}$$

$$\rho=\frac{E[X(t)X(s)]}{[EX^2(t)EX^2(s)]^{1/2}}=\frac{R_X(t-s)}{R_X(0)}$$

$$\cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$\tan^2\phi=\frac{1+\rho}{1-\rho},\quad 1+\tan^2\phi=\frac{2}{1-\rho}$$

$$\rho=1-\frac{2}{1+\tan^2\phi}\Rightarrow\frac{1-\tan^2\phi}{1+\tan^2\phi}=-\rho$$

$$\therefore\ \cos 2\phi=-\rho,\quad 2\phi=\cos^{-1}(-\rho)$$

$$\sin^{-1}(x)+\cos^{-1}(x)=\frac{\pi}{2}$$

$$\Rightarrow 2\phi=\cos^{-1}(-\rho)=\frac{\pi}{2}-\sin^{-1}(-\rho)=\frac{\pi}{2}+\sin^{-1}(\rho)$$

$$\therefore\ P(X(t)X(s)\ge 0)=\frac{1}{2}+\frac{\sin^{-1}(\rho)}{\pi}$$

"Arcsin Law"

$$\begin{cases}R_Y(t,s)=2\cdot\left[\frac{1}{2}+\frac{\sin^{-1}(\rho)}{\pi}\right)-1=\frac{2}{\pi}\sin^{-1}(\rho)\\ \rho=\frac{R_X(t-s)}{R_X(0)}\end{cases}$$

## Useful Tool dealing with Gaussian Linearity:

$g(x_1,x_2)$ general non-linear function, $(X_1,X_2)\sim N(0,0,\sigma_1^2,\sigma_2^2,\rho)$

$$\frac{\partial E[g(X_1,X_2)]}{\partial\rho}=\sigma_1\sigma_2\,E\left[\frac{\partial^2 g(X_1,X_2)}{\partial X_1\,\partial X_2}\right]$$

Price Theorem

Use Price Thm. to solve $E[\operatorname{sgn}(X(t))\operatorname{sgn}(X(s))]$ 07-11