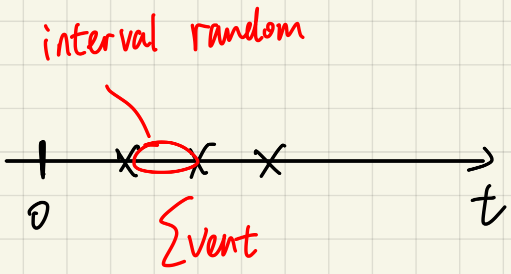
Poisson Processes



interval & occurrences $N(t)$ are two sides of the same coin

$N(t)$: Event Occurences in $[0, t]$ $\Rightarrow$ Stochastic Processes $N(t)$

Counting Process ; Point Process (Insurance ; Network packet) $\quad$ $P(N(t)=k)$

① $N(0)=0$ ② Independent Increment $\forall t_1<t_2\le t_3<t_4$, $N(t_4)-N(t_3) \perp\!\!\!\perp N(t_2)-N(t_1)$

③ Stationary Increments $N(t)-N(s) \sim P(t-s)$ ④ Sparsity

$\Rightarrow P(N(t)=k)=?$ $\quad$ Z-transform

Moment Generating Function (MGF) $G_{N(t)}(z) = E[z^{N(t)}] = \sum_k z^k P(N(t)=k)$

$$\frac{d}{dt}G_{N(t)}(z) = \frac{1}{\Delta t}\left[G_{N(t+\Delta t)}(z) - G_{N(t)}(z)\right] = \frac{1}{\Delta t}\left\{E[z^{N(t+\Delta t)}] - E[z^{N(t)}]\right\} = \frac{1}{\Delta t}E[z^{N(t+\Delta t)} - z^{N(t)}]$$

$$= \frac{1}{\Delta t}E[(z^{N(t+\Delta t)-N(t)}-1)z^{N(t)}] \xrightarrow{\text{indep. increment}} \frac{1}{\Delta t}E[z^{N(t)}]\,E[z^{N(t+\Delta t)-N(t)}-1] \xrightarrow[\text{increment}]{\text{Stationary}} E[z^{N(t)}]\cdot\frac{1}{\Delta t}E[z^{N(\Delta t)-N(0)}-1]$$

$$= G_{N(t)}(z)\cdot \underset{\sim\sim}{?}$$

$$E[z^{N(\Delta t)}-1] = \sum_k z^k\cdot P(N(\Delta t)=k) - 1 = \underline{P(N(\Delta t)=0)-1} + z P(N(\Delta t)=1) + \sum_{k\ge 2} z^k P(N(\Delta t)=k)$$

$$P(N(t)=0) = P(N(s)=0,\ N(t)-N(s)=0),\quad \forall\, 0<s<t$$

$$g(t) \quad = P(N(S)=0)\cdot P(N(t)-N(S)=0) \xrightarrow[\text{Increment}]{\text{Stationary}} P(N(S)=0)\cdot P(N(t-s)=0)$$

trivial

$\Rightarrow g(t) = g(s)\cdot g(t-s)$ $\quad$ i.e. $g(t+s) = g(t)g(s)$ $\forall t,s>0$

Prove: $g(t)>0$, $\quad g(t) = g(\frac{t}{2})^2 \ge 0$, $\quad$ if $\exists\, t_0$, s.t $g(t_0)=0$ $\Rightarrow g(\frac{t_0}{2})=0$, $g(\frac{t_0}{4}) = \cdots = g(\frac{t_0}{2^n})=0 \xrightarrow{\text{continuity}} g(0)=0 \rightarrow g(t)=0$

$\therefore g(t)\ne 0$, $g(t)\ge 0 \Rightarrow g(t)>0$, let $h(t) = \log g(t)$

$h(t+s) = h(t) + h(s) \Rightarrow h(t) = \lambda t$

$\lambda = h(1)$

① $t\in\mathbb{N}$, $t=n$: $h(n) = h(n-1) + h(1) = h(n-2) + 2h(1) = \cdots = n h(1)$

② $t\in\mathbb{Z}$, $h(0) = h(0) + h(0) \Rightarrow h(0) = 0 \Rightarrow h(-n) = h(0) - h(n) = -h(n) = -n h(1)$

③ $t\in\mathbb{Q}$, $h(\frac{n}{m}) = n h(\frac{1}{m})$, $h(1) = m h(\frac{1}{m}) \Rightarrow h(\frac{n}{m}) = \frac{n}{m}h(1)$

④ $t\in\mathbb{R}$, $h(x) = h(\lim_{n\to\infty} q_n)$ (approximate irrational number from rational numbers) $= \lim_{n\to\infty} h(q_n) = \lim_{n\to\infty} q_n\cdot h(1) = x\cdot h(1)$

$\Rightarrow$ $h(t) = \lambda t$

$g(t) = \exp(\lambda t)$

$P(N(\Delta t)=0) = \exp(-\lambda \Delta t)$ $(\lambda > 0)$

深刻的三个条件：陈述很简单，结论很直观，证明很困难

Profoundness = Simple statement + intuitive conclusions + challenging proofs.

$$z P(N(\Delta t)=1) + \sum_{k\ge 2} z^k P(N(\Delta t)=k) = P(N(\Delta t)=1)\left[z + \sum_{k\ge 2} z^k \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}\right]$$

ROC of z-transform? $\sum z^k P(N(t)=k)$, $|z|\le 1$ ✓ $|z|>1$? ROC: $|z|\le 1$ for any distribution.

Consider z in unit circle:

$$\left|\sum_{k\ge 2} z^k \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}\right| \le \sum_{k\ge 2} |z|^k \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)} \le \sum_{k\ge 2} \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)} = \frac{P(N(\Delta t)\ge 2)}{P(N(\Delta t)=1)} \xrightarrow{\Delta t\to 0} 0 \quad \text{(Sparsity!)}$$

$$\frac{1}{\Delta t} P(N(\Delta t)=1)\left(z+\sum_{k\ge 2} z^k \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}\right) = \frac{1}{\Delta t}P(N(\Delta t)=1)\cdot z = \frac{1}{\Delta t}\left[1-P(N(\Delta t)=0)\right]\cdot z = \frac{1-\exp(-\lambda\Delta t)}{\Delta t}\cdot z = \lambda z$$

$$\therefore \frac{1}{\Delta t} E\left[z^{N(\Delta t)}-1\right] = \frac{1}{\Delta t}\left[P(N(\Delta t)=0) + P(N(\Delta t)=1)\left(z+\sum_{k=2}^{\infty} z^k \frac{P(N(\Delta t)=k)}{P(N(\Delta t)=1)}\right)\right] = \frac{\exp(-\lambda\Delta t)}{\Delta t} + \lambda z = \lambda(z-1)$$

$$\therefore \begin{cases} \frac{d}{dt} G_{N(t)}(z) = G_{N(t)}(z)\cdot \lambda(z-1) \\ G_{N(0)}(z) = 1 \quad (N(0)=0) \end{cases} \Rightarrow G_{N(t)}(z) = G_{N(0)}(z)\cdot \exp(\lambda(z-1)t) = \exp(-\lambda t)\cdot\sum_{k=0}^{\infty}\frac{(\lambda z t)^k}{k!} = \sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\exp(-\lambda t)\cdot z^k$$

$$G_{N(t)}(z) = \sum_{k=0}^{\infty} z^k P(N(t)=k)$$

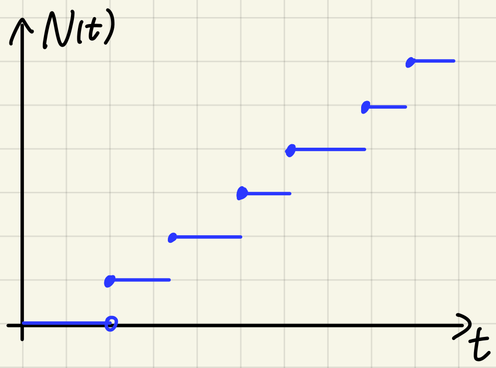
$\therefore P(N(t)=k) = \frac{(\lambda t)^k}{k!}\exp(-\lambda t) \Rightarrow \lambda$: Intensity

$E[N(t)] = \lambda t$, $\lambda = \frac{E[N(t)]}{t}$ $var(N(t)) = \lambda t$

$$E[N(t)N(s)] = E[(N(t)-N(s)+N(s))N(s)] = E[N(t)-N(s)]\,E[N(s)] + E[N^2(s)] = \lambda(t-s)\cdot\lambda s + (\lambda s)^2 + \lambda s = \lambda^2 ts + \lambda s \quad (t>s)$$

$$E[N(t)N(s)] = \lambda^2 ts + \lambda \min(t,s)$$

Sample Path of Poisson Processes:



Waiting + Jumping

Life is as such Markov!

We have already acquired the distribution of occurences in a given time range,

Statistical properties of intervals between occurrences. (derive from knowledge of occurrences)

Interval: $F_{T_1}(t) = P(T_1 \le t) = P(N(t)\ge 1) = 1 - P(N(t)=0) = 1 - \frac{(\lambda t)^0}{0!}\cdot\exp(-\lambda t) = 1-\exp(-\lambda t)$

$$\Rightarrow f_{T_1}(t) = \frac{d}{dt}F_{T_1}(t) = \begin{cases}\lambda\exp(-\lambda t), & t\ge 0\\ 0, & t<0\end{cases}$$ Memoryless

$\therefore (T_1, \dots, T_n) \overset{i.i.d.}{\sim} Exp(\lambda)$ (Stationary Increments)

$S_k = T_1 + \cdots + T_k \Rightarrow$ Characteristic function. $\left(\frac{1}{\lambda + j\omega}\right)^k$ 留数. Complex Analysis

$$F_{S_k}(t) = P(S_k \le t) = P(N(t) \ge k) = \sum_{k}^{\infty} \frac{(\lambda t)^k}{k!} \exp(-\lambda t)$$

$$f_{S_k}(t) = \frac{d F_{S_k}(t)}{dt} = \sum_{k}^{\infty} \left[\frac{\lambda (\lambda t)^{n-1}}{(n-1)!} \exp(-\lambda t) + (-\lambda) \frac{(\lambda t)^n}{n!} \exp(-\lambda t) \right] = \frac{(\lambda t)^{k-1}}{(k-1)!} \exp(-\lambda t) - \lim_{n \to \infty} \lambda \frac{(\lambda t)^n}{n!} \exp(-\lambda t)$$

$$= \begin{cases} \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} \exp(-\lambda t), & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Gamma Distribution $\Gamma(x) = \int_0^\infty t^x \exp(-t) dt$

S_k：事件发生时刻

$$\underbrace{\Gamma(n+1)}_{n!} = \int_0^\infty t^n \exp(-t) dt = \int_0^\infty t^n \, d[-\exp(-t)]$$

$$= \int_0^\infty d(-\exp(-t) t^n) - \int_0^\infty [-\exp(-t)] \, d(t^n)$$

$$= n \int_0^\infty t^{n-1} \exp(-t) \, dt$$

$$= n \underbrace{\Gamma(n)}_{(n-1)!}$$

$$0! = \Gamma(1) = \int_0^\infty 1 \cdot \exp(-t) dt = 1$$