Continuous-Time Markov Chain

Discrete State Markov Processes ⇒ Chain.

$P_{ij}(t) = P(X(t)=j \mid X(0)=i)$ $\quad P_{ij}(n) = P(X_n = j \mid X_0 = i)$ $\quad$ C-K Equation: $P_{ij}(t+s) = \sum_k P_{ik}(s)\cdot P_{kj}(t)$

$P(m+n) = P(m)\cdot P(n) \Rightarrow$ 1-step transition prob. mat. $\mathbb{P}(1)$, $P(n) = [\mathbb{P}(1)]^n$

Matrix Form: $P(t+s) = P(t)\cdot P(s)$, $P(?)$ basic unit? $\quad$ Continuous-Time $P(1)$?

$P(t) \xrightarrow{t\to 0} I$, $\quad P(n+1) = P(n)P(1)$ (difference eqn.) $\quad \frac{1}{\Delta t}[P(t+\Delta t) - P(t)] = ?$

continuous-time $P(1)$ (Q-Matrix)

$$\frac{1}{\Delta t}[P(t+\Delta t) - P(t)] = \frac{1}{\Delta t}[P(t)\cdot P(\Delta t) - P(t)] = P(t)\cdot\frac{1}{\Delta t}(P(\Delta t) - I), \quad \frac{d}{dt}P(t) = P(t)\cdot \boxed{\lim_{\Delta t\to 0}\frac{1}{\Delta t}(P(\Delta t)-I)} = P(t)\cdot Q$$

Generator (生成元)

Forward

$$\begin{cases} \frac{d}{dt}P(t) = P(t)\cdot Q \\ P(0) = I \end{cases} \Rightarrow P(t) = \exp(Qt)$$

$P_{ij}(t) \geq 0$, $\sum_j P_{ij}(t) = 1$ $\quad$ v.s. $\quad Q_{ij}$. $\quad (P(\Delta t) - I)$

$$\begin{cases} \frac{d}{dt}P(t) = Q\cdot P(t) \\ P(0) = I \end{cases} \Rightarrow P(t) = \exp(Qt)$$

Backward

$$Q_{ij}\begin{cases} \geq 0 & \text{on non-diagonal entry } (i\neq j) \\ \leq 0 & \text{on diagonal entry } (i=j) \end{cases}$$

$\sum_j Q_{ij} = 0$

E.g.

Poisson Process: $P_{ij}(t) = P\{N(t) = j-i\} = \begin{cases} \frac{(\lambda t)^{j-i}}{(j-i)!}\exp(-\lambda t), & j-i \geq 0 \\ 0, & j-i < 0 \end{cases}$

diag. entry: $\lim_{\Delta t\to 0}\frac{1}{\Delta t}(P_{ii}(\Delta t) - I) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}[\exp(-\lambda\Delta t) - I] = -\lambda < 0$

non-diag. entry: $\lim_{\Delta t\to 0}\frac{1}{\Delta t}P_{ij}(\Delta t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\frac{(\lambda\Delta t)^{j-i}}{(j-i)!}\exp(-\lambda\Delta t) = \begin{cases}\lambda, & j = i+1 \\ 0, & j\geq i+2\end{cases}$

$$\Rightarrow Q = \begin{bmatrix} -\lambda & \lambda & & & \\ & -\lambda & \lambda & & \\ & & -\lambda & \lambda & \\ & & & \ddots & \ddots \end{bmatrix}$$

最简单的Q-Matrix

Poisson: 最简单的MC.

CT → Sojourn (short stay) + Jumping

$$P_{ij}(t) \xrightarrow{t\to\infty} P_j \cong \frac{\text{Sojourn Time}}{\text{Total Time}}$$

irreducible. (对初值不敏感很重要. All states are commutable)

No period in cont.-time Markov Process.

(No $P(1)$ in Cont.-time) ⇒ No GCD.

Continuous-Time: irreducible ⇒ $P_{ij}(t) \xrightarrow{t\to\infty} P_j$

period: $d_i = GCD\{k: P_{ii}(k) > 0\}$ $\quad P_{ii}[1] > 0 \Rightarrow$ non-periodic.

$\exists\, t_0$, $P_{ij}(t_0) > 0 \Rightarrow \forall t$, $P_{ij}(t) > 0$

C-K Equation:

$$QP(t) = Q\begin{bmatrix} P_1 & P_2 & \cdots & P_n \\ P_1 & P_2 & \cdots & P_n \\ \vdots & \vdots & & \vdots \\ P_1 & P_2 & \cdots & P_n \end{bmatrix} \Rightarrow \text{trivial}$$

row sum = 0

$\frac{d}{dt}P(t) = P(t)\,Q = \cancel{Q\cdot P(t)}$

$t\to\infty$, $P \to P$'s limit, $\frac{d}{dt}P \to 0 \Rightarrow 0 = P\cdot Q \ (\pi Q)$

sojourn & jump → sojourn & jump → sojourn & jump.

(Random) $\quad$ (不耗时间)

sojourn time is random. (Exponential Distribution) — Poisson (more complex)

jump (跳跃转关系确定且不耗时)

Discrete Time: $\pi = \pi P$

① Transition Relation ② Transition Prob. $P(1)$ ③ $\pi = \pi P \Rightarrow \pi = ?$

Continuous Time: $0 = \pi Q \Rightarrow$ limit distribution

① Transition Relation ② Generator Q ③ $0 = \pi Q \Rightarrow \pi = ?$

Queuing: Service Model $M/M/1 \to 1$ counter

M ↓ arriving, M ↓ leaving (service $\sim \exp(\cdot)$)

Model: Birth-Death Model

$$\begin{cases} 1 \text{ birth}: \lambda_i \Delta t + o(\Delta t) \\ \geq 2 \text{ birth}: o(\Delta t) \end{cases} \xleftrightarrow{\text{independent}} \begin{cases} 1 \text{ death}: \mu_i \Delta t + o(\Delta t) \\ \geq 2 \text{ death}: o(\Delta t) \end{cases}$$

$P_{ij}(t)$

$$Q = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(P(\Delta t) - I)$$

$$\begin{cases} P_{ii}(\Delta t) = (1 - \lambda_i \Delta t + o(\Delta t))(1 - \mu_i \Delta t + o(\Delta t)) + (\lambda_i \Delta t + o(\Delta t))(\mu_i \Delta t + o(\Delta t)) + o(\Delta t) \\ P_{i,i+1}(\Delta t) = (\lambda_i \Delta t + o(\Delta t))(1 - \mu_i \Delta t + o(\Delta t)) + o(\Delta t) \\ P_{i,i-1}(\Delta t) = (\mu_i \Delta t + o(\Delta t))(1 - \lambda_i \Delta t + o(\Delta t)) + o(\Delta t) \\ P_{ij}(\Delta t) = o(\Delta t), \quad |i-j| \geq 2 \end{cases}$$

$Q_{ii} = -(\lambda_i + \mu_i)$

$Q_{i,i+1} = \lambda_i$

$Q_{i,i-1} = \mu_i$

$Q_{i,j} = 0, \quad |i-j| \geq 2$

$i = 1, 2, \cdots$

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\pi Q = 0 \Rightarrow \begin{array}{l} -\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \\ \lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2 = 0 \\ \lambda_1 \pi_1 - (\lambda_2 + \mu_2)\pi_2 + \mu_3 \pi_3 = 0 \\ \vdots \\ \lambda_{k-1}\pi_{k-1} - (\lambda_k + \mu_k)\pi_k + \mu_{k+1}\pi_{k+1} = 0 \\ \vdots \end{array}$$

$$\begin{array}{l} \pi_1 = \frac{\lambda_0}{\mu_1}\pi_0 \\ \pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}\pi_0 \\ \pi_3 = \frac{\lambda_2}{\mu_3}\pi_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1}\pi_0 \\ \vdots \\ \pi_{k+1} = \frac{\lambda_k \cdots \lambda_0}{\mu_{k+1} \cdots \mu_1}\pi_0 = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}\pi_0 \\ \vdots \end{array}$$

$$\sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k}\pi_0\right) + \pi_0 = 1 \Rightarrow \left(1 + \sum_{n=1}^{\infty}\prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k}\right)\pi_0 = 1$$

$$\Rightarrow \begin{cases} \pi_0 = \left(1 + \sum_{n=1}^{\infty}\prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k}\right)^{-1} \\ \pi_n = \prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k} \cdot \left(1 + \sum_{n=1}^{\infty}\prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k}\right)^{-1} \end{cases} \quad n = 1, 2, \cdots$$

$M/M/1$: arriving → Markov (Poisson); ↕ decoupled; leaving → Markov

$\lambda_i = \lambda$

$\mu_i = \mu$

Birth: $1: \lambda \Delta t + o(\Delta t)$; $\geq 2: o(\Delta t)$

Death: $1: \mu \Delta t + o(\Delta t)$; $\geq 2: o(\Delta t)$

$$\prod_{k=1}^{n} \frac{\lambda_{k-1}}{\mu_k} = \prod_{k=1}^{n}\left(\frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^n \Rightarrow \pi_0 = \left(1 + \sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \left(1 + \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}}\right)^{-1} = \frac{\mu - \lambda}{\mu}$$

Convergence: $\frac{\lambda}{\mu} < 1 \Leftrightarrow \lambda < \mu$, arriving < leaving ⇒ Dynamic Balance

if $\lambda = \mu$ / $\lambda > \mu$, the queue goes infinitely long ⇒ no non-trivial limit distribution (non-Recurrent)

$\pi_0 = 0 \Rightarrow \pi_k = 0 \quad k = 1, 2, \cdots$

$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{\mu - \lambda}{\mu}$. Average Queue Length. $= \sum_{n=1}^{\infty} n \cdot \pi_n = \sum_{n=1}^{\infty} n \cdot \left(\frac{\lambda}{\mu}\right)^n \frac{\mu - \lambda}{\mu}$

$= \frac{\mu - \lambda}{\mu} \frac{\lambda \mu}{(\mu - \lambda)^2} = \frac{\lambda}{\mu - \lambda}$

$S = \sum_{n=1}^{\infty} n \cdot \left(\frac{\lambda}{\mu}\right)^n$

$\frac{\mu}{\lambda} S = \sum_{m=0}^{\infty} (m+1)\left(\frac{\lambda}{\mu}\right)^m$

$\left(\frac{\mu}{\lambda} - 1\right)S = \sum_{m=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^m = \frac{1}{1 - \frac{\lambda}{\mu}}$

$\Rightarrow \frac{\mu - \lambda}{\lambda} S = \frac{\mu}{\mu - \lambda}$, $S = \frac{\lambda \mu}{(\mu - \lambda)^2}$

Average staying time: Expected length × Expected service time + Expected service time (yourself)

$$T = \frac{\lambda}{\mu - \lambda} \cdot \frac{1}{\mu} + \frac{1}{\mu} = \frac{1}{\mu - \lambda}$$

$\therefore \quad L = \frac{\lambda}{\mu - \lambda} = \frac{1}{\mu - \lambda} \cdot \lambda = T \cdot \lambda$ Little Formula

$L = \lambda \cdot T$