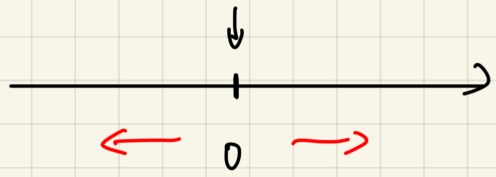
Gaussian Everywhere

ink pour on the axis at $t=0$



Spatial distribution of ink ?

① Diffusion. (Micro & Macro scopic View)

diffuse gradually

Suppose Distribution $f(x,t)$ $t \to$ variance (spread over time)

Introduce $\rho(y,\tau)$: describe the velocity of ink diffusing

random

percentage of ink diffusing distance y in duration τ also a distribution.

i) $f(x,t+\tau) = \int_{\mathbb{R}} \rho(y,\tau)\cdot f(x-y,t)\,dy$ key equation

ii) $\int_{\mathbb{R}} \rho(y,\tau)\,dy = 1$

iii) $\int_{\mathbb{R}} y\rho(y,\tau)\,dy = 0$

iv) $\int_{\mathbb{R}} y^2\rho(y,\tau)\,dy = D(\tau)$

have to eliminate $\rho(y,\tau)$ in the process of derivation to arrive at $f(x,t)$:

Taylor Expansion of $f(x-y,t)$:

$$f(x-y,t) = f(x,t) + (-y)\frac{\partial}{\partial x} f(x,t) + \frac{y^2}{2}\frac{\partial^2}{\partial x^2} f(x,t) + \cdots$$

(y small enough [Taylor] $\neq$ $y \in \mathbb{R}$ [Integration range])

just ignore this imperfection

$$\therefore \int_{\mathbb{R}} \left[f(x,t) + (-y)\frac{\partial}{\partial x} f(x,t) + \frac{(-y)^2}{2}\frac{\partial}{\partial x} f(x,t) \right]\cdot \rho(y,\tau)\,dy = f(x,t+\tau)$$

$$\Rightarrow f(x,t) + \frac{D}{2}\frac{\partial^2}{\partial x^2} f(x,t) = f(x,t+\tau), \quad f(x,t+\tau) - f(x,t) = \frac{D}{2}\frac{\partial^2}{\partial x^2} f(x,t) \quad \frac{f(x,t+\tau) - f(x,t)}{\tau} = \frac{D}{2\tau}\frac{\partial^2}{\partial x^2} f(x,t)$$

iv) $\int_{\mathbb{R}} y^2\rho(y,\tau)\,dy = D(\tau)$ when $\tau \to 0$, $D(\tau) \to 0$ (no time for diffusing, hence no spreading)

Suppose $\lim_{\tau \to 0} \frac{D(\tau)}{\tau} = D \Rightarrow \frac{\partial f}{\partial t} = \frac{D}{2}\frac{\partial^2 f}{\partial x^2}$ Diffusion Equation

$$f(x,0) = \delta(x) \Rightarrow f(x,t) = \frac{1}{\sqrt{2\pi D t}} \exp\left(-\frac{x^2}{2Dt}\right)$$

Discrete Interpretation of this work: