Markov Processes

Recap. on filtered Poisson. $Y(t)=\sum_{k=1}^{N(t)} h(t,S_k,A_k)$, $\{A_k\}$ i.i.d r.v.

$$\begin{cases} \Phi_Y(t) = E[\exp(j\omega Y(t))] = E\left[\prod_{k=1}^{N(t)} \frac{1}{t}\int_0^t B(t,S_k)dS_k\right] = E_{N(t)}\left[\frac{1}{t}\int_0^t B(t,S_k)dS_k\right]^{N(t)} = G_{N(t)}(z)\Big|_{z=\frac{1}{t}\int_0^t B(t,s)ds} \\ B(t,s) = E_{A_k}[\exp(j\omega h(t,s,A_k))] \end{cases}$$

When $\{A_k\}$ is no longer independent, filtered Poisson is not applicable $\Rightarrow$ Markov Processes

Markov Chains

$X_0, X_1, \cdots, X_n \cdots \Rightarrow P_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = ?$

Joint Dist. $= P(X_n|X_0,X_1,\cdots,X_{n-1})\cdot P(X_0,X_1,\cdots,X_{n-1}) = \cdots = P(X_n|\underbrace{X_{n-1},\cdots,X_0}_{\text{Condition} \Leftrightarrow \text{Constraint}})\cdot P(X_{n-1}|\cdot\cdot)\cdots P(X_1|X_0)\cdot P(X_0)$

Markov's Assumption: $P(X_n|X_{n-1},\cdots,X_0) = P(X_n|X_{n-1})$ 好假设：陈述很简单，推进很迅速，应用很广泛

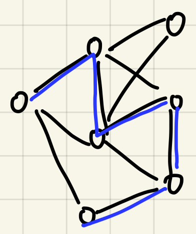
$$P(X_n, X_{n-1},\cdots,X_0) = \prod_{k=1}^{n} P(X_k|X_{k-1})\cdot P(X_0)$$

P(Future | Present, Past) = P(Future | Present) 抓住当下，就把握住了未来

$\Updownarrow$ Equivalent.

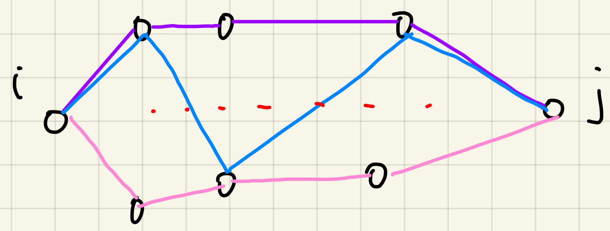
P(Future, Past | Present) = P(Future | Present) · P(Past | Present) Future & Past is symmetric to Present

Markov $\to$ Discrete Time, Discrete States. Chain



Transition Probability $P_{ij}(m,n) = P(X_n=j|X_m=i)$

$t=m$, i^{th} state $\Rightarrow$ $t=n$, j^{th} state



$\downarrow$

Stationary on Transition Prob.: $P_{ij}(m,n) = P_{ij}(n-m)$ $P_{ij}(n) = ?$

Chapman-Kolmogrov Equation : $P_{ij}(n) = \sum_k P_{ik}(m)\cdot P_{kj}(n-m)$, $0<m<n$

Proof: $P_{ij}(n) = P(X_n=j|X_0=i) = \sum_k P(X_n=j, X_m=k|X_0=i) = \sum_k P(X_m=k|X_0=i)\cdot P(X_n=j|X_m=k, X_0=i) \xlongequal{\text{Markov}} \sum_k P(X_m=k|X_0=i)\cdot P(X_n=j|X_m=k)$

$\xlongequal{\text{Stationary}} \sum_k P_{ik}(m)\cdot P_{kj}(n-m)$ matrix multiplication

Let $P(n) = [P_{ij}(n)]_{ij}$, then $P(n) = P(m)\cdot P(n-m)$, $0<m<n$

$P(n) = P(1)\cdot P(n-1) = P(1)\cdot P(1)\cdot P(n-2) = \cdots = [P(1)]^n \to$ One step determines everything ✱

$P(X_{n_k}=i_k, \cdots, X_{n_1}=i_1) = \underbrace{P(X_{n_k}=i_k|X_{n_{k-1}}=i_{k-1})}_{P_{i_{k-1},i_k}(n_k-n_{k-1})}\cdots P(X_{n_2}=i_2|X_{n_1}=i_1)\, P(X_{n_1}=i_1)$

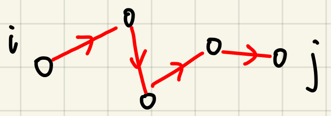
$\lim_{n\to\infty} P_{ij}(n) = ?$ Asymptotic Behavior (Steady State)

$\lim_{n\to\infty} P_{ij}(n) = \pi_{ij} = \pi_j$

$\boxed{\text{Steady State}}$

Stochastic process $\rightarrow$ r.v.

① Reachable: $i\to j$, $\exists n$, $P_{ij}(n) > 0$



$i\to j,\ j\to k \Rightarrow i\to k$

not necessarily the same route

② Commutable $\{i\leftrightarrow j\} \Leftrightarrow \{i\to j,\ j\to i\}$

Equivalent Relation: 1) $i\leftrightarrow i$ 2) $i\leftrightarrow j \Leftrightarrow j\leftrightarrow i$ 3) $i\leftrightarrow j,\ j\leftrightarrow k \Leftrightarrow i\leftrightarrow k$

③ Closed Set: S: state space, $C\subseteq S$ C is closed: $\forall i\in C$, $j\notin C \Rightarrow i\not\to j$ (j is not reachable to i)

transient state

$\Downarrow$

Reduction: 1) only study the transition inside the closed set

$\Downarrow$

2) to ulterior, closed set can be regarded as 1 state

④ Irreducible: No closed true subset

Irreducible $\Leftrightarrow$ All states are commutable

$\Leftarrow$: trivial $\forall i\in S$, not exists a $j\in S$. $i\not\to j$, i.e. $\forall i,j\in S$. $i\to j$

$\Rightarrow$: $\forall i\in S$, define set $A_i=\{j: i\to j\}$. A_i must be closed. (no closed true subset $\overset{\text{irreducible}}{\Rightarrow}$ $A_i = S$)

$\forall j\in A_i$ $k\notin A_i$. Assume $j\to k$. $\because i\to j (j\in A_i)$ $\therefore i\to k$. Contradict $\therefore j\not\to k$. A_i is a closed set

$(i\not\to k)$

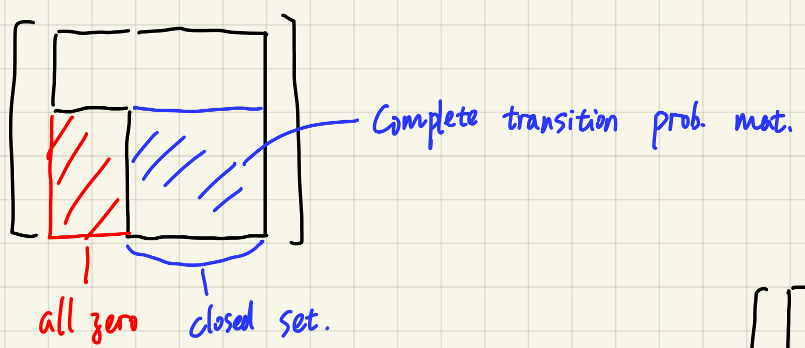
$\because$ no closed true subset in S $\therefore A_i = S$. $\forall i,j\in S$. $i\to j$ $\therefore$ all states are commutible.

Transition Probability Matrix (1 step) P

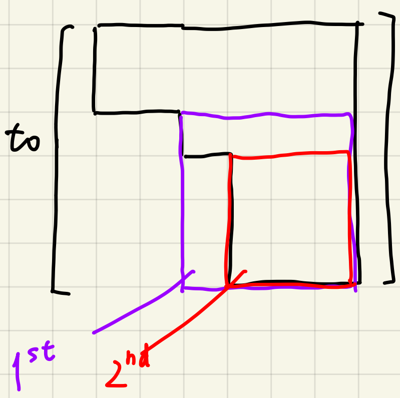
① $P_{ij}\geq 0$ $\boxed{② \sum_j P_{ij} = 1}$

if closed true subset exists in S, through proper column/row exchange (re-index the states),

TP Matrix can be converted to the following form



If closed true subset exists in the 1st closed true subset, TP mat. can be converted to



⑤ Recurrent (常返) state $i \Leftrightarrow \sum_{n=1}^{\infty} f_{ii}(n) = 1$ 1st passage seems to have little to do with "recurrent"

⑥ 1st Passage Probability: $f_{ij}(n) = P(X_n = j, X_k \neq j \, (k=1 \sim n-1) \mid X_0 = i)$

Relation between 1st Passage & Transition Prob.

$\sum_{n=1}^{\infty} f_{ij}(n) \le 1$, $\sum_{n=1}^{\infty} P_{ij}(n)$ may be $+\infty$

paths are mutual exclusive

$P_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k)$ Temporal Decomposition

C-K Equation: $P_{ij}(n) = \sum_k P_{ik}(m) \cdot P_{kj}(n-m)$ Spatial Decomposition.

Proof: $P_{ij}(n) = P(X_n = j \mid X_0 = i)$

Def. T_j: Random Time, $\{T=k\} = \{X_1 \neq j, X_2 \neq j, \cdots, X_k \neq j\}$ $\Rightarrow$ 1st Passage

$P_{ij}(n) = P(X_n = j \mid X_0 = i) = \sum_{k=1}^{n} P(X_n = j, T_j = k \mid X_0 = i) = \sum_{k=1}^{n} P(T_j = k \mid X_0 = i) \cdot P(X_n = j \mid T_k = j, X_0 = i) = \sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k)$

$P_{ij}(n) = \sum_k f_{ij}(k) \cdot P_{jj}(n-k)$ $\rightarrow$ Convolution

$P_{ij}(0) = \delta_{ij}$, $f_{ii}(0) = 1$, $\sum_{n=0}^{\infty} P_{ij}(n) \cdot z^n = \delta_{ij} + \sum_{n=1}^{\infty} P_{ij}(n) \cdot z^n = \delta_{ij} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k) \right) \cdot z^n$

$= \delta_{ij} + \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k) \cdot z^n = \delta_{ij} + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_{ij}(k) \cdot z^k \cdot P_{jj}(n-k) \cdot z^{n-k}$

$= \delta_{ij} + \sum_{k=1}^{\infty} f_{ij}(k) \cdot z^k \cdot \sum_{n'=0}^{\infty} P_{jj}(n') \cdot z^{n'} = \delta_{ij} + P_{jj}(z) \cdot F_{ij}(z)$

$\boxed{\therefore \; P_{ij}(z) = \delta_{ij} + P_{jj}(z) \cdot F_{ij}(z)}$

let $i = j$,

$P_{ii}(z) = 1 + P_{ii}(z) \cdot F_{ii}(z) \Rightarrow P_{ii}(z) = \frac{1}{1 - F_{ii}(z)} \xRightarrow{z \to 1} \sum_{n=1}^{\infty} P_{ii}(n) = \frac{1}{1 - \sum_{n=1}^{\infty} f_{ii}(n)}$ $\begin{cases} \text{Divergent} & \text{state } i \text{ Recurrent} \\ \text{Convergent} & i \text{ not Recurrent} \end{cases}$

$i \neq j$. $P_{ij}(z) = \delta_{ij} + P_{jj}(z) \cdot F_{ij}(z) = P_{jj}(z) \cdot F_{ij}(z)$

non-recurrent $\Rightarrow$ marginal

$z \to 1$: $\sum_{n=0}^{\infty} P_{ij}(n) = \sum_{n=1}^{\infty} f_{ij}(n) \cdot \sum_{n=0}^{\infty} P_{jj}(n)$ $\begin{cases} j \text{ Non-Recurrent}, \sum_{n=0}^{\infty} P_{jj}(n) < \infty \\ \sum_{n=1}^{\infty} f_{ij}(n) \le 1 \end{cases}$ $\Rightarrow \sum_{n=0}^{\infty} P_{ij}(n) < \infty$, $P_{ij}(n) \xrightarrow{n \to \infty} 0 \; (\forall i \neq j)$

Property: $i \Leftrightarrow j$ i Recurrent $\Rightarrow$ j Recurrent

$\exists m$, $P_{ij}(m) > 0$; $\exists n$, $P_{ji}(n) > 0$. if $\sum_{k=0}^{\infty} P_{jj}(k) = \infty$ (j Recurrent).

$P_{ii}(m+n+k) \ge P_{ij}(m) \cdot P_{jj}(k) \cdot P_{ji}(n)$

$\sum_{k=0}^{\infty} P_{ii}(m+n+k) \ge P_{ij}(m) \cdot \sum_{k=0}^{\infty} P_{jj}(k) \cdot P_{ji}(n) = \infty$ $\Rightarrow \sum_{l=0}^{\infty} P_{ii}(l) = \infty$ (i.e. if $i \Leftrightarrow j$, j recurrent, then i recurrent)

Property: Finite States $\Rightarrow$ Recurrent States exist.

$\sum_{j=1}^{N} P_{ij}(n) = 1 \Rightarrow \lim_{n \to \infty} \sum_{j=1}^{N} P_{ij}(n) = 1 \xRightarrow[\text{finite states}]{} \sum_{j=1}^{N} \lim_{n \to \infty} P_{ij}(n) = 1 \xRightarrow[\text{recurrent states}]{\text{Suppose no}} \sum_{j=1}^{N} 0 = 1$ Impossible!

Property: Finite States + Irreducible $\Rightarrow$ All States are Recurrent.

Ex. N_i r.v. $\Rightarrow$ re-visiting times $E[N_i|X_0=i]=?$

$$E[N_i|X_0=i] = E[\sum_{k=0}^{\infty} \underbrace{I(X_k=i)}_{\text{Indicator Function}} | X_0=i] = \sum_{k=0}^{\infty} E[I(X_k=i)|X_0=i] = \sum_{k=0}^{\infty} P(X_k=i|X_0=i) = \sum_{k=0}^{\infty} P_{ii}(k) \begin{cases} \infty, & \text{Recurrent.} \\ <\infty \end{cases}$$

Ex. $P(N_i=\infty|X_0=i)=?$ infinite re-visiting time's Prob. $=?$

Let $g_{ii}(m) = P(N_i \geq m|X_0=i)$ revisiting m times. $\Rightarrow$ must have a first (1^{st} Passage)

$$\{T_i=k\} = \{X_1\neq i, X_2\neq i, \cdots, X_{k-1}\neq i, X_k=i\} \Rightarrow P(N_i\geq m|X_0=i) = \sum_{k=1}^{\infty} P(N_i\geq m, T_i=k|X_0=i) = \sum_{k=1}^{\infty} P(T_i=k|X_0=i)\cdot P(N_i\geq m|T_i=k, X_0=i)$$

$$g_{ii}(m) = P(N_i\geq m|X_0=i) = \sum_{k=1}^{\infty} f_{ii}(k)\cdot \underbrace{P(N_i\geq m-1|X_0=i)}_{g_{ii}(m-1)} = g_{ii}(m-1)\cdot\sum_{k=1}^{\infty} f_{ii}(k)$$

re-visit -1

$$\therefore \begin{cases} g_{ii}(m) = g_{ii}(m-1)\cdot\sum_{k=1}^{\infty} f_{ii}(k) \\ g_{ii}(m) = P(N_i\geq m|X_0=i) \end{cases}$$

$$\Rightarrow g_{ii}(m) = \left[\sum_{k=1}^{\infty} f_{ii}(k)\right]^m \xrightarrow{m\to\infty} P(N_i=\infty|X_0=i) = \begin{cases} 1, & i \text{ Recurrent} \\ 0, & i \text{ non Recurrent} \end{cases}$$

Ex. i Recurrent, $i\to j \Rightarrow j$ Recurrent (i Recurrent, $i\to j \Rightarrow j\to i$ ($i\leftrightarrow j \Rightarrow j$ Recurrent))

Let $g_{ij}(m) = P(N_j\geq m|X_0=i)$, i Recurrent $\Leftrightarrow g_{ii}(\infty)=1$ $\quad g_{ii}(\infty) = \sum_k P_{ik}(m)\cdot g_{ki}(\infty)$ (Spatial Decomposition).

$i\to j$: $\exists\, m>0$, $P_{ij}(m)>0$,

$$\begin{cases} \sum_k P_{ik}(m) = 1 \\ \sum_k P_{ik}(m)\, g_{ki}(\infty) = g_{ii}(\infty) = 1 \end{cases} \Rightarrow \sum_k \underbrace{P_{ik}(m)}_{\geq 0}[\underbrace{1-g_{ki}(\infty)}_{\geq 0}] = 0 \quad \therefore 每一项都是零$$

$k=j$时 $P_{ij}(m)>0 \Rightarrow 1-g_{ji}(\infty)=0$, $g_{ji}(\infty)=1$