Markov Processes

Recap. on filtered Poisson. $Y(t)=\sum_{k=1}^{N(t)} h(t,S_k,A_k)$, $\{A_k\}$ i.i.d r.v.

$$\begin{cases} \Phi_Y(t) = E[\exp(j\omega Y(t))] = E\left[\prod_{k=1}^{N(t)} \frac{1}{t}\int_0^t B(t,S_k)dS_k\right] = E_{N(t)}\left[\frac{1}{t}\int_0^t B(t,S_k)dS_k\right]^{N(t)} = G_{N(t)}(z)\Big|_{z=\frac{1}{t}\int_0^t B(t,s)ds} \\ B(t,s) = E_{A_k}[\exp(j\omega h(t,s,A_k))] \end{cases}$$

When $\{A_k\}$ is no longer independent, filtered Poisson is not applicable $\Rightarrow$ Markov Processes

Markov Chains

$X_0, X_1, \cdots, X_n \cdots \Rightarrow P_{X_1,\cdots,X_n}(x_1,\cdots,x_n)=?$

Joint Dist. $= P(X_n|X_0,X_1,\cdots,X_{n-1})\cdot P(X_0,X_1,\cdots,X_{n-1}) = \cdots = P(X_n|\underbrace{X_{n-1},\cdots,X_0}_{\text{Condition} \Leftrightarrow \text{Constraint}})\cdot P(X_{n-1}|\cdot\cdot)\cdots P(X_1|X_0)\cdot P(X_0)$

Markov's Assumption: $P(X_n|X_{n-1},\cdots,X_0) = P(X_n|X_{n-1})$ 好假设：陈述很简单，推进很迅速，应用很广泛

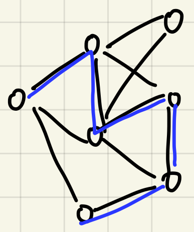
$$P(X_n,X_{n-1},\cdots,X_0) = \prod_{k=1}^{n} P(X_k|X_{k-1})\cdot P(X_0)$$

$P(\text{Future}\,|\,\text{Present},\text{Past}) = P(\text{Future}\,|\,\text{Present})$ 抓住当下，就把握住了未来

$\Updownarrow$ Equivalent.

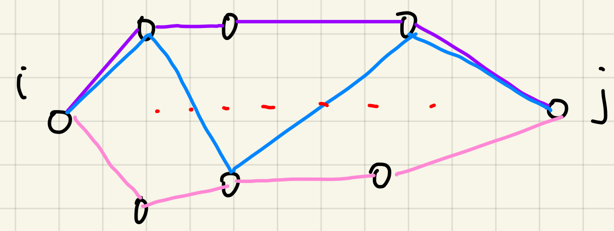
$P(\text{Future},\text{Past}\,|\,\text{Present}) = P(\text{Future}\,|\,\text{Present})\cdot P(\text{Past}\,|\,\text{Present})$ Future & Past is symmetric to Present

Markov $\to$ Discrete Time, Discrete States. Chain



Transition Probability $P_{ij}(m,n) = P(X_n=j\,|\,X_m=i)$

$t=m$, i^{th} state $\Rightarrow$ $t=n$, j^{th} state



Stationary on Transition Prob.: $P_{ij}(m,n) = P_{ij}(n-m)$ $\qquad P_{ij}(n)=?$

Chapman-Kolmogrov Equation : $P_{ij}(n) = \sum_k P_{ik}(m)\cdot P_{kj}(n-m)$, $0<m<n$

Proof: $P_{ij}(n) = P(X_n=j|X_0=i) = \sum_k P(X_n=j,X_m=k|X_0=i) = \sum_k P(X_m=k|X_0=i)\cdot P(X_n=j|X_m=k,X_0=i) \xlongequal{\text{Markov}} \sum_k P(X_m=k|X_0=i)\cdot P(X_n=j|X_m=k)$

$\xlongequal{\text{Stationary}} \sum_k P_{ik}(m)\cdot P_{kj}(n-m)$ matrix multiplication

Let $P(n) = [P_{ij}(n)]_{ij}$, then $P(n) = P(m)\cdot P(n-m)$, $0<m<n$

$P(n) = P(1)\cdot P(n-1) = P(1)\cdot P(1)\cdot P(n-2) = \cdots = [P(1)]^n \to$ One step determines everything ✱

$P(X_{n_k}=i_k,\cdots,X_{n_1}=i_1) = \underbrace{P(X_{n_k}=i_k|X_{n_{k-1}}=i_{k-1})}_{P_{i_{k-1},i_k}(n_k-n_{k-1})}\cdots P(X_{n_2}=i_2|X_{n_1}=i_1)\,P(X_{n_1}=i_1)$

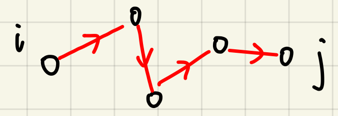
$\lim_{n\to\infty} P_{ij}(n) = ?$ Asymptotic Behavior (Steady State)

$$\lim_{n\to\infty} P_{ij}(n) = \pi_{ij} = \pi_j$$

$\boxed{\text{Steady State}}$

Stochastic process $\to$ r.v.

① Reachable: $i \to j$, $\exists n$, $P_{ij}(n) > 0$



$i\to j,\ j \to k \Rightarrow i \to k$

② Commutative $\{i \leftrightarrow j\} \Leftrightarrow \{i\to j,\ j \to i\}$ not necessarily the same route

Equivalent Relation: 1) $i \leftrightarrow i$ 2) $i \leftrightarrow j \Leftrightarrow j \leftrightarrow i$ 3) $i\leftrightarrow j,\ j \leftrightarrow k \Leftrightarrow i \leftrightarrow k$

③ Closed Set: S: state space, $C \subseteq S$ C is closed: $\forall i \in C$, $j \notin C \Rightarrow i \not\to j$ (j is not reachable to i)

transient state

$\Downarrow$

Reduction: 1) only study the transition inside the closed set

$\Downarrow$

2) to ulterior, closed set can be regarded as 1 state

④ Irreducible: No closed true Subset

Irreducible $\Leftrightarrow$ All states are commutative

$\Leftarrow$: trivial $\forall i \in S$, not exists a $j \in S$. $i \not\to j$, i.e. $\forall i,j \in S$. $i \to j$

$\Rightarrow$: $\forall i \in S$, define set $A_i = \{j : i \to j\}$. A_i must be closed. (no closed true subset $\Rightarrow A_i = S$) irreducible

$\forall j \in A_i$ $k \notin A_i$ ($i \not\to k$). Assume $j \to k$. $\because i \to j (j \in A_i)$ $\therefore i \to k$. Contradict $\therefore j \not\to k$. A_i is a closed set

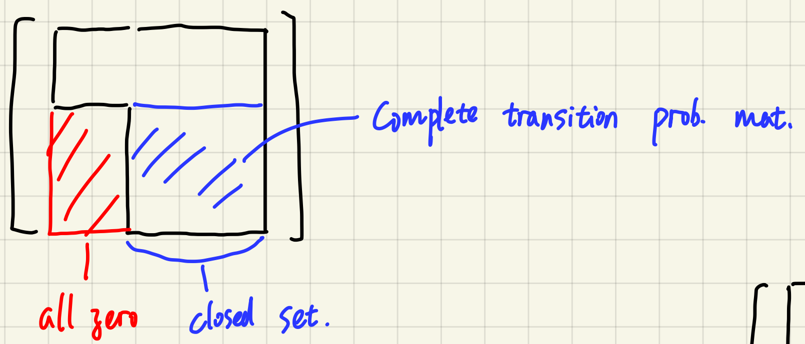
$\because$ no closed true subset in S $\therefore A_i = S$. $\forall i,j \in S$. $i \to j$ $\therefore$ all states are commutative.

Transition Probability Matrix (1 step) P

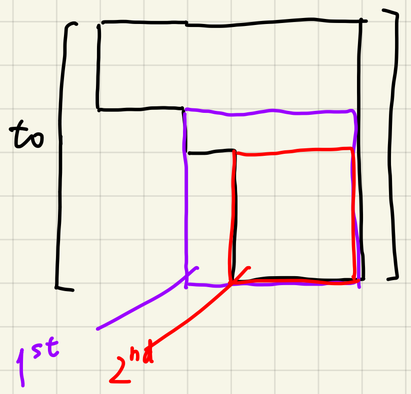
① $P_{ij} \geq 0$ $\boxed{② \sum_j P_{ij} = 1}$

if closed true subset exists in S, through proper column/row exchange (re-index the states),

TP Matrix can be converted to the following form



If closed true subset exists in the 1st closed true subset. TP mat. can be converted to



⑤ Recurrent (常返)

State $i \Leftrightarrow \sum_{n=1}^{\infty} f_{ii}(n) = 1$

1^{st} passage seems to have little to do with "recurrent"

⑥ 1^{st} Passage Probability: $f_{ij}(n) = P(X_n = j, X_k \neq j\ (k=1 \sim n-1) \mid X_0 = i)$

Relation between 1^{st} Passage & Transition Prob.

$\sum_{n=1}^{\infty} f_{ij}(n) \le 1$, $\sum_{n=1}^{\infty} P_{ij}(n)$ may be $+\infty$

paths are mutual exclusive

$$P_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k)$$ Temporal Decomposition

C-K Equation: $P_{ij}(n) = \sum_k P_{ik}(m) \cdot P_{kj}(n-m)$ Spatial Decomposition.

Proof: $P_{ij}(n) = P(X_n = j \mid X_0 = i)$

Def. T_j: Random Time, $\{T=k\} = \{X_1 \neq j, X_2 \neq j, \cdots, X_k \neq j\}$ ⇒ 1^{st} Passage

$$P_{ij}(n) = P(X_n = j \mid X_0 = i) = \sum_{k=1}^{n} P(X_n = j, T_j = k \mid X_0 = i) = \sum_{k=1}^{n} P(T_j = k \mid X_0 = i) \cdot P(X_n = j \mid T_k = j, X_0 = i) = \sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k)$$

$$P_{ij}(n) = \sum_k f_{ij}(k) \cdot P_{jj}(n-k) \rightarrow \text{Convolution}$$

$P_{ij}(0) = \delta_{ij}$, $f_{ii}(0) = 1$,

$$\sum_{n=0}^{\infty} P_{ij}(n) \cdot z^n = \delta_{ij} + \sum_{n=1}^{\infty} P_{ij}(n) \cdot z^n = \delta_{ij} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k)\right) \cdot z^n$$

$$= \delta_{ij} + \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{ij}(k) \cdot P_{jj}(n-k) \cdot z^n = \delta_{ij} + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} f_{ij}(k) \cdot z^k \cdot P_{jj}(n-k) \cdot z^{n-k}$$

$$= \delta_{ij} + \sum_{k=1}^{\infty} f_{ij}(k) \cdot z^k \cdot \sum_{n'=0}^{\infty} P_{jj}(n') \cdot z^{n'} = \delta_{ij} + P_{jj}(z) \cdot F_{ij}(z)$$

$$\boxed{\therefore\ P_{ij}(z) = \delta_{ij} + P_{jj}(z) \cdot F_{ij}(z)}$$

let $i = j$,

$$P_{ii}(z) = 1 + P_{ii}(z) \cdot F_{ii}(z) \Rightarrow P_{ii}(z) = \frac{1}{1 - F_{ii}(z)} \xrightarrow{z \to 1} \sum_{n=1}^{\infty} P_{ii}(n) = \frac{1}{1 - \sum_{n=1}^{\infty} f_{ii}(n)}$$

$\begin{cases} \text{Divergent} & \text{state } i \text{ Recurrent} \\ \text{Convergent} & i \text{ not Recurrent} \end{cases}$

$i \neq j$, $P_{ij}(z) = \delta_{ij} + P_{jj}(z) \cdot F_{ij}(z) = P_{jj}(z) \cdot F_{ij}(z)$

non-recurrent ⇒ marginal

$z \to 1$: $\sum_{n=0}^{\infty} P_{ij}(n) = \sum_{n=1}^{\infty} f_{ij}(n) \cdot \sum_{n=0}^{\infty} P_{jj}(n)$ $\begin{cases} j \text{ Non-Recurrent}, \sum_{n=0}^{\infty} P_{jj}(n) < \infty \\ \sum_{n=1}^{\infty} f_{ij}(n) \le 1 \end{cases}$ $\Rightarrow \sum_{n=0}^{\infty} P_{ij}(n) < \infty$, $P_{ij}(n) \xrightarrow{n \to \infty} 0\ (\forall i \neq j)$

Property: $i \Leftrightarrow j$ j Recurrent ⇒ i Recurrent (Commutative ⇒ Properties same)

$\exists m$, $P_{ij}(m) > 0$; $\exists n$, $P_{ji}(n) > 0$. if $\sum_{k=0}^{\infty} P_{jj}(k) = \infty$ (j Recurrent).

$$P_{ii}(m+n+k) \ge P_{ij}(m) \cdot P_{jj}(k) \cdot P_{ji}(n)$$

$$\sum_{k=0}^{\infty} P_{ii}(m+n+k) \ge P_{ij}(m) \cdot \sum_{k=0}^{\infty} P_{jj}(k) \cdot P_{ji}(n) = \infty \Rightarrow \sum_{l=0}^{\infty} P_{ii}(l) = \infty$$ (i.e. if $i \Leftrightarrow j$, j recurrent, then i recurrent)

Property: Finite States ⇒ Recurrent States exist.

$$\sum_{j=1}^{N} P_{ij}(n) = 1 \Rightarrow \lim_{n\to\infty} \sum_{j=1}^{N} P_{ij}(n) = 1 \xrightarrow[\text{finite states}]{} \sum_{j=1}^{N} \lim_{n\to\infty} P_{ij}(n) = 1 \xrightarrow[\text{recurrent states}]{\text{Suppose no}} \sum_{j=1}^{N} 0 = 1$$ Impossible!

Property: Finite States + Irreducible $\Rightarrow$ All States are Recurrent.

Ex. N_i r.v. $\Rightarrow$ re-visiting times $E[N_i|X_0=i]=?$

$$E[N_i|X_0=i]=E\Big[\sum_{k=0}^{\infty}I(X_k=i)\Big|X_0=i\Big]=\sum_{k=0}^{\infty}E[I(X_k=i)|X_0=i]=\sum_{k=0}^{\infty}P(X_k=i|X_0=i)=\sum_{k=0}^{\infty}P_{ii}(k)\begin{cases}\infty, & \text{Recurrent.}\\ <\infty\end{cases}$$

Counting $\Rightarrow$ Sum of Indicator Function

Ex. $P(N_i=\infty|X_0=i)=?$ infinite re-visiting time's Prob. = ? 从概率层面认识 Recurrent.

Let $g_{ii}(m)=P(N_i\geq m|X_0=i)$ revisiting m times. $\Rightarrow$ must have a first (1^{st} Passage)

$$\{T_i=k\}=\{X_1\neq i, X_2\neq i,\cdots,X_{k-1}\neq i, X_k=i\}\Rightarrow P(N_i\geq m|X_0=i)=\sum_{k=1}^{\infty}P(N_i\geq m, T_i=k|X_0=i)=\sum_{k=1}^{\infty}P(T_i=k|X_0=i)\cdot P(N_i\geq m|T_i=k,X_0=i)$$

$$g_{ii}(m)=P(N_i\geq m|X_0=i)=\sum_{k=1}^{\infty}f_{ii}(k)\cdot \underbrace{P(N_i\geq m-1|X_0=i)}_{g_{ii}(m-1)}=g_{ii}(m-1)\cdot\sum_{k=1}^{\infty}f_{ii}(k)$$

re-visit -1

$$\therefore\begin{cases}g_{ii}(m)=g_{ii}(m-1)\cdot\sum_{k=1}^{\infty}f_{ii}(k)\\ g_{ii}(m)=P(N_i\geq m|X_0=i)\end{cases}$$

$$\Rightarrow g_{ii}(m)=\Big[\sum_{k=1}^{\infty}f_{ii}(k)\Big]^m\xrightarrow{m\to\infty}P(N_i=\infty|X_0=i)=\begin{cases}1, & i\ \text{Recurrent}\\ 0, & i\ \text{non Recurrent}\end{cases}$$

Ex. i Recurrent, $i\to j\Rightarrow j$ Recurrent (i Recurrent, $i\to j\Rightarrow j\to i$ ($i\leftrightarrow j\Rightarrow j$ Recurrent))

Let $g_{ij}(m)=P(N_j\geq m|X_0=i)$, i Recurrent $\Leftrightarrow g_{ii}(\infty)=1$ $\quad g_{ii}(\infty)=\sum_k P_{ik}(m)\cdot g_{ki}(\infty)$ (Spatial Decomposition).

$i\to j$: $\exists\, m>0$, $P_{ij}(m)>0$,

$$\begin{cases}\sum_k P_{ik}(m)=1\\ \sum_k P_{ik}(m)\,g_{ki}(\infty)=g_{ii}(\infty)=1\end{cases}\Rightarrow \sum_k \overset{\geq 0}{P_{ik}(m)}\,\overset{\geq 0}{[1-g_{ki}(\infty)]}=0$$

$\therefore$ 每一项都是零

$k=j$时 $P_{ij}(m)>0\Rightarrow 1-g_{ji}(\infty)=0$, $g_{ji}(\infty)=1$ 往来无白丁

$j\to i$