Spectral Representation

$X(t)$ w.s.s. $X(t) = \int \exp(j\omega t) dF_X(\omega) \cdot \frac{1}{2\pi}$

$F_X(\omega)$: spectral process (Random) → Orthogonal Increments,

$\forall\ \omega_1 \le \omega_2 \le \omega_3 \le \omega_4$, $F_X(\omega_2) - F_X(\omega_1) \perp F_X(\omega_4) - F_X(\omega_3)$, $E[(F_X(\omega_2)-F_X(\omega_1))(F_X(\omega_4)-F_X(\omega_3))] = 0$ Uncorrelated.

$E[dF_X(\omega_0)(dF_X(\omega_1))^*] = S_X(\omega_0)\, d\omega_0 \cdot \delta_{01}$

if $X(t) = X(t+T)$, then $X(t) = \sum_k \alpha_k \exp(jk\frac{2\pi}{T}t)$, $\alpha_k = \frac{1}{T}\int_T X(t)\exp(-jk\frac{2\pi}{T}t)\,dt$

Same expression

Spectral Representation: $X(t) = \int \exp(j\omega t) dF_X(\omega)$

Orthogonality between Fourier Series index: $E[\alpha_k \alpha_m^*] = 0$ $k \ne m$. (Orthogonal Increment $F_X(\omega)$)

$\alpha_k = \frac{1}{T}\int_T X(t)\exp(-jk\frac{2\pi}{T}t)\,dt$

$E[\alpha_k \alpha_m^*] = \frac{1}{T^2}\int_T\int_T E[X(t)X^*(s)] \exp(-jk\frac{2\pi}{T}t)\exp(jm\frac{2\pi}{T}s)\,dt\,ds$

$= \frac{1}{T^2}\int_T\int_T \exp(-j\frac{2\pi}{T}(kt-ms))\, R_X(t-s)\,dt\,ds$

Why we use Fourier Series to prove orthogonality?

i) FS representation & Spectral representation are basically the same thing.

ii) Orthogonality of complex exponential functions & Periodicity are better visualized on a limited interval of integration.

$X(t)$ periodic : $E|X(t)-X(t+T)|^2 \Leftrightarrow R_X(t) = R_X(t+T)$

2^{nd} Lecture.

Let $t' = t-s$. Jacobian $= \frac{\partial(t',s)}{\partial(t,s)} = 1$

$= \frac{1}{T^2}\int_T \int_{-\frac{T}{2}-s}^{\frac{T}{2}-s} R_X(t')\cdot \exp\left(-j\frac{2\pi}{T}(kt'-(m-k)s)\right) dt'\,ds$

$= \frac{1}{T^2}\int_T \exp(j\frac{2\pi}{T}(m-k)s)\left[\underbrace{\int_{-\frac{T}{2}-s}^{\frac{T}{2}-s} R_X(t')\exp(-j\frac{2\pi}{T}kt')\,dt'}\right] ds$

$R_X(t)$ periodic : $\int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t')\cdot \exp(-j\frac{2\pi}{T}kt')\,dt$

$= \frac{1}{T}\int_T S_X(k\frac{2\pi}{T})\cdot \exp(j\frac{2\pi}{T}(m-k)s)\,ds$

$= \frac{1}{T} S_X(k\frac{2\pi}{T}) \int_T \exp(j\frac{2\pi}{T}(m-k)s)\,ds$

when $k \ne m$, $E[\alpha_k\alpha_m^*] = 0$, α_k is equivalent to $dF_X(\omega)$ in Spectral Representation.

$E[dF_X(\omega)\cdot dF_X^*(\omega')] = 0$

$\frac{1}{2\pi} E|dF_X(\omega)|^2 = S_X(\omega)\,d\omega$.

$R_X(\tau) = \frac{1}{2\pi}\int \exp(j\omega\tau) S_X(\omega)\,d\omega$.

$$R_X(\tau) = E[X(t)X^*(t-\tau)] = E\left[\frac{1}{4\pi^2}\int \exp(j\omega t)\,dF_X(\omega)\left(\int \exp(j\omega'(t-\tau))\,dF_X(\omega')\right)^*\right]$$

$$= \frac{1}{4\pi^2}\iint \exp(j\omega t - j\omega'(t-\tau))\, E[dF_X(\omega)\,(dF_X(\omega'))^*]$$

Orthogonal Increments $\Rightarrow \omega = \omega'$: $\frac{1}{4\pi^2}\int \exp(j\omega'\tau)\left[\int E[dF_X(\omega)\,dF_X^*(\omega')]\exp(j(\omega-\omega')t)\right]$

$$= \frac{1}{4\pi^2}\int \exp(j\omega'\tau)\cdot 2\pi\, S_X(\omega)\cdot d\omega'$$

$$= \frac{1}{2\pi}\int S_X(\omega)\exp(j\omega\tau)\,d\omega = \frac{1}{2\pi}\int \exp(j\omega\tau)\cdot E|dF_X(\omega)|^2$$

$X(t)$ W.S.S. $\quad X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{\exp(j\omega t)}_{\text{Orthogonality}}\, \underbrace{dF_X(\omega)}_{\text{Randomness.}}$

$X(t) \Leftrightarrow \exp(j\omega t)$

Oscillating (W.S.S.)

$\underbrace{X(\omega, t)}_{\text{Sample Space}} \Leftrightarrow \exp(\underbrace{j\omega t}_{\text{Frequency.}})$

Define 2 Distances:

$$\|X(t) - X(s)\|_1 = \|\exp(j\omega t) - \exp(j\omega s)\|_2$$

Isometry between Stochastic Processes & Complex Exponential Func