Markov Processes.

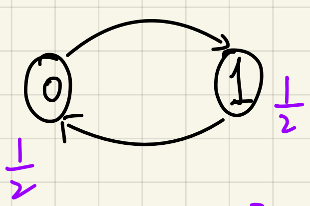
i. Recurrent, Long-Run Behavior $\Rightarrow$ j Non-Recurrent : $\forall i$, $P_{ij}(n) \xrightarrow{n\to\infty} 0$

$P_{ij}(n) \xrightarrow{n\to\infty} \pi_j$ Exist ?

Long-term behavior (limit) : Process Property $\downarrow$, Randomness stays

(on diff. states , prob. $\leftarrow$ freq. = $\frac{\#\text{ of this states}}{\text{length of the chain}}$)

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$P_{10}(n) = P_{01}(n) = \{1, 0, 1, 0, \cdots\}$

$P_{00}(n) = P_{11}(n) = \{0, 1, 0, 1, \cdots\}$

does not converge (oscillating periodically)

$B \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

显然过程特性退化，只有r.v.特性，但这个例子不在现有的理论体系中

Define a concept about the existence of $\vec{\pi} = [\pi_1, \cdots, \pi_n]^T \Rightarrow$ Periodic Property

To incorporate this

Define a concept encompassing the above example into long-term behavior.

Periodic Property:

i, $d_i = GCD\{k : P_{ii}(k) > 0\}$ Greatest Common Divisor

$d_i = 1 \Rightarrow i$: Non-Periodic

Theorem.

$i \leftrightarrow j \iff d_i = d_j$ (S is irreducible $d \to S$ | d is a property of S)

$i \leftrightarrow j$, $\exists m$, $P_{ij}(m) > 0$, $\exists n$, $P_{ji}(n) > 0$. $\Rightarrow P_{ii}(m+n) > 0$, Denote $\{k : P_{ii}(k) > 0\}$ as R_i

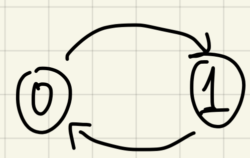
i) $P_{ii}(m+n) > 0 \Rightarrow m+n \in R_i \Rightarrow d_i \mid (m+n)$

ii) $\forall k \in R_j$, $P_{ii}(m+k+n) > 0 \Rightarrow m+k+n \in R_i \Rightarrow d_i \mid (m+n+k)$

iii) $d_i \mid (m+n+k) - (m+n)$, i.e. $d_i \mid k \Rightarrow \forall k \in R_j$, $d_i \mid k$, d_i is a common divisor of R_j ($d_i = CD\{R_j\}$)

iv) $d_i = CD\{R_j\}$. $d_j = GCD\{R_j\} \Rightarrow d_i \mid d_j$

vice versa ($d_j \mid d_i$) $\Rightarrow \begin{cases} d_i \mid d_j \\ d_j \mid d_i \end{cases} \Rightarrow d_i = d_j$



$P_{11}(n) = P_{00}(n) = \{0, 1, 0, 1, 0, 1, \cdots\}$

period : 2 (limit is not explicit)

Theorem.

前提（所有状态都相通）

Markov Chain S irreducible, non-periodic ($\forall i \in [1, n]$, $d_i = 1$)

链的性质

$\Rightarrow P_{ij}(n) \xrightarrow{n\to\infty} \pi_j$ (limit exists)

上面那个例子被解释了

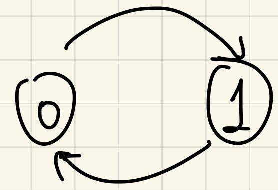
periodic is not a general property unless S is irreducible. (if S is reducible, period is only a local property)

$\begin{cases} \text{Positive Recurrent} \\ \text{Null Recurrent} \end{cases}$ 稍弱

$i \xrightarrow{\text{Criterion}}$ 研究状态 j

$\forall i \in n, \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n) = ?$

$\lim_{N\to\infty} P_{ij}(N)$ 更强的极限



definitely Recurrent.

$(a_n \to a \Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_i a_i = a)$ $\cancel{\Leftarrow}$

做一个较弱的定义

(weaker definition)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n) = \begin{cases} >0, & j \text{ Positive Recurrent} \\ =0, & j \text{ Null Recurrent} \end{cases}$$

$P_{i0}(n) \longrightarrow ?\ (i=0,1)$ diverge

$\frac{1}{N}\sum_{n=1}^{N} P_{i0}(n) \xrightarrow{N\to\infty} \frac{1}{2}$, Positive Recurrent

Weak Ergodic

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n) = \frac{1}{\sum_{k=1}^{\infty} k f_{ii}(k)}$$

Cessaro Sum

$f_{00}(k) = \{0, 1, 0, 0, \cdots\}$ $\sum_{k=1}^{\infty} k f_{00}(k) = 2$

Even if Transitional Prob. is not convergent. we can go to Cessaro Sum / Weak Ergodic (Pos./Null Recurrent)

How to calculate $\lim_{n\to\infty} P_{ij}(n)$? (if limit does exist)

$\begin{cases} P(n) = P(1)P(n-1) & \text{后退方程} \\ P(n) = P(n-1)P(1) & \text{前进方程} \end{cases}$

Suppose $P_{ij}(n) \xrightarrow{n} \Pi_j$

前进方程更本质!

$$P(n) \to \begin{bmatrix} \Pi_1 & \Pi_2 & \cdots & \Pi_n \\ \Pi_1 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \Pi_1 & \Pi_2 & \cdots & \Pi_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [\Pi_1 \cdots \Pi_n]$$

$\begin{cases} \Pi = P(1)\cdot\Pi = P(1)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}[\Pi_1, \cdots, \Pi_n] \\ \Pi = \Pi\cdot P(1) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}[\Pi_1 \cdots \Pi_n] P(1) \end{cases}$

行和为1, trivial

$\left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - P(1)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right)(\Pi_1, \cdots, \Pi_n) = 0$

$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\left([\Pi_1 \cdots \Pi_n] - [\Pi_1 \cdots \Pi_n]P(1)\right) = 0$ ✓

Let $\Pi = (\Pi_1, \cdots, \Pi_n)$ row vector

$\Pi = \Pi P$

not always exist

$\Pi \to$ Limit Distribution ($\sum_k \Pi_k = 1$, finite states)

$\Pi = \Pi P$

Limits don't exist

$\Pi = \Pi P \to \lim_{N\to\infty} \frac{1}{N}\sum P_{ij}(n)$

i) Stationary Distribution.

$P(X_0 = k) = \Pi_k \Rightarrow P(X_n = k) = \Pi_k,\ \forall n$

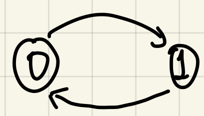
$$P(X_n = k) = \sum_l P(X_n = k, X_{n-1} = l) = \sum_l P(X_{n-1} = l)\,P(X_n = k \mid X_{n-1} = l) = \sum_l P_{lk}\,P(X_{n-1} = l)$$

Suppose $P(X_0 = k) = \Pi_k \Rightarrow \overset{\Pi_k}{P(X_1 = k)} = \sum_l P_{lk} P(X_0 = l) = \sum_l P_{lk}\Pi_l$

$$\Pi = \Pi P \Rightarrow [\Pi_1 \cdots \Pi_k \cdots \Pi_n] = [\Pi_1 \cdots \Pi_n]\begin{bmatrix} P_{11} & \cdots & P_{1k} & \cdots & P_{1n} \\ \vdots & & \vdots & & \vdots \\ P_{n1} & \cdots & P_{nk} & \cdots & P_{nn} \end{bmatrix} \Rightarrow \forall k \in [1,n],\ \Pi_k = \sum_i \Pi_i \cdot P_{ik} \Rightarrow P(X_1 = k) = \Pi_k$$

if Limit Distribution does not exist, what's the implication of Π ($\Pi = \Pi P$)?

$\Rightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n)$



$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{cases} \Pi = \Pi P \\ \sum \Pi_i = 1 \end{cases} \Rightarrow \Pi = [\frac{1}{2}, \frac{1}{2}] = \lim_{N\to\infty} \frac{1}{N}\sum P_{ij}(n)$

$\downarrow$

Π_j does not exist

ii) $\{\exists \pi \neq 0,\ \pi = \pi P\} \iff \{\exists \pi \neq 0,\ P^T\pi^T = \pi^T\} \iff \{\exists \pi \neq 0,\ \underbrace{(P-I)^T}_{\text{Singular}}\pi^T = 0\} \iff \det(P-I)=0 \iff 1 = \operatorname{eig}(P)$

$\{\exists \lambda = 1,\ \det(\lambda I - P) = 0\}$

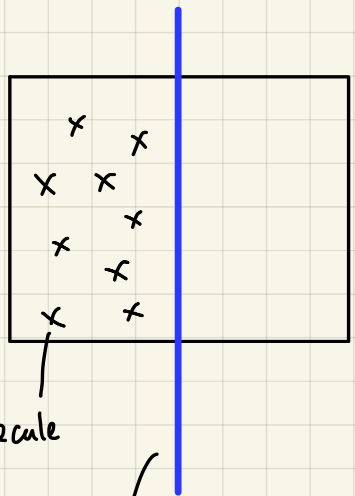
$\Uparrow$

$\left\{P \cdot \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = I \cdot \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right\}$

行和为1 $\Rightarrow$ 每行减1，一定奇异

$\sum_j P_{ij}(1) = 1$

Example: Ehrenfest Model (Markov Chain: characterize molecular motion)



molecule

撤去板子，如何演化?

i) $S = \{1, 2, \cdots, N\}$ # of particles. State Space: # of particles in left half space.

ii) Discrete Time: only 1 particle change its state in a time unit.

X_n: # of particles in left half on time n

$\{X_n\}$: Markov Chain

左半边的粒子个数

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & \cdots & N \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ N \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \frac{1}{N} & 0 & \frac{N-1}{N} & & \vdots \\ & \frac{2}{N} & 0 & \frac{N-2}{N} & \\ \vdots & & \ddots & & \\ & & \frac{N-1}{N} & 0 & \frac{1}{N} \\ 0 & \cdots & & 1 & 0 \end{bmatrix} \end{array}$$

Transition Prob. P

Assumption: equal chance of moving to the other side.

$\pi = \pi P$

$P^T\pi^T = \pi^T$

$$\begin{bmatrix} 0 & \frac{1}{N} & & & \\ 1 & 0 & \frac{2}{N} & & \\ & \frac{N-1}{N} & 0 & \ddots & \\ & & \frac{N-2}{N} & \ddots & \frac{N-1}{N} \\ & & & \ddots & 0 & 1 \\ & & & & \frac{1}{N} & 0 \end{bmatrix} \pi^T = \pi^T$$

$$P - I = \begin{bmatrix} -1 & \frac{1}{N} & & & \\ 1 & -1 & \frac{2}{N} & & \\ & \frac{N-1}{N} & -1 & \frac{3}{N} & \\ & & \ddots & \ddots & \\ & & \frac{2}{N} & -1 & 1 \\ & & & \frac{1}{N} & -1 \end{bmatrix}$$

$\pi_0 = \frac{1}{N}\pi_1 \quad (\pi_1 = \frac{N}{1}\pi_0) \qquad \pi_1 = \binom{N}{1}\pi_0$

$\pi_0 + \frac{2}{N}\pi_2 = \pi_1 \Rightarrow \pi_2 = \frac{N-1}{2}\pi_1 \qquad \pi_2 = \frac{N(N-1)}{2!}\pi_0 = \binom{N}{2}\pi_0$

$\frac{N-1}{N}\pi_1 + \frac{3}{N}\pi_3 = \pi_2 \Rightarrow \pi_3 = \frac{N-2}{3}\pi_2 \qquad \pi_3 = \binom{N}{3}\pi_0$

$\vdots$

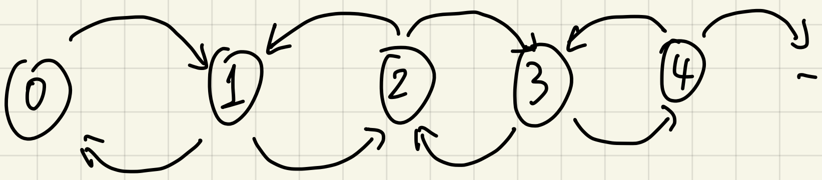
$\cdots\cdots \Rightarrow \pi_k = \frac{N-k+1}{k}\pi_{k-1}$

$\pi_N = \frac{1}{N}\pi_{N-1} \qquad \pi_N = \binom{N}{N}\pi_0$

$\therefore \pi_0 = \frac{1}{\sum_{k=0}^{N}\binom{N}{k}} = \frac{1}{2^N}$

$\therefore i \in [0, N],\ \pi_i = \frac{\binom{N}{i}}{\sum_{k=0}^{N}\binom{N}{k}} = \frac{\binom{N}{i}}{2^N}$

$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n)$

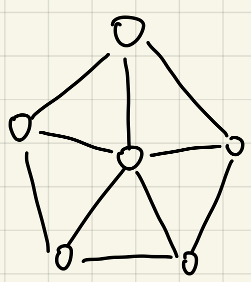


$\cdots\cdots$ $d=2$, periodic $\therefore$ Limit Distribution does not exist.

$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{i0}(n) = \frac{1}{\sum_{k=1}^{\infty} k f_{00}(k)} = \pi_0 = \frac{1}{2^N}$ $\Rightarrow$ Expectation of 1st Passage $= 2^N$，很漫长

N: 阿伏加德罗常数.

Example: Random Walk on Graph. $\Rightarrow$ PageRank Algorithm (Google)



degree: 边数 d_i, prob. of each vertex $= \frac{1}{d_i}$

N nodes. $\pi_k = \frac{d_k}{\sum_i d_i}$. Examine whether $\pi = \pi P$

$\sum_{k=1}^{N} \pi_k P_{kl} = \sum_{k=1}^{N} \frac{d_k}{\sum_i d_i} \cdot P_{kl}$

Uniform Dist. on node k

$$= \sum_{k=1,\, P_{k\ell}\neq 0}^{N} \frac{d_k}{\sum_i d_i} \cdot \frac{1}{d_k} \quad (P_{k\ell} = 0 \cdot \frac{1}{d_k}, \text{ non-zero term } \# = d_\ell)$$

$$= d_\ell \cdot \frac{d_k}{\sum_i d_i} \cdot \frac{1}{d_k} = \frac{d_\ell}{\sum_i d_i} = \pi_\ell \qquad \text{End of Proof.}$$

Detailed Balance Relation: $\pi = (\pi_0, \cdots, \pi_k, \cdots)$, $\pi_i P_{ij} = \pi_j P_{ji}$

$$\sum_k \pi_k P_{k\ell} = \sum_k \pi_\ell P_{\ell k} = \pi_\ell$$, Markov Chain Monte Carlo

Construct P from π:

i) $\forall P_{ij}$, $\pi_i P_{ij} \overset{?}{=} \pi_j P_{ji}$ $\quad \tilde{P}_{ij} = P_{ij} \min\left(1, \frac{\pi_j P_{ji}}{\pi_i P_{ij}}\right)$ → Satisfy Detailed Balance Relation.

$$\pi_i \tilde{P}_{ij} = \pi_i P_{ij} \min\left(1, \frac{\pi_j P_{ji}}{\pi_i P_{ij}}\right) = \min(\pi_i P_{ij}, \pi_j P_{ji}) = \pi_j P_{ji} \min\left(1, \frac{\pi_i P_{ij}}{\pi_j P_{ji}}\right) = \pi_j \tilde{P}_{ji}$$

Metropolis - Hastings Algorithms.

MCMC

Next Lecture: Continuous - Time Markov Chain

Birth & Death Process