

Non-Stationary Processes

- 1) Cyclostationary
- 2) Orthogonal Increment

$$X(t) \text{ w.s.s.} \Leftrightarrow E[X(t)] = m, \quad E[X(t)X(s)] = R_X(t-s)$$

$$R_X(t+T, s+T) = R_X(t, s), \quad \forall T \in \mathbb{R} \quad (\text{w.s.s.}) \xrightarrow[\text{condition}]{\text{relax this}} \quad R_X(t+T, s+T) = R_X(t, s). \quad \exists T \in \mathbb{R} \quad (\text{Cyclo-Stationary})$$

$$R_X(t+mT, s+mT) = R_X(t, s). \quad \forall m$$

Relation between WSS & cyclostationary:

Randomization $\Theta \sim U(0, T)$. $\Theta, X(t)$ indep.

$$Y(t) = X(t + \Theta) : \quad X(t) \text{ Cyclo} \Rightarrow Y(t) \text{ w.s.s.}$$

$$R_Y(t, s) \approx E[Y(t)Y(s)] = E[X(t+\Theta)X(s+\Theta)] = E_{\Theta} \left[E_X[X(t+\Theta)X(s+\Theta) | \Theta=\theta] \right] = E_{\Theta} [R_X(t+\Theta, s+\Theta)]$$

$$\underline{\underline{\Theta \sim U(0, T)}} \quad \frac{1}{T} \int_0^T R_X(t+\theta, s+\theta) d\theta$$

$$\begin{aligned} \forall T', \quad R_Y(t+T', s+T') &= \frac{1}{T} \int_0^T R_X(t+T'+\theta, s+T'+\theta) d\theta \xrightarrow{\theta=T+\theta'} \frac{1}{T} \int_{T'}^{T+T'} R_X(t+\theta', s+\theta') d\theta' \\ &= \frac{1}{T} \int_0^T R_X(t+\theta', s+\theta') d\theta' = R_Y(t, s) \end{aligned}$$

$\Rightarrow Y(t)$ w.s.s.

Constellation (Phase + Amp. \rightarrow Symbol / Bitstream)

PAM (Pulse Amplitude Modulation) ^{Complex} BPSK, QPSK, QAM

$$X(t) = \sum_{-\infty}^{+\infty} \alpha_k \phi(t-kT) \quad \alpha_k: \text{r.v. Information Symbol}$$

$\phi(t)$: Baseband Waveform.

$$\beta_0 = 0$$

$$R_X(t, s) = E[X(t)X(s)] = E \left[\sum_k \sum_n \alpha_k \alpha_n \phi(t-kT) \phi(s-nT) \right] = \sum_k \sum_n E[\alpha_k \alpha_n] \phi(t-kT) \phi(s-nT)$$

$$\{\alpha_n\} \text{ WSS.} \rightarrow R_\alpha(k-n) \Rightarrow \sum_k \sum_n R_\alpha(k-n) \cdot \phi(t-kT) \phi(s-nT)$$

Generally: $R_X(t+T', s+T') \neq R_X(t, s)$

$$\text{But when } T' = T, \quad R_X(t+T, s+T) = \sum_k \sum_n R_\alpha(k, n) \phi(t+T-kT) \phi(s+T-nT) \xrightarrow[k'=k-1]{n'=n-1} \sum_{k', n'} R_\alpha(k'+1, n'+1) \phi(t-kT) \phi(s-nT)$$

$\phi(t-kT)$

$R_\alpha(k'-n')$

$X(t)$: Cyclostationary Stochastic processes.

$$\begin{matrix} || \\ R_X(t, s) \end{matrix}$$

$$\bar{X}(t) = X(t + \Theta), \quad (\Theta) \sim U(0, T)$$

$$R_{\bar{X}}(t, s) = \frac{1}{T} \int_0^T R_X(t+\theta, s+\theta) d\theta = \frac{1}{T} \int_0^T \sum_k \sum_n R_\alpha(k-n) \phi(t-kT+\theta) \phi(s-nT+\theta) d\theta$$

$$k' = k-n, \quad n' = n$$

$$= \frac{1}{T} \int_0^T \sum_{k'} \sum_{n'} R_\alpha(k') \phi(t+\theta-(n'+k')T) \phi(s+\theta-n'T) d\theta$$

$$\text{Let } \theta' = \theta - n'T : \quad \frac{1}{T} \sum_{k'} \sum_{n'} R_\alpha(k') \int_{-n'T}^{-(n'-1)T} \phi(t+\theta'-kT) \phi(s+\theta') d\theta' \quad \begin{matrix} \text{Time Correlation:} \\ \uparrow \end{matrix} \quad R_X(T) = \int_{IR} X(t+\tau) X(t) dt$$

$$= \frac{1}{T} \sum_{k'} R_\alpha(k') \cdot \sum_{n=-\infty}^{+\infty} \int_{-n'T}^{-(n'-1)T} \phi(t+\theta'-kT) \phi(s+\theta') d\theta' = \frac{1}{T} \sum_{k'} R_\alpha(k') \underbrace{\int_{IR} \phi(t+\theta'-kT) \phi(s+\theta') d\theta'}_{\theta'' = s+\theta'} \quad \theta'' = s+\theta'$$

$$= \frac{1}{T} \sum_{k'} R_\alpha(k') \int_{IR} \phi(t-kT-s+\theta'') \phi(\theta'') d\theta''$$

$$= \frac{1}{T} \sum_{k'} R_\alpha(k') R_\phi(t-s-kT) = R_{\bar{X}}(t-s)$$

$$S_{\bar{X}}(w) = \int_{-\infty}^{+\infty} R_{\bar{X}}(\tau) \exp(-jw\tau) d\tau = \frac{1}{T} \int_{-\infty}^{+\infty} \left[\sum_k R_\alpha(k) R_\phi(\tau-kT) \right] \exp(-jw\tau) d\tau = \frac{1}{T} \sum_k R_\alpha(k) \cdot \int_{IR} R_\phi(\tau-kT) \exp(-jw\tau) d\tau$$

$$= \frac{1}{T} \sum_k R_\alpha(k) \exp(-jwkT) \int_{IR} R_\phi(\tau') \exp(-jw\tau') d\tau'$$

$$= \frac{1}{T} S_\alpha(w) \cdot S_\phi(w) = \frac{1}{T} S_\alpha(w) |\Phi(w)|^2$$

$$Z(t) = \sum_k \alpha_k \delta(t-kT) \quad Z(t) \rightarrow \boxed{\phi(t)} \rightarrow ? \quad Z(t) * \phi(t) = \sum \alpha_k \phi(t-kT)$$

2) Orthogonal Increment

$$X(t), \quad \forall t_1 < t_2 \leq t_3 < t_4, \quad X(t_4) - X(t_3) \perp X(t_2) - X(t_1) : \quad E[(X(t_4) - X(t_3))(X(t_2) - X(t_1))] = 0$$

$$R_X(t, s) = E[X(t)X(s)] \xrightarrow[\substack{s < t \\ X(0)=0}]{} E[(X(t) - X(s) + X(s) - X(0))(X(s) - X(0))] = E[X^2(s)]$$

$$R_X(t, s) = E[X^2(\min(t, s))] = g(\min(t, s)) \rightarrow \text{Characteristic Property of Orthogonal Increment.}$$

i.e. if $R_X(t, s) = g(\min(t, s))$, $X(t)$ has Orthogonal Increment
($\forall t, s$)

$$E[(X(t_4) - X(t_3))(X(t_2) - X(t_1))] = R_X(t_4, t_2) + R_X(t_3, t_1) - R_X(t_3, t_2) - R_X(t_4, t_1)$$

$$\begin{aligned} t_4 > t_3 > t_2 > t_1 \quad & \text{if } R_X(t, s) = g(\min(t, s)), \text{ then} \\ & = g(t_2) + g(t_1) - g(t_3) - g(t_4) = 0 \end{aligned}$$

\therefore Orthogonal Increment

Example of Orthogonal Increment: Brownian Motion

def. of Brownian Motion:

- ① $B(0) = 0$ ② Orthogonal Increment ③ $B(t) - B(s) \sim N(0, 6^2(t-s))$

$$R_B(t,s) = E(B^2(s)) = 6^2 s = 6^2 \min(t,s)$$

$$Y(t) = \frac{d}{dt} B(t) \text{ (Not Rigorous)}$$

$$B(s) - B(0) \sim N(0, 6^2(s-0)) = N(0, 6^2 s)$$

$$R_Y(t,s) = E[Y(t) Y(s)]$$

$$E(B(s)) = 0, E(B^2(s)) = \text{var}(B(s)) = 6^2 s$$

$$= E\left[\frac{d}{dt} Y(t) \frac{d}{ds} Y(s)\right]$$

$$= \frac{\partial^2}{\partial t \partial s} E[Y(t) Y(s)]$$

$$= \frac{\partial^2}{\partial t \partial s} R_B(t,s) = 6^2 \frac{\partial^2}{\partial t \partial s} \min(t,s) = 6^2 \frac{\partial^2}{\partial t \partial s} \left[\frac{1}{2}(s+t-|s-t|) \right]$$

$$= -\frac{1}{2} 6^2 \frac{\partial^2}{\partial t \partial s} |t-s|$$

$$R_Y(t,s) = -\frac{6^2}{2} \frac{\partial^2}{\partial t \partial s} |t-s| \quad \left(\frac{d}{dx} |x| = \text{sgn}(x) \right)$$

$$= -\frac{6^2}{2} \frac{\partial}{\partial s} \left(\frac{\partial}{\partial t} |t-s| \right)$$

$$= -\frac{6^2}{2} \frac{\partial}{\partial s} (\text{sgn}(t-s)) \quad \left(\frac{d}{dx} \text{sgn}(x) = 2\delta(x), \quad \frac{d}{dx} \text{sgn}(x) = 2\delta(x) \right)$$

$$= -\frac{6^2}{2} [2\delta(t-s)]$$

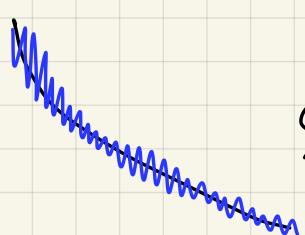
$$= 6^2 \delta(t-s)$$

derivative is $\delta(\cdot)$ function \Rightarrow White Noise

(Brownian Motion)

$$V(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \frac{dV(x)}{dx} = \delta(x), \quad \text{sgn}(x) = V(x) - V(-x) \quad \left(\frac{d}{dx} V(x) = \delta(x) \right)$$

Orthogonal Increment $\xrightarrow{\text{differential}}$ W.S.S.



Stationary lies in

~~envelope~~ (Trend \Rightarrow Non-Stationary)

glitches

(Brownian Motion $\xrightarrow{\text{Derivative}} \text{W.S.S.}$)

High-Pass Filter

Isometry with Complex Exponentials
(Oscillating)

\therefore W.S.S. \rightarrow High-Frequency

- ① $B(0) = 0$ ② Orthogonal Increment ③ $B(t) - B(s) \sim N(0, 6^2(t-s))$

Consider function $f(t, B(t))$, $df(t, B(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB(t)$

$$df(t, B(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} (dB)^2$$

Not Quite Right.

$\frac{\partial f}{\partial B} \approx \frac{1}{dt}$

$$(dt \cancel{\rightarrow} df) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} dt$$

It's Formula.

$$B(t) \sim N(0, 6^2 t)$$

$$dB(t) \sim N(0, 6^2 dt)$$

$$B(t+dt) - B(t) \sim N(0, 6^2 dt)$$

$$\Rightarrow E[dB(t)] = 6^2 dt$$

$$(dB(t))^2 \sim dt$$

$$dB(t) \sim (dt)^{1/2}$$

$\therefore dB \rightarrow$ has to go to 2nd order.

$$\int_0^1 B(t) dB(t) \neq \int_0^1 d\left(\frac{1}{2} B^2(t)\right)$$

$$d\left(\frac{1}{2} B^2(t)\right) = B(t) dB(t) + \frac{1}{2} dt$$

$$\therefore \int_0^1 B(t) dB(t) = \int_0^1 d\left(\frac{1}{2} B^2(t) - \frac{1}{2} t\right)$$

Stochastic Calculus \Rightarrow Option Pricing (期权定价)

$B(t)$ Brownian Motion \rightarrow Stock Price Modelling

$B(t)$ may be less than 0 \Rightarrow Stock Price: $\exp(B(t))$
 $\exp(\mu t + B(t))$

任何事都是有道理的，
区别是有没有必要知道它，
和有没有能力去知道它。

Everything in existence carries a purpose,

the distinction lies in whether
there is a necessity to comprehend it
and a capacity to do so.

Option Pricing: Stock price: $\exp(\mu t + \beta(t))$ Geometric Brown Motion Samuelson (Economics)

$V(t, S(t))$ \rightarrow Portfolio (Heritage) Sell Stock for Option

$$V(t, S(t)) - \alpha S(t) = P(t) \text{ (Property)}$$

$$dP(t) \geq \text{interest rate} \times P(t) = rP(t) \quad (dP(t) = rP(t)dt)$$

how to set proper V ? $dP(t) = dV(t, S(t)) - \alpha dS(t)$ $(S(t) = \exp(\mu t + \beta(t)),$

$$dP = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \cdot (dS)^2 - \alpha dS \leftarrow dS = S \cdot \mu dt + S(t) dB(t) + \frac{1}{2} S \cdot dt$$

$$= r(V - \alpha S) dt$$

$$\Rightarrow \left(\frac{\partial V}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(\frac{\partial V}{\partial S} - \alpha \right) dS = r(V - \alpha S) dt$$

Random Term.

$$\Rightarrow \begin{cases} \alpha = \frac{\partial V}{\partial S} \text{ (Heritage)} \\ \frac{\partial V}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} = r(V - \frac{\partial V}{\partial S} \cdot S) \end{cases} \quad \text{i.e. } \frac{\partial V}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad \text{Black-Scholes Equation.}$$

2023 Spring Non-Stationary Processes

Strict-Sense Stationary: $\forall t_1, t_2, \dots, t_n, \forall T \quad (X(t_1), \dots, X(t_n))^T \stackrel{F}{\sim} (X(t_1+T), \dots, X(t_n+T))^T$

$$(\text{W.S.S. : } E[X(t)] = \mu, R_X(t, s) = g(t-s))$$

Conditional Expectation:

$$X. \quad EX = c. \quad EX = \int x f_X(x) dx, \quad X \sim f_X(x)$$

$$X, Y, E[X|Y] \rightarrow \text{r.v.} \quad Y \rightarrow \text{provisionally deterministic} \quad E[X|Y]$$

$$E_{x,y}[g(x,y)] = E_{y|x}[E[g(x,y)|Y]]$$

Conditional expectation is a r.v.

我们从来不曾懂得过什么(爱)

我们只是不断学习

$$\{X_k\} \text{ i.i.d. } N \text{ r.v.}$$

$$E[X_1 + \dots + X_N] = E[E[X_1 + \dots + X_N | N]] = E[N E[X_k]] = E[X_k] \cdot E[N]$$

$X \rightarrow \text{data}, Y \rightarrow \text{target.}$ $\min_g E|Y - g(X)|^2 \Rightarrow g(X) = E[Y|X]$ optimal choice.

$$\forall g. E|Y - g(X)|^2 = E|Y - E[Y|X] + E[Y|X] - g(X)|^2 \quad \text{Prove. this is } 0$$

$$= E[Y - E[Y|X]]^2 + E[E[Y|X] - g(X)]^2 + 2 \underbrace{E[(Y - E[Y|X])(E[Y|X] - g(X))]}_{\approx 0}$$

$$E[(Y - E[Y|X])(E[Y|X] - g(X))] = E \left[E[(Y - E[Y|X])(E[Y|X] - g(X))|X] \right] = E[(E[Y|X] - g(X)) E[(Y - E[Y|X])|X]]$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$$= E[(E[Y|X] - g(X))(E[Y|X] - E[Y|X])] = 0$$

$$\Rightarrow E[Y - g(X)]^2 = E[Y - E[Y|X]]^2 + E[g(X) - E[Y|X]]^2$$

$$\Rightarrow g(X) = E[Y|X] \quad \text{misleading} \quad E[Y|X] \text{ is also a r.v.}$$

$$\text{In textbooks, } E[g(x, y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y) f_{XY}(xy) dx dy = \underbrace{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x, y) f_{Y|X}(y|x) dy \right) f_X(x) dx}_{\text{misleading}}$$