Gaussian & Nonlinearity

① Square Law:

$X(t)$ Gaussian $\longrightarrow \boxed{x^2} \longrightarrow Y(t) = X^2(t) \geq 0$, not Gaussian.

$$E[Y(t)] = EX^2 = R_X(t,t) = R_X(0)$$

$R_Y(t,s) = E[Y(t)Y(s)] = E[X^2(t)X^2(s)]$ High-Order Moments (Gaussian) → did $E[X^n]$ in 9th lecture

we want to make it more general

to calculate moments between different r.v.s

Example: $(X_1, X_2, X_3, X_4) \sim N$, $E[X_k] = 0$, $E[X_1X_2X_3X_4] = ?$

$$E[X_1X_2X_3X_4] = \int x_1x_2x_3x_4 \, f_{X_1X_2X_3X_4} \, dx_1 dx_2 dx_3 dx_4$$

more ideal than Σ^{-1}

approach moments from $\phi_X(w)$ (Σ) rather than distribution (Σ^{-1}) ???

$$\exp\left(-\frac{1}{2}(x_1, x_2, x_3, x_4)\Sigma^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) \Rightarrow \phi_X(w) = \exp\left(jw^T\mu - \frac{1}{2}w^T\Sigma w\right) \xlongequal{E[X]=0} \exp\left(-\frac{1}{2}w^T\Sigma w\right)$$

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$$X = (X_1, \cdots, X_n)^T \Rightarrow \phi_X(w) = E[\exp(j(w_1X_1 + \cdots + w_nX_n))]$$

$$EX_1 = \frac{1}{j}\frac{\partial}{\partial w_1}\phi_X(w_1 \cdots w_n)\Big|_{w_1 = \cdots = w_n = 0}$$

Moment

Distribution	Characteristic Function
⇓	⇓
Integral	Differential (Easier)

$$\therefore EX_k^\alpha = \frac{1}{j^\alpha}\frac{\partial^\alpha}{\partial w_k^\alpha}\phi_X(w_1 \cdots w_n)\Big|_{w_1 = \cdots = w_n = 0}$$

$$E[X_1^{\alpha_1} \cdots X_n^{\alpha_n}] = \frac{1}{j^{\alpha_1 + \cdots + \alpha_n}} \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial w_1^{\alpha_1} \cdots \partial w_n^{\alpha_n}} \phi_X(w_1 \cdots w_n)\Big|_{w_1 = \cdots = w_n = 0}$$

$$E[X_1X_2X_3X_4] = j^{-4}\frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\phi_X(w)\Big|_{w=0} = \cdots = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$$

$$\frac{\partial^4}{\partial x_1\partial x_2\partial x_3\partial x_4}E[\exp(jw^TX)] = \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\exp\left(jw^T\mu_X - \frac{1}{2}w^T\Sigma w\right) \xlongequal[\text{normalized}]{\text{Suppose } \mu = 0} \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\exp\left(-\frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4 w_i\Sigma_{ij}w_j\right)$$

$$= \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\exp\left(-\frac{1}{2}\sum_i w_i^2\Sigma_{ii} - \sum_{i<j}w_iw_j\Sigma_{ij}\right) = \frac{\partial^3}{\partial w_2\partial w_3\partial w_4}\left(-\sum_i w_i\Sigma_{i1}\right)\exp\left(-\frac{1}{2}\sum_i w_i^2\Sigma_{ii} - \sum_{i<j}w_iw_j\Sigma_{ij}\right)$$

$$= \frac{\partial^2}{\partial w_3\partial w_4}\left[-\Sigma_{21} + \left(-\sum_i w_i\Sigma_{i1}\right)\cdot\left(-\sum_j w_j\Sigma_{j2}\right)\right]\exp(\cdots) = \frac{\partial^2}{\partial w_3\partial w_4}\left[-\Sigma_{21} + \sum_i w_i\Sigma_{i1}\cdot\sum_j w_j\Sigma_{j2}\right]\exp(\cdots)$$

$$= \frac{\partial}{\partial w_4}\left[\Sigma_{31}\cdot\sum_j w_j\Sigma_{j2} + \Sigma_{32}\cdot\sum_i w_i\Sigma_{i1} + \left(-\Sigma_{21} + \sum_i w_i\Sigma_{i1}\cdot\sum_j w_j\Sigma_{j2}\right)\left(-\sum_k w_k\Sigma_{k3}\right)\right]\exp(\cdots)$$

Since we substitute (w_1, w_2, w_3, w_4) with $(0, 0, 0, 0)$, we only search 1st-order w_4 terms.

$$E[X_1X_2X_3X_4] = \Sigma_{31}\cdot\Sigma_{42} + \Sigma_{32}\Sigma_{41} + (-\Sigma_{21})\cdot(-\Sigma_{43}) = \Sigma_{12}\Sigma_{34} + \Sigma_{13}\Sigma_{24} + \Sigma_{14}\Sigma_{23}$$ Proved!

$$E[X^2(t)X^2(s)] = R_X(t,t)R_X(s,s) + 2\cdot R_X^2(t,s) \xrightarrow{\text{if } X \text{ is WSS}} R_X^2(0) + 2\cdot R_X^2(t-s), \text{ PSD: } \cdots$$

② Sign Function

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

Gaussian

$$X(t) \to \boxed{\operatorname{sgn}(x)} \to Y(t)$$

$$E[X(t)] = 0$$

$$EY = 1 \cdot P(X(t) \geq 0) + (-1) P(X(t) < 0) = 0 \quad (f_X : \text{even function})$$

$$R_Y(t,s) = E[Y(t)Y(s)] = 1\, P(Y(t)Y(s) = 1) + (-1)\, P(Y(t)Y(s) = -1) = 2P(Y(t)\cdot Y(s) = 1) - 1 = 2P(X(t)X(s) \geq 0) - 1$$

$$P(X(t)X(s) \geq 0) = \left(\int_{-\infty}^{0}\int_{-\infty}^{0} + \int_{0}^{\infty}\int_{0}^{\infty}\right) \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1}{\sigma_1}\right)^2 + \left(\frac{x_2}{\sigma_2}\right)^2 - 2\rho\frac{x_1x_2}{\sigma_1\sigma_2}\right)\right] dx_1 dx_2$$

let $x_1' = \frac{x_1}{\sigma_1}$, $x_2' = \frac{x_2}{\sigma_2}$

$$= \left(\int_{-\infty}^{0}\int_{-\infty}^{0} + \int_{0}^{\infty}\int_{0}^{\infty}\right) \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x_1'^2 + x_2'^2 - 2\rho x_1'x_2'\right)\right) dx_1' dx_2'$$

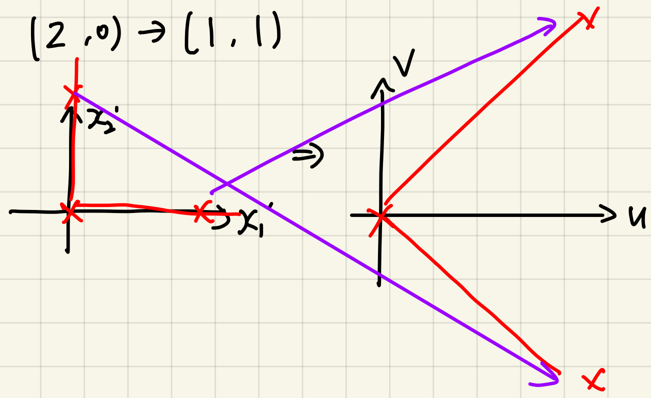
$$= 2\cdot \int_0^\infty\int_0^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x_1'^2 + x_2'^2 - 2\rho x_1'x_2'\right)\right) dx_1' dx_2'$$

看关键点

linear transform: $(0,0) \to (0,0)$

$(0,2) \to (1,-1)$

$(2,0) \to (1,1)$



let $\begin{cases} x_1' = u + v \\ x_2' = u - v \end{cases}$ $\quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$, $|J| = \left|\frac{\partial(x_1', x_2')}{\partial(u,v)}\right| = \left|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right| = 2$

$$= 2\cdot\int_0^\infty\int_{-u}^{u} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(2u^2 + 2v^2 - 2\rho(u^2 - v^2)\right)\right) 2\,du\,dv$$

$$= 4\int_0^\infty\int_{-u}^{u} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{1-\rho^2}\left((1-\rho)u^2 + (1+\rho)v^2\right)\right) du\,dv$$

$$= 4\int_0^\infty\int_{-u}^{u} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2}{1+\rho} - \frac{v^2}{1-\rho}\right) du\,dv$$

let $u' = \frac{u}{\sqrt{1+\rho}}$, $v' = \frac{v}{\sqrt{1-\rho}}$,

$(1,1) \to \left(\frac{1}{\sqrt{1+\rho}}, \frac{1}{\sqrt{1-\rho}}\right)$

$(1,-1) \to \left(\frac{1}{\sqrt{1+\rho}}, \frac{-1}{\sqrt{1-\rho}}\right)$

$$= 4\int_0^\infty\int_{-\sqrt{\frac{1-\rho}{1+\rho}}u'}^{\sqrt{\frac{1-\rho}{1+\rho}}u'} \frac{1}{2\pi} \exp(-u'^2 - v'^2)\, du'dv'$$

polar coord. $\begin{cases} u' = r\cos\theta \\ v' = r\sin\theta \end{cases}$

$$= 4\int_0^\infty\int_{-\phi}^{\phi} \frac{1}{2\pi} \exp(-r^2)\, r\,dr\,d\theta \quad \left(\phi = \tan^{-1}\sqrt{\frac{1+\rho}{1-\rho}}\right)$$

$$= 4\cdot\frac{1}{2\pi}\cdot 2\phi\cdot\frac{1}{2} = \frac{2\phi}{\pi}$$

$$\rho = \frac{E[X(t)X(s)]}{[EX^2(t)EX^2(s)]^{1/2}} = \frac{R_X(t-s)}{R_X(0)}$$

$$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

$\tan^2\phi = \frac{1+\rho}{1-\rho}$, $\quad 1 + \tan^2\phi = \frac{2}{1-\rho}$

$$\rho = 1 - \frac{2}{1+\tan^2\phi} \Rightarrow \frac{1-\tan^2\phi}{1+\tan^2\phi} = -\rho$$

$\therefore \cos 2\phi = -\rho$, $\quad 2\phi = \cos^{-1}(-\rho)$

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

$$\Rightarrow 2\phi = \cos^{-1}(-\rho) = \frac{\pi}{2} - \sin^{-1}(-\rho) = \frac{\pi}{2} + \sin^{-1}(\rho)$$

$$\therefore P(X(t)X(s) \geq 0) = \frac{1}{2} + \frac{\sin^{-1}(\rho)}{\pi}$$

"Arcsin Law"

$$\begin{cases} R_Y(t,s) = 2\cdot\left[\frac{1}{2} + \frac{\sin^{-1}(\rho)}{\pi}\right] - 1 = \frac{2}{\pi}\sin^{-1}(\rho) \\ \rho = \frac{R_X(t-s)}{R_X(0)} \end{cases}$$

Useful Tool dealing with Gaussian Linearity:

$g(x_1, x_2)$ general non-linear function, $(X_1, X_2) \sim N(0, 0, \sigma_1^2, \sigma_2^2, \rho)$

$$\frac{\partial E[g(X_1, X_2)]}{\partial\rho} = \sigma_1\sigma_2\, E\left[\frac{\partial^2 g(X_1, X_2)}{\partial X_1\,\partial X_2}\right]$$

Price Theorem

Use Price Thm. to solve $E[\mathrm{sgn}(X(t))\,\mathrm{sgn}(X(s))]$ 07-11

① $g(x_1, x_2) = \mathrm{sgn}(x_1)\,\mathrm{sgn}(x_2)$ ② $\dfrac{\partial^2 g(x_1, x_2)}{\partial x_1 \partial x_2} = 2\delta(x_1)\cdot 2\delta(x_2)$

③ $E\left[\dfrac{\partial^2 g(X_1, X_2)}{\partial X_1 \partial X_2}\right] = \iint 4\delta(x_1)\delta(x_2)\,\dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[\dfrac{-1}{2(1-\rho^2)}\left(\dfrac{x_1^2}{\sigma_1^2}+\dfrac{x_2^2}{\sigma_2^2}-2\rho\left(\dfrac{x_1}{\sigma_1}\right)\left(\dfrac{x_2}{\sigma_2}\right)\right)\right] dx_1 dx_2$

$$= \frac{2}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$\Rightarrow \frac{\partial E[g(X_1, X_2)]}{\partial \rho} = \sigma_1\sigma_2\cdot E\left[\frac{\partial^2 g(X_1, X_2)}{\partial X_1 \partial X_2}\right] = \frac{2}{\pi}\frac{1}{\sqrt{1-\rho^2}}$$

$$\Rightarrow E[g(X_1, X_2)] = \int_0^\rho \frac{2}{\pi}\frac{1}{\sqrt{1-\rho^2}}\,d\rho + \underbrace{E[g(X_1, X_2)]\Big|_{\rho=0}}_{E[\mathrm{sgn}(X_1)\cdot\mathrm{sgn}(X_2)]=0} = \frac{2}{\pi}\sin^{-1}\rho$$

Apply Price Thm. to 1st Example: $E[X^2(t)\,X^2(s)]$

$g(X_1, X_2) = X_1^2 X_2^2$ $\quad \dfrac{\partial^2 g}{\partial X_1 \partial X_2} = 4X_1X_2$ $\quad E\left[\dfrac{\partial^2 g}{\partial X(t)\partial X(s)}\right] = 4E[X(t)X(s)] = 4R_X(t-s)$

$$\begin{cases} \sigma_1^2 = EX_1^2 = E[X^2(t)] = R_X(0) \\ \sigma_2^2 = R_X(0) \end{cases}$$

$$\therefore \frac{\partial E[g]}{\partial \rho} = \sigma_1\sigma_2\, E\left[\frac{\partial^2 g}{\partial X(t)\partial X(s)}\right] = R_X(0)\cdot 4R_X(t-s) \xrightarrow{\rho = \frac{R_X(t-s)}{R_X(0)}} 4R_X^2(0)\cdot\rho$$

$$\therefore E[g] = \int_0^\rho 4R_X^2(0)\,\rho\, d\rho + \underbrace{E[g(X(t), X(s))]\Big|_{\rho=0}}_{E[X^2(t)]\,E[X^2(s)] = R_X^2(0)} = 2R_X^2(0)\rho^2 + R_X^2(0) = R_X^2(0) + 2R_X^2(t-s)$$

$$\mathrm{ReLU}(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Price Thm. can be applied to $\mathrm{ReLU}(x)$ as well. $\left(\dfrac{\partial \mathrm{ReLU}(x)}{\partial x} = u(x)\right)$

Prove Price Theorem:

$$E[g(X_1, X_2)] = \int_{\mathbb{R}}\int_{\mathbb{R}} g(x_1, x_2)\cdot\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}-2\rho\frac{x_1x_2}{\sigma_1\sigma_2}\right)\right] dx_1 dx_2$$

$\dfrac{\partial E[g(X_1, X_2)]}{\partial \rho}$ can't be computed directly from $E[g(X_1, X_2)]$, since ρ is all mingled together.

Complexity comes from Σ^{-1} (inversion of covariance matrix)

To obviate such complexity, we should harness characteristic function $\phi_X(w) = E[\exp(jw^TX)]$

$$\phi(w_1, w_2) = \exp\left(-\frac{1}{2}(w_1, w_2)\, R \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \exp\left(-\frac{1}{2}(\sigma_1^2 w_1^2 + \sigma_2^2 w_2^2 + 2\rho\sigma_1\sigma_2 w_1 w_2)\right)$$

ρ's differential is doable

$$E[g(X_1,X_2)] = \int_{\mathbb{R}}\int_{\mathbb{R}} g(x_1,x_2) \cdot \left[\iint_{\mathbb{R}^2} \phi_{X_1X_2}(w_1,w_2) \exp(-j(w_1x_1+w_2x_2))\, dw_1 dw_2\right] dx_1 dx_2$$

$$\frac{\partial E[g(X_1,X_2)]}{\partial \rho} = \iint_{\mathbb{R}^2} g(x_1,x_2) \cdot \left[\iint_{\mathbb{R}^2} -\sigma_1\sigma_2 w_1 w_2 \cdot \phi_{X_1X_2}(w_1,w_2) \exp(-j(w_1x_1+w_2x_2))\, dw_1 dw_2\right] dx_1 dx_2$$

$$= \sigma_1\sigma_2 \cdot \iint_{\mathbb{R}^2} g(x_1,x_2) \cdot \frac{\partial^2}{\partial x_1 \partial x_2}\left[\iint_{\mathbb{R}^2} \phi_{X_1X_2}(w_1,w_2) \exp(-j(w_1x_1+w_2x_2))\, dw_1 dw_2\right] dx_1 dx_2$$

$$= \sigma_1\sigma_2 \iint_{\mathbb{R}^2} g(x_1,x_2) \cdot \frac{\partial^2}{\partial x_1 \partial x_2} f_{X_1X_2}(x_1,x_2)\, dx_1 dx_2$$

Assumption: $\lim_{\substack{x_1\to\infty,\\ x_2\to\infty}} g(x_1,x_2) f(x_1,x_2) = 0$, 边缘是0

$\frac{\partial}{\partial x_i} g \cdot f = 0$, $g \cdot \frac{\partial}{\partial x_i} f = 0$ $(i=1,2)$

$$= \sigma_1\sigma_2 \iint_{\mathbb{R}^2} \left[\frac{\partial^2}{\partial x_1 \partial x_2} g(x_1,x_2)\right] \cdot f_{X_1X_2}(x_1,x_2)\, dx_1 dx_2$$

2x Integration by parts, $(-1)^2 = 1$ (minus sign goes away)

$$= \sigma_1\sigma_2\, E\left[\frac{\partial^2}{\partial X_1 \partial X_2} g(X_1,X_2)\right]$$

$$\therefore \frac{\partial E[g(X_1,X_2)]}{\partial \rho} = \sigma_1\sigma_2\, E\left[\frac{\partial^2}{\partial X_1 \partial X_2} g(X_1,X_2)\right]$$

Proved

Exercise:

$X(t) \rightarrow \boxed{\exp(X)} \rightarrow Y(t)$

Gaussian

$R_Y(t,s) = ?$

$\exp(X_1 X_2)\exp(-X_1^2 - X_2^2) \to 0$ (Satisfies Price Thm. assumption)

1) $g(X_1,X_2) = \exp(X_1)\exp(X_2)$ 2) $\frac{\partial^2 g}{\partial X_1 \partial X_2} = \exp(X_1)\exp(X_2)$ 3) $E\left[\frac{\partial^2 g}{\partial X_1 \partial X_2}\right] = \exp(X_1)\exp(X_2)$

4) $\frac{\partial E[g]}{\partial \rho} = E[g]$ 5) $E[g] = g_0 \exp(\rho)$

Through $\Phi_X(w)$:

$$E[\exp(X_1+X_2)] = \phi_X(w_1 = w_2 = -j) = \exp\left(j w^T \mu - \frac{1}{2} w^T \Sigma_X w\right) = \exp\left(\mu_1 + \mu_2 + \frac{1}{2}(\Sigma_{11} + \Sigma_{12} + \Sigma_{21} + \Sigma_{22})\right) \checkmark$$

Nonlinear with Gaussian

Polynomial: Partial Derivative on Charac. Func.

Triangular: Decompose into Complex Exponential (Charac. Func.)

ReLU: Price Thm.

Mostly Every Func. (satisfies its assumption) $\Longrightarrow$ Price Thm.