Gaussian Processes

$X(t)$ Gaussian Process : $\forall n, \forall t_1 \cdots t_n$ $\quad X = (X(t_1), \cdots, X(t_n))^T$, $X \sim N(\mu, \Sigma)$. $\begin{cases} \mu = E[X] \\ \Sigma = E[(X-\mu)(X-\mu)^T] \end{cases}$

$n=1$, $X \sim N(\mu, \sigma^2)$. $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

$n=2$, $X \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, $f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left((\frac{x_1-\mu_1}{\sigma_1})^2 + (\frac{x_2-\mu_2}{\sigma_2})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2})\right)\right]$

Matrix-Vector Notation. $X \in \mathbb{R}^n$, $X \sim N(\mu, \Sigma)$. $\mu \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$.

$$f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}} (\det\Sigma)^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right]$$

Examine $f_X(x)$ is indeed a distribution : $\quad f_X(x) \geq 0$. $\int_{\mathbb{R}^n} f_X(x)\,dx = 1$

i) $f_X(x) \geq 0$. Σ pos. def. $\Rightarrow \det\Sigma > 0$, $f_X(x) \geq 0$ $\Rightarrow$? $\frac{1}{2}\sum \alpha_k x_k^2 + \sum x_i x_j k_{ij}$

ii) $\int_{\mathbb{R}^n} f_X(x)\,dx = 1$, $\int_{\mathbb{R}^n} f_X(x)\,dx = \frac{1}{(2\pi)^{\frac{n}{2}} (\det\Sigma)^{-1}} \int_{\mathbb{R}^n} \exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu))\,dx$

Diagonalize Σ: Σ Covariance Matrix , $\Sigma = \Sigma'$. Σ pos. def.

$\begin{cases} \Sigma = U^T \Lambda U \text{ (Eigen-Decomposition)} \\ U U^T = U^T U = I \end{cases}$ $\quad \therefore \Sigma^{-1} = (U^T \Lambda U)^{-1} = U^{-1} \Lambda^{-1} (U^T)^{-1} = U^T \Lambda^{-1} U$

$\lambda_1 \sim \lambda_n \geq 0 \Rightarrow \sqrt{\lambda_k}$ exists

$= U^T \Lambda^{-1/2} \Lambda^{-1/2} U = B^T B$

$$\therefore \frac{1}{(2\pi)^{n/2} (\det\Sigma)^{1/2}} \int_{\mathbb{R}^n} \exp(-\frac{1}{2}\underbrace{(x-\mu)^T B^T}_{Y^T} \underbrace{B(x-\mu)}_{Y})\,dx \overset{Y = B(X-\mu)}{=\!=} \frac{1}{(2\pi)^{\frac{n}{2}} (\det\Sigma)^{1/2}} \int_{\mathbb{R}^n} \exp(-\frac{1}{2} Y^T Y) (\det\Sigma)^{1/2}\,dY$$

① ② ③

② Jacobian

$\begin{cases} dX = \frac{\partial X}{\partial Y} dY \\ \frac{\partial Y}{\partial X} = B \end{cases}$ $\quad dX = |\det B|^{-1} dY = \left|(\prod_k \lambda_k)^{-1/2}\right|^{-1} dY = (\prod_k \lambda_k)^{1/2} dY = (\det\Sigma)^{1/2} dY$

$\underbrace{\det(\Sigma^{-1})}_{= [\det(B)]^2} = \det(U) \cdot \det(\Lambda^{-1}) \det(U^T) = \det(\Lambda^{-1}) = (\prod_k \lambda_k)^{-1}$

③ Integration Range: still $\mathbb{R}^n$

$$\therefore \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(-\frac{1}{2}\sum_k y_k^2)\,dy_1 \cdots dy_n = \prod_k \underbrace{\left[\frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \exp(-\frac{1}{2}y_k^2)\,dy_k\right]}_{=1}$$

$= 1$ $\rightarrow$ is a distribution

$I = \int_{\mathbb{R}} \exp(-\frac{y^2}{2})\,dy$

$I^2 = \int_{\mathbb{R}^2} \exp(-\frac{x^2+y^2}{2})\,dx\,dy$ In polar coord.

$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp(-\frac{\rho^2}{2})\,\rho\,d\rho\,d\theta = 2\pi \cdot \int_0^{\infty} \exp(-\frac{\rho^2}{2})\,\rho\,d\rho$

$= 2\pi$

$\Rightarrow I = (2\pi)^{-1/2}$

n-dimensional Characteristic Function:

$X \in \mathbb{R}^n$ r.v. $\quad \phi_X(w) = E[\exp(jw^T X)]$

$w = (w_1, \cdots, w_n)^T$

$X \sim N(\mu, \Sigma)$, $\phi_X(w) = \int_{\mathbb{R}^n} \exp(jw^T x) \cdot f_X(x)\,dx = \frac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}} \int_{\mathbb{R}^n} \exp(jw^T x)\exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))\,dx$

Informal : 1-Dim

$$-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) + jw^T x \xrightarrow{\text{1-Dim}} -\frac{1}{2\sigma^2}(x-\mu)^2 + jwx = -\frac{1}{2\sigma^2}(x-\mu-j\sigma^2 w)^2 - \frac{1}{2}\sigma^2 w^2 + jw\mu$$

$$\underbrace{-\frac{1}{2}(x-\mu-j\Sigma w)^T \Sigma^{-1}(x-\mu-j\Sigma w)}_{=1} \underbrace{-\frac{1}{2}w^T \Sigma w + jw^T \mu}_{\text{indep. to } x.} \xleftarrow{\text{n-Dim}} -\frac{1}{2\sigma^2}(x-\mu-j\sigma^2 w)^2 - \frac{1}{2}\sigma^2 w^2 + jw\mu$$

remains

$$\frac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}} \int_{\mathbb{R}^n} \exp(-\frac{1}{2}(x-\mu-j\Sigma w)^T \Sigma^{-1}(x-\mu-j\Sigma w))\,dx = 1$$

$$\therefore \phi_X(w) = \exp(jw^T\mu - \frac{1}{2}w^T \Sigma w)$$

Linearity Property. $X \in \mathbb{R}^n$, $X \sim N(\mu, \Sigma)$, $A \in \mathbb{R}^{m\times n}$.
Universal.

$Y = AX \in \mathbb{R}^m \Rightarrow Y \sim N(A\mu, A\Sigma A^T)$

$$\phi_Y(w) = E[\exp(jw^T Y)] = E[\exp(jw^T AX)] = E[\exp(j(A^T w)^T X)] = \phi_X(A^T w) = \exp(j(A^T w)^T\mu - \frac{1}{2}(A^T w)^T \Sigma (A^T w))$$

$$= \exp(jw^T(A\mu) - \frac{1}{2}w^T(A\Sigma A^T)w) \Rightarrow \text{Gaussian Dist.}$$

$\therefore \mu_Y = A\mu$, $\Sigma_Y = A\Sigma A^T$, $Y \sim N(A\mu, A\Sigma A^T)$

$X = (X_1, \cdots, X_n)^T \sim N$, $\tilde{X} = (X_{n_1}, \cdots, X_{n_k})$, $(n_1, \cdots, n_k) \subseteq (1, \cdots, n)$ $\quad \tilde{X} \sim N$ ($\forall$ margin $\rightarrow$ normal dist)

$$\tilde{X} = \begin{bmatrix} X_{n_1} \\ \vdots \\ X_{n_k} \end{bmatrix} = \underbrace{\begin{bmatrix} \cdots & \cdots & 1 & \cdots \\ \cdots & 1 & \cdots & \cdots \\ & & & \\ \cdots & \cdots & 1 & \cdots \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad \tilde{X} \sim N(A\mu, A\Sigma A^T)$$

n_k

Joint Gaussian $\Longrightarrow$ Boundary Gaussian
$\overset{?}{\Longleftarrow}$

Example: $f(x_1, x_2) = \frac{1}{2\pi}\exp(-\frac{1}{2}(x_1^2 + x_2^2)) + g(x_1, x_2)$

$$\begin{cases} \int_{\mathbb{R}} g(x_1, x_2)\,dx_1 = 0 \\ \int_{\mathbb{R}} g(x_1, x_2)\,dx_2 = 0 \end{cases} \qquad g(x_1, x_2) = \sin x_1 \sin x_2.$$

$f'(x_1, x_2) = \frac{1}{2\pi}\exp(-\frac{1}{2}(x_1^2 + x_2^2))[1 + \sin x_1 \sin x_2] > 0$

Boundary ✓, Joint ✗

Criterion for Joint Gaussian:

$X \in \mathbb{R}^n,\ X \sim N \Leftrightarrow \{\forall \alpha \in \mathbb{R}^n,\ \alpha^T X \sim N\}$

?

$\phi_X(w) = E[\exp(jw^T X)] = \phi_{w^T X}(1) = \exp(j 1 \cdot \mu_{w^T X} - \frac{1}{2} 1 \cdot \sigma^2_{w^T X} \cdot 1) = \exp(j \mu_{w^T X} - \frac{1}{2}\sigma^2_{w^T X})$

$w^T X \sim N$

$$\begin{cases} \mu_{w^T X} = E[w^T X] = w^T \mu_X \\ \sigma^2_{w^T X} = E[w^T X - w^T \mu_X]^2 \\ \quad = E[w^T (X - \mu_X)]^2 \\ \quad = w^T \Sigma_X \cdot w \end{cases}$$

$\therefore\ \phi_X(w) = \exp(jw^T \mu_X - \frac{1}{2} w^T \Sigma_X w) \Rightarrow X \sim N(\mu_X, \Sigma_X)$

$X = (X_1, \cdots, X_n)^T \sim N,\ E[X_i X_j] = EX_i\, EX_j \Rightarrow X_i, X_j$ independence.

$\Sigma_{ij} = E(X_i - EX_i)(X_j - EX_j) = E(X_i X_j) - EX_i\, EX_j = 0\ (i \neq j) \Rightarrow \Sigma$ is diagonal. $(\Sigma = \mathrm{diag}(\sigma_1^2, \cdots, \sigma_n^2))$

$$f_{X_1 \cdots X_n} = \frac{1}{(2\pi)^{n/2} (\prod_k \sigma_k^2)^{1/2}} \exp(-\frac{1}{2} \sum \frac{(x_k - \mu_k)^2}{\sigma_k^2})$$

$$= \prod_k \left[\frac{1}{\sqrt{2\pi \sigma_k^2}} \exp(-\frac{1}{2} \frac{(x_k - \mu_k)^2}{\sigma_k^2})\right] = \prod_k f_{X_k}(x_k)$$

In Gaussian.

Uncorrelated $\Leftrightarrow$ Independent.

PCA $\Rightarrow$ ICA (Independent)

Uncorrelated

Blind Source Separation

AIGC

Diffusion MidJourney.

highly relevant to Stochastic Processes

$X \sim N(0, I) \Rightarrow \tilde{X} = \Sigma^{1/2}(X + \Sigma^{-1/2}\mu) \sim N(\mu, \Sigma)$

$X_0, X_1, X_2, \cdots, X_N \in \mathbb{R}^n\ (X_k = \sqrt{1-\alpha_k}\, X_{k-1} + \sqrt{\alpha_k}\, \varepsilon_k,\ \varepsilon_k \sim N(0, I),$ i.i.d.$)$

$X_k \sim N\,! \rightarrow$ Determine $X_m.!$

$\beta_k = 1 - \alpha_k,\ X_k = \sqrt{\beta_k}\, X_{k-1} + \sqrt{1-\beta_k}\, \varepsilon_k = \sqrt{\beta_k}(\sqrt{\beta_{k-1}}\, X_{k-2} + \sqrt{1-\beta_{k-1}}\, \varepsilon_{k-1}) + \sqrt{1-\beta_k}\, \varepsilon_k$

"Reparametric Trick"

$= \sqrt{\beta_k \beta_{k-1}}\, X_{k-2} + \sqrt{\beta_k(1-\beta_{k-1})}\, \varepsilon_{k-1} + \sqrt{1-\beta_k}\, \varepsilon_k$

indep. Gaussian

$N(0, \sqrt{\beta_k(1-\beta_{k-1})}\, I) + N(0, \sqrt{1-\beta_k}\, I) \Rightarrow N(0, \sqrt{1-\beta_k \beta_{k-1}}\, I)$

$= \sqrt{\beta_k \beta_{k-1} \beta_{k-2}}\, X_{k-2} + \sqrt{1-\beta_k \beta_{k-1}}\, \varepsilon_0 = \cdots$

$$\therefore \begin{cases} X_k = \sqrt{\bar\beta}\, X_0 + \sqrt{1-\bar\beta}\, \varepsilon_0 \\ \bar\beta = \prod_k \beta_k \end{cases}$$