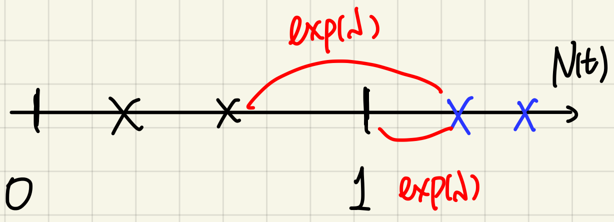
Poisson Processes 3

$E[S_4 | N(1)=2]$

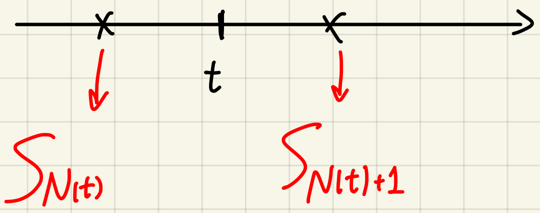


$1+\frac{2}{\lambda}$ suggests we can interpret the question as $1+2$ expected intervals, where we have 2 occurrences (intervals) after $t=1$ indeed; and the memoryless property of $\exp(\lambda)$ seems to be supportive to our hypothesis.

However, $\exp(\cdot)$ is computed for S_n (n deterministic), but not $S_{N(1)}$;

In which sense, time difference between 2^{nd} and 3^{rd} occurrence $\sim \exp(\lambda)$, but not between $t=1$ and 3^{rd} occurrence; while our hypothesis totally disagree. So, what went wrong?

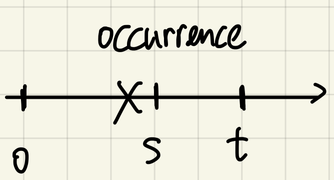
$$\forall n,\ S_{n+1}-S_n \sim \exp(\lambda); \qquad S_{N(t)+1}-S_{N(t)} \sim ?$$



$$T = S_{N(t)+1}-S_{N(t)},\quad F_T(\tau) = P(T\le\tau) = P(S_{N(t)+1}-S_{N(t)}\le\tau) = \sum_n \underbrace{P(S_{n+1}-S_n\le\tau \mid N(t)=n)}_{?}\, P(N(t)=n)$$

$P(S_{n+1}-S_n\le\tau \mid N(t)=n)$, $N(t)=n$ will affect $S_{n+1}-S_n$, the 2 statements are not indep.

Suppose $N(t)=1$, $F_{T|N(t)=1}(s|N(t)=1) = P(T\le s \mid N(t)=1) = P(N(s)=1,\ N(t)-N(s)=0 \mid N(t)=1)$



$$= P(N(s)=1)\cdot P(N(t)-N(s)=0)/P(N(t)=1) = \frac{\exp(-\lambda s)\cdot\frac{\lambda s}{1}\cdot\exp(-\lambda(t-s))}{\lambda t\,\exp(-\lambda t)}$$

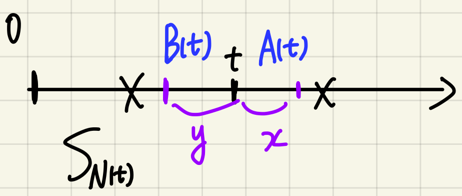
$$= \frac{s}{t}$$

$$\therefore\ f_{T|N(t)=1}(s|N(t)=1) = \frac{dF_{\cdot\cdot}}{ds} = \frac{1}{t} \qquad \text{Uniform Distribution}$$

After conditioning the occurrences in a time range, the distribution of occurrence time becomes uniform.

不是指数分布

$\therefore\ S_{N(t)+1}-S_{N(t)} \not\sim \exp(\lambda)$ (Inspection Paradox)



$$\begin{cases} A(t) = S_{N(t)+1}-t \\ B(t) = t - S_{N(t)} \end{cases}$$

$F_{A(t),B(t)}(x,y) = P(A(t)\le x,\ B(t)\le y)$

$P(A(t)>x) - P(A(t)>x,\ B(t)>y) = P(A(t)>x,\ B(t)\le y)$

$P(A(t)>x,\ B(t)>y) \Rightarrow P(A(t)>x)$ (set y to 0) $\Rightarrow P(A(t)\le x)$

$\Rightarrow P(B(t)>y)$ (set x to 0) $\Rightarrow P(B(t)\le y)$

$P(B(t)\le y) - P(A(t)>x,\ B(t)\le y) = P(A(t)\le x,\ B(t)\le y)$

$\Updownarrow$

$P(N[t-y,\ t+x]=0) = P(N(x+y)=0) = \exp(-\lambda(x+y))$

Stationary

$P(A(t)>x) = \exp(-\lambda x) \rightarrow P(A(t)>x,\ B(t)\le y) = \exp(-\lambda x)(1-\exp(-\lambda y))$

$P(B(t)>y) = \exp(-\lambda y) \rightarrow P(B(t)\le y) - P(A(t),B(t)\le y) = 1-\exp(-\lambda y) - \exp(-\lambda x)(1-\exp(-\lambda y))$

(truncated by $t=0$)

$= [1-\exp(-\lambda x)][1-\exp(-\lambda y)]$

Exponential distribution! $\Rightarrow$ $t=1 \sim 3^{rd}$ occurrence!

$N(t)=n$, $F_{S_1,S_2,\cdots,S_n|N(t)=n}(x_1,\cdots,x_n) = ?$

$= P(S_1\le x_1,\ \cdots\cdots,\ S_2\le x_n \mid N(t)=n)$

$A(t)\sim\exp(\lambda) \Rightarrow 1+\frac{1}{\lambda}+\frac{1}{\lambda}$

$A(t)$ (Inspection Paradox)

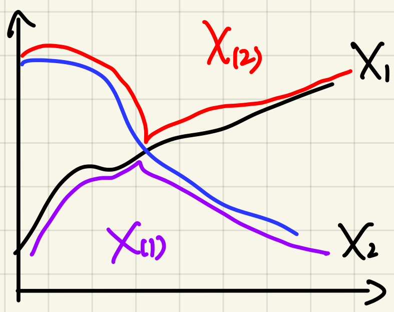
Order Statistics: $X_1, \cdots, X_n$ i.i.d. $\to (X_{(1)} \le \cdots \le X_{(n)})$

$X_{(1)} = \min(X_1, \cdots, X_n)$

$X_{(2)} = 2^{nd} \min(X_1, \cdots, X_n)$

$\vdots$

$X_{(n)} = \max(X_1, \cdots, X_n)$



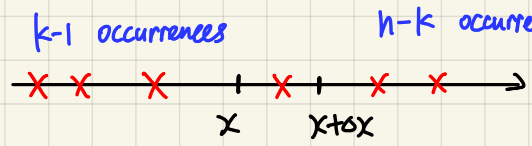
不用CDF微分,而用微元求解

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(\max(X_1 \cdots X_n) \le x) = P(X_1 \le x, \cdots, X_n \le x)$$
$$= \prod_k P(X_k \le x) = [P(X_1 \le x)]^n$$
$$f_{X_{(n)}}(x) = \frac{d}{dx} F_{X_{(n)}}(x) = n F_{X_1}^{n-1}(x) \cdot f_{X_1}(x)$$

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, \cdots, X_n > x)$$
$$= 1 - [1 - F_{X_1}(x)]^n$$
$$f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x) = n[1 - F_{X_1}(x)]^{n-1} \cdot f_{X_1}(x)$$

$F_{X_{(k)}}(x) = P(X_{(k)} \le x) = ?$ Micro-Cell $P(x < X \le x + \Delta x) = F_X(x+\Delta x) - F_X(x) \Rightarrow f_X = \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} = \frac{\cdots}{\Delta x}$

$$P(x < X_{(k)} \le x+\Delta x) = F_{X_{(k)}}(x+\Delta x) - F_{X_{(k)}}(x)$$
$$= [F_{X_1}(x)]^{k-1} \cdot [1 - F_{X_1}(x)]^{n-k} \cdot [F_{X_1}(x+\Delta x) - F_{X_1}(x+\Delta x)] \cdot \binom{n}{n-k}\binom{k}{1}$$



$$f_{X_{(k)}}(x) = \frac{1}{\Delta x} P(x \le X_{(k)} < x+\Delta x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \binom{n}{n-k}\binom{k}{1} \cdot [F_X(x)]^{k-1} [1 - F_X(x+\Delta x)]^{n-k} \cdot [F_X(x+\Delta x) - F_X(x)]$$
$$= \binom{n}{n-k}\binom{k}{1} [F_X(x)]^{k-1} [1 - F_X(x)]^{n-k} \cdot f_X(x)$$

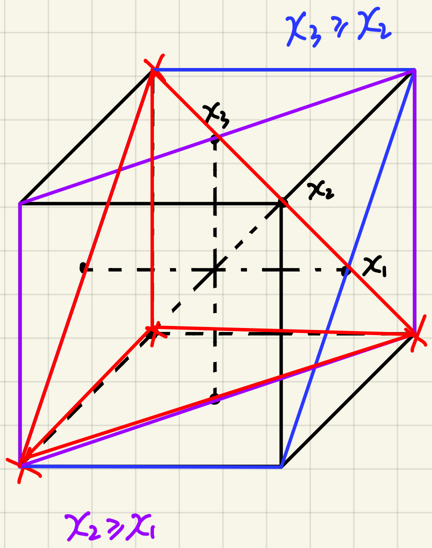
$x_k \le x_m$.

$$f_{X_{(k)} X_{(m)}}(x_k, x_m) = \binom{n}{k-1}\binom{n-k+1}{1}\binom{n-k}{m-k-1}\binom{n-m+1}{1} [1 - F_X(x_k)]^{k-1} \cdot f_X(x_k) [F_X(x_m) - F_X(x_k)]^{m-k-1} f_X(x_m) [1 - F_X(x_m)]^{n-m}$$

$$\boxed{f_{X_{(1)}, \cdots X_{(n)}}(x_1, \cdots, x_n) = n!\, f_X(x_1) \cdots f_X(x_n) = n! \prod_k f_X(x_k)}$$

$x_1 \le x_2 \le \cdots \le x_n$

$$\int f_{X_{(1)} \cdots X_{(n)}}(x_1 \cdots x_n) dx_1 \cdots dx_n \begin{cases} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \cdots \int_{-\infty}^{x_2} n!\, f_{X_1}(x_1) \cdots f_{X_1}(x_n)\, dx_1 \cdots dx_n \\ \underbrace{\int_{-\infty}^{+\infty} \int_{x_1}^{+\infty} \int_{x_2}^{+\infty} \cdots \int_{x_{n-1}}^{+\infty}}_{\text{Integration volume} = \frac{1}{n!} \cdot \mathbb{R}^n} n!\, \underbrace{f_{X_1}(x_1) \cdots f_{X_1}(x_n)}_{\text{Symmetric Function}}\, dx_n \cdots dx_1 \end{cases}$$



$g(x_1 \cdots x_n) = \prod_k f_X(x_k)$

$g(x_1, \cdots, x_n) = g(x_{6_1}, \cdots, x_{6_n})$

Symmetric

$\{6_1, \cdots, 6_n\}: \{1 \sim n\}$ 任意排序

$(-1,-1,-1), (-1,1,-1)$

$(1,1,-1), (-1,1,1)$

$\Downarrow$

$x_3 \ge x_2 \ge x_1$, $\frac{1}{6} = \frac{1}{3!} \mathbb{R}^3$

$n=2$时:

$x_1 < x_2$ $\quad$ $x_2 < x_1$

$$g(x_1, x_2) = g(x_2, x_1) \Rightarrow \int_{\mathbb{R}} \int_{-\infty}^{x_2} g(x_1, x_2) dx_1 dx_2 = \int_{\mathbb{R}} \int_{-\infty}^{x_1} g(x_1, x_2) dx_2 dx_1$$

$\downarrow$

x_1, x_2 change order

$$\int_{\mathbb{R}} \int_{-\infty}^{x_1} g(x_2, x_1) dx_2 dx_1 \quad g(x_2, x_1) = g(x_1, x_2)$$

$$\int_{\mathbb{R}} \int_{-\infty}^{x_2} g(x_1, x_2) dx_1 dx_2 = \frac{1}{2} \int_{\mathbb{R}^2} g(x_1, x_2) dx_1 dx_2$$

$$\int_{\mathbb{R}} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} n!\, f_{X_1}(x_1) \cdots f_{X_1}(x_n) dx_1 \cdots dx_n$$
$$= \frac{1}{n!} \int_{\mathbb{R}^n} n!\, f_{X_1}(x_1) \cdots f_{X_n}(x_n) dx_1 \cdots dx_n$$
$$= \int_{\mathbb{R}^n} f_{X_1}(x_1) \cdots f_{X_n}(x_n) dx_1 \cdots dx_n$$
$$= 1$$

$N(t)=n,\quad F_{S_1,\cdots,S_n|N(t)=n}(x_1,\cdots,x_n)=P(S_1\le x_1,\cdots;S_n\le x_n\,|\,N(t)=n)$

$$f_{S_1\cdots S_n|N(t)=n}(x_1,\cdots,x_n)=\lim_{\Delta x_k\to 0}\frac{P(x_1<S_1\le x_1+\Delta x_1,\cdots,x_n<S_n\le x_n+\Delta x_n\,|\,N(t)=n)}{\Delta x_1\cdots\Delta x_n}=\lim_{\Delta x_k\to 0}\frac{1}{\prod_k\Delta x_k}\frac{P(N(\Delta x_1)=1,N(\Delta x_2)=1,\cdots,N(\Delta x_n)=1,N(t-\sum\Delta x_k)=0)}{P(N(t)=n)}$$

$$=\lim_{\Delta x_k\to 0}\frac{1}{\prod_k\Delta x_k}\frac{\prod_k[\lambda\Delta x_k\exp(-\lambda\Delta x_k)]\cdot\exp(-\lambda(t-\sum_k\Delta x_k))}{\frac{(\lambda t)^n}{n!}\exp(-\lambda t)}$$

$$=\begin{cases}\dfrac{n!}{(\lambda t)^n} & 0\le x_1\le x_2\le\cdots\le x_n\le t\\[2ex] 0. & \text{others.}\end{cases}$$

$S_1\cdots S_n|N(t)=n\sim(U_{(1)},\cdots,U_{(n)})$ Uniform Distribution of Order Statistics

n个均匀分布 r.v. 的顺序统计量

$\Sigma x.$:

n Buses in $[0,T]$, interval: $T_1,\cdots,T_n$, $\sum_k T_k=T$, passengers $\sim$ Poisson(λ).

Each bus is large enough to carry every waiting passenger at the bus stop.

give the set of $T_1\sim T_n$ s.t. Σ waiting time minimum.

$$E\Big[\sum_1^{N(T_1)}S_k\,\Big|\,N(T_1)=n\Big]=E\Big[\sum_i U_{(i)}\,\Big|\,N(t)=n\Big]=E\Big[\sum U_i\,\Big|\,N(t)=n\Big]$$

Each Uniform Distribution $\Rightarrow E[WT]=\frac{T_1}{2}$

$=\frac{T_1}{2}\cdot n$

$$T_1:\ E\Big[\sum_{k=1}^{N(T_1)}(T_1-S_k)\Big]=E\Big[\sum_1^{N(T_1)}T_1\Big]-E\Big[\sum_1^{N(T_1)}S_k\Big]=\lambda T_1\cdot T_1-E\Big[E\Big[\sum_1^{N(T_1)}S_k\,\Big|\,N(T_1)=n\Big]\Big]$$

$$=\lambda T_1^2-E\Big[\frac{T_1}{2}N(T_1)\Big]$$

$$=\lambda T_1^2-\frac{T_1}{2}\cdot\lambda T_1$$

$$=\frac{\lambda}{2}T_1^2$$

$\Rightarrow$ for $(T_1,T_2,\cdots,T_n)$, Expected Sum of Waiting Time: $\frac{\lambda}{2}\sum T_i^2$ (Optimal: $T_1=\cdots=T_n=\frac{T}{n}$)

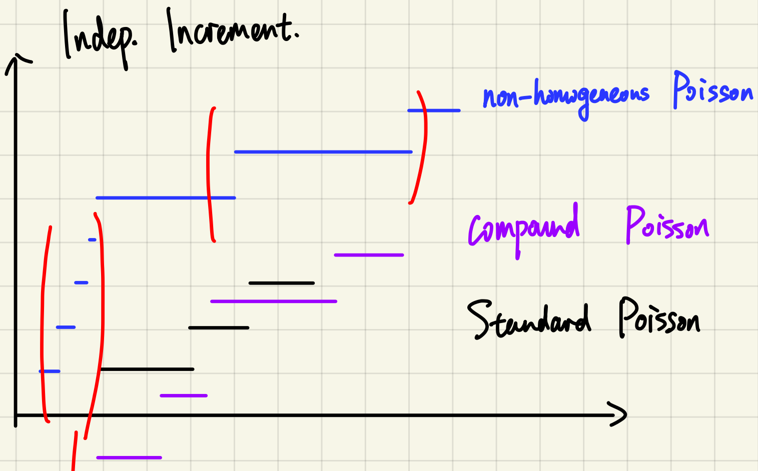
$S_1\cdots S_n.\quad S_n-S_{n-1}\overset{\text{i.i.d}}{\sim}\exp(\lambda),\quad S_n\sim\Gamma(\lambda)$

$$S_1\cdots S_n|N(t)=n\sim(U_{(1)},\cdots,U_{(n)}),\quad f_{S_1\cdots S_n|N(t)=n}(x_1,\cdots,x_n)=\begin{cases}\dfrac{n!}{t^n}, & 0\le x_1\le\cdots\le x_n\le t\\[2ex] 0, & \text{others.}\end{cases}$$

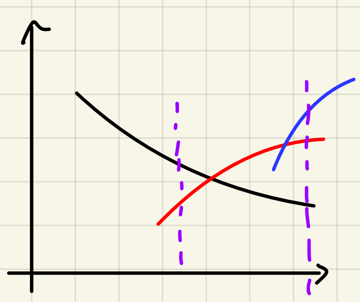
What if Poisson is not indep. incremental?

$$G_{N(t+\Delta t)}(z)-G_{N(t)}(z)=E[z^{N(t+\Delta t)}-z^{N(t)}]=E[z^{N(t)}(z^{N(t+\Delta t)-N(t)}-1)]$$
$$\ne E[z^{N(t)}]\,E[z^{N(t+\Delta t)-N(t)}-1]$$

Indep. Increment.



no indep. increment. $\Rightarrow$



$$Y(t)=\sum_{k=1}^{N(t)}h_k(t)=\sum_{k=1}^{N(t)}h(t,S_k,A_k)$$

A_k $\downarrow$ Amplitude

$\{A_k\}$ r.v. i.i.d.

not affecting occurrences afterwards (this event is time-invariant once occurred)

$$Y(t)=\sum_{k=1}^{N(t)} h(t,S_k,\underset{\text{具体数值}}{A_k}) \quad \boxed{\{A_k\}\ \text{r.v., i.i.d.}}$$

Filtered Poisson Processes. 不再仅取整数，选用 Charac. Func 而非 MGF

发生时刻

$$\Phi_{Y(t)}=E[\exp(j\omega Y(t))]=E\Big[\exp\big(j\omega\sum_{k=1}^{N(t)}h(t,S_k,A_k)\big)\Big]=\underset{N(t)}{E}\Big[\underset{S_k}{E}\Big[\underset{A_k}{E}\Big[\exp\big(j\omega\sum_{k=1}^{N(t)}h(t,S_k,A_k)\big)\Big|N(t)=n,\{S_k\}\Big]\Big|N(t)=n\Big]\Big]$$

1^{st}, $E_{A_k}\Big[\exp\big(j\omega\sum^{N(t)}h(t,S_k,A_k)\big)\Big|N(t)=n,\{S_k\}\Big]$

let $B(t,s)=E[\exp(j\omega h(t,s,A_k))] \Rightarrow \prod_{k=1}^{N(t)}B(t,S_k)$

2^{nd} $E_{S_k}\Big[E\big[\exp(j\omega\sum^{N(t)}h(t,S_k,A_k))\big|N(t)=n,\{S_k\}\big]\Big|N(t)=n\Big]=E\Big[\prod_{k=1}^{N(t)}B(t,S_k)\Big|N(t)=n\Big]$

$S_1\sim S_n|N(t)=n \sim (U_{(1)}\cdots U_{(n)})$

$$f_{S_1\sim S_n|N(t)=n}(x_1\cdots x_n|N(t)=n)=\begin{cases}\frac{n!}{t^n}, & x_1\le x_2\le\cdots\le x_n\\ 0, & \text{others}\end{cases}$$

$$\Rightarrow \int_0^t\int_0^{S_n}\cdots\int_0^{S_2}\prod_{k=1}^{N(t)}B(t,S_k)\,\frac{n!}{t^n}\,dS_1dS_2\cdots dS_n$$

3^{rd}. $E_{N(t)}\Big[\int_0^t\int_0^{S_n}\cdots\int_0^{S_2}\underbrace{\prod_{k=1}^{N(t)}B(t,S_k)}_{\text{Symmetric}}\cdot\frac{n!}{t^n}dS_1dS_2\cdots dS_n\Big] = E_{N(t)}\Big[\frac{1}{n!}\int_0^t\int_0^t\cdots\int_0^t\prod_{k=1}^{N(t)}B(t,S_k)\cdot\frac{n!}{t^n}dS_1\cdots dS_n\Big]$

$$=E\Big[\prod_{k=1}^{N(t)}\frac{1}{t}\int_0^t B(t,S_k)dS_k\Big]=E_{N(t)}\Big[\Big(\frac{1}{t}\int_0^t B(t,S_k)dS_k\Big)^{N(t)}\Big]=G_{N(t)}(z)\Big|_{z=\frac{1}{t}\int_0^t B(t,s)ds}$$

$$=\exp(\lambda t(z-1))\Big|_{z=\frac{1}{t}\int_0^t B(t,s)ds}=\exp\Big(\lambda t\Big(\frac{1}{t}\int_0^t B(t,s)ds-1\Big)\Big)=\exp\Big(\lambda\int_0^t(B(t,s)-1)ds\Big)$$

$$\because B(t,s)=E[\exp(j\omega h(t,s,A_k))] \Rightarrow \exp\Big(\lambda\int_0^t\big(E_A(\exp(j\omega h(t,s,A)))-1\big)ds\Big)$$

$$E[Y(t)]=\frac{1}{j}\frac{d}{d\omega}\Phi_Y(\omega)\Big|_{\omega=0}=\frac{1}{j}\cdot\lambda\int_0^t\big(E_A(\exp(j\omega h(t,s,A)))-1\big)ds\cdot\int_0^t E_A\big(jh(t,s,A)\exp(j\omega h(t,s,A))\big)ds\cdot \sim\Big|_{\omega=0}$$

$$=\lambda\int_0^t E_A(h(t,s,A))ds$$

Queueing : Service Model (serving status, queue length / customer arrival), # of counter)

2 r.v. 1 deterministic

Kendall: M/M/K M: Markov
G G G: General

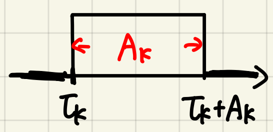
$M/G/\infty$ infinite counter (M: arrival, G: serve time)

$Y(t)$: # in Service System, Queue length.

$E[Y(t)]=?$ Filtered Poisson Process

$$Y(t)=\sum_{k=1}^{N(t)}h(t,S_k,A_k)$$

$h(\cdot)$: rect window,



$h(t,\tau_k,A_k)=u(\tau_k)-u(\tau_k+A_k)$

$$\Phi_{Y(t)}=E[\exp(j\omega Y(t))]=E\Big[\exp\big(j\omega\sum_{k=1}^{N(t)}h(t,S_k,A_k)\big)\Big]=E\Big[E\big[\exp(j\omega\sum_{N(t)=n}h(t,S_k,A_k))\big|N(t)\big]\Big]=E\Big[E\Big[E\big[\exp(j\omega\sum_{N(t)=n}h(t,S_k,A_k))\big|S_k,N(t)=n\big]\Big|N(t)=n\Big]\Big]$$

$$=E\Big[E\Big[E\Big[\exp\big(j\omega\sum_{k=1}^{n}(u(\tau_k)-u(\tau_k+A_k))\big)\Big|S_k=\tau_k,N(t)=n\Big]\Big|N(t)=n\Big]\Big]$$

if # of counter is limited, filtered Poisson is no longer applicable. (A_k is not independent, waiting time $\neq 0$)

where in filtered Poisson, $\{A_k\}$ has to be independent. $\Rightarrow$ Markov Process. (Next Lecture)