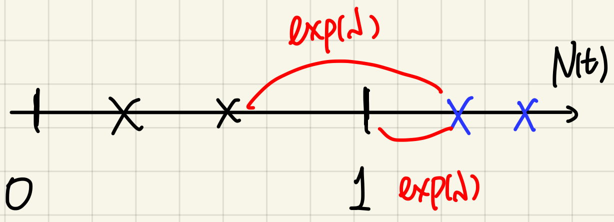
Poisson Processes 3

$E[S_4 | N(1) = 2]$

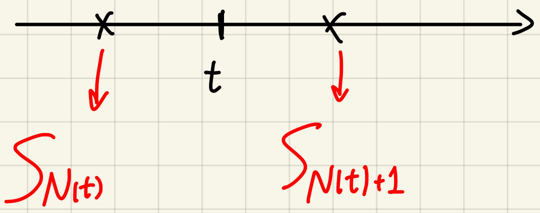


$1 + \frac{2}{\lambda}$ suggests we can interpret the question as $1+2$ expected intervals, where we have 2 occurrences (intervals) after $t=1$ indeed; and the memoryless property of exp(λ) seems to be supportive to our hypothesis.

However, exp(·) is computed for S_n (n deterministic), but not $S_{N(1)}$;

In which sense, time difference between 2^{nd} and 3^{rd} occurrence $\sim exp(\lambda)$. but not between $t=1$ and 3^{rd} occurrence; while our hypothesis totally disagree. So, what went wrong?

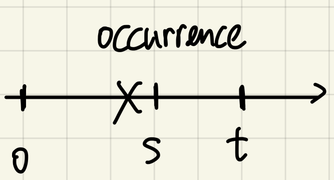
$\forall n,\ S_{n+1} - S_n \sim exp(\lambda)$; $\qquad S_{N(t)+1} - S_{N(t)} \sim ?$



$T = S_{N(t)+1} - S_{N(t)}$, $F_T(\tau) = P(T \le \tau) = P(S_{N(t)+1} - S_{N(t)} \le \tau) = \sum_n \underbrace{P(S_{n+1} - S_n \le \tau | N(t) = n)}_{?} P(N(t) = n)$

$P(S_{n+1} - S_n \le \tau | N(t) = n)$, $N(t) = n$ will affect $S_{n+1} - S_n$, the 2 statements are not indep.

Suppose $N(t) = 1$, $F_{T/N(t)=1}(s | N(t) = 1) = P(T \le s | N(t) = 1) = P(N(s) = 1, N(t) - N(s) = 0 | N(t) = 1)$



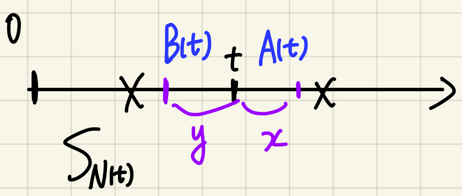
$$= P(N(s) = 1) \cdot P(N(t) - N(s) = 0) / P(N(t) = 1) = \frac{exp(-\lambda s) \cdot \frac{\lambda s}{1} \cdot exp(-\lambda(t-s))}{\lambda t\ exp(-\lambda t)}$$

$$= \frac{s}{t}$$

$\therefore f_{T|N(t)=1}(s | N(t) = 1) = \frac{dF_{\cdot}}{ds} = \frac{1}{t}$ Uniform Distribution

After conditioning the occurrences in a time range, the distribution of occurrence time becomes uniform.

$\therefore S_{N(t)+1} - S_{N(t)} \not\sim exp(\lambda)$ (Inspection Paradox)



$$\begin{cases} A(t) = S_{N(t)+1} - t \\ B(t) = t - S_{N(t)} \end{cases}$$

$F_{A(t), B(t)}(x, y) = P(A(t) \le x, B(t) \le y)$

$P(A(t) > x, B(t) > y) \Rightarrow P(A(t) > x)$ (set y to 0) $\Rightarrow P(A(t) \le x)$

$P(A(t) > x, B(t) > y) \Rightarrow P(B(t) > y)$ (set x to 0) $\Rightarrow P(B(t) \le y)$

$P(A(t) > x) - P(A(t) > x, B(t) > y) = P(A(t) > x, B(t) \le y)$

$P(B(t) > y) - P(A(t) > x, B(t) > y) = P(A(t) \le x, B(t) > y)$

$\Updownarrow$

$P(N(t-y, t+x] = 0) = P(N(x+y) = 0) = exp(-\lambda(x+y))$

$\rightarrow P(A(t) > x) = exp(-\lambda x) \rightarrow P(A(t) > x, B(t) \le y) = exp(-\lambda x)(1 - exp(-\lambda y))$

$\rightarrow P(B(t) > y) = exp(-\lambda y) \rightarrow P(A(t) \le x, B(t) > y) = exp(-\lambda y)(1 - exp(-\lambda x))$

(not truncated by $t=0$)

Exponential distribution! $\Rightarrow t=1 \sim 3^{rd}$ occurrence!

$N(t) = n$, $F_{S_1, S_2, \cdots, S_n | N(t) = n}(x_1, \cdots, x_n) = ?$

$= P(S_1 \le x_1, \cdots, S_2 \le x_n | N(t) = n)$

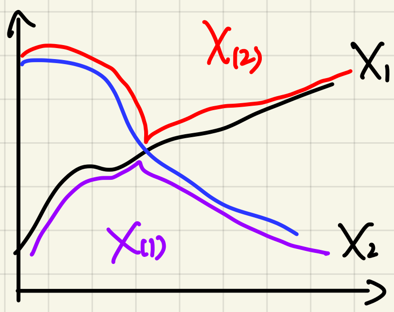
Order Statistics: $X_1,\cdots,X_n$ i.i.d. $\to (X_{(1)}\le\cdots\le X_{(n)})$

$X_{(1)} = \min(X_1,\cdots,X_n)$

$X_{(2)} = 2^{nd}\ \min(X_1,\cdots,X_n)$

$\vdots$

$X_{(n)} = \max(X_1,\cdots,X_n)$

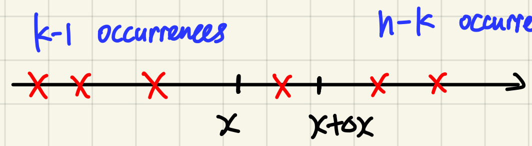


$$F_{X_{(n)}}(x) = P(X_{(n)}\le x) = P(\max(X_1\cdots X_n)\le x) = P(X_1\le x,\cdots,X_n\le x)$$
$$= \prod_k P(X_k\le x) = [P(X_1\le x)]^n$$
$$f_{X_{(n)}}(x) = \frac{d}{dx}F_{X_{(n)}}(x) = n\,F_{X_1}^{n-1}(x)\cdot f_{X_1}(x)$$

$$F_{X_{(1)}}(x) = P(X_{(1)}\le x) = 1-P(X_{(1)}>x) = 1-P(X_1>x,\cdots,X_n>x)$$
$$= 1-[1-F_{X_1}(x)]^n$$
$$f_{X_{(1)}}(x) = \frac{d}{dx}F_{X_{(1)}}(x) = n[1-F_{X_1}(x)]^{n-1}\cdot f_{X_1}(x)$$

$F_{X_{(k)}}(x) = P(X_{(k)}\le x) = ?$ Micro-Cell $P(x<X\le x+\Delta x) = F_X(x+\Delta x)-F_X(x) \Rightarrow f_X = \dfrac{F_X(x+\Delta x)-F_X(x)}{\Delta x} = \dfrac{\cdots}{\Delta x}$

$$P(x<X_{(k)}\le x+\Delta x) = F_{X_{(k)}}(x+\Delta x)-F_{X_{(k)}}(x)$$
$$= [F_{X_1}(x)]^{k-1}\cdot[1-F_{X_1}(x)]^{n-k}\cdot[F_{X_1}(x+\Delta x)-F_{X_1}(x+\Delta x)]\cdot\binom{n}{n-k}\binom{k}{1}$$



$$f_{X_{(k)}}(x) = \frac{1}{\Delta x}P(x\le X_{(k)}<x+\Delta x) = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\binom{n}{n-k}\binom{k}{1}\cdot[F_X(x)]^{k-1}[1-F_X(x+\Delta x)]^{n-k}\cdot[F_X(x+\Delta x)-F_X(x)]$$
$$= \binom{n}{n-k}\binom{k}{1}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}\cdot f_X(x)$$

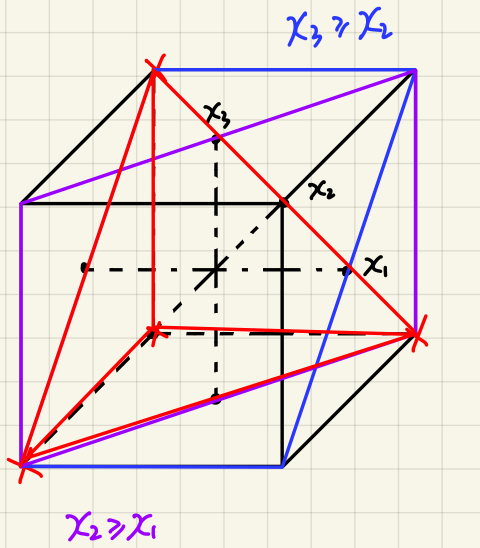
$x_k\le x_m$.

$$f_{X_{(k)}X_{(m)}}(x_k,x_m) = \binom{n}{k-1}\binom{n-k+1}{1}\binom{n-k}{m-k-1}\binom{n-m+1}{1}[1-F_X(x_k)]^{k-1}\cdot f_X(x_k)[F_X(x_m)-F_X(x_k)]^{m-k-1}f_X(x_m)[1-F_X(x_m)]^{n-m}$$

$$\boxed{f_{X_{(1)},\cdots X_{(n)}}(x_1,\cdots,x_n) = n!\,f_X(x_1)\cdots f_X(x_n) = n!\prod_k f_X(x_k)}$$

$x_1\le x_2\le\cdots\le x_n$

$$\int f_{X_{(1)}\cdots X_{(n)}}(x_1\cdots x_n)dx_1\cdots dx_n$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_n}\int_{-\infty}^{x_{n-1}}\cdots\int_{-\infty}^{x_2} n!\,f_{X_1}(x_1)\cdots f_{X_1}(x_n)\,dx_1\cdots dx_n$$
$$\underbrace{\int_{-\infty}^{+\infty}\int_{x_1}^{+\infty}\int_{x_2}^{+\infty}\cdots\int_{x_{n-1}}^{+\infty}}_{\text{Integration volume} = \frac{1}{n!}\cdot\mathbb{R}^n} n!\,\underbrace{f_{X_1}(x_1)\cdots f_{X_1}(x_n)}_{\text{Symmetric Function}}\,dx_n\cdots dx_1$$



$(-1,-1,-1),\ (-1,1,-1)$

$(1,1,-1),\ (-1,1,1)$

$\Downarrow$

$x_3\ge x_2\ge x_1,\ \frac{1}{6} = \frac{1}{3!}\mathbb{R}^3$

$g(x_1\cdots x_n) = \prod_k f_X(x_k)$

$g(x_1,\cdots,x_n) = g(x_{\sigma_1},\cdots,x_{\sigma_n})$ Symmetric

$\{\sigma_1,\cdots,\sigma_n\}:\{1\sim n\}$ 任意排序

$n=2$时:

$$g(x_1,x_2) = g(x_2,x_1) \Rightarrow \underbrace{\int_{\mathbb{R}}\int_{-\infty}^{x_2} g(x_1,x_2)dx_1dx_2}_{x_1<x_2} = \underbrace{\int_{\mathbb{R}}\int_{-\infty}^{x_1} g(x_1,x_2)dx_2dx_1}_{x_2<x_1}$$

$\downarrow$ x_1,x_2 change order

$$\int_{\mathbb{R}}\int_{-\infty}^{x_1} g(x_2,x_1)dx_2dx_1 \qquad g(x_2,x_1) = g(x_1,x_2)$$

$$\int_{\mathbb{R}}\int_{-\infty}^{x_2} g(x_1,x_2)dx_1dx_2 = \frac{1}{2}\int_{\mathbb{R}^2} g(x_1,x_2)dx_1dx_2$$

$$\int_{\mathbb{R}}\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_3}\int_{-\infty}^{x_2} n!\,f_{X_1}(x_1)\cdots f_{X_1}(x_n)dx_1\cdots dx_n$$
$$= \frac{1}{n!}\int_{\mathbb{R}^n} n!\,f_{X_1}(x_1)\cdots f_{X_n}(x_n)dx_1\cdots dx_n$$
$$= \int_{\mathbb{R}^n} f_{X_1}(x_1)\cdots f_{X_n}(x_n)dx_1\cdots dx_n$$
$$= 1$$

$N(t)=n, \quad F_{S_1,\dots,S_n|N(t)=n}(x_1,\dots,x_n) = P(S_1\le x_1,\dots;S_n\le x_n \,|\, N(t)=n)$

$$f_{S_1\cdots S_n|N(t)=n}(x_1,\dots,x_n) = \lim_{\Delta x_k\to 0}\frac{P(x_1<S_1\le x_1+\Delta x_1,\dots,x_n<S_n\le x_n+\Delta x_n \,|\, N(t)=n)}{\Delta x_1\cdots\Delta x_n} = \lim_{\Delta x_k\to 0}\frac{1}{\prod_k \Delta x_k}\frac{P(N(\Delta x_1)=1,\,N(\Delta x_2)=1,\dots,N(\Delta x_n)=1,\,N(t-\sum\Delta x_k)=0)}{P(N(t)=n)}$$

$$= \lim_{\Delta x_k\to 0}\frac{1}{\prod_k \Delta x_k}\frac{\prod_k[\lambda\Delta x_k\exp(-\lambda\Delta x_k)]\cdot\exp(-\lambda(t-\sum_k\Delta x_k))}{\frac{(\lambda t)^n}{n!}\exp(-\lambda t)}$$

$$= \begin{cases}\dfrac{n!}{(\lambda t)^n} & 0\le x_1\le x_2\le\cdots\le x_n\le t\\[2ex] 0. & \text{others.}\end{cases}$$

$S_1\cdots S_n|N(t)=n \sim (U_{(1)},\dots,U_{(n)})$ Uniform Distribution of Order Statistics

Σx. :

n Buses in $[0,T]$, interval: $T_1,\dots,T_n$, $\sum_k T_k = T$, passengers $\sim$ Poisson(λ).

Each bus is large enough to carry every waiting passenger at the bus stop.

give the set of $T_1\sim T_n$ s.t. Σ waiting time minimum.

$$T_1:\ E\Big[\sum_{k=1}^{N(T_1)}(T_1-S_k)\Big] = E\Big[\sum_1^{N(T_1)}T_1\Big]-E\Big[\sum_1^{N(T_1)}S_k\Big] = \lambda T_1\cdot T_1 - E\Big[E\Big[\sum_1^{N(T_1)}S_k\,\Big|\,N(T_1)=n\Big]\Big]$$

$$= \lambda T_1^2 - E\Big[\frac{T_1}{2}N(T_1)\Big]$$

$$= \lambda T_1^2 - \frac{T_1}{2}\cdot\lambda T_1$$

$$= \frac{\lambda}{2}T_1^2$$

$E\Big[\sum_1^{N(T_1)}S_k\,\Big|\,N(T_1)=n\Big] = E\Big[\sum_i U_{(i)}\,\Big|\,N(t)=n\Big] = E\Big[\sum U_i\,\Big|\,N(t)=n\Big]$

Each Uniform Distribution $\Rightarrow E[WT]=\frac{T_1}{2}$

$= \frac{T_1}{2}\cdot n$

$\Rightarrow$ for $(T_1,T_2,\dots,T_n)$, Expected Sum of Waiting Time: $\frac{\lambda}{2}\sum T_i^2$ (Optimal: $T_1=\cdots=T_n=\frac{T}{n}$)

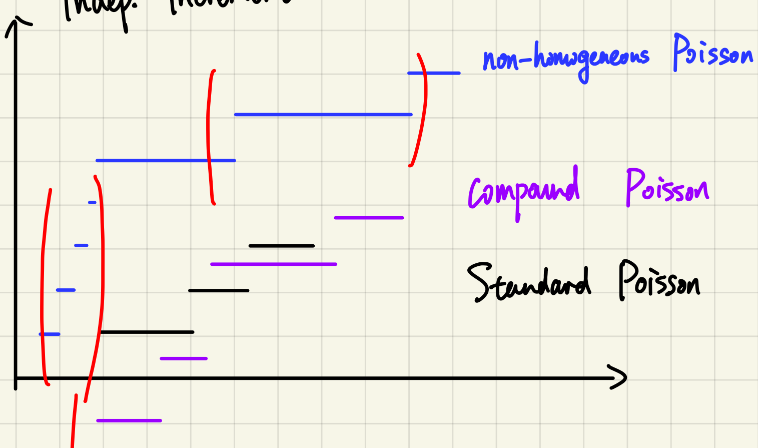
$S_1\cdots S_n$. $\quad S_n-S_{n-1}\overset{\text{i.i.d}}{\sim}\exp(\lambda)$. $\quad S_n\sim\Gamma(\lambda)$

$S_1\cdots S_n|N(t)=n \sim (U_{(1)},\dots,U_{(n)})$. $\quad f_{S_1\cdots S_n|N(t)=n}(x_1,\dots,x_n) = \begin{cases}\frac{n!}{t^n}, & 0\le x_1\le\cdots\le x_n\le t\\ 0, & \text{others.}\end{cases}$

What if Poisson is not indep. incremental?

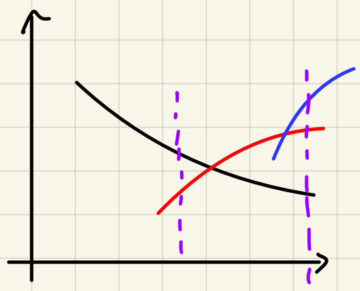
$G_{N(t+\Delta t)}(z)-G_{N(t)}(z) = E[z^{N(t+\Delta t)}-z^{N(t)}] = E[z^{N(t)}(z^{N(t+\Delta t)-N(t)}-1)]$
$\neq E[z^{N(t)}]\,E[z^{N(t+\Delta t)-N(t)}-1]$

Indep. Increment.



not affecting occurrences afterwards (this event is time-invariant once occurred)

no indep. increment. $\Rightarrow$



$$Y(t) = \sum_{k=1}^{N(t)} h_k(t) = \sum_{k=1}^{N(t)} h(t, S_k, A_k)$$

A_k ↓ Amplitude

$\{A_k\}$ r.v. i.i.d.

$Y(t) = \sum_{k=1}^{N(t)} h(t, S_k, A_k)$ $\boxed{\{A_k\} \text{ r.v., i.i.d.}}$ Filtered Poisson Processes.

$$\phi_{Y(t)} = E[\exp(jwY(t))] = E[\exp(jw\sum_{k=1}^{N(t)} h(t, S_k, A_k))] = E_{N(t)}[E_{S_k}[E_{A_k}[\exp(jw\sum_{k=1}^{N(t)} h(t, S_k, A_k)) | N(t)=n, \{S_k\}] | N(t)=n]]$$

1^{st}, $E_{A_k}[\exp(jw\sum^{N(t)} h(t, S_k, A_k)) | N(t)=n, \{S_k\}]$

let $B(t, s) = E[\exp(jwh(t, s, A_k))]$ $\Rightarrow \prod_{k=1}^{N(t)} B(t, S_k)$

2^{nd} $E_{S_k}[E[\exp(jw\sum^{N(t)} h(t, S_k, A_k)) | N(t)=n, \{S_k\}] | N(t)=n] = E[\prod_{k=1}^{N(t)} B(t, S_k) | N(t)=n]$

$S_1 \sim S_n | N(t)=n \sim (U_{(1)} \cdots U_{(n)})$

$$f_{S_1 \sim S_n | N(t)=n}(x_1 \cdots x_n | N(t)=n) = \begin{cases} \frac{n!}{t^n}, & x_1 \le x_2 \le \cdots \le x_n \\ 0, & \text{others} \end{cases}$$

$$\Rightarrow \int_0^t \int_0^{S_n} \cdots \int_0^{S_2} \prod_{k=1}^{N(t)} B(t, S_k) \frac{n!}{t^n} dS_1 dS_2 \cdots dS_n$$

3^{rd}. $E_{N(t)}[\int_0^t \int_0^{S_n} \cdots \int_0^{S_2} \underbrace{\prod_{k=1}^{N(t)} B(t, S_k)}_{\text{Symmetric}} \cdot \frac{n!}{t^n} dS_1 dS_2 \cdots dS_n] = E_{N(t)}[\frac{1}{n!} \int_0^t \int_0^t \cdots \int_0^t \prod_{k=1}^{N(t)} B(t, S_k) \cdot \frac{n!}{t^n} dS_1 \cdots dS_n]$

$$= E[\prod_{k=1}^{N(t)} \frac{1}{t} \int_0^t B(t, S_k) dS_k] = E_{N(t)}[(\frac{1}{t}\int_0^t B(t, S_k) dS_k)^{N(t)}] = G_{N(t)}(z)\Big|_{z=\frac{1}{t}\int_0^t B(t,s)ds}$$

$$= \exp(\lambda t(z-1))\Big|_{z=\frac{1}{t}\int_0^t B(t,s)ds} = \exp(\lambda t(\frac{1}{t}\int_0^t B(t,s)ds - 1)) = \exp(\lambda \int_0^t (B(t,s)-1) ds)$$

$\because B(t, s) = E[\exp(jwh(t, s, A_k))] \Rightarrow \exp(\lambda \int_0^t (E_A(\exp(jwh(t,s,A))) - 1) ds)$

$$E[Y(t)] = \frac{1}{j} \frac{d}{dw} \phi_Y(w)\Big|_{w=0} = \frac{1}{j} \cdot \lambda \int_0^t (E_A(\exp(jwh(t,s,A)) - 1) ds \cdot \int_0^t E_A(jh(t,s,A)\exp(jwh(t,s,A))) ds \cdot \sim \Big|_{w=0}$$

$$= \lambda \int_0^t E_A(h(t,s,A)) ds$$

Queueing : Service Model (serving status, queue length / customer arrival), # of counter)

2 r.v. 1 deterministic

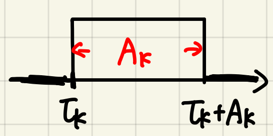
Kendall : M/M/K M: Markov
G G G: General

M/G/∞ infinite counter
arrival serve time

$Y(t)$: # in Service System, Queue length.

$E[Y(t)] = ?$ Filtered Poisson Process

$$Y(t) = \sum_{k=1}^{N(t)} h(t, S_k, A_k)$$

$h(\cdot)$: rect window, 

$h(t, \tau_k, A_k) = u(t) - u(t+A_k)$

if # of counter is limited, filtered Poisson is no longer applicable. (A_k is not independent, waiting time $\neq 0$)

where in filtered Poisson, $\{A_k\}$ has to be independent. $\Rightarrow$ Markov Process. (Next Lecture)