Markov Processes.

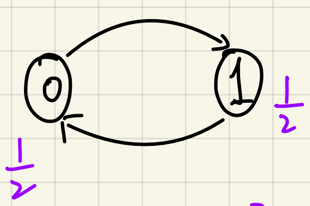
i. Recurrent, Long-Run Behavior $\Rightarrow$ j Non-Recurrent : $\forall i$, $P_{ij}(n) \xrightarrow{n\to\infty} 0$

$P_{ij}(n) \xrightarrow{n\to\infty} \Pi_j$ Exist ?

Long-term behavior (limit) : Process Property $\downarrow$, Randomness stays

(on diff. states , prob. $\Leftarrow$ freq. $= \dfrac{\#\text{ of this states}}{\text{length of the chain}}$)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$P_{10}(n) = P_{01}(n) = \{1, 0, 1, 0, \cdots\}$

$P_{00}(n) = P_{11}(n) = \{0, 1, 0, 1, \cdots\}$

does not converge (oscillating periodically)

$$B\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Define a concept about the existence of $\vec{\Pi} = [\Pi_1, \cdots, \Pi_n]^T \Rightarrow$ Periodic Property

Define a concept encompassing the above example into long-term behavior.

Periodic Property: i , $d_i = \underbrace{GCD}_{}\{k : P_{ii}(k) > 0\}$ Greatest Common Divisor

$d_i = 1 \Rightarrow i$: Non-Periodic

Theorem.

$i \leftrightarrow j \iff d_i = d_j$ (S is irreducible $d \to S$ | d is a property of S))

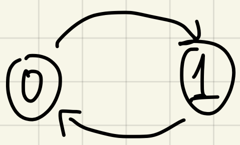
$i \leftrightarrow j$, $\exists m$, $P_{ij}(m) > 0$, $\exists n$, $P_{ji}(n) > 0$. $\Rightarrow P_{ii}(m+n) > 0$, Denote $\{k : P_{ii}(k) > 0\}$ as R_i

i) $P_{ii}(m+n) > 0 \Rightarrow m+n \in R_i \Rightarrow d_i \mid (m+n)$

ii) $\forall k \in R_j$, $P_{ii}(m+k+n) > 0 \Rightarrow m+k+n \in R_i \Rightarrow d_i \mid (m+n+k)$

iii) $d_i \mid (m+n+k) - (m+n)$, i.e. $d_i \mid k \Rightarrow \forall k \in R_j$. $d_i \mid k$, d_i is a common divisor of R_j ($d_i = CD\{R_j\}$)

iv) $d_i = CD\{R_j\}$. $d_j = GCD\{R_j\} \Rightarrow d_i \mid d_j$



$P_{11}(n) = P_{00}(n) = \{0, 1, 0, 1, 0, 1, \cdots\}$

period : 2 (limit is not explicit)

vice versa ($d_j \mid d_i$) $\Rightarrow \begin{cases} d_i \mid d_j \\ d_j \mid d_i \end{cases} \Rightarrow d_i = d_j$

Theorem. Markov Chain S irreducible, non-periodic ($\forall i \in [1, n]$, $d_i = 1$)

$\Rightarrow P_{ij}(n) \xrightarrow{n\to\infty} \Pi_j$ (limit exists)

periodic is not a general property unless S is irreducible. (if S is reducible, period is only a local property)

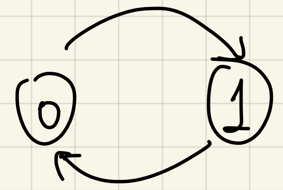
$\begin{cases} \text{Positive Recurrent} \\ \text{Null Recurrent} \end{cases}$ 稍弱

$i \xrightarrow{\text{Criterion}}$ 研究状态 j

$\forall i \in n, \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n) = ?$

$\left(a_n \to a \Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_i a_i = a\right)$ $\not\Leftarrow$

做一个较弱的定义 (weaker definition)



definitely Recurrent.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n) = \begin{cases} >0, & j \text{ Positive Recurrent} \\ =0, & j \text{ Null Recurrent} \end{cases}$$

$P_{i0}(n) \longrightarrow ?\ (i=0,1)$ diverge

$\frac{1}{N}\sum_{n=1}^{N} P_{i0}(n) \xrightarrow{N\to\infty} \frac{1}{2}$, Positive Recurrent

Weak Ergodic

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n) = \frac{1}{\sum_{k=1}^{\infty} k f_{ii}(k)}$$ Cessaro Sum

$f_{00}(k) = \{0, 1, 0, 0, \cdots\}$ $\quad \sum_{k=1}^{\infty} k f_{00}(k) = 2$

Even if Transitional Prob. is not convergent, we can go to Cessaro Sum / Weak Ergodic (Pos./Null Recurrent)

How to calculate $\lim_{n\to\infty} P_{ij}(n)$? (if limit does exist)

$\begin{cases} P(n) = P(1)P(n-1) & \text{后退方程} \\ P(n) = P(n-1)P(1) & \text{前进方程} \end{cases}$ ✓

Suppose $P_{ij}(n) \xrightarrow{n} \pi_j$

前进方程更本质!

$$P(n) \to \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [\pi_1 \cdots \pi_n]$$

行和为1, trivial

$$\begin{cases} \Pi = P(1)\cdot\Pi = P(1)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}[\pi_1, \cdots, \pi_n] & \left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - P(1)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right)(\pi_1, \cdots, \pi_n) = 0 \\ \Pi = \Pi\cdot P(1) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}[\pi_1 \cdots \pi_n] P(1) & \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\left([\pi_1 \cdots \pi_n] - [\pi_1 \cdots \pi_n]P(1)\right) = 0 \ ✓ \end{cases}$$

Let $\pi = (\pi_1, \cdots, \pi_n)$ row vector

$\pi = \pi P$

not always exist

$\pi \to$ Limit Distribution ($\sum_k \pi_k = 1$, finite states)

i) Stationary Distribution.

$P(X_0 = k) = \pi_k \Rightarrow P(X_n = k) = \pi_k,\ \forall n$

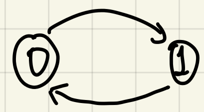
$$P(X_n = k) = \sum_l P(X_n = k, X_{n-1} = l) = \sum_l P(X_{n-1} = l)\, P(X_n = k \mid X_{n-1} = l) = \sum_l P_{lk}\, P(X_{n-1} = l)$$

Suppose $P(X_0 = k) = \pi_k \Rightarrow P(X_1 = k) = \sum_l P_{lk} P(X_0 = l) = \sum_l P_{lk}\pi_l$

$$\pi = \pi P \Rightarrow [\pi_1 \cdots \pi_k \cdots \pi_n] = [\pi_1 \cdots \pi_n]\begin{bmatrix} P_{11} & \cdots & P_{1k} & \cdots & P_{1n} \\ \vdots & & \vdots & & \vdots \\ P_{n1} & \cdots & P_{nk} & \cdots & P_{nn} \end{bmatrix} \Rightarrow \forall k \in [1, n],\ \pi_k = \sum_i \pi_i \cdot P_{ik} \Rightarrow P(X_1 = k) = \pi_k$$

if Limit Distribution does not exist, what's the implication of π ($\pi = \pi P$)?

$\Rightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}(n)$



$\downarrow$

π_j does not exist

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \begin{cases} \pi = \pi P \\ \sum \pi_i = 1 \end{cases} \Rightarrow \pi = [\frac{1}{2}, \frac{1}{2}] = \lim_{N\to\infty} \frac{1}{N}\sum P_{ij}(n)$

ii) $\{\exists \pi \neq 0,\ \pi = \pi P\} \Leftrightarrow \{\exists \pi \neq 0,\ P^T\pi^T = \pi^T\} \Leftrightarrow \{\exists \pi \neq 0,\ (P-I)^T\pi^T = 0\} \Leftrightarrow \det(P-I)=0 \Leftrightarrow 1=\mathrm{eig}(P)$

Singular

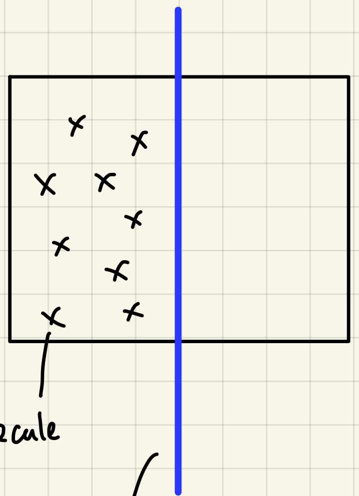
$\{\exists \lambda = 1,\ \det(\lambda I - P) = 0\}$

$\Uparrow$

$\left\{P\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = I\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right\}$

行和为1 $\sum_j P_{ij}(1)=1$ ⇒ 每行减1，一定奇异

Example: Ehrenfest Model (Markov Chain: characterize molecular motion)



molecule

撤去板子，如何演化？

i) $S=\{1,2,\dots,N\}$ # of particles. State Space: # of particles in left half space.

ii) Discrete Time: only 1 particle change its state in a time unit.

X_n: # of particles in left half on time n

$\{X_n\}$: Markov Chain

$$\begin{array}{c|c}
 & \begin{matrix}0 & 1 & 2 & \cdots & N\end{matrix} \\ \hline
\begin{matrix}0\\1\\ \vdots \\ \\ N\end{matrix} & \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\frac{1}{N} & 0 & \frac{N-1}{N} & & \vdots \\
 & \frac{2}{N} & 0 & \frac{N-2}{N} & \\
\vdots & & \ddots & \ddots & \\
 & & \frac{N-1}{N} & 0 & \frac{1}{N} \\
0 & \cdots & & 1 & 0
\end{bmatrix}
\end{array}$$

Transition Prob. P

Assumption: equal chance of moving to the other side.

$\pi = \pi P$

$P^T\pi^T = \pi^T$

$$\begin{bmatrix}
0 & \frac{1}{N} & & & \\
1 & 0 & \frac{2}{N} & & \\
 & \frac{N-1}{N} & 0 & \ddots & \\
 & & \frac{N-2}{N} & \ddots & \frac{N-1}{N} \\
 & & & \ddots & 0 & 1 \\
 & & & & \frac{1}{N} & 0
\end{bmatrix}\pi^T = \pi^T$$

$$P - I = \begin{bmatrix}
-1 & \frac{1}{N} & & & \\
1 & -1 & \frac{2}{N} & & \\
 & \frac{N-1}{N} & -1 & \frac{3}{N} & \\
 & & \ddots & \ddots & \\
 & & \frac{2}{N} & -1 & 1 \\
 & & & \frac{1}{N} & -1
\end{bmatrix}$$

$\pi_0 = \frac{1}{N}\pi_1 \quad (\pi_1 = \frac{N}{1}\pi_0) \qquad \pi_1 = \binom{N}{1}\pi_0$

$\pi_0 + \frac{2}{N}\pi_2 = \pi_1 \Rightarrow \pi_2 = \frac{N-1}{2}\pi_1 \qquad \pi_2 = \frac{N(N-1)}{2!}\pi_0 = \binom{N}{2}\pi_0$

$\frac{N-1}{N}\pi_1 + \frac{3}{N}\pi_3 = \pi_2 \Rightarrow \pi_3 = \frac{N-2}{3}\pi_2 \qquad \pi_3 = \binom{N}{3}\pi_0$

$\vdots$

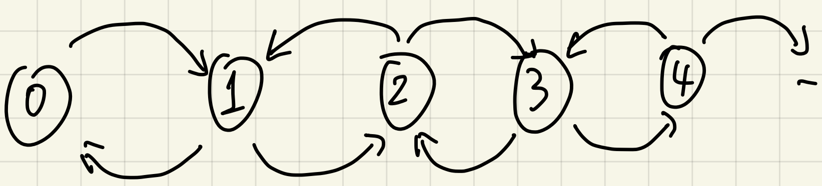
$\cdots\cdots \Rightarrow \pi_k = \frac{N-k+1}{k}\pi_{k-1}$

$\pi_N = \frac{1}{N}\pi_{N-1} \qquad \pi_N = \binom{N}{N}\pi_0$

$\therefore \pi_0 = \frac{1}{\sum_{k=0}^{N}\binom{N}{k}} = \frac{1}{2^N}$

$\therefore i\in[0,N],\ \pi_i = \frac{\binom{N}{i}}{\sum_{k=0}^{N}\binom{N}{k}} = \frac{\binom{N}{i}}{2^N}$

$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P_{ij}(n)$

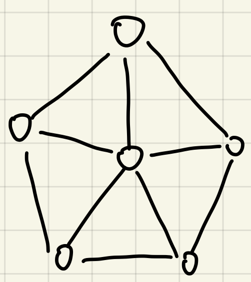


$\cdots\cdots$ $d=2$, periodic $\therefore$ Limit Distribution does not exist.

$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P_{i0}(n) = \frac{1}{\sum_{k=1}^{\infty}kf_{00}(k)} = \pi_0 = \frac{1}{2^N}$ Expectation of 1st Passage $=2^N$，很漫长

N: 阿伏加德罗常数.

Example: Random Walk on Graph ⇒ PageRank Algorithm (Google)



degree: 边数 d_i, prob. of each vertex $= \frac{1}{d_i}$

N nodes. $\pi_k = \frac{d_k}{\sum_i d_i}$. Examine whether $\pi = \pi P$

$\sum_{k=1}^{N}\pi_k P_{k\ell} = \sum_{k=1}^{N}\frac{d_k}{\sum_i d_i}\cdot P_{k\ell}$

Uniform Dist. on node k

$$= \sum_{k=1,\, P_{kl}\neq 0}^{N} \frac{d_k}{\sum_i d_i} \cdot \frac{1}{d_k} \quad (P_{kl} = 0 \text{ or } \frac{1}{d_k},\ \text{non-zero term } \# = d_l)$$

$$= d_l \cdot \frac{d_k}{\sum_i d_i} \cdot \frac{1}{d_k} = \frac{d_l}{\sum_i d_i} = \pi_l$$

End of Proof.

Detailed Balance Relation: $\pi = (\pi_0, \cdots, \pi_k, \cdots)$, $\pi_i P_{ij} = \pi_j P_{ji}$

$$\sum_k \pi_k P_{kl} = \sum_k \pi_l P_{lk} = \pi_l$$

, Markov Chain Monte Carlo

Construct P from π:

i) $\forall P_{ij}$, $\pi_i P_{ij} \overset{?}{=} \pi_j P_{ji}$ $\quad \tilde{P}_{ij} = P_{ij} \min\left(1, \frac{\pi_j P_{ji}}{\pi_i P_{ij}}\right) \rightarrow$ Satisfy Detailed Balance Relation.

$$\pi_i \tilde{P}_{ij} = \pi_i P_{ij} \min\left(1, \frac{\pi_j P_{ji}}{\pi_i P_{ij}}\right) = \min(\pi_i P_{ij}, \pi_j P_{ji}) = \pi_j P_{ji} \min\left(1, \frac{\pi_i P_{ij}}{\pi_j P_{ji}}\right) = \pi_j \tilde{P}_{ji}$$

Metropolis - Hastings Algorithms.

MCMC

Next Lecture: Continuous - Time Markov Chain

Birth & Death Process