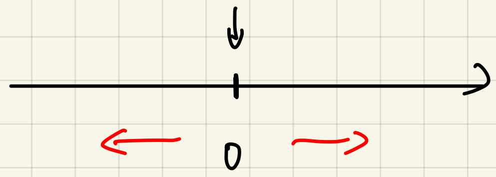
Gaussian Everywhere

① Diffusion. (Micro & Macroscopic View)

ink pour on the axis at $t=0$



diffuse gradually

Spatial distribution of ink ?

Suppose Distribution $f(x,t)$ $t \to$ variance (spread over time)

Introduce $p(y,\tau)$: describe the velocity of ink diffusing

random

percentage of ink diffusing distance y in duration τ also a distribution.

i) $f(x,t+\tau) = \int_{\mathbb{R}} p(y,\tau)\cdot f(x-y,t)\,dy$ key equation

ii) $\int_{\mathbb{R}} p(y,\tau)\,dy = 1$

iii) $\int_{\mathbb{R}} y\,p(y,\tau)\,dy = 0$

iv) $\int_{\mathbb{R}} y^2 p(y,\tau)\,dy = D(\tau)$

have to eliminate $p(y,\tau)$ in the process of derivation to arrive at $f(x,t)$:

Taylor Expansion of $f(x-y,t)$:

$$f(x-y,t) = f(x,t) + (-y)\frac{\partial}{\partial x} f(x,t) + \frac{y^2}{2}\frac{\partial^2}{\partial x^2} f(x,t) + \cdots$$

(y small enough $\underset{\text{Taylor}}{}\neq$ $\underset{\text{Integration range}}{y\in\mathbb{R}}$)

just ignore this imperfection

$$\therefore \int_{\mathbb{R}} \left[f(x,t) + (-y)\frac{\partial}{\partial x} f(x,t) + \frac{(-y)^2}{2}\frac{\partial}{\partial x} f(x,t)\right]\cdot p(y,\tau)\,dy = f(x,t+\tau)$$

$$\Rightarrow f(x,t) + \frac{D}{2}\frac{\partial^2}{\partial x^2} f(x,t) = f(x,t+\tau), \quad f(x,t+\tau) - f(x,t) = \frac{D}{2}\frac{\partial^2}{\partial x^2} f(x,t) \quad \frac{f(x,t+\tau)-f(x,t)}{\tau} = \frac{D}{2\tau}\frac{\partial^2}{\partial x^2} f(x,t)$$

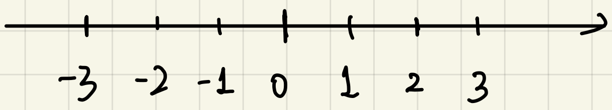
iv) $\int_{\mathbb{R}} y^2 p(y,\tau)\,dy = D(\tau)$ when $\tau\to 0$, $D(\tau)\to 0$ (no time for diffusing, hence no spreading)

Suppose $\lim\limits_{\tau\to 0}\frac{D(\tau)}{\tau} = D \Rightarrow \frac{\partial f}{\partial t} = \frac{D}{2}\frac{\partial^2 f}{\partial x^2}$ Diffusion Equation

$$f(x,0) = \delta(x) \Rightarrow f(x,t) = \frac{1}{\sqrt{2\pi D t}}\exp\left(-\frac{x^2}{2Dt}\right)$$

Discrete Interpretation of this work:

$P(m,n)$: position m, time n, Particle Number.



Random Walk. $P_{left} = P_{right} = \frac{1}{2}$, $P(m,n+1) = ?$

does not linger (always moving)

$$P(m,n+1) = \frac{1}{2}P(m+1,n) + \frac{1}{2}P(m-1,n)$$

2nd order difference on space

$$\underbrace{P(m,n+1) - P(m,n)}_{\text{1st order difference on time.}} = \frac{1}{2}\left[P(m+1,n) - P(m,n)\right] - \frac{1}{2}\left[P(m,n) - P(m-1,n)\right]$$

$\boxed{\frac{\partial P}{\partial t} = C\cdot\frac{\partial^2 P}{\partial x^2}}$ $\to$ Diffusion Equation

$\Downarrow$

Gaussian Distribution

② Maximum Entropy:

distribution $f(x)$. $f(x) \geq 0$. $\int_{\mathbb{R}} f(x)dx = 1$

Define Entropy for $f(x)$:

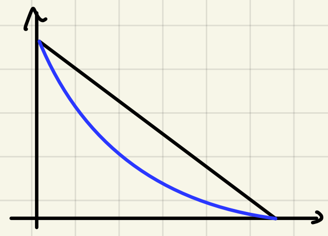
$H(f) = -\int_{\mathbb{R}} f(x) \log f(x) dx$ ← Measure of Randomness

$\max_{f} H(f) = ?$, $f(x)$?

functional analysis

Uniform Distribution only works in bounded case ($|x| \leq B$)

eg. of functional analysis: Brachistochrone Curve



Methodology: Variational Method.

$\max_{f} H(f) \Rightarrow f_0(x)$: optimal solution, $g(t) = H(f_0 + t \cdot h)$ $h(x)$: arbitrary function

$t \in \mathbb{R}$

Since $f_0(x)$ is optimal, $g(0) \geq g(t) \Rightarrow \left.\frac{d\,g(t)}{dt}\right|_{t=0} = 0$

$g(t) = H(f_0 + th) = -\int_{\mathbb{R}} (f_0 + th) \log(f_0 + th)\, dx \Rightarrow$ function of t

has to optimize the variance and Scaling factor as well $\Rightarrow \max_{f} H(f)$, s.t. $E[X] = m$, $E[X^2] = 6^2$

characterize randomness

1st, 2nd order moment

$\therefore \max_{f} H(f)$, $E[X] = m$, $E[X^2] = 6^2$

$$L(t, \lambda_1, \lambda_2, \lambda_3) = \int_{\mathbb{R}} (f_0 + t \cdot h) \log(f_0 + t \cdot h) dx + \lambda_1 \left[\int_{\mathbb{R}} x (f_0 + t \cdot h) dx - m\right] + \lambda_2 \left[\int_{\mathbb{R}} x^2 (f_0 + t \cdot h) dx - 6^2\right] + \lambda_3 \left[\int_{\mathbb{R}} (f_0 + th) dx - 1\right]$$

$$\frac{\partial}{\partial t} L(t, \lambda_1, \lambda_2, \lambda_3) = 0$$

$$\frac{\partial}{\partial t} L = \int_{\mathbb{R}} (h \log(f_0 + th) + h) dx + \lambda_1 \int_{\mathbb{R}} x \cdot h\, dx + \lambda_2 \int_{\mathbb{R}} x^2 \cdot h\, dx + \lambda_3 \int_{\mathbb{R}} h\, dx = 0$$

$$= \int_{\mathbb{R}} h(x) \left[\log(f_0 + th) + \lambda_1 x + \lambda_2 x^2 + \lambda_3'\right] dx$$

$\left.\frac{d\,g(t)}{dt}\right|_{t=0} = 0 \Rightarrow \left.\frac{\partial L}{\partial t}\right|_{t=0} = 0$, $\therefore \forall h$, $\int_{\mathbb{R}} h(x) \left[\log f_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3'\right] dx = 0$

Gaussian

$f_0 = \exp(-\lambda_2 x^2 - \lambda_1 x + \lambda_3') \cdot C$

Central Limit Theorem: $E[X_k] = 0$. $E[X_k^2] = 1$, $\{X_n\}$ i.i.d. Then $\frac{1}{\sqrt{n}} \sum X_k \to N(0,1)$

Characteristic Function: $X \sim f_X(x)$. $\phi_X(w) = E[\exp(jwX)] = \int_{\mathbb{R}} \exp(jwx) f_X(x) dx$.

X_1, X_2 indep. $Y = X_1 + X_2$, $Y \sim f_{X_1} * f_{X_2}$

Proof: $f_Y = f_{X_1} * f_{X_2}$

$$\phi_Y(w) = E[\exp(jw(X_1+X_2))] = E[\exp(jwX_1)\exp(jwX_2)] = E[\exp(jwX_1)]\,E[\exp(jwX_2)] = \phi_{X_1}\cdot\phi_{X_2}$$

$$\phi_Y(w) = \phi_{X_1}\phi_{X_2} \xleftrightarrow{F.T.} f_Y = f_{X_1} * f_{X_2}$$

CLT

$$\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{\sqrt{n}}, \quad E[X_k]=0,\ \text{var}(X_k)=1,\ k=1\cdots n$$

$$\phi_{\frac{\sum X_k}{\sqrt{n}}} = E\left[\exp\left(jw\frac{\sum X_k}{\sqrt{n}}\right)\right] = E\left[\prod_k \exp\left(j\frac{w}{\sqrt{n}}X_k\right)\right] = \prod_k E\left[\exp\left(j\frac{w}{\sqrt{n}}X_k\right)\right] = \prod_k \phi_{X_k}\left(\frac{w}{\sqrt{n}}\right) \xlongequal{i.i.d.} \left[\phi_{X_1}\left(\frac{w}{\sqrt{n}}\right)\right]^n$$

$$\phi_{X_1}\left(\frac{w}{\sqrt{n}}\right) = E\left[\exp\left(j\frac{w}{\sqrt{n}}X_1\right)\right] = E\left[1+j\frac{w}{\sqrt{n}}X_1 + \underbrace{\frac{1}{2!}\left(j\frac{w}{\sqrt{n}}X_1\right)^2 + o\left(\frac{1}{n}\right)}_{\sim\frac{1}{n}}\right]$$

$$= E\left[1+j\frac{w}{\sqrt{n}}X_1 + \left(\frac{1}{2}\cdot-\frac{w^2}{n}X_1^2\right)+o\left(\frac{1}{n}\right)\right]$$

$$= E\left[1-\frac{w^2}{2}X_1^2\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)\right]$$

$$= 1-\frac{w^2}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \exp(a)$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}+o\left(\frac{1}{n}\right)\right)^n = \exp(a)$$

$$\lim_{n\to\infty}\left(1+\frac{a}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)^n \quad \text{unknown}$$

Gaussian FT Pair:

$$\exp\left(-\frac{t^2}{2\sigma^2}\right) \Leftrightarrow \exp\left(-\frac{\sigma^2}{2}w^2\right)$$

$$\therefore \phi_{\frac{\sum X_k}{\sqrt{n}}} = \left[1-\frac{w^2}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)\right]^n \xrightarrow[n\to\infty]{} \exp\left(-\frac{w^2}{2}\right) \quad \text{Gaussian}$$

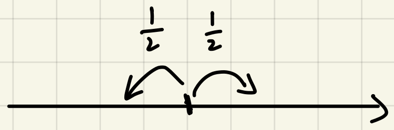
$$\phi_{\frac{X_1+\cdots+X_n}{n}}(w) = (\cdots)^n, \quad E[X_k]=m,\ \text{i.i.d.}$$

$$= E\left[1+j\frac{w}{n}X+o\left(\frac{1}{n}\right)\right]^n = \left[1+j\frac{w}{n}m+o\left(\frac{1}{n}\right)\right]^n \xrightarrow{n\to\infty} \exp(jwm)$$

Constant m's characteristic func. $P(X=m)=1$

$$\therefore \frac{X_1+\cdots+X_n}{n} \xrightarrow{D.} m \iff \frac{X_1+\cdots+X_n}{n} \xrightarrow{P.} m \quad (\text{convergence to a constant})$$

Random Walk:



$$S_n = X_1+\cdots+X_n \qquad X_k \sim \begin{pmatrix}\Delta x & 0\\ 1/2 & 1/2\end{pmatrix}.$$

discrete $\to$ continuous ($\Delta t\to 0$, $\Delta x\to 0$)

$$E[X_k]=\frac{\Delta x}{2},\ \text{var}(X_k)=\frac{\Delta x^2}{4}$$

in $[0,t]$, $n=\frac{t}{\Delta t}$

$$\tilde{X}_k = \frac{X_k-\frac{\Delta x}{2}}{\sqrt{\frac{(\Delta x)^2}{4}}}, \quad X_k = \frac{\Delta x}{2}\tilde{X}_k+\frac{\Delta x}{2} \Rightarrow S_n = \sum_k X_k = \frac{\Delta x}{2}\sum_k\tilde{X}_k + \underbrace{\frac{n\Delta x}{2}}_{\text{deterministic}}$$

$$\frac{S_n-\frac{n\Delta x}{2}}{\frac{\Delta x}{2}} = \sum_1^n \tilde{X}_k \Rightarrow \frac{S_n-\frac{n\Delta x}{2}}{\frac{\Delta x}{2}\sqrt{n}} = \frac{\sum_1^n\tilde{X}_k}{\sqrt{n}} \xrightarrow{n\to\infty} N(0,1)$$

$$n = t/\Delta t \Rightarrow \frac{\Delta x}{2}\sqrt{n} = \frac{\Delta x}{2}\sqrt{\frac{t}{\Delta t}} = \sqrt{t}\,\frac{\Delta x}{2\sqrt{\Delta t}} \to \sqrt{D}$$

$\lim_{\Delta t\to 0}\Delta x = 0$ (when time slot approaches 0, $\Delta x\to 0$)

in Diffusion (1st example), $\frac{D(\tau)}{\tau}\xrightarrow{\tau\to 0} D$

$$\frac{\frac{\Delta x}{2}}{\sqrt{\Delta t}} \xrightarrow[\substack{n\to\infty\\ \Delta t\to 0}]{} \sqrt{D}, \quad \text{then} \quad \frac{S_n-\frac{n\Delta x}{2}}{\sqrt{t}\sqrt{D}} \sim N(0,1), \quad \text{var}(S_n) = Dt$$

$\sqrt{\text{var}(X_k)}$