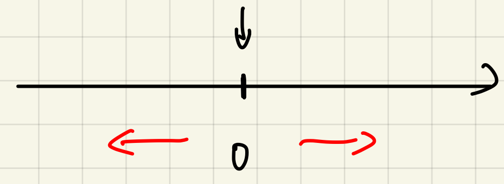
Gaussian Everywhere

ink pour on the axis at $t=0$



Spatial distribution of ink ?

① Diffusion. (Micro & Macroscopic View)

Suppose Distribution $f(x,t)$ $t \to$ variance (spread over time)

Introduce $P(y,\tau)$: describe the velocity of ink diffusing

random $\quad$ percentage of ink diffusing distance y in duration τ $\quad$ also a distribution.

i) $f(x,t+\tau) = \int_{\mathbb{R}} P(y,\tau)\cdot f(x-y,t)\,dy$ $\quad$ key equation

ii) $\int_{\mathbb{R}} P(y,\tau)\,dy = 1$

iii) $\int_{\mathbb{R}} y P(y,\tau)\,dy = 0$

iv) $\int_{\mathbb{R}} y^2 P(y,\tau)\,dy = D(\tau)$

have to eliminate $P(y,\tau)$ in the process of derivation to arrive at $f(x,t)$:

Taylor Expansion of $f(x-y,t)$:

$$f(x-y,t) = f(x,t) + (-y)\frac{\partial}{\partial x}f(x,t) + \frac{y^2}{2}\frac{\partial^2}{\partial x^2}f(x,t) + \cdots$$

(y small enough $\underset{\text{Taylor}}{\neq}$ $\underset{\text{Integration range}}{y\in\mathbb{R}}$)

just ignore this imperfection

$$\therefore \int_{\mathbb{R}} \left[f(x,t) + (-y)\frac{\partial}{\partial x}f(x,t) + \frac{(-y)^2}{2}\frac{\partial}{\partial x}f(x,t)\right]\cdot P(y,\tau)\,dy = f(x,t+\tau)$$

$$\Rightarrow f(x,t) + \frac{D}{2}\frac{\partial^2}{\partial x^2}f(x,t) = f(x,t+\tau), \quad f(x,t+\tau) - f(x,t) = \frac{D}{2}\frac{\partial^2}{\partial x^2}f(x,t) \quad \frac{f(x,t+\tau)-f(x,t)}{\tau} = \frac{D}{2\tau}\frac{\partial^2}{\partial x^2}f(x,t)$$

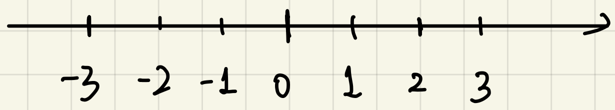
iv) $\int_{\mathbb{R}} y^2 P(y,\tau)\,dy = D(\tau)$ when $\tau\to 0$, $D(\tau)\to 0$ (no time for diffusing, hence no spreading)

Suppose $\lim\limits_{\tau\to 0}\frac{D(\tau)}{\tau} = D \Rightarrow \frac{\partial f}{\partial t} = \frac{D}{2}\frac{\partial^2 f}{\partial x^2}$ $\quad$ Diffusion Equation

$$f(x,0) = \delta(x) \Rightarrow f(x,t) = \frac{1}{\sqrt{2\pi Dt}}\exp\left(-\frac{x^2}{2Dt}\right)$$

Discrete Interpretation of this work:

$P(m,n)$: position m, time n, Particle Number.



Random Walk. $\quad P_{left} = P_{right} = \frac{1}{2}$, $P(m,n+1) = ?$

does not linger (always moving)

$$P(m,n+1) = \frac{1}{2}P(m+1,n) + \frac{1}{2}P(m-1,n)$$

2nd order difference on space

$$\underbrace{P(m,n+1) - P(m,n)}_{\text{1st order difference on time.}} = \frac{1}{2}\left[P(m+1,n) - P(m,n)\right] - \frac{1}{2}\left[P(m,n) - P(m,n-1)\right]$$

$\boxed{\frac{\partial P}{\partial t} = C\cdot\frac{\partial^2 P}{\partial x^2}}$ $\to$ Diffusion Equation

$\Downarrow$

Gaussian Distribution

② Maximum Entropy:

distribution $f(x)$. $f(x) \geq 0$. $\int_{\mathbb{R}} f(x)\,dx = 1$

Define Entropy for $f(x)$:

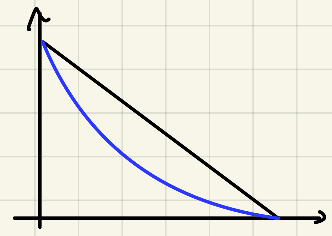
$H(f) = -\int_{\mathbb{R}} f(x) \log f(x)\,dx$ ← Measure of Randomness

$\max\limits_{f} H(f) = ?$, $f(x)$?

← functional analysis

Uniform Distribution only works in bounded case ($|x| \leq B$)

eg. of functional analysis: Brachistochrone Curve



Methodology: Variational Method.

$\max\limits_{f} H(f) \Rightarrow f_0(x)$: optimal solution, $g(t) = H(f_0 + t \cdot h)$ $h(x)$: arbitrary function

$t \in \mathbb{R}$

Since $f_0(x)$ is optimal, $g(0) \geq g(t) \Rightarrow \left.\frac{d\,g(t)}{dt}\right|_{t=0} = 0$

$g(t) = H(f_0 + th) = -\int_{\mathbb{R}} (f_0 + th) \log(f_0 + th)\,dx \Rightarrow$ function of t

has to optimize the variance and scaling factor as well $\Rightarrow \max\limits_{f} H(f)$, s.t. $E[X] = m$, $E[X^2] = \sigma^2$

characterize randomness

1st, 2nd order moment

$\therefore \max\limits_{f} H(f)$, $E[X] = m$, $E[X^2] = \sigma^2$

$$L(t, \lambda_1, \lambda_2, \lambda_3) = \int_{\mathbb{R}} (f_0 + t \cdot h) \log(f_0 + t \cdot h)\,dx + \lambda_1 \left[\int_{\mathbb{R}} x (f_0 + t \cdot h)\,dx - m\right] + \lambda_2 \left[\int_{\mathbb{R}} x^2 (f_0 + t \cdot h)\,dx - \sigma^2\right] + \lambda_3 \left[\int_{\mathbb{R}} (f_0 + th)\,dx - 1\right]$$

$$\frac{\partial}{\partial t} L(t, \lambda_1, \lambda_2, \lambda_3) = 0$$

$$\frac{\partial}{\partial t} L = \int_{\mathbb{R}} (h \log(f_0 + th) + h)\,dx + \lambda_1 \int_{\mathbb{R}} x \cdot h\,dx + \lambda_2 \int_{\mathbb{R}} x^2 \cdot h\,dx + \lambda_3 \int_{\mathbb{R}} h\,dx = 0$$

$$= \int_{\mathbb{R}} h(x) \left[\log(f_0 + th) + \lambda_1 x + \lambda_2 x^2 + \lambda_3'\right] dx$$

$\left.\frac{d\,g(t)}{dt}\right|_{t=0} = 0 \Rightarrow \left.\frac{\partial L}{\partial t}\right|_{t=0} = 0$, $\therefore \forall h$, $\int_{\mathbb{R}} h(x) \left[\log f_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3'\right] dx = 0$

Gaussian

$f_0 = \exp(-\lambda_2 x^2 - \lambda_1 x + \lambda_3') \cdot C$

Central Limit Theorem: $E[X_k] = 0$. $E[X_k^2] = 1$, $\{X_n\}$ i.i.d. , Then $\frac{1}{\sqrt{n}} \sum X_k \to N(0,1)$

Characteristic Function: $X \sim f_X(x)$, $\phi_X(w) = E[\exp(jwX)] = \int_{\mathbb{R}} \exp(jwx) f_X(x)\,dx$.

X_1, X_2 indep. $Y = X_1 + X_2$, $Y \sim f_{X_1} * f_{X_2}$

Proof: $f_Y = f_{X_1} * f_{X_2}$

$$\phi_Y(w) = E[\exp(jw(X_1+X_2))] = E[\exp(jwX_1)\exp(jwX_2)] = E[\exp(jwX_1)]\,E[\exp(jwX_2)] = \phi_{X_1}\cdot\phi_{X_2}$$

$$\phi_Y(w) = \phi_{X_1}\phi_{X_2} \xleftrightarrow{F.T.} F_Y = F_{X_1} * F_{X_2}$$

CLT

$$\lim_{n\to\infty}\frac{X_1+\cdots+X_n}{\sqrt n}, \quad E[X_k]=0, \quad var(X_k)=1, \quad k=1\cdots n$$

$$\phi_{\frac{\sum X_k}{\sqrt n}} = E\left[\exp\left(jw\frac{\sum X_k}{\sqrt n}\right)\right] = E\left[\prod_k \exp\left(j\frac{w}{\sqrt n}X_k\right)\right] = \prod_k E\left[\exp\left(j\frac{w}{\sqrt n}X_k\right)\right] = \prod_k \phi_{X_k}\left(\frac{w}{\sqrt n}\right) \overset{i.i.d.}{=\!=\!=} \left[\phi_{X_1}\left(\frac{w}{\sqrt n}\right)\right]^n$$

$$\phi_{X_1}\left(\frac{w}{\sqrt n}\right) = E\left[\exp\left(j\frac{w}{\sqrt n}X_1\right)\right] = E\left[1+j\frac{w}{\sqrt n}X_1 + \underbrace{\frac{1}{2!}\left(j\frac{w}{\sqrt n}X_1\right)^2}_{\frac{1}{n}} + o\left(\frac1n\right)\right]$$

$$\lim_{n\to\infty}\left(1+\frac an\right)^n = \exp(a)$$

$$\lim_{n\to\infty}\left(1+\frac an + o\left(\frac1n\right)\right)^n = \exp(a)$$

$$\lim_{n\to\infty}\left(1+\frac{a}{\sqrt n} + o\left(\frac{1}{\sqrt n}\right)\right)^n \quad \text{unknown}$$

$$= E\left[1+j\frac{w}{\sqrt n}X_1 + \left(\frac12 - \frac{w^2}{n}X_1^2\right) + o\left(\frac1n\right)\right]$$

$$= E\left[1-\frac{w^2}{2}X_1^2\cdot\frac1n + o\left(\frac1n\right)\right]$$

$$= 1-\frac{w^2}{2}\cdot\frac1n + o\left(\frac1n\right)$$

Gaussian FT Pair:

$$\exp\left(-\frac{t^2}{2\sigma^2}\right) \Leftrightarrow \exp\left(-\frac{\sigma^2}{2}w^2\right)$$

$$\therefore \phi_{\frac{\sum X_k}{\sqrt n}} = \left[1-\frac{w^2}{2}\cdot\frac1n + o\left(\frac1n\right)\right]^n \xrightarrow[n\to\infty]{} \exp\left(-\frac{w^2}{2}\right) \quad \text{Gaussian}$$

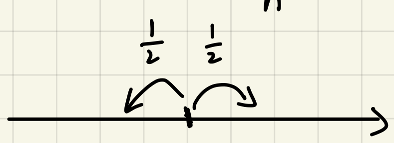
$$\phi_{\frac{X_1+\cdots+X_n}{n}}(w) = (\cdots)^n. \quad E[X_k]=m, \ i.i.d.$$

$$= E\left[1+j\frac{w}{n}X + o\left(\frac1n\right)\right]^n = \left[1+j\frac{w}{n}m + o\left(\frac1n\right)\right]^n \xrightarrow{n\to\infty} \exp(jwm)$$

Constant m's characteristic func. $P(X=m)=1$

$$\therefore \frac{X_1+\cdots+X_n}{n} \xrightarrow{D.} m \Leftrightarrow \frac{X_1+\cdots+X_n}{n} \xrightarrow{P.} m \quad (\text{Convergence to a constant})$$

Random Walk:



$S_n = X_1+\cdots+X_n$ $\quad X_k \sim \begin{pmatrix}\Delta x & 0 \\ 1/2 & 1/2\end{pmatrix}$.

discrete → Continuous ($\Delta t \to 0$, $\Delta x \to 0$)

$$E[X_k] = \frac{\Delta x}{2}, \quad var(X_k) = \frac{\Delta x^2}{4}$$

in $[0,t]$, $n = \frac{t}{\Delta t}$ $\quad \tilde X_k = \frac{X_k - \frac{\Delta x}{2}}{\sqrt{\frac{(\Delta x)^2}{4}}}$, $\quad X_k = \frac{\Delta x}{2}\tilde X_k + \frac{\Delta x}{2} \Rightarrow S_n = \sum_k X_k = \frac{\Delta x}{2}\sum_k \tilde X_k + \underline{\frac{n\Delta x}{2}}$ deterministic

$$\frac{S_n - \frac{n\Delta x}{2}}{\frac{\Delta x}{2}} = \sum_1^n X_k \Rightarrow \frac{S_n - \frac{n\Delta x}{2}}{\frac{\Delta x}{2}\sqrt n} = \frac{\sum_1^n \tilde X_k}{\sqrt n} \xrightarrow{n\to\infty} N(0,1)$$

$$n = t/\Delta t \Rightarrow \frac{\Delta x}{2}\sqrt n = \frac{\Delta x}{2}\sqrt{\frac{t}{\Delta t}} = \sqrt t\,\frac{\Delta x}{2\sqrt{\Delta t}} \to \sqrt D$$

$\lim_{\Delta t\to 0}\Delta x = 0$ (when time slot approaches 0, $\Delta x \to 0$)

in Diffusion (1st example), $\frac{D(\tau)}{\tau} \xrightarrow{\tau\to 0} D$ $\quad \frac{\underbrace{\frac{\Delta x}{2}}_{\sqrt{var(X_k)}}}{\sqrt{\Delta t}} \xrightarrow[\Delta t\to 0]{n\to\infty} \sqrt D$, then $\frac{S_n - \frac{n\Delta x}{2}}{\sqrt t\sqrt D} \sim N(0,1)$, $var(S_n) = Dt$