Poisson Processes 2

Sparsity may not be necessary during the derivation. of MGF. $G_{N(t)}(z)$.

$$\frac{1}{\Delta t}\left[G_{N(t+\Delta t)}(z) - G_{N(t)}(z) \right] = \frac{1}{\Delta t} G_{N(t)}(z) E[z^{N(\Delta t)} - 1]$$

$$E[z^{N(\Delta t)} - 1] = \underbrace{P(N(\Delta t) = 0) - 1}_{\exp(-\lambda \Delta t)} + P(N(\Delta t) = 1) \cdot z + \sum_{k \geq 2} z^k \cdot P(N(\Delta t) = k)$$

$M(t) = P(N(t) = 1)$, $[0, t] = [0, s] \cup [s, t] \Rightarrow P(N(t) = 1) = P(N(s) = 1, N(t) - N(s) = 0) + P(N(s) = 0, N(t) - N(s) = 1)$

$$\xrightarrow[\text{increment}]{\text{indep.}} P(N(s) = 1) \cdot P(N(t) - N(s) = 0) + P(N(s) = 0) \cdot P(N(t) - N(s) = 1)$$

$$\xrightarrow[\text{increment.}]{\text{stationary}} \underset{M(s)}{P(N(s) = 1)} \cdot \underset{Z(t-s)}{P(N(t-s) = 0)} + \underset{Z(s)}{P(N(s) = 0)} \cdot \underset{M(t-s)}{P(N(t-s) = 1)}$$

$$M(t) \overline{\overline{}} M(s) \cdot Z(t-s) + Z(s) M(t-s) \qquad Z(t) = P(N(t)=0) = \exp(-\lambda t)$$

$$\Rightarrow M(t+s) = M(t) \cdot Z(s) + M(s) Z(t) \Rightarrow \frac{M(t+s)}{Z(t)Z(s)} = \frac{M(t)}{Z(t)} + \frac{M(s)}{Z(s)} \quad (Z(t+s) = \exp(-\lambda(t+s)) = Z(t) Z(s))$$

$$\Rightarrow \frac{M(t+s)}{Z(t+s)} = \frac{M(t)}{Z(t)} + \frac{M(s)}{Z(s)} \quad \lambda(s) \Rightarrow \lambda(t+s) = \lambda(t) + \lambda(s) \quad \therefore \lambda(t) = \alpha t$$

$$\therefore M(t) = Z(t) \cdot \lambda(t) = \alpha t \cdot \exp(-\lambda t) \qquad P(N(\Delta t) = 0) + P(N(\Delta t) = 1) + P(N(\Delta t) \geq 2) = 1$$

$$\frac{\exp(-\lambda \Delta t)}{1 - P(N(\Delta t) = 0)} \frac{\alpha \Delta t \cdot \exp(-\lambda \Delta t)}{\Delta t} = \frac{P(N(\Delta t) = 1)}{\Delta t}\left(1 + \frac{P(N(\Delta t) \geq 2)}{P(N(\Delta t) = 1)}\right)$$

$$\lambda \exp(-\lambda \Delta t) \quad \alpha \exp(-\lambda \Delta t)$$

$$\text{let } \alpha = \lambda \Rightarrow \frac{P(N(\Delta t) \geq 2)}{P(N(\Delta t) = 1)} \xrightarrow{\Delta t \to 0} 0$$

i) Non-Stationary Increment: $\Rightarrow$ Need Another Assumption. $\displaystyle\lim_{\Delta t \to 0} \frac{1 - P(N(t+\Delta t) - N(t) = 0)}{\Delta t} = \lambda(t)$ time-dependent.

$$\begin{cases} \dfrac{d}{dt} G_{N(t)}(z) = G_{N(t)}(z) \lambda(t)(z-1) \\ G_{N(0)}(z) = E[z^{N(0)}] = E[z^0] = 1 \end{cases}$$

$$\Rightarrow G_{N(t)}(z) = \exp\left((z-1)\int_0^t \lambda(s)ds\right), \quad P(N(t) = k) = \frac{\int_0^t \lambda(s)ds}{k!} \exp\left(-\int_0^t \lambda(s)ds\right)$$

Non-Homogeneous Poisson. $\lambda \equiv \lambda(t)$ $\left(\displaystyle\lim_{\Delta t \to 0} \frac{1 - P(N(t+\Delta t) - N(t) = 0)}{\Delta t} = \lambda(t) \right)$

Counting Events. $\Rightarrow$ Effect Statistics

— Compound Poisson Processes

$$\boxed{Y(t) = \sum_{k=1}^{N(t)} X_k}$$, $N(t)$: Standard Poisson. X_k i.i.d. , indep. of $N(t)$

$$X_k \equiv 1 \Rightarrow Y(t) = N(t)$$

$$G_{Y(t)}(z) = E[z^{Y(t)}] = E\left[z^{\sum_1^{N(t)} X_k}\right] = E\left[E\left[z^{\sum_1^{N(t)} X_k} \mid N(t) = n\right]\right] = E\left[E\left[z^{\sum_1^n X_k} \mid N(t) = n\right]\right] = E\left[E\left[z^{n X_k} \mid N(t) = n\right]\right] = E\left[G_{X_1}^{N(t)}(z)\right] = E[z^{N(t)}]\Big|_{z = G_{X_1}(z)}$$

$$= G_{N(t)}(G_{X_1}(z)) \quad \left(G_{N(t)}(z) = \exp(\lambda t(z-1)) \right) = \exp\left(\lambda t (G_{X_1}(z) - 1)\right)$$

Ex.1:

$N(t)$, $X_k \sim B\begin{pmatrix} M & F \\ P & 1-P \end{pmatrix}$ only focuses on 'M' $\Rightarrow$ still Standard Poisson? : $X_k \overset{i.i.d.}{\sim} \begin{pmatrix} 1 & 0 \\ P & 1-P \end{pmatrix}$

$G_{Y(t)}(z) = G_{N(t)}(G_X(z)) = \exp(\lambda t(G_X(z)-1))$

$\color{red}{\lambda' = \lambda P}$

$\Rightarrow G_{Y(t)}(z) = \exp(\lambda t(Pz+1-P-1)) = \exp(\lambda t \cdot P(z-1)) = \exp(\lambda P \cdot t \cdot (z-1)) \color{red}{\Rightarrow Poisson}$

$G_X(z) = z \cdot P + 1 \cdot (1-P) = Pz + 1 - P$

Randomization. (Load Balance, Ethernet (CSMA/CD))

Ex. 2: $N_1(t): \lambda_1.$ $N_2(t): \lambda_2$ N_1 indep. N_2 , $N = N_1 + N_2$ Poisson?

$G_{N(t)}(z) = E[z^{N(t)}] = E[z^{N_1+N_2}] = E[z^{N_1(t)}] \cdot E[z^{N_2(t)}] = \exp(\lambda_1 t(z-1)) \exp(\lambda_2 t(z-1)) = \exp((\lambda_1+\lambda_2)t(z-1))$

$\Rightarrow N_k(t): \lambda_k,$ $k = 1 \sim n,$ $Y(t) = \sum N_k(t) \Rightarrow Y(t)$ Poisson, $\lambda = \sum\limits_k \lambda_k$

10min

Ex. 3.

2 Counters occupied. Serve time $\overset{i.i.d.}{\sim} \exp(\lambda)$, comes another Customer 3 $(t=0)$

$P\{$ cusmoter 3 leaves latest $\} = ?$ $\color{red}{\frac{1}{2}}$

Distribution of 1st leaving customer's leaving time $\sim \exp(\cdot)$

$\color{red}{\lambda' = \lambda_1 + \lambda_2 = 2\lambda}$

$T_1 \sim \exp(\lambda),$ $T_2 \sim \exp(\lambda),$ $T = \min(T_1, T_2) \sim \exp(2\lambda)$

$\underline{}$ $\underline{}$ 1st occurrence of $N_1 \cup N_2 \iff 1^{st}$ occurrence of Poisson Process N. intensity $= \lambda_1 + \lambda_2$

1st occurrence of a Poisson Process N_1, N_2 indep. ,

When one is vacant, and customer 3 filled in $\Rightarrow$ Still $\exp(\cdot)$, memoryless?

Memoryless: $P(X > x+y \mid X > y) = P(X > x)$, if y is r.v. (Y), X still memoryless?

Prove: if $X \sim \exp(\lambda_1),$ $Y \sim \exp(\lambda_2).$ $P(X > x+Y \mid X > Y) = P(X > x)$

$P(X > Y) = \int_0^\infty P(Y=y) \cdot P(X>Y \mid Y=y) \, dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \cdot \int_y^\infty \lambda_1 \exp(-\lambda_1 x) dx \cdot dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \cdot \exp(-\lambda_1 y) dy = \lambda_2 \cdot \int_0^\infty \exp(-(\lambda_1+\lambda_2)y) dy = \frac{\lambda_2}{\lambda_1+\lambda_2}$

$P(X > Y+x) = \int_0^\infty P(Y=y) \cdot P(X > Y+x \mid Y-y) \, dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \cdot \int_{x+y}^\infty \lambda_1 \exp(-\lambda_1 x) dx \cdot dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \exp(-\lambda_1(x+y)) dy = \lambda_2 \exp(-\lambda_1 x) \cdot \frac{1}{\lambda_1+\lambda_2}$

$= \frac{\lambda_2}{\lambda_1+\lambda_2} \exp(-\lambda_1 x)$

$P(X > x) = \int_x^\infty \lambda_1 \exp(-\lambda_1 s) ds = \exp(-\lambda_1 x).$

$\therefore P(X > x+Y \mid X>Y) = \frac{P(X > x+Y)}{P(X>Y)} = \exp(-\lambda_1 x) = P(X>x)$

$\Rightarrow$ Customer 3's leaving time $\sim \exp(\lambda)$, $P(\cdots) = \frac{1}{2}$

Ex. 4. $N(t) = N_1(t) - N_2(t).$ Still Poisson? $\times$ $P(N_1(t) - N_2(t) < 0) > 0$

Compound Poisson.

Compound Poisson: $\exp(\lambda t(G_X(z)-1))$

$G_{N(t)}(z) = E[z^{N(t)}] = E[z^{N_1-N_2}] = E[z^{N_1(t)}] \cdot E[z^{-N_2(t)}] = G_{N_1}(z) \cdot G_{N_2}(\frac{1}{z}) = \exp(\lambda_1 t(z-1)) \exp(\lambda_2 t(z^{-1}-1)) = \exp((\lambda_1+\lambda_2)t(\frac{\lambda_1 z + \lambda_2 z^{-1}}{\lambda_1+\lambda_2} - 1))$

$$G_N(z) = \exp\left[(\lambda_1+\lambda_2)t\left(\frac{\lambda_1 z+\lambda_2 z^{-1}}{\lambda_1+\lambda_2}-1\right)\right] \implies \text{Compound Poisson}$$

intensity

$$X \sim \begin{pmatrix} 1 & -1 \\ \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} \end{pmatrix}$$

Ex. 5: 2 counters $\sim \exp(\lambda_1), \exp(\lambda_2)$ are occupied. customer 3 comes in (t=0)

Expected leaving time of customer 3 = ?

Waiting + Serving:

Waiting: $\sim \exp(\lambda_1+\lambda_2)$, $E[\text{Waiting}] = \frac{1}{\lambda_1+\lambda_2}$

$\iint f_{T_1 T_2} dt_1 dt_2$ $t_1 < t_2$

Serving:
- #1 $P(T_1<T_2) \, E[T_1 | T_1<T_2] = P(T_1<T_2)\cdot\frac{1}{\lambda_1} = \frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{1}{\lambda_1}$
- #2 $P(T_2<T_1) \, E[T_2 | T_2<T_1] = P(T_1>T_2)\cdot\frac{1}{\lambda_2} = \frac{\lambda_2}{\lambda_1+\lambda_2}\cdot\frac{1}{\lambda_2}$

$\frac{2}{\lambda_1+\lambda_2}$

$$\implies E[T_3] = \frac{3}{\lambda_1+\lambda_2}$$

My Solution:

$3 \nearrow$ #1 $P(T_1<T_2) \implies t_1+E[T_1]$ $P(T_1<T_2) = P(T_1=t_1)\cdot P(T_2>T_1 | T_1=t) \implies P(T_1=t_1)\,P(T_2>t_1)\cdot(t_1+\frac{1}{\lambda_1})\int_0^\infty dt_1$

$\searrow$ #2 $P(T_1>T_2) \implies t_2+E[T_2]$ $P(T_1>T_2) \implies P(T_2=t_2)\,P(T_1>t_2)\cdot(t_2+\frac{1}{\lambda_2})\cdot\int_0^\infty dt_2$

$$\int_0^\infty \lambda_1 \exp(-\lambda_1 t_1)\cdot\int_{t_1}^\infty \lambda_2 \exp(-\lambda_2 s)ds\,(t_1+\frac{1}{\lambda_1})dt_1$$

$$= \int_0^\infty \lambda_1 \exp(-\lambda_1 t_1)\cdot\exp(-\lambda_2 t_1)(t_1+\frac{1}{\lambda_1})dt_1$$

$$\therefore \int_0^\infty \lambda_2 \exp(-\lambda_2 t_2)\int_{t_2}^\infty \lambda_1 \exp(-\lambda_1 s)ds\,(t_2+\frac{1}{\lambda_2})dt_2 = \frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{(\lambda_1+\lambda_2)^2}$$

$$= \int_0^\infty \lambda_1 \exp(-(\lambda_1+\lambda_2)t_1)(t_1+\frac{1}{\lambda_1})dt_1$$
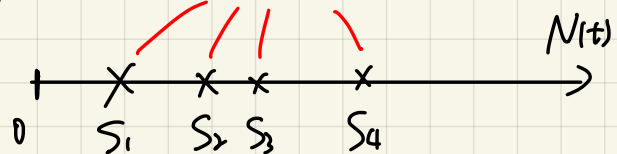
$$= \frac{1}{\lambda_1+\lambda_2}+\lambda_1\int_0^\infty t_1 \exp(-(\lambda_1+\lambda_2)t_1)dt_1$$

$$E[T_3] = \frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_1}{(\lambda_1+\lambda_2)^2}+\frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{(\lambda_1+\lambda_2)^2} = \frac{3}{\lambda_1+\lambda_2}$$

$$= \frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_1}{(\lambda_1+\lambda_2)^2}$$

Ex. 6: Event Occurrence

$N(t)$

$0 \quad S_1 \quad S_2\,S_3 \quad S_4$

$E[S_4 | N(t)=2] = ?$

$1 - P(N(t-1)=0) - P(N(t-1)=1) = 1 - \exp(-\lambda(t-1))[1+\frac{\lambda(t-1)}{1!}]$

$f_{S_4}(t|N(1)=2) = ? \implies F_{S_4}(t|N(1)=2) = P(S_4\le t | N(1)=2) = \frac{P(S_4\le t, N(1)=2)}{P(N(1)=2)} = \frac{P(N(t)-N(1)\ge 2, N(1)=2)}{P(N(1)=2)} = \frac{P(N(t)-N(1)\ge 2)\,P(N(1)=2)}{P(N(1)=2)}$

indep increment.

stationary increment. $\frac{P(N(t-1)\ge 2)\,P(N(1)=2)}{P(N(1)=2)} = 1-(\lambda(t-1)+1)\exp(-\lambda(t-1))$

$f_{S_4}(t|N(1)=2) = \frac{d F_{S_4}(t|N(1)=2)}{dt} = -\lambda \exp(-\lambda(t-1))+\lambda(\lambda(t-1)+1)\exp(-\lambda(t-1)) = \lambda^2(t-1)\exp(-\lambda(t-1))$

$= 1+\frac{2}{\lambda}$

$E[S_4 | N(1)=2] = \int_0^\infty t\cdot f_{S_4}(t|N(1)=2)dt = \int_0^\infty t\,\lambda^2(t-1)\exp(-\lambda(t-1))dt = \int_{-1}^\infty (t+1)\lambda^2 t\exp(-\lambda t)dt = \int_{-1}^\infty (\lambda^2 t^2+\lambda^2 t)\exp(-\lambda t)dt$

$N(1)=2$. 2 occurrences afterwards. each $\sim \exp(\lambda) \implies t = 1+2\cdot E[T(\lambda)] = 1+2\cdot\frac{1}{\lambda}$

$2^{nd} 3^{rd}: \frac{1}{\lambda}$
$1-3^{rd} < \frac{1}{\lambda}$

However $\quad 0 \quad 1$ $1\to 3^{rd}$ occurrence should be less than $E[T(\lambda)]=\frac{1}{\lambda}$, expectation should $1+$ portion of $\frac{1}{\lambda}+\frac{1}{\lambda}$

interval $\sim \exp(\lambda)$

not $1+\frac{1}{\lambda}+\frac{1}{\lambda}$