

# Continuous-Time Markov Chain

Discrete State Markov Processes  $\Rightarrow$  Chain.

$$P_{ij}(t) = P(X(t)=j | X(0)=i) \quad P_{ij}(n) = P(X_n=j | X_0=i) \quad C-K \text{ Equation: } P_{ij}(t+s) = \sum_k P_{ik}(s) \cdot P_{kj}(t)$$

$$P_{(m+n)} = P_{(m)} \cdot P_{(n)} \Rightarrow 1\text{-step transition prob. mat. } P_{(1)}, P_{(n)} = [P_{(1)}]^n$$

Matrix Form:  $P(t+s) = P(t) \cdot P(s)$ ,  $P(?)$  basic unit? Continuous-Time  $P(?)$ ?

$$P(t) \xrightarrow{t \rightarrow 0} I, \quad P(n+1) = P(n) P(1) \text{ (difference eqn.)} \quad \frac{1}{\Delta t} [P(t+\Delta t) - P(t)] = ?$$

$$\frac{1}{\Delta t} [P(t+\Delta t) - P(t)] = \frac{1}{\Delta t} [P(t) \cdot P(\Delta t) - P(t)] = P(t) \cdot \frac{1}{\Delta t} (P(\Delta t) - I), \quad \frac{d}{dt} P(t) = P(t) \cdot \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (P(\Delta t) - I) = P(t) \cdot Q$$

Forward

$$\begin{cases} \frac{d}{dt} P(t) = P(t) \cdot Q \\ P(0) = I \end{cases} \Rightarrow P(t) = \exp(Q t)$$

Generator (生成元)

Continuous-time  $P(?)$  (Q-Matrix)

$$\begin{cases} \frac{d}{dt} P(t) = Q \cdot P(t) \\ P(0) = I \end{cases} \Rightarrow P(t) = \exp(Q t)$$

Backward

$$P_{ij}(t) \geq 0, \quad \sum_j P_{ij}(t) = 1 \quad \text{v.s.} \quad Q_{ij} \cdot (P(\Delta t) - I)$$

$$Q_{ij} \begin{cases} \geq 0 & \text{on non-diagonal entry } (i \neq j) \\ \leq 0 & \text{on diagonal entry } (i=j) \end{cases}$$

$$\sum_j Q_{ij} = 0$$

$\Sigma.g.$  Poisson Process:  $P_{ij}(t) = P\{N(t)=j-i\} = \begin{cases} \frac{(\lambda t)^{j-i}}{(j-i)!} \exp(-\lambda t), & j-i \geq 0 \\ 0, & j-i < 0 \end{cases}$

diag. entry:  $\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (P_{ii}(\Delta t) - I) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\exp(-\lambda \Delta t) - I] = -\lambda < 0$

non-diag. entry:  $\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P_{ij}(\Delta t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \frac{(\lambda \Delta t)^{j-i}}{(j-i)!} \exp(-\lambda \Delta t) = \begin{cases} \lambda, & j=i+1 \\ 0, & j \geq i+2 \end{cases}$

$$\Rightarrow Q = \begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & -\lambda & \lambda \\ & & & \ddots \end{bmatrix}$$

最简单的 Q-Matrix

Poisson: 最简单的 MC

CT  $\rightarrow$  Sojourn (short stay) + Jumping

$$P_{ij}(t) \xrightarrow{t \rightarrow \infty} P_j \approx \frac{\text{Sojourn Time}}{\text{Total Time}}$$

irreducible. (对初值不敏感很重要, All states are commutable)

No period in cont.-time Markov Process.

(No  $P(?)$  in Cont.-time)  $\Rightarrow$  No GCD.

$$\text{period: } d_i = \text{GCD}\{k: P_{ii}(k) > 0\} \quad P_{ii}(1) > 0 \Rightarrow \text{non-periodic.}$$

Continuous-Time: irreducible  $\Rightarrow P_{ij}(t) \xrightarrow{t \rightarrow \infty} P_j$

C-K Equation:

$$Q P(t) = Q \begin{bmatrix} P_1 & P_2 & \dots & P_n \\ P_1 & P_2 & \dots & P_n \\ \vdots & \vdots & \ddots & \vdots \\ P_1 & P_2 & \dots & P_n \end{bmatrix} \Rightarrow \text{trivial}$$

$$\frac{d}{dt} P(t) = P(t) Q = Q \cdot P(t)$$

row sum = 0

$$t \rightarrow \infty, P \rightarrow P's \text{ limit, } \frac{d}{dt} P \rightarrow 0 \Rightarrow 0 = P \cdot Q (\pi Q)$$

Sojourn & jump  $\rightarrow$  Sojourn & jump  $\rightarrow$  Sojourn & jump.  
(Random) (不耗时间)

Sojourn time is random (Exponential Distribution)

jump (状态转移关系确定且不耗时)

Poisson (more complex)

Discrete Time:  $\pi = \pi P$

① Transition Relation

② Transition Prob.  $P(1)$

③  $\pi = \pi P \Rightarrow \pi = ?$

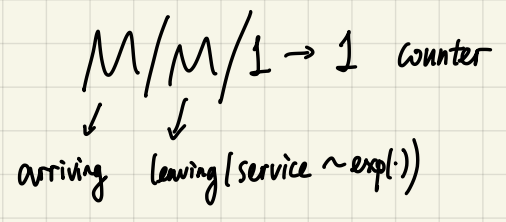
Continuous Time:  $0 = \pi Q \Rightarrow$  limit distribution

① Transition Relation

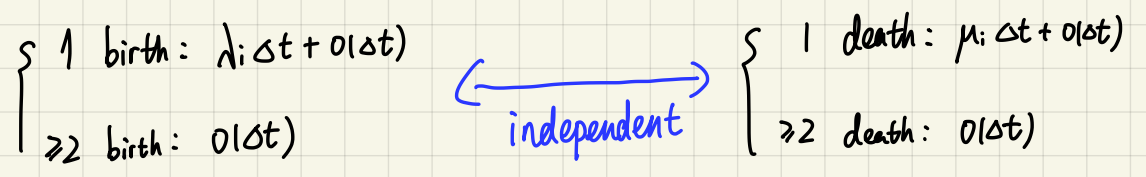
② Generator  $Q$

③  $0 = \pi Q \Rightarrow \pi = ?$

Queuing: Service Model



Model: Birth-Death Model



$P_{ij}(t)$

$$Q = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (P(\Delta t) - I)$$

$$\begin{cases} P_{ii}(\Delta t) = (1 - \lambda_i \Delta t + o(\Delta t))(1 - \mu_i \Delta t + o(\Delta t)) + (\lambda_i \Delta t + o(\Delta t))(\mu_i \Delta t + o(\Delta t)) + o(\Delta t) \\ P_{i,i+1}(\Delta t) = (\lambda_i \Delta t + o(\Delta t))(1 - \mu_i \Delta t + o(\Delta t)) + o(\Delta t) \\ P_{i,i-1}(\Delta t) = (\mu_i \Delta t + o(\Delta t))(1 - \lambda_i \Delta t + o(\Delta t)) + o(\Delta t) \\ P_{ij}(\Delta t) = o(\Delta t), |i-j| \geq 2 \end{cases}$$

$$\begin{aligned} Q_{ii} &= -(\lambda_i + \mu_i) \\ Q_{i,i+1} &= \lambda_i \\ Q_{i,i-1} &= \mu_i \\ Q_{ij} &= 0, |i-j| \geq 2 \\ &i=1, 2, \dots \end{aligned}$$

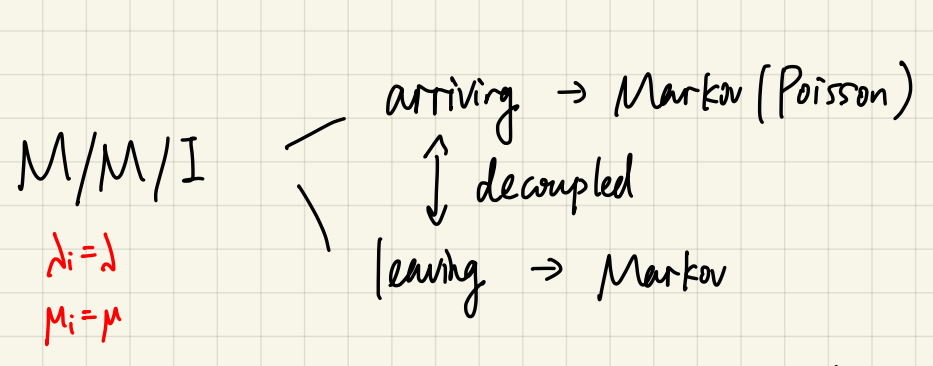
$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\pi Q = 0 \Rightarrow \begin{aligned} -\lambda_0 \pi_0 + \mu_1 \pi_1 &= 0 \\ \lambda_0 \pi_0 - (\lambda_1 + \mu_1) \pi_1 + \mu_2 \pi_2 &= 0 \\ \lambda_1 \pi_1 - (\lambda_2 + \mu_2) \pi_2 + \mu_3 \pi_3 &= 0 \\ &\vdots \\ \lambda_{k-1} \pi_{k-1} - (\lambda_k + \mu_k) \pi_k + \mu_{k+1} \pi_{k+1} &= 0 \\ &\vdots \end{aligned}$$

$$\begin{aligned} \pi_1 &= \frac{\lambda_0}{\mu_1} \pi_0 \\ \pi_2 &= \frac{\lambda_1}{\mu_2} \pi_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} \pi_0 \\ \pi_3 &= \frac{\lambda_2}{\mu_3} \pi_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} \pi_0 \\ &\vdots \\ \pi_{k+1} &= \frac{\lambda_k \dots \lambda_0}{\mu_{k+1} \dots \mu_1} \pi_0 = \prod_{i=1}^k \frac{\lambda_{i-1}}{\mu_i} \pi_0 \end{aligned}$$

$$\sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \sum_{n=1}^{\infty} \left( \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} \pi_0 \right) + \pi_0 = 1 \Rightarrow \left( 1 + \sum_{n=1}^{\infty} \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} \right) \pi_0 = 1$$

$$\Rightarrow \begin{cases} \pi_0 = \left( 1 + \sum_{n=1}^{\infty} \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} \right)^{-1} \\ \pi_n = \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} \cdot \left( 1 + \sum_{n=1}^{\infty} \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} \right)^{-1} \end{cases} \quad n=1, 2, \dots$$



Birth:  $1: \lambda \Delta t + o(\Delta t)$   
 $\geq 2: o(\Delta t)$

Death:  $1: \mu \Delta t + o(\Delta t)$   
 $\geq 2: o(\Delta t)$

$$\prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} = \prod_{k=1}^n \left( \frac{\lambda}{\mu} \right) = \left( \frac{\lambda}{\mu} \right)^n \Rightarrow \pi_0 = \left( 1 + \sum_{n=1}^{\infty} \left( \frac{\lambda}{\mu} \right)^n \right)^{-1} = \left( 1 + \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} \right)^{-1} = \frac{\mu - \lambda}{\mu}$$

Convergence:  $\frac{\lambda}{\mu} < 1 \Leftrightarrow \lambda < \mu$ , arriving < leaving  $\Rightarrow$  Dynamic Balance

if  $\lambda = \mu$  /  $\lambda > \mu$ , the queue goes infinitely long  $\Rightarrow$  no non-trivial limit distribution (non-Recurrent)

$$\pi_0 = 0 \Rightarrow \pi_k = 0 \quad k=1, 2, \dots$$

$$\pi_n = \left( \frac{\lambda}{\mu} \right)^n \cdot \frac{\mu - \lambda}{\mu} \quad \text{Average Queue Length} = \sum_{n=1}^{\infty} n \cdot \pi_n = \sum_{n=1}^{\infty} n \cdot \left( \frac{\lambda}{\mu} \right)^n \cdot \frac{\mu - \lambda}{\mu}$$

$$= \frac{\mu - \lambda}{\mu} \cdot \frac{\lambda \mu}{(\mu - \lambda)^2} = \frac{\lambda}{\mu - \lambda}$$

$$\begin{aligned} S &= \sum_{n=1}^{\infty} n \cdot \left( \frac{\lambda}{\mu} \right)^n \\ \frac{\lambda}{\mu} S &= \sum_{m=0}^{\infty} (m+1) \left( \frac{\lambda}{\mu} \right)^m \\ \Rightarrow \frac{\mu - \lambda}{\lambda} S &= \frac{\mu}{\mu - \lambda} \quad S = \frac{\lambda \mu}{(\mu - \lambda)^2} \end{aligned}$$

Average staying time: Expected length  $\times$  Expected service time + Expected service time (yourself)

$$T = \frac{\lambda}{\mu - \lambda} \cdot \frac{1}{\mu} + \frac{1}{\mu} = \frac{1}{\mu - \lambda}$$

$$\therefore L = \frac{\lambda}{\mu - \lambda} = \frac{1}{\mu - \lambda} \cdot \lambda = T \cdot \lambda \quad \text{Little Formula}$$

$$L = \lambda \cdot T$$