Stochastic Calculus

Calculus: number. Limit essence of calculus. Distance / Metric.

$\{x_n\}$, $x_n \to x \ (n\to\infty) \Leftrightarrow \forall \varepsilon>0,\ \exists N \in \mathbb{N}$, s.t. $\forall n>N$, $|x_n - x| \le \varepsilon$ Approximation $\Leftarrow$ "Nearness"

$x_n \not\to x \ (n\to\infty) \Leftrightarrow \exists \varepsilon>0,\ \forall N \in \mathbb{N}$, s.t. $\exists n>N$, $|x_n - x| > \varepsilon$ Absolute Value $\Leftrightarrow$ Euclidean.

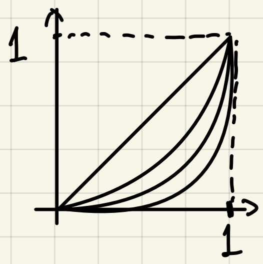
$X_n \in \mathbb{R}^n$, $\forall \varepsilon>0,\ \exists N>0$, s.t. $\forall n>N$, $\|X_n - X\| \le \varepsilon$, $\quad \|X_n - X\| = \left(\sum_{k=1}^{N}(X_n(k)-X(k))^2\right)^{1/2}$

$\{X_n\}$ Numbers $\xrightarrow{\text{Stochastic Sense}}$ Random Variables.

r.v.: $\Omega \to \mathbb{R}$. Function (Limit of Functions, Distance, Metric Stochastically)
Sample Space

Limits of Functions in #'s case: $\{f_n(x)\}$, $f_n(x) \to f(x)\ (n\to\infty)$ $f_n, f: A \to B$.

i) Pointwise: $\forall x \in A$, $\forall \varepsilon>0,\ \exists N \in \mathbb{N}$. s.t. $n>N$. $|f_n(x) - f(x)| \le \varepsilon$,
$N(x, \varepsilon)$



$f_n(x) = x^n$.
$f_n(x) \to \delta(x-1)$
Pointwise

ii) Uniformly: $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$, $\forall x \in A$, s.t. $n>N$, $|f_n(x) - f(x)| \le \varepsilon$
$N(\varepsilon)$

iii) Integration: $\|f_n - f\|^2 = \int_A |f_n(x) - f(x)|^2 dx \to 0 \quad (n\to\infty)$ Fourier Series.

$\Rightarrow$ Understand Stochastic Approximation based on previous function approximation.

$\{X_n\}$ r.v.: ① Mean Square Convergence (Distance $d(X,Y) = (E|X-Y|^2)^{1/2}$)

$X_n \xrightarrow[\text{sense}]{\text{mean square}} X$: $\forall \varepsilon>0,\ \exists N>0,\ \forall n>N,\ d(X_n, X) \le \varepsilon$

Purpose of Mean Square Distance: Triangle Inequality

$X_n \xrightarrow{m.s.} X$, $Y_n \xrightarrow{m.s.} Y$. i) $X_n + Y_n \xrightarrow{m.s.} X+Y$. $d(X_n+Y_n, X+Y) \le d(X_n, X) + d(Y_n, Y) \xrightarrow{n\to\infty} 0$

ii) $X_n Y_n \xrightarrow{m.s.} XY$ $\|X_nY_n - XY\| = \|X_nY_n - X_nY + X_nY - XY\| \le \|X_nY_n - X_nY\| + \|X_nY - XY\|$

$= \|X_n\|\,\|Y_n - Y\| + \|X_n - X\|\,\|Y\| \to 0$

Cauchy Criterion $\|X_n - X_m\| \to 0\ (n, m \to \infty) \Rightarrow \exists X,\ X_n \xrightarrow{m.s.} X$

(works in Stochastic calculus as well)

Introducing Pointwise into Stochastic Calculus:

② **Almost Surely Convergence** (Almost Everywhere): $P(\{w\in\Omega \mid X_n(w) \to X(w)\}) = 1$, $\{X_n\} \xrightarrow{a.s.} X$

on X_n

approximation inside "P" probability

Almost Surely $\not\Leftrightarrow$ Mean Square.

e.g.: $\Omega=[0,1]$, (Ω,Σ,P) Borel Probability: $P([c,d])=d-c$

Construct. $\{X_n\}:[0,1]\to\mathbb{R}$, s.t. $X_n \xrightarrow{a.s.} 0$. $X_n \cancel{\xrightarrow{m.s.}} 0$

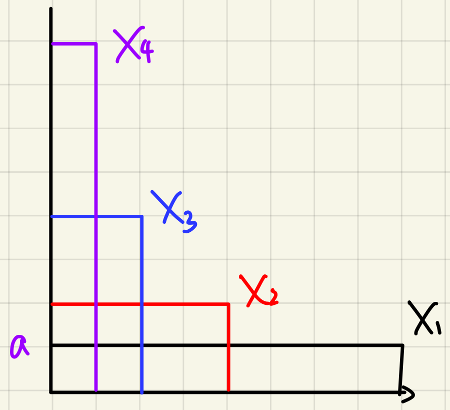
$X_1: [0,1]\to\{a_1\}$ (constant value)

$X_2: [0,1]\to\{a_2, 0\}$

$X_3: [0,1]\to\{a_3, 0\}$

$X_4: [0,1]\to\{a_4, 0\}$

⋮



$\forall w\in(0,1]$. $\exists N$, $n>N$, $X_n(w)=0$

$0\to$ neglect (prob.$=0$) : Almost Surely

$\therefore X_n \xrightarrow{a.s.} 0$

$$E|X_n(w)-0|^2 = E|X_n(w)|^2 = \int_\Omega |X_n(w)|^2 P(dw) = \int_\Omega |X_n(w)|^2 dw = \int_0^1 |X_n(w)|^2 dw$$

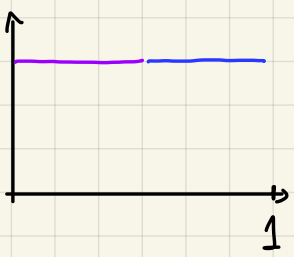
$$= \int_0^{2^{1-n}} a_n^2 dw = 2^{1-n}\cdot a_n^2$$

Let $a_n = 2^n$, $\lim_{n\to\infty} E|X_n(w)|^2 = 2^{1-n}\cdot 2^{2n} = 2^{n+1} = \infty$

almost surely: 局部 (local) Almost Surely + bounded ($|X_n|\le C$) contains Mean Square.

mean square: 整体 (entire)

e.g. 2.



$X_1: u(0)-u(\frac12)$

$X_2: u(\frac12)-u(1)$

$X_3: u(0)-u(\frac14)$

$X_4: u(\frac14)-u(\frac12)$

$X_5: u(\frac12)-u(\frac34)$

$X_6: u(\frac34)-u(1)$

$X_7: u(0)-u(\frac18)$

$X_8: u(\frac18)-u(\frac14)$

⋮

$\forall w\in[0,1]$, $\forall N\ge 0$, $\exists n>N$, $|X_n(w)|=1$, limit is not 0 (not AS. convergent)

while $X_n \xrightarrow{m.s.} 0$ (the area under $X_n \to 0$)

③ **Convergence in Probability**: $\forall \varepsilon>0$, $P(\{w\in\Omega: |X_n(w)-X(w)|\ge\varepsilon\}) \to 0\ (n\to\infty)$

not directly on $\{X_n\}$

Prove: Almost Surely $P\{w\in\Omega: X_n(w)\to X(w)\}=1 \Leftrightarrow P\{w\in\Omega: X_n(w)\not\to X(w)\}=0$

$X_n \not\to X \Leftrightarrow \underline{\exists\varepsilon>0}$. $\underline{\forall N\in\mathbb{N}}$, $\underline{\exists n>N}$, $|x_n-x|>\varepsilon \Leftrightarrow P(\bigcup_{\varepsilon>0}\bigcap_{N\in\mathbb{N}}\bigcup_{n>N}\{w\in\Omega: |X_n(w)-X(w)|>\varepsilon\})=0$

Interpret this clause in "Group" sense:

$\exists$: $\omega \in A \cup B$ (exists) $\quad \cup$

$\forall$: $\omega \in A \cap B$ (intersect) $\quad \cap$

$$P\left(\bigcup_{\varepsilon>0} \bigcap_{N\in\mathbb{N}} \bigcup_{n>N} \{\omega\in\Omega : |X_n(\omega) - X(\omega)| > \varepsilon\}\right) = 0$$

$$\xrightarrow[\varepsilon>0]{\text{fixed } \varepsilon} P\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n>N} \{\omega\in\Omega : |X_n(\omega) - X(\omega)| > \varepsilon\}\right) = 0 \quad (\text{subset of the above set})$$

intersection of $B_1, B_2, B_3 \cdots$

B_N , then $B_N \supseteq B_{N+1} \supseteq B_{N+2} \supseteq B_{N+3} \cdots$ (decreasing)

intersection of a set of decreasing sets $\Rightarrow B_n \ (n\to\infty)$

$$\Rightarrow \lim_{N\to\infty} P\left(\bigcup_{n>N} \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}\right) = 0 \quad \text{Almost Surely Convergence}$$

$$\xrightarrow[n=N+1]{\text{take the subset of}} \lim_{N\to\infty} P(\{\omega\in\Omega : |X_{N+1}(\omega) - X_n(\omega)| > \varepsilon\}) = 0$$

$\Updownarrow$ Equivalent

$\therefore$ Convergence in Probability $\subseteq$ Almost Surely

Convergence in Probability : $\forall \varepsilon > 0, \ \lim_{n\to\infty} P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = 0$

eg. 2 $\Rightarrow$ not Almost Surely $\to 0$. but convergence to 0 in prob. $\left(\begin{array}{l}\text{non-zero range decreasing } (\to 0), \\ \lim_{n\to\infty} P\{\omega\in\Omega : |X_n(\omega) - X(\omega)| > \varepsilon\} = 0, \ \forall \varepsilon > 0\end{array}\right)$

Convergence in prob. $\not\Rightarrow$ almost surely convergence

$$X_1, \cdots, X_n \text{ i.i.d.} \quad \frac{1}{n}(X_1 + \cdots + X_n) = \frac{1}{n}\sum_k X_k \xrightarrow{n\to\infty} E[X_1] \quad \begin{cases}\text{Strong Law of Large Numbers SLLN} \\ \text{Weak Law of Large Numbers WLLN}\end{cases}$$

Strong : almost surely (a.s.)

Weak : in probability (p.)

$$E\left|\frac{1}{n}\sum_1^n X_k - EX_1\right|^2 = \frac{1}{n^2} E\left|\sum_k (X_k - EX_1)\right|^2 = \frac{1}{n^2}\left[\sum_k E(X_k - EX_1)^2 + 2\sum_{i<j} E[\underbrace{(X_i - EX_1)(X_j - EX_1)}_{=0}]\right]$$

$$= \frac{1}{n^2}\sum_k \mathrm{var}(X_k) = \frac{1}{n}\mathrm{var}(X_1) \xrightarrow{n\to\infty} 0 \quad (\text{m.s. convergence})$$

To prove WLLN. we need Tchebyshev Inequality : $\forall a > 0, \quad P(|X| \geq a) \leq \frac{E|X|^2}{a^2}$ (Concentration)

Proof of Tchebyshev : $E|X|^2 = \int_{\mathbb{R}} x^2 f_X(x)\,dx = \left(\int_{|x|\geq a} + \int_{|x|<a}\right) x^2 f_X(x)\,dx \geq \int_{|x|\geq a} x^2 f_X(x)\,dx \geq \int_{|x|\geq a} a^2 f_X(x)\,dx$

$$= a^2 P(|X| \geq a)$$

$\therefore E|X|^2 \geq a^2 P(|X| \geq a)$, let $X = X' - E[X'] \Rightarrow P(|X' - EX'| \geq a) \leq \frac{\mathrm{var}(X')}{a^2}$

代入上式 : $P\left(\left|\frac{1}{n}\sum_k X_k - EX_1\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \cdot \frac{1}{n}\mathrm{var}(X_1), \ \forall \varepsilon > 0$

i.e. $P\left(\left|\frac{1}{n}\sum_k X_k - EX_1\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\cdot\frac{1}{n}\,\mathrm{var}(X_1) \xrightarrow{n\to\infty} 0$ **WLLN**

④ Convergence in Distribution:

$X_n \sim F_{X_n}(x) = P(X_n \leq x)$

$X \sim F_X(x) = P(X \leq x)$

$X_n \xrightarrow{F.} X \Leftrightarrow F_{X_n}(x) \xrightarrow{n\to\infty} F_X(x)$ at $F_X(x)$ continuous pts.

e.g. $X \sim N(0,1)$, $X_n = (-1)^n X = (-X, X, -X, X, \cdots)$ preserves some randomness

$\Leftrightarrow -X \sim N(0,1)$

Central Limit Theorem: $X_1 \cdots X_n$, $E(X_i)=0$, $\mathrm{var}(X_i)=1$, $i=1\sim n$. $\frac{1}{\sqrt{n}}\sum_k X_k \xrightarrow{F.} N(0,1)$

Convergence in Distribution

$\frac{1}{n}$: no randomness

$\frac{1}{\sqrt{n}}$: normal distribution (some randomness)

transition ? $\sqrt{n\ln(\ln n)}$

$\{X_n\} \xrightarrow{P.} X \Rightarrow \{X_n\} \xrightarrow{F.} X \Leftrightarrow F_{X_n}(x) \to F_X(x)$

probability distribution

$(y > x)$

$F_{X_n}(x) = P(X_n \leq x) = P(X_n \leq x,\ X \in \mathbb{R}) = P(X_n \leq x,\ X > y) + P(X_n \leq x,\ X \leq y) \leq P\{|X_n - X| > y - x\} + \underbrace{P\{X \leq y\}}_{F_X(y)}$

$\Rightarrow |X_n - X| > y - x$

i.e. $\{X_n \leq x,\ X > y\} \subseteq \{|X_n - X| > y - x\}$ $\therefore$ $P\{\cdots\} \leq P\{|X_n - X| > y - x\}$

$\limsup_n F_{X_n}(x) \leq F_X(y)\quad (y > x)$

$F_X(z) = P\{X \leq z\} = P\{X \leq z,\ X_n \in \mathbb{R}\} = P\{X_n \leq z,\ X > x\} + P\{X_n \leq z,\ X \leq x\} \leq P(|X_n - X| > x - z) + P(X_n \leq x)$

$F_X(z) \leq \liminf_n F_{X_n}(x)\quad (z < x)$

$F_X(z) \leq \liminf_n F_{X_n}(x) \leq \limsup_n F_{X_n}(x) \leq F_X(y)$ $\begin{cases} z \to x \\ y \to x \end{cases}$ $\therefore$ $n\to\infty$ $F_{X_n}(x) \to F_X(x)$

$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P.} X \Rightarrow X_n \xrightarrow{F.} X$

SLLN WLLN CLT

$P(X_n \xrightarrow{a.s.} X) = 1$ $P(|X_n - X| \geq \varepsilon) \to 0\ \forall \varepsilon > 0$ $F_{X_n}(x) \to F_X(x)$

$X_n \xrightarrow{m.s.} X \Rightarrow X_n \xrightarrow{P.} X$; $\not\Rightarrow X_n \xrightarrow{a.s.} X$

$|EX| \leq B$

Engineering.

$E|X_n - X|^2 \to 0$

$X_n \xrightarrow{m.s.} X \Rightarrow X_n \xrightarrow{P.} X$, Tchebyshev: $\underset{\substack{\downarrow \\ 0 \\ P.}}{P(|X_n-X| \le \varepsilon)} \le \frac{1}{\varepsilon^2} \underset{\substack{\downarrow \\ 0 \\ m.s.}}{E|X_n-X|^2}$