Course Maxim:

No Reading, No Learning ⇒ Understanding

No Writing, No Reading ⇒ Effective Reading

No Data, No Truth

No Analytic, No Understanding (High-level)

No Programming, No Cognition (Matlab)

(落地)

Stochastic Processes. ← Group of R.V.s

(Gambling Implication) (Relationship)

3 Types

Correlation (Linear) → Time Domain: Correlation Function

→ Freq. Domain: Spectrum

(Gaussian Process)

Markov Property (M. Processes) → Discrete Time

→ Continuous Time

Maturity (不懂搞懂的信心)

⇓

Poisson Process

Martingale → Optional Theorem

→ Financial Application

r.v. X, Y → $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$ Joint Dist.

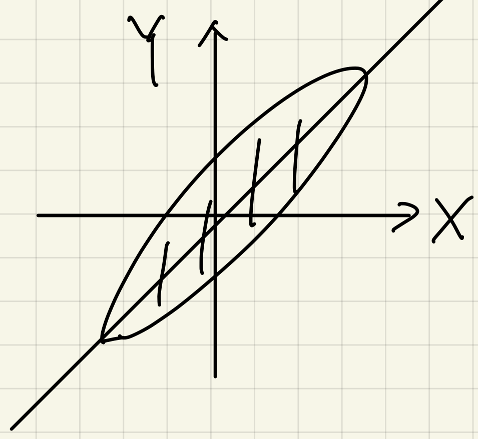
$F_{XY}(x,y) = P(X \le x, Y \le y)$

Independence − Correlated − Linear Correlation

$Y \overset{?}{=} \alpha X$ (Linear Correlation)

$\Uparrow$

$E[Y-\alpha X]^2$ Mean Square Error



$$f_{XY}(x,y) = \begin{cases} \frac{1}{|\Omega|}, & (x,y) \in \Omega \\ 0, & \text{others} \end{cases}$$

$\min_{\alpha} E[Y-\alpha X]^2 \Rightarrow \alpha_{opt} = \frac{E[XY]}{E[X^2]}$ $\mathbb{R}$. $E[XY]$ Correlation

$E[(X-E(X))(Y-E(Y))] = E[XY] - \underbrace{EXEY}$ 和 R.V. 关联设关系

Uncorrelated : $E[XY]=0$ ($E[XY] = EXEY$)

$\Uparrow$

Independence : more rigorous.

$\Theta \sim U(0, 2\pi)$

$X = \cos\Theta$, $Y = \sin\Theta$ $\Rightarrow X^2+Y^2=1$ not independent

$E[XY] = E[\cos\Theta \sin\Theta] = E[\frac{1}{2}\sin 2\Theta] = \int_0^{2\pi} \frac{1}{2\pi} \cdot \frac{1}{2} \sin 2\theta \, d\theta$

$= \frac{1}{4\pi} \int_0^{4\pi} \sin\theta' \, d\theta' \cdot \frac{1}{2} = 0 = EXEY$

Uncorrelated but Dependent.

Geometrical View of Correlation : $E[XY] = \langle X, Y \rangle$

$\langle X, Y \rangle : H \times H \to \mathbb{R}$ (二元操作) $\Rightarrow$ Does these properties still apply in r.v. context ?

1. $\langle X, Y \rangle = \langle Y, X \rangle$
2. $\langle X, X \rangle \geq 0$ ($\langle X, X \rangle = 0 \Leftrightarrow X = 0$)
3. Bilinear : $\langle X, \alpha Y + \beta Z \rangle = \alpha \langle X, Y \rangle + \beta \langle X, Z \rangle$

$E[X^2] = 0 \overset{?}{\Rightarrow} X = 0$

$\Downarrow$

$P(X=0) = 1$ (Not necessarily be 0 all the time, but other cases are negligible)

Inner Product $\Rightarrow$ Angle (Projection)

$\cos \angle(X,Y) = \frac{\langle X, Y \rangle}{\sqrt{\langle X,X \rangle \langle Y,Y \rangle}}$ $\Rightarrow$ introducing geometrical perspective into Stochastic Processes

R.V. $\Rightarrow$ Vector.

$|\cos \angle(X,Y)| \leq 1$ Cauchy-Schwarz Inequalities

$\cup$ parabola.

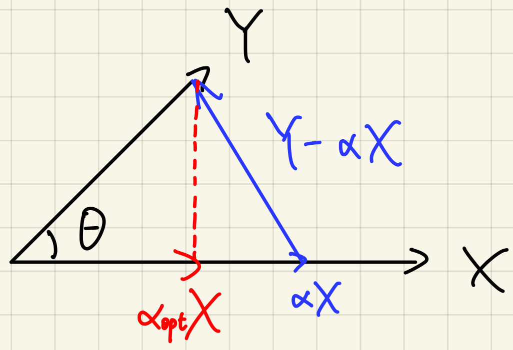
Proof : auxiliary func. $0 \leq g(\lambda) = \langle X+\lambda Y, X+\lambda Y \rangle = \langle Y,Y \rangle \lambda^2 + 2\langle X,Y \rangle \lambda + \langle X,X \rangle$ (λ arbitrary)

(3^{rd} property)

2^{nd}-order function : $\Delta \leq 0$, i.e., $4\langle X,Y \rangle^2 - 4\langle X,X \rangle \langle Y,Y \rangle \leq 0$, i.e. $\frac{\langle X,Y \rangle^2}{\langle X,X \rangle \langle Y,Y \rangle} \leq 1$.

Other Form: $\left|\sum x_k y_k\right| \le \left(\sum x_k^2\right)^{1/2}\left(\sum y_k^2\right)^{1/2}$ ----

Previous Q: $\alpha_{opt} \Rightarrow \min E[Y-\alpha X]^2$



$\alpha_{opt} \Rightarrow$ Y's projection onto X.

$$\begin{cases} \|Y\|\cdot\cos\theta\cdot\frac{X}{\|X\|} = \left(\frac{\|Y\|}{\|X\|}\cos\theta\right)X \\ \cos\theta = \frac{E[XY]}{\|X\|\|Y\|}\end{cases} \Rightarrow \alpha_{opt} = \frac{\|Y\|}{\|X\|}\cdot\frac{E[XY]}{\|X\|\|Y\|} = \frac{E[XY]}{\|X\|^2\,(EX^2)}$$

(Stochastic Process)

Correlation Function $X(t) \to$ a group, series of r.v., t: Index (t 2-Dim. Random Field)

$R_X(t,s) = E(X(t)X(s))$ Binary → Univariate (Under what assumption?)

① $R_X(t,s) = R_X(s,t)$

Stationary (平稳) ⇐ Invariance Infinite types of invariance ⇓ Infinite types of stationary

② $|R_X(t,s)|^2 \le |R_X(t,t)R_X(s,s)|^{1/2}$

Wide-Sense Stationary

deterministic

① $E[X(t)] = m(t) \equiv m$ $\quad (X'(t) = X(t) - m(t))$

② $R_X(t,s) = R_X(t+D, s+D)$. $\forall D \in \mathbb{R}$

$\Rightarrow R_X(t,s) = R_X(t-s) = R_X(\tau)$, $\tau = t-s$

① Modulated Signal.

$X(t) = A(t)\cos(2\pi f_0 t + \Theta)$, $A(t)$: r.v. $\Theta \sim U(0,2\pi)$, A, Θ indep.

$$E[X(t)] \underset{\text{indep.}}{=} E[A(t)]\,E[\cos(2\pi f_0 t+\Theta)] = E[A(t)]\cdot\int_0^{2\pi}\frac{1}{2\pi}\cos(2\pi f_0 t+\theta)\,d\theta = 0$$

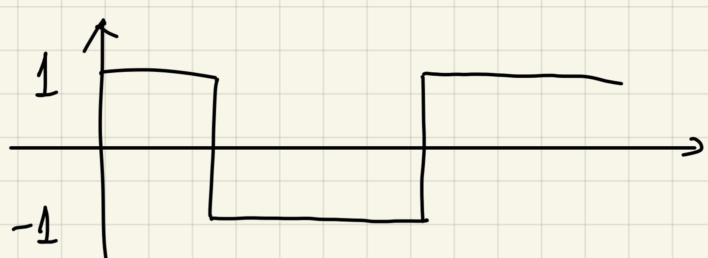
$$R_X(t,s) = E[A(t)A(s)]\,E[\cos(2\pi f_0 t+\Theta)\cos(2\pi f_0 s+\Theta)]$$

$$= E[A(t)A(s)]\,\frac{1}{2}E[\underbrace{\cos(2\pi f_0(t+s)+2\Theta)}_{0} + \cos(2\pi f_0(t-s))]$$

$$= E[A(t)\cdot A(s)]\,\frac{1}{2}\cos(2\pi f_0(t-s))$$ (if $A(t)$ wide-sense stationary) W.S.S

$$= R_A(t-s)\cdot\frac{1}{2}\cos(2\pi f_0(t-s))$$

② Random Telegraph Signal:



$[s,t]$ switch k times prob. $P = \frac{(\lambda(t-s))^k}{k!} \exp(-\lambda(t-s))$ Poisson Distribution.

$\{+1, -1\}$

2nd-order Moment: $E[X(t)X(s)] = R_X(t,s)$

$$= 1 \cdot P_1 + (-1) \cdot P_{-1}$$

$$P_1 = P([s,t], \text{even switches}) = \sum_{k \text{ even}} \frac{(\lambda(t-s))^k}{k!} \exp(-\lambda(t-s)) \rightarrow S_{even} \exp(-\lambda(t-s))$$

$$P_{-1} = P([s,t], \text{odd switches}) = \sum_{k \text{ odd}} \frac{(\lambda(t-s))^k}{k!} \exp(-\lambda(t-s)) \rightarrow S_{odd} \cdot \exp(-\lambda(t-s))$$

$$\sum_{k=0}^{\infty} \frac{(\lambda(t-s))^k}{k!} = \exp(\lambda(t-s)) = S_{even} + S_{odd}$$

$$\sum_{k=0}^{\infty} \frac{(-\lambda(t-s))^k}{k!} = \exp(-\lambda(t-s)) = S_{even} - S_{odd}$$

$$\Rightarrow S_{even} = \frac{1}{2}\left[\exp(\lambda(t-s)) + \exp(-\lambda(t-s))\right] \Rightarrow P_1 = \frac{1}{2}\left[1 + \exp(-2\lambda(t-s))\right]$$

$$P_{-1} = \frac{1}{2}\left[1 - \exp(-2\lambda(t-s))\right]$$

$$\therefore R_X(t,s) = P_1 - P_{-1} = \frac{1}{2}\left[1 + \exp(-2\lambda(t-s))\right] - \frac{1}{2}\left[1 - \exp(-2\lambda(t-s))\right] = \exp(-2\lambda(t-s))$$