Poisson Processes 2

Sparsity may not be necessary during the derivation of MGF. $G_{N(t)}(z)$.

$$\frac{1}{\Delta t}\left[G_{N(t+\Delta t)}(z)-G_{N(t)}(z)\right]=\frac{1}{\Delta t}G_{N(t)}(z)\,E[z^{N(\Delta t)}-1]$$

$$E[z^{N(\Delta t)}-1]=\underbrace{P(N(\Delta t)=0)-1}_{\exp(-\lambda\Delta t)}+P(N(\Delta t)=1)\cdot z+\sum_{k\geq 2}z^k\cdot P(N(\Delta t)=k)$$

$M(t)=P(N(t)=1)$, $[0,t]=[0,s]\cup[s,t]\Rightarrow P(N(t)=1)=P(N(s)=1,\ N(t)-N(s)=0)+P(N(s)=0,\ N(t)-N(s)=1)$

$$\overset{\text{indep. increment}}{=}P(N(s)=1)\cdot P(N(t)-N(s)=0)+P(N(s)=0)\cdot P(N(t)-N(s)=1)$$

$$\overset{\text{stationary increment.}}{=}\underset{M(s)}{P(N(s)=1)}\cdot\underset{Z(t-s)}{P(N(t-s)=0)}+\underset{Z(s)}{P(N(s)=0)}\cdot\underset{M(t-s)}{P(N(t-s)=1)}$$

$$M(t)=M(s)\cdot Z(t-s)+Z(s)\,M(t-s)\qquad Z(t)=P(N(t)=0)=\exp(-\lambda t)$$

$$\Rightarrow M(t+s)=M(t)\cdot Z(s)+M(s)Z(t)\Rightarrow\frac{M(t+s)}{Z(t)Z(s)}=\frac{M(t)}{Z(t)}+\frac{M(s)}{Z(s)}\quad\left(Z(t+s)=\exp(-\lambda(t+s))=Z(t)Z(s)\right)$$

$$\Rightarrow\frac{M(t+s)}{Z(t+s)}=\frac{M(t)}{Z(t)}+\underbrace{\frac{M(s)}{Z(s)}}_{}\ \lambda(s)\quad\Rightarrow\quad\lambda(t+s)=\lambda(t)+\lambda(s)\qquad\therefore\ \lambda(t)=\alpha t$$

$$\therefore\ M(t)=Z(t)\cdot\lambda(t)=\alpha t\cdot\exp(-\lambda t)$$

$$P(N(\Delta t)=0)+P(N(\Delta t)=1)+P(N(\Delta t)\geq 2)=1$$

$$\frac{1-\overset{\exp(-\lambda\Delta t)}{P(N(\Delta t)=0)}}{\Delta t}=\frac{\overset{\alpha\Delta t\cdot\exp(-\lambda\Delta t)}{P(N(\Delta t)=1)}}{\Delta t}\left(1+\frac{P(N(\Delta t)\geq 2)}{P(N(\Delta t)=1)}\right)$$

$$\lambda\exp(-\lambda\Delta t)\qquad\alpha\exp(-\lambda\Delta t)$$

Let $\alpha=\lambda\Rightarrow\dfrac{P(N(\Delta t)\geq 2)}{P(N(\Delta t)=1)}\xrightarrow{\Delta t\to 0}0$

i) Non-Stationary Increment: $\Rightarrow$ Need Another Assumption. $\displaystyle\lim_{\Delta t\to 0}\frac{1-P(N(t+\Delta t)-N(t)=0)}{\Delta t}=\lambda(t)$ time-dependent.

$$\begin{cases}\dfrac{d}{dt}G_{N(t)}(z)=G_{N(t)}(z)\,\lambda(t)(z-1)\\ G_{N(0)}(z)=E[z^{N(0)}]=E[z^0]=1\end{cases}\Rightarrow G_{N(t)}(z)=\exp\left((z-1)\int_0^t\lambda(s)ds\right),\quad P(N(t)=k)=\frac{\left[\int_0^t\lambda(s)ds\right]^k}{k!}\exp\left(-\int_0^t\lambda(s)ds\right)$$

Non-Homogeneous Poisson. $\lambda\equiv\lambda(t)$ $\left(\displaystyle\lim_{\Delta t\to 0}\frac{1-P(N(t+\Delta t)-N(t)=0)}{\Delta t}=\lambda(t)\right)$

Counting Events. $\Rightarrow$ Effect Statistics

$\boxed{Y(t)=\sum_{k=1}^{N(t)}X_k}$ — Compound Poisson Processes

, $N(t)$: Standard Poisson. X_k i.i.d., indep of $N(t)$

$X_k\equiv 1\Rightarrow Y(t)=N(t)$

$$G_{Y(t)}(z)=E[z^{Y(t)}]=E\left[z^{\sum^{N(t)}X_k}\right]=E\left[E\left[z^{\sum_1^{N(t)}X_k}\middle|N(t)=n\right]\right]=E\left[E\left[z^{\sum_1^{n}X_k}\middle|N(t)=n\right]\right]=E\left[E\left[z^{nX_k}\middle|N(t)=n\right]\right]=E\left[G_{X_1}(z)^{N(t)}\right]=E[z^{N(t)}]\Big|_{z=G_{X_1}(z)}$$

$$=G_{N(t)}(G_{X_1}(z))\quad\left(G_{N(t)}(z)=\exp(\lambda t(z-1))\right)=\exp\left(\lambda t(G_{X_1}(z)-1)\right)$$

Ex. 1: $N(t)$, $X_k \sim B\begin{pmatrix} M & F \\ p & 1-p \end{pmatrix}$ only focuses on 'M' ⇒ still Standard Poisson? : $X_k \overset{i.i.d.}{\sim} \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}$

$$G_{Y(t)}(z) = G_{N(t)}(G_X(z)) = \exp(\lambda t(G_X(z)-1))$$

$$\Rightarrow G_{Y(t)}(z) = \exp(\lambda t(pz+1-p-1)) = \exp(\lambda t \cdot p(z-1)) = \exp(\underbrace{\lambda p}_{\lambda'=\lambda p} \cdot t \cdot (z-1)) \Rightarrow \text{Poisson}$$

$$G_X(z) = z \cdot p + 1 \cdot (1-p) = pz + 1 - p$$

Randomization. (Load Balance, Ethernet (CSMA/CD))

Ex. 2: $N_1(t): \lambda_1$. $N_2(t): \lambda_2$ N_1 indep. N_2, $N = N_1 + N_2$ Poisson?

$$G_{N(t)}(z) = E[z^{N(t)}] = E[z^{N_1+N_2}] = E[z^{N_1(t)}] \cdot E[z^{N_2(t)}] = \exp(\lambda_1 t(z-1)) \exp(\lambda_2 t(z-1)) = \exp((\lambda_1+\lambda_2) t(z-1))$$

$\Rightarrow N_k(t): \lambda_k$, $k = 1 \sim n$, $Y(t) = \sum N_k(t) \Rightarrow Y(t)$ Poisson, $\lambda = \sum_k \lambda_k$

10min

Ex. 3. 2 Counters occupied, Serve time $\overset{i.i.d.}{\sim} \exp(\lambda)$, Comes another customer 3 $(t=0)$

$P\{$cusmoter 3 leaves latest$\} = ?$ $\frac{1}{2}$

Distribution of 1^{st} leaving customer's leaving time $\sim \exp(\cdot)$

$$\underbrace{T_1 \sim \exp(\lambda)}, \ \underbrace{T_2 \sim \exp(\lambda)}, \ T = \min(T_1, T_2) \sim \exp(\underbrace{2\lambda}_{\lambda'=\lambda_1+\lambda_2=2\lambda})$$

1^{st} occurrence of $N_1 \cup N_2 \Leftrightarrow 1^{st}$ occurrence of Poisson Process N, intensity $= \lambda_1 + \lambda_2$

1^{st} occurrence of a Poisson Process N_1, N_2 indep.,

When one is vacant, and customer 3 filled in ⇒ still $\exp(\cdot)$, memoryless?

Memoryless: $P(X > x+y \mid X > y) = P(X > x)$, if y is r.v. (Y), X still memoryless?

Prove: if $X \sim \exp(\lambda_1)$, $Y \sim \exp(\lambda_2)$. $\quad P(X > x+Y \mid X > Y) = P(X > x)$

$$P(X>Y) = \int_0^\infty P(Y=y) \cdot P(X>Y \mid Y=y)\, dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \cdot \int_y^\infty \lambda_1 \exp(-\lambda_1 x)\, dx \cdot dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \cdot \exp(-\lambda_1 y)\, dy = \lambda_2 \cdot \int_0^\infty \exp(-(\lambda_1+\lambda_2) y)\, dy = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

$$P(X > Y+x) = \int_0^\infty P(Y=y) \cdot P(X > Y+x \mid Y=y)\, dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \cdot \int_{x+y}^\infty \lambda_1 \exp(-\lambda_1 x)\, dx \cdot dy = \int_0^\infty \lambda_2 \exp(-\lambda_2 y) \exp(-\lambda_1(x+y))\, dy = \lambda_2 \cdot \exp(-\lambda_1 x) \cdot \frac{1}{\lambda_1+\lambda_2}$$

$$= \frac{\lambda_2}{\lambda_1+\lambda_2} \exp(-\lambda_1 x)$$

$$\therefore P(X > x+Y \mid X > Y) = \frac{P(X > x+Y)}{P(X>Y)} = \exp(-\lambda_1 x) = P(X > x)$$

$$P(X > x) = \int_x^\infty \lambda_1 \exp(-\lambda_1 s)\, ds = \exp(-\lambda_1 x).$$

⇒ Customer 3's leaving time $\sim \exp(\lambda)$, $P(\cdots) = \frac{1}{2}$

Ex. 4. $N(t) = N_1(t) - N_2(t)$. still Poisson? × $\quad P(N_1(t) - N_2(t) < 0) > 0$

Compound Poisson.

Compound Poisson: $\exp(\lambda t(G_X(z)-1))$

$$G_{N(t)}(z) = E[z^{N(t)}] = E[z^{N_1-N_2}] = E[z^{N_1(t)}] \cdot E[z^{-N_2(t)}] = G_{N_1}(z) \cdot G_{N_2}\left(\frac{1}{z}\right) = \exp(\lambda_1 t(z-1)) \exp(\lambda_2 t(z^{-1}-1)) = \exp\left((\lambda_1+\lambda_2) t \left(\frac{\lambda_1 z + \lambda_2 z^{-1}}{\lambda_1+\lambda_2} - 1\right)\right)$$

$$G_N(z) = \exp\left(\underbrace{(\lambda_1+\lambda_2)}_{\text{intensity}} t\left(\frac{\lambda_1 z+\lambda_2 z^{-1}}{\lambda_1+\lambda_2}-1\right)\right) \Rightarrow \text{Compound Poisson}$$

$$X \sim \begin{bmatrix} 1 & -1 \\ \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} \end{bmatrix}$$

Ex. 5: 2 counters $\sim \exp(\lambda_1), \exp(\lambda_2)$ are occupied. customer 3 comes in $(t=0)$

Expected leaving time of customer $3 = ?$

Waiting + Serving:

Waiting: $\sim \exp(\lambda_1+\lambda_2)$, $E[\text{Waiting}] = \frac{1}{\lambda_1+\lambda_2}$

$\iint_{t_1<t_2} f_{T_1T_2}\, dt_1 dt_2$

Serving:
- #1 $P(T_1<T_2)\, E[T_1 | T_1<T_2] = P(T_1<T_2)\cdot \frac{1}{\lambda_1} = \frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{1}{\lambda_1}$
- #2 $P(T_2<T_1)\, E[T_2 | T_2<T_1] = P(T_1>T_2)\cdot \frac{1}{\lambda_2} = \frac{\lambda_2}{\lambda_1+\lambda_2}\cdot\frac{1}{\lambda_2}$

$\frac{2}{\lambda_1+\lambda_2}$

$\Rightarrow E[T_3] = \frac{3}{\lambda_1+\lambda_2}$

My Solution:

3 →
- #1 $P(T_1<T_2) \Rightarrow t_1+E[T_1]$ $\quad P(T_1<T_2) = P(T_1=t_1)\cdot P(T_2>T_1 | T_1=t) \Rightarrow P(T_1=t_1)P(T_2>t_1)\cdot(t_1+\frac{1}{\lambda_1}) \quad \int_0^\infty dt_1$
- #2 $P(T_1>T_2) \Rightarrow t_2+E[T_2]$ $\quad P(T_1>T_2) \Rightarrow P(T_2=t_2)P(T_1>t_2)(t_2+\frac{1}{\lambda_2})\cdot \quad \int_0^\infty dt_2$

$$\int_0^\infty \lambda_1 \exp(-\lambda_1 t_1)\cdot\int_{t_1}^\infty \lambda_2\exp(-\lambda_2 s)ds\,(t_1+\tfrac{1}{\lambda_1})dt_1$$

$$= \int_0^\infty \lambda_1\exp(-\lambda_1 t_1)\cdot\exp(-\lambda_2 t_1)(t_1+\tfrac{1}{\lambda_1})dt_1$$

$$= \int_0^\infty \lambda_1 \exp(-(\lambda_1+\lambda_2)t_1)(t_1+\tfrac{1}{\lambda_1})dt_1$$

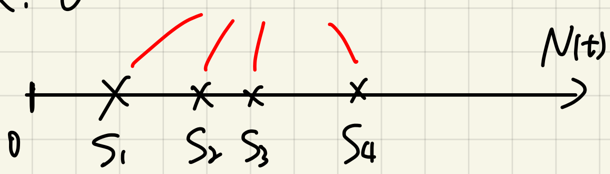
$$= \frac{1}{\lambda_1+\lambda_2} + \lambda_1\int_0^\infty t_1\exp(-(\lambda_1+\lambda_2)t_1)dt_1$$

$$= \frac{1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{(\lambda_1+\lambda_2)^2}$$

$$\therefore \int_0^\infty \lambda_2\exp(-\lambda_2 t_2)\int_{t_2}^\infty \lambda_1\exp(-\lambda_1 s)ds\,(t_2+\tfrac{1}{\lambda_2})dt_2 = \frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{(\lambda_1+\lambda_2)^2}$$

$$E[T_3] = \frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_1}{(\lambda_1+\lambda_2)^2}+\frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{(\lambda_1+\lambda_2)^2} = \frac{3}{\lambda_1+\lambda_2}$$

Ex. 6: Event Occurrence



$E[S_4 | N(t)=2] = ?$

$1-P(N(t-1)=0)-P(N(t-1)=1) = 1-\exp(-\lambda(t-1))[1+\frac{\lambda(t-1)}{1!}]$

indep increment.

$$f_{S_4}(t|N(1)=2) = ? \Rightarrow F_{S_4}(t|N(1)=2) = P(S_4\le t | N(1)=2) = \frac{P(S_4\le t,\ N(1)=2)}{P(N(1)=2)} = \frac{P(N(t)-N(1)\ge 2,\ N(1)=2)}{P(N(1)=2)} = \frac{P(N(t)-N(1)\ge 2)\,P(N(1)=2)}{P(N(1)=2)}$$

Stationary increment.

$$\frac{P(N(t-1)\ge 2)\,P(N(1)=2)}{P(N(1)=2)} = 1-(\lambda(t-1)+1)\exp(-\lambda(t-1))$$

$$f_{S_4}(t|N(1)=2) = \frac{dF_{S_4}(t|N(1)=2)}{dt} = -\lambda\exp(-\lambda(t-1)) + \lambda(\lambda(t-1)+1)\exp(-\lambda(t-1)) = \lambda^2(t-1)\exp(-\lambda(t-1))$$

$= 1+\frac{2}{\lambda}$

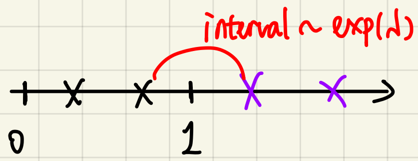
$$E[S_4|N(1)=2] = \int_1^\infty t\cdot f_{S_4}(t|N(1)=2)dt = \int_0^\infty t\,\lambda^2(t-1)\exp(-\lambda(t-1))dt = \int_{-1}^\infty (t+1)\lambda^2 t\exp(-\lambda t)dt = \int_{-1}^\infty(\lambda^2t^2+\lambda^2 t)\exp(-\lambda t)dt$$

$N(1)=2$, 2 occurrences afterwards. each $\sim\exp(\lambda) \Rightarrow t = 1+2\cdot E[T(\lambda)] = 1+2\cdot\frac{1}{\lambda}$

2^{nd}-3^{rd}: $\frac{1}{\lambda}$

$1-3^{rd} < \frac{1}{\lambda}$

However



$1\to 3^{rd}$ occurrence should be less than $E[T(\lambda)] = \frac{1}{\lambda}$, expectation should $1+$ portion of $\frac{1}{\lambda}+\frac{1}{\lambda}$

not $1+\frac{1}{\lambda}+\frac{1}{\lambda}$