Smith Chart Intro.

Yiming Zuo
August 2023

The Smith chart, shown on the last page, is a graphical aid that can be very useful for solving transmission line problems. Although there are a number of other impedance and reflection coefficient charts that can be used for such problems, the Smith chart is probably the best known and most widely used. It was developed in 1939 by P. Smith at the Bell Telephone Laboratories. The reader might feel that, in this day of personal computers and computer-aided design (CAD) tools, graphical solutions have no place in modern engineering. The Smith chart, however, is more than just a graphical technique. Besides being an integral part of much of the current CAD software and test equipment for microwave design, the Smith chart provides a useful way of visualizing transmission line phenomenon without the need for detailed numerical calculations. We can develop a good intuition about transmission line and impedance-matching problems by learning to think in terms of the Smith chart.

The impedance matching between an antenna and a transmission line is mostly realized and visualized in **Smith Chart**. The antenna must be impedance matched when assembled for the end user environment so that it operates in the desired frequency band with maximum efficiency. Optimal efficiency results in maximum range, minimum power consumption, reduced heating and reliable data throughput. It is good to understand that an antenna itself can be considered an impedance transformer. The antenna transitions power received from the RF circuitry through the Tx line (matched to an impedance of 50 Ω in most cases) to free space (impedance of 377 Ω).

At first glance the Smith chart may seem intimidating, but the key to its understanding is to realize that it is based on a polar plot of the voltage reflection coefficient,Γ. Let the reflection coefficient be expressed in magnitude and phase (polar) form as $\Gamma = |\Gamma|e^{j\theta}$. Then the magnitude $|\Gamma|$ is plotted as a radius ($|\Gamma| \leq 1$) from the center of the chart, and the angle $\theta(-180^\circ \leq \theta \leq 180^\circ)$ is measured counterclockwise from the right-hand side of

1 Telegrapher Equations

To grasp the essence of Smith Chart, we first need to introduce Telegrpher Equations and resulting wave solution so later we could get a better idea of rotating around the Smith Chart. Elements in transmission lines are comparable to the operating wavelength, making it a distributed-parameter network. Voltages and currents vary in magnitde and phase over space.

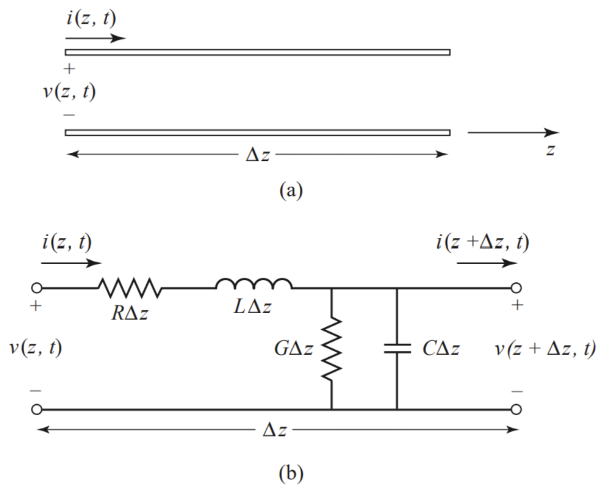


Figure 1: Voltage and current definitions and equivalent circuit for an incremental length of transmission line. (a) Voltage and current definitions. (b) Lumped-element equivalent circuit.

Hence a transmission line is often schematically represented as a two-wire line since transmission lines (for transverse electromagnetic [TEM] wave propagation) always have at least two conductors. The piece of line of infinitesimal length Δz of Figure 1a can be modeled as a lumped-element circuit, as shown in Figure 1b, where R, L, G, and C are per-unit-length quantities defined as follows:

- R = series resistance per unit length, for both conductors, in Ω/m.
- L = series inductance per unit length, for both conductors, in H/m.
- G = shunt conductance per unit length, in S/m.
- C = shunt capacitance per unit length, in F/m.

Applying Kirchhoff's voltage law and Kirchhoff's current law to the signals, we get

$$\begin{aligned} v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \\ i(z,t) - G\Delta z v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0 \end{aligned} \tag{1}$$

Dividing both equation by Δz, we have

$$\begin{aligned} \frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t} \\ \frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t} \end{aligned} \tag{2}$$

The equations above are known as **telegrapher equations** (time-domain form). For the sinusoidal steady-state condition, we can express it in the form of phasors:

$$\begin{aligned} \frac{dV(z)}{dz} = -(R + j\omega L)I(z) \\ \frac{dI(z)}{dz} = -(G + j\omega C)V(z) \end{aligned} \tag{3}$$

The two variables $V(z), I(z)$ could be decoupled into two separate 2nd-order differential equations:

$$\begin{aligned} \frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \\ \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \end{aligned} \tag{4}$$

where,

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{5}$$

is the complex propagation constant, the 2 solutions to this 2nd-order equation is exponential functions (traveling wave) apparantly. We can express this in terms of 2 traveling waves.

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \tag{6}$$

By applying this to the phasor differential equation, we yield the expression of the current,

$$I(z) = \frac{\gamma}{R + j\omega L}(V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}) \tag{7}$$

We define the quantity $\frac{R+j\omega L}{\gamma} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ as *characteristic impedance* Z_0. Then the current can be rewritten as

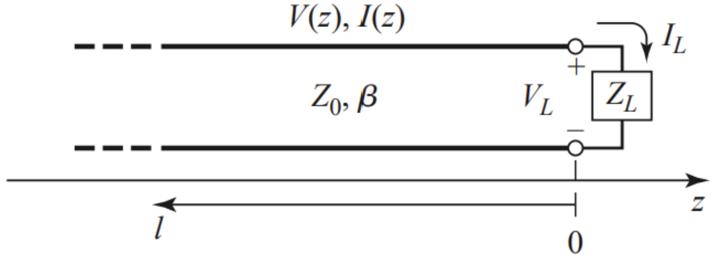
$$I(z) = \frac{V_o^+}{Z_0} e^{-\gamma z} - \frac{V_o^-}{Z_0} e^{\gamma z} \tag{8}$$

both the quantity $\frac{V_o^+}{Z_0}$ and $e^{-\gamma z}$ can be complex value at the same time, but for the sake of simplicity, we limit the discussion to the scope of lossless case (the resistance R and conductance G are both 0). Propagation constant γ is purely imaginary($\gamma = j\beta$) and characteristic impedance Z_0 is real ($Z_0 = \sqrt{\frac{L}{C}}$). And we can restate the expression to be

$$\begin{aligned} V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z} \\ I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{j\beta z} \end{aligned} \tag{9}$$

Hence, traveling along the transmission line would only result in a phase shift.The lossy case is similar, but with an attenuation factor $e^{-\alpha z}$ (if travelling in +z direction).

When we terminate the transmission line with some load impedance Z_L, we have an interface between two different "medium". We are still limiting our discussion to the lossless case. Then we can set the origin (z=0) at the interface,



Assume that an incident wave of the form $V_o^+ e^{-j\beta z}$ is generated from a source at $z < 0$. We have seen that the ratio of voltage to current for such a traveling wave is Z_0, the characteristic impedance of the line. However, when the line is terminated in an arbitrary load $Z_L \neq Z_0$, the ratio of voltage to current at the load must be Z_L . Thus, a reflected wave must be excited with the appropriate amplitude to satisfy this condition.

The total voltage and current at the load are related by the load impedance, so at z = 0 we must have

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_o^+ e^{-j\beta*0} + V_o^- e^{j\beta*0}}{\frac{V_o^+}{Z_0} e^{-j\beta*0} - \frac{V_o^-}{Z_0} e^{j\beta*0}} = \frac{V_o^+ + V_o^-}{V_o^+ - V_o^-} Z_0 \tag{10}$$

Solving for reflection V_o^-,

$$V_o^- = V_o^+ * \frac{Z_L - Z_0}{Z_L + Z_0} \tag{11}$$

We can define the *voltage reflection coefficient* Γ as the ratio of reflection over incidence, i.e.

$$\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{12}$$

Then we can restate the travelling-wave form of voltage and current as:

$$\begin{aligned} V(z) &= V_o^+ (e^{-j\beta z} + \Gamma e^{-j\beta z}) \\ I(z) &= \frac{V_o^+}{Z_0} (e^{-j\beta z} - \Gamma e^{-j\beta z}) \end{aligned} \tag{13}$$

The impedance along the transmission line (input impedance) can be expressed in the form of:

$$\begin{aligned} Z_{in} &= \frac{V(z)}{I(z)} = Z_0 \frac{e^{-j\beta z} + \Gamma e^{-j\beta z}}{e^{-j\beta z} - \Gamma e^{-j\beta z}} \\ &= Z_0 \frac{1 + \Gamma e^{-2j\beta l}}{1 - \Gamma e^{-2j\beta l}} \end{aligned} \tag{14}$$

We can generalize the expression of reflection coefficient as well (moving along the transmission line)

$$\Gamma(z) = \frac{V_o^- e^{j\beta z}}{V_o^- e^{j\beta z}} = \Gamma e^{2j\beta z} \tag{15}$$

Travelling along the line doesnot change the amplitude of the coefficient, but only exerts a phase shift.

The above only shows the tip of transmission-line calculations, determination of input impedance, reflection coefficient, standing-wave ratio and so on, oftern involve tedious manipulation of complex numbers. This boredum can be alleviated by introducing the graphical method, **Smith Chart**. And a bonus point of using it is the much easier realization of impedance matching.

2 Smith Chart

Unlike most methods, Smith chart depends on a not so straight-forward parameter, the reflection coefficient Γ. To understand how the Smith chart for a lossless TX line is constructed, we have to go back the expression of voltage reflection coefficient of load impedance:

$$\begin{aligned}\Gamma &= \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma} \\ Z_0 &= \sqrt{\frac{L}{C}}\end{aligned} \tag{16}$$

we could normalize the load Z_L with characteristic impedance Z_0, $z_L = \frac{Z_L}{Z_0} = \frac{R_L}{Z_0} + j\frac{X_L}{Z_0} = r + jx(Dimensionless)$

Then the reflection coefficient can be rewritten as,

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \tag{17}$$

The inverse of the above equation is:

$$\begin{aligned}z_L &= \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}} \\ r + jx &= \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}\end{aligned} \tag{18}$$

We rationalize the denominator and get

$$\begin{aligned}r &= \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} \\ x &= \frac{2*\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}\end{aligned} \tag{19}$$

or we can restate as

$$\begin{aligned}(\Gamma_r - \frac{r}{1+r})^2 + \Gamma_i^2 &= (\frac{1}{1+r})^2 \\ (\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{x})^2 &= (\frac{1}{x})^2\end{aligned} \tag{20}$$

Hence, we got a one-to-one mapping, from (r, x) to (Γ_r, Γ_i). Or we can say that we are mapping a whole plane onto a circle of radius 1(The total circle). And we refer the trace of a specific radius in this chart as $|\Gamma|$-circle.

We already learned from previous section that travelling along the lossless transmission line would only cause a phase shift on Γ but no amplitude change. Hence, this operation is like rotating around the $|\Gamma|$-circle. Equation (20) also indicates that loads with same normalized resistance r lie on the circle with radius 1/(1+r) snd centered at $(\Gamma_r, \Gamma_i) = (r/(1+r), 0)$ **(r-circle)**
and loads with same normalized reactance x lie on the circle with radius 1/x snd centered at $(\Gamma_r, \Gamma_i) = (1, 1/x)$ **(x-circle)**

The most direct usage of this chart is to realize impedance matching (making Γ go to zero). By introducing series or shunt stubs, we can make the reflection coefficient rotate around **(r-circle)** or **(x-circle)**, at last going back to the origin ($\Gamma = 0$).

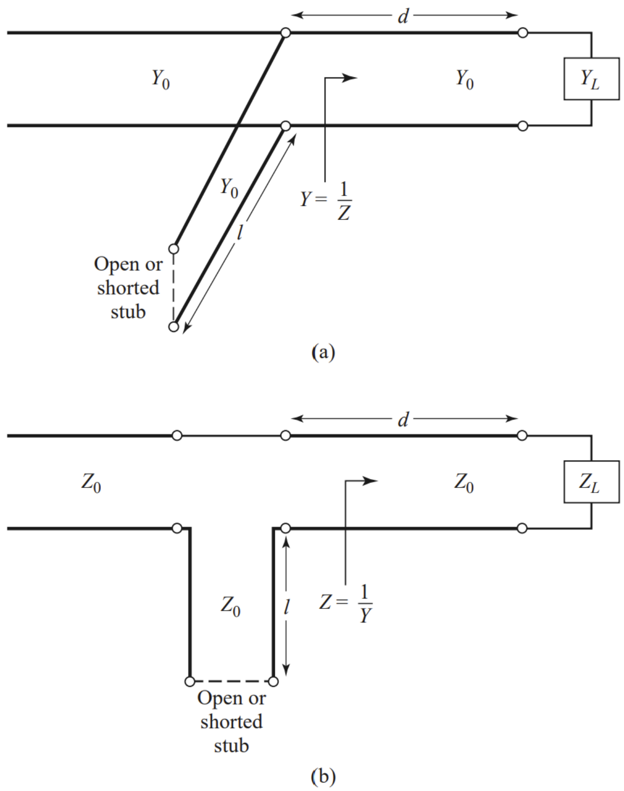
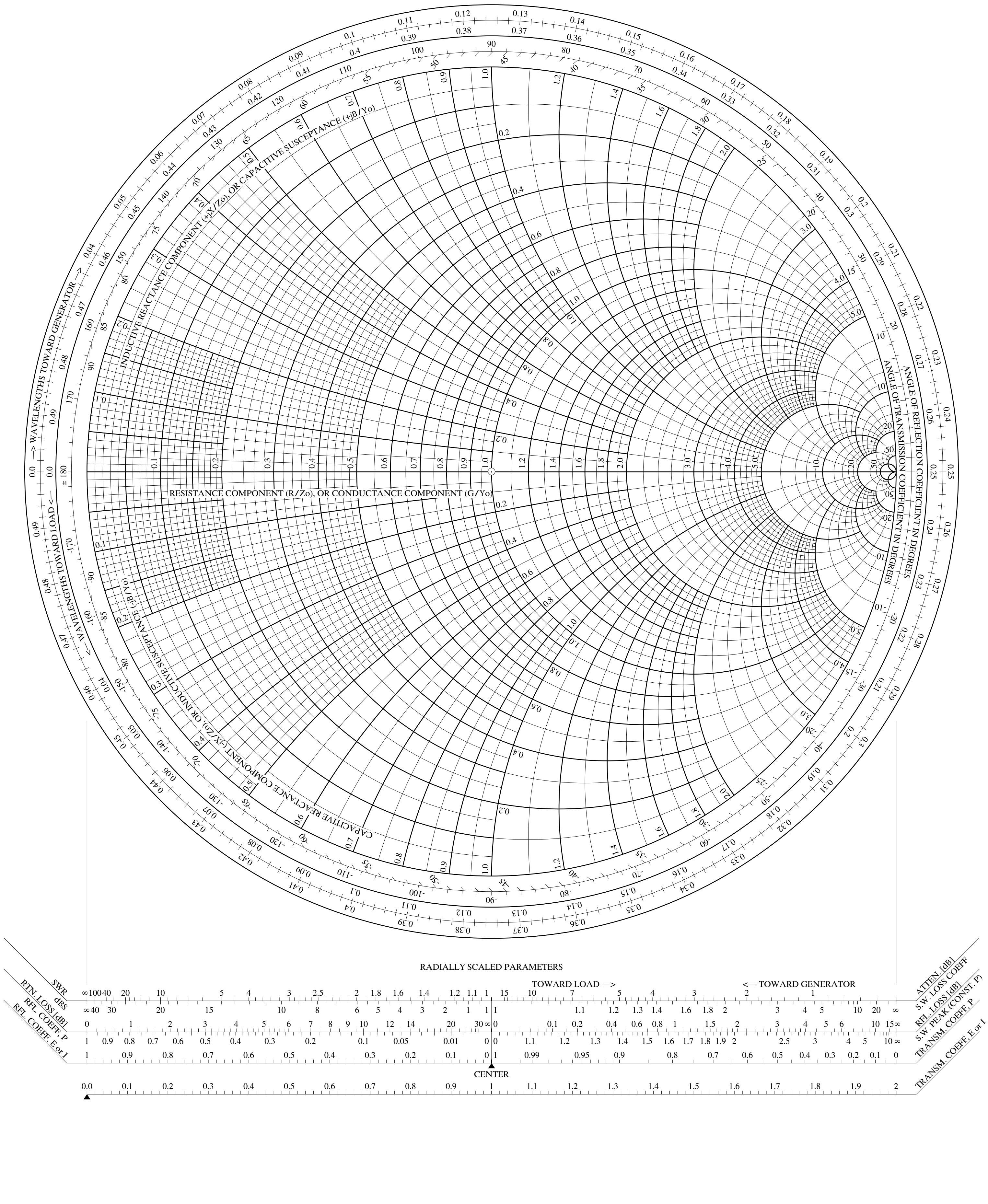


Figure 2: Single-stub tuning circuits. (a) Shunt stub. (b) Series stub

This tutorial only serves as an introduction to the rotating and theoretical fundation of **Smith Chart**. There are other methods for impedance matching other than single-stub tuning that can be visualized in **Smith Chart**. For more information, you can refer to Field and Wave Electromagnetics (David K. Cheng) and Microwave Engineering (David M. Pozar).



—> WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD <—
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
SWR
dBS
RTN. LOSS [dB]
RFL. COEFF, P
RFL. COEFF, E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF, P
TRANSM. COEFF, E or I
CENTER