Inverse Probability Weighting (IPW)

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Notation Recap

Let $W \in \{0, 1\}$ be a binary treatment or policy.

Let $Y(1)$ and $Y(0)$ denote a unit's *potential* outcome under active treatment $(W = 1)$ and control $(W = 0)$ respectively.

The *observed* outcome, Y, equals

$$\boxed{Y = (1 - W)\,Y(0) + WY(1)}$$

For each unit we observe either $Y(1)$ or $Y(0)$, but never both.

Notation Recap (continued)

1. The *Average Treatment Effect* (ATE) equals

$$\boxed{\beta^{\mathsf{ATE}} = \mathbb{E}\left[Y\left(1\right) - Y\left(0\right)\right]}$$

2. The *Propensity Score* equals

$$\boxed{e\left(x\right) = \Pr\left(W = 1 \middle| X = x\right)}$$

Two Key Assumptions

A.1 Exogeneity:

$$(Y(0), Y(1)) \perp W | X = x, \; x \in \mathbb{X}$$

A.2 Overlap:

$$0 < \kappa \leq e(x) \leq 1 - \kappa < 1, \; x \in \mathbb{X}$$

Inverse Probability Weighting (IPW): Heuristics

Selection into treatment implies that the distribution of covariates, X, varies across treatment units ($W = 1$) and control units ($W = 0$).

We can use the propensity score to characterize how the distribution of covariates differs across treated and control units.

IPW: Heuristics (continued)

Some values of $X = x$ are rarely observed among treated units.

These values correspond to $X = x$ where the propensity score, $e(x)$, is closer to zero.

This is because units with these values of $X = x$ are almost never treated (hence rarely seen among the treated).

IPW: Heuristics (continued)

Likewise some values of $X = x$ are rarely observed among control units.

These values correspond to $X = x$ where the propensity score, $e(x)$, is closer to one.

This is because units with these values of $X = x$ are almost always treated (hence rarely seen among the controls).

IPW: Heuristics (continued)

Can we construct a re-weighting procedure using the propensity score?

1. Correct averages of the outcome among *treated* units by up-weighting under-represented units (i.e., those with *low* propensity scores).

2. Correct averages of the outcome among *control* units by up-weighting under-represented units (i.e., those with *high* propensity scores).

IPW: Identification

By the law of iterated expectations (LIE) we have

$$\mathbb{E}\left[\frac{WY}{e(X)}\right] = \mathbb{E}\left[\mathbb{E}\left[\left.\frac{WY}{e(X)}\right| X\right]\right]$$
$$= \mathbb{E}\left[\frac{1}{e(X)}\mathbb{E}\left[WY \mid X\right]\right], \quad (1)$$

where we condition on X in the inner expectation.

This allows us to factor out the propensity score which is a function of X alone.

IPW: Identification (continued)

Applying LIE a second time, now to the inner expectation, we have that

$$\begin{aligned}\mathbb{E}\left[\left.WY\right|X\right] =& \mathbb{E}\left[\left.\mathbb{E}\left[\left.WY\right|X,W\right]\right|X\right] \\ =& e\left(X\right)\mathbb{E}\left[\left.1\cdot Y\right|X,W=1\right] \\ &+\left(1-e\left(X\right)\right)\mathbb{E}\left[\left.0\cdot Y\right|X,W=0\right] \\ =& e\left(X\right)\mathbb{E}\left[\left.Y\left(1\right)\right|X,W=1\right] \\ \overset{A.1}{=}& e\left(X\right)\mathbb{E}\left[\left.Y\left(1\right)\right|X\right].\end{aligned} \tag{2}$$

First and second equalities are implications of LIE.

Last equality is an implication of selection on observables (A.2).

IPW: Identification (continued)

Combining (1) and (2) yields

$$
\begin{aligned}
\mathbb{E}\left[\frac{WY}{e(X)}\right] &= \mathbb{E}\left[\frac{1}{e(X)}\mathbb{E}\left[\left.WY\right|X\right]\right] \\
&= \mathbb{E}\left[\frac{e(X)\,\mathbb{E}\left[\left.Y(1)\right|X\right]}{e(X)}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left.Y(1)\right|X\right]\right] \\
&= \mathbb{E}\left[Y(1)\right] \qquad (3)
\end{aligned}
$$

Weighting by the *inverse* of the propensity score recovers the mean of $Y(1)$ from the observed values of Y among treated units.

IPW: Interpretation Redux

Consider the mean outcome among the treated:

$$
\begin{aligned}
\mathbb{E}\left[Y|\,W=1\right] &= \mathbb{E}\left[\frac{WY}{Q}\right] \\
&= \mathbb{E}\left[Y\left(1\right)|\,W=1\right] \\
&\neq \mathbb{E}\left[Y\left(1\right)\right]
\end{aligned}
$$

with $Q = \Pr\left(W=1\right)$ the *marginal* probability of treatment.

Replacing Q with $e\left(X\right)$ "adjusts" the naive mean outcome of the treated to account for covariate differences.

IPW: Main Result

Analogous calculations give

$$\mathbb{E}\left[\frac{(1-W)\,Y}{1-e\,(X)}\right]=\mathbb{E}\left[Y\,(0)\right]. \tag{4}$$

Combining (3) and (4) yields our main identification result

$$\boxed{\beta^{\mathsf{ATE}}=\mathbb{E}\left[\frac{WY}{e\,(X)}\right]-\mathbb{E}\left[\frac{(1-W)\,Y}{1-e\,(X)}\right]}$$

IPW: Estimation

We will begin with the case where the propensity score is *known*.

We will cover the case where the propensity score is unknown later.

The obvious *analog estimator* replaces population means with their sample counterparts.

IPW: Estimation (continued)

Available is a simple random sample of size N, $\{(W_i, X_i, Y_i)\}_{i=1}^{N}$.

Our initial estimate of the ATE is

$$\widehat{\beta}^{\mathsf{ATE}} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{W_i Y_i}{e\left(X_i\right)} - \frac{\left(1 - W_i\right) Y_i}{1 - e\left(X_i\right)}\right).$$

IPW: Estimation (continued)

In practice it is better to use instead the estimate

$$\widehat{\beta}^{\mathsf{ATE}} = \sum_{i=1}^{N} \left(\frac{\frac{W_i Y_i}{e(X_i)}}{\sum_{j=1}^{N} \frac{W_j}{e(X_j)}} - \frac{\frac{(1-W_i)Y_i}{1-e(X_i)}}{\sum_{j=1}^{N} \frac{1-W_j}{1-e(X_j)}} \right) \tag{5}$$

This ensures that the IPW weights sum to one (this is only true in expectation for the initial estimate).

See Graham, Pinto and Egel (2012, *Review of Economic Studies*) for some relevant theory.

Weighted Least Squares (WLS) Approach

1. Define the weight

$$V_i = \left\{ \frac{W_i}{e\left(X_i\right)} - \frac{1 - W_i}{1 - e\left(X_i\right)} \right\}$$
$$= \left\{ \frac{W_i - e\left(X_i\right)}{e\left(X_i\right)\left(1 - e\left(X_i\right)\right)} \right\}$$

   for $i = 1, \ldots, N$.

2. Compute WLS fit of Y_i onto a constant and W_i using V_i as the weight.

3. The coefficient on W_i coincides with $\widehat{\beta}^{\mathsf{ATE}}$ as defined in (5) above.

Inference

We will use the Efron's percentile bootstrap to construct a confidence interval for β^{ATE}.

This is just one recipe/approach.

Efficiency theory in this problem is interesting; with practical and theoretical open questions remaining.

See Imbens and Wooldridge (2009, *Journal of Economic Literature*) for a recent survey.

Bootstrap

Main idea: we use our sample and simulation to learn about the underlying population from which it was drawn.

Algorithm:

1. Using the original sample, draw a *bootstrap sample* of N observations with replacement.

2. Using this bootstrap sample, compute the WLS fit described earlier. Let $\widehat{\beta}^{\text{ATE}}_{(b)}$ be the associated ATE estimate.

3. Repeat steps #1 and #2 $b = 1, \ldots, B$ times.

Bootstrap (continued)

Say we chose to perform $B = 1,000$ bootstrap replications.

To construct a confidence interval for β^{ATE} we begin by sorting our bootstrap estimates $\widehat{\beta}^{\text{ATE}}_{(1)}, \ldots, \widehat{\beta}^{\text{ATE}}_{(B)}$ from smallest to largest.

The 25^{th} and 975^{th} sorted bootstrap ATE estimates, which correspond to the 0.025 and 0.0975 quantiles of the bootstrap distribution, can be used as the lower and upper bounds of an (approximate) 95 percent confidence interval.

More exotic bootstrap procedures also possible (e.g., Bayesian Bootstrap).

Wrapping Up

There are many other consistent ATE estimators (literally hundreds)!

IPW generally works well when overlap is good (i.e., there are not too many very small or very large values of the propensity score).

In practice modified versions of IPW work even better.

Wrapping Up

Next time we will discuss logistic regression.

We will use logistic regression to construct estimates of the propensity score.

These are required when the propensity score is unknown, as is typically the case in observational settings.