EECS 501 Homework 3 Yuzhan JIANG

P1

$5 \to 6 \to$ win $\qquad 6\text{-}1$ — Lose

Based on the first outcomes of the game, the following events can be observed:

a) 1, 2, 3, 4 $\to P = \frac{2}{3}$. repeat the game

b) 5 $\to P = \frac{1}{6}$

b_1: 1, 2, 3, 4, $P = \frac{1}{6} \times \frac{4}{6} = \frac{4}{36} = \frac{1}{9}$ repeat the game

b_2: 6 win $P = \frac{1}{36}$

b_3: 5 stay in the stage $\frac{1}{36}$

c) 6 $\to P = \frac{1}{6}$

c_1: 1 Lose $P = \frac{1}{36}$

c_2: 2, 3, 4 $P = \frac{3}{6} \times \frac{1}{6} = \frac{1}{12}$, repeat the game

c_3: 6 $P = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ stay in the state

c_4: 5 go to state b, $P = \frac{1}{36}$

By Law of total Expectation:

$$E[X] = \sum_{i=1}^{m} E[X|B_i]\,P(B_i)$$

$$= E[X|a]P(a) + E[X|b]P(b) + E[X|c]P(c)$$

Where

① $E[X|a] = 1 + E[X]$, $P(a) = \frac{2}{3}$

② $E[X|b] = E[X|b_1]P(b_1|b) + E[X|b_2]P(b_2|b) + E[X|b_3]P(b_3|b)$ (By law of total Expecations)

(Note that $E[X|b_1] = 2 + E[X]$, $P(b_1|b) = \frac{2}{3}$ $E[X|b_2] = 2$, $P(b_2|b) = \frac{1}{6}$, $E[X|b_3] = 1 + E[X|b]$, $P(b_3|b) = \frac{1}{6}$

$$\Rightarrow E[X|b] = \frac{2}{3}(2 + E[X]) + 2 \times \frac{1}{6} + \frac{1}{6}(E[X|b] + 1)$$

$$= \left(\frac{4}{3} + \frac{1}{3} + \frac{1}{6}\right) + \frac{2}{3}E[X] + \frac{1}{6}E[X|b]$$

$$\frac{5}{6}E[X|b] = \frac{11}{6} + \frac{2}{3}E[X]$$

$$E[X|b] = \frac{11}{5} + \frac{4}{5}E[X]$$

(By law of total Expecations)

③ $E[X|C] = E[X|C_1]P(C_1|C) + E[X|C_2]P(C_2|C) + E[X|C_3]P(C_3|C) + E[X|C_4]P(C_4|C)$

Note that, $E[X|C_1] = 2$, $P[C_1|C) = \frac{1}{6}$, $E[X|C_2] = 2 + E[X]$, $P(C_2|C) = \frac{1}{2}$

$E[X|C_3] = 1 + E[X|C]$, $P(C_3|C) = \frac{1}{6}$, $P(C_4|C) = \frac{1}{6}$, $E[X|C_4] = 1 + E[X|b]$

$$\therefore \quad E[X|C] = 2\times\frac{1}{6} + \frac{1}{2}(2+E[X]) + \frac{1}{6}(1+E[X|C]) + \frac{1}{6}(1+E[X|b])$$

$$E[X|C] = \left(\frac{1}{3}+1+\frac{1}{6}+\frac{1}{6}\right) + \frac{1}{2}E[X] + \frac{1}{6}E[X|C] + \frac{11}{30} + \frac{4}{30}E[X]$$

$$\frac{5}{6}E[X|C] = \frac{5}{3} + \frac{11}{30} + \frac{19}{30}E[X]$$

$$\frac{5}{6}E[X|C] = \frac{61}{30} + \frac{19}{30}E[X]$$

$$E[X|C] = \frac{61}{25} + \frac{19}{25}E[X]$$

Overall,

$$E[X] = (1+E[X])\frac{2}{3} + \left(\frac{11}{5}+\frac{4}{5}E[X]\right)\frac{1}{6} + \left(\frac{61}{25}+\frac{19}{25}E[X]\right)\frac{1}{6}$$

$$= \left(\frac{2}{3}+\frac{11}{30}+\frac{61}{150}\right) + \left(\frac{2}{3}+\frac{2}{15}+\frac{19}{150}\right)E[X]$$

$$\Rightarrow E[X] = \frac{216}{11}$$

P2: We are given that,

$$P(\text{Congested}) = P(C) = \frac{1}{5} \Rightarrow P(C^c) = \frac{4}{5}$$

$$P(ACK|C) = \frac{5}{10} \Rightarrow P(NACK|C) = 1 - \frac{5}{10} = \frac{1}{2}$$

$$P(ACK|C^c) = \frac{9}{10} \Rightarrow P(NACK|C^c) = 1 - \frac{9}{10} = \frac{1}{10}$$

$$P(C|X=3) = ?$$

By the law of total Probability and Bayes' Law

$$P(C|X=3) = \frac{P(X=3|C)\,P(C)}{P(X=3)}$$

$$= \frac{P(X=3|C)\,P(C)}{P(X=3|C)P(C) + P(X=3|C^c)P(C^c)}$$

$$= \frac{\left(\frac{5}{10}\right)^3 \frac{1}{5}}{\left(\frac{5}{10}\right)^3 \cdot \frac{1}{5} + \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^2 \frac{4}{5}}$$

$$= \frac{\frac{1}{40}}{\frac{1}{40} + \frac{4}{5} \times \frac{9}{1000}}$$

$$= \frac{125}{161}$$

$$\therefore P(C|X=3) = \frac{125}{161}$$

P3. Roll once:

(a) $6 \to$ win $\quad 1 \to$ lose

Based on the first outcomes, we can consider following situations:

a) $6 \to$ win $\quad P = \frac{1}{6}$

b) $1 \to$ lose $\quad P = \frac{1}{6}$

c) $2, 3, 4, 5.$ $\quad P = \frac{2}{3}$

c_1: $1, 6 \to$ Continue roll untill $2,3,4,5$, $P = \frac{2}{3} \times \frac{2}{6} = \frac{2}{9}$

c_2: $2, 3, 4 \to$ win, $P = \frac{2}{3} \times \frac{3}{6} = \frac{1}{3}$

c_3: $5 \to$ Lose. $P = \frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$

By the law of total expecation:

$$E[X] = \sum_{i=1}^{m} E[X|B_i] P[B_i]$$

$$= E[X|a] P(a) + E[X|b] P(b) + E[X|C] P(C)$$

Note that:

$$E[X|a] = 1 \quad P(a) = \frac{1}{6}; \quad E[X|b] = 1, \ P(b) = \frac{1}{6}$$

Again by the law of total Expecation:

$$E[X|C] = E[X|C_1] P(C_1|C) + E[X|C_2] P(C_2|C) + E[X|C_3] P(C_3|C)$$

Note that, $E[X|C_1] = 1 + E[X|C]$, $P(C_1|C) = \frac{1}{3}$, $E[X|C_2] = E[X|C_3] = 2$, $P(C_2|C) = \frac{1}{2}$

$P(C_3|C) = \frac{1}{6}$

$$\therefore \quad E[X|C] = (1 + E[X|C]) \frac{1}{3} + 2 \times \frac{1}{2} + 2 \times \frac{1}{6}$$

$$= \left(\frac{1}{3} + 1 + \frac{1}{3}\right) + \frac{1}{3} E[X|C]$$

$$\frac{2}{3} E[X|C] = \frac{5}{3}$$

$$E[X|C] = \frac{5}{2}$$

$$\therefore \quad E[X] = 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + \frac{2}{3} \times \frac{5}{2}$$

$$= \frac{1}{3} + \frac{5}{3}$$

$$= 2$$

$$\therefore \quad E[X] = 2$$

b) Using Law of total Probability, we condition on the events a, b, c.

$$P(X=1) = P(a \cup b)$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{1}{3}$$

$$P(X=2) = P(C_2 \cup C_3)$$
$$= \frac{2}{3} \times \frac{3}{6} + \frac{2}{3} \times \frac{1}{6}$$
$$= \frac{1}{3} + \frac{1}{9}$$
$$= \frac{4}{9}$$

For $k > 2$, we have,

$$P(X=k) = \underbrace{P(X=k|a)}_{0} P(a) + \underbrace{P(X=k|b)}_{0} P(b) + P(X=k|c)P(c)$$

$$P(X=k) = P(X=k|c)P(c)$$

And $P(X=k|c) = P(X=k|C_1)P(C_1|c) + \underbrace{P(X=k|C_2)}_{0} P(C_2|c) + \underbrace{P(X=k|C_3)}_{0} P(C_3|c)$

$$\therefore P(X=k) = P(X=k-1|c) \cdot \frac{1}{3} \cdot \frac{2}{3}$$
$$= \frac{2}{9} P(X=k-1|c)$$

Note that,

$$P(X=k-1|c) = \frac{P(X=k-1 \cap c)}{P(c)} = \frac{3}{2} P(X=k-1)$$

$$\therefore P(X=k) = \frac{2}{9} \cdot \frac{3}{2} \cdot P(X=k-1)$$
$$= \frac{1}{3} P(X=k-1)$$

$$\therefore P(X=k) = \begin{cases} \frac{1}{3}, & k=1 \\ \frac{4}{9}, & k=2 \\ \frac{1}{3} P(X=k-1), & k \geq 3 \end{cases}$$

P4:

(a) K denote the numbers of confirming and showing up for fixed N, $K \le N$

$$P_K(k) = P(K=k) = \binom{N}{k}(0.92)^k(0.08)^{N-k}$$

D denotes that the profit that airline makes for each flight:

① If $K \le 100$, $N \ge K$

$$P_D(d) = \sum_{k: g(k)=d} P_K(k) \quad \text{where } g(k) \text{ is function of } k \text{ to compute the profit}$$

$$= \binom{N}{k}(0.92)^k(0.08)^{N-k}, \quad d = 75k$$

② If $N > 100$

a. If $K \le 100$

$$P_D(d) = \sum_{k: g(k)=d} P_K(k)$$

$$= \binom{N}{k}(0.92)^k(0.08)^{N-k}, \quad d = 75k$$

b. If $K > 100$

$$P_D(d) = \sum_{k: g(k)=d} P_K(k)$$

$$= \binom{N}{k}(0.92)^k(0.08)^{N-k}, \quad d = 75 \times 100 - (k-100) \times 100$$

$$d = 7500 - 100k + 10000$$

$$= 17500 - 100k$$

$\therefore$ overall,

$$P(D=d) = \begin{cases} \binom{N}{k}(0.92)^k(0.08)^{N-k}, & k < 100, \quad d = 75k \Rightarrow k = \frac{d}{75} \\ \binom{N}{k}(0.92)^k(0.08)^{N-k}, & k > 100, \quad d = 17500 - 100k \Rightarrow k = \frac{17500-d}{100} \end{cases}$$

$$\Rightarrow P(D=d) = \begin{cases} \binom{N}{\frac{d}{75}}(0.92)^{\frac{d}{75}}(0.08)^{N-\frac{d}{75}}, & k = \frac{d}{75} \\ \binom{N}{\frac{17500-d}{100}}(0.92)^{\frac{17500-d}{100}}(0.08)^{N-\frac{17500-d}{100}}, & k = \frac{17500-d}{100} \end{cases}$$

(b) $E[D] = \sum_{k} g(k) P(K=k)$ $\qquad D = g(k)$

$$E[D] = \sum_{k=101}^{N} (17500 - 100k)\binom{N}{k} 0.92^{k}\, 0.08^{N-k} + \sum_{k=0}^{100} \binom{N}{k}(75k)(0.92)^{k}(0.08)^{N-k}$$

(c)

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```
function compute(N)
    res = 0
    for k in 0:100
        res = res + binomial(BigInt(N), BigInt(k)) * 75* k *  (0.92)^k * (0.08)^(N-k)
    end
    for k in 101:N
        res = res  + binomial(BigInt(N), BigInt(k)) * (17500 - 100* k)  *  (0.92)^k * (0.08)^(N-k)
    end
    return res
end

function Find_max_N()
    res = 0
    Max_N = 0
    for N in 0:150
        cur_res =  compute(N)
        if cur_res > res
            Max_N = N
            res = cur_res
        end
    end
    return Max_N
end

display(Find_max_N())
```

```
julia> display(Find_max_N())
108
```

When $N = 108$, it maximizes the expected profit.

P5:

(a) X, Y are two independent geometric random variables

cdf of X: $1-(1-P_1)^k$ cdf of Y: $1-(1-P_2)^k$

$P(X>k) = (1-P_1)^k$ $P(Y>k) = (1-P_2)^k$

$P(Z=\min(X,Y)>k) = P(X>k, Y>k) = P(X>k)P(Y>k)$

$= (1-P_1)^k(1-P_2)^k = [(1-P_1)(1-P_2)]^k$

$\therefore P(Z=\min(X,Y) \le k) = 1-[(1-P_1)(1-P_2)]^k$ where Z is geometric r.v with $p=1-((1-P_1)(1-P_2))$

$\therefore P(Z=z) = [(1-P_1)(1-P_2)]^z[1-(1-P_1)(1-P_2)]$, $z=0,1,2\ldots$

(b) X, Y are two independent binomial random variables.

Let $Z=X+Y$, where $X\sim B(n_1,p)$ $Y\sim B(n_2,p)$

The moment generating function of X, Y are

$M_X(t) = (q+pe^t)^{n_1}$ and $M_Y(t) = (q+pe^t)^{n_2}$

$\Rightarrow M_Z(t) = E[e^{tZ}] = E[e^{t(X+Y)}]$

$= E[e^{tX}\cdot e^{tY}]$

$= E[e^{tX}]\cdot E[e^{tY}]$ (due to X and Y are independent r.v)

$= (q+pe^t)^{n_1}(q+pe^t)^{n_2}$

$= (q+pe^t)^{n_1+n_2} \Rightarrow Z\sim B(n_1+n_2,p)$

$\therefore P(Z=z) = \binom{n_1+n_2}{z}p^z q^{n_1+n_2-z}$

5(C) X_1, X_2, X_3 are geometric random variables

The moment generating function

$$M(t) = \frac{p}{1-qe^t}$$

$$P(X=x) = p^k q^x \binom{x+k-1}{k-1} e^{tx} \quad X \sim \text{Negative Binomial}(k, p)$$

$$Z = X_1 + X_2 + X_3 \sim \text{Negative Binomial}(3, p)$$

$$\therefore P(Z=z) = p^3 q^z \binom{z+2}{2}$$