Lecture 3: Goals

- Sinusoids
- Vectors to signals and signals to vectors

Sinusoids

Aprop freq phase.

$A\cos(2\pi f_c t + \theta)$

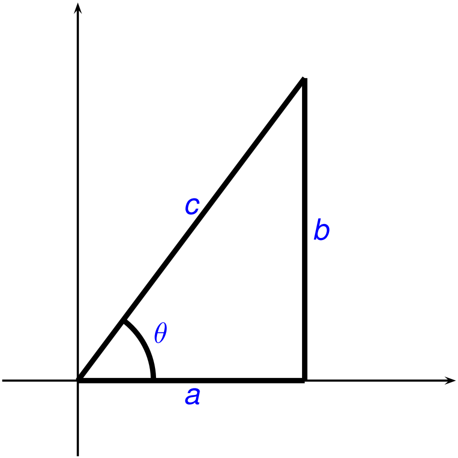
$$
\begin{aligned}
s(t) &= c\cos(2\pi f_c t + \theta) \\
&= c\cos(\theta)\cos(2\pi f_c t) - c\sin(\theta)\sin(2\pi f_c t) \\
&= a\cos(2\pi f_c t) - b\sin(2\pi f_c t)
\end{aligned}
$$

where $a = c\cos(\theta)$ and $b = c\sin(\theta)$.

in-phase.

Relation between sinusoidal parameters

$$
\begin{aligned}
c &= \sqrt{a^2 + b^2} \\
\theta &= \tan^{-1}\left(\frac{b}{a}\right) \\
a &= c\cos(\theta) \\
b &= c\sin(\theta)
\end{aligned}
$$



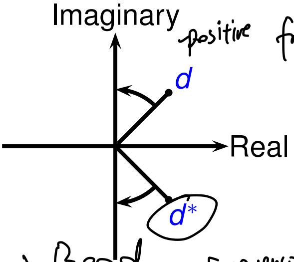
Complex Exponentials/Rotating Vectors

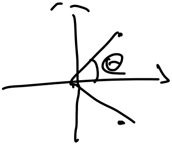
$\cos\alpha = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha})$

$\frac{1}{2}e$

$$
\begin{aligned}
s(t) &= \underline{c\cos(2\pi f_c t + \theta)} \\
&= c\left[\frac{e^{j(2\pi f_c t+\theta)} + e^{-j(2\pi f_c t+\theta)}}{2}\right] \\
&= \frac{ce^{j\theta}}{2}e^{j2\pi f_c t} + \frac{ce^{-j\theta}}{2}e^{-j2\pi f_c t} \\
&= de^{j2\pi f_c t} + d^* e^{-j2\pi f_c t} \\
&= \Re\left[2de^{j2\pi f_c t}\right]
\end{aligned}
$$

positive freq





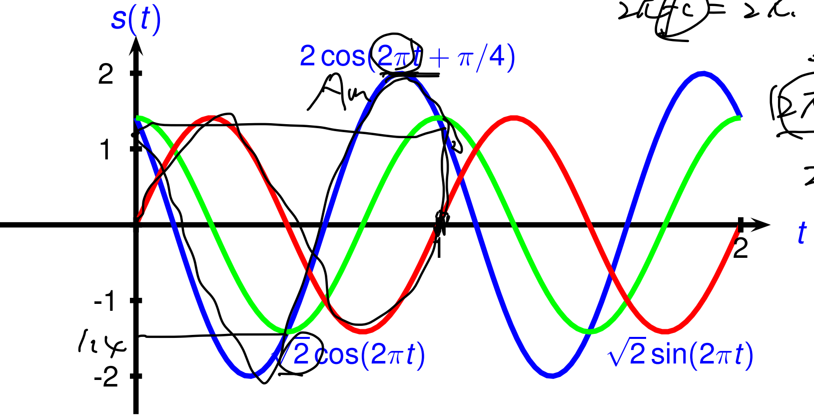
Real

neg frequency

Example: Sinusoidal Signals

A max

$2\pi f_c = 2\pi.$



$= 1$

$(2\pi f_c t + \theta)$

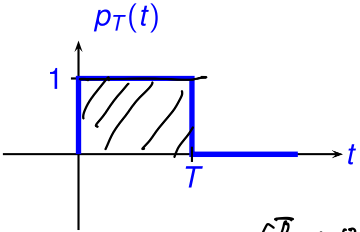
$2\pi = 2\pi f_c$

$f_c = 1$

$\frac{1}{f_c} = 1$

Pulse Function

$$p_T(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$



$\int_0^T |1|^2 \, dt = T$

Signals and Vectors: Definitions

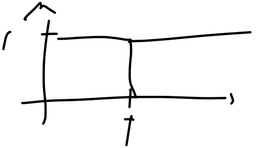
magnitude

- The **energy** of a (possibly complex) **signal** $x(t)$ is $E = ||x(t)||^2 = \int |x(t)|^2 dt$.
- The **energy** of a (possibly complex) **vector** $\mathbf{x} = (x_0, x_1, x_2, ..., x_{N-1})$ is $E = ||\mathbf{x}||^2 = \sum_{\mathbf{i=0}}^{\mathbf{N-1}} |\mathbf{x_i}|^2$.

- The **inner product** of two **signals** $x(t)$ and $y(t)$ is $(x(t), y(t)) = \int x(t)y^*(t)dt$.
- The **inner product** of two **vectors** $\mathbf{x} = (x_0, x_1, ..., x_{N-1})$ and $\mathbf{y} = (y_0, y_1, ..., y_{N-1})$ is $(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{N-1} x_i y_i^*$.

Signals and Vectors: Definitions

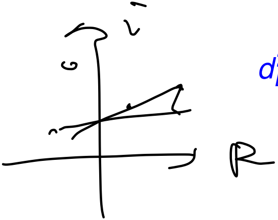
- Two **signals** $x(t)$ and $y(t)$ or two **vectors** $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal** if the inner product is 0.
- A **signal** or a **vector** $\mathbf{x}$ is said to be **normalized** if the energy is 1.
- A set of **signals** or a set of **vectors** are said to be **orthonormal** if they are pair-wise orthogonal and normalized.



- The **squared Euclidean distance** between two possibly complex **signals** $x(t)$ and $y(t)$ is defined as

$$d_E^2(x(t), y(t)) = \int |x(t) - y(t)|^2 dt.$$

The **squared Euclidean distance** between two possibly complex **vectors** $\mathbf{x}$ and $\mathbf{y}$ is defined as



$$d_E^2(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{N-1} |x_n - y_n|^2.$$

Signals and Vectors: Definitions

Def.: A set of N (possibly complex) functions $\varphi_i(t), i = 0, 1, ..., N-1$ is said to be orthonormal over the time interval I if

$$\int_I \varphi_i(t)\varphi_j^*(t)dt = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Note that the orthonormal signals have unit energy.

$\varphi_1(t)\varphi_2^*(t) =$ $\varphi_1 \varphi_r$

Orthonormal Set 1

$N=2$

- Assume $2f_cT$ is an integer. That is, there is an integer number of cycles of a sinusoid of frequency f_c in T seconds.

$$\varphi_0(t) = +\sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t)$$
$$\varphi_1(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t)$$

- The reason for the negative sign in $\varphi_1(t)$ will be clear later.
- These signals are said to be orthogonal in phase.

$$\cos^2(2\pi f_c t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi 2 f_c t)$$

Proof of orthogonality

$\int \cos x\, dx = \sin x$

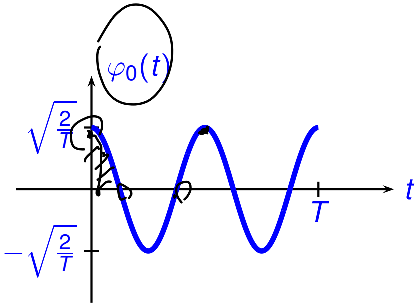
$$
\begin{aligned}
(\varphi_0(t), \varphi_1(t)) &= \int \varphi_0(t)\varphi_1^*(t)dt \\
&= -\int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\sqrt{\frac{2}{T}}\sin(2\pi f_c t)dt \\
&= -\frac{2}{T}\int_0^T \cos(2\pi f_c t)\sin(2\pi f_c t)dt \\
&= -\frac{1}{T}\int_0^T \sin(2\pi(2f_c)t)dt \\
&= \frac{\cos(2\pi(2f_c)t)}{2\pi(2f_c)T}\Big|_0^T \\
&= \left[\frac{\cos(2\pi(2f_c)T) - 1}{2\pi(2f_c)T}\right]
\end{aligned}
$$

$2f_cT$ is an interger.

$= \left(\frac{1-1}{2\pi(2f_c)T}\right) = 0$ or ≈ 0 if $2f_cT \gg 1$

- As long as $f_c T \gg 1$ the signals $\varphi_0(t)$ and $\varphi_1(t)$ are approximately orthogonal. The $\sin(2\pi(2f_c)t)$ term is called the double frequency term. We will often (always?) ignore such terms.
- For example if $f_c = 2.4GHz$ and $T = 0.1\mu s$ then $f_c T = 240$ and the magnitude of the inner product will be no larger than 6.6×10^{-4} which is negligible compared to the energy (1) of each signal.

Orthonormal Set 1



Orthonormal Set 2

Assume $2(f_1 - f_0)T$ is an integer. That is, there is an integer number of cycles in the difference frequency in T seconds.

$$\varphi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)$$

$$\varphi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t)$$

These signals are said to be orthogonal in frequency.

$\int \varphi_0(t)\cdot\varphi_1^*(t)\,dt$

$= \frac{2}{T}\int \cos(2\pi f_0 t)\cos(2\pi f_1 t)\,dt$

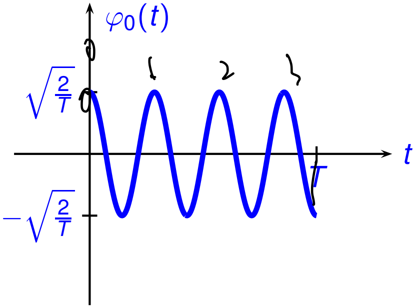
$= \frac{2}{T}\int \frac{1}{2}[\cos(2\pi t(f_0-f_1)) + \cos(2\pi t(f_0+f_1))]\,dt$

$= \frac{1}{T}\left(\frac{\sin(2\pi t(f_0-f_1))}{2\pi(f_0-f_1)}\Big|_0^T + \left(\frac{\sin(2\pi t(f_0+f_1))}{2\pi(f_0+f_1)}\right)\Big|_0^T\right)$

$= 0$

$2fT$

Orthonormal Set 2



Orthonormal Set 3

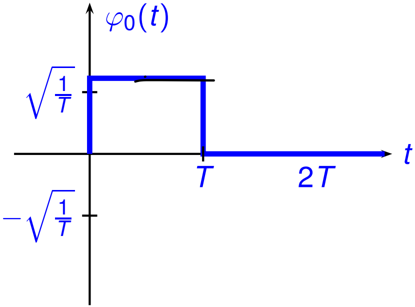
In this example the signals are orthonormal over the time interval $I = [0, 2T]$.

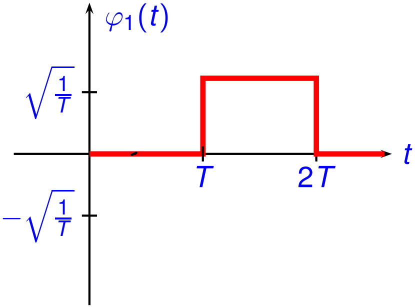
$$\varphi_0(t) = \sqrt{\frac{1}{T}} p_T(t)$$
$$\varphi_1(t) = \sqrt{\frac{1}{T}} p_T(t - T)$$

These signals are said to be orthogonal in time.

$$\int \varphi_0(t) \cdot \varphi_1^*(t)\, dt$$
$$= \int \frac{1}{T} p_T(t) \cdot p_T(t-T) \cdot dt$$
$$= \int_0^{\frac{T}{2}} \sqrt{\tfrac{1}{T}} \cdot 0 \; dt + \int_{\frac{T}{2}}^{T} 0 \cdot \sqrt{\tfrac{1}{T}} \cdot dt$$
$$= 0$$

Orthonormal Set 3



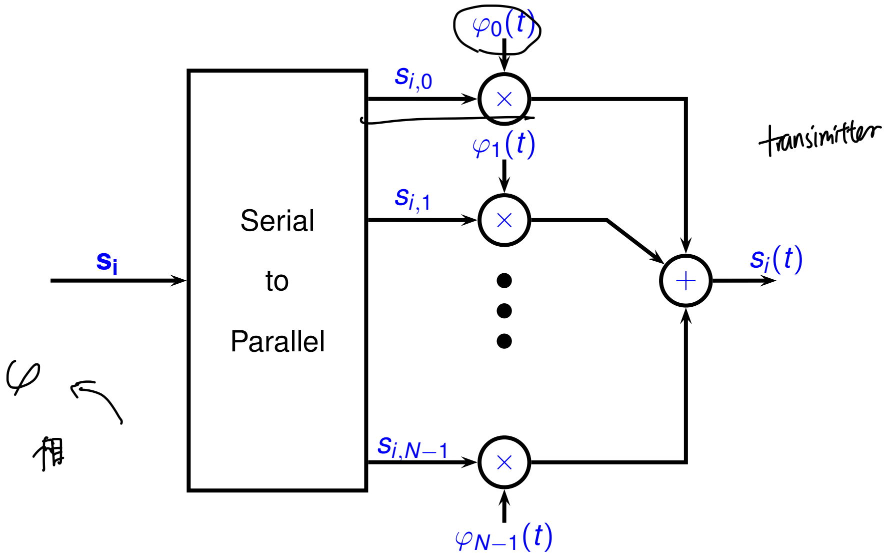


Signals and Vectors

Given a set of orthogonal signals $\{\varphi_i(t), i = 0, 1, ..., N-1\}$ and a set of M vectors $\mathbf{s_m} = (s_{m,0}, ..., s_{m,N-1}), m = 0, 1, ..., M-1$ we can construct a set of M signals as

$$s_m(t) = \sum_{i=0}^{N-1} s_{m,i}\varphi_i(t), m = 0, 1, ..., M-1$$

From a Vector to a Signal (Signal Composition)



Signals and Vectors

Given a signal $s_m(t)$ constructed from a set of orthonormal waveforms as above, we can determine the vector as

$$s_{m,n} = \int_I s_m(t)\varphi_n^*(t)dt, n = 0, 1, ..., N-1, m = 0, 1, ..., M-1.$$

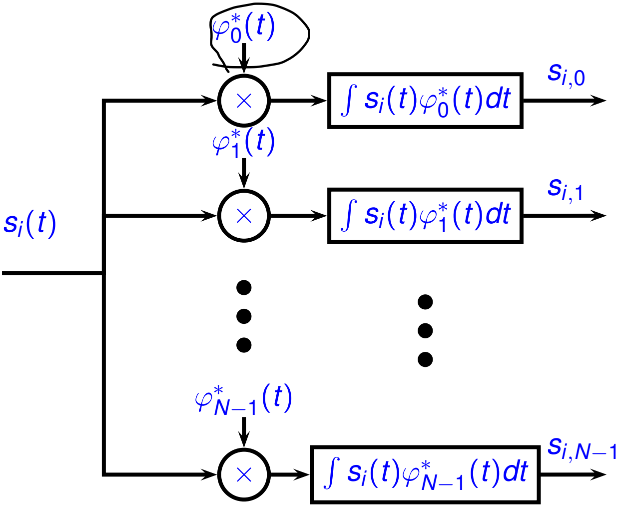
Proof: $s_{m,n} = \int s_m(t)\varphi_n^* dt$

$$\begin{aligned}
\int_I s_m(t)\varphi_n^*(t)dt &= \int_I \sum_{i=0}^{N-1} s_{m,i}\varphi_i(t)\varphi_n^*(t)dt \\
&= \sum_{i=0}^{N-1} s_{m,i} \underbrace{\int_I \varphi_i(t)\varphi_n^*(t)dt} \\
&= \sum_{i=0}^{N-1} s_{m,i}\delta_{i,n} \\
&= s_{m,n}
\end{aligned}$$

$= 0$ if $i \neq n$

1 if $i = n$

From a Signal to a Vector (Signal Decomposition)



Recie ver.

Properties

Energy, inner product and distance of signal waveforms constructed from orthonormal waveforms and signal vectors are the same.

1. $||s_m(t)||^2 = \int |s_m(t)|^2 dt = \sum_{n=0}^{N-1} |s_{m,n}|^2 = ||\mathbf{s_m}||^2$
2. $(s_m(t), s_l(t)) = \int s_m(t) s_l^*(t) dt = \sum_{n=0}^{N-1} s_{m,n} s_{l,n}^* = (\mathbf{s_m}, \mathbf{s_l})$
3. $d_E^2(s_m(t), s_l(t)) = ||s_m(t) - s_l(t)||^2 = \int |s_m(t) - s_l(t)|^2 dt = \sum_{n=0}^{N-1} |s_{m,n} - s_{l,n}|^2 = ||\mathbf{s_m} - \mathbf{s_l}||^2 = d_E^2(\mathbf{s_m}, \mathbf{s_l})$

- It is instructive to prove one of these relationships.
- The others are proved in a similar manner.
- The proof technique involves changing the order of summation and integration.
- If we think of the vector as the "frequency domain" and the signal as the "time domain" then the first property is the same as Parseval's Theorem.

Proof of 2

$$
\begin{aligned}
(s_m(t), s_l(t)) &= \int s_m(t) s_l^*(t) dt \\
&= \int \sum_{n=0}^{N-1} s_{m,n} \varphi_n(t) \sum_{p=0}^{N-1} s_{l,p}^* \varphi_p^*(t) dt \\
&= \sum_{n=0}^{N-1} s_{m,n} \sum_{p=0}^{N-1} s_{l,p}^* \int \varphi_n(t) \varphi_p^*(t) dt \\
&= \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} s_{m,n} s_{l,p}^* \delta_{n,p} \\
&= \sum_{n=0}^{N-1} s_{m,n} s_{l,n}^* = (\mathbf{s_m}, \mathbf{s_l})
\end{aligned}
$$

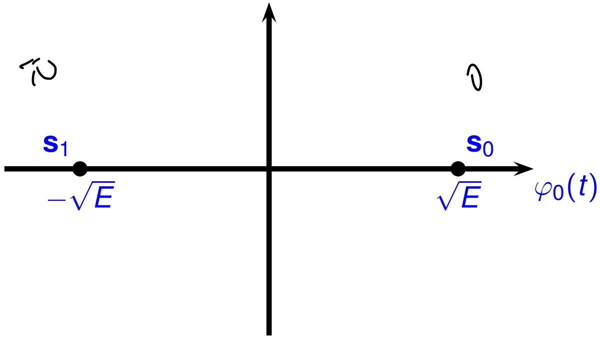
where $\delta_{n,p} = 1$ if $n = p$ and is zero otherwise. This is called the Kronecker delta function.

Example 1

- Two signals in one dimension: $N = 1, M = 2$

BPSK

$$\mathbf{s}_0 = \sqrt{E}(+1)$$
$$\mathbf{s}_1 = \sqrt{E}(-1)$$



Example 1

- These signal vectors with orthogonal set 1 but just using one of the two orthogonal waveforms generates the following signals

$$
\begin{aligned}
s_0(t) &= \sqrt{E}\varphi_0(t) \\
&= +\sqrt{2E/T}\cos(2\pi f_c t)p_T(t) \\
&= \sqrt{2P}\cos(2\pi f_c t)p_T(t) \\
s_1(t) &= -\sqrt{E}\varphi_0(t) \\
&= -\sqrt{2E/T}\cos(2\pi f_c t)p_T(t) \\
&= \sqrt{2P}\cos(2\pi f_c t + \pi)p_T(t)
\end{aligned}
$$

since E/T is power P and $-\cos(x) = \cos(x+\pi)$.

Example 1

- Minimum squared Euclidean distance is

$$d_E^2(\mathbf{s}_0, \mathbf{s}_1) = (\sqrt{E} - (-\sqrt{E}))^2 = 4E.$$

- The average energy per bit is $E_b = E$.
- The normalized minimum squared Euclidean distance is

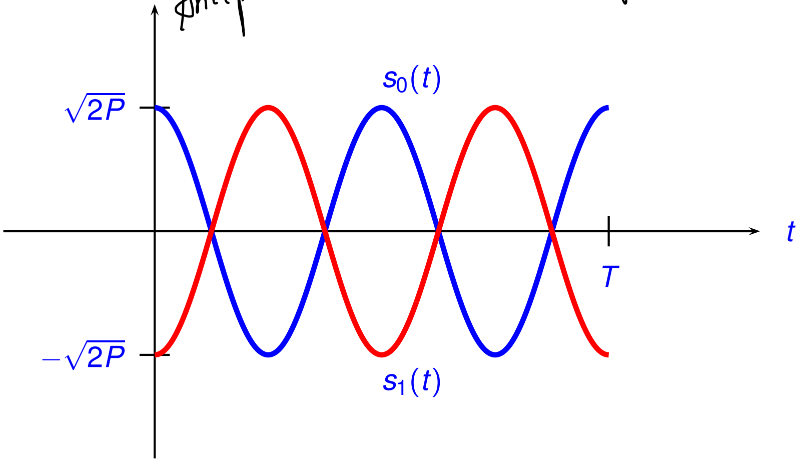
$$\frac{d_E^2(\mathbf{s}_0, \mathbf{s}_1)}{E_b} = \frac{4E}{E} = 4$$

- The rate is $r = 1$ bit/1 dimension.
- This signal set with orthonormal set 1 is called binary phase shift keying (BPSK).

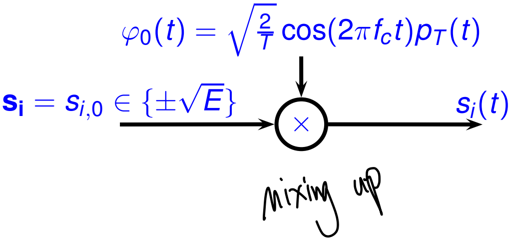
Example 1

Antipodal

$P T = E$

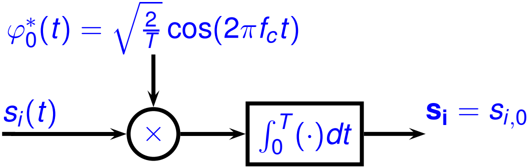


From a Vector to a Signal (Signal Composition, Transmitter): Example 1



This is called binary phase shift keying since the information (one bit) is sent via the phase of the signal.

From a Signal to a Vector (Signal Decomposition, Receiver): Example 1

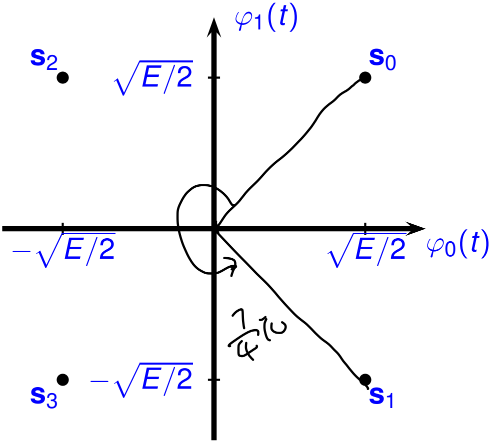


Example 2

QPSK

- Four signals in two dimensions: $N = 2, M = 4$, orthogonal waveform set 1.

$$
\begin{aligned}
\mathbf{s}_0 &= \sqrt{E/2}(+1,+1) \\
\mathbf{s}_1 &= \sqrt{E/2}(+1,-1) \\
\mathbf{s}_2 &= \sqrt{E/2}(-1,+1) \\
\mathbf{s}_3 &= \sqrt{E/2}(-1,-1)
\end{aligned}
$$



Example 2

- Minimum squared Euclidean distance is

$$d_E^2(\mathbf{s}_0, \mathbf{s}_1) = (\sqrt{E/2} - (-\sqrt{E/2}))^2 = 2E.$$

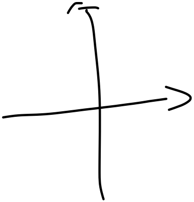
- Each signal has energy E.
- The average energy per bit is $E_b = E/2$.
- The normalized minimum squared Euclidean distance is

$$\frac{d_E^2(s_0, s_1)}{E_b} = \frac{4E}{E} = 4$$

- The rate is $r = 2$ bits/2 dimensions.
- This signal constellation with orthonormal set 1 is called quadrature phase shift keying (QPSK).

Example 2

- This constellation with orthogonal set 1 generates these four signals.

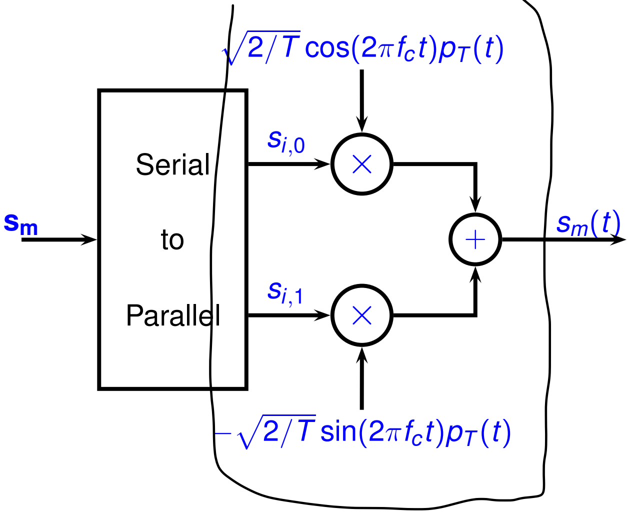


$$
\begin{aligned}
s_0(t) &= \sqrt{2P}\cos(2\pi f_c t + \pi/4)p_T(t) \\
s_1(t) &= \sqrt{2P}\cos(2\pi f_c t + 7\pi/4)p_T(t) \\
s_2(t) &= \sqrt{2P}\cos(2\pi f_c t + 3\pi/4)p_T(t) \\
s_3(t) &= \sqrt{2P}\cos(2\pi f_c t + 5\pi/4)p_T(t).
\end{aligned}
$$

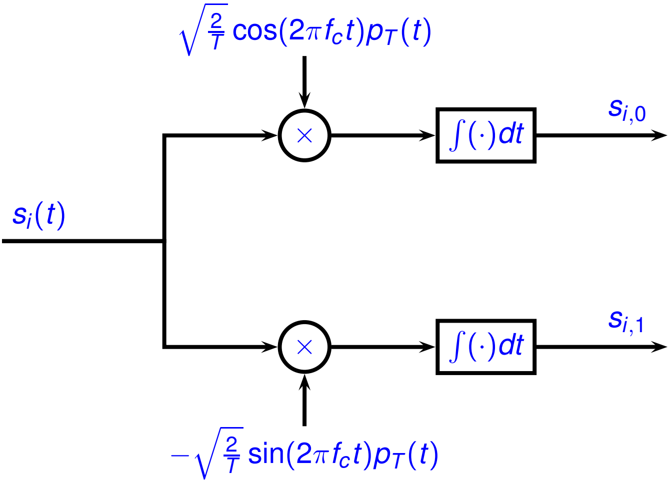
Note that because of the negative sign in $\varphi_1(t)$ the phase of the signal waveform is the same as the phase angle to horizontal of the signal vector in the two dimensional plane.

From a Vector to a Signal (Signal Composition, Transmitter): Example 2

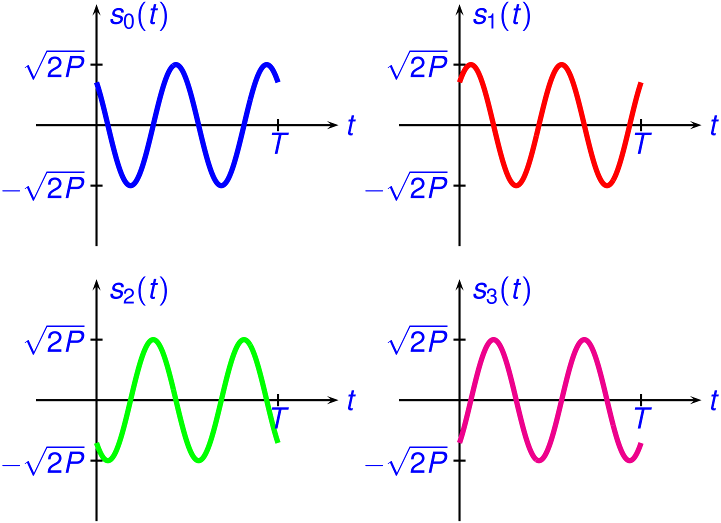
IQ modulater



From a Signal to a Vector (Signal Decomposition, Receiver): Example 2



Example 2


$s_0(t)$
$\sqrt{2P}$
$-\sqrt{2P}$
t
T
$s_1(t)$
$\sqrt{2P}$
$-\sqrt{2P}$
t
T
$s_2(t)$
$\sqrt{2P}$
$-\sqrt{2P}$
t
T
$s_3(t)$
$\sqrt{2P}$
$-\sqrt{2P}$
t
T

Example 3

- Complex version of orthonormal set 1 with complex signal vectors
- Example: $N = 1$, $M = 4$.
- $\varphi_0(t) = \sqrt{\frac{1}{T}} e^{j2\pi f_c t} p_T(t)$.
- Signal vectors (actually complex scalars)

$$\begin{aligned} s_{0,0} &= \sqrt{E/2}(+1 + j) \\ s_{1,0} &= \sqrt{E/2}(+1 - j) \\ s_{2,0} &= \sqrt{E/2}(-1 + j) \\ s_{3,0} &= \sqrt{E/2}(-1 - j). \end{aligned}$$

- Complex signals

$$s_m(t) = s_{m,0}\varphi_0(t), \quad m = 0, 1, 2, 3;$$

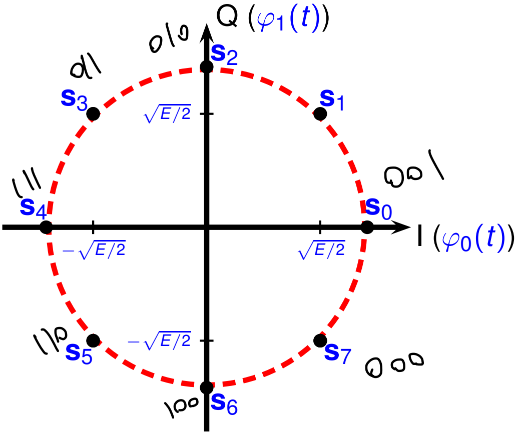
- Actual “real” signal is $u_m(t) = \Re\{\sqrt{2} s_m(t)\}$.

Example 4

8PSK

- Eight signals in two dimensions: $N = 2, M = 8$

$$
\begin{aligned}
s_0 &= (\sqrt{E}, 0) \\
s_1 &= (\sqrt{E/2}, \sqrt{E/2}) \\
s_2 &= (0, \sqrt{E}) \\
s_3 &= (-\sqrt{E/2}, \sqrt{E/2}) \\
s_4 &= (-\sqrt{E}, 0) \\
s_5 &= (-\sqrt{E/2}, -\sqrt{E/2}) \\
s_6 &= (0, -\sqrt{E}) \\
s_7 &= (+\sqrt{E/2}, -\sqrt{E/2})
\end{aligned}
$$



Example 4

- Minimum squared Euclidean distance is

$$\begin{aligned}\min_{m\neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l) &= (\sqrt{E} - \sqrt{E/2})^2 + (0 - \sqrt{E/2})^2 \\ &= [(1 - \sqrt{1/2})^2 + (1/\sqrt{2})^2]E \\ &= (2 - \sqrt{2})E = .5857E\end{aligned}$$

- The average energy per bit is $E_b = E/3$.
- The normalized minimum squared Euclidean distance is

$$\frac{d_E^2(s_0, s_1)}{E_b} = \frac{.5857E}{E/3} = 1.757$$

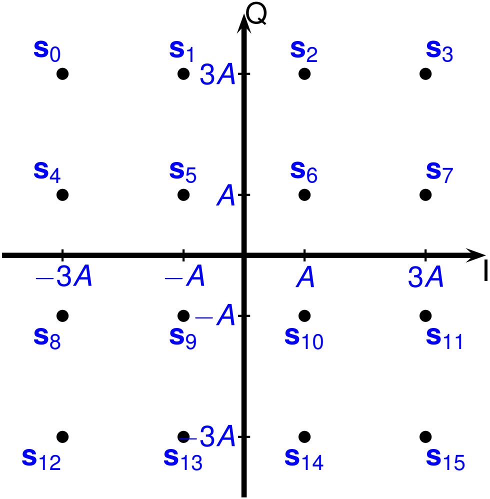
- The rate is $r = 3$ bits$/2$ dimensions $=1.5$ bits/dimension.
- This signal constellation with orthonormal set 1 is called 8-ary phase shift keying (8PSK).

Example 5

- Sixteen signals in two dimensions: $N = 2, M = 16$

$\mathbf{s}_0 = A(-3, +3)$	$\mathbf{s}_1 = A(-1, +3)$	$\mathbf{s}_2 = A(+1, +3)$	$\mathbf{s}_3 = A(+3, +3)$
$\mathbf{s}_4 = A(-3, +1)$	$\mathbf{s}_5 = A(-1, +1)$	$\mathbf{s}_6 = A(+1, +1)$	$\mathbf{s}_7 = A(+3, +1)$
$\mathbf{s}_8 = A(-3, -1)$	$\mathbf{s}_9 = A(-1, -1)$	$\mathbf{s}_{10} = A(+1, -1)$	$\mathbf{s}_{11} = A(+3, -1)$
$\mathbf{s}_{12} = A(-3, -3)$	$\mathbf{s}_{13} = A(-1, -3)$	$\mathbf{s}_{14} = A(+1, -3)$	$\mathbf{s}_{15} = A(+3, -3)$

Example 5


Q
I
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_3$
$\mathbf{s}_4$
$\mathbf{s}_5$
$\mathbf{s}_6$
$\mathbf{s}_7$
$\mathbf{s}_8$
$\mathbf{s}_9$
$\mathbf{s}_{10}$
$\mathbf{s}_{11}$
$\mathbf{s}_{12}$
$\mathbf{s}_{13}$
$\mathbf{s}_{14}$
$\mathbf{s}_{15}$
$3A$
A
$-A$
$-3A$
$-3A$
$-A$
A
$3A$

Example 5

- Minimum squared Euclidean distance is

$$\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l) = (3A - A)^2 + (0)^2 = 4A^2$$

- The average energy of a signal is $E = (2A^2 + 10A^2 + 10A^2 + 18A^2)/4 = 10A^2$.
- The average energy per bit is $E_b = E/4 = 10A^2/4 = 5A^2/2$.
- The normalized minimum squared Euclidean distance is

$$\frac{\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{4A^2}{5A^2/2} = 8/5 = 1.6$$

- The rate is $r = 4$ bits$/2$ dimensions $= 2$ bits/dimension.
- This signal constellation with orthonormal set 1 is called 16-ary quadrature amplitude modulation (16QAM).

Peak-to-Average Power Ratio

- Peak-to-average power ratio (PAPR) of constellation or set of vectors

$$\Gamma_v = \frac{\max_m |\mathbf{s_m}|^2}{\sum_{m=0}^{M-1} |\mathbf{s_m}|^2 / M}$$

- A small peak-to-average power ratio allows energy efficient amplification of a signal. That is, the efficiency of converting DC or battery energy to RF or radiated energy is better for a signal with a low peak-to-average power ratio (PAPR).
- Peak-to-average power ratio of a set of signal waveforms is

$$\Gamma_w = \frac{\max_{m,t} |s_m(t)|^2}{\sum_{m=0}^{M-1} |P_m| / M}$$

where $P_m = \int |s_m(t)|^2 dt / T$ is the power of signal $s_m(t)$ of duration T (as in the examples above).

Summary

More R, more bandwidth efficiency

More $\frac{d_E^2}{\bar{E}}$, more energy efficiency

Modulation	$\frac{\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b}$	rate r	PAPR
BPSK	4	1 (0.5)	0
QPSK	4	1	0
8PSK	1.7574	1.75	0
16 QAM	1.6	2	2.55
64 QAM	.5713	3	3.68
256 QAM	.1882	4	4.23

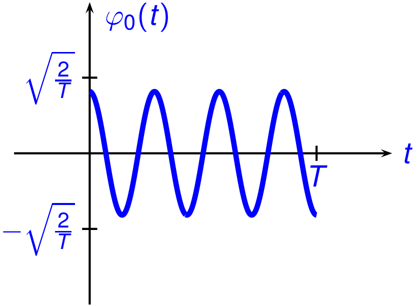
- For BPSK the rate r is more realistically 1/2 because there is an extra dimension that can be used without increasing the bandwidth.

Orthonormal waveform set 2

- For the next few examples we will consider the following orthonormal signals (orthonormal signal set 2). Here $2\pi f_0 T$, $2\pi f_1 T$ and $2\pi(f_1 - f_0)T$ are integers.

$$\begin{aligned} \varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t) \\ \varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t) \end{aligned}$$

Orthonormal waveform set 2



- Here $\varphi_0(t)$ has frequency $7/2T$ while φ_1 has frequency $4T$.

Example 6: Signal vectors

- Four signals in two dimensions ($M = 4, N = 2$).

$$\begin{aligned} \mathbf{s}_0 &= \sqrt{E/2}(+1,+1) \\ \mathbf{s}_1 &= \sqrt{E/2}(+1,-1) \\ \mathbf{s}_2 &= \sqrt{E/2}(-1,+1) \\ \mathbf{s}_3 &= \sqrt{E/2}(-1,-1) \end{aligned}$$

- Same normalized squared Euclidean distance, same rate.

Example 6: Signals

OFDM orthogonal Frequency division

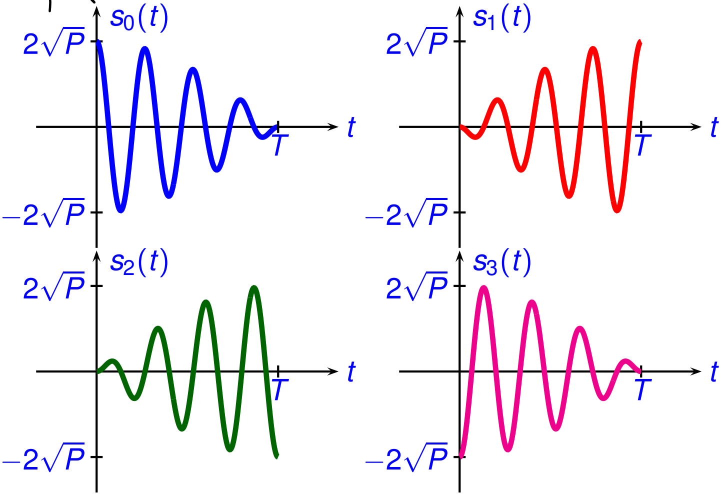
$$
\begin{aligned}
s_0(t) &= +\sqrt{P}\cos(2\pi f_0 t)p_T(t) + \sqrt{P}\cos(2\pi f_1 t)p_T(t) \\
s_1(t) &= +\sqrt{P}\cos(2\pi f_0 t)p_T(t) - \sqrt{P}\cos(2\pi f_1 t)p_T(t) \\
s_2(t) &= -\sqrt{P}\cos(2\pi f_0 t)p_T(t) + \sqrt{P}\cos(2\pi f_1 t)p_T(t) \\
s_3(t) &= -\sqrt{P}\cos(2\pi f_0 t)p_T(t) - \sqrt{P}\cos(2\pi f_1 t)p_T(t).
\end{aligned}
$$

Example 6: Signal vectors

- Four signals in two dimensions ($M = 4, N = 2$).
- Signals have non-constant envelope (bad for energy efficient operation of amplifier).
- This is a simple example of “orthogonal frequency division multiplexing” (OFDM).
- Each of two frequencies is modulated by a bit.

Example 6: Signals

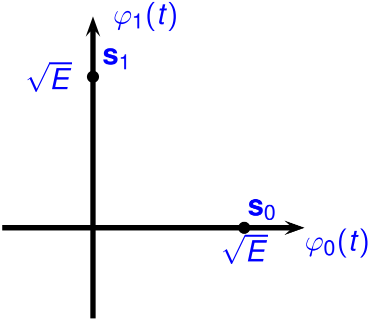
$f_0 T = 3.5$, $f_1 T = 4$



Example 7

- Orthonormal set 2 (frequency orthogonal).
- Two signals in two dimensions ($M = 2, N = 2$).

$$
\begin{aligned}
\mathbf{s}_0 &= \sqrt{E}(1, 0) \\
\mathbf{s}_1 &= \sqrt{E}(0, 1).
\end{aligned}
$$



Example 7

- Minimum squared Euclidean distance is

$$\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l) = (\sqrt{E})^2 + (\sqrt{E})^2 = 2E$$

- The average energy per bit is $E_b = E$.
- The normalized minimum squared Euclidean distance is

$$\frac{\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{2E}{E} = 2$$ not good

- The rate is $r = 1$ bits$/2$ dimensions $=0.5$ bits/dimension.

Example 7

$$
\begin{aligned}
s_0(t) &= \sqrt{E}\varphi_0(t) + 0\varphi_1(t) \\
&= \sqrt{2P}\cos(2\pi f_0 t)p_T(t) \\
s_1(t) &= 0\varphi_0(t) + \sqrt{E}\varphi_1(t) \\
&= \sqrt{2P}\cos(2\pi f_1 t)p_T(t).
\end{aligned}
$$

- This modulation (signal vectors + orthonormal waveforms) is called binary frequency shift keying.
- This modulation can be demodulated without knowing the phase of the signal (called noncoherent demodulation).

Example 8

- Extend the set of orthonormal waveforms in set 2 to have N orthogonal signals.

$$
\begin{aligned}
\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t) \\
\varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t) \\
\varphi_2(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)p_T(t) \\
\cdots & \quad \cdots \\
\varphi_{M-1}(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_{M-1} t)p_T(t)
\end{aligned}
$$

Example 8

- Extend signal set in Example 7 to N dimensions.
- M signals in M dimensions ($M = N$).

$$
\begin{aligned}
\mathbf{s}_0 &= \sqrt{E}(1,0,0,0,\cdots,0) \\
\mathbf{s}_1 &= \sqrt{E}(0,1,0,0,\cdots,0) \\
\mathbf{s}_2 &= \sqrt{E}(0,0,1,0,\cdots,0) \\
\ldots & \qquad \ldots \\
\ldots & \qquad \ldots \\
\ldots & \qquad \ldots \\
\mathbf{s}_{M-1} &= \sqrt{E}\underbrace{(0,0,0,0,\cdots,1)}_{N}
\end{aligned}
$$

Example 8

- Minimum squared Euclidean distance is

$$\min_{m\neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l) = (\sqrt{E})^2 + (\sqrt{E})^2 = 2E$$

- The average energy per bit is $E_b = E/\log_2(M)$.
- The normalized minimum squared Euclidean distance is

$$\frac{\min_{m\neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{2E}{E/\log_2(M)} = 2\log_2(M)$$

- The rate is $r = \log_2(M)$ bits$/M$ dimensions $=\frac{\log_2(M)}{M}$ bits/dimension.

$M \to \infty$

$\frac{\log_2(M)}{M} \to 0$

- Normalized distance is growing with M but rate is going to zero as M gets larger.

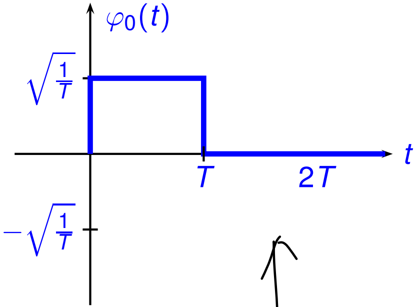
Example 9

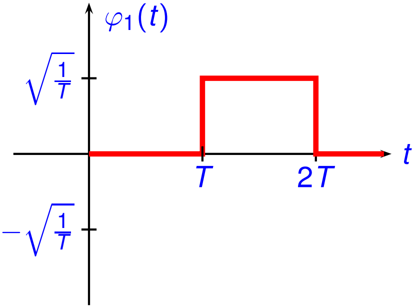
- Use orthogonal set 3.
- These waveforms are time orthogonal.

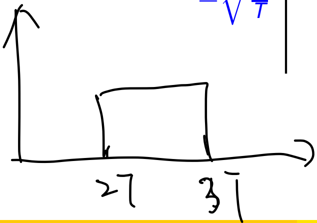
$$\begin{aligned} \varphi_0(t) &= \sqrt{\frac{1}{T}} p_T(t) \\ \varphi_1(t) &= \sqrt{\frac{1}{T}} p_T(t-T) = \varphi_0(t-T) \end{aligned}$$

- This is a simple example of orthonormal signals that are time shifts. That is, $\varphi_1(t) = \varphi_0(t-T)$.

Example 9


$\varphi_0(t)$
$\sqrt{\frac{1}{T}}$
$-\sqrt{\frac{1}{T}}$
t
T
$2T$


$\varphi_1(t)$
$\sqrt{\frac{1}{T}}$
$-\sqrt{\frac{1}{T}}$
t
T
$2T$


2T
3T

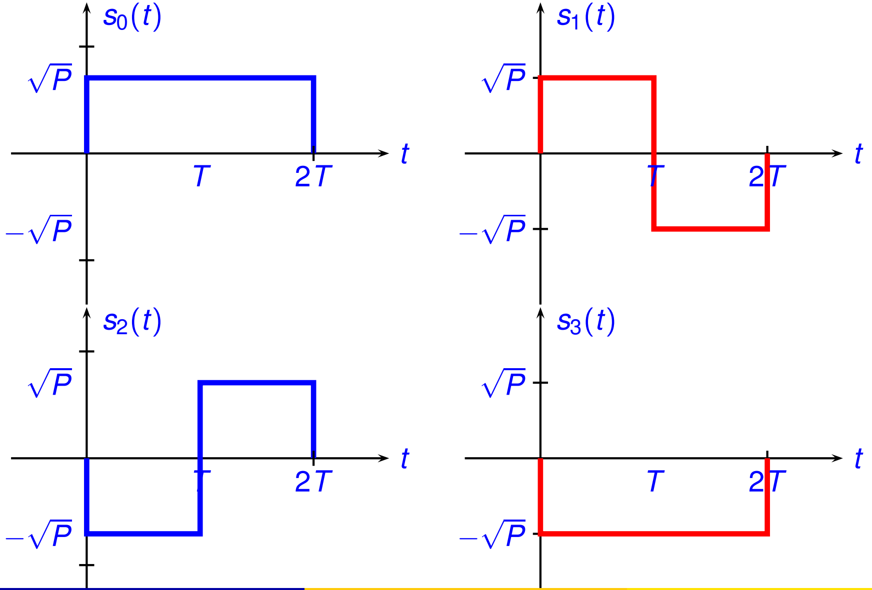
Example 9: Signals

QPSK

$$
\begin{aligned}
\mathbf{s}_0 &= \sqrt{E/2}(+1,+1) \\
\mathbf{s}_1 &= \sqrt{E/2}(+1,-1) \\
\mathbf{s}_2 &= \sqrt{E/2}(-1,+1) \\
\mathbf{s}_3 &= \sqrt{E/2}(-1,-1)
\end{aligned}
$$

- Same constellation as Example 2.
- The minimum squared Euclidean distance is sample as Example 2 ($2E$)
- The energy per bit E_b is sample as Example 2 ($E/2$)
- The normalized minimum squared Euclidean distance is sample as Example 2 (4)
- The rate is same as Example 2.

Example 9: Signals

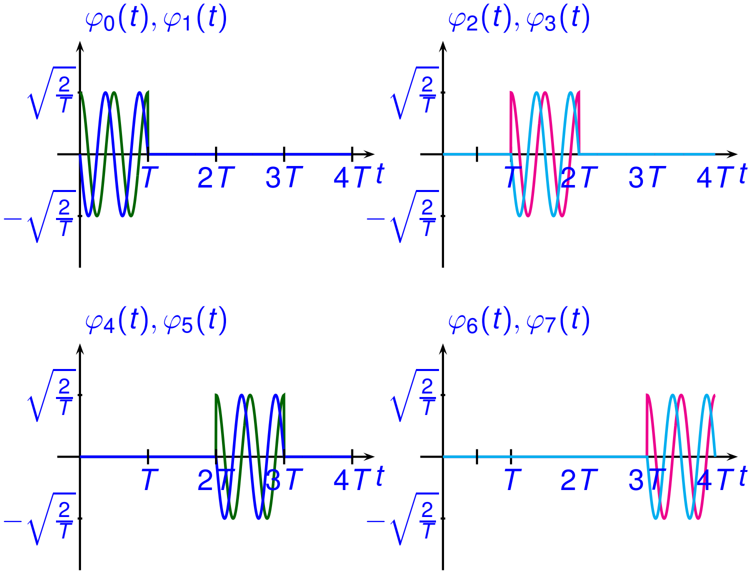


Example 10

- We can combine orthogonality over phase and time.
- Consider the $N = 8$ orthonormal waveforms below. These signals are orthogonal over the time interval $I = [0,\ 4T]$.

$$\varphi_0(t) = \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t), \qquad \varphi_4(t) = \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t-2T)$$
$$\varphi_1(t) = -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t), \qquad \varphi_5(t) = -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t-2T)$$
$$\varphi_2(t) = \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t-T), \qquad \varphi_6(t) = \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t-3T)$$
$$\varphi_3(t) = -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t-T), \qquad \varphi_7(t) = -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t-3T).$$

Example 10


$\varphi_0(t), \varphi_1(t)$
$\varphi_2(t), \varphi_3(t)$
$\varphi_4(t), \varphi_5(t)$
$\varphi_6(t), \varphi_7(t)$
$\sqrt{\frac{2}{T}}$
$-\sqrt{\frac{2}{T}}$
T
$2T$
$3T$
$4T$
t

Example 10: Signal Vectors

- $M = 16$ signals in $N = 8$ dimensions.

$\mathbf{s}_0 = \sqrt{\frac{E}{8}}(+1,+1,+1,+1,+1,+1,+1,+1)$	$\mathbf{s}_1 = \sqrt{\frac{E}{8}}(+1,+1,+1,-1,+1,-1,-1,-1)$
$\mathbf{s}_2 = \sqrt{\frac{E}{8}}(+1,+1,-1,+1,-1,-1,-1,+1)$	$\mathbf{s}_3 = \sqrt{\frac{E}{8}}(+1,+1,-1,-1,-1,+1,+1,-1)$
$\mathbf{s}_4 = \sqrt{\frac{E}{8}}(+1,-1,+1,+1,-1,-1,+1,-1)$	$\mathbf{s}_5 = \sqrt{\frac{E}{8}}(+1,-1,+1,-1,-1,+1,-1,+1)$
$\mathbf{s}_6 = \sqrt{\frac{E}{8}}(+1,-1,-1,+1,+1,+1,-1,-1)$	$\mathbf{s}_7 = \sqrt{\frac{E}{8}}(+1,-1,-1,-1,+1,-1,+1,+1)$
$\mathbf{s}_8 = \sqrt{\frac{E}{8}}(-1,+1,+1,+1,-1,+1,-1,-1)$	$\mathbf{s}_9 = \sqrt{\frac{E}{8}}(-1,+1,+1,-1,-1,-1,+1,+1)$
$\mathbf{s}_{10} = \sqrt{\frac{E}{8}}(-1,+1,-1,+1,+1,-1,+1,-1)$	$\mathbf{s}_{11} = \sqrt{\frac{E}{8}}(-1,+1,-1,-1,+1,+1,-1,+1)$
$\mathbf{s}_{12} = \sqrt{\frac{E}{8}}(-1,-1,+1,+1,+1,-1,-1,+1)$	$\mathbf{s}_{13} = \sqrt{\frac{E}{8}}(-1,-1,+1,-1,+1,+1,+1,-1)$
$\mathbf{s}_{14} = \sqrt{\frac{E}{8}}(-1,-1,-1,+1,-1,+1,+1,+1)$	$\mathbf{s}_{15} = \sqrt{\frac{E}{8}}(-1,-1,-1,-1,-1,-1,-1,-1)$

Example 10

- There are 14 vectors ($\mathbf{s}_1, \cdots, \mathbf{s}_{14}$) that differ from $\mathbf{s}_0$ in 4 places and one vector ($\mathbf{s}_{15}$) that differs from $\mathbf{s}_0$ in 8 places.
- In each component where two vectors differ the squared Euclidean distance is $4E/8 = E/2$.
- The distance structure is the same if we chose any other signal vector rather than $\mathbf{s}_0$. (Geometricaly uniform)
- Minimum squared Euclidean distance is

$$\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l) = 4(E/2) = 2E$$

$E/4$

- The average energy per bit is $E_b = E/\log_2(M) = E/4$.

$\log_2 16 = 4$

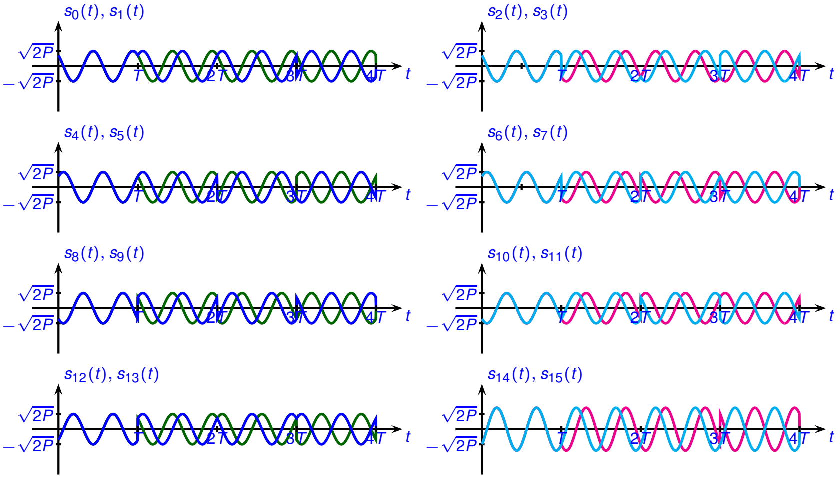
Example 10

- The normalized minimum squared Euclidean distance is

$$\frac{\min_{m\neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{2E}{E/4} = 8$$

- The rate is $r = 4$ bits$/8$ dimensions $=1/2$ bits/dimension.

Example 10: Signals



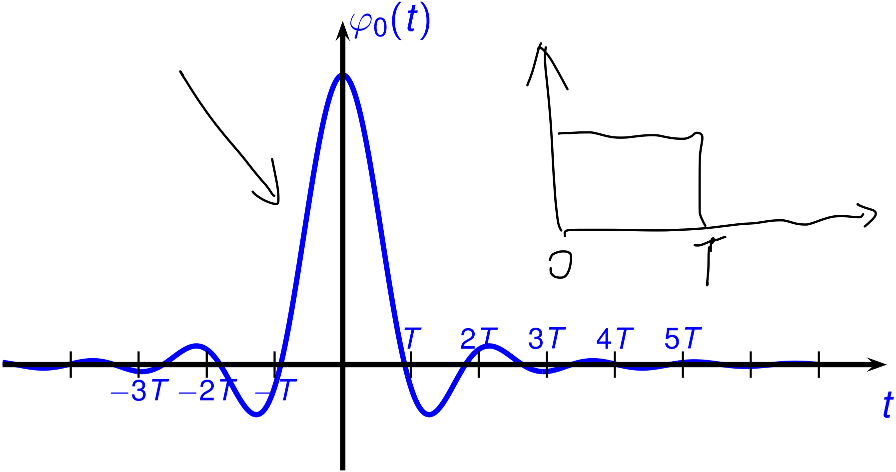
Example 11

- Orthonormal waveform set 4: Based on square-root raised cosine pulses.
- These are time-shifted orthonormal. $\int_I \varphi_0(t)\varphi_0(t-nT)dt = \delta_{n,0}$.
- Interval is $I = [-\infty, \infty]$.

$$\varphi_0(t) = \frac{1}{\sqrt{T}}\left(\frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T\cos(\pi(1+\alpha)t/T)}{[1-(4\alpha t/T)^2]\pi t/T}\right),$$
$$\varphi_n(t) = \varphi_0(t-nT),\ n = 1, 2, ..., N-1$$

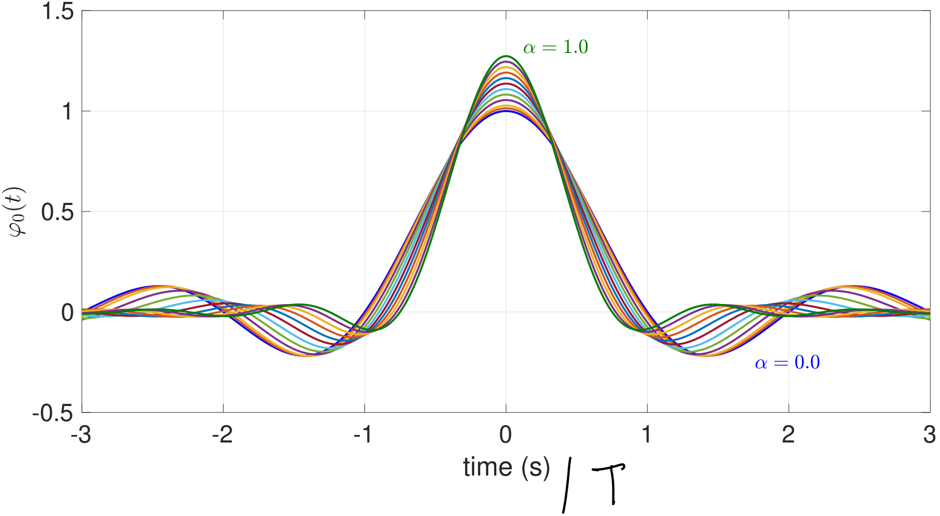
time-shifted

Example 11

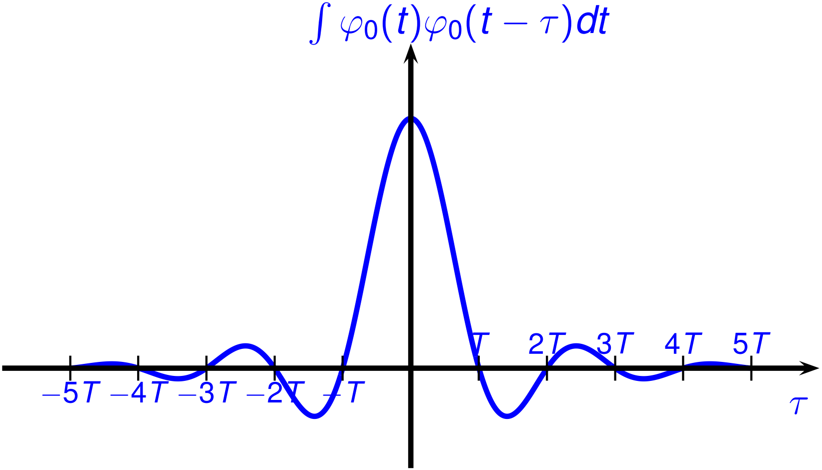


This pulse is continuous, with continuous derivative. Has better bandwidth occupancy than rectangular pulse.

Example 11: $\alpha = 0.1, 0.2, \ldots, 0.9, 1.0$

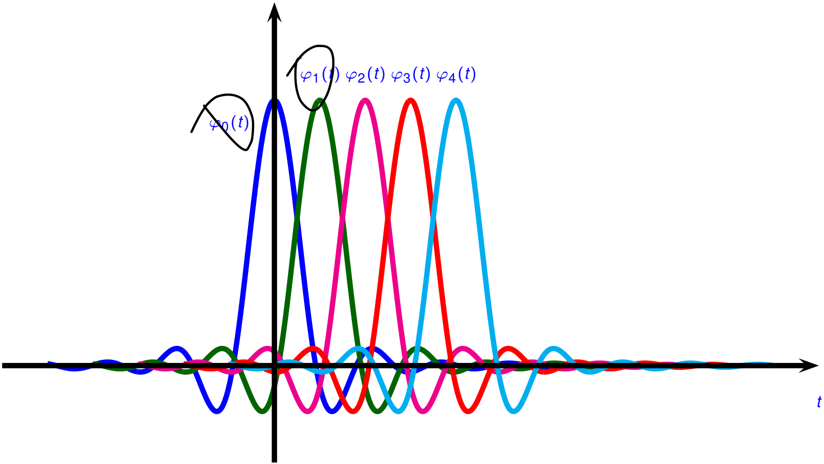


Example 11



This integral (autocorrelation function) is zero at offsets (τ) that are nonzero multiples of T ($\tau = nT,\ n \neq 0$).

Example 11



Example 11

$N=3 \quad M=8$

- Constellation

$$
\begin{aligned}
s_0 &= \sqrt{E/3}(+1,+1,+1),\\
s_1 &= \sqrt{E/3}(+1,+1,-1),\\
s_2 &= \sqrt{E/3}(+1,-1,+1),\\
s_3 &= \sqrt{E/3}(+1,-1,-1),\\
s_4 &= \sqrt{E/3}(-1,+1,+1),\\
s_5 &= \sqrt{E/3}(-1,+1,-1),\\
s_6 &= \sqrt{E/3}(-1,-1,+1),\\
s_7 &= \sqrt{E/3}(-1,-1,-1).
\end{aligned}
$$

Example 11

- Minimum squared Euclidean distance is

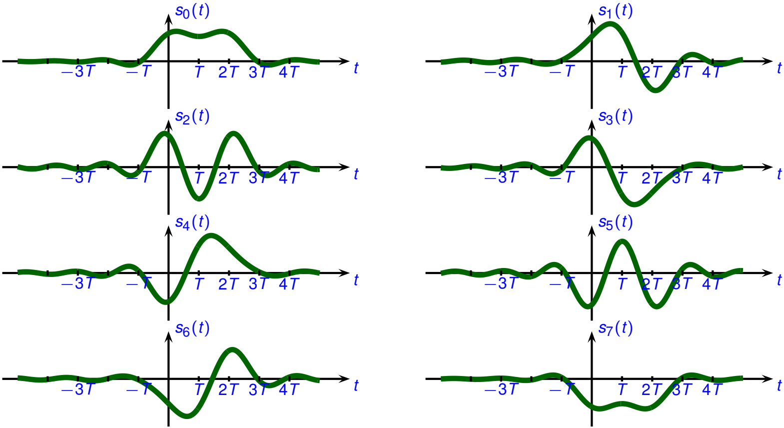
$$\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l) = (2\sqrt{E/3})^2 = 4E/3$$

- The average energy per bit is $E_b = E/3$.
- The normalized minimum squared Euclidean distance is

$$\frac{\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{4E/3}{E/3} = 4$$

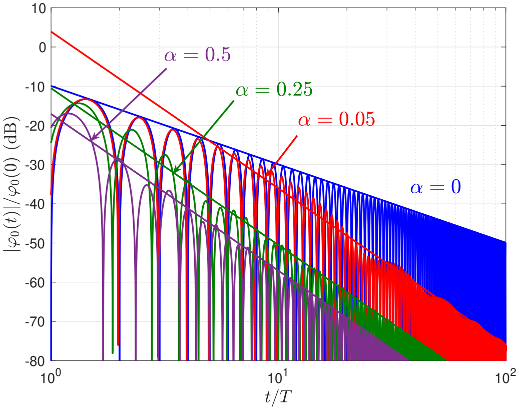
- The rate is $r = 3$ bits$/3$ dimensions $=1$ bits/dimension. This is the same as Example 1. We will see later has this set of orthonormal waveforms has advantage of having a smaller bandwidth but has a larger peak-to-average power ratio.

Example 11



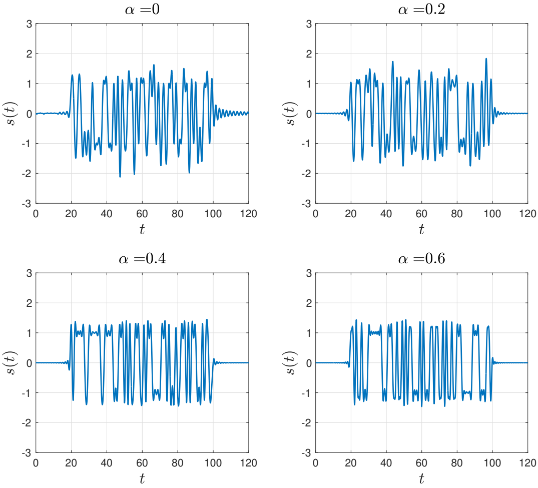
Example 11: Asymptotic Decay

- These signals are theoretically infinity in duration.
- In practice we need to truncate to a certain time limit (e.g. when the signal is 40dB below the peak).

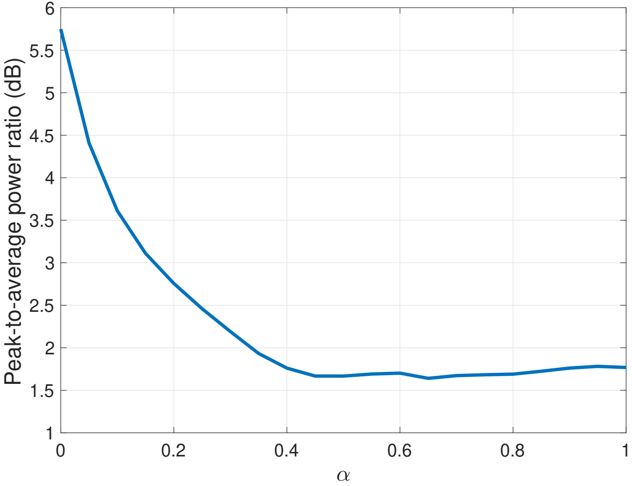


Largar α mean larger bandwidth.

Example 11: $\alpha = 0.0, 0.2, 0.4, 0.6$



Example 11: Peak-to-average power ratio



Example 11: Asymptotic Decay

- Smaller α requires longer time for the signal decay to a level below a certain level.
- Smaller α corresponds to a narrower bandwidth (next lecture).
- Smaller α has a smaller peak-to-average power ratio (larger peak-to-average power ratio makes the energy efficient amplification more difficult).