EECS 455: Solutions to Problem Set 2

1. (a) Consider two signals of duration T seconds.

$$\phi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)$$
$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t).$$

Determine the minimum separation between f_0 and f_1 so that $\phi_0(t)$ and $\phi_1(t)$ are orthogonal.

Solution:

$$\begin{aligned}
(\phi_0(t),\phi_1(t)) &= -\int_0^T \frac{2}{T}\cos(2\pi f_0 t)\cos(2\pi f_1 t)dt \\
&= -\frac{1}{T}\int_0^T \cos(2\pi(f_1-f_0)t)+\cos(2\pi(f_0+f_1)t)dt \\
&= -\frac{1}{T}\left[\frac{\sin(2\pi(f_1-f_0)t)}{2\pi(f_1-f_0)}+\frac{\sin(2\pi(f_0+f_1)t)}{2\pi(f_0+f_1)}\right]_0^T \\
&= -\left[\frac{\sin(2\pi(f_1-f_0)T)}{2\pi(f_1-f_0)T}+\frac{\sin(2\pi(f_0+f_1)T)}{2\pi(f_0+f_1)T}\right]
\end{aligned}$$

For this to be 0 it is required that $2\pi(f_1-f_0)T = n\pi$ for some integer n and $2\pi(f_0+f_1)T = m\pi$. So if $(f_1-f_0)T = n/2$ for some integer n then the signals will be orthogonal. The smallest is $n=1$ in which case $f_1-f_0 = 1/(2T)$. Of course we also need $(f_0+f_1) = m/2$ for some integer. But if $f_0 = k/(2T)$ and $f_1 = (k+1)/(2T)$ for an integer k then both conditions will be satisfied and the minimum separation is $1/(2T)$.

(b) Consider two signals of duration T seconds.

$$\phi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)$$
$$\phi_1(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_1 t)p_T(t).$$

Determine the minimum (non-zero) separation between f_0 and f_1 so that $\phi_0(t)$ and $\phi_1(t)$ are orthogonal.

Solution:

$$\begin{aligned}
(\phi_0(t),\phi_1(t)) &= \int_0^T \frac{2}{T}\cos(2\pi f_0 t)\sin(2\pi f_1 t)dt \\
&= \frac{1}{T}\int_0^T \sin(2\pi(f_1+f_0)t)-\sin(2\pi(f_0-f_1)t)dt \\
&= -\frac{1}{T}\left[\frac{\cos(2\pi(f_0+f_1)t)}{2\pi(f_0+f_1)}-\frac{\cos(2\pi(f_0-f_1)t)}{2\pi(f_0-f_1)}\right]_0^T \\
&= \left[\frac{1-\cos(2\pi(f_0+f_1)T)}{2\pi(f_0+f_1)T}-\frac{1-\cos(2\pi(f_0-f_1)T)}{2\pi(f_0-f_1)T}\right]
\end{aligned}$$

The separation required for orthogonality is $2\pi(f_1 - f_0)T = 2n\pi$ for some integer n. The minimum separation is $f_1 - f_0 = 1/T$.

(c) Consider two sets of orthogonal signals of duration T seconds. Signal set one consists of signals $\phi_0(t)$ and $\phi_1(t)$.

$$\phi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)$$
$$\phi_1(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_0 t)p_T(t).$$

Signal set two consists of signals $\psi_0(t)$ and $\psi_1(t)$.

$$\psi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t)$$
$$\psi_1(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_1 t)p_T(t).$$

Determine the minimum separation between f_0 and f_1 so that either signal in signal set 1 is orthogonal to any signal in signal set 2.

Solution: From the previous parts the minimum separation is $1/T$.

(d) Define the bandwidth of a set of orthonormal signals as follows. Consider two different signal sets with different frequencies (like part b). Define the bandwidth to be the minimum separation in frequency between signals sets so that any signal from the first signal set is orthogonal to any signal in the second signal set. Determine the bandwidth as a function of T for the signal sets defined in part b.

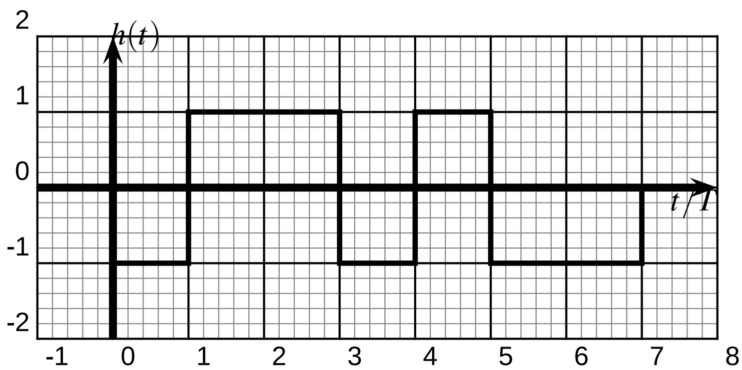
Solution: The bandwidth is $1/T$.

(e) From part (a) and (c) compute the time-bandwidth product.

Signals per frequency	WT
One orthogonal signal per frequency (part a)	1/2
Two orthogonal signals per frequency (part c)	1

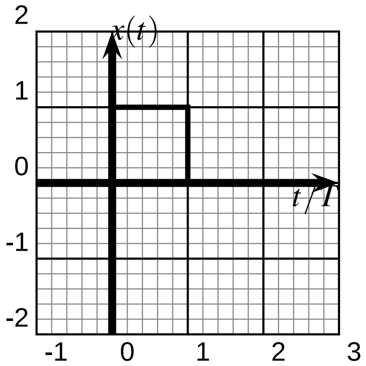
2. A filter has impulse response $h(t)$ shown in the figure below.

$$h(t) = -p_T(t) + p_T(t-T) + p_T(t-2T) - p_T(t-3T) + p_T(t-4T) - p_T(t-5T) - p_T(t-6T)$$



Note that each pulse lasts T seconds since the plot has a scale of t/T.

(a) The input $x(t)$ to the filter is a single rectangular pulse of duration T.



Find the output of the filter. Plot this from time $-T$ to time $9T$.

Solution:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau
\end{aligned}
$$

Note that $x(t-\tau) = 1$ for $0 \leq t-\tau \leq T$ and is zero elsewhere. Equivalently $x(t-\tau) = 1$ for $t-T < \tau < t$. So

$$
y(t) = \int_{t-T}^{t} h(\tau)d\tau.
$$

This implies that the output at time t is just the integral of $h(\tau)$ for the last T seconds (i.e. from $t-T$ to t). If $h(t)$ was a single pulse the output would be a triangular pulse. Since

$$
h(t) = -p_T(t) + p_1(t-1) + p_1(t-2) - p_T(t-3T) + p_T(t-4T) - p_T(t-5T) - p_T(t-6T)
$$

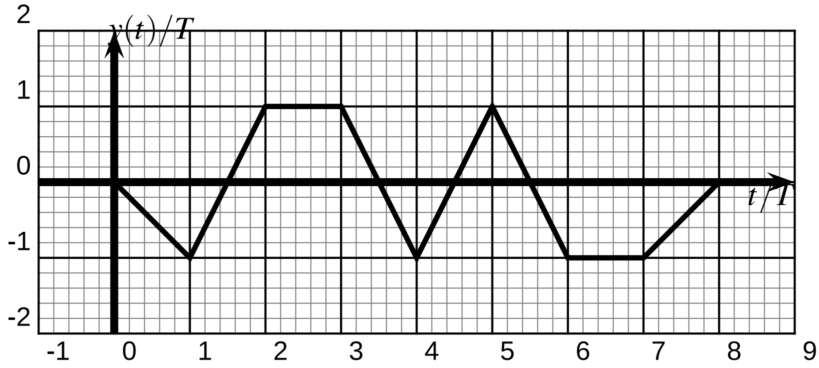
the output is the sum of delayed versions of triangular pulses. Let $\Lambda_T(t)$ be a triangular pulse which achieves peak of 1 at time T. Then

$$
\int_{t-T}^{t} p_T(\tau)d\tau = T\Lambda_T(t)
$$

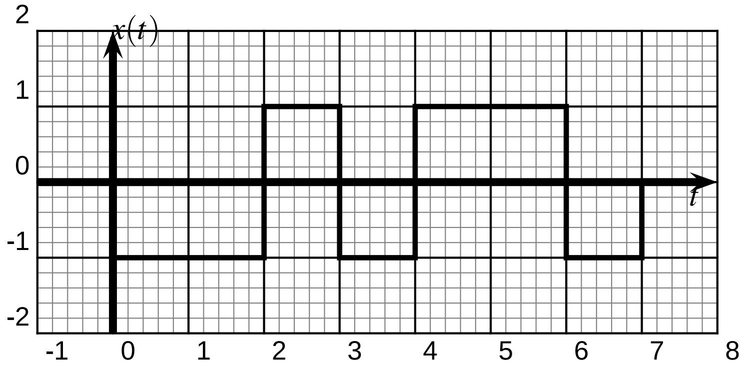
Thus

$$
\begin{aligned}
y(t) &= \int_{t-T}^{t} h(\tau)d\tau \\
&= \int_{t-T}^{t} [-p_T(t) + p_T(t-T) + p_T(t-2T) - p_T(t-3T) + p_T(t-4T) - p_T(t-5T) - p_T(t-6T)]d\tau \\
&= T[-\Lambda(t) + \Lambda(t-T) + \Lambda(t-2T) - \Lambda(t-3T) + \Lambda(t-4T) - \Lambda(t-5T) - \Lambda(t-6T)]
\end{aligned}
$$

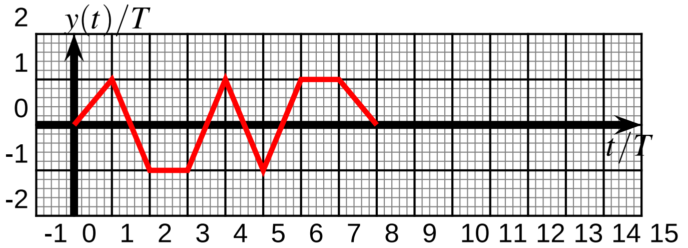
This is plotted below.



(b) Using the result above and superposition (linearity principle) find the output due to a sequence of pulses shown below. Plot the output from time -1 to time 15.

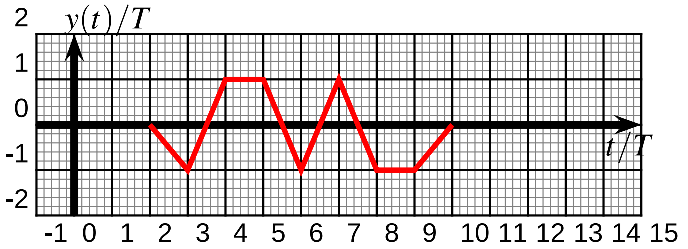


Solution: Output due to $-p_T(t)$.

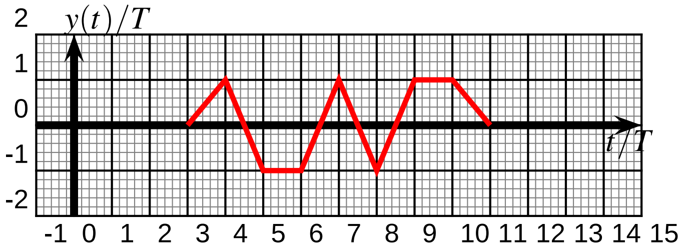


Output due to $-p_T(t-T)$.

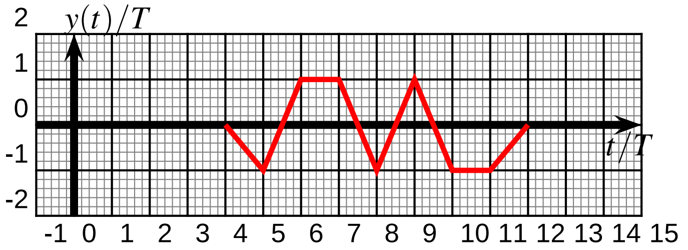
Output due to $p_T(t-2T)$.



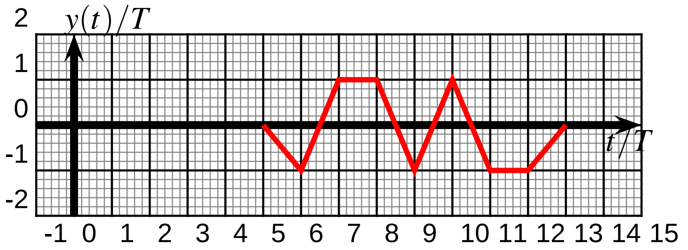
Output due to $-p_T(t-3T)$.



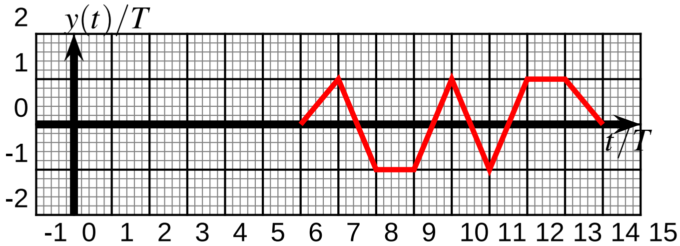
Output due to $p_T(t-4T)$.



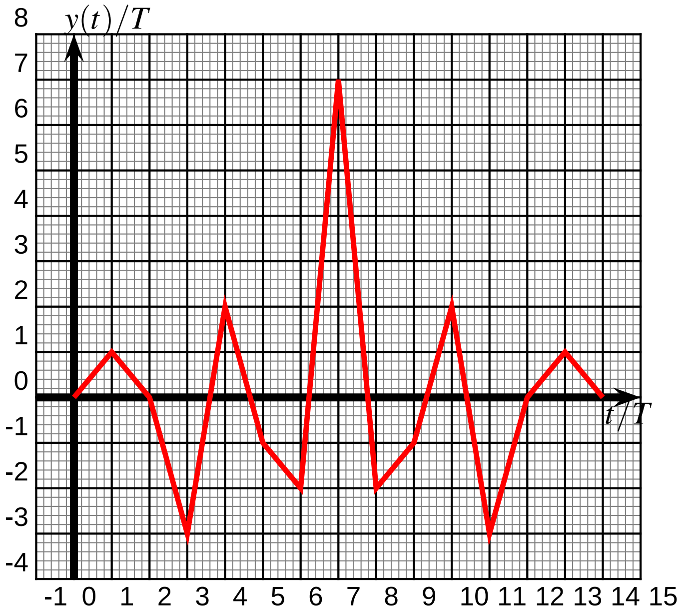
Output due to $p_T(t-5T)$.



Output due to $-p_T(t-6T)$.



Output due to $x(t)$ (add up the individual outputs).



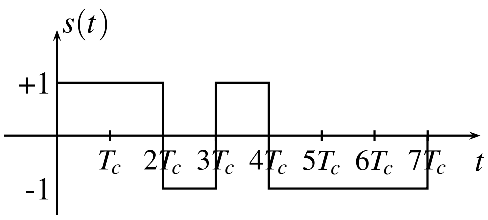
3. (a) Find the filter output when the input is a sequence of four pulses each of duration $T_c = T/7$ as shown below

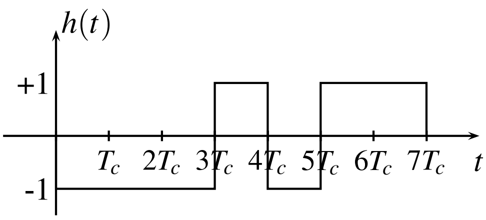
$$s(t) = p_{T_c}(t) + p_{T_c}(t-T_c) - p_{T_c}(t-2T_c) + p_{T_c}(t-3T_c) - p_{T_c}(t-4T_c) - p_{T_c}(t-5T_c) - p_{T_c}(t-6T_c)$$

and the filter is given by.

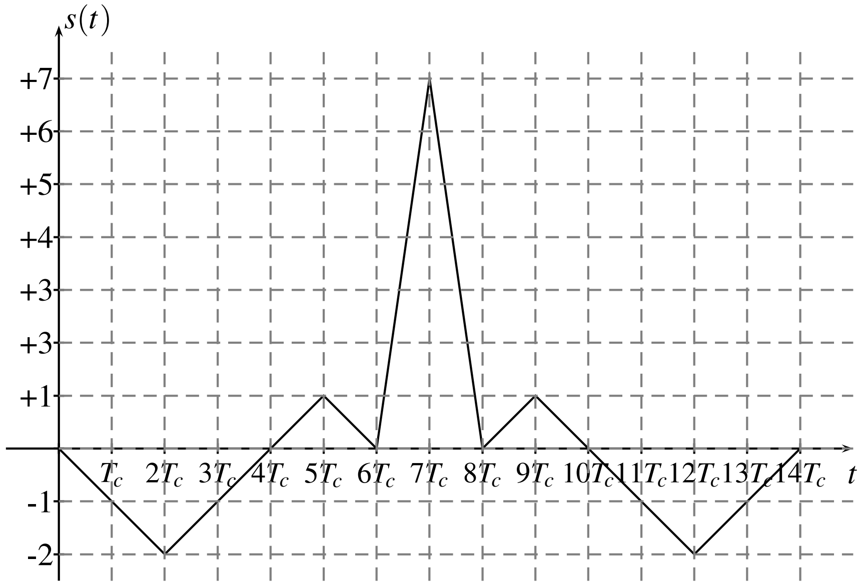
$$h(t) = -p_{T_c}(t) - p_{T_c}(t-T_c) - p_{T_c}(t-2T_c) + p_{T_c}(t-3T_c) - p_{T_c}(t-4T_c) + p_{T_c}(t-5T_c) + p_{T_c}(t-6T_c)$$

as shown below. The output should be a function of time beginning at time 0 and ending at time $2T = 14T_c$.

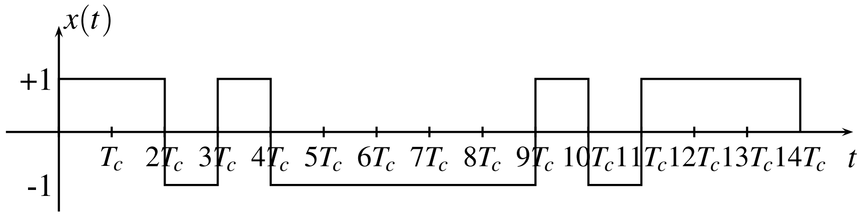




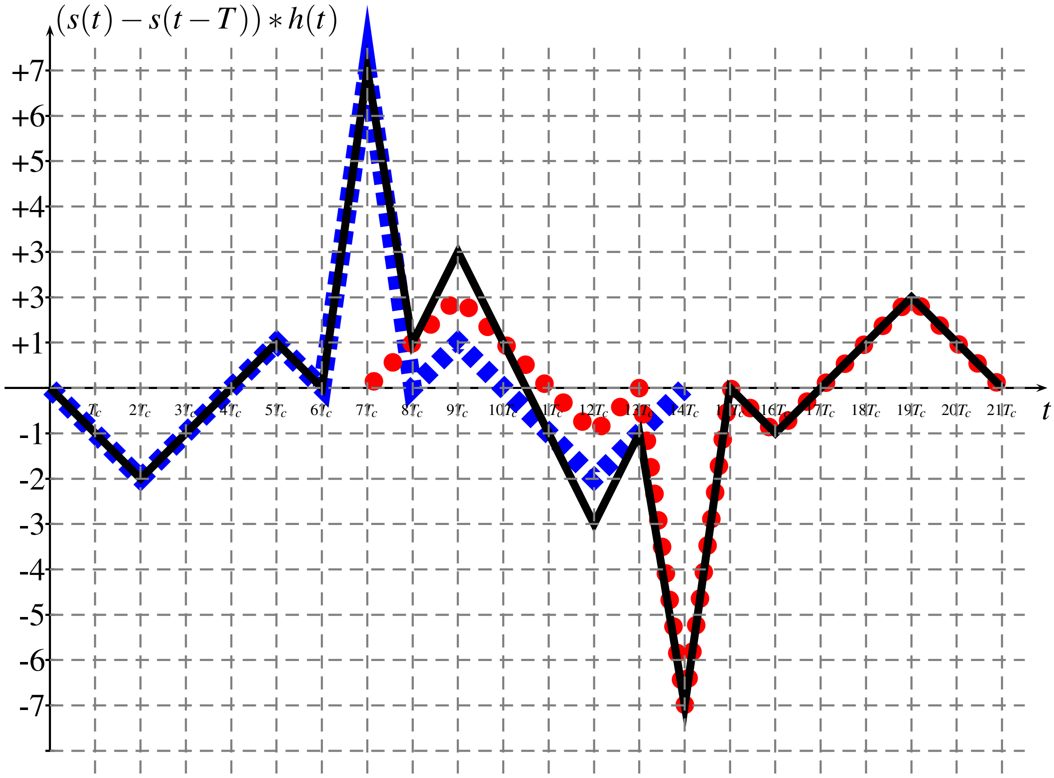
Solution:



(b) Find the filter output (for the same filter) when the input is $x(t) = s(t) - s(t-T)$. The output is a function beginning at time 0 and ending at time $21T_c = 3T$.



Solution: The blue (dashed) curve represents the output due to $s(t)$ and the red (dotted) curve represents the output due to $-s(t-T)$. The total output is the sum of the two and is show by the solid black line.



4. (a) Consider a signal $f(t)$ with unit energy defined over the time interval $[0,T]$ that is zero outside that time interval. For example $f(t) = \sqrt{1/T}p_T(t)$. Consider two signals of duration NT.

$$
\begin{aligned}
s_0(t) &= \sum_{i=0}^{N-1} s_{0,i} f(t-iT)p_T(t-iT) \\
s_1(t) &= \sum_{i=0}^{N-1} s_{1,i} f(t-iT)p_T(t-iT)
\end{aligned}
$$

Determine $(s_0(t), s_1(t))$ in terms of the sequence $s_{0,i}, i=0,1,...,N-1$ and $s_{1,i}, i=0,1,...,N-1$.

Solution:

$$
\begin{aligned}
(s_0(t), s_1(t) &= \int_0^T s_0(t)s_1(t)dt \\
&= \int_0^{NT} \sum_{i=0}^{N-1} s_{0,i} f(i-iT)p_T(t-iT) \sum k = 0^{N-1} s_{1,j}^* f^*(i-jT)p_T(t-jT)dt \\
&= \sum_{i=0}^{N-1} s_{0,i}s_{1,i} \int_0^{NT} |f(i-iT)|^2 p_T(t-iT)dt \\
&= \sum_{i=0}^{N-1} s_{0,i}s_{1,i} \int_0^{T} |f(i)|^2 dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{N-1} s_{0,i} s_{1,i} \\
&= (s_0, s_1)
\end{aligned}
$$

(b) Consider the two signals of duration $2T$ generated from $f(t)$ and the two vectors $s_0 = (+1,+1)$ and $s_1 = (+1,-1)$. We will form a matrix of vectors. In this case $N = 2$.

$$
\begin{aligned}
H_2 &= \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \\
&= \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}.
\end{aligned}
$$

where the first row can be used to generate a first signal and the second row used to generate a second signal. Using part (a) show that the two signals

$$
\begin{aligned}
s_0(t) = s_{0,0} f(t) p_T(t) + s_{0,1} f(t-T) p_T(t-T) \\
s_1(t) = s_{1,0} f(t) p_T(t) + s_{1,1} f(t-T) p_T(t-T)
\end{aligned}
$$

are orthogonal.

Solution:

$$
\begin{aligned}
(s_0(t), s_1(t)) &= (s_0, s_1) \\
&= \sum_{i=0}^{1} s_{0,i} s_{1,i}^* \\
&= (+1)(+1) + (+1)(-1) = 1 - 1 \\
&= 0
\end{aligned}
$$

(c) Now consider four signals $s_0(t)$, $s_1(t)$, $s_2(t)$, $s_3(t)$ generated with a matrix H_4 where

$$
H_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}.
$$

Each row of the matrix represents one of four signals. The components of a row are the coefficients of $f(t)$ in different time intervals. Show that using part (a) that any distinct pair of signals is orthogonal.

Solution: Call the four signals s_0, s_1, s_2, and s_3.

$$
\begin{aligned}
(s_0, s_1) = (+1)(+1) + (+1)(-1) + (+1)(+1) + (+1)(-1) = 0 \\
(s_0, s_2) = (+1)(+1) + (+1)(+1) + (+1)(-1) + (+1)(-1) = 0 \\
(s_0, s_3) = (+1)(+1) + (+1)(-1) + (+1)(-1) + (+1)(+1) = 0 \\
(s_1, s_2) = (+1)(+1) + (-1)(+1) + (+1)(-1) + (-1)(-1) = 0 \\
(s_1, s_3) = (+1)(+1) + (-1)(-1) + (+1)(-1) + (-1)(-1) = 0 \\
(s_2, s_3) = (+1)(+1) + (+1)(-1) + (-1)(-1) + (-1)(+1) = 0
\end{aligned}
$$

(d) Consider H_8

$$
\begin{aligned}
H_8 &= \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} \\
&= \begin{bmatrix}
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\
+1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1 & -1 & +1 & +1 & -1
\end{bmatrix}
\end{aligned}
$$

Show that the signals in the set generated by H_8 are orthogonal using the fact that the signals in set generated by h_4 are orthogonal.

Solution:

Clearly any two (distinct) signals in the top half are orthogonal because they are orthogonal over the first half and they are orthogonal over the second half (which is a repeat of the first half). Similarly for any two signals in the bottom half. Consider a signal from the top half and a signal from the bottom half of the matrix. The first signal in the top half and the first signal in the bottom half are orthogonal because they are identical in the left half but opposite in the right half. Similarly for the second, third and fourth signals in each half. A signal in the top half is orthogonal to a different signal in the bottom half (e.g. the top signal in the top half and the bottom signal in the bottom half) because they are orthogonal over the left half and right half. So any two distinct signal are orthogonal.