EECS 551 Homework 3 YUZHAN JIANG

P1. Prove that if Q is a $M \times K$ matrix for which $\|Qx\|_2 = \|x\|_2$ for all $x \in C^K$, then $Q^*Q = I_K$

proof:

Let $Q = [u_1\ u_2\ u_3 \cdots u_K]$ where $u_i \in C^M$

Since $\|Qx\|_2 = \|x\|_2$,

$$\Rightarrow \|Qx\|_2^2 = \|x\|_2^2$$

$$(Qx)^*(Qx) = x^*x \qquad (a)$$

$$x^*Q^*Qx = x^*x$$

let $e_i = (0, 0, 0, \cdots 1, \cdots 0)$ for $i \in 1, 2, \cdots k$, we have $e_i^* e_i = 1$

(1 at the i position)

① Let $x = e_i$,

$Qx = Qe_i = u_i$ where u_i is the column of Q

$\therefore (Qx)^*(Qx) = x^*x$ by (a)

$$u_i^* u_i = e_i^* e_i$$

$$u_i^* u_i = 1$$

$$\therefore \|u_i\|_2^2 = 1$$

$$\therefore \|u_i\|_2 = 1$$

Thus, every column of Q has norm 1.

② let $x = e_j + e_i$ for $\forall i, j \in 1, 2, \cdots k$

$$Qx = Q(e_j + e_i) = u_i + u_j$$

$\therefore (Qx)^*(Qx) = x^*x$ by (a)

$$(u_i + u_j)^*(u_i + u_j) = (e_j + e_i)^*(e_j + e_i)$$

$$u_i^* u_i + 2u_i^* u_j + u_j^* u_j = e_j^* e_j + 2e_j^* e_i + e_i^* e_i$$

$2u_i^* u_j + 2 = 2$ by some results in ①

$$u_i^* u_j = 0$$

(to be continued....)

$\therefore$ columns of Q are orthogonal to each other.

Finally, $Q'Q = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_k^* \end{bmatrix} [u_1 \ u_2 \ \cdots \ u_k]$

$$= \begin{bmatrix} u_1^* u_1 & u_1^* u_2 & \cdots & u_1^* u_k \\ u_2^* u_1 & u_2^* u_2 & & \vdots \\ \vdots & & \ddots & \\ u_k^* u_1 & \cdots & \cdots & u_k^* u_k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= I_k$$

P2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(a) let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, by def of nullspace:

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad x_1 + 0 = 0$

$x_2 = x_2$

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ nullspace of A is span$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Range of A is span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ = span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Overall, the $N(A)$ is span$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $R(A)$ is span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(b) No, they are not equal.

The answer does not hold in general. Counter example:

let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

let $Ax = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = -x_1 \end{matrix} \quad \therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore \quad N(A) =$ span$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Where the Range of A is span$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ = span$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$\therefore \quad N(A) = R(A)$ in this case

P3:

(a) $A \in F^{m \times n}$, and W and Q are unitary matrix

By definition, the singular value of A are the square root of non-zero eigenvalues of A^TA

and the singular values of WAQ are the square root of non-zero eigenvalues of

$$(WAQ)^T(WAQ)$$
$$= Q^TA^TW^TWAQ$$
$$= Q^TA^TAQ \qquad \text{where } W^TW = I$$

Thus, A^TA is similar to Q^TA^TAQ,

A and $C \triangleq WAQ$ have the same eigenvalues

(b) Suppose that W and Q are nonsingular but not necessarily unitary matrix

$\text{rank}(WAQ) \le \min(\text{rank}(W), \text{rank}(A), \text{rank}(Q))$ (By property of rank)

$\Rightarrow \text{rank}(WAQ) \le \text{rank}(A)$

Since W, Q are nonsingular $\Rightarrow$ There exist W^{-1} and Q^{-1} such that $W^{-1}W = I$, $Q^{-1}Q = I$

Then, $A = W^{-1}WAQQ^{-1}$

$\therefore \text{rank}(A) = \text{rank}(W^{-1}WAQQ^{-1}) \le \min(\text{rank}(W^{-1}), \text{rank}(WAQ), \text{rank}(Q^{-1})) \le \text{rank}(WAQ)$

$\because \text{rank}(WAQ) \le \text{rank}(A)$ and $\text{rank}(A) \le \text{rank}(WAQ)$

$\therefore \text{rank}(A) = \text{rank}(WAQ)$

(c) Let $A = [1] \in F^{1 \times 1}$ and $W = Q = [3] \in F^{1 \times 1}$

The singular value of A is 1, W, Q are nonsingular but not unitary matrix

$WAQ = [9]$, the singular value of WAQ is 9

$\therefore$ The singular value of A and WAQ are not the same

P4: prove that

$$R^{\perp}(A) = N(A')$$

proof:

① $\Leftarrow$:

let $x \in N(A')$, then $A'x = 0$

So, $\forall y$, we have $y'(A'x) = 0$

$$\Rightarrow x'(yA') = 0$$
$$x'(Ay) = 0$$
$$\Rightarrow x \in R^{\perp}(A)$$

Therefore, if $x \in N(A')$, then $x \in R^{\perp}(A)$

② $\Rightarrow$:

let $x \in R^{\perp}(A)$, then $\forall y$, $x'(Ay) = 0$

Take $y = A'x$

$$x'(Ay) = x'AA'x = 0$$
$$= (A'x)'(A'x)$$
$$\Rightarrow ||A'x|| = 0$$

which implies $A'x = 0$

$\therefore$ $x \in N(A')$

There, if $x \in R^{\perp}(A)$, then $x \in N(A')$

Overall, $R^{\perp}(A) = N(A')$

P5:

$$A = Q\Lambda Q' \quad \text{where } A \text{ is Hermitian matrix}$$

$$A = V\Lambda V' = \sum_{i=1}^{N} \lambda_n V_n V_n' = \sum_{i=1}^{N} |\lambda_n| \, \text{sign}(\lambda_n) V_n V_n' = U\Sigma V'$$

Let $\Sigma = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & 0 & \\ & & \ddots & & & \\ & & & \lambda_k & & \\ & 0 & & & |\lambda_{k+1}| & \\ & & & & & \ddots & \\ & & & & & & |\lambda_N| \end{bmatrix}$

Let $Q = [Q_1 \ Q_2 \cdots Q_K, Q_{K+1} \cdots Q_N]$

Let $U = [Q_1 \ Q_2 \cdots Q_K, -Q_{K+1} \cdots -Q_N]$

$V' = Q'$

$\therefore$ An SVD of A is $U\Sigma V'$

P6.

$$\begin{array}{cccc} & 1\times 1 & 1\times 1 & 1\times 2 \end{array}$$

① $A_1 = [0 \quad 1]$ where $U = [-1]$ $\Sigma = [1]$ $V' = [0 \quad -1]$

② $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ where $\lambda_1 = \lambda_2 = 1$ $V = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $AV = V\Lambda$

$(1-\lambda)(1-\lambda) = (\lambda-1)^2 = 0$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

③ $A_3 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ where $\lambda_1 = 1$ $\lambda_2 = -1$ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

④ $A_4 = \begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix}$ where $\Lambda = \begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix}$ $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $V' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

⑤ $A_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ A_5 is Hermitian and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $V^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

P7. $A = xy'$ where x nor y is 0

(a) let $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$\therefore A = xy' = [y_1 x \quad y_2 x \quad \cdots \quad y_n x]$

By def of Linearly independent, the equation:

$a_1 \, {}_1 x + a_2 y_2 x + \cdots + a_n y_n x = 0$

when $a_2 = -\frac{y_1}{y_2}$ $a_4 = -\frac{y_2}{y_4}$, $a_6 = -\frac{y_4}{y_6}$ $\cdots$, are the Solution

Therefore, there is one linearly independent column.

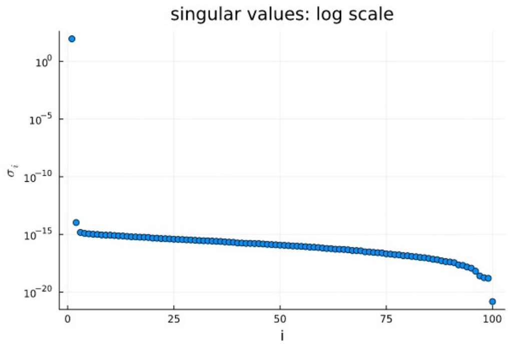
(b) Since there is only one linearly independent column,

The rank(A) is 1

P7 (c)

In [6]:
```
LinearAlgebra: svdvals
LinearAlgebra
Plots
LaTeXStrings
00
andn(n); y = randn(n); A = x*y'
vdvals(A)
er(s, yscale = :log10, label= "", title = "singular values: log scale",xlabel="i", ylabel=L"\sigma_i") # to make  i
```

Out[6]:



In [8]:
```
display(rank(A))
display(s)
```

```
1

100-element Vector{Float64}:
 89.92505159881593
  1.0692779808940873e-14
  1.4953523502218396e-15
  1.2331802536314417e-15
  1.1253515484362254e-15
  1.0402240272670478e-15
  9.94998789125674e-16
  9.077372012081195e-16
  8.720198900135821e-16
  8.645777498707103e-16
  8.104546757147691e-16
  7.590962267861637e-16
  7.349731864389871e-16
  ⋮
  4.420943511411403e-18
  4.0814207714459556e-18
  3.533256532007776e-18
  2.2977498109765776e-18
  2.074131554783197e-18
  1.5009541860233865e-18
  1.1955979278269644e-18
  6.569215307740453e-19
  2.5350004444418053e-19
  1.7901250602458462e-19
  1.5770013276207504e-19
  1.5118637398392276e-21
```

P7

(d)

```
function rank(A::AbstractMatrix; atol::Real = 0.0, rtol::Real = (min(size(A)...)*eps(real(float(one(eltype(A))))))*iszero(atol))
    isempty(A) && return 0 # 0-dimensional case
    s = svdvals(A)
    tol = max(atol, rtol*s[1])
    count(x -> x > tol, s)
end
rank(x::Number) = x == 0 ? 0 : 1
```

The threshold formula:

$$tol = max(atol, rtol * s[1])$$

Where "atol" and "rtol" are the absolute and relative tolerances, respectly
"s[1]" is the largest singular

(e) $A = [9 \ 9] \in F^{1\times 2}$

The tolerance threshold is $2\epsilon = 2 \times 2.204 \times 10^{-16}$

$= 4.408 \times 10^{-16}$

P8.

(a) $A = U\Sigma V'$ $A^+ = V\Sigma^+ U'$

where Σ is rectangular diagonal matrix containing $\min(m,n)$ non-zero singular of A

By the def of pseudo-inverso Σ^+

Σ^+ is rectangular matrix whose non-zeros are the reciprocals of non-zeros of Σ

$\therefore \Sigma^+\Sigma = I$

$A'A = (V\Sigma'U')(U\Sigma V') = V\Sigma'\Sigma V'$

$$\begin{aligned}\therefore \quad A'AA^+ &= (V\Sigma'\Sigma V')V\Sigma^+U' \\ &= V\Sigma'\Sigma\Sigma^+U' \\ &= V\Sigma'U' \\ &= (U\Sigma V')' \\ &= A'\end{aligned}$$

$\therefore \quad A^+ = (A'A)^{-1}A'$

when $A = xy'$,

$$\begin{aligned}A^+ &= \left((xy')'(xy')\right)^{-1}(xy')' \\ &= (yx'xy')^{-1}(yx')\end{aligned}$$

(b) By Part (a), we have $A^+ = (A'A)^{-1}A'$

when $A = xx'$

$$\begin{aligned}A^+ &= ((xx')'(xx'))^{-1}(xx')' \\ &= (xx'xx')^{-1}(xx')\end{aligned}$$

P9:

(a) $R(V) = \text{span}\left\{\begin{bmatrix}0\\2\\0\end{bmatrix}\right\}$ $\quad R^{\perp}(V) = N(V')$ (by P4)

let $x \in V'$ then $V'x = 0 \Rightarrow \begin{bmatrix}0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \Rightarrow \quad x_2 = 0$

$\therefore \quad x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$

$\therefore \quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is the orthonormal basis for the orthogonal complement of $\text{span}\left\{\begin{bmatrix}0\\2\\0\end{bmatrix}\right\}$

(b) $R(V) = \text{span}\left\{\begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$, $R^{\perp}(V) = N(V')$ (by P4)

let $x \in V'$, then $V'x = 0 \Rightarrow \begin{bmatrix}0 & 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$

$\therefore \quad 2x_2 + 2x_3 = 0$

$x_2 + x_3 = 0$

$x_2 = -x_3$

$\Rightarrow \quad x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\-x_3\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$\therefore \quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$ is the orthonormal basis for the orthogonal complement of $\text{span}\left\{\begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$

P9:

(c) The projection of the vector $y = \begin{bmatrix}1\\2\\4\end{bmatrix}$ onto the orthogonal complement of $S = \text{span}\{[8]\}$

$$P_{S^\perp} y = (I - P_S) y$$

$$= y - P_S y$$

$$= y - z(z'z)^{-1} z' y$$

$$= y - \frac{z'y}{z'z} z \quad \text{where} \begin{cases} z'y = \begin{bmatrix}0 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix} = 4+8 = 12 \\ z'z = \begin{bmatrix}0 & 2 & 2\end{bmatrix}\begin{bmatrix}0\\2\\2\end{bmatrix} = 4+4 = 8 \end{cases}$$

$$= \begin{bmatrix}1\\2\\4\end{bmatrix} - \frac{12}{8}\begin{bmatrix}0\\2\\2\end{bmatrix}$$

$$= \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

(e)

orthcomp1.jl U ×

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```julia
"""
z = orthcomp1(y, x)
Project `y` onto the orthogonal complement of `Span({x})`
In:
* `y` vector
* `x` nonzero vector of same length, both possibly very long
Out:
* `z` vector of same length
For full credit, your solution should be computationally efficient.
"""
function orthcomp1(y, x)
    z = y - ((x' * y) / ( x' * x)) * x
    return z
end
```