Last (family) name:__________________

First (given) name:__________________

I have neither given nor received aid on this examination, nor concealed any violation of the Honor Code.

Signature: __________________

ID Number: __________________

EECS 551 Midterm 1, 2021-10-25 6-8PM

- There are 18 problems for a total of 100 points.
- This part of the exam has 10 pages. Make sure your copy is complete.
- Write your name in the upper right of every page!
- You must complete the exam entirely on your own.
- You may use without rederiving any of the results presented in the course notes.
- During the exam you may use two pieces of 8.5×11" paper with notes on both sides, but no electronic devices.
- Clearly $\boxed{\text{box}}$ your final answers. For full credit, show your complete work clearly and legibly in the space provided.
- For full credit, ~~cross out~~ any incorrect intermediate steps.
- For multiple-choice questions, select *all* correct answers.
- To "disprove" any statement, provide a concrete counter-example. For maximum credit, make that counter-example as small and simple as possible, *e.g.*, having the smallest possible matrix dimensions and using the simplest numbers like "0" and "1" as much as possible. For example, to disprove the statement "any square matrix $\boldsymbol{A}$ is invertible," the smallest and simplest counter-example is the 1×1 matrix $\boldsymbol{A} = [0]$.
- For any True-False question, box $\boxed{\text{True}}$ *and* give a short justification if the statement always holds, otherwise box $\boxed{\text{False}}$ *and* give a simple counter-example.
- For multiple-choice questions, $\boxed{\text{box}}$ *all* of the correct given answer choices *and* write a short explanation.
  If none of the answers is always correct, then write in $\boxed{\text{None}}$ and explain briefly.
  If multiple choices are incorrect, it suffices to explain why *one* of them is incorrect.
- You may use the back side of each exam page for scratch work, but that work will not be graded.
  Only the answers in the designated areas will be scanned into gradescope and graded.
- For all JULIA code, assume that the following code has been invoked already:
  `using LinearAlgebra, MIRTjim, FFTW, Random, Plots`

DO NOT TURN OVER THIS PAGE UNTIL TOLD TO BEGIN!

(Solutions)

Regrade policy

- For any student who submits a regrade request, we may regrade the entire exam, and your final score may increase **or decrease**.
- Submit any regrade request on gradescope, with a clear description of why you think the problem should be regraded. Saying just "please look at it again" is insufficient and will not be considered.
- Regrade requests must be within 3 days (72 hours) of when the exam scores were released on gradescope.
- After scores are returned, discussing your solutions with a professor or GSI nullifies the opportunity to submit a regrade request. Your request needs to be based on your answer at the time of the exam. (Wait to discuss until *after* requesting a regrade, if you plan to make such a request.) (Submitting reports of errors in the solution to Canvas is fine.)
- Some questions were graded by student graders who may not have been very discriminating about the precision of your justifications for your answers. If you submit a regrade request, that means you think you were unfairly given too few points on some part of the exam, so it is logical for us to see if you were unfairly given too many points on some other part of the exam! Of course we want fairness overall. Minor mistakes are inevitable when grading numerous exams, and those mistakes go in both directions. One point on the midterm is only 0.3% of your overall final score, and unlikely to affect your final grade. You should look over the solutions and your answers to all problems carefully *before* submitting a regrade request to make sure that you really want to have your whole exam reevaluated.
- For elaboration on these solutions, please come to office hours.

tf/cauchy1

1. (4 points)

For $\boldsymbol{A} = \begin{bmatrix} 3 & -4 \end{bmatrix}$, $\|\boldsymbol{A}\boldsymbol{x}\|_2$ is maximized for unit norm $\boldsymbol{x}$ when $\boldsymbol{x} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

False. It is maximized when $\boldsymbol{x} = e^{\imath\phi} \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$ for any $\phi \in \mathbb{R}$.

(q06, q13) [71%; mean 3.1/4]

tf/pinv/pinv3b

2. (4 points)

If $\boldsymbol{A} \in \mathbb{F}^{9\times 4}$ has 4 nonzero singular values, then $\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{I}$.

True. For $N = 4$: $\boldsymbol{A} = \boldsymbol{U}_N \boldsymbol{\Sigma}_N \boldsymbol{V}' \implies \boldsymbol{A}^+ = \boldsymbol{V}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{U}_N'$ so $\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{U}_N'\boldsymbol{U}_N\boldsymbol{\Sigma}_N\boldsymbol{V}' = \boldsymbol{I}$.

(f17 student) [84%; mean 3.4/4]

tf/ls1

3. (4 points)

If $\boldsymbol{A}$ is $M \times N$ with rank N and $M \geq N$, then $\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2 = 0$.

False. $\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{y}$ only if $\boldsymbol{y} \in \mathcal{R}(\boldsymbol{A})$.

(q11) [56%; mean 2.5/4]

tf/ls3

4. (4 points)

If $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}$ then $\min_{\boldsymbol{x}} \|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{y}\|_2 > \min_{\boldsymbol{z}} \|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{y}\|_2$, assuming dimensions match appropriately.

False. This is like in the JULIA homework problem where you found that fitting a higher order polynomial led to a "better" fit (smaller residual), so $\min_{\boldsymbol{x}} \|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{y}\|_2 \leq \min_{\boldsymbol{z}} \|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{y}\|_2$. Also $\mathcal{R}(\boldsymbol{A}) \subseteq \mathcal{R}(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{C} \end{bmatrix})$.
For example, take $\boldsymbol{A} = \boldsymbol{0}$ and $\boldsymbol{B} = \boldsymbol{I}$ and $\boldsymbol{y} = \boldsymbol{1}$, then $\min_{\boldsymbol{x}} \|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{y}\|_2 = 0 < \|\boldsymbol{y}\|_2 = \min_{\boldsymbol{z}} \|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{y}\|_2$.

(f17 student) [47% (10% 'true' and 6% no answer, the rest had explanation issues); mean 2.7/4]

mc/max1

5. (4 points)

When $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \end{bmatrix}$, a unit norm vector $\boldsymbol{x}$ that maximizes $\|\boldsymbol{A}\boldsymbol{x}\|_2$ is:

a: $\begin{bmatrix} 1 & 2 \end{bmatrix}' / \sqrt{5}$

b: $\begin{bmatrix} 6 & 4 & 2 \end{bmatrix}' / \sqrt{14}$

c: $\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}' / \sqrt{14}$

d: $\begin{bmatrix} 2 & 1 \end{bmatrix}' / \sqrt{5}$

e: $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}' / \sqrt{14}$

$\boldsymbol{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ so $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} / \sqrt{1^2 + 2^2 + 3^2}$.

(q06, q13) [84%; mean 3.5/4]

mc/left1

6. (4 points)

A left singular vector of $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 4 & 2 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$, corresponding to its largest singular value is:

a: $\begin{bmatrix} 0 & 2 & 1 \end{bmatrix}'$

b: $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}'$

c: $\begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}'$

d: $\begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}'$

. . . .

$$\boldsymbol{A} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{\boldsymbol{u}_1} \sqrt{70} \underbrace{\frac{1}{\sqrt{14}} \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix}}_{\boldsymbol{v}_1'}.$$

(q09) [79%; mean 3.4/4]

tf/eigsig1

7. (4 points)

If $\boldsymbol{A}$ is Hermitian symmetric, then any left or right singular vector of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{A}$.

. . . .

False. Consider $\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$, $\quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$. Here $\boldsymbol{x}$ is a right singular vector of $\boldsymbol{A}$ because $\boldsymbol{A}'\boldsymbol{A}\boldsymbol{x} = 9\boldsymbol{x}$, but clearly $\boldsymbol{x}$ is not an eigenvector of $\boldsymbol{A}$. In fact, every $\boldsymbol{x} \in \mathbb{F}^2$ having unit norm is a singular vector of this $\boldsymbol{A}$ because $\boldsymbol{A}'\boldsymbol{A} = 9\boldsymbol{I}$. See Ch. 2 section relating SVD and eigdec.

(q06) [15% (35% 'true' 46% had explanation issues); mean 1.1/4]

mc/unitary1

8. (4 points)

Let $\boldsymbol{A}$ be a $M \times M$ unitary matrix and let $\boldsymbol{B}$ denote the last N columns of $\boldsymbol{A}$, where $1 \leq N \leq M$.
What is $\|\boldsymbol{B}\|_2$?

a: 0

b: 1

c: N

d: M

e: $\min(N, M)$

. . . .

Using compact SVD: $\boldsymbol{B} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_N \end{bmatrix} = \underbrace{\boldsymbol{A} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_N \end{bmatrix}}_{\boldsymbol{U}_N} \underbrace{\boldsymbol{I}_N}_{\boldsymbol{\Sigma}_N} \underbrace{\boldsymbol{I}_N}_{\boldsymbol{V}} \Longrightarrow \|\boldsymbol{B}\|_2 = \sigma_1(\boldsymbol{B}) = 1.$

Alternative explanation: $\sigma_i^2(\boldsymbol{B}) = \lambda_{(i)}(\boldsymbol{B}'\boldsymbol{B}) = \lambda_{(i)}(\boldsymbol{I}_N) = 1.$

(q10) [51% (14% wrong choice, rest had explanation issues); mean 2.5/4]

mc/gd1

9. (4 points)
Nesterov's accelerated gradient method for solving the LS problem $\arg\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2$ will converge if $0 < \mu \leq 1/\sigma_1(\boldsymbol{A}'\boldsymbol{A})$. When $\boldsymbol{A} = \mathbf{1}_2\mathbf{1}_8'$, where $\mathbf{1}_n =$ `ones(n)`, the maximum value of the step size μ is:

a: 1
b: $1/2$
c: $1/4$
d: $1/8$
e: $1/16$

....

$\sigma_1(\boldsymbol{A}) = \sqrt{2 \cdot 8} = 4 \Longrightarrow \sigma_1(\boldsymbol{A}'\boldsymbol{A}) = 4^2$ so $\mu \leq 1/4^2$.
(h06, q10) [70% (27% wrong choice); mean 2.8/4]

mc/ls2

10. (4 points)

$$\min_{\boldsymbol{x}\in\mathbb{R}^3} \left\| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\|_2^2 = ?$$

a: 0
b: 1
c: 2
d: 3
e: 4

....

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^+\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \boldsymbol{r} = \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{y} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \|\boldsymbol{r}\|_2^2 = 4.$$

(q11) [70% (20% forgot square?); mean 3.2/4]

mc/norm2

11. (4 points)

For $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$, the value of $\|\boldsymbol{A}\|_2$ is:

a: 0

b: 1

c: 5

d: 10

e: 20

$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}'$, $\boldsymbol{C} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix}'$, where $\|\boldsymbol{B}\|_2 = \sqrt{5 \cdot 5} = 5$, $\|\boldsymbol{C}\|_2 = \sqrt{5 \cdot 20} = 10$.
So $\|\boldsymbol{A}\|_2 = \max(\|\boldsymbol{B}\|_2, \|\boldsymbol{C}\|_2) = 10$.

Alternative: $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{X}'$, $\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 2 \end{bmatrix} \Longrightarrow \|\boldsymbol{X}\|_2 = \|\boldsymbol{X}\boldsymbol{X}'\|_2 = \|\boldsymbol{X}'\boldsymbol{X}\|_2 = \left\| \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \right\|_2 = 10.$

Alternative: examine $\det\{z\boldsymbol{I} - \boldsymbol{A}\}$ because $\boldsymbol{A}$ is Hermitian so $\sigma_k = \left|\lambda_{(k)}\right|$. In fact $\boldsymbol{A}$ is PSD so $\sigma_k = \lambda_{(k)}$.

Alternative: $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 4 \end{bmatrix}' = \cdots$

(f17 student) [51% (35% wrong choice); mean 2.2/4]

mc/rank1

12. (4 points)

If $\boldsymbol{B}$ is a 200×400 matrix of rank 100, then:

a: $\dim(\mathcal{R}(\boldsymbol{B})) = 100$

b: $\dim\left(\mathcal{R}^\perp(\boldsymbol{B})\right) = 100$

c: $\dim(\mathcal{N}(\boldsymbol{B})) = 100$

d: $\dim\left(\mathcal{N}^\perp(\boldsymbol{B})\right) = 100$

e: The number of distinct singular values is at least 2.

All of the statements are true except $\dim(\mathcal{N}(\boldsymbol{B})) = 300 \neq 100$.
The rank is the number of nonzero singular values and the rank is 100 so there is at least one nonzero singular value (possibly repeated 100 times). But $\min(M, N) = 200$ so there are at least 100 singular values equal to zero. So there is at least one zero and one nonzero σ.

(q08) [30% fully correct; 30% had 3/4 correct selected; mean 2.6/4]

mc/p1p2

13. (4 points)

Let $\boldsymbol{A}$ be a tall matrix having rank $r > 0$ with SVD given by $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}' = \begin{bmatrix}\boldsymbol{U}_r & \boldsymbol{U}_0\end{bmatrix} \begin{bmatrix}\boldsymbol{\Sigma}_r \\ \boldsymbol{0}\end{bmatrix} \boldsymbol{V}'$.

Define $\boldsymbol{P}_r^{\perp} \triangleq \boldsymbol{I} - \boldsymbol{U}_r\boldsymbol{U}_r'$ and $\boldsymbol{P}_0^{\perp} \triangleq \boldsymbol{I} - \boldsymbol{U}_0\boldsymbol{U}_0'$ and $\boldsymbol{B} \triangleq \boldsymbol{P}_0^{\perp}\boldsymbol{P}_r^{\perp}$. Then:

a: $\boldsymbol{B}$ is a unitary matrix

b: $\boldsymbol{B}$ is not a unitary matrix

c: Need more information to assess

$\boldsymbol{U}_0'\boldsymbol{U}_r = \boldsymbol{0} \Longrightarrow \boldsymbol{B} = \boldsymbol{P}_0^{\perp}\boldsymbol{P}_r^{\perp} = (\boldsymbol{I} - \boldsymbol{U}_0\boldsymbol{U}_0')(\boldsymbol{I} - \boldsymbol{U}_r\boldsymbol{U}_r') = (\boldsymbol{U}_r\boldsymbol{U}_r')(\boldsymbol{U}_0\boldsymbol{U}_0') = \boldsymbol{U}_r(\boldsymbol{U}_r'\boldsymbol{U}_0)\boldsymbol{U}_0' = \boldsymbol{0} \Longrightarrow$ not unitary.

(q09) [51% (28% wrong choice); mean 2.3/4]

code/basis1

14. (10 points)

The vectors $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ form a orthonormal basis for a subspace $\mathcal{S}$ of $\mathbb{R}^N$, for $N \geq 5$. Complete the following JULIA function so that, given input vector $\boldsymbol{x} \in \mathbb{R}^N$, it returns the nearest vector in $\mathcal{S}$.
For full credit, your code must use as few floating-point calculations as possible. Explain briefly.

```
function nearest(x, b1, b2, b3)
    B = [b1 b2 b3]
    return B * (B' * x)
end
```

The projection onto the subspace is $\boldsymbol{B}\boldsymbol{B}'\boldsymbol{x}$. The parentheses ensure minimum computation.

(HW 4.3) [54% correct and efficient; mean 7.0/10;
18% used expensive `rank(A)` *or* `rank(diag(s))` *; 13% formed $N \times N$ projection matrix]*

code/nullbasis1

15. (8 points)

Complete the following JULIA function so that it returns an orthonormal basis (as a matrix) for the null space of a matrix argument. Assume $\dim(\mathcal{N}(\boldsymbol{A})) > 0$.

```
function nullbasis(A)
    (_,s,V) = svd(A, full=true)
    r = rank(Diagonal(s)) # avoids redundant SVD call
    return V[:,(r+1):end] # V0
end
```

The columns of the matrix $\boldsymbol{V}_0$, *i.e.*, the last $N-r$ columns of the right singular vectors $\boldsymbol{V}$, form an orthonormal basis for the null space of $\boldsymbol{A}$. (See anatomy of SVD notes.) Because $\dim(\mathcal{N}(\boldsymbol{A})) > 0$, we have $r < N$ so the returned matrix always has at least one column. The code above works fine even when $r = 0$.

If $r = N$, then the null space is just $\mathbf{0}$, which has no orthonormal basis.

The JULIA code for `nullspace` is similar to this.

(f17 student) [51% correct; mean 6.3/8; 37% used inefficient `rank(A)` *; 14% had* `r:end` *]*

code/lsuv4

16. (10 points)

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ denote unitary $N \times N$ matrices. Complete the following JULIA function so that, given input vector $\boldsymbol{y} \in \mathbb{C}^N$, it returns a linear least-squares solution $\underset{\boldsymbol{x} \in \mathbb{C}^{2N}}{\arg\min} \left\| \boldsymbol{y} - \begin{bmatrix} 2\boldsymbol{U} & 3\boldsymbol{V} \end{bmatrix} \boldsymbol{x} \right\|_2$, computed as efficiently as possible.

. . . .

Here $\boldsymbol{A} = \begin{bmatrix} 2\boldsymbol{U} & 3\boldsymbol{V} \end{bmatrix}$ has linearly independent rows, so a (minimum norm) LS solution is:

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^+ \boldsymbol{y} = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{A}')^{-1} \boldsymbol{y} = \begin{bmatrix} 2\boldsymbol{U}' \\ 3\boldsymbol{V}' \end{bmatrix} \left(\begin{bmatrix} 2\boldsymbol{U} & 3\boldsymbol{V} \end{bmatrix} \begin{bmatrix} 2\boldsymbol{U}' \\ 3\boldsymbol{V}' \end{bmatrix} \right)^{-1} \boldsymbol{y} = \begin{bmatrix} 2\boldsymbol{U}' \\ 3\boldsymbol{V}' \end{bmatrix} (13\boldsymbol{I}_N)^{-1} \boldsymbol{y} = \begin{bmatrix} (2/13)(\boldsymbol{U}'\boldsymbol{y}) \\ (3/13)(\boldsymbol{V}'\boldsymbol{y}) \end{bmatrix} = \frac{2}{13}(\boldsymbol{A}'\boldsymbol{y}).$$

Because $\boldsymbol{A}$ is wide, of course there are infinitely many other (non-minimum norm) solutions, such as these (more compute efficient) options:

$$\hat{\boldsymbol{x}} = \begin{bmatrix} (1/2)\boldsymbol{U}'\boldsymbol{y} \\ \boldsymbol{0} \end{bmatrix}, \quad \hat{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0} \\ (1/3)\boldsymbol{V}'\boldsymbol{y} \end{bmatrix}.$$

A compact SVD of $\boldsymbol{A}$ is not required to solve this problem, but some students tried to find it so here it is for reference:

$$\boldsymbol{A} = \begin{bmatrix} 2\boldsymbol{U} & 3\boldsymbol{V} \end{bmatrix} = \boldsymbol{I}_N (\sqrt{13}\boldsymbol{I}_N) \begin{bmatrix} \frac{2}{\sqrt{13}}\boldsymbol{U}' \\ \frac{3}{\sqrt{13}}\boldsymbol{V}' \end{bmatrix}'.$$

```julia
function lsuv(y, U, V)
    return [(2/13) * (U'y); (3/13) * (V'y)]
end
```

Using code that has `A = [2U 3V]` is less efficient than the solution above because it requires multiplying scalars with matrices, rather than just with vectors.

(clicker 4#16, HW 6.2 ?)

[19 % correct and efficient; 14% used `pinv, rank, svd` *; 26% used backslash (yet wide!?); mean 4.3/10]*

norm/frob1

17. (10 points)

A $M \times N$ matrix having rank 2 has Frobenius norm = 5 and spectral norm = 4.
Determine all of the singular values of this matrix.

. . . .

$\sigma_1 = 4$ and $\sqrt{\sigma_1^2 + \sigma_2^2} = 5$ so $\sigma_2 = 3$. If $M > 2$ and $N > 2$, then $\sigma_3 = \cdots = \sigma_{\min(M,N)} = 0$.

(HW 2.6) [30% (43% miss the zero singular values); mean 6.5/10. Several answers said "only 2 singular values."]

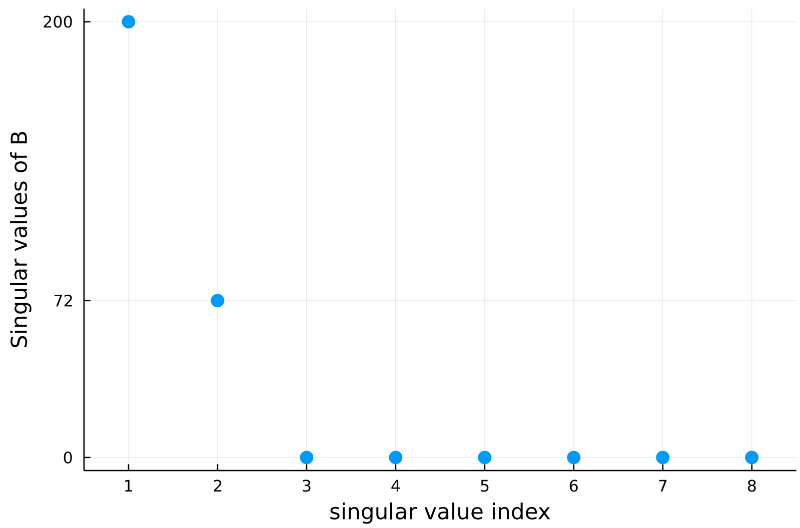
code/scree1

18. (10 points)

The following JULIA code produces a scatter plot. Sketch by hand that plot, with labeled values. Add appropriate axis labels relevant to this plot. As always, explain your answer.

```
x = (-1).^(1:8) # vector of alternating 1, -1 entries
y = ones(8)
A = [3x 5y]; B = A * A'
scatter(svdvals(B), label="")
```

$\boldsymbol{B} = \boldsymbol{A}\boldsymbol{A}' = 3^2 \boldsymbol{x}\boldsymbol{x}' + 5^2 \boldsymbol{y}\boldsymbol{y}'$ where $\boldsymbol{x} \perp \boldsymbol{y}$ and $\|\boldsymbol{x}\|_2^2 = \|\boldsymbol{y}\|_2^2 = 8$ so $\sigma_1(\boldsymbol{B}) = 8 \cdot 5^2 = 200$ and $\sigma_2(\boldsymbol{B}) = 8 \cdot 3^2 = 72$, and the remaining $8 - 2 = 6$ singular values are zero.



(HW 3.7, clicker 3.25) [17% correct; mean 4.2/10; 44% missed the zeros; 44% did not correctly calculate σ_1 or σ_2.]

Exam scores with *approximate* grades.

128 students; min 9.0, max 99.8/100, std dev 21.8, mean 64.6, median 63.6