EECS551: HW2 SOLUTIONS

Problem 1

Let $A = Q\Lambda Q^T$ be the eigendecomposition of A. Then we may write

$$
\begin{aligned}
B &= A - 10I \\
&= Q\Lambda Q^T - 10QQ^T \qquad (I = QQ^T \text{ since } Q \text{ is orthogonal}) \\
&= Q(\Lambda - 10I)Q^T \qquad (0.1)
\end{aligned}
$$

Notice that $\Lambda - 10I$ is a diagonal matrix, and since Q is orthogonal, the right hand side of (0.1) gives us the eigendecomposition of B. So we conclude that the eigenvectors of B are identical to the eigenvectors of A. Further, if λ_B is an eigenvalue of B, and λ_A is an eigenvalue of A, $\lambda_B = \lambda_A - 10$.

Problem 2

Let A and B be the companion matrices associated with the degree n polynomial $p_A(t) = a_0 + a_1 t + \ldots + a_{n-1}t^{n-1} + t^n$ and degree m polynomial $p_B(t) = b_0 + b_1 t + \ldots + b_{m-1}t^{m-1} + t^m$, respectively.
Consider the matrix:

$$C = A \otimes I_m + I_n \otimes (-B) = A \otimes I_m - I_n \otimes -B$$

.
By Theorem 13.16 of Laub, the eigenvalues of C will equal $\lambda_i(A) - \lambda_j(B)$ so that if A and B have common roots, at least one of the eigenvalues of C will be identically zero. Thus if $\det(C) = 0$ we can declare that the polynomials associated with the companion matrices A and B will have common roots. For randomly chosen Gaussian polynomial coefficients, there is an exceedingly small probability of having common roots.
I had the following MATLAB code:

```
n = 5; tol = 1e-10;
% MATLAB uses convention that first entry of vector be the coefficient of the highest degree term
pa = [1; randn(n-1,1)]; pb = [1; randn(n-1,1)];
A = compan(pa); B = compan(pb); % In-build MATLAB command
C = kron(A,eye(size(B)))- kron(eye(size(A)),B);
if(abs(det(C)) < tol), ['Common root'], end
```

Problem 3

Here $A \in \mathbb{R}^{n\times n}$ have the SVD $A = U\Sigma V^T$.

$$B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

We have that:

$$
\begin{aligned}
\det(B - zI) &= \begin{bmatrix} -zI & A \\ A^T & -zI \end{bmatrix} \\
&= \det(-zI) \cdot \det(-zI - A^T(-zI)^{-1}A) \quad \text{By Property 16 of Laub} \\
&= (-z)^n \cdot \det((z^2 I - A^T A)/(-z)) \quad \text{By Property 10 of Laub, } \det(-zI) = (-z)^n \\
&= \det(z^2 I - A^T A)
\end{aligned}
$$

Recall that the eigenvalues of $A^T A$, which are simply $\sigma_1^2, \ldots, \sigma_n^2$ are the solutions of the equation $\det(A^T A - zI) = 0$ or equivalently $\det(zI - A^T A) = 0$. Thus the solutions of the equation $\det(z^2 I - A^T A) = 0$, which are exactly the eigenvalues of B, are exactly $\sigma_1, \ldots, \sigma_n, -\sigma_1, \ldots, -\sigma_n$.

Let $A = U\Sigma V^T$. We know from above that the eigenvalues of B are exactly $\pm\sigma_i$. Notice that:

$$B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & U\Sigma V^T \\ V\Sigma U^T & 0 \end{bmatrix} \quad (\ \Sigma^T = \Sigma \text{ since } A \text{ is square }\ )$$

Hence

$$B \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V^T \\ V\Sigma U^T & 0 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix}$$
$$= \begin{bmatrix} U\Sigma V^T v_i \\ V\Sigma U^T u_i \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_i u_i \\ \sigma_i v_i \end{bmatrix}$$
$$= \sigma_i \begin{bmatrix} u_i \\ v_i \end{bmatrix}$$

Thus $[u_i^T v_i^T]$ is a eigenvector of B corresponding to the eigenvalue σ_i. Similarly we note that

$$B \begin{bmatrix} u_i \\ -v_i \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V^T \\ V\Sigma U^T & 0 \end{bmatrix} \begin{bmatrix} u_i \\ -v_i \end{bmatrix}$$
$$= \begin{bmatrix} -U\Sigma V^T v_i \\ V\Sigma U^T u_i \end{bmatrix}$$
$$= \begin{bmatrix} -\sigma_i u_i \\ \sigma_i v_i \end{bmatrix}$$
$$= -\sigma_i \begin{bmatrix} u_i \\ -v_i \end{bmatrix}$$

so that $[u_i^T - v_i^T]$ is an eigenvector of B corresponding to the eigenvalue $-\sigma_i$. Note that $[u_i^T - v_i^T][u_i^T v_i^T]^T = 0$.

We have to normalize the eigenvectors to have unit norm so that unit-norm eigenvectors corresponding to the eigenvalue σ_i are $[u_i^T v_i^T]/\sqrt{2}$ (Check that this is unit norm!) while the unit-norm eigenvectors corresponding to the eigenvalues $-\sigma_i$ are $[u_i^T - v_i^T]/\sqrt{2}$ where u_i and v_i are the left and right singular vectors associated with the singular value σ_i.

Notice how I solved the eigenvector part of the problem. I spotted (or guessed) the structure right away instead of doing it algebraically. This takes practice and it is okay if you do not see this right away. By the end of the semester you will - that is our end goal.

Problem 4

If:

Given that $v_1, v_2, \ldots, v_n$ are orthonormal and $Av_i, i = 1, \ldots, n$ are orthonormal. Then, the matrix $A\Big[v_1, v_2, \ldots, v_n\Big] := AV$ is orthogonal. We obtain:

$$(AV)^T(AV) = I$$
$$V^T A^T AV = I$$
$$A^T A = VV^T$$
$$A^T A = I.$$

Using similar steps, we can show that $AA^T = I$, and we conclude that A is orthogonal.

Only if:

Given that $v_1, v_2, \ldots, v_n$ are orthonormal and A is orthogonal. Then

$$\begin{aligned}(Av_i)^T(Av_j) &= v_i^T \underbrace{A^T A}_{=I} v_j \\ &= v_i^T v_j \\ &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{(since } v_i \text{ are orthonormal)},\end{aligned}$$

so that Av_i are also orthonormal.

Problem 5

Here T is an invertible (square) matrix and $B = TAT^{-1}$. Let λ be an eigenvalue of B associated with the eigenvector v. Thus by definition we have:

$$Bv = \lambda v \Rightarrow TAT^{-1}v = \lambda v \Rightarrow \underbrace{T^{-1}T}_{=I} AT^{-1}v = \lambda T^{-1}v$$

In other words, we have that $AT^{-1}v = \lambda T^{-1}v$. Let $u = T^{-1}v$ then this is equivalent to saying that $Au = \lambda u$. Or, equivalently, that u is an eigenvector of A associated with the eigenvalue λ. Since this is true for *every* eigenvector v of B associated with *every* eigenvalue λ, this implies that the eigenvalues of A and B are identical.
We obtained that if v is an eigenvector of B associated with the eigenvalue λ then $u = T^{-1}v$ is an eigenvector of A associated with the same eigenvalue λ. Thus for $i = 1, \ldots, n$, the unit-norm eigenvector v_i of B associated with the eigenvalue λ_i is related to the unit-norm eigenvector u_i of A associated with the same λ_i by :

$$v_i = \frac{Tu_i}{||Tu_i||_2}.$$

It is important when speaking of the *set* of eigenvectors associated with a particular eigenvalue that we normalize the eigenvectors to have unit-norm. If we did not normalize v_i as above then we would have to multiply the i-th eigenvalue of B by $||Tu_i||_2^2$ so that $B = \sum_{i=1}^n \lambda_i v_i v_i^T$ as needed.

Problem 6

Here $A = xy^T$ is written in outerproduct form. We can write $A = \sigma uv^T$ where $\sigma = \sqrt{\langle x, x\rangle \cdot \langle y, y\rangle}$, $u = x/\sqrt{\langle x, x\rangle}$ and $v = y/\sqrt{\langle y, y\rangle}$. Comparing this with the SVD form $A = U\Sigma V^T$ we have that the first column of U and V are equal to the u and v derived. The remaining columns of U and V can be any set of orthogonal vectors that is in the ortho-complement of u and v. Further, Σ has only one non zero element, given by σ. Since there is only one non-zero singular value, the rank of A is 1, which implies A has only one linearly independent column.

Problem 7

Let $A \in \mathbb{R}_n^{n\times n}$ be a square matrix with (full) rank n. The SVD of A always exists and is given by $A = U\Sigma V^T$. Since the rank of A is n, Σ is an $n \times n$ diagonal matrix with strictly positive entries. Since V is an orthogonal matrix, $V^TV = I$ and hence we can express A as :

$$\begin{aligned}A &= U\Sigma V^T \\ &= UI\Sigma V^T \quad \text{Note that since } A \text{ is square } U\ , V \text{ and } I \text{ have the same dimensions} \\ &= \underbrace{UV^T}_{=Q}\underbrace{V\Sigma V^T}_{=P}.\end{aligned}$$

We note that $Q = UV^T$ is an orthogonal matrix since $Q^TQ = VU^TUV^T = VIV^T = I$ and $QQ^T = UV^TVU^T = UIU^T = I$. The matrix $P = V\Sigma V^T$ is symmetric since $P^T = V\Sigma V^T = P$. It is positive definite since its eigenvalues, which are exactly equal to the n strictly positive singular values of A, are strictly positive (Note this is a sufficient and necessary test for positive-definiteness of a matrix).

Problem 8

This is a straight-forward application of the properties of the SVD. Let $X = U\Sigma V^T$. Then $X^T X = V\Sigma^T\Sigma V^T = 0 \Rightarrow \Sigma^T\Sigma = 0$. Since the singular values of X are non-negative numbers, the matrix $\Sigma^T\Sigma$ is a diagonal matrix with non-negative numbers equal to the singular values squared along the diagonal. Thus $\Sigma^T\Sigma = 0 \Rightarrow \Sigma = 0 \Rightarrow X = U\Sigma V^T = 0$

Problem 9

From the Matlab experiment, we observe that `eig(A)` are equal to `1./eig(Atil)`.
Given a vector $u = \begin{bmatrix} u_1, u_2, \ldots, u_n \end{bmatrix}^T$, the characteristic equation of the companion matrix A is
$p_1(z) = z^n + \frac{u_2}{u_1} z^{n-1} + \frac{u_3}{u_1} z^{n-2} + \ldots + \frac{u_n}{u_1} = 0.$
`flipud` inverts the vector u, so that the characteristic equation of the matrix $\widetilde{A}$ is
$p_2(z) = z^n + \frac{u_{n-1}}{u_n} z^{n-1} + \frac{u_{n-2}}{u_n} z^{n-2} + \ldots + \frac{u_1}{u_n} = 0.$
Using the transformation of variables $y = \frac{1}{z}$ in $p_2(z)$, we get the equation
$p_2(y) = \left(\frac{1}{y}\right)^n + \frac{u_{n-1}}{u_n}\left(\frac{1}{y}\right)^{n-1} + \frac{u_{n-2}}{u_n}\left(\frac{1}{y}\right)^{n-2} + \ldots + \frac{u_1}{u_n} = 0$
Multiplying throughout by $\frac{u_n}{u_1} y^n$, we get $p_2(y) = \frac{u_n}{u_1} + \frac{u_{n-1}}{u_1} y + \frac{u_{n-2}}{u_1} y^2 \ldots + y^n = 0$,
which has the same roots as $p_1(z)$. But the roots of $p_1(z)$ are the eigenvalues of A, and the roots of $p_2(y)$ are the reciprocals of the eigenvalues of $\widetilde{A}$, due to the transformation $z = \frac{1}{y}$. Since the roots of the two equations are equal, $\text{eig}(A) = \frac{1}{\text{eig}(\widetilde{A})}$.

Problem 10

Let z be one of the n roots of unity, i.e. $z \in \{e^{-\frac{j2\pi k}{n}} := \omega^k \,|\, k = 0, \ldots, n-1\}$, where $\omega := e^{-\frac{j2\pi}{n}}$. Let v_k denote the vector v when $z = \omega^k$.
Notice that:

$$\begin{aligned} v_k^H v_l &= \sum_{i=0}^{n-1} \omega^{-ik}\omega^{il} \\ &= \sum_{i=0}^{n-1} \omega^{i(l-k)} \\ &= \begin{cases} 0 & l \neq k \\ \sum_{i=0}^{n-1} 1 & l = k \end{cases} \\ &= \begin{cases} 0 & l \neq k \\ n & l = k \end{cases}. \end{aligned} \tag{0.2}$$

This may be computed as the sum of the geometric series $\{1, \omega^{l-k}, \omega^{2(l-k)}, \ldots, \omega^{(n-1)(l-k)}\}$, or by using the fact that the n roots of unit solve $z^n - 1 = 0$, and therefore sum to $\frac{\text{coefficient of (n−1) order term}}{\text{coefficient of n order term}} = 0$.
Now,

$$
Cv_k = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(n-1)k} \end{bmatrix}
$$

$$
= \begin{bmatrix} c_0 + c_{n-1}\omega^k + \ldots + c_2\omega^{(n-2)k} + c_1\omega^{(n-1)k} \\ c_1 + c_0\omega^k + c_{n-1}\omega^{2k} + \ldots + c_2\omega^{(n-1)k} \\ \vdots \\ c_{n-1} + c_{n-2}\omega^k + \ldots + c_1\omega^{(n-2)k} + c_0\omega^{(n-1)k} \end{bmatrix}
$$

$$
= \begin{bmatrix} c_0 + c_{n-1}\omega^k + \ldots + c_2\omega^{(n-2)k} + c_1\omega^{(n-1)k} \\ \omega^k(c_1\omega^{(n-1)k} + c_0 + c_{n-1}\omega^k + \ldots + c_2\omega^{(n-2)k}) \\ \vdots \\ \omega^{(n-1)k}(c_{n-1}\omega^k + c_{n-2}\omega^{2k} + \ldots + c_1\omega^k + c_0) \end{bmatrix}
$$

$$
= \left(c_0 + c_{n-1}\omega^k + \ldots + c_2\omega^{(n-2)k} + c_1\omega^{(n-1)k}\right) \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(n-1)k} \end{bmatrix}
$$

$$
= \lambda_k v_k
$$

Therefore, v_k is an eigenvector of C corresponding to eigenvalue λ_k.
Relation to DFT: Notice that the k^{th} eigenvalue λ_k is the expression for the DFT of the time reversed version of the signal $c = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix}^T$.
We have also shown that $\{v_k, k = 0, \ldots, n-1\}$ are orthogonal to each other in (0.2). In order to ensure unit norm for each of the eigenvectors, we divide each vector v_k by $\sqrt{v_k^H v_k} = \sqrt{n}$, so that the orthogonal matrix of eigenvectors becomes $V = \frac{1}{\sqrt{n}} \begin{bmatrix} v_0 & v_1 & \dots & v_{n-1} \end{bmatrix}$. If we define a diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ such that $\Lambda_{ii} = \lambda_{i-1}$, $i = 1, \ldots, n$, we can write the eigendecomposition of C as $C = V\Lambda V^{-1}$.
NOTE: If we define a new matrix $\widetilde{C}$ as:

$$
\widetilde{C} = C^T = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ & \vdots & \vdots & \\ c_1 & c_2 & \dots & c_0 \end{bmatrix}
$$

then the eigenvalues of $\widetilde{C}$ will be exactly the DFT coefficients of the vector $\begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix}^T$.
`MATLAB`'s computation of eigenvalues might not correspond to the ordering defined by the DFT. Recollect that the DFT of the time reversed version of a signal is a time reversal of the DFT of the original signal. Hence, while the *set* of eigenvalues returned by the `eig` command will be the same, the eigenvalues of C and $\widetilde{C}$ will correspond to different eigenvectors. The following code illustrates a few of these concepts:

```
% Create c vector : [c_0 ... c_{n-1}]
n = 3;
c = randn(n,1);

% Generate circulant matrix C
C = zeros(n);
for idx = 1:n
    C(:,idx) = circshift(c,(idx-1));
end

% Compute the eigendecomposition of C
%
% Sort the eigenvalues in descending order, and reorder the eigenvectors
% correspondingly, to make comparison with DFT coefficients easier.
[QC,LambdaC] = eig(C);
[lambdaC_ord,qCidx] = sort(diag(LambdaC),'descend');
QC = QC(:,qCidx);


% Since eigenvalues are the DFT of the time reversed signal version of c,
% compute its time reversal ctil, and then its DFT.
ctil = [c(1); flipud(c(2:end))];
dftctil = fft(ctil);
[ctilft_ord, ctilftidx] = sort(dftctil, 'descend');

disp('-----------------------------------------------------------');
disp('Eigenvalues of C (in descending order): ');
disp(lambdaC_ord);
disp('DFT of ctil = [c_0, c_{n-1} ... c_1] (in descending order): ');
disp(ctilft_ord);


% Generate the matrix V of eigenvectors, and reorder them so that they
% correspond to the order of the sorted DFT values.
e = ones(n,1);
gridkn = (e*(0:n-1)).*((0:n-1)'*e');
V = 1/sqrt(n)*exp((-1j*2*pi/n).*gridkn);

Vctil = V(:,ctilftidx);

% Compare with the eigenvectors computed using eig()
disp('');
disp('abs(inner product) of eigenvectors using eig() and solution');
disp(num2str(abs(QC'*Vctil)));


% Repeat the process for Ctil = transpose(C) as in 'NOTE'
Ctil = transpose(C);
[QCtil, LambdaCtil] = eig(Ctil);
[lambdaCtil_ord, qCtilidx] = sort(diag(LambdaCtil),'descend');
QCtil = QCtil(:,qCtilidx);

dftc = fft(c);
[cft_ord, cftidx] = sort(dftc, 'descend');
```

```
disp('----------------------------------------------------------');
disp('Eigenvalues of Ctil = transpose(C) (in descending order): ');
disp(lambdaCtil_ord);
disp('DFT of c = [c_0, c_1, ... c_{n-1}] (in descending order): ');
disp(cft_ord);

Vc = V(:,cftidx);
disp('');
disp('abs(inner product) of eigenvectors using eig() and solution');
disp(num2str(abs(QCtil'*Vc)));


% Compare the eigenvector matrices for C and Ctil:
disp('');
disp('----------------------------------------------------------');
disp('abs(inner product) of eigenvectors of C and transpose(C):');
disp(abs(QCtil'*QC));
```