EECS 455 Exam I: Fall 2016

Instructions: Print your name and sign the honor code.

Print Name: ______________________

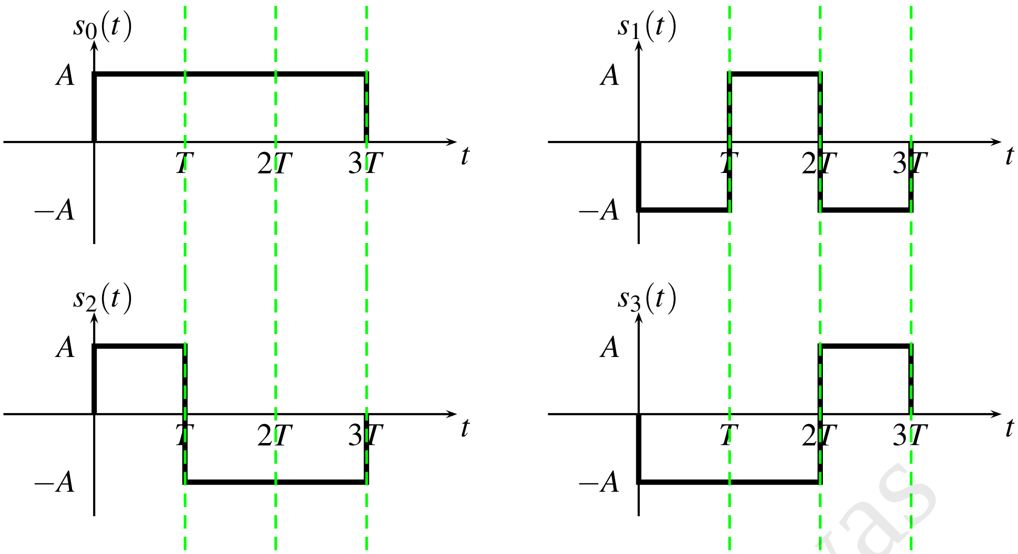
Honor Code: I have neither given nor received unauthorized aid on this examination, nor have I concealed any violations of the Honor Code

Signature ________________________

Problem 1 is worth 28 points.
Problem 2 is worth 25 points.
Problem 3 is worth 30 points.
Problem 4 is worth 30 points.
Total of 113 points.

Problem	Score
1	
2	
3	
4	
Total	

1. Consider the four signals shown below used to transmit two bits over an additive white Gaussian noise channel with power spectral density $N_0/2$.



(a) Find the energy E of each signal in terms of A and T.

Solution:

$$E_i = \int |s_i(t)|^2 dt = 3A^2T, \ i = 0,1,2,3.$$

(b) Find the energy E_b per bit of information that can be commuicated using these signals in terms of A and T.

Solution:

$$E_b = E_i/2 = \frac{3A^2T}{2}.$$

(c) Find a set of three orthonormal signals for which the signals $s_0(t)$, $s_1(t)$, $s_2(t)$, and $s_3(t)$ can be represented as a linear combination. That is find $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$ such that

$$s_i(t) = \sum_{k=0}^{2} s_{i,k}\phi_k(t).$$

Solution:

$$\begin{aligned}
\phi_0(t) &= \frac{1}{\sqrt{T}} p_T(t) \\
\phi_1(t) &= \frac{1}{\sqrt{T}} p_T(t-T) \\
\phi_2(t) &= \frac{1}{\sqrt{T}} p_T(t-2T)
\end{aligned}$$

$$
\begin{aligned}
s_0(t) &= +A\sqrt{T}\phi_0(t) + A\sqrt{T}\phi_1(t) + A\sqrt{T}\phi_2(t) \\
s_1(t) &= -A\sqrt{T}\phi_0(t) + A\sqrt{T}\phi_1(t) - A\sqrt{T}\phi_2(t) \\
s_2(t) &= +A\sqrt{T}\phi_0(t) - A\sqrt{T}\phi_1(t) - A\sqrt{T}\phi_2(t) \\
s_3(t) &= -A\sqrt{T}\phi_0(t) - A\sqrt{T}\phi_1(t) + A\sqrt{T}\phi_2(t)
\end{aligned}
$$

(d) Find the rate of communication in terms of the number of bits/dimension that is communicated.

Solution: Rate =2 bits/3 dimensions.

(e) Find the pairwise distance between a pair (any pair) of distinct signals in terms of A and T.

Solution:

$$
\begin{aligned}
d^2(s_0, s_1) &= \int_0^{3T} (s_0(t) - s_1(t))^2 dt \\
&= \int_0^{T} (2A)^2 dt + \int_{2T}^{3T} (2A)^2 dt \\
&= (2A)^2 T + (2A)^2 T \\
&= 8A^2 T \\
d(s_0, s_1) &= \sqrt{8T}A
\end{aligned}
$$

(f) Find the ratio of the squared distance between a pair of distinct signals and the energy per bit.

Solution:

$$
\begin{aligned}
\frac{d^2(s_0, s_1)}{E_b} &= \frac{8A^2T}{3A^2T/2} \\
&= \frac{16}{3}
\end{aligned}
$$

(g) Determine the pairwise error probability between signal s_0 and s_1. That is, what is the probability that the received signal is closer to s_1 given that s_0 is transmitted. Express your answer in terms of the Q function and E_b/N_0.

Solution:

$$
\begin{aligned}
P(s_0 \to s_1) = Q(\frac{d(s_0, s_1)}{2\sigma}) \\
= Q(\frac{\sqrt{8TA^2}}{2\sqrt{N_0/2}})
\end{aligned}
$$

$$= Q(\sqrt{\frac{8TA^2}{2N_0}})$$
$$= Q(\sqrt{\frac{4TA^2}{N_0}})$$
$$= Q(\sqrt{\frac{4(2E_b/3)}{N_0}})$$
$$= Q(\sqrt{\frac{8E_b}{3N_0}})$$

2. Consider a commuication system that transmits one of four signals using one orthonormal waveform.

$$\begin{aligned} s_0(t) &= -3A\psi(t) \\ s_1(t) &= -A\psi(t) \\ s_2(t) &= +A\psi(t) \\ s_3(t) &= +3A\psi(t) \end{aligned}$$

where $\int |\psi(t)|^2 dt = 1$. The received signal is the transmitted signal plus white Gaussian noise with power spectral density $N_0/2$. Let H_i be the hypothesis that signal s_i is trasmitted, $i = 0, 1, 2, 3$.

(a) Determie the average energy of the four signals.

Solution:

$$E_0 = E_3 = 9A^2, \; E_1 = E_2 = A^2$$

$$\begin{aligned} \bar{E} &= \frac{E_0 + E_1 + E_2 + E_3}{4} \\ &= 5A^2. \end{aligned}$$

(b) Determine the average energy per bit of information, E_b.

Solution:

$$\begin{aligned} E_b &= \frac{\bar{E}}{2} \\ &= \frac{5A^2}{2} \end{aligned}$$

The receiver multiplies the received signal with the orthonormal signal to generate a decision statistic. Let

$$Z = \int r(t)\psi(t)dt$$

denote the receiver output.

(c) Determine variance of the noise at the output of the receiver

Solution:

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2}\int |\psi(t)|^2 dt \\
&= \frac{N_0}{2}
\end{aligned}$$

(d) The receiver decides hypothesis H_0 if $Z < -2A$. Determine the probability of error given H_0. Express your answer in terms of the Q function and E_b/N_0.

Solution:

$$\begin{aligned}
P_{e,0} &= P\{Z > -2A|H_0\} \\
&= P\{-3A+\eta > -2A|H_0\} \\
&= P\{-3A+\eta > -2A|H_0\} \\
&= P\{\eta > A|H_0\} \\
&= Q(\frac{A}{\sigma}) \\
&= Q(\sqrt{\frac{2A^2}{N_0}}) \\
&= Q(\sqrt{\frac{4E_b}{5N_0}})
\end{aligned}$$

(e) The receiver decides hypothesis H_1 if $-2A < Z < 0$. Determine the probability of error given H_1. Express your answer in terms of the Q function and E_b/N_0.

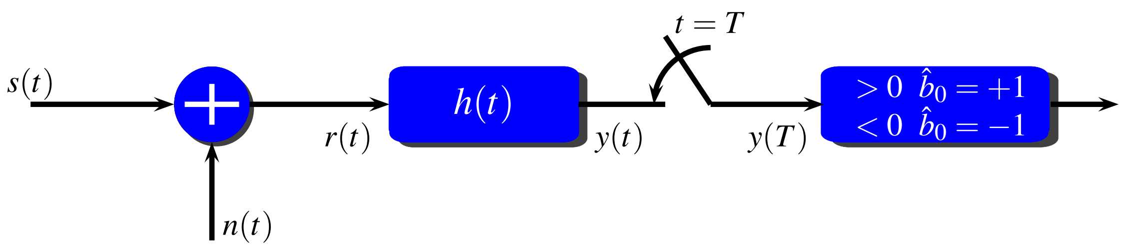
Solution:

$$\begin{aligned}
P_{e,1} &= P\{Z < -2A OR Z > 0|H_0\} \\
&= P\{Z < -2A|H_0\} + P\{Z > 0|H_0\} \\
&= P\{-A+\eta < -2A\} + P\{-A+\eta > 0|H_0\} \\
&= P\{\eta < -A\} + P\{\eta > A|H_0\} \\
&= 2Q(\frac{A}{\sigma}) \\
&= 2Q(\sqrt{\frac{4E_b}{5N_0}})
\end{aligned}$$

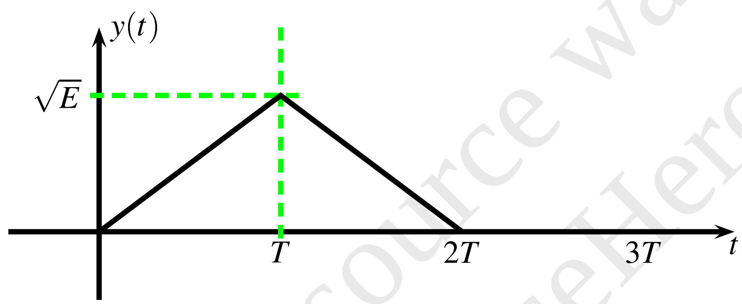
3. A communication system transmits a bit of information b_0 using a single pulse of duration T.

$$s(t) = b_0\sqrt{P}p_T(t).$$

At the receiver a filter with impulse response $h(t) = \frac{1}{\sqrt{T}}p_T(t)$ is used and is sampled at time T.



The output of the filter (without noise) as a function of time due to the transmitted signal if $b_0 = +1$ is shown below.

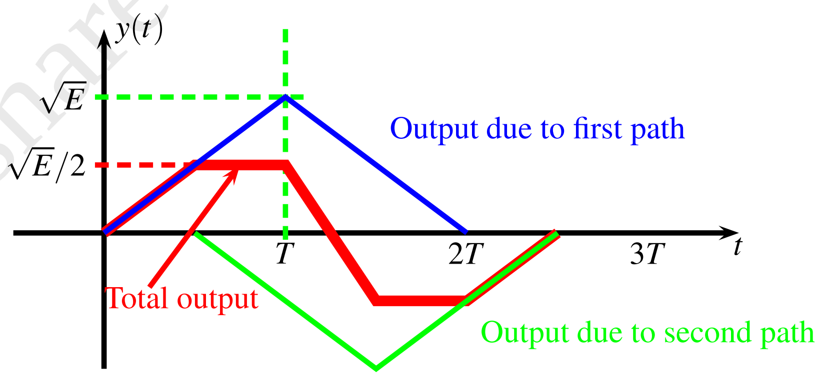


Now consider that there is a multipath channel between the transmitter and receiver. Now the received signal is $r(t) = s(t) - s(t - T/2) + n(t)$ where $n(t)$ is the additive white Gaussian noise with power spectral density $N_0/2$. That is, the transmitted signal is received twice: the first time as above and a second time corresponding to a delay of $T/2$ and the sign changed.

(a) Determine (e.g. draw below) the output of the filter due to a single pulse (no noise) with $b_0 = +1$ and the multipath channel. Be sure to label all relevant points in the diagram.
Solution: See below.

(b) Determine the output at time T.

Solution: $y(T) = \sqrt{E}/2$.



(c) Now consider the effect with noise. Determine the probaiblity of error with and without multipath. Your answers should be in terms of the Q function and E_b/N_0.

Solution: Without multipath

$$\begin{aligned}
P_e &= P\{y(T) < 0|b_0 = +1\} \\
&= P\{\sqrt{E} + \eta < 0|b_0 = +1\} \\
&= P\{\eta < -\sqrt{E}\} \\
&= Q(\frac{\sqrt{E}}{\sigma}) \\
&= Q(\frac{\sqrt{2E}}{N_0}).
\end{aligned}$$

With multipath

$$\begin{aligned}
P_e &= P\{y(T) < 0|b_0 = +1\} \\
&= P\{\sqrt{E}/2 + \eta < 0|b_0 = +1\} \\
&= P\{\eta < -\sqrt{E}/2\} \\
&= Q(\frac{\sqrt{E}/2}{\sigma}) \\
&= Q(\frac{\sqrt{E}}{2N_0}).
\end{aligned}$$

4. (a) A communication system using BPSK requires an error probability of $P_e = Q(\sqrt{20})$. The noise is additive white Gaussian noise with power spectral density $N_0/2$ with $N_0 = 4\times10^{-21}$ W/Hz. The received power is $P = 4\times10^{-12}$ Watts. The (null-to-null) bandwidth is $W = 25$MHz. Find the maximum data rate that can achieve the given error probability within the given bandwidth.

Solution:

$$\begin{aligned}
\frac{E_b}{N_0} &= 10 \\
\frac{PT}{N_0} &= 10 \\
\frac{P}{10N_0} &= \frac{1}{T} \\
R &= \frac{1}{T} = \frac{P}{10N_0} \\
&= \frac{4\times10^{-12}}{10(4\times10^{-21}} \\
&= \frac{\times10^{-12}}{\times10^{-20}}
\end{aligned}$$

$$
\begin{aligned}
&= 10^8 \\
&= 100 \text{ Mbps}
\end{aligned}
$$

However, because $W = 25$ MHz the data rate is limited by the bandwidth, not the energy. Thus the largest data rate is 12.5 Mbps which has a null-to-null bandwidth of 25MHz.

(b) Compare the E_b/N_0 of part (a) with the minimal E_b/N_0 for the same bandwidth efficiency (R/W) with any modulation scheme.

Solution:

There are two possible ways of doing this problem. The first is to assume that in the above system the duration of the signals is reduced to meet the bandwidth requirement and then the power level is reduced to maintain the same $E_b/N_0 = 10$. In this case the bandwidth efficiency of part (a) is $R/W = 0.5$. The minimum E_b/N_0 for this bandwidth efficiency is

$$
\begin{aligned}
E_b/N_0 &> \frac{2^{R/W} - 1}{R/W} \\
&= \frac{2^{1/2} - 1}{1/2} \\
&= 0.83 \\
&= -.82\text{dB}
\end{aligned}
$$

Compared with part (a) which has $E_b/N_0 = 10 = 10$dB this is about 10.82 smaller E_b/N_0. The second way to do this problem is to assume the power level stays the same in part (a) but since the transmission time is 8 times longer the energy per bit goes up by a factor of 8 (9.03 dB) and now $E_b/N_0 = 19.03$ dB. The error probability will be much less than what is required from part (a). With this assumption then optimal system would require 19.8dB more sigal-to-noise ratio.