EECS 455 Exam III: Fall 2020

Instructions:
Print your name and sign the honor code.

Name ____________________

Honor code: _____________________

Problem	Points	Score
1	30	
2	40	
3	30	
Total	100	

1. Consider the finite field with three elements $\{0, 1, 2\}$ with addition and multiplication modulo 3. Over this field the equation $x^2 + x + 2 = 0$ which is equivalent to $x^2 = 2x + 1$ does not have a solution. Let α be the solution to this equation. The elements of the field are listed below.

$\alpha^{-\infty}$	$0\alpha + 0$
α^0	$0\alpha + 1$
α^1	$1\alpha + 0$
α^2	$2\alpha + 1$
α^3	$2\alpha + 2$
α^4	2
α^5	2α
α^6	$\alpha + 2$
α^7	$\alpha + 1$
α^8	1

The generator polynomial for the RS code with 6 information symbols $\in$ GF(9) and 8 code symbols $\in$ GF(9) is

$$g(x) = (x - \alpha)(x - \alpha^2)$$

(a) How many symbol errors can this code correct.

Solution: The minimum distance of the code is $d = N - K + 1 = 3$ which means it can correct 1 symbol error.

(b) How many received vectors are within the error correcting capability of each codeword.

Solution: The received vectors within the error correcting capability of each codeword includes the codeword itself and all vectors distance 1 from the codeword. There are 8*8=64 vectors that are distance 1 corresponding to the 8 positions and 8 possible symbols different from the codeword. So a total of 65 received vectors distance 1 or less from a codeword.

(c) How many received vectors within the error correcting capability of any codeword.

Solution: There are 9^6 codewords so the total number of vectors distance 1 or less is 34,543,655.

(d) Determine the fraction of received vectors within distance 1 of any codeword.

Solution: The total number of possible received vectors is $9^8 = 43,046,721$ Thus the fraction that are in the decoding region of a codeword is $34543655/43046721 = 0.802$

(e) The received polynomial is

$$r(x) = x^3 + x + 2\alpha + 2$$

Determine the codeword within distance 1 of this received polynomial.

Solution:

$$
\begin{aligned}
S_1 &= r(\alpha) = \alpha^3 + \alpha + 2\alpha + 2 \\
&= (2\alpha + 2) + \alpha + 2\alpha + 2 \\
&= 2\alpha + 1 \\
&= \alpha^2 \\
S_2 &= r(\alpha^2) = \alpha^6 + \alpha^2 + 2\alpha + 2 \\
&= (\alpha + 2) + 2\alpha + 1 + 2\alpha + 2 \\
&= 2\alpha + 2 \\
&= \alpha^3 \\
r(x) &= c(x) + e(x) \\
S_1 &= r(\alpha) \\
&= c(\alpha) + e(\alpha) \\
&= e_j \alpha^j \\
S_2 &= r(\alpha^2) = c(\alpha^2) + e(\alpha^2) \\
&= e_j \alpha^{2j} \\
\alpha^j &= \frac{S_2}{S_1} \\
\alpha^j &= \frac{\alpha^3}{\alpha^2} \\
&= \alpha^1 \\
j &= 1 \\
e_j &= S_1/\alpha \\
&= \alpha^2/\alpha \\
&= \alpha \\
e(x) &= \alpha x \\
\hat{c}(x) &= r(x) - e(x) \\
&= r(x) = x^3 + x + 2\alpha + 2 - \alpha x \\
&= x^3 + (2\alpha + 1)x + 2\alpha + 2 \\
&= x^3 + \alpha^2 x + 2\alpha + 2
\end{aligned}
$$

2. THIS PROBLEM IS SPLIT OVER TWO PAGES. BE SURE TO ANSWER ALL 10 PARTS.

Consider the (6,3) linear code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

This is the same code as in the notes. You can do calculations in Matlab but you can also do them manually. You can also use the Standard Array shown in Table 1.

s	Coset Leader							
000	000000	100101	010111	110010	001011	101110	011100	111001
101	100000	000101	110111	010010	101011	001110	111100	011001
111	010000	110101	000111	100010	011011	111110	001100	101001
011	001000	101101	011111	111010	000011	100110	010100	110001
100	000100	100001	010011	110110	001111	101010	011000	111101
010	000010	100111	010101	110000	001001	101100	011110	111011
001	000001	100100	010110	110011	001010	101111	011101	111000
110	101000	001101	111111	011010	100011	000110	110100	010001

Table 1: Standard array for the (6,3) code

(a) Determine the minimum distance, rate and error correcting capability of this code.

Solution: The code has minimum distance 3, rate 1/2 and can correct any pattern of one error or less.

(b) Suppose the codewords are mapped to the alphabet +1, -1 with the usual mapping ($0 \to +1, 1 \to -1$) and a codeword is transmitted over the additive white Gaussian noise channel. Suppose that the received vector is $r_1 = (-1,-4,1,-3,2,-2)$. Determine the codeword most likely to have caused this received vector.

Solution: By correlating with each of the 8 vectors the correlations are

$$\begin{aligned} (r,s_0) &= -7 \\ (r,s_1) &= +5 \\ (r,s_2) &= +7 \\ (r,s_3) &= -1 \\ (r,s_4) &= -9 \\ (r,s_5) &= -5 \\ (r,s_6) &= 5 \\ (r,s_7) &= 5 \end{aligned}$$

The largest correlation is with signal s_2. This corresponds to codeword $c_2 = (010111)$

(c) Suppose that hard decisions are made of the received vector of part (e). That is, $\hat{r}_1(i) = 0$ if $r_1(i) > 0$ and $\hat{r}_1(i) = 1$ if $r_1(i) < 0$. So $\hat{r}_1 = (1,1,0,1,0,1)$ for the received vector of part (b). Determine the codeword most likely to have caused the hard decision version of the received vector. (Assume errors on each coded bit have probability less than 1/2).

Solution: The decoder finds the vector closest in Hamming distance to $\hat{r}$. The Hamming distance are

$$\begin{aligned} d_H(\hat{r}, c_0) &= 4 \\ d_H(\hat{r}, c_1) &= 1 \\ d_H(\hat{r}, c_2) &= 2 \\ d_H(\hat{r}, c_3) &= 3 \\ d_H(\hat{r}, c_4) &= 5 \\ d_H(\hat{r}, c_5) &= 4 \\ d_H(\hat{r}, c_6) &= 3 \\ d_H(\hat{r}, c_7) &= 2 \end{aligned}$$

The codeword with the smallest Hamming distance is $c_1 = (100101)$. Equivalently the Standard Array has the vector (1, 1, 0, 1, 0, 1) in the decoding region of the codeword at the top of the column in which it appears.

(d) For the same hard decision vector of part (c) what would be the output bounded distance decoder?

Solution: The bounded distance decoder would also result in $c_1 = (100101)$ since this is the result of 1 or fewer errors.

(e) Suppose now that the received vector is $r_2 = (-1,-4,-1,-3,-2,-2)$ What is the output of the optimum receiver (soft decisions decoder).

Solution: In this case the correlations are

$$\begin{aligned} (r, s_0) &= -13 \\ (r, s_1) &= -1 \\ (r, s_2) &= +9 \\ (r, s_3) &= +1 \\ (r, s_4) &= -3 \\ (r, s_5) &= +1 \\ (r, s_6) &= +3 \\ (r, s_7) &= +3 \end{aligned}$$

The largest correlation is still with signal s_2. This corresponds to codeword $c_2 = (010111)$

(f) Suppose that hard decisions are made on the received vector of part (b). That is, $\hat{r}_2(i) = 0$ if $r_2(i) > 0$ and $\hat{r}_2(i) = 1$ if $r_2(i) < 0$. So $\hat{r}_2 = (1,1,1,1,1,1)$ for the received vector of part (e). Determine the codeword(s) most likely to have caused the hard decision version of the received signal.

Solution: In this case there is a three way tie. There are 3 codewords that are distance 2 to the received vector after hard decisions. These codewords are (010111), (101110), and (111001). The optimum receiver (after hard decisions) could choose any of these three codewords.

(g) For the same hard decision vector of part (f) what would be the output of the bounded distance decoder.

Solution: A bounded distance decoder would output a failure. That is no codeword is distance 1 or less from the received vector.

(h) For the additive white Gaussian noise channel what is the union bound on the codeword error probability for the optimum (soft decision) receiver. That is, the BI-AWGN channel. Express your answer in terms of E_b/N_0 in dB and the Q function. As usual, E_b is the energy per *information* bit and $N_0/2$ is the noise power spectral density.

Solution: The weight enumerator for this code is $A_0 = 1, A_3 = 4, A_4 = 3$. So the union bound for AWGN is

$$P_e \leq 4Q(\sqrt{\frac{2E_b r3}{N_0}}) + 3Q(\sqrt{\frac{2E_b r4}{N_0}})$$

where $r = 1/2$.

(i) For the binary symmetric channel with crossover probability p, what is the probability that the bound distance decoder choose the correct codeword. Express your answer in terms of p.

Solution: The probability of correct is just the probability of one or fewer errors.

$$P_e = \sum_{m=0}^{1} \binom{6}{1} p^m (1-p)^{6-m}.$$

(j) This part is just to make the number of parts easily divisible by 10. Which of the following are true:

(i) Shannon was a distant relative of Thomas Edison.
(ii) Shannon built a rocket powered Frisbee.
(iii) Shannon built a juggling clown.
(iv) Shannon built a Roman Numeral calculator (THROBAC).
(v) Shannon built a maze solving mouse.
(vi) Shannon suspended a student from his feet (upside down) to investigate bounce juggling while upside down.
(vii) **All of the above**

3. (a) Consider the convolutional code with encoder shown below. Assume that the code is truncated at depth 4 so that four information bits enter the encoder (which is initially in the all zero state) followed by two zeroes to clear out the shift register. Assume the codeword generated consists of the top output of the encoder followed by the bottom output for each input bit. That is, the order of transmitted bits is $c_0^{(0)}, c_0^{(1)}, c_1^{(0)}, c_1^{(1)}, c_2^{(0)}, c_2^{(1)}, \ldots$. The coded information is punctured by not transmitting one coded bit out of every four encoded bits. That is, if the input to the encoder is the sequence 1 0 1 1 0 0 where the last two bits represent the tail bits then the output of the encoder is 11 01 00 10 10 11. However, after puncturing the output becomes 11 0x 00 1x 10 1x where x represents a bit that is not transmitted.

(a) Determine the exact rate of the truncated and punctured code taking into account the tail bits and the puncturing.

Solution: Four information bits are transmitted using one of 16 codewords of length 9. Thus the exact rate of the code is 4/9.

(b) Find the Hamming distance between the codeword corresponding to the input sequence 0 0 0 0 0 0 and the codeword corresponding to the input sequence 1 0 0 0 0 0 where the last two bits represent the tail bits. That is, determine the Hamming distance between the codewords after puncturing.

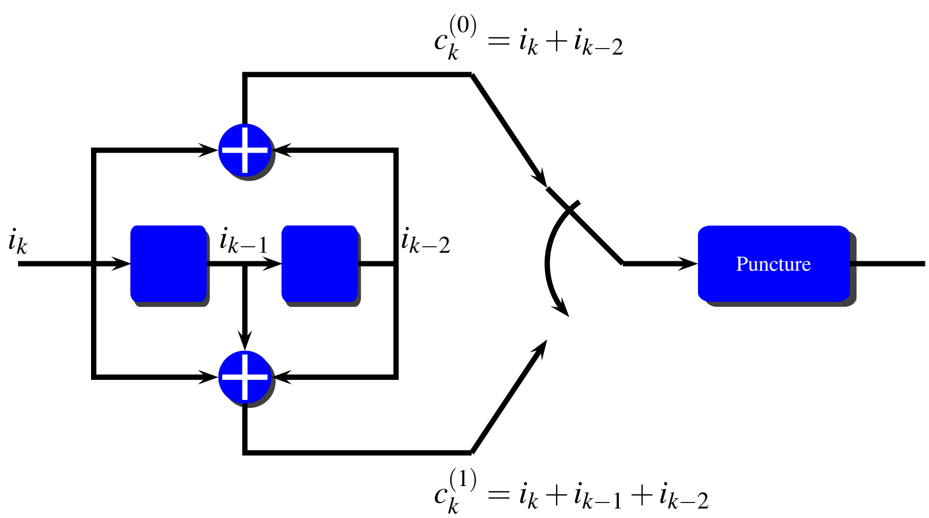
Solution: The output of the encoder due to the input sequence 0000 00 is 000000000000. After puncturing the output is 000x000x000x. The output of the encoder due to the input sequence 1000 00 is 110111000000. After puncturing the output becomes 110x110x000x. The distance between the two codewords after puncturing is 4 (it was 5 before puncturing).

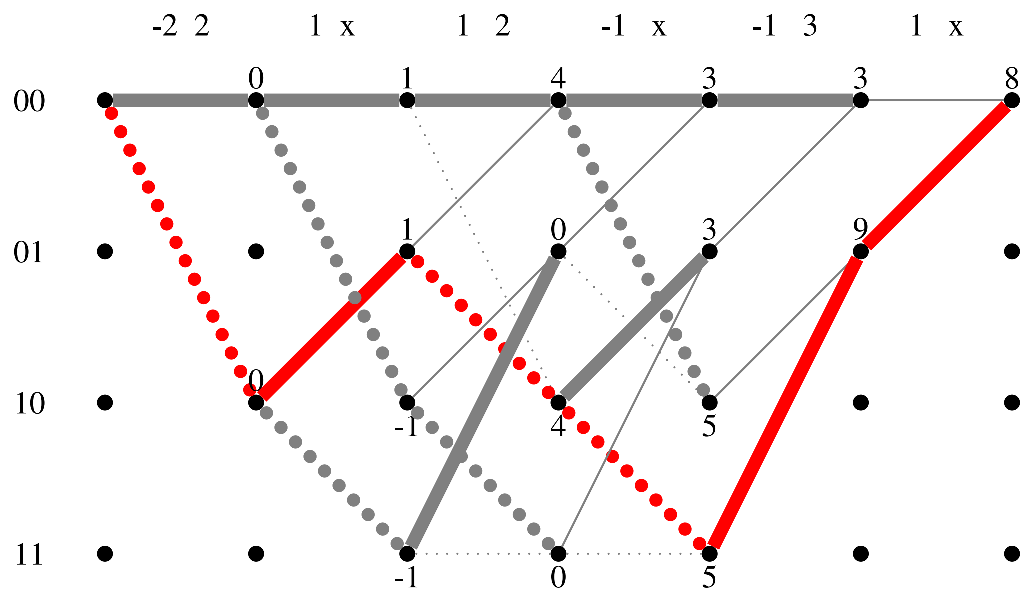
(c) Assume the modulator maps $0 \to +1$ and $1 \to -1$. Find the squared Euclidean distance between the same two codewords after they are mapped to ± 1.

Solution: The squared Euclidean distance is 4 $\times$ the Hamming distance. So the squared Euclidean distance is 16.

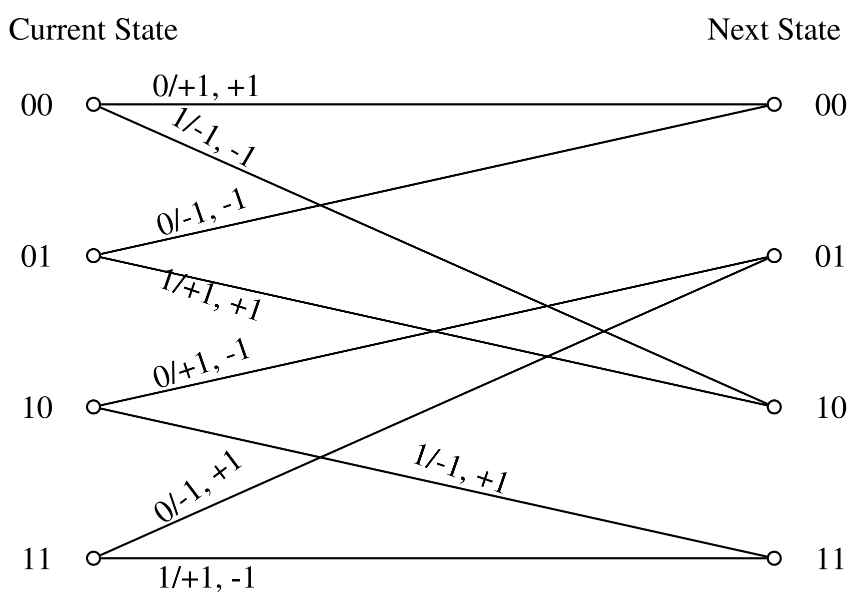
(d) Decode the following sequence received over the AWGN.

$$r = (-2, 2, 1, x, 1, 2, -1, x, -1, 3, 1, x)$$

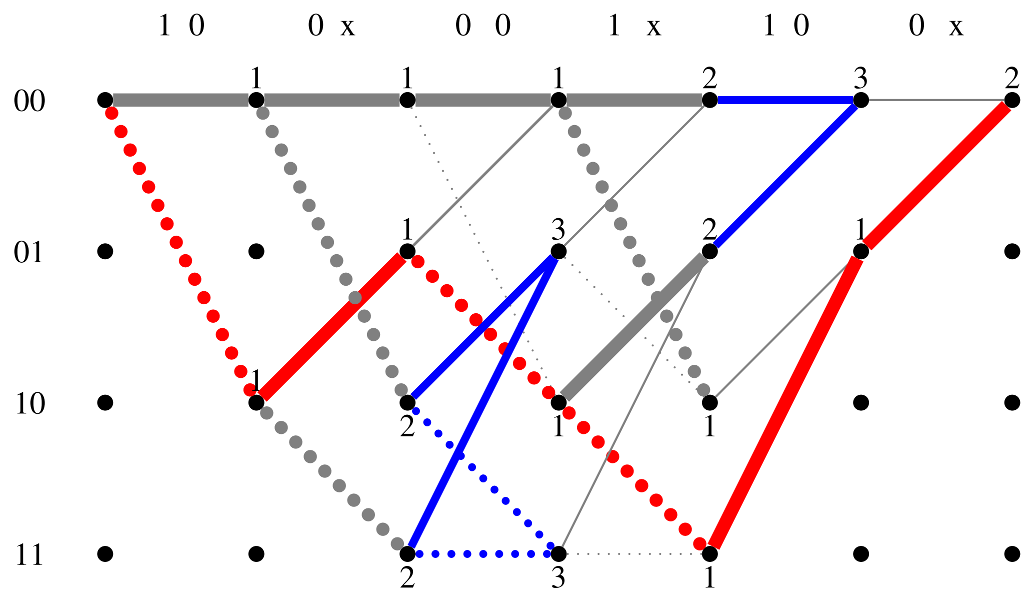




Dotted lines correspond to the encoder input being 1 while gray lines correspond to the input being 0. Below is the state transition diagram of the convolutional code without puncturing but with the data mapped to ± 1.



So the four information bits would be 1011.



So the four information bits would still be 1011.