Last (family) name:________________

First (given) name:________________

I have neither given nor received aid on this examination, nor concealed any violation of the Honor Code. The only online resources I have used during this exam are those listed below as well as the following sites (list URL and exam problem #):

Signature: ________________

ID Number: ________________

EECS 551 Midterm 3, 2020-11-19 EST (24 hour online)

- There are 6 problems for a total of 70 points.
- This part of the exam has 8 pages. Make sure your copy is complete.
- This is a 24-hour "online" exam. Many of the exam questions will be on Canvas; this pdf has additional questions that you should answer by submitting to gradescope unless instructed otherwise in each problem.
- This is an "open book" exam. During the exam, you may use all of the course materials on the Canvas EECS 551 site including the course notes on google drive, as well as wikipedia, the built-in JULIA help, and the online JULIA manual. If you use any other resources for solving the CanvasQuiz questions, then you must cite the source along with your honor code statement above. *If you use any other resources for solving the gradescope questions, then cite the source as part of your solution submitted to gradescope.* Be sure to *sign* the honor code above and scan (e.g., photograph) the top part of this page and submit to gradescope.
- You may use without rederiving any of the results presented in the course notes.
- You must complete the exam entirely on your own.
- If you need an exam question to be clarified, post a private question to the instructors on piazza.
- Clearly $\boxed{\text{box}}$ your final answers. For full credit, show your complete work clearly and legibly. Answers must be submitted properly to gradescope to earn credit.
- For multiple-choice questions, select *all* correct answers.
- To "disprove" any statement, provide a concrete counter-example. For maximum credit, make that counter-example as small and simple as possible, *e.g.*, having the smallest possible matrix dimensions and using the simplest numbers like "0" and "1" as much as possible. For example, to disprove the statement "any square matrix $\boldsymbol{A}$ is invertible," the smallest and simplest counter-example is the 1×1 matrix $\boldsymbol{A} = [0]$.

autograder instructions

For any problem involving writing a JULIA function, carefully follow these instructions.

- Submit a mathematical explanation of your solution to gradescope.
- Submit a readable screenshot of your code to gradescope for grading *efficiency*.
  Solutions must be as efficient as possible to earn full credit for any problem involving JULIA.
- Test your code thoroughly *before* submitting, to maximize credit earned.
  (Passing on the 1st or 2nd submission will earn full credit; partial credit will decrease rapidly after that.)
- Submit your tested code by email attachment to `eecs551@autograder.eecs.umich.edu` as usual.
  Incorporate any necessary `using` statements.
  Name your function correctly.
  Name the file attachment like `thefunction.jl` where `thefunction` is the function name.

For any prohibited function, you may not use the "in place" version either. For example, if `sort` is not allowed, then the corresponding in-place version `sort!` is also not allowed.

[7] **1**. Determine $\|\boldsymbol{A}\|_2$ when $\boldsymbol{A}^+ = \boldsymbol{A}'$ and $\mathsf{rank}(\boldsymbol{A}) \geq 1$. (Use SVD components of $\boldsymbol{A}$ as needed and explain.)

[15] **2**. Complete the following JULIA function so that it returns all the eigenvalues of the matrix $(\boldsymbol{X}'\boldsymbol{X} + \beta \boldsymbol{I})^+$ given any matrix $\boldsymbol{X} \in \mathbb{F}^{M \times N}$ and any $\beta \geq 0$. Your code must return a vector of the eigenvalues in *ascending* order.
To earn full credit, the code must be as efficient as possible, and should work for both real and complex inputs. You may not use (or replicate) any of the functions `eigen`, `eigvals`, `eigvecs`, `svd`, `inv`, `pinv`, "\", `sort`, `sortperm`, or `sortslices`. You may use any function *not* listed here. Hint: your code may need an `if` statement or equivalent.
Template:

```
"""
    e = eigtiki(X, beta)
Return the eigenvalues in ascending order of `(X'X + beta I)^+` for `β ≥ 0`,
without calling (or duplicating) functions listed in the problem statement.
The output should be a Vector.
"""
function eigtiki(X::AbstractMatrix, beta::Real)::Vector
```

See submission instructions near front of exam.

[9] **3**. Let $\boldsymbol{Y}$ be a 6×7 matrix with singular values $0, 2, 3, 8, 9, 13$ and define

$$\hat{\boldsymbol{X}} = \underset{\boldsymbol{X} \in \mathbb{R}^{6\times 7}}{\arg\min} \frac{1}{2} \|\boldsymbol{X} - \boldsymbol{Y}\|_{\mathrm{F}}^2 + 8 \|\boldsymbol{X}\|_*$$

$$\hat{\boldsymbol{Z}} = \underset{\boldsymbol{Z} \in \mathbb{R}^{6\times 7}}{\arg\min} \frac{1}{2} \|\boldsymbol{Z} - \boldsymbol{Y}\|_{\mathrm{F}}^2 + 8 \, \mathsf{rank}(\boldsymbol{Z}) \, .$$

Determine the ratio $\|\hat{\boldsymbol{X}} - \boldsymbol{Y}\|_* / \|\hat{\boldsymbol{Z}} - \boldsymbol{Y}\|_*$. Explain.

[15] **4**. Let $\boldsymbol{A} \in \mathbb{F}^{N \times N}$ denote a circulant matrix and define

$$\hat{\boldsymbol{A}}_K \triangleq \underset{\boldsymbol{B} \in \mathbb{F}^{N \times N} : \, \mathsf{rank}(\boldsymbol{B}) \leq K}{\arg\min} \|\boldsymbol{A} - \boldsymbol{B}\|_{\mathrm{F}}$$

for $K \in \{0, 1, 2, \ldots\}$. Complete the following function so that it returns $\hat{\boldsymbol{A}}_K$ given $\boldsymbol{A}$ and K, *without* calling or duplicating `svd` or `eigen` or their relatives.
Template:

```
"""
    Ah = lr_circ(A, K::Int)

Compute the rank-at-most-`K` best approximation to circulant matrix `A`

In:
- `A` : `N × N` circulant matrix
- `K` : rank constraint (nonnegative integer: 0, 1, 2, ...)

Out:
- `Ah` : `N × N` best approximation to `A` having rank ≤ `K`
"""
function lr_circ(A, K::Int)::Matrix
```

Hint. To make complex circulant test matrices, you may use
`using SpecialMatrices: Circulant`
`Circulant(rand(ComplexF64, N))`
See submission instructions near front of exam.

[9] **5**. A $M \times N$ matrix $\boldsymbol{B}$ has nonzero singular values $\{1, 2, 4, 8, 16, 24\}$.
Determine $\|\hat{\boldsymbol{B}} - \boldsymbol{B}\|_{\mathrm{F}}^2$, where $\hat{\boldsymbol{B}} \triangleq \underset{\boldsymbol{X} \in \mathbb{C}^{M \times N}}{\arg\min} \, \frac{1}{2} \|\boldsymbol{X} - \boldsymbol{B}\|_{\mathrm{F}}^2 + \frac{9}{2} \, \mathsf{rank}(\boldsymbol{X})$. Explain.

[15] **6**. Consider the set of $N \times M$ Parseval tight frames: $\mathcal{T} = \left\{ \boldsymbol{T} \in \mathbb{F}^{N \times M} \;:\; \boldsymbol{T} \text{ is a Parseval tight frame} \right\}$.
Write a JULIA function that, given a $N \times M$ matrix $\boldsymbol{X}$, returns the closest (in the Frobenius norm sense) Parseval tight frame to $\boldsymbol{X}$, *i.e.*, $\hat{\boldsymbol{T}} = \underset{\boldsymbol{T} \in \mathcal{T}}{\arg\min} \|\boldsymbol{T} - \boldsymbol{X}\|_{\mathrm{F}}$. Assume $N \leq M$.

Template:

```julia
"""
    T = nearptf(X)

Find nearest (in Frobenius norm sense) Parseval tight frame to matrix `X`.

In:
* `X` : `N × M` matrix

Out:
* `T` `N × M` matrix that is the nearest Parseval tight frame to `X`
"""
function nearptf(X::AbstractMatrix)
```

See submission instructions near front of exam.

end