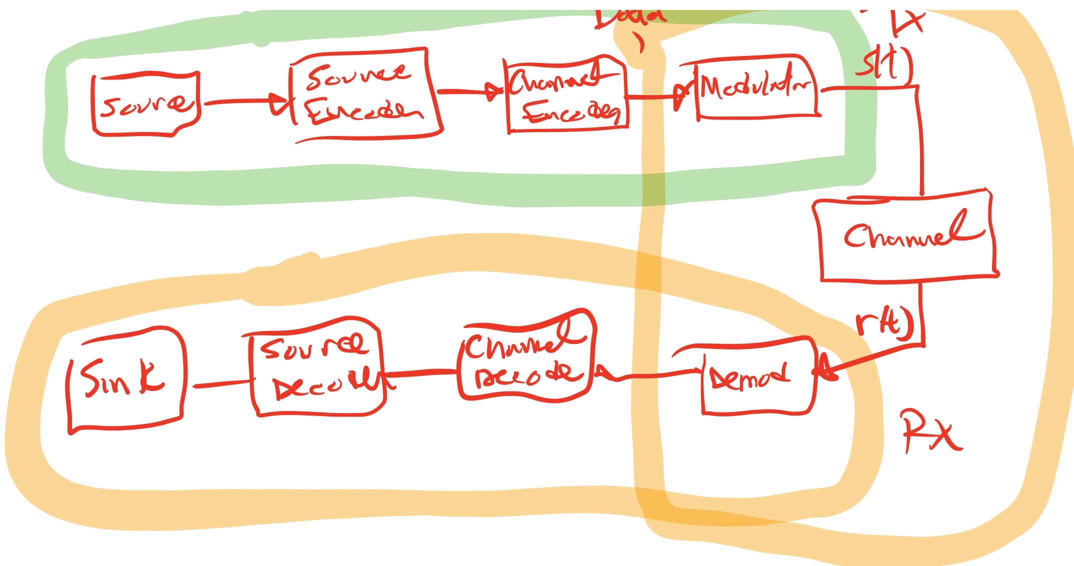


Signals as vectors

M vectors of length N (dimension) $s_0, s_1, \ldots, s_{M-1}$

$$\text{rate} = \frac{\log_2 M}{N} \text{ bits/dim}$$

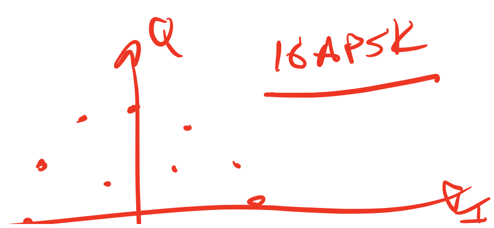
$$d_E^2(s_i, s_j) = \sum_{l=0}^{N-1} |s_{i,l} - s_{j,l}|^2$$

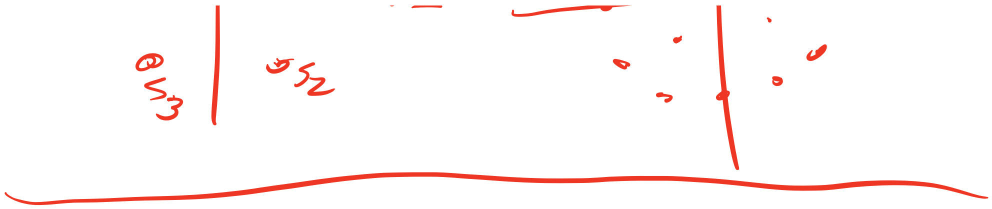
$$d_{E,min}^2 = \min_{s_i \neq s_j} d_E^2(s_i, s_j)$$

$$E_i = \sum_{l=0}^{N-1} |s_{i,l}|^2$$ energy for signal i

$$\bar{E} = \frac{1}{M} \sum_{i=0}^{M-1} E_i, \quad \bar{E}_b = \bar{E}/\log_2 M$$

$$\frac{d_{E,min}^2}{\bar{E}_b}$$ — normalized measure of distinguishability

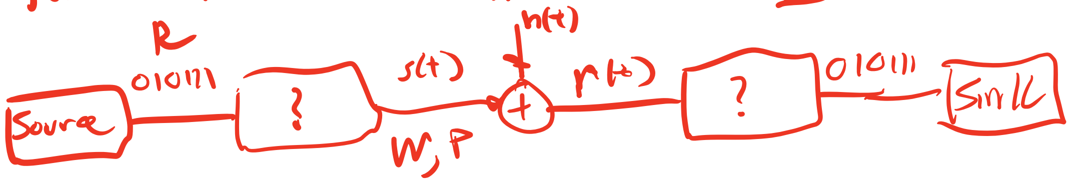




Fundamental Limits

AWGN PSD $N_0/2$

1)



R bps

Shannon $\quad R < W \log_2\left(1+\frac{P}{N_0 W}\right)$

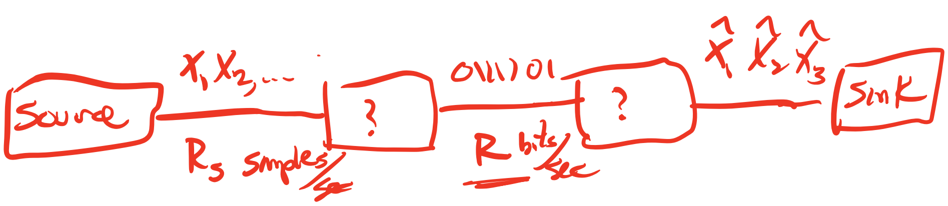
$\Rightarrow$ reliable communication possible

$R > W \log_2\left(1+\frac{P}{N_0 W}\right) \Rightarrow$ error prob is bounded above 0.

$E_b = P/R \qquad \frac{\text{joules/sec}}{\text{bits/sec}}$

$\frac{E_b}{N_0} > \frac{2^{R/W}-1}{R/W} \Rightarrow$ reliable communication possible

2)



Case 1: X_i are i.i.d. Gaussian, mean zero, variance σ^2

$\underline{D} = E[(\hat{X}_i - X_i)^2]$ — mean squared error

$R \quad \geq \quad \frac{1}{2}\log_2\left(\frac{\sigma^2}{D}\right) \quad D<\sigma^2$

$\frac{R}{R_s}$ bits/sample $\Big\}$ 0 $\qquad D > \sigma^2$

Case 2: X_i are i.i.d binary $P(X_i=0)=P(X_i=1)$

$$D = P(X_i \neq \hat{X}_i) = P_e$$

$$\underline{\frac{R}{R_s}} > \underline{1 - H_2(P_e)}$$

Combine 1) & 2)

AWGN channel + binary source

$$\underbrace{R_s(1-H_2(P_e))}_{R} < W \log_2\left(1+\frac{P}{N_0 W}\right)$$

$x(t) \to$ Energy $= \int |x(t)|^2 dt$

$x = (x_0, x_1, \ldots, x_{N-1}) \to$ Energy $\sum_{i=0}^{N-1} x_i^2$

$$(x(t), y(t)) = \int x(t) y^*(t) dt$$

$$(x, y) = \sum x_i y_i^*$$

$$d_E^2(x(t), y(t)) = \int |x(t) - y(t)|^2 dt$$

$$d_E^2(x, y) = \sum_{i=0}^{N-1} |x_i - y_i|^2$$

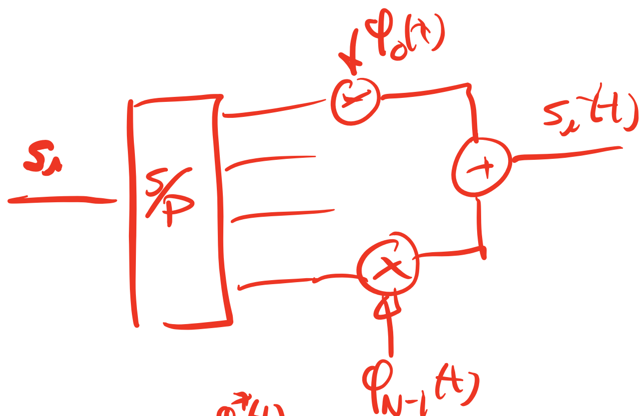
$\varphi_0(t), \dots, \varphi_{N-1}(t)$

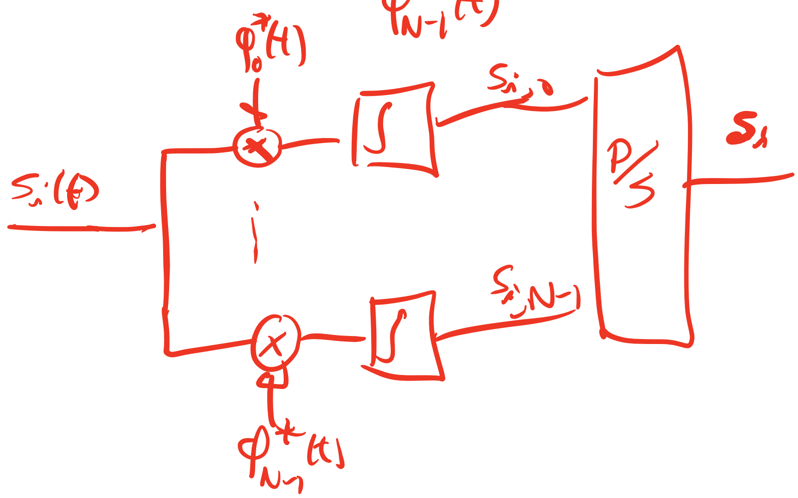
$$\int \varphi_i(t)\,\varphi_j^*(t)\,dt = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \quad \text{— orthonormal}$$

Vector $\mathbf{s}_i = (s_{i,0}, \dots, s_{i,N-1}) \quad i = 0, 1, \dots, M-1$

Waveform $\quad s_i(t) = \sum_{l=0}^{N-1} s_{i,l}\,\varphi_l(t)$

$$\underbrace{d_E^2(s_i(t), s_j(t))}_{\text{waveforms}} = \underbrace{d_E^2(\mathbf{s}_i, \mathbf{s}_j)}_{\text{vectors}}$$

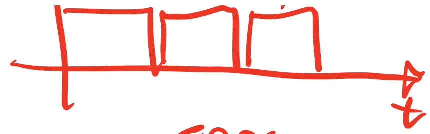


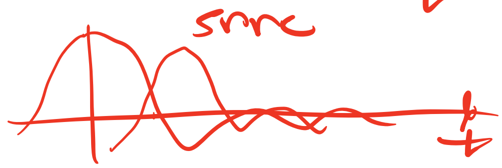


Time shifted orthonormal

$\varphi_0(t)$, $\varphi_1(t)$, $\varphi_2(t)$

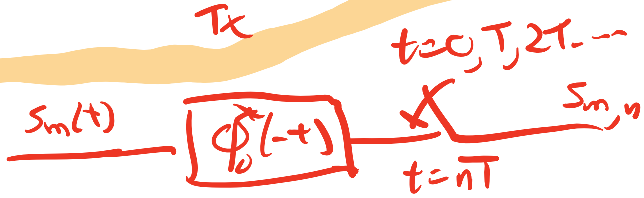
$\varphi_i(t) = \varphi_0(t - iT)$





This simplifies Tx, Rx

Tx



Rx

Time shifted orthonormal $\Longleftrightarrow$ No ISI

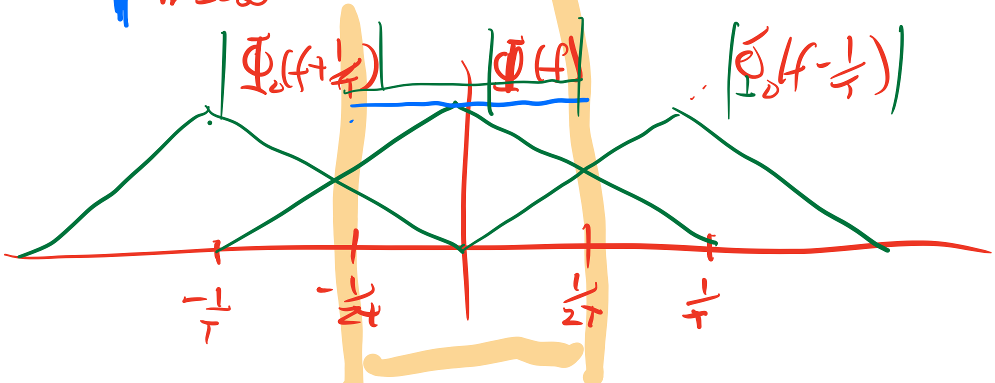
$\varphi_0(t)$, $\varphi_n(t) = \varphi_0(t-nT)$

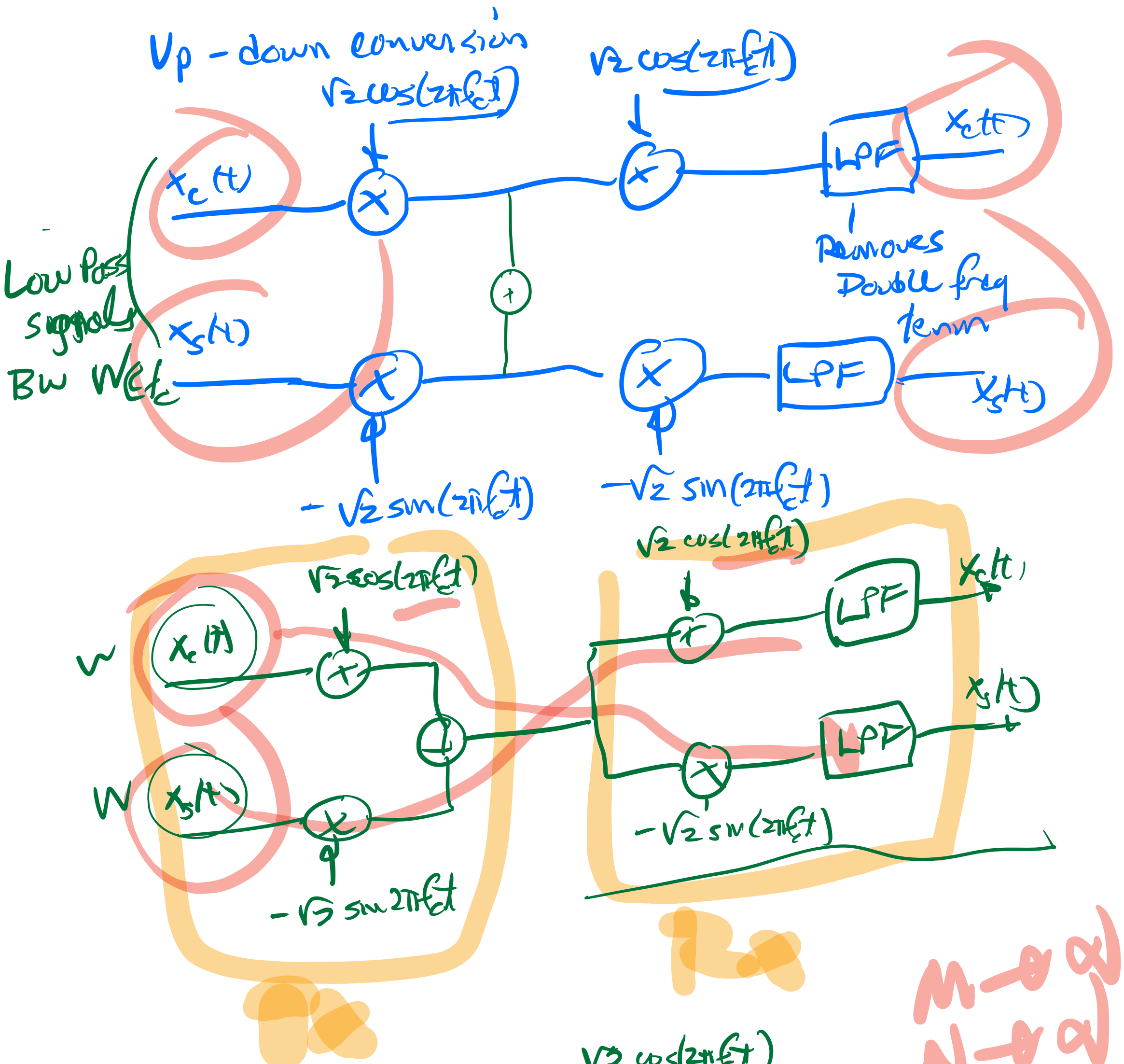
$$\int \varphi_0(t)\varphi_n^*(t)\,dt = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

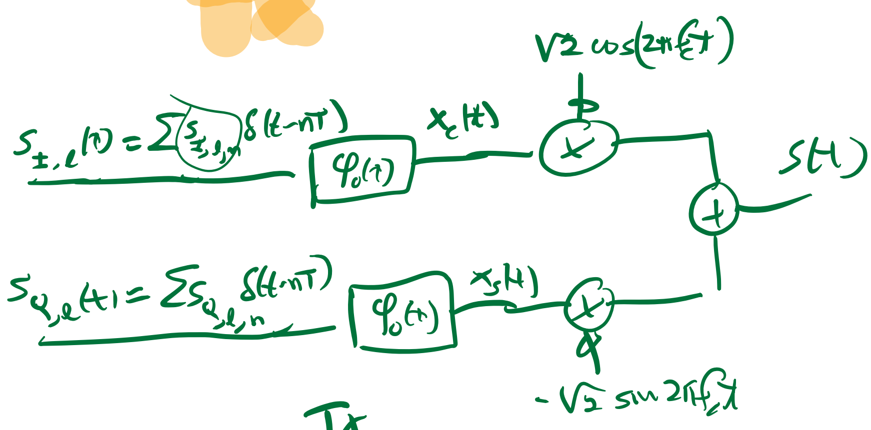
$\Phi_0(f) = \mathcal{F}\{\varphi_0(t)\}$ —.

$$\sum_{m=-\infty}^{\infty} \left|\Phi_0\left(f+\frac{m}{T}\right)\right|^2 = \begin{cases} T & -\frac{1}{2T} < f < \frac{1}{2T} \\ 0 & \text{o.w.} \end{cases}$$

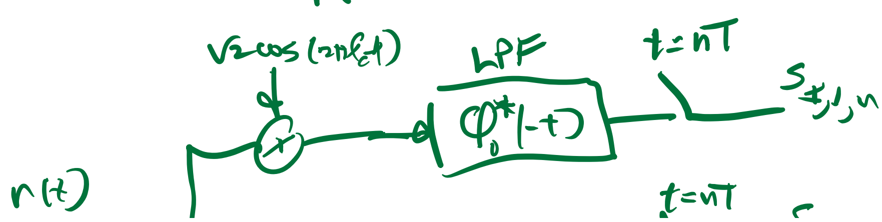
Don't care

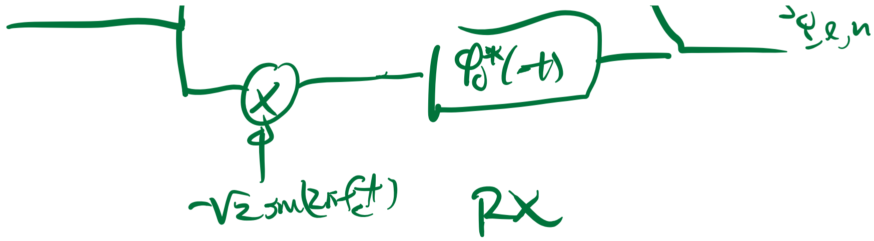



Up - down conversion
$\sqrt{2}\cos(2\pi f_c t)$
$\sqrt{2}\cos(2\pi f_c t)$
$x_c(t)$
$x_s(t)$
Low Pass signals BW $W \ll f_c$
LPF
LPF
$x_c(t)$
$x_s(t)$
Removes Double freq term
$-\sqrt{2}\sin(2\pi f_c t)$
$-\sqrt{2}\sin(2\pi f_c t)$
$\sqrt{2}\cos(2\pi f_c t)$
$\sqrt{2}\cos(2\pi f_c t)$
W
W
$x_c(t)$
$x_s(t)$
$-\sqrt{2}\sin 2\pi f_c t$
$-\sqrt{2}\sin(2\pi f_c t)$
LPF
LPF
$x_c(t)$
$x_s(t)$
Tx
Rx
$M \to \infty$
$N \to \infty$


$s_{\pm,\ell}(t) = \sum s_{\pm,\ell,n}\,\delta(t-nT)$
$\varphi_0(t)$
$x_c(t)$
$\sqrt{2}\cos(2\pi f_c t)$
$s(t)$
$s_{q,\ell}(t) = \sum s_{q,\ell,n}\,\delta(t-nT)$
$\varphi_0(t)$
$x_s(t)$
$-\sqrt{2}\sin 2\pi f_c t$

Tx

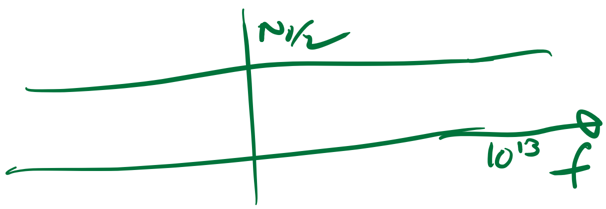

$\sqrt{2}\cos(2\pi f_c t)$
LPF
$\varphi_0^*(-t)$
$t = nT$
$s_{\pm,\ell,n}$
$r(t)$
$t = nT$

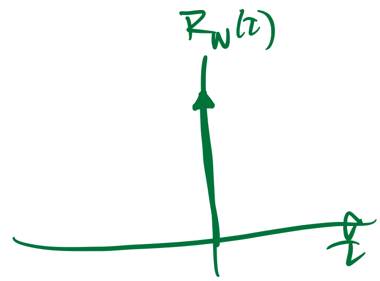


RX

<u>Noise</u> AWGN

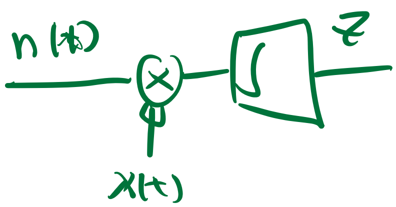
PSD $S_N(f)$





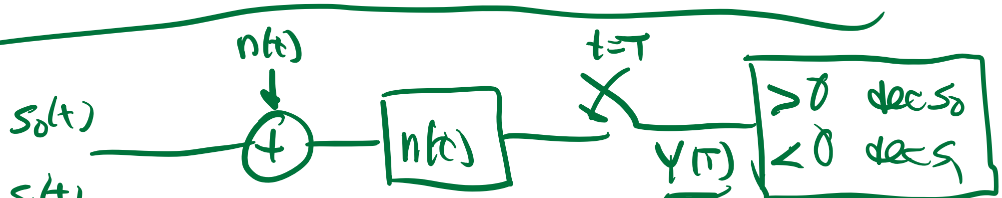
$\text{mean}(Y(t)) = 0$

$\text{variance}(Y(t)) = \frac{N_0}{2}\int h^2(t)\,dt$



$\text{mean}(Z) = 0$

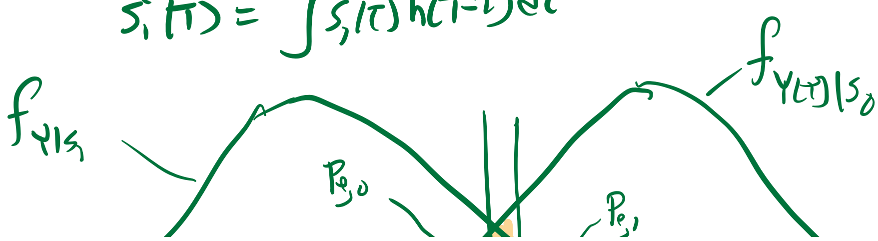
$\text{variance}(Z) = \frac{N_0}{2}\int x^2(t)\,dt$

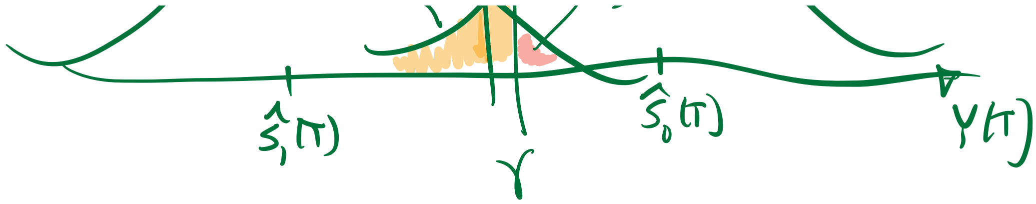


LTI

$$\hat{s}_0(T) = \int s_0(\tau)h(T-\tau)\,d\tau$$

$$\hat{s}_1(T) = \int s_1(\tau)h(T-\tau)\,d\tau$$





$\bar{P}_e = \pi_0 P_{e,0} + \pi_1 P_{e,1}$ $\pi_0 = P(s_0 \text{ trans})$
$\pi_1 = P(s_1 \text{ trans})$

$P_{e,0} = Q\left(\frac{\hat{s}_0(T) - \gamma}{\sigma}\right)$

$P_{e,1} = Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma}\right)$ — any signals filter $h(t)$ γ

Optimize over γ, h, s_0, s_1

$\pi_0 = \pi_1$

Step 1 $\min_{\gamma} \bar{P}_e(\gamma, h, s_0, s_1) = Q(\alpha\lambda)$ $\gamma_{opt} = \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}$

$\alpha = \sqrt{\frac{E(1-r)}{N_0}}$ $r = (s_0, s_1)/E$

$\lambda = \frac{(h, s_T)}{\|h\| \|s_T\|}$ $s_T = s_0(T-t) - s_1(T-t)$

Step 2 $\min_{\gamma, h} \bar{P}_e(\gamma, h, s_0, s_1) = Q(\alpha)$ $\alpha = \sqrt{\frac{E(1-r)}{N_0}}$

Matched filter — $h_{opt}(t) = s_0(T-t) - s_1(T-t)$ $r = \frac{(s_0, s_1)}{E}$

$\gamma_{opt} = \frac{1}{2}(E_0 - E_1)$

Step 3 $\min \bar{P}_e(\gamma, h, s_0, s_1) = Q\left(\sqrt{\frac{2E}{N_0}}\right)$

r, h, s_0, s_1

$s_0(t) = -s_1(t)$

Bandwidth

$$W(t) = \sum_{\ell=-\infty}^{\infty} b_\ell \, \underline{x(t-\ell T)}$$

pulse function

b_ℓ: data, random

$W(t)$ is not WSS

$\Rightarrow R_W(t,\tau) = E[W(t)W(t+\tau)]$ depends on t and τ

$\Rightarrow$ Can't find PSD of W

$Y(t) = W(t-U)$

$U \sim$ uniform on $[0,T]$

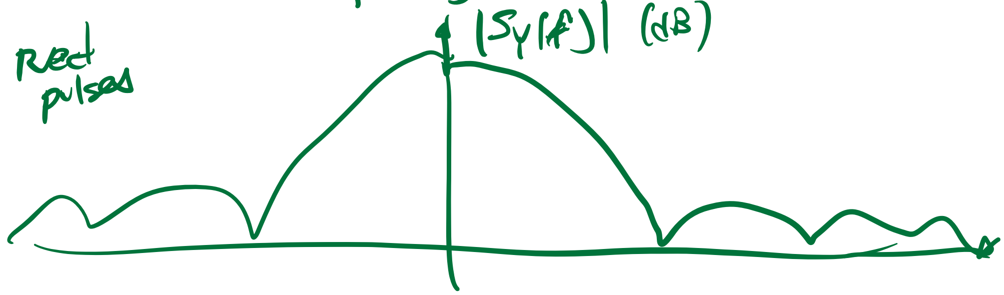
$Y(t)$ is WSS

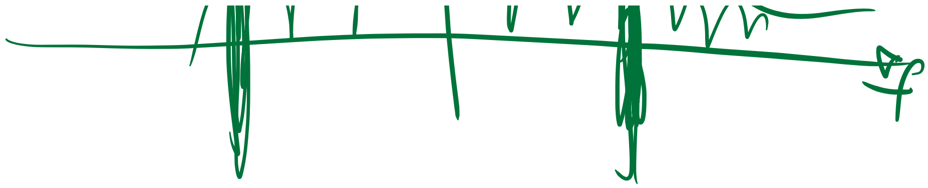
$$\underline{S_Y(f)} = \frac{1}{T}\underline{|X(f)|^2} \quad \text{for } P(b_\ell=+1)=P(b_\ell=-1)=\tfrac{1}{2}$$

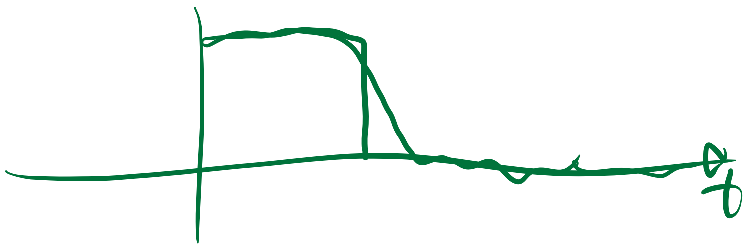
$\underline{\text{i.i.d}}$

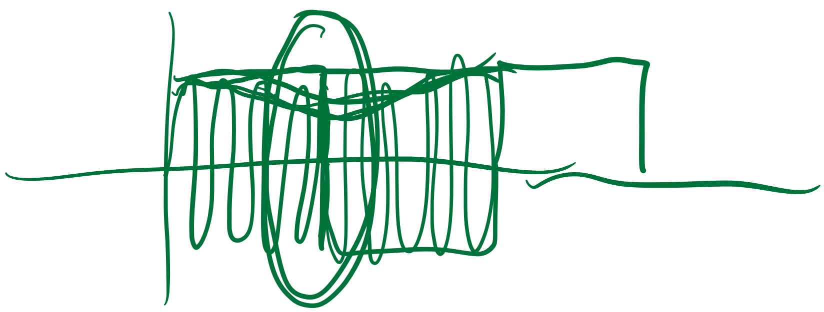
$X(f) = \mathcal{F}\{x(t)\}$

Rect pulses









Bandlimited or Energy limited

$s_0(t) = -s_1(t)$
$\Rightarrow$ optimal

$P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)$

BPSK Power P rcvd

$P_e = 10^{-5} = Q\left(\sqrt{\frac{2E}{N_0}}\right)$

$\Rightarrow \frac{E}{N_0} = 9.6\,\text{dB}$

$N_0 = 4 \times 10^{-21}$ Watts/Hz

$E = N_0 \times \left(\frac{E}{N_0}\right)_{\text{required}}$

$E = PT$

$T = 1\,\text{msec}$

$W = 10\,\text{kHz} \Rightarrow$ Bandwidth limited

≈ 200 MHz $\Rightarrow$ Energy limited