Pr. 1.
Type in the following `Julia` commands that make some **companion matrix** examples.

```
using LinearAlgebra: eigvals, I
n = rand(4:7)
a = randn(n+1) # random vector of polynomial coefficients (a_0,a_1,...,a_n)
b = reverse(a) # reverse the coefficient order
# companion matrix maker:
compan = c -> [-transpose(reverse(c)); [I zeros(length(c)-1)]]
A = compan(a[1:end-1] / a[end])
B = compan(b[1:end-1] / b[end])
[eigvals(A) eigvals(B) 1 ./ eigvals(B)] # study the columns of this array
```

What pattern do you see? Be sure to also try *complex* coefficients using `rand(ComplexF64, n)`. Repeat multiple times to form a conjecture on how the **eigenvalues** of `A` and `B` are related. Prove your conjecture theoretically.

Pr. 2.
Let $\boldsymbol{A}$ denote a $N \times N$ **Vandermonde** matrix corresponding to the roots of the polynomial $z^N = 1$.
Let $\boldsymbol{B}$ denote a $N \times N$ **Vandermonde** matrix corresponding to $\left\{\mathrm{e}^{-\imath 2\pi(k+1/2)/N} : k = 0, \ldots, N-1\right\}$.
Define the matrix $\boldsymbol{X} = \begin{bmatrix} a\boldsymbol{A} & b\boldsymbol{B} \end{bmatrix}$, for $a, b \in \mathbb{C}$.

(a) Determine whether the matrix $\boldsymbol{X}$ is a **frame**, a **tight frame**, a **Parseval tight frame**, or none of the above.

(b) If it is some type of frame, determine its **frame bound(s)**.

(c) Write a `Julia` function with inputs b, c and $\boldsymbol{y} \in \mathbb{C}^N$, that computes efficiently the minimum-norm solution to the LS problem $\arg\min_{\boldsymbol{x} \in \mathbb{C}^{2N}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2$.
Hint. For maximum efficiency, use a Vandermonde matrix form corresponding to the **DFT**. SIPML students should strive for a solution with $O(N \log_2 N)$ computation!

Pr. 3.

(a) Show that all **circulant matrices** (of the same size) **commute**.

(b) Show that all **circulant matrices** are **normal**.

(c) Determine the eigenvalues of the following $N \times N$ circulant matrix:

$$\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 & -1 \end{bmatrix}.$$

Give a simple expression that holds for any $N \in \mathbb{N}$. Multiplying this matrix by a vector performs **finite differences** of the vectors elements with **periodic end conditions**.

(d) Check your expression for the previous part numerically for the case $N = 4$.
Report the eigenvalues from your expression and from the `eigvals` command in `Julia`.

(e) Write a very short `Julia` function that computes the **nuclear norm** of a **circulant matrix** `C` *without* calling expensive functions like `eig` or `svd`. Suggestion: check your function using the previous part.

(f) Optional. In image processing, matrices that are **block circulant with circulant block** (BCCB) are particularly important. Such matrices have the form $\boldsymbol{B} = \boldsymbol{C}_2 \otimes \boldsymbol{C}_1$, where $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are both circulant matrices.
Express the eigenvalues of $\boldsymbol{B}$ in terms of the first columns of $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$.
Challenge. Express the eigenvalues of $\boldsymbol{B}$ in terms of the first column of $\boldsymbol{B}$.

Pr. 4.

Low-rank matrix denoising using non-convex Schatten p-"norm"

The **Schatten p-norm**: $\|A\|_{S,p} = \left(\sum_{k=1}^r \sigma_k^p\right)^{1/p}$, where σ_k denotes the singular values of A, is a proper matrix norm for $p \geq 1$. Values of $p \in (0, 1)$ are also useful as regularizers for **low-rank matrix denoising** problems even though it is not a proper norm for $p < 1$.

(a) For $Y = X + \varepsilon \in \mathbb{C}^{M \times N}$, where we think X is low rank, find an expression for the solution of the regularized low-rank matrix denoising method that uses the following **Schatten-norm regularizer** for $p = 1/2$:

$$\hat{X} = \underset{X \in \mathbb{C}^{M \times N}}{\arg\min} \frac{1}{2} \|Y - X\|_F^2 + \beta R(X, 1/2), \quad R(X, p) = \|X\|_{S,p}^p = \sum_{k=1}^r \sigma_k^p(X).$$

Hint: a previous HW problem will be helpful. Submit your written answer to this part to gradescope.

(b) Write a `Julia` function that performs regularized low-rank matrix denoising using the above cost function.

In `Julia`, your file should be named `lr_schatten.jl` and should contain the following function:

```
"""
    lr_schatten(Y, reg::Real)

Compute the regularized low-rank matrix approximation as the minimizer over `X`
of `1/2 \\|Y - X\\|^2 + reg R(x)`
where `R(X)` is the Schatten p-norm of `X` raised to the pth power, for `p=1/2`,
i.e., `R(X) = \\sum_k (\\sigma_k(X))^{1/2}`

In:
- `Y` : `M × N` matrix
- `reg` regularization parameter

Out:
- `Xh` : `M × N` solution to above minimization problem
"""
function lr_schatten(Y, reg::Real)
```

Email your solution as an attachment to `eecs551@autograder.eecs.umich.edu`.

Think about your solution and consider whether it seems to be a good method for denoising.

(c) Apply your denoising method to the noisy 100×30 Block M image `Y` in the demo notebook `06_optshrink1.ipynb` at `http://web.eecs.umich.edu/~fessler/course/551/julia/demo/` using $\beta = 1000$.

Report the NRMSE of your estimate $\hat{X}$ and submit to gradescope a picture of $\hat{X}$ and a scatter plot of the singular values of $\hat{X}$, X and Y. All of the plotting commands are in the notebook already; you simply need to `include` your `lr_schatten.jl` solution.

Pr. 5.
Testing for common roots of polynomials

(a) Given two polynomials $p(z) = \sum_{i=0}^{m} \alpha_i z^i$ and $q(z) = \sum_{j=0}^{n} \beta_j z^j$, for $n \geq 0$ not necessarily equal to $m \geq 0$, and the definition of **Kronecker product**, how would you check whether p and q have a common (possibly complex) root? In other words, we want to check whether the algebraic curves $p(z) = 0$ and $q(z) = 0$ intersect in the complex plane.

You are may invoke the determinant and/or the trace to ascertain the desired property. You may **not** explicitly compute the eigenvalues or eigenvectors of the respective companion matrices to check for common roots. Other matrix decompositions (QR, LU) are not allowed either.

Hint: Use the relationship between the eigenvalues of the **Kronecker sum** of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

(b) Write a function called `common_root` that takes as input two vectors (not necessarily of equal dimension) whose elements represent the coefficients of the polynomials $p(z)$ and $q(z)$ and returns `true` if the two polynomials share a common root and `false` otherwise.

Caution: the leading coefficient is not necessary 1 here, but you can assume it is nonzero.

In `Julia`, your file should be named `common_root.jl` and should contain the following function:

```
"""
    haveCommonRoot = common_root(a, b ; atol)

Determine if the polynomials described by input coefficient vectors `a`
and `b` share a common root, to within an absolute tolerance parameter `atol`.
Assume leading coefficients `a[end]` and `b[end]` are nonzero.

In:
- `a` : vector of length `m + 1` with `a[m+1] != 0` and `m ≥ 0`
defining a degree `m` polynomial of the form:
`p(z) = a[m+1] z^m + a[m] z^(m - 1) + ... + a[2] z + a[1]`

- `b` : vector of length `n + 1` with `b[n+1] != 0` and `n ≥ 0`
defining a degree `n` polynomial of the form:
`q(z) = b[n+1] z^n + b[n] z^(n - 1) + ... + b[1] z + b[1]`

Option:
- `atol::Real` absolute tolerance for calling `isapprox`

Out:
- `haveCommonRoot` = `true` when `p` and `q` share a common root, else `false`
"""
function common_root(a::AbstractVector, b::AbstractVector ; atol::Real=1e-6)
```

Email your solution as an attachment to `eecs551@autograder.eecs.umich.edu`.

Hint 1: Use the `Julia` function `kron` .

Hint 2: To consider finite numerical precision, use the `isapprox` function with the input absolute tolerance parameter `atol` to decide whether some value is "close enough" to zero. See the `Julia` manual for `isapprox` to see why `atol` is essential here.

Test your code with some polynomials of your own design before submitting to the autograder. Be sure to test some cases that have roots that are zero or complex. Be sure to handle cases where $m = 0$ and/or $n = 0$.

For randomly chosen Gaussian polynomial coefficients, there is an exceedingly small probability of having common roots.

Non-graded problem(s) below
(Solutions will be provided for self check; do not submit to gradescope.)

Pr. 6.
The lecture notes show that the set $\left\{G_N^0, G_N^1, \dots, G_N^{N-1}\right\}$ forms a **basis** for the subspace of $N \times N$ circulant matrices in the vector space $\mathbb{F}^{N \times N}$ of all $N \times N$ matrices, where G_N denotes the "generator" for circulant matrices.
Is this set an **orthogonal basis** for that subspace? Explain.
Is this set an **orthonormal basis** for that subspace? Explain.

Pr. 7.
Prove or disprove (by a counter-example) the following statement. If T is any matrix for which $T^k = I$ for some natural number $k > 1$, then T is **normal**.

Pr. 8.
Let $t_k = k\Delta$, for some integer k, and $\Delta \in \mathbb{R}$. Consider the sum of sinusoids signal

$$\boldsymbol{y}(t_k) = \sum_{i=1}^{r} \boldsymbol{b}_i e^{\imath w_i t_k},$$

where $\boldsymbol{y}_i \in \mathbb{C}^n$, $\boldsymbol{b}_1, \dots, \boldsymbol{b}_r \in \mathbb{C}^n$ are linearly independent vectors, and $w_i \in [0, 1]$ are distinct, and $\imath = \sqrt{-1}$. Express the recursion in the following simple matrix form:

$$\boldsymbol{y}(t_{k+1}) = \boldsymbol{A}\, \boldsymbol{y}(t_k).$$

(a) Determine the matrix $\boldsymbol{A}$.

(b) Determine the eigenvalues and eigenvectors of $\boldsymbol{A}$.

Hint: For a matrix $\boldsymbol{B}$ of full column rank r, we have that $\boldsymbol{B}^+ \boldsymbol{B} = \boldsymbol{I}_r$, where $\boldsymbol{I}_r$ is an identity matrix of rank r.

Pr. 9.
Define $\mathbb{R}_+ \triangleq [0, \infty)$. A **symmetric gauge function** $\phi(\boldsymbol{x})$ is a mapping from $\mathbb{R}^n$ into $\mathbb{R}_+$ that satisfies the following four properties for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

- $\phi(\boldsymbol{x}) > 0$ if $\boldsymbol{x} \neq \boldsymbol{0}$ (positivity)
- $\phi(\alpha \boldsymbol{x}) = |\alpha|\, \phi(\boldsymbol{x})$ for all $\alpha \in \mathbb{R}$ (homogeneity)
- $\phi(\boldsymbol{x} + \boldsymbol{y}) \leq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ (triangle inequality)
- $\phi(s_1 x_{[1]}, \dots, s_n x_{[n]}) = \phi(\boldsymbol{x})$ for all $s_k = \pm 1$ and for any permutation $\left(x_{[1]}, \dots, x_{[n]}\right)$ of the elements of $\boldsymbol{x}$. (symmetry)

Some examples are:

- $\phi(\boldsymbol{x}) = \|\boldsymbol{x}\|_1$
- $\phi(\boldsymbol{x}) = \max_i |x_i| = \|\boldsymbol{x}\|_\infty$
- $\phi(\boldsymbol{x}) = 7\left|x_{(1)}\right| + 5\left|x_{(2)}\right|$, where $x_{(1)}$ and $x_{(2)}$ denote the first and second largest elements of $\boldsymbol{x}$ in magnitude.

Such symmetric gauge functions are at the heart of many matrix norm properties.
Let $\phi(\cdot)$ be any symmetric gauge function and define a matrix norm by $\|\boldsymbol{A}\|_\phi = \phi(\sigma_1, \dots, \sigma_{\min(M,N)})$ where $\{\sigma_k\}$ denote the singular values of a $M \times N$ matrix $\boldsymbol{A}$.

(a) Verify for yourself that this definition does indeed define a proper matrix norm.

(b) Now prove or disprove (by counter-example) that any such matrix norm $\|\boldsymbol{A}\|_\phi$ is **unitarily invariant**.

Pr. 10.

This problem explores extensions of **unitary invariance** of norms. Throughout this problem, let $\boldsymbol{A} \in \mathbb{F}^{M \times N}$ and assume $\boldsymbol{X}$ is a $K \times M$ matrix with orthonormal columns and $\boldsymbol{Y}$ is a $N \times L$ matrix with orthonormal rows. (Do not assume $\boldsymbol{X}$ or $\boldsymbol{Y}$ are square.)

(a) Prove that $\|\boldsymbol{X}\boldsymbol{A}\boldsymbol{Y}\|_{\mathrm{F}} = \|\boldsymbol{A}\|_{\mathrm{F}}$

This is a kind of generalization of unitary invariance. (Maybe it should be called "orthonormal invariance"?)

(b) For any matrix $\boldsymbol{B}$ with $J \in \mathbb{N}$ rows, define the function $f(\boldsymbol{B}) = J \|\boldsymbol{B}\|_*$. Show that $f(\boldsymbol{B})$ is a unitarily invariant norm.

(c) Prove or disprove: $f(\boldsymbol{X}\boldsymbol{A}\boldsymbol{Y}) = f(\boldsymbol{A})$

(d) Consider the following "unified" matrix norm defined in Ch. 5: $\|\boldsymbol{A}\|_{K,p} \triangleq \left(\sum_{k=1}^{K} \sigma_k^p\right)^{1/p}$ where $K \in \mathbb{N}$ and $1 \leq p < \infty$.

(e) It is easy to verify that this "unified" norm is unitarily invariant.

Prove or disprove: $\|\boldsymbol{A}\|_{K,p} = \|\boldsymbol{X}\boldsymbol{A}\boldsymbol{Y}\|_{K,p}$.