University of Michigan Fall 2021

EECS501: Homework 8

Assigned: Oct 31, 2021
Due: Nov 9, 2021 at 11:59PM on gradescope
Text: "Probability and random processes" by J. A. Gubner

Reading assignment: Please read Chapters 4, 5 and 7.

1. State TRUE or FALSE by giving reasons [5 points each]

You must state a correct reason to get credit. No partial credit.

(a) Let Y_1, Y_2 and Y_3 be three continuous random variables with the following joint PDF:

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \begin{cases} e^{-y_3} & \text{if } y_3 \geq y_2 \geq y_1 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The MMSE estimate of Y_1 from Y_2 and Y_3 is given by $Y_2/2$.

(b) Consider a sequence of independent and identically distributed random variables X_1, X_2, $X_3, \ldots,\ X_n, \ldots$. The PMF of X_i is given by $P(X_i = 1) = P(X_i = -1) = \frac{1}{4}$, and $P(X_i = 0) = \frac{1}{2}$. Let $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$. The CDF of Z_n converges to that of Gaussian with zero mean and unit variance.

2. Estimation of Gaussian Vector [10 points]

Consider an n-dimensional signal $\mathbf{X}$ modeled as a jointly Gaussian vector with zero mean and covariance matrix Σ_X. Let $\mathbf{Y}$ denote the observation which is an n-dimensional vector and is related to X as

$$\mathbf{Y} = H\mathbf{X} + \mathbf{N},$$

where H is a known deterministic matrix of size $n \times m$, and $\mathbf{N}$ is the noise vector which is modeled as jointly Gaussian with zero mean and covariance matrix Σ_N. H can be thought of as the transfer function of the noisy medium that takes $\mathbf{X}$ and produces $\mathbf{Y}$. It is assumed that $\mathbf{X}$ and $\mathbf{N}$ are independent. Find the MMSE estimate of $\mathbf{X}$ from $\mathbf{Y}$.

3. Chernoff Inequality [10 points, 10 points]

(a) Let $X_1, X_2, \ldots, X_n$ denote n independent Bernoulli random variables with parameter p. Let $Z = \frac{1}{n}\sum_{i=1}^n X_i$. Use Chernoff inequality and show the following for any $\delta > 0$:

$$P(Z \geq p + \delta) \leq e^{-nD(p+\delta\|p)},$$

where for any p and q such that $0 \leq p \leq 1$ and $0 \leq q \leq 1$, we have

$$D(q\|p) = q\log(q/p) + (1-q)\log((1-q)/(1-p)).$$

Note that $D(q\|p)$ is called Kullback-Leibler divergence.

(b) Similarly, show the following

$$P(Z \leq p - \delta) \leq e^{-nD(p-\delta\|p)}.$$

Combine the two results and conclude that $P(|Z - p| \geq \delta)$ decays exponentially in n.

4. Concentration Bounds [5 points, 5 points, 5 points]

A gambler bets one dollar in each game. She wins two dollars with probability 0.3, one dollar with probability 0.2 and zero dollar with probability 0.5. Please use Chebyshev's inequality, the central limit theorem and the Chernoff bound to estimate the probability that she is ahead after 400 games.

5. Wick's theorem (also Isserli's theorem)[10 points]

Let X be a multivariate normal random vector with covariance matrix C. Use Wick's theorem (stated below) to evaluate $E[X_1 X_3^2 X_4]$, $E[X_1^2 X_2^2]$ and $E[X_1^6]$.

Wick's theorem: Let $\mathbf{X}$ be a jointly Gaussian zero-mean $2n$-dimensional vector with covariance matrix C. Then

$$E(X_1 X_2 \ldots X_{2n}) = \sum \prod E(X_i X_j)$$

$\sum \prod$ stands for summing over all distinct ways of partitioning the $2n$ random variables (possibly perfectly correlated, i.e., $X_l = X_k$ for some $l \neq k$) into n pairs of (X_i, X_j), and each summand is the product of the n pairs. For example, if $2n = 4$, we have $E(X_1 X_2 X_3 X_4) = C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}$, and similarly, $E(X_1^4) = E(X_1 X_1 X_1 X_1) = C_{11}C_{11} + C_{11}C_{11} + C_{11}C_{11} = 3\left[E(X_1^2)\right]^2$.

6. MMSE and LMMSE [10 points, 10 points]

Let $Y = X + N$, where X has the exponential distribution with parameter λ and N is Gaussian with mean 0 and variance σ^2. The variables X and N are independent, and the parameters λ and σ^2 are strictly positive. Find $\hat{E}[X|Y]$ (the LMMSE estimator) and the mean square error for estimating X using $\hat{E}[X|Y]$.