Lecture 6: Goals

- Review basic concepts of probability
- Review basic concepts of random variables
- Determine the noise variance out of a linear system when the input is white Gaussian noise
- Determine the bandwidth of a digitally modulated signal

Probability

- Because we can not know the value of the noise ahead of time we can only characterize it by its statistical properties such as average values.
- Using probability we can determine likelihoods of events.

Probability

- Probability starts with an experiment.
- Probability is a mapping from a set of outcomes (events) of an experiment to numbers in the interval $[0, 1]$.
- Probability has axioms (assumed truths)
  - The probability of any event is non-negative: $P(A) \geq 0$ for an event A,
  - The probability of the event of the all outcomes is 1: $P(S) = 1$ where S is the set of possible outcomes,
  - The probability of a union of disjoint events (A and B) is the sum of the probabilities: for $A \cap B = \phi$, $P(A \cup B) = P(A) + P(B)$ where ϕ is the empty set.

Consequences of the Axioms

1. $P(\phi) = 0$
2. $P(\bar{A}) = 1 - P(A)$
3. $P(A) \leq 1$
4. If $A \subset B$ then $P(A) \leq P(B)$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
6. $P(A \cup B) \leq P(A) + P(B)$
7. $P(A) = P(A \cap B) + P(A \cap \bar{B})$
8. If $A_i, i = 1, 2, ..., n$ is a partition then $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$
9. $$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$
10. $P(A \cup B \cup C) = P(A) + P(B \cap \bar{A}) + P(C \cap \bar{A} \cap \bar{B})$

Independence

- Two events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

- Independence means that knowing whether or not one event occurred does not change the likelihood of the other event occurring
- Cloudy and rainy are not independent but cloudy and the day being Wednesday are independent.
- Example: Flip a coin twice. Let H_1 be event of heads on the first toss. Let H_2 be event of heads on the first toss. One would naturally assume that H_1 and H_2 are independent.

Example

- Experiment: Flip a biased coin 10 times with probability of heads p.
- Probability: $P(HTHTTTHTHHH) = p^6(1-p)^4$.
- $P(k \text{ heads in } N \text{ flips }) = \binom{N}{k} p^k (1-p)^{N-k}$, $k = 0, 1, ..., N$
- Probability of the event A of 6 or more heads in 10 flips.

$$P(A) = \sum_{k=6}^{10} \binom{10}{k} p^k (1-p)^{10-k}$$

Conditional Probability: Definition

Definition

For two events A and B with $P(B) > 0$ the **conditional** probability of A **given** the event B occurred is given by

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0}$$

Conditional Probability: Example

Given: Experiment is tossing two die. Event A is the event that one of the die is six. Event B is the event that the total of the two die is seven. Find probability of one die being a six given that the total is seven.
Solution:

$$\begin{aligned}
A &= \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),\\
&\qquad (6,5),(6,4),(6,3),(6,2),(6,1)\}\\
B &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}\\
A \cap B &= \{(1,6),(6,1)\}\\
P(A \cap B) &= 2/36, \qquad P(B) = 6/36\\
P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{2/36}{6/36} = 1/3
\end{aligned}$$

Example

- Experiment: Two coins in a hat. One has probability of heads p_1 and one has probability of heads p_2. Choose a coin at random (equally likely) and flip it twice.
- Determine the probability of two heads.
- Solution:
- Let C_1 be the event the first coin is chosen.
- The event the second coin is chosen is $C_2 = \bar{C}_1$, the complement of C_1.
- Let H_1 be the event the first toss is heads and H_2 the event the second toss is heads.

Example

- The events H_1 and H_2 are not independent ($p_1 \neq p_2$). Consider that $p_1 = 0.99$ and $p_2 = 0.01$. Then getting a heads on the first toss gives a strong indication that coin 1 was chosen and so then it is very likely to get a heads on the second toss.

$$\begin{aligned} P(H_1 \cap H_2) &= P(H_1 \cap H_2|C_1)P(C_1) + P(H_1 \cap H_2|C_2)P(C_2) \\ &= P(H_1|C_1)P(H_2|C_1)P(C_1) + P(H_1|C_2)P(H_2|C_2)P(C_2) \\ &= p_1^2(1/2) + p_2^2(1/2) \end{aligned}$$

- We have used the fact that given C_1 or C_2 the events of heads on the two flips are independent. This is known as conditionally independent.

Example

- Experiment: Two coins in a hat. One has probability of heads p_1 and one has probability of heads p_2. Choose a coin at random (equally likely) and flip it once. After the first flip it is put back in the hat and choose a coin at random and flip it.
- Determine the probability of two heads.
- Solution:

$$
\begin{aligned}
P(H_1 \cap H_2) &= P(H_1)P(H_2) \\
P(H_1) &= P(H_1|C_1)P(C_1) + P(H_2|C_2)P(C_2) \\
&= p_1(1/2) + p_2(1/2) \\
P(H_1 \cap H_2) &= (\frac{p_1 + p_2}{2})^2 \\
&= \frac{p_1^2}{4} + \frac{p_1 p_2}{2} + \frac{p_2^2}{4}
\end{aligned}
$$

Random Variables: Example

- A random variable is a mapping from experimental outcomes to real numbers.
- Experiment: Flip an unbiased coin 10 times.
- Probability: $P(HTHTTHTHHT) = 2^{-10}$.
- Random Variable: $X =$ number of heads. $X(HTHTTHTHHT) = 5$.
- This is an example of a discrete random variable (finite or countable number of possible values).
- $P(X = 8) = \binom{10}{8}(1/2)^8(1/2)^2$.

Random Variables: Example

- A discrete random variable is characterized by a probability mass function

$$\begin{aligned} p_X(k) &= P(X = k) \\ \sum_k p_X(k) &= 1 \end{aligned}$$

- A random variable (discrete or otherwise) is characterized by a cumulative probability distribution (cdf).

$$F_X(x) = P(X \leq x)$$

Random Variables: Expectation

- Example: Flip a coin N times.

$$p_X(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$F_X(x) = \sum_{k \le x} \binom{N}{k} p^k (1-p)^{N-k}$$

Random Variables: Expectation

- The expected value (average, mean) of a discrete random variable is

$$\mu_X = E[X] = \sum_k k p_X(k)$$

- Example: Flip a coin N times and count the number of heads.

$$\begin{aligned}\mu_X = E[X] &= \sum_{k=0}^{N} kP\{X = k\} \\ &= \sum_{k=0}^{N} k \binom{N}{k} p^k p^{N-k} \\ &= Np\end{aligned}$$

Random Variables: Variance

- A function of a random variable is another random variable.
- If $Y = g(X)$ then Y is a mapping from outcomes of the experiment to real numbers.
- The expected value of Y can be found from the probability mass function of X.

$$E[Y] = \sum_k g(k) p_X(k)$$

- The variance of X measures the spread of a random variable. The variance is the expected value of the squared error between the random variable and the mean.

$$\begin{aligned} \text{Var}[X] &= E[(X - \mu_X)^2] \\ &= E[X^2] - \mu_X^2 \end{aligned}$$

Random Variables: Variance

- Example: Flip a coin N times

$$
\begin{aligned}
E[X^2] &= \sum_{k=0}^{N} k^2 P\{X = k\} \\
&= \sum_{k=0}^{N} k^2 \binom{N}{k} (p)^k (1-p)^{N-k} \\
&= Np(1-p) + N^2 p^2 \\
\sigma_X^2 = \mathrm{Var}[X] &= E[X^2] - \mu_X^2 \\
&= Np(1-p)
\end{aligned}
$$

- Maximum variance ($N/4$) when $p = 0.5$. Minimum variance (0) when $p = 0$ or $p = 1$.

Continuous Random Variables

- A continuous random variable is characterized by a probability density function or pdf $f_X(x)$ such that

$$P(X \in A) = \int_A f_X(u)du$$

- The distribution of a continuous random variable is

$$F_X(x) = \int_{u=-\infty}^{x} f_X(u)du.$$

- The expected value of a continuous random variable is

$$E[X] = \int x f_X(x)dx$$

Continuous Random Variables

- The expected value of a function $Y = g(X)$ of a continuous random variable X is

$$E[Y] = \int_x g(x) f_X(x) dx$$

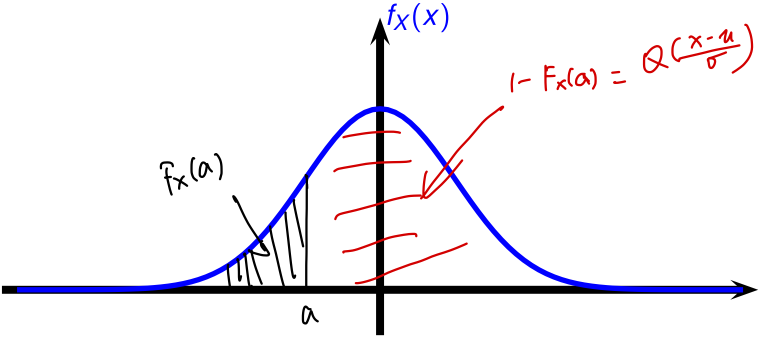
- The variance of a continuous random variable is

$$\begin{aligned} \sigma_X^2 = \mathrm{Var}[X] &= \int_x (x - \mu_x)^2 f_X(x) dx \\ &= E[X^2] - \mu_x^2 \end{aligned}$$

$$= E[(x - \mu_x)^2]$$

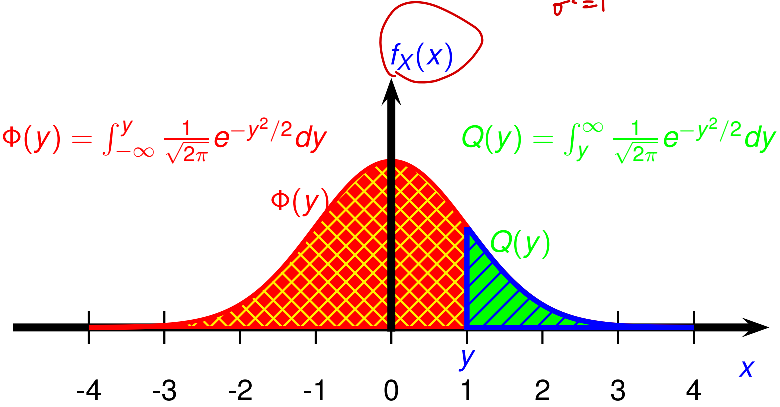
Gaussian Random Variables

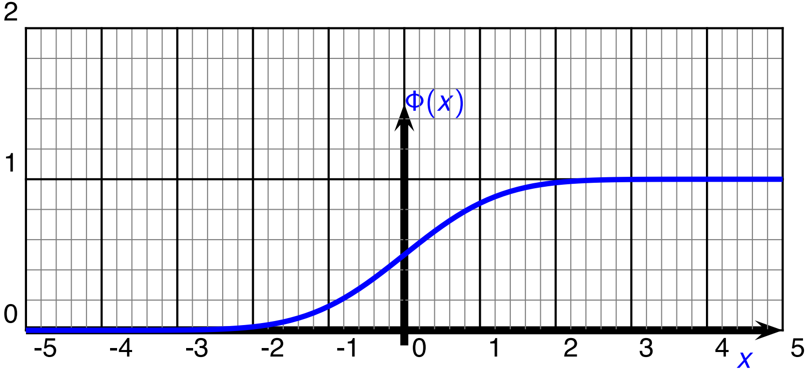
- A Gaussian random variable X with mean μ and variance σ^2 has density $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$
- $P\{X \leq x\} = \Phi(\frac{x-\mu}{\sigma})$
- $P\{X > x\} = Q(\frac{x-\mu}{\sigma})$

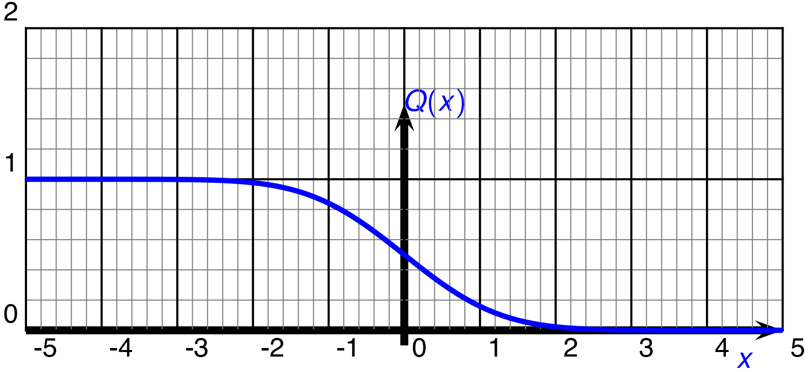


Cummulative Distribution and Complementary Cummulative Distribution Function

$\mu = 0$
$\sigma^2 = 1$

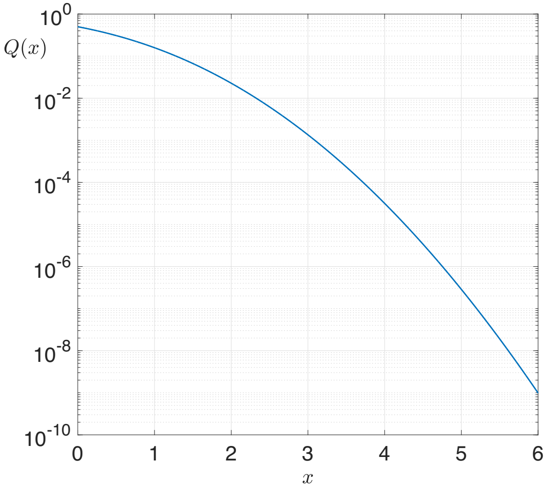



2
1
0
$\Phi(x)$
-5
-4
-3
-2
-1
0
1
2
3
4
5
x



2
1
0
Q(x)
-5
-4
-3
-2
-1
0
1
2
3
4
5
x

Q function



Inverse Q function

$Q(x)$	x
10^{-1}	1.2816
10^{-2}	2.3264
10^{-3}	3.0903
10^{-4}	3.7190
10^{-5}	4.2649
10^{-6}	4.7534
10^{-7}	5.1993
10^{-8}	5.6120
10^{-9}	5.9978
10^{-10}	6.3614

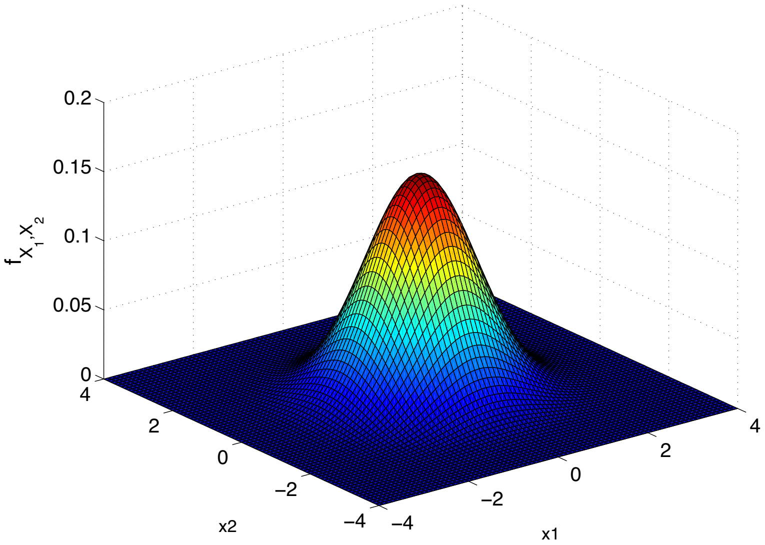
Two independent Gaussian random variables

- The joint density for two jointly Gaussian random variables is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\{-\frac{(\tilde{x}^2 - 2\rho\tilde{x}\tilde{y} + \tilde{y}^2)}{2(1-\rho^2)}\}$$

  where $\tilde{x} = (x - \mu_X)/\sigma_X$ and $\tilde{y} = (y - \mu_Y)/\sigma_Y$.
- μ_X, μ_Y are the means and σ_X^2, σ_Y^2 are the variances.
- Two Gaussian random variables (real) are independent if they are uncorrelated ($\rho = 0$).
- The joint density of two independent, identically distributed Gaussian random variables is circularly symmetric.
- The (weighted) sum of two jointly Gaussian random variables is a Gaussian random variable.

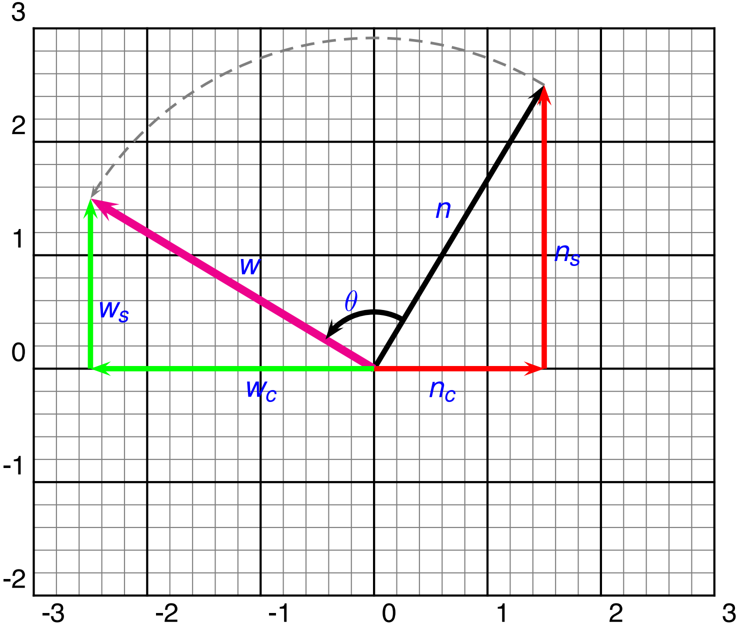
Two dimensional Gaussian



Rotation of a random vector

Consider a pair of random variables n_c and n_s that are independent, zero mean and jointly Gaussian each with variance σ^2. Because they are independent and zero mean $E[n_c n_s] = 0$. Also $E[n_c^2] = E[n_s^2] = \sigma^2$. Now consider a transformation of those random variables that is just a rotation. That is if $n = (n_c + jn_s)$ and $w = ne^{j\theta}$ then

$$
\begin{aligned}
w = (w_c + jw_s) &= (n_c + jn_s)(\cos(\theta) + j\sin(\theta)) \\
&= n_c\cos(\theta) - n_s\sin(\theta) + j(n_s\cos(\theta) + n_c\sin(\theta)) \\
w_c &= n_c\cos(\theta) - n_s\sin(\theta) \\
w_s &= n_s\cos(\theta) + n_c\sin(\theta)
\end{aligned}
$$


3
2
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0
-1
-2
-3
-2
-1
0
1
2
3
n
n_s
w
w_s
θ
w_c
n_c

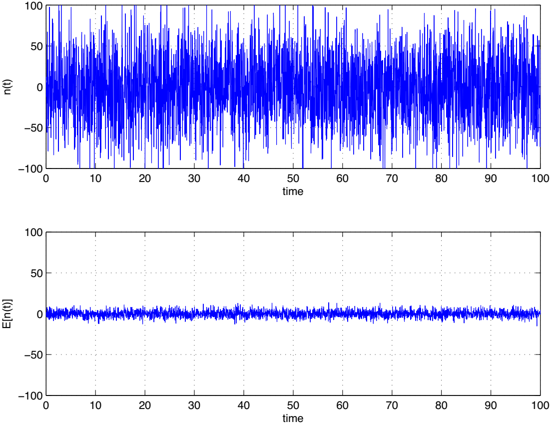
Statistics of w_c and w_s

$$
\begin{aligned}
E[w_c] &= E[n_c\cos(\theta) - n_s\sin(\theta)] \\
&= E[n_c]\cos(\theta) - E[n_s]\sin(\theta) \\
&= 0 \\
E[w_s] &= E[n_s\cos(\theta) + n_c\sin(\theta)] \\
&= E[n_s]\cos(\theta) + E[n_c]\sin(\theta) \\
&= 0 \\
\mathrm{Var}[w_c] &= \mathrm{Var}[n_c\cos(\theta) - n_s\sin(\theta)] \\
&= E[(n_c\cos(\theta) - n_s\sin(\theta))^2] \\
&= E[n_c^2\cos^2(\theta)] - E[n_c n_s]\cos(\theta)\sin(\theta) + E[n_s^2\sin^2(\theta)] \\
&= E[n_c^2]\cos^2(\theta) + E[n_s^2]\sin^2(\theta) \\
&= \sigma^2\cos^2(\theta)] + \sigma^2\sin^2(\theta) \\
&= \sigma^2 \\
E[w_c w_s] &= E[(n_c\cos(\theta) - n_s\sin(\theta))(n_s\cos(\theta) + n_c\sin(\theta))] \\
&= E[n_c n_s]\cos^2(\theta) + E[n_c^2]\cos(\theta)\sin(\theta) - E[n_s^2]\sin(\theta)\cos(\theta) - E[n_c n_s]\sin^2(\theta) \\
&= 0\cos^2(\theta) + \sigma^2\cos(\theta)\sin(\theta) - \sigma^2\sin(\theta)\cos(\theta) - 0\sin^2(\theta) \\
&= 0
\end{aligned}
$$

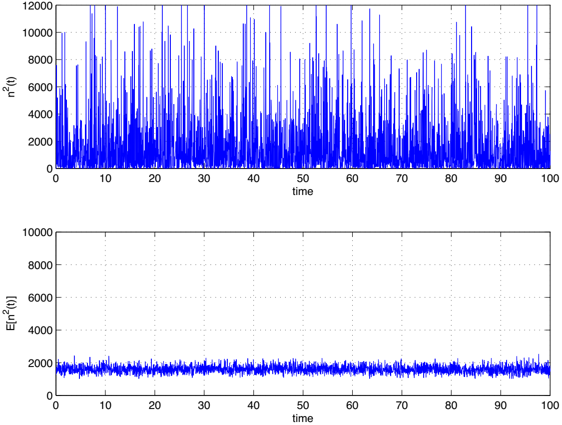
Random Processes

- Random processes are indexed (by time) random variables.
- $X(t)$ is a random process if for each time t, $X(t)$ is a random variable.
- Example: $X(t)$ is Gaussian distributed for each t with mean 0 and variance σ^2 and $X(t)$ and $X(s)$ are independent for $t \neq s$.
- We can characterize the statistical properties of these random variables.
- $E[X(t)] = 0$, $\text{Var}[X(t)] = \sigma^2$, $E[X^2(t)] = \sigma^2$, $E[X(t)X(s)] = 0$ for $t \neq s$.

Noise Mean


100
50
0
−50
−100
n(t)
0
10
20
30
40
50
60
70
80
90
100
time
100
50
0
−50
−100
E[n(t)]
0
10
20
30
40
50
60
70
80
90
100
time

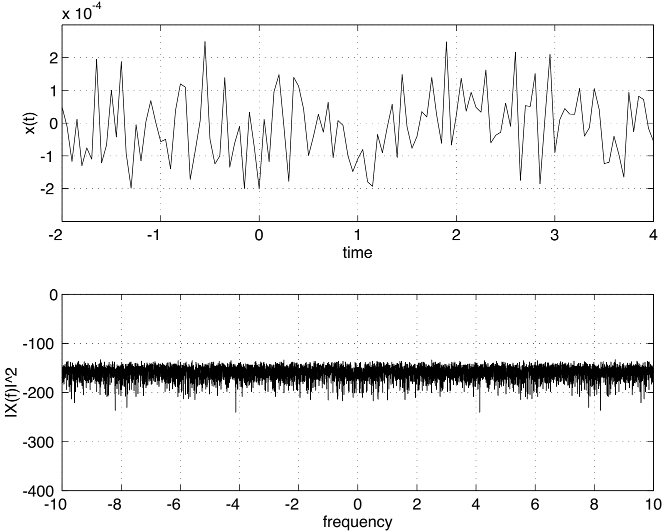
Noise Variance



Power Spectral Density

- The power spectral density function of a random signal is the amount of power in the signal as a function of frequency.
- The autocorrelation measures the correlation between the noise at different points in time.
- For noise like signals the autocorrelation does not depend on the time but just the time difference between two samples.
- In this case (and assuming zero mean) the process is called wide-sense stationary.

Sample Path and Frequency Content of Noise



Random Process Definitions

- The autocorrelation of a random process $X(t)$ is

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

- The power of a (wide-sense stationary) random process $X(t)$ is

$$P_X = E[X^2(t)] = R_X(0).$$

To see the meaning of this definition we start with the time average power of a particular realization of a random process over a finite interval. Then take the (probabilistic) average to get the average power.

$$
\begin{aligned}
P_X &= E\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|X(t)|^2 dt\right] \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[|X(t)|^2]dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_X(0)dt \\
&= R_X(0)
\end{aligned}
$$

Power Spectral Density

- Define the power spectral density as the Fourier transform of the autocorrelation function.

$$\begin{aligned} S_X(f) &= \int_{\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \\ R_X(\tau) &= \int_{\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df \end{aligned}$$

- Then the average power of a random process is

$$P_X = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df.$$

Power Spectral Density

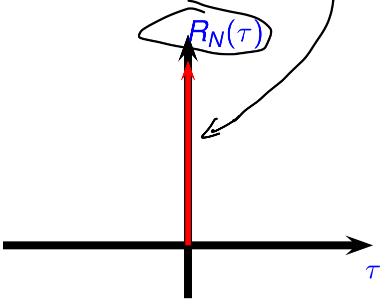
- Thus the power spectral density is a measure of how the power is distributed over frequency.
- Another way to see that the Fourier transform of autocorrelation is power spectral density is to put a random process through a linear time-invariant filter which has a very narrow bandpass transfer function.
- Then the power at the output of the filter is the power spectral density of the input evaluated at the frequency that the filter passes.

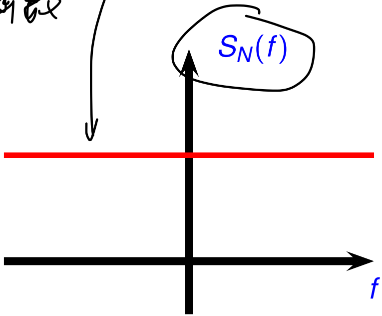
Example: White Noise

$$\delta(\tau) \leftrightarrow 1$$

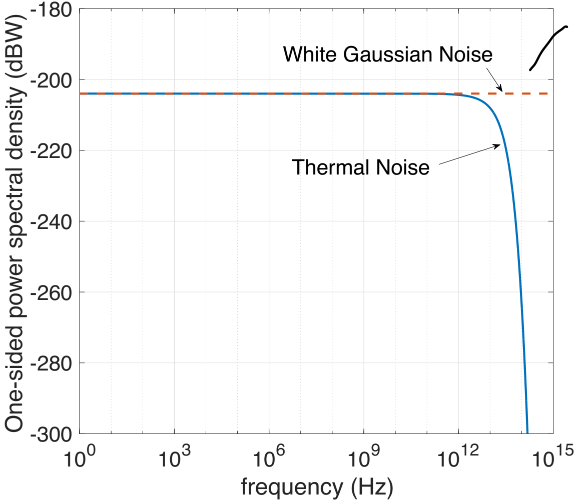
$$R_N(\tau) = \frac{N_0}{2}\delta(\tau), \quad S_N(f) = \frac{N_0}{2}$$

冲击函数





Thermal Noise Power Spectral Density



filter out

Thermal Noise: Power Spectral Density

- White Gaussian noise (WGN) sometimes also called thermal noise or Johnson noise has a flat power spectral density (same noise power at all frequencies).
- The power spectral $\underbrace{\text{density is } N_0/2}$ where $N_0 = kT_o$, $k = 1.38 \times 10^{-23}$ Joules/$^\circ$K (Boltzmann's constant) and T_o is the temperature in Kelvin.
- For room temperatures $T_o = 290$K which makes $N_0 = 4 \times 10^{-21}$ Watts/Hz.
- In dBs this is $N_0 = -174$ dBm/Hz or $N_0 = -204$ dBW/Hz. The power spectral density is $N_0/2 = -177$ dBm/Hz or $N_0/2 = -207$ dBW/Hz.
- The distribution of the noise (at any time) is Gaussian.

White Gaussian Noise

- When using power spectral density the noise has equal power at positive and negative frequencies. "One-sided" power spectral density is the sum of the power at the positive and negative frequencies. The one-sided power spectral density of white noise is N_0.
- Notice that the power (not the power spectral density) of white noise is infinite. Since there is the same power at all frequencies the total power must be infinite. White noise is only a model for what happens in practice. When operating at frequencies below the optical band this is a very accurate model.
- Since we always filter the signal and noise, the noise at the output of the filter becomes finite.

Noise Power Example

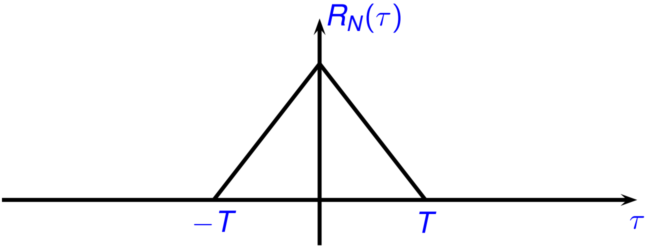
What is the noise power in the frequency band from 2400 MHz to 2483 MHz at room temperature?

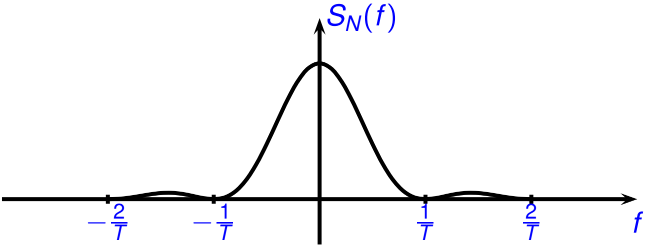
$$
\begin{aligned}
P_N &= \int_{2.4\times10^6}^{2.483\times10^6} \frac{N_0}{2}\,df + \int_{-2.483\times10^6}^{-2.4\times10^6} \frac{N_0}{2}\,df \\
&= \int_{2.4\times10^6}^{2.483\times10^6} N_0\,df = \underbrace{83\times10^6}_{W} \times \underbrace{4\times10^{-21}}_{N_0} \\
&= 3.32\times10^{-13}\text{Watts}
\end{aligned}
$$

Correlated Noise

$$
\begin{aligned}
R_N(\tau) &= \Lambda(\tau/T) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \leq T \\ 0 & |\tau| > T \end{cases}, \\
S_N(f) &= T\,\mathrm{sinc}^2(fT) \\
&= T\frac{\sin^2(\pi fT)}{(\pi fT)^2}
\end{aligned}
$$

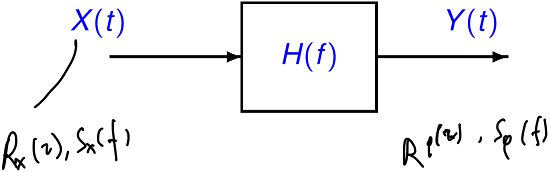
Correlated Noise





Noise into linear systems

Now consider noise at the input to the receiver.



The power spectral density of the output of the filter is determined from the power spectral density at the input to the filter and the transfer function of the filter.

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

$R_X(\tau) = E[X(t) X(t+\tau)]$

Autocorrelation

The autocorrelation is given by

Det

y(t)

$$
\begin{aligned}
R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\
&= E[\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha \int_{-\infty}^{\infty} \underline{X(t+\tau-\beta)h(}\beta)d\beta] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\alpha)X(t+\tau-\beta)]h(\alpha)h(\beta)d\alpha d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\gamma-\beta)\tilde{h}(\gamma)h(\beta)d\gamma d\beta \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_X((\tau-\beta)-\gamma)\tilde{h}(\gamma)d\gamma\right] h(\beta)d\beta
\end{aligned}
$$

Conclusion.

$$\boxed{R_Y(\tau) = R_X(\tau) * \tilde{h} * h,}$$

double hit convolution

where $\tilde{h}(t) = h(-t)$.

$$\begin{aligned}
S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \int_{-\infty}^{\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_X((\tau-\beta)-\gamma)\tilde{h}(\gamma)d\gamma\right] h(\beta)e^{-j2\pi f\tau} d\beta d\tau
\end{aligned}$$

Let $u = \tau - \beta - \gamma$. Then

$$\begin{aligned}
S_Y(f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_X(u)\tilde{h}(\gamma)d\gamma\right] h(\beta)e^{-j2\pi f(u+\beta+\gamma)} d\beta du \\
&= \int_{-\infty}^{\infty} R_X(u)e^{-j2\pi fu}du \int_{-\infty}^{\infty} \tilde{h}(\gamma)e^{-j2\pi f\gamma}d\gamma \int_{-\infty}^{\infty} h(\beta)e^{-j2\pi f(\beta)}d\beta \\
&= S_X(f)H^*(f)H(f) \\
&= S_X(f)|H(f)|^2
\end{aligned}$$

$$S_y(f) = S_x(f)\,|H(f)|^2 \qquad \mu_y(t) = E[Y(t)]$$

$$= [\int h(t-\tau)\, x(\tau)]$$

At any particular time the output due to noise alone is a random variable with a certain density function. The mean of the output is the convolution of the mean of the input signal with the impulse response of the system. The variance of the output is

$$\begin{aligned}\sigma^2 = \text{Var}[Y(t)] = R_Y(0) &= \int_{-\infty}^{\infty} R_X(\beta-\gamma)\tilde{h}(\gamma)h(\beta)d\gamma d\beta \\ &= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df\end{aligned}$$

If $H(f) = 1$ for $|f| \in [f_0, f_0 + \Delta f]$ for Δf small and 0 otherwise then

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df \\ &= S_X(f_0)\Delta f\end{aligned}$$

So the power in the frequency interval $[f_0, f_0 + \Delta f]$ is $S_X(f_0)\Delta f$. So the interpretation of $S_X(f)$ as the power spectral density is appropriate.

White Noise into a Filter

For the case when the noise is white with power spectral density $N_0/2$ the variance of the output is

Variance at output of filter due to WGN

$$
\begin{aligned}
\sigma^2 &= \mathrm{Var}[Y(t)] \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(\gamma)d\gamma \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df.
\end{aligned}
$$

WGN

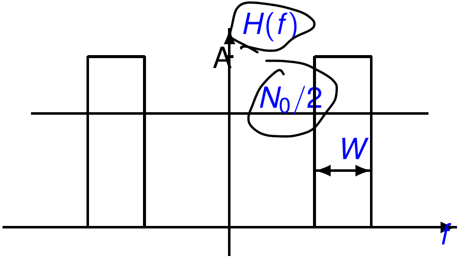
1) $\mu_x = 0 \Rightarrow \mu_y = 0$

2) $R_y(t) = \frac{N_0}{2}\delta(t)$

$\Rightarrow \sigma_y^2 = \frac{N_0}{2}\int h^2(\gamma)d\gamma$

$= \frac{N_0}{2}\int |H(f)|^2 df$

Example: Ideal Brickwall Filter



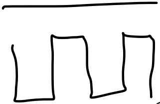
$$\underbrace{\sigma^2 = A^2 WN_0}$$

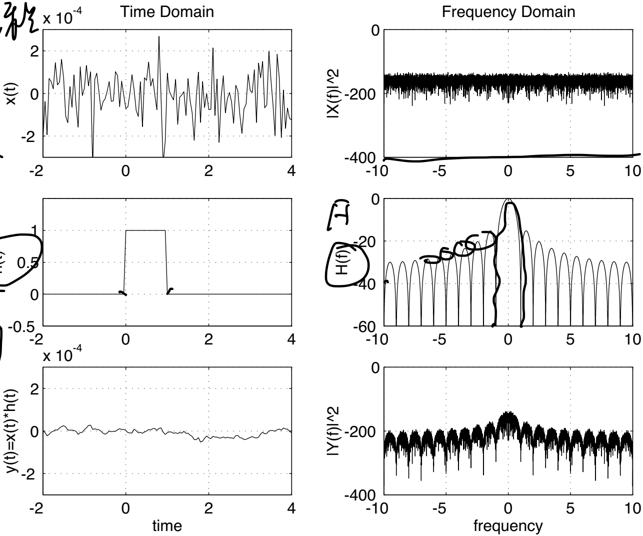
$w = \frac{\sigma^2}{A^2 N_0}$

(A filter for which the noise variance is σ^2 but does not have the brickwall shape is said to have noise bandwidth $\sigma^2/(A^2 N_0)$ where A is the peak output).

Filtering of Gaussian Noise

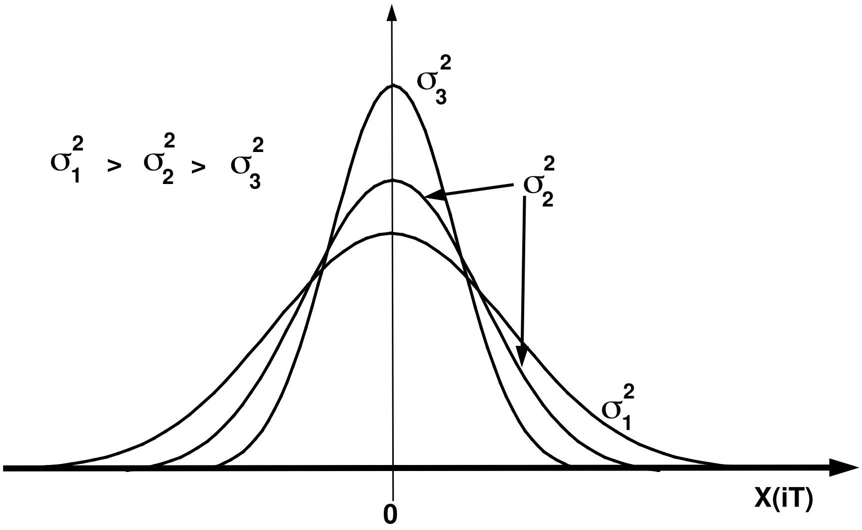
时域的卷积
= 频域
的相乘





LPF

Gaussian Density



Gaussian Density

If η is a Gaussian distributed random variable with mean μ and variance σ^2 then we can calculate various probabilities involving η. In particular

$$\begin{aligned} P\{\eta < x\} &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(w-\mu)^2/2\sigma^2} dw \\ &= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \\ &= \Phi(\frac{x-\mu}{\sigma}) = Q(-\frac{x-\mu}{\sigma}) \end{aligned}$$

$N(0,1)$

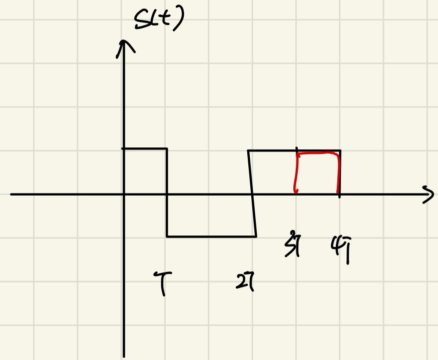
where

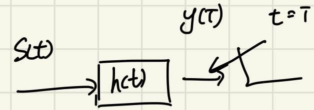
$$\begin{aligned} \Phi(\alpha) &= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \\ Q(\alpha) &= \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = \frac{1}{2}\mathrm{erfc}(\alpha/\sqrt{2}) \end{aligned}$$

Matlab has a built in Q function.

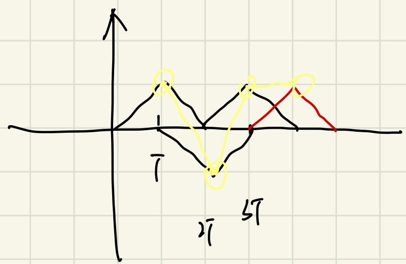
```
>> x=0:5
x =
     0     1     2     3     4     5
>> y=qfunc(x)
y =
  Columns 1 through 4
   0.500000000000000   0.158655253931457   0.022750131948179   0.001349898031630

  Columns 5 through 6
   0.000031671241833   0.000000286651572
```

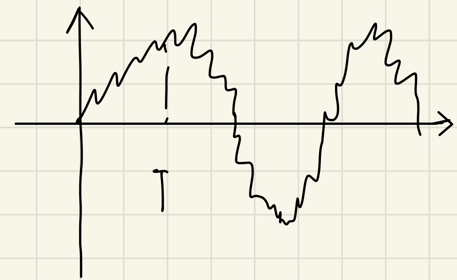




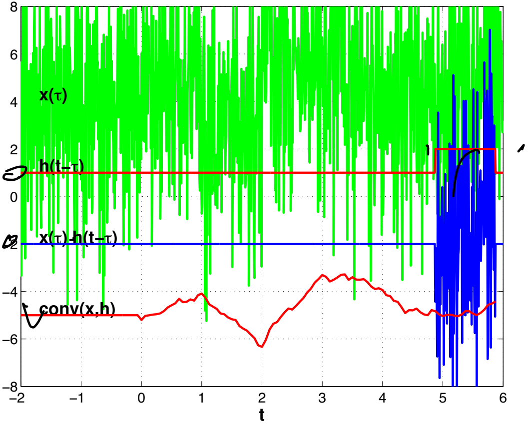
$h_K(t) = f_i(-t)$



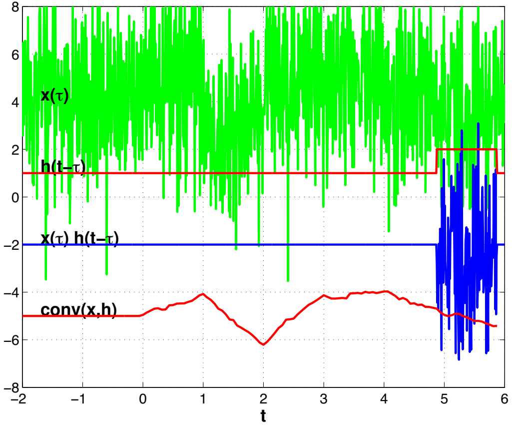
with noise



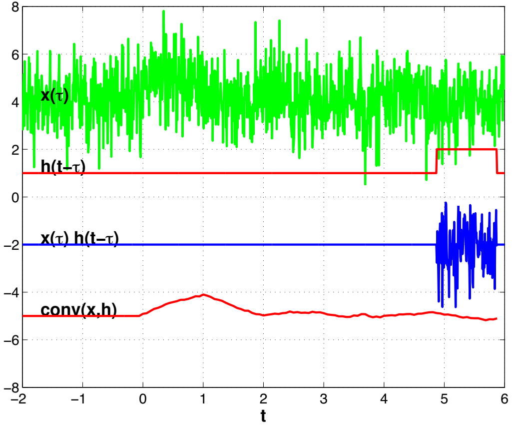
Example 1



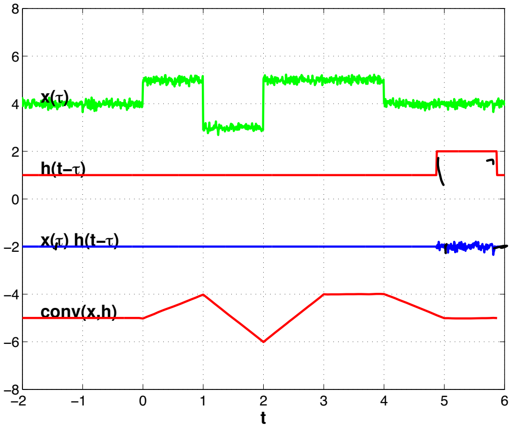
Example 2



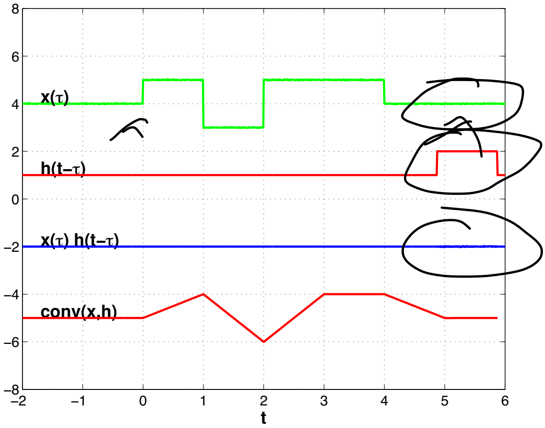
Example 3



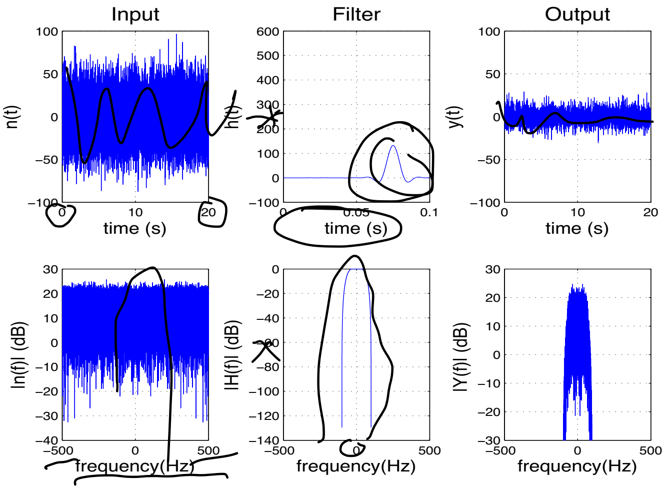
Example 4



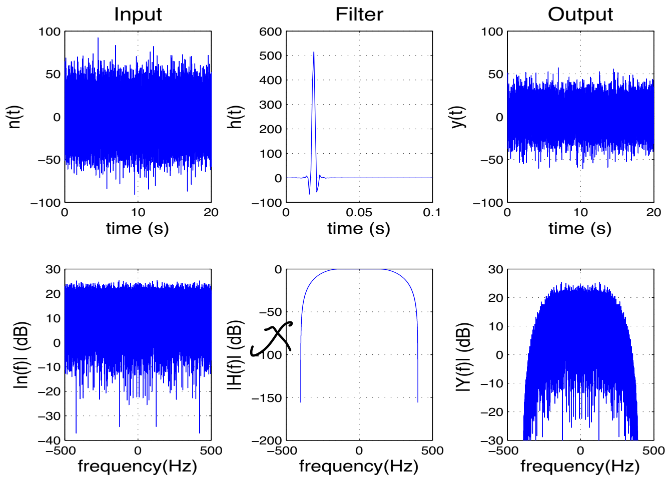
Example 5



Example 6


Input
Filter
Output
n(t)
h(t)
y(t)
time (s)
|n(f)| (dB)
|H(f)| (dB)
|Y(f)| (dB)
frequency(Hz)

Example 7


Input
Filter
Output
n(t)
h(t)
y(t)
time (s)
time (s)
time (s)
|n(f)| (dB)
|H(f)| (dB)
|Y(f)| (dB)
frequency(Hz)
frequency(Hz)
frequency(Hz)

Example 8

$$E[\eta] = E\left[\int_0^T n(t) A_0 \cos(2\pi f_c t) dt\right] = \int_0^T A\cos(2\pi f_c t) \underbrace{E[n(t)]}_{0} dt = 0$$

$$Var(\eta) = E[\eta^2] - E^2[\eta] = E[\eta^2] = E\left[\int_0^T n(t) A\cos(2\pi f_c t) dt \int_0^T n(s) A\cos(2\pi f_c s) ds\right]$$

- Consider noise multiplied by a sinusoid into an integrator. That is,

$$= \int_0^t \int_0^T A^2 \cos^2(2\pi f_c t) E[n(t)n(s)] dt\, ds$$

$$\eta = \int_0^T n(t) A\cos(2\pi f_c t) dt$$

$$= \int_0^T A^2 \cos^2(2\pi f_c t) \frac{N_0}{2} dt$$

$$= \frac{A^2}{2}\frac{N_0}{2}\int_0^T [1+\cos(2 \cdot 2\pi f_c t)] dt$$

where $n(t)$ is white Gaussian noise with two sided power spectral density $N_0/2$. Assume that $f_c T \gg 1$.

$$= \frac{A^2}{4} N_0 T$$

- Because we are performing a linear operation on Gaussian noise, the resulting random variable η is Gaussian. Since the mean on $n(t)$ is zero, the mean of η is zero.
- The variance is determined as follows.

Example 8

$$
\begin{aligned}
\text{Var}[\eta] &= E[\eta^2] \\
&= E[\int_0^T n(t)A\cos(2\pi f_c t)dt \int_0^T n(s)A\cos(2\pi f_c s)ds] \\
&= E[\int_0^T \int_0^T n(t)n(s)A^2\cos(2\pi f_c t)\cos(2\pi f_c s)dtds] \\
&= \int_0^T \int_0^T E[n(t)n(s)]A^2\cos(2\pi f_c t)\cos(2\pi f_c s)dtds \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)A^2\cos(2\pi f_c t)\cos(2\pi f_c s)dtds \\
&= \int_0^T \frac{N_0}{2}A^2\cos^2(2\pi f_c t)dt \\
&= \frac{A^2N_0}{4}\int_0^T (1+\cos(2\pi 2 f_c t))dt = \frac{A^2TN_0}{4}
\end{aligned}
$$

Example 9

Consider zero mean white Gaussian noise with power spectral density $S_x(f) = N_0/2$ or autocorrelation $R_x(\tau) = \frac{N_0}{2}\delta(\tau)$. If the noise is multiplied by a waveform $x(t)$ and integrated then the variance can be calucated as follows.

$E[\eta] = E[\int n(t)x(t)dt$

$= \int \underbrace{E[n(t)]}_{0} x(t)$

$= 0$

$$
\begin{aligned}
\eta &= \int n(t)x(t)dt \\
\mathrm{Var}[\eta] &= E[\eta^2] \\
&= E[\int n(t)x(t)dt \int n(s)x(s)ds] \\
&= \int \int E[n(t)n(s)]x(t)x(s)dtds \\
&= \int \int \frac{N_0}{2}\delta(t-s)x(t)x(s)dtds \\
&= \int \frac{N_0}{2}x^2(t)dt = \frac{N_0}{2}\int x^2(t)dt
\end{aligned}
$$

$\mathrm{Var}[\eta] = E[\eta^2] - \underbrace{E[\eta]}_{0}{}^2$

$= E[\int n(t)x(t)dt \int n(s)x(s)ds]$

$= \int_t \int_s E[n(t)n(s)]\, x(t)x(s)dtds$

$= \int_t \int_s \frac{N_0}{2}\delta(t-s)x(t)x(s)dtds$

Summary: White Gaussian noise

Let $n(t)$ be white Gaussian noise. Then

$$\eta_1 = \int n(t)x(t)dt \quad \Rightarrow \quad \mathrm{Var}[\eta_1] = \frac{N_0}{2}\int x^2(t)dt$$

$$\eta_2 = \int_{-\infty}^{\infty} n(t)h(T-t)dt \quad \Rightarrow \quad \mathrm{Var}[\eta_2] = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt$$

$$\eta_3 = \int_0^T n(t)\cos(2\pi f_c t)dt \quad \Rightarrow \quad \mathrm{Var}[\eta_3] = \frac{N_0 T}{4}.$$

$E[\eta_1] = E[\eta_2] = E[\eta_3] = 0$

Time- shifted orthonormal

Bandwidth of Digital Data Signals

- In many digital communication systems the transmitted signal $W(t)$ is an infinite sequence of amplitude modulated pulses or waveforms i.e.

$$W(t) = \sum_{\ell=-\infty}^{\infty} b_\ell x(t - \ell T) .$$

- This signal is a random process because the data sequence b_ℓ are random.
- The random process $W(t)$, however, is not wide sense stationary. That is, the autocorrelation $R_W(t, \tau) = E[W(t)W(t + \tau)]$ typically depends on both t and τ.
- This means we can not take the Fourier transform of the autocorrelation function to get the power spectral density (because the autocorrelation is a function of both t and τ).

Bandwidth of Digital Data Signals

- A digital data signal is modeled as a random process $Y(t)$ which is a stationary (wide sense) version of a process $W(t)$;

$$Y(t) = W(t - U)$$

  where U is a random variable needed inorder to make $Y(t)$ wide sense stationary.
- In this case if U is uniformly distributed between 0 and T then $Y(t)$ is a wide snese stationary random process. We desire then to compute the auto correlation of $Y(t)$ and also the spectrum of $Y(t)$.
- Assume that $\{b_\ell\}_{\ell=-\infty}^{\infty}$ is a sequences of i.i.d. random variables with zero mean and variance σ^2 (e.g. $P\{b_\ell = +1\} = 1/2 \; P\{b_\ell = -1\} = 1/2$).
- Also assume U and b_ℓ are independent.

Bandwidth of Digital Data Signals

Claim:

$$R_Y(\tau) = \frac{\sigma^2}{T}\int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

$$S_Y(f) = \frac{\sigma^2}{T}|X(f)|^2 \text{ where } X(f) = \mathcal{F}\{x(t)\}$$

power spectral density

Bandwidth of Digital Data Signals

Derivation:

For any t and τ

$$\begin{aligned}
E[Y(t)Y(t+\tau)] &= E\left[\overset{y(t)}{\sum_{\ell=-\infty}^{\infty} b_\ell x(t-\ell T-U)} \overset{y(t+\tau)}{\sum_{m=-\infty}^{\infty} b_m x(t+\tau-mT-U)}\right] \\
&= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \underbrace{E\{b_\ell b_m\}}_{\delta_{\ell m}=\begin{cases}\sigma^2, & \ell=m \\ 0, & \ell\neq m\end{cases}} E\left[x(t-\ell T-U)x(t+\tau-mT-U)\right] \\
&= \sum_{\ell=-\infty}^{\infty} \sigma^2 E\left[x(t-\ell T-U)x(t+\tau-\ell T-U)\right] \\
&= \sum_{\ell=-\infty}^{\infty} \sigma^2 \frac{1}{T}\int_{u=0}^{T} x(t-\ell T-u)x(t+\tau-\ell T-u)du \\
&= \frac{1}{T}\sum_{\ell=-\infty}^{\infty} \sigma^2 \int_{\ell T}^{(\ell+1)T} x(t-v)x(t+\tau-v)dv \quad (v=\ell T+u, \quad dv=du) \\
&= \frac{\sigma^2}{T}\int_{-\infty}^{\infty} x(t-v)x(t+\tau-v)dv \qquad (w=t-v) \\
&= \frac{\sigma^2}{T}\int_{-\infty}^{\infty} x(w)x(w+\tau)dw = \frac{\sigma^2}{T}\int_{-\infty}^{\infty} x(t)x(t+\tau)dt.
\end{aligned}$$

Bandwidth of Digital Data Signals

Derivation: (cont.)

Thus $Y(t)$ is wide sense stationary with autocorrelation

$$R_Y(\tau) = \frac{\sigma^2}{T} \int_{-\infty}^{\infty} x(t)x(t+\tau)dt.$$

Now let $g_1(t) = x(t)$ and $g_2(t) = x(-t)$ then

$$g_1 * g_2(\tau) = \int_{-\infty}^{\infty} g_1(t)g_2(\tau - t)dt = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau)x(t)dt.$$

So

$$\begin{aligned}
R_Y(\tau) &= \frac{\sigma^2}{T}(g_1 * g_2)(\tau) \\
S_Y(f) &= \mathcal{F}\left\{\frac{\sigma^2}{T}(g_1 * g_2)(\tau)\right\} = \frac{\sigma^2}{T} G_1(f)G_2(f) \\
G_1(f) &= \mathcal{F}\{x(t)\} = X(f) \\
G_2(f) &= \mathcal{F}\{x(-t)\} = X^*(f) \\
S_Y(f) &= \frac{\sigma^2}{T}|X(f)|^2
\end{aligned}$$

Example 1: Rectangular Pulses

$$x(t) = \sqrt{P} p_T(t) = \begin{cases} \sqrt{P} & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

$b_\ell \in \{\pm 1\} \Rightarrow \sigma^2 = 1.$

$$\int_{-\infty}^{\infty} x(t)x(t+\tau)dt = P\int_0^T p_T(t+\tau)dt = \begin{cases} P(T-\tau) & 0 \leq \tau \leq T \\ P(T+\tau) & -T \leq \tau \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$p_T(t+\tau) = \begin{cases} 1 & 0 \leq t+\tau \leq T \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 1 & -\tau \leq t \leq T-\tau \\ 0 & \text{elsewhere} \end{cases}$$

Example 1: Rectangular Pulses

$$R_Y(\tau) = \begin{cases} \frac{P}{T}(T - |\tau|) & |\tau| \le T \\ 0 & \text{elsewhere} \end{cases}$$

$$S_Y(f) = \mathcal{F}\{R_Y(\tau)\} = P[\mathcal{F}\{\Lambda_T(t)\}]$$

$$X(f) = \mathcal{F}\{\sqrt{P}p_T(t)\} = \sqrt{P}T\left[\frac{\sin(2\pi fT/2)}{2\pi fT/2}\right] e^{-j2\pi fT/2}$$

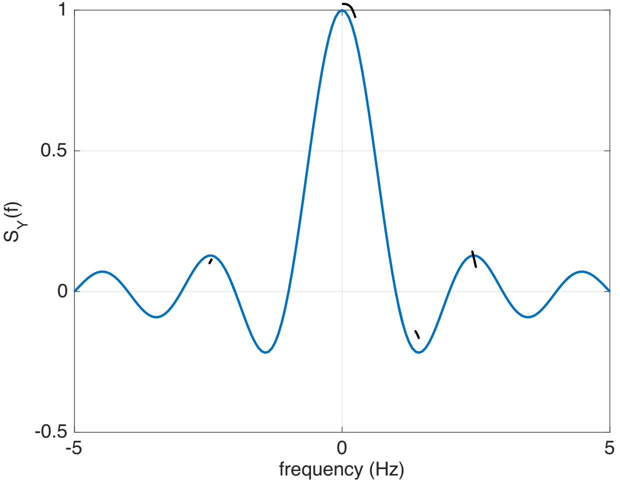
$$|X(f)|^2 = PT^2\frac{\sin^2(2\pi fT/2)}{(2\pi fT/2)^2}$$

$$S_Y(f) = \frac{1}{T}|X(f)|^2 = T\frac{\sin^2(2\pi fT/2)}{(2\pi fT/2)^2}$$

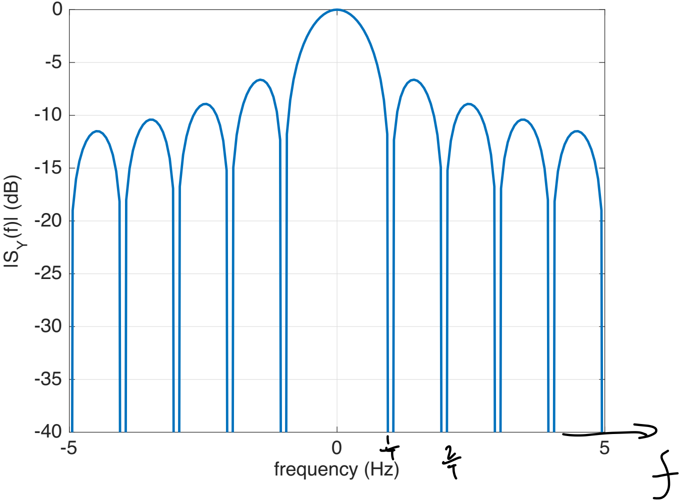
$$S_Y(f) = PT\frac{\sin^2(\pi fT)}{(\pi fT)^2} = PT\text{sinc}^2(fT)$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Spectrum for Rectangular Pulses

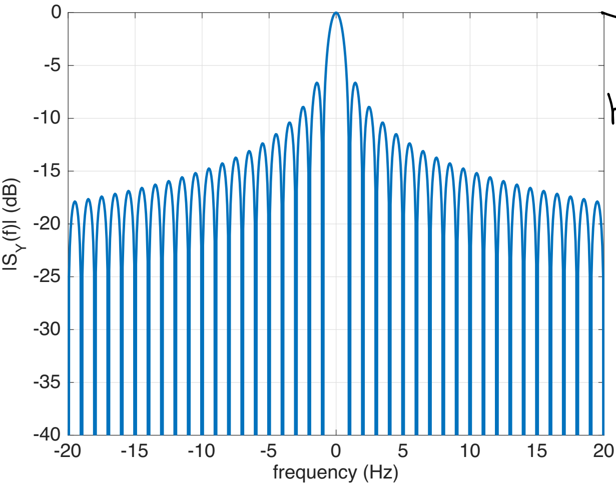


Spectrum for Rectangular Pulses



Spectrum for Rectangular Pulses

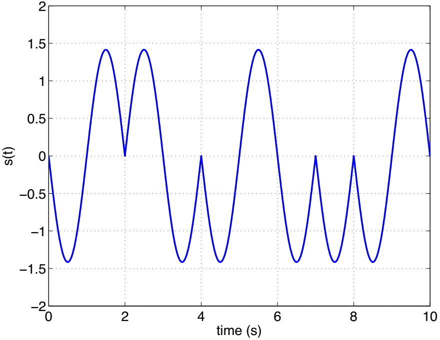
high frequency



Example 2: Half Cosine/Sine Pulses

Consider a pulse that consists of a half sinusoid.

$$x(t) = \sqrt{2P}\sin(\pi t/T)p_T(t)$$



Spectrum for Half Cosine Pulses

The half cosine pulse is $x(t) = \sqrt{2P}\sin(\pi t/T)p_T(t)$ The Fourier transform is given by

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} \sqrt{2P}\sin(\pi t/T)p_T(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \frac{\sqrt{2P}}{2j}(e^{j\pi t/T} - e^{-j\pi t/T})p_T(t)e^{-j2\pi ft}dt \\
&= \frac{\sqrt{2P}}{2j}\int_0^T (e^{j\pi t/T} - e^{-j\pi t/T})e^{-j2\pi ft}dt \\
&= \frac{\sqrt{2P}}{2j}\int_0^T (e^{j\pi t/T(1-2fT)} - e^{-j\pi t/T(1+2fT)})dt \\
&= \frac{\sqrt{2P}}{2j}\left[\frac{e^{j\pi t/T(1-2fT)}}{j\pi(1-2fT)/T} + \frac{e^{-j\pi t/T(1+2fT)}}{j\pi(1+2fT)/T}\right]_0^T \\
&= \frac{\sqrt{2P}}{2j}\left[\frac{e^{j\pi(1-2fT)}-1}{j\pi(1-2fT)/T} + \frac{e^{-j\pi(1+2fT)}-1}{j\pi(1+2fT)/T}\right] \\
&= \frac{\sqrt{2P}T}{2j}\left[\frac{-e^{-j2\pi fT}-1}{j\pi(1-2fT)} + \frac{-e^{-j2\pi fT}-1}{j\pi(1+2fT)}\right] \\
&= \frac{\sqrt{2P}T}{2\pi}(1+e^{-j2\pi fT})\left[\frac{1}{(1+2fT)} + \frac{1}{(1-2fT)}\right]
\end{aligned}
$$

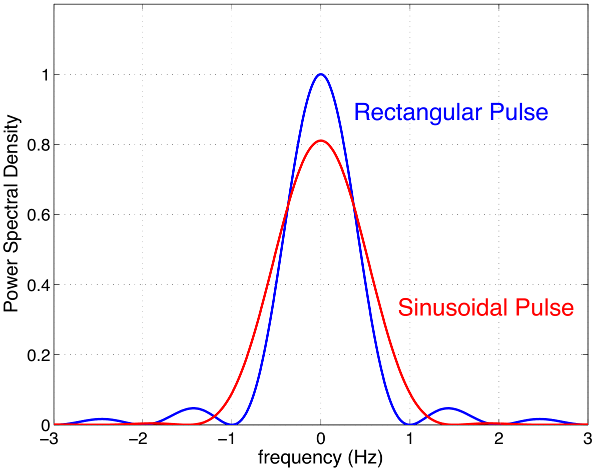
Spectrum for Half Cosine Pulses

$$
\begin{aligned}
X(f) &= \frac{\sqrt{2P}T}{2\pi}(1+e^{-j2\pi fT})\left[\frac{2}{(1-4f^2T^2)}\right] \\
&= \frac{\sqrt{2P}T}{2\pi}(1+e^{-j2\pi fT})\left[\frac{2}{(1-4f^2T^2)}\right] \\
&= \frac{\sqrt{2P}T}{2\pi}e^{-j2\pi fT/2}(e^{j2\pi fT/2}+e^{-j2\pi fT/2})\left[\frac{2}{(1-4f^2T^2)}\right] \\
&= \frac{\sqrt{2P}T}{2\pi}e^{-j2\pi fT/2}(2\cos(2\pi fT/2))\left[\frac{2}{(1-4f^2T^2)}\right] \\
&= \frac{2\sqrt{2P}T}{\pi}e^{-j2\pi fT/2}(\cos(2\pi fT/2))\left[\frac{1}{(1-4f^2T^2)}\right] \\
|X(f)|^2 &= \frac{8PT^2}{(\pi^2)}\left[\frac{\cos^2(2\pi fT/2)}{(1-4f^2T^2)^2}\right] \\
S_Y(f) &= \frac{1}{T}|X(f)|^2 \\
&= \frac{8PT}{(\pi^2)}\left[\frac{\cos^2(2\pi fT/2)}{(1-4f^2T^2)^2}\right]
\end{aligned}
$$

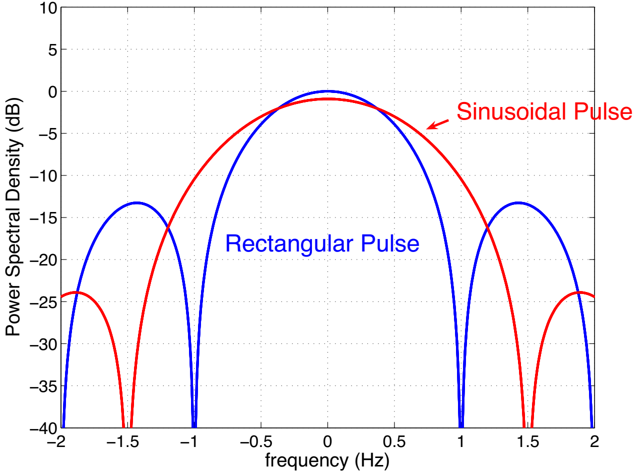
Summary for Half Cosine Pulses

- $x(t) = \sqrt{2P}\sin(\pi t/T)p_T(t)$
- $X(f) = \frac{8PT^2\cos^2(\pi fT)}{\pi^2(1-4f^2T^2)^2}$
- The energy of the pulse is $E = \int x^2(t)dt = \int |X(f)|^2 df = PT$
- The power spectral density of the pulse is $S_Y(f) = \frac{8PT\cos^2(\pi fT)}{\pi^2(1-4f^2T^2)^2}$.
- The power $\int S_Y(f)df = P$
- Notice that the spectrum falls of as $1/f^4$ rather than $1/f^2$.

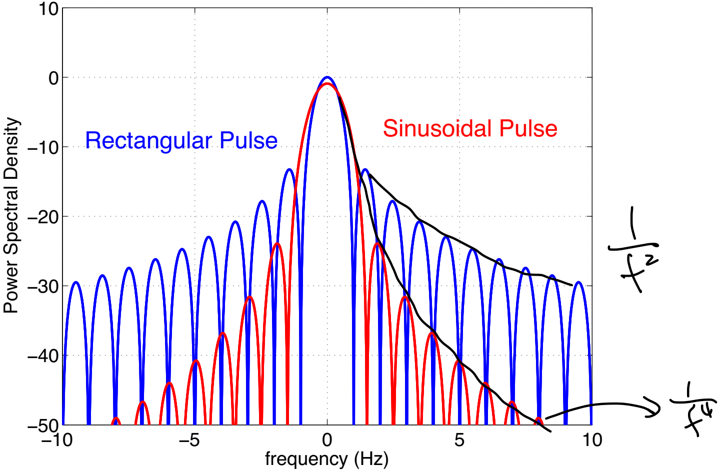
Spectrum for Sine Pulses



Spectrum for Sine Pulses



Spectrum for Sine Pulses



Example 3: Square-Root Raised-Cosine Pulses

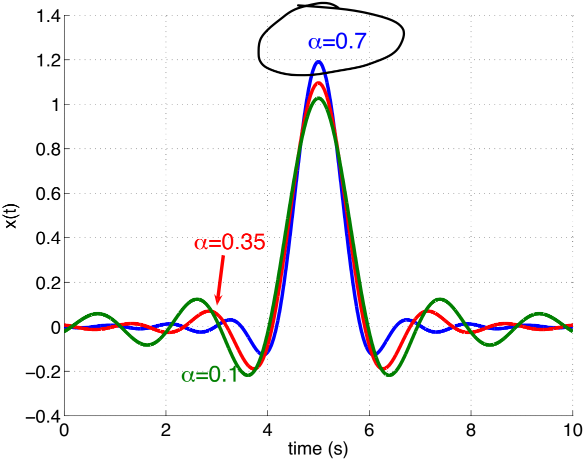
$$x(t) = \sqrt{PT}\frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T\cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T}.$$

$$X(f) = \begin{cases} \sqrt{P}T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ T\sqrt{\frac{P}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \end{cases}$$

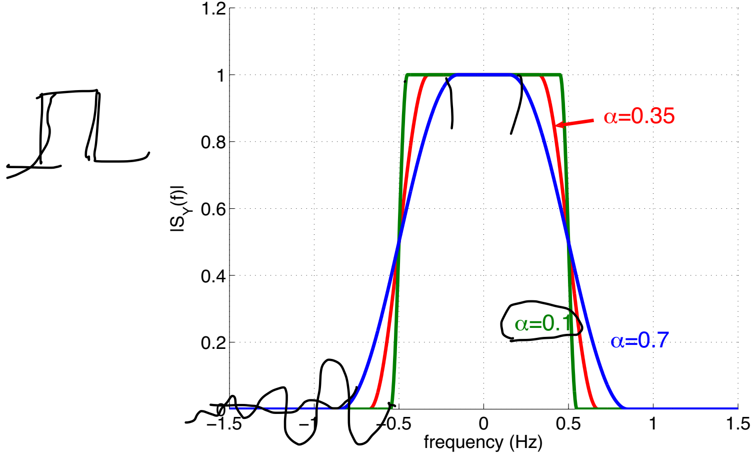
$$|X(f)|^2 = \begin{cases} PT^2, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{PT^2}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

$$S_Y(f) = \begin{cases} PT, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{PT}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

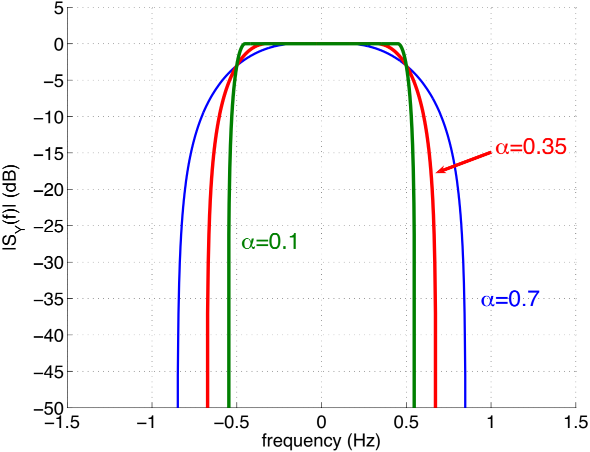
Square-Root Raised-Cosine Pulses



Spectrum for Square-Root Raised-Cosine Pulses



Spectrum for Square-Root Raised-Cosine Pulses



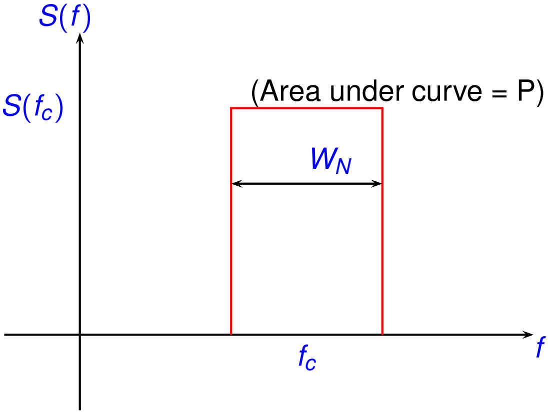
Square-Root Raised-Cosine Pulses

- The smaller α the longer the time for the pulse to die out.
- The smaller α the narrower the spectrum
- When $\alpha \to 0$ the spectrum becomes flat and concentrated over the interval $[-\frac{1}{2T}, \frac{1}{2T}]$. This is called the Nyquist pulse.

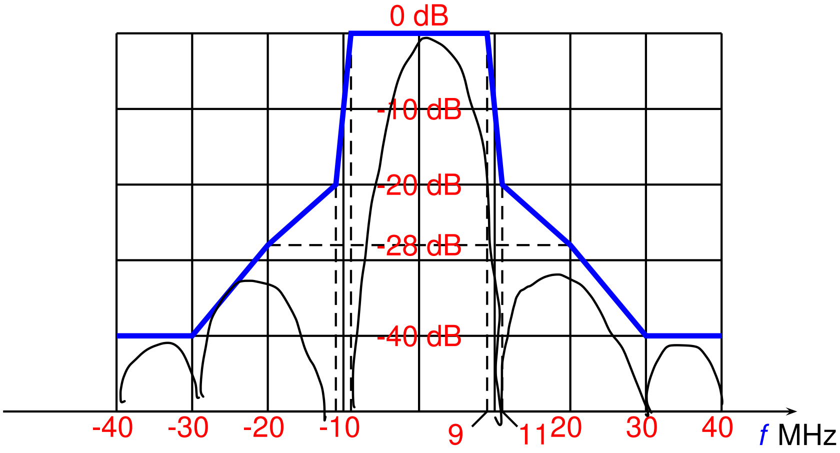
Definitions of Bandwidth for Digital Signals

1. Null-to-Null bandwidth $\triangleq$ bandwidth of main lobe of power spectral density
2. 99% power bandwidth containtment $\triangleq$ bandwidth such that 1/2% of power lies above upper band limit and 1/2% lies below lower band limit
3. x dB bandwidth $\triangleq$ bandwidth such that spectrum is x dB below spectrum at center of band (e.g. 3dB bandwidth)
4. Noise bandwidth $\triangleq W_N = P/S(f_c)$ where P is total power and $S(f_c)$ is value of spectrum at $f = f_c$.
   f_c
5. Gabor bandwidth $\triangleq \sigma$ where $\sigma^2 = \frac{\int_{-\infty}^{\infty}(f-f_c)^2 S(f)df}{\int_{-\infty}^{\infty} S(f)df}$
6. Absolute bandwidth $\triangleq W_A = \min\{W : S(f) = 0 \forall |f| > U\}$

Definitions of Bandwidth for Digital Signals



Wi-Fi Spectral Mask



Wi-Fi Spectral Mask with Sine Pulses

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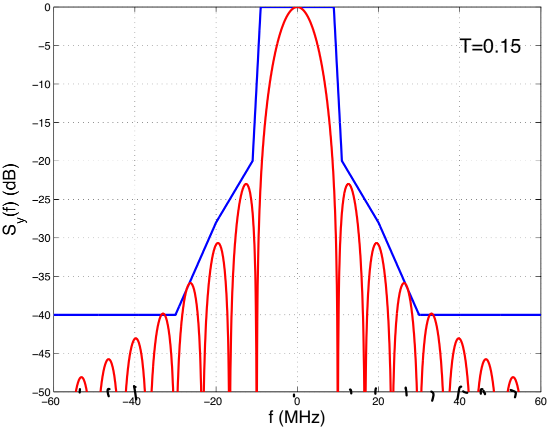

Rectangular Pulse Example

- $$S_Y(f) = T\frac{\sin^2 \pi f T}{(\pi f T)^2} = T\mathrm{sinc}^2(fT)$$
- $\mathrm{sinc}((f - f_c)T = 0$ at $\pi(f - f_c)T = h\pi \; h = \pm 1, \pm 2, \pm 3, \ldots$
  $f = f_c + \frac{n}{T}, \quad n = \pm 1, \pm 2, \ldots$
- Null to null bandwidth $\stackrel{\Delta}{=}$ width of main lobe of spectral density.
- For PSK null to null bandwidth = $\frac{2}{T}$
- Fractional Power containment Bandwidth $\stackrel{\Delta}{=}$ width of frequency band which leaves 1/2% of signal power above upper band limit and 1/2% of signal power below band limit
- For PSK 99% energy bandwidth = $\frac{20.56}{T}$
- We would like to find modulation schemes which decrease the bandwidth while retaining acceptable performance

Bandwidths for Different Pulse Shapes

Should know

Pulse Shape	Bandwidth Definition				
	Null-to-null	99% Power	35dB	Noise	3dB
Rectangular	2.0	20.56	35.12	1.00	0.88
Sinusoidal	3.00	1.18	3.24	1.23	0.59
Sinc pulse (square-root raised cosine with $\alpha = 0$	1.00	1.00	1.00	1.00	1.00
Square-Root Raised Cosine ($\alpha = 0.05$)	1.05	1.03	1.05	1.0	1.0
Square-Root Raised Cosine ($\alpha = 0.25$)	1.25	1.10	1.24	1.0	1.0
Square-Root Raised Cosine ($\alpha = 0.50$)	1.50	1.27	1.49	1.0	1.0

Bandwidths for Different Pulse Shapes

Pulse Shape	Bandwidth Definition: Null-to-null	99% Power	35dB	Noise	3dB
Rectangular	2.0	20.56	35.12	1.00	0.88
Sinusoidal	3.00	1.18	3.24	1.23	0.59
Sinc pulse (square-root raised cosine with $\alpha = 0$	1.00	1.00	1.00	1.00	1.00
Square-Root Raised Cosine ($\alpha = 0.05$)	1.05	1.03	1.05	1.0	1.0
Square-Root Raised Cosine ($\alpha = 0.25$)	1.25	1.10	1.24	1.0	1.0
Square-Root Raised Cosine ($\alpha = 0.50$)	1.50	1.27	1.49	1.0	1.0