EECS 501
Discussion 4
Solution

Review

- Variance
  - $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$
  - $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
  - If X and Y are independent, $Var(X + Y) = Var(X) + Var(Y)$
  - Law of Total Variance $Var(X) = Var(E(X|Y)) + E(Var(X|Y))$
- Cumulative Distribution Function (CDF)
  - The CDF of a random variable is defined as follows,

$$F_X(x) = P\{X \leq x\}$$

  - Properties of CDF:
    * Range of distribution function: $0 \leq F_X(x) \leq 1 \qquad \forall x \in \mathbb{R}$
    * Value at ∞: $F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1$
    * Value at $-\infty$: $F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0$
    * Probability of being in $(a, b]$: $P\{a < X \leq b\} = F_X(b) - F_X(a)$
    * CDF is an increasing function.
- Continuous Random Variable:
  - Definition: A random variable X is said to be continuous if it's CDF is continuous everywhere in the real line,
- Probability Density Function (PDF):
  - For a continuous random variable, PDF is defined as the derivative of CDF

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}$$

  - Properties of density:

    The probability of $\mathbb{R}$ is 1: $\int_{-\infty}^{\infty} f_X(x)dx = 1$
    Probability density function is non-negative:$f_X(x) \geq 0$
    Probability of an event B: $P\{X \in B\} = \int_B f_X(x)dx$
    $P(x \leq X \leq x + \delta) \approx \delta f_X(x) \qquad$ for small δ
- Relation between CDF and PDF,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx \qquad\qquad f_X(x) = \frac{\partial F_X(x)}{\partial x}$$

- Important Remark: for a **continuous** random variable:

$$
\begin{aligned}
P(X \le a) &= P(X < a) \qquad \forall a \in \mathbb{R} \\
P(X \ge b) &= P(X > b) \qquad \forall b \in \mathbb{R}
\end{aligned}
$$

- Measurability
  - Measurable function: A function $f : X \to Y$ is measurable with respect to the measurable spaces (X, Σ_X) and (Y, Σ_Y) if for every $B \in \Sigma_Y$, we have $f^{-1}(B) \in \Sigma_X$.
  - Measurable random variable: a random variable is a function from Ω to Ω_X.

Practice Problems

Problem 1. Calculate the variance of a binomial random variable X with parameters N and p.

Solution:

First approach:

$$
\begin{aligned}
Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \sum_{x=0}^{N} x^2 \binom{N}{x} (1-p)^{N-x} p^x - (Np)^2 \\
&= \sum_{x=2}^{N} x(x-1) \frac{N!}{(N-x)!x!} (1-p)^{N-x} p^x + \sum_{x=1}^{N} x \binom{N}{x} (1-p)^{N-x} p^x - (Np)^2 \\
&= p^2 \sum_{x=2}^{N} \frac{N!}{(N-x)!(x-2)!} (1-p)^{N-x} p^{x-2} + Np - (Np)^2 \\
&= p^2 N(N-1) \sum_{x'=0}^{N-2} \frac{(N-2)!}{(N-2-x')!(x')!} (1-p)^{N-2-x'} p^{x'} + Np - (Np)^2 \\
&= p^2 N(N-1) + Np - (Np)^2 = Np(1-p)
\end{aligned}
$$

Second approach:

Show X as a summation of N independent Bernoulli random variables, $X = X_1 + X_2 + \cdots + X_N$. We have

$$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_N) = Np(1-p)$$

Problem 2. Calculate the variance of a geometric random variable X with parameter p.

Solution:

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p - \frac{1}{p^2}$$

For $|a| < 1$, we have

$$\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$$

By taking derivative of the above equation w.r.t. a, we have

$$
\begin{aligned}
\sum_{x=1}^{\infty} x a^{x-1} &= \frac{1}{(1-a)^2} \\
\sum_{x=1}^{\infty} x^2 a^{x-1} &= \frac{1+a}{(1-a)^3}
\end{aligned}
$$

Therefore, we have

$$Var(X) = \sum_{x=1}^{\infty} x^2(1-p)^{x-1}p - \frac{1}{p^2} = p\frac{1+(1-p)}{(1-(1-p))^3} - \frac{1}{p^2} = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Problem 3. The number of cars waiting for a traffic light is a geometric random variable with parameter q. You know the occupants of each car with probability p; $0 < p < 1$. Let N be the total number of cars whose occupants you are familiar with.

- What is the conditional pmf of N given the number of cars waiting is k?
- Find the expected value of N.

Solution:

- Let K denote the random variable of the number of cars waiting. Given the number of cars waiting is k, the situation for N is exactly the same as the binomial random variable. So

$$P_{N|\{K=k\}}(n) = \binom{k}{n} p^n(1-p)^{k-n} \quad \text{for } n \leq k$$

- By law of total expectation,

$$E[N] = \sum_{k=1}^{\infty} E[N|K=k] \cdot P_K(k)$$

Since given $K = k$, the random variable N is $Binom(k, p)$, $E[N|K = k] = kp$. So we have

$$E[N] = \sum_{k=1}^{\infty} kp \cdot P_K(k) = p\sum_{k=1}^{\infty} k \cdot P_K(k) = pE[K] = \frac{p}{q}$$

Note that since K is $Geometric(q)$, $E[K] = \frac{1}{q}$.

Problem 4. Assume X is a random variable with mean 1 and Y is a random variable with mean 1 and variance X^2. We have $Z \sim Geometric(p)$. Find $E(YE(Z|X))$.

Solution:

Using law of iterated expectation,

$$E(YE(Z|X)) = E(E(YE(Z|X)|X)) = E(E(Z|X)E(Y|X)) = E(E(Z|X)) = E(Z) = \frac{1}{p}$$

The second equality is due to the fact that $E(Z|X)$ is a function of random variable X. The third equality is because $E(Y|X) = 1$.

Problem 5. Assume we have $Y = X_1X_2...X_N$, where $\mathbb{E}[X_i] = \frac{1}{i}$ and $N \sim Geo(p)$. Assume X_i's and N are mutually independent. Find $E(Y)$ (Hint: $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$).

Solution:

Using law of iterated expectation,

$$E(Y) = E(E(Y|N)) = E(E(X_1X_2...X_N|N)) = E(E(X_1)E(X_2)...E(X_N)) = E(\frac{1}{1}\frac{1}{2}...\frac{1}{N})$$
$$= E(\frac{1}{N!}) = \sum_{k=1}^{\infty}\frac{1}{k!}(1-p)^{k-1}p = \frac{p}{1-p}\sum_{k=1}^{\infty}\frac{1}{k!}(1-p)^k = \frac{p}{1-p}(\sum_{k=0}^{\infty}\frac{1}{k!}(1-p)^k - 1)$$
$$= \frac{p}{1-p}(e^{(1-p)} - 1)$$

Problem 6. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas trains headed for destination B arrive at 15-minute intervals starting at $7:05$ a.m. If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 a.m. he/she gets on the first train that arrives. What is the probability that he/she goes to destination A?

Solution:

The arrival times of train A is $\{7, 7:15, 7:30, 7:45, 8\}$ and the arrival times of train B is $\{7:05, 7:20, 7:35, 7:50\}$. In order to make sure that the passenger gets on train A, he needs to arrive at the time intervals $\{[7:05, 7:15], [7:20, 7:30], [7:35, 7:45], [7:50, 8]\}$. The sum of lengths of these intervals is 40 mins and the length of the whole time interval is 60 mins. So the probability of him arriving at these intervals is $\frac{40}{60} = \frac{2}{3}$.

Problem 7. Let X be a continuous random variable with PDF

$$f_X(x) = x^2(2x + \frac{3}{2}), \quad \text{for } 0 \leq x \leq 1$$

- Find the CDF of X.
- Let $Y = \frac{2}{X} + 3$. Find $Var(Y)$.

Solution:

- $F_X(x) = \int_{u=0}^{x} u^2(2u + \frac{3}{2})du = \frac{1}{2}(x^4 + x^3)$, for $0 \leq x \leq 1$.
- $Var(Y) = 4Var(\frac{1}{X}) = 4\mathbb{E}[\frac{1}{X^2}] - 4\mathbb{E}[\frac{1}{X}]^2$. We have

$$\mathbb{E}[\frac{1}{X^2}] = \int_{x=0}^{1} \frac{1}{x^2}x^2(2x + \frac{3}{2})dx = \int_{x=0}^{1} (2x + \frac{3}{2})dx = \frac{5}{2}$$

$$\mathbb{E}[\frac{1}{X}] = \int_{x=0}^{1} \frac{1}{x}x^2(2x + \frac{3}{2})dx = \int_{x=0}^{1} x(2x + \frac{3}{2})dx = \frac{17}{12}$$

Therefore, $Var(Y) = \frac{71}{36}$.

Problem 8. Measurability

Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{a, b, \ldots, k, l\}$, $\mathcal{F}$ is the smallest sigma-algebra that contains the events $A = \{a, b, c, d, e, f, g, h\}$ and $B = \{e, f, g, h, i, j, k, l\}$, and $P(A) = \frac{1}{3}$, $P(B) = \frac{4}{5}$. Consider the following random variable

$$X(w) = \begin{cases} 1 & \text{if } w = a, b \\ 2 & \text{if } w = c, d \\ 3 & \text{if } w = e, f, g, h \\ 4 & \text{if } w = i, j, k, l \end{cases}$$

Note that $\Omega_X = \{1, 2, 3, 4\}$.

- Is P a valid probability assignment?
- Show that X is not measurable with respect to $(\Omega, \mathcal{F})$, $(\Omega_X, \mathcal{F}_X)$, where $\mathcal{F}_X = 2^{\Omega_X}$.
- Find the largest sigma-algebra $\mathcal{F}_X^*$ on Ω_X such that X measurable with respect to $(\Omega, \mathcal{F})$, $(\Omega_X, \mathcal{F}_X^*)$.

Solution:

- We note that $P(A) = \frac{1}{3}$ and $P(B) = \frac{4}{5}$. Moreover, we have $A \cup B = \Omega$, and hence $P(A \cup B) = 1$. Let us use the inclusion-exclusion principle. We have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{2}{15} \geq 0,$$

and $P(A \cap B) \leq 1$. So we infer that P is a valid probability assignment. The smallest sigma-algebra that contains A and B is given below:

$$\mathcal{F} = \{\phi, \Omega, A, B, A \cap B, A \cap B^c, A^c \cap B, A^c \cup B^c\}.$$

We note that $P(A \cap B) = \frac{2}{15}$, $P(A \cap B^c) = \frac{1}{5}$, $P(A^c \cap B) = \frac{2}{3}$, and $P(A^c \cup B^c) = \frac{13}{15}$.

- The given function is not measurable. Because, we are unable to find $P(X = 1)$.
- We have
$$\mathcal{F}^* = \{\phi, \Omega, \{1,2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{3,4\}\},$$
because, we can find the probability of every event in $\mathcal{F}^*$.

Problem 9. Measure on the power set

Consider the following measure on the set of all subsets of the unit interval $[0,1]$: for all $A \subset [0,1]$,

$$P(A) = \begin{cases} 1 & \text{if } \frac{1}{\pi} \in A \\ 0 & \text{otherwise} \end{cases}$$

Show that this measure satisfies all the three axioms of probability.

Solution: The first two axioms are clearly satisfied. To show the third axiom, let $A_1, A_2, ..., A_n, ...$ be a disjoint collection of events.

Case 1: $(\bigcup\limits_{i=1}^{\infty} A_i)$ does not contain $\frac{1}{\pi}$. This implies that none of A_i's contain $\frac{1}{\pi}$.

$$P(\bigcup_{i=1}^{\infty} A_i) = 0$$

$$\sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} 0 = 0$$

$$\Rightarrow P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Case 2: $(\bigcup\limits_{i=1}^{\infty} A_i)$ contains $\frac{1}{\pi}$. Since A_i's is a disjoint collection, $\frac{1}{\pi}$ can only belong to one of A_i's. Let $\frac{1}{\pi} \in A_j$. Then $P(A_j) = 1$ and $P(A_i) = 0, \forall i \neq j$.

$$P(\bigcup_{i=1}^{\infty} A_i) = 1$$

$$\sum_{i=1}^{\infty} P(A_i) = \sum_{i \neq j}^{\infty} P(A_i) + P(A_j) = 1$$

$$\Rightarrow P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$