University of Michigan Fall 2021

EECS501: Homework 4

Assigned: September 27, 2021
Due: October 5, 2021 at 11:59PM on gradescope
Text: "Probability and random processes" by J. A. Gubner

Reading assignment: Please read Chapter 1 and Chapter 2. In lectures we are covering the material in this order: 2.1 - 2.4 till Indicator functions.

1. Law of Iterated Expectation [10 points]

Let Y be a uniform random variable over the interval $[0, 1]$. Let X be a Binomial random variable with parameters N and Y, i.e., $X \sim Binomial(N, Y)$. Find the pdf of $E(X|Y)$ and find $E(X)$.

2. Search Engine [10 points]

A search engine is looking for a particular document. Any given website carries a copy of the document with probability p independently of others. The engine spends a random amount of time, that is exponentially distributed with parameter λ, searching for a copy of the document in a given website until either the document is found and then the search is stopped or the document is not found and the next website is visited. Assume that the websites are listed as $1, 2, 3, \ldots,$ all the way to infinity (a reasonable model for the internet which has around 1 billion websites), and let U_i denote the time spent on website i. Let V denote the time spent by the engine to find a copy of the document. Observe that $V = U_1 + U_2 + \ldots + U_N$, $U_i \sim \exp(\lambda)$, and N is a geometric random variable with parameter p, and $U_1, U_2, \ldots,$ are mutually independent. Find $E(V)$.

3. Fluorescent Lamps [10 points]

The lifetime in hours of a certain kind of fluorescent lamps is a random variable with PDF

$$f(x) = \begin{cases} 0 & x \leq 100 \\ \frac{100}{x^2} & x > 100 \end{cases}.$$

What is the probability that exactly 2 of the 5 lamps fail within the first 150 hours of operation. Assume that the events of failure (within the first 150 hours) are independent.

4. CDF of non-negative random variable [5 points each]

(a) Show that for any non-negative random variable X, we have

$$E(X) = \int_0^\infty P(X > t)dt$$

(b) Let the CDF of X be given by $1 - e^{-x^2}$ for $x \geq 0$. Find $E(X)$.

5. Variance [5 points each]

(a) Let X and Y be independent random variables with $\mathrm{Var}(X) = 1$, and $\mathrm{Var}(Y) = 2$. Let $Z = 3X + 4Y$. Find $\mathrm{Var}(Z)$.

(b) Consider two random variables X and Y, where Y is uniformly distributed over the set $\{1, 2, 3\}$. Moreover, it is given that $E(X|Y = i) = i$ and $E(X^2|Y = i) = i^2 + 1$. Find $\text{Var}(X)$.

(c) Consider n independent random variables $X_1, X_2, \ldots, X_n$, where all them have the same variance σ^2. These random variables do not necessarily have the same mean. Let $Y = \frac{1}{n}(X_1 + X_2 + \ldots X_n)$. Show $\text{Var}(Y) = \frac{\sigma^2}{n}$.

6. Uniform Distribution [5 points each]

(a) A point is chosen at random and uniformly on a yard stick, thus dividing it into two pieces. Find the probability that the ratio of the shorter to the longer piece is less than $\frac{1}{4}$.

(b) Let Y be a uniformly distributed random variable over the interval $(0, 5)$. What is the probability that the roots of the following equation are both real?

$$4x^2 + 4xY + Y + 2 = 0$$