Homework 2 YUZHAN JIANG

P1:

We have $B = A - 10I$ where $A, B \in F^{N \times N}$

① let λ_A be the eigenvalue of A, then there exists a vector x such that $Ax = \lambda x$

Therefore, $Bx = (A - 10I)x = Ax - 10x = \lambda_A x - 10x = (\lambda_A - 10)x$

$\Rightarrow Bx = (\lambda_A - 10)x$

$\therefore$ $\lambda_A - 10$ is the eigenvalue of B, and the eigenvector is also the x

② let λ_B be the eigenvalue of B, then there exists a vector x such that $Bx = \lambda_B x$

Since $B = A - 10I \Rightarrow A = B + 10I$

$Ax = (B + 10I)x = Bx + 10x = \lambda_B x + 10x = (\lambda_B + 10)x$

$\therefore$ $\lambda_B + 10$ is the eigenvalue of A, and the eigenvector is also the x.

Thus, A, B have the same eigenvector and the relationship between A and B is $\lambda_A = \lambda_B + 10$

$$A = \underset{N\times N}{[\quad]}\ \underset{N\times 1}{[\ ]} = \underset{N\times 1}{[\ ]}$$

P2: we have $v_1, v_2, \cdots v_N$ denote orthonormals vectors in R^N

proof: ① $\longrightarrow$: If $Av_1, Av_2, \cdots, Av_N$ are orthonormal vectors, then $A \in R^{N\times N}$ is an orthogonal matrix.

By the definition of orthonormal vectors:

$$\begin{cases} (Av_1)'Av_1 = (Av_2)'Av_2 = (Av_3)'(Av_3) = \cdots = (Av_N)'Av_N = 1 \Rightarrow v_1'A'Av_1 = \cdots v_N'A'Av_N = 1 \\ v_1'v_1 = v_2'v_2 = v_3'v_3 = \cdots = v_N'v_N = 1 \end{cases}$$

Let $V = [v_1, v_2, v_3 \cdots v_n]$ matrix, $(AV)'AV = V'A'AV = \begin{bmatrix} v_1'A'Av_1 & & \\ & \ddots & \\ & & v_N'A'Av_N \end{bmatrix} = I$

$$V'V = \begin{bmatrix} v_1'v_1 & 0 & 0 \\ 0 & v_2'v_2 & \\ 0 & & \ddots \\ & & & v_n'v_n \end{bmatrix} = I$$

$\therefore$ AV and V is orthogonal

Next, $I = V'A'AV$

$\Rightarrow VIV' = VV'A'AVV'$

$\Rightarrow VV' = (VV')A'AVV'$

$I = IA'AI$

$\therefore A'A = I$

$\Rightarrow A$ is an orthogonal matrix

② $\longleftarrow$: if $A \in R^{N\times N}$ is an orthogonal matrix, then $Av_1, Av_2, \ldots Av_N$ are orthonormal vectors

we have $A'A = I$ by the def of orthogonal

we are given $v_1, v_2 \ldots v_N$ are orthornormal vectors

$$\therefore (Av_i)'(Av_j) = v_i'A'Av_j = v_i'Iv_j = v_i'v_j = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

$\therefore Av_1, Av_2 \cdots Av_N$ are orthonormal vectors

P3. $A \in R^{m\times n}$ Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$

$$\underset{n\times n}{A'}\underset{m\times n}{A} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & & & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}^2+a_{21}^2+\cdots a_{m1}^2 & & \\ & \ddots & \\ & & a_{1n}^2+a_{2n}^2+\cdots a_{mn}^2 \end{bmatrix}$$

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2} = \text{sqrt}\,(a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2 +$$
$$a_{21}^2 + a_{22}^2 + \cdots + a_{2n}^2 +$$
$$\vdots$$
$$a_{m1}^2 + a_{m2}^2 + \cdots + a_{mn}^2)$$
$$= \sqrt{tr(A'A)}$$

let $A = U\Sigma V'$ by SVD

$$\Rightarrow \|A\|_F = \sqrt{tr((U\Sigma V')'(U\Sigma V'))}$$

$$= \sqrt{tr(V\Sigma' U' U \Sigma V')}$$

$$= \sqrt{tr(V\Sigma'\Sigma V')}$$

$$= \sqrt{tr(V'V\Sigma'\Sigma)} \quad \text{(by the cyclic commutative property)}$$

$$= \sqrt{tr(\Sigma'\Sigma)}$$

$$= \sqrt{\sum_{i=1}^{r}\sigma_i^2} \quad \text{where } r \text{ is the rank of matrix } A$$

P4. In Problem 3:

we have $||A||_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2}$ where r is the rank of A

$\sigma_1, \sigma_2, \sigma_3, \cdots\cdots, \sigma_r$ are the singular values of A and they have such following relationship.

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots\cdots \sigma_r > 0$$

$$\therefore \quad ||A||_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots \sigma_r^2} \geq \sqrt{\sigma_1^2} = \sigma_1$$

r is the rank of A, it is also the # of linearly independent rows

$$\therefore \quad r \leq \min(n, m)$$

$$\therefore \quad ||A||_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots \sigma_r^2} \leq \sqrt{\sum_{i=1}^{r} \sigma_1^2} \leq \sqrt{\sum_{i=1}^{\min(n,m)} \sigma_1^2} = \sqrt{\min(n,m)}\, \sigma_1$$

Therefore, $\sigma_1 \leq ||A||_F \leq \sqrt{\min(M,N)}\ \sigma_1$

P5: we have $x \in F^{M}$ and $y \in F^{N}$, So that $xy^T \in F^{M\times N}$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$y \otimes x = \begin{bmatrix} y_1 x \\ y_2 x \\ \vdots \\ y_n x \end{bmatrix} = \begin{bmatrix} y_1 x_1 \\ \vdots \\ y_1 x_m \\ \vdots \\ y_n x_1 \\ \vdots \\ y_n x_m \end{bmatrix}$$ Length of $y \otimes x$ is MN

$$\underset{M\times 1}{xy^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}} \underset{1\times N}{[y_1\, y_2 \cdots y_n]} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & & & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x\, y_1, & x\, y_2 & \cdots & x\, y_n \\ | & | & & | \end{bmatrix}$$

$$\text{vec}(xy^T) = \begin{bmatrix} xy^T[:1] \\ xy^T[:2] \\ \vdots \\ xy^T[:n] \end{bmatrix} = \begin{bmatrix} y_1 x_1 \\ y_1 x_2 \\ \vdots \\ y_1 x_m \\ \vdots \\ y_n x_1 \\ \vdots \\ y_n x_m \end{bmatrix} = y \otimes x$$

(by the def of vec(C) operation)

Therefore, $\text{vec}(xy^T) = y \otimes x \in F^{MN}$

P6.

Let $A = U\Sigma V'$ by SVD, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2 \cdots \sigma_{\min(n,m)})$ and U, V are unitary

$$\Rightarrow A = U I \Sigma V'$$
$$= U(V'V)\Sigma V' \quad \text{since } V'V = I$$
$$= (UV')(V\Sigma V')$$

Let $Q = UV'$ and $S = V\Sigma V'$

$A = QS$, we need to prove that Q is unitary and $S = S' \geq 0$

① $Q'Q = (UV')'(UV')$ $\quad$ $QQ' = (UV')(UV')'$

$= VU'UV'$ $\quad$ $= UV'VU'$

$= VV'$ $\quad$ $= UU'$

$= I$ $\quad$ $= I$

$\therefore$ Q is unitary

② $S = V\Sigma V'$

$S' = (V\Sigma V')'$

$= V\Sigma' V'$

$= V\Sigma V'$ Since Σ is diagonal matrix, $\Sigma' = \Sigma$

Thus, we can write $A = QS$ where $Q = UV'$ and $V = V'\Sigma V$

P7:

(a) No. Counter example:

$a = 2 - \sqrt{3} \qquad b = \sqrt{3}$

$a+b = 2 \in \{ \mathbb{R} - \mathbb{Q} \cup \{0\} \}$

(b) Yes

(c) No. Counter example:

let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A+B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ whose diagonals are not all zeros or non-zeros

(d) Yes

(e) No.

Counter example:

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is vertible

P8.

$$B = \begin{bmatrix} 0 & A \\ A' & 0 \end{bmatrix}$$

$$B - \lambda I = \begin{bmatrix} -\lambda I & A \\ A' & -\lambda I \end{bmatrix}$$

$\det(B-\lambda I) = \det(-\lambda I) \quad \det(-\lambda I - A'(-\lambda I)^{-1} A)$ by the property of Laub

$= (-\lambda)^n \cdot \quad \det((\lambda^2 I - A'A) | -(\lambda))$ since $\det(-\lambda I) = (-\lambda)^n$

$= \det(\lambda^2 I - A'A)$

let $\det(B-\lambda I) = \det(\lambda^2 I - A'A) = 0$

$\det(\lambda^2 I - A'A)$ the solution is the eigenvalues of $A'A$: $\sigma_1^2, \sigma_2^2 \cdots \sigma_n^2$

which are exactly the eigenvalues of B, $\sigma_1, \sigma_2, \cdots \sigma_n, -\sigma_1, \cdots, -\sigma_n$

Let $A = U\Sigma V'$ by SVD and we know that eigenvalues of B are $\pm\sigma_i$ for $i \in 1, n$

$$B = \begin{bmatrix} 0 & A \\ A' & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & U\Sigma V' \\ (U\Sigma V)' & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & U\Sigma V' \\ V\Sigma U' & 0 \end{bmatrix}$$ ($\Sigma' = \Sigma$ since A is diagonal matrix)

Thus, $B\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V' \\ V\Sigma U' & 0 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix}$

$$= \begin{bmatrix} U\Sigma V' v_i \\ V\Sigma U' u_i \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_i u_i \\ \sigma_i v_i \end{bmatrix}$$

$$= \sigma_i \begin{bmatrix} u_i \\ v_i \end{bmatrix}$$

Therefore, $\begin{bmatrix} u_i \\ v_i \end{bmatrix}$ is eigenvector of B with the eigenvalue σ_i.

Similarly,

$$B\begin{bmatrix} u_i \\ -v_i \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V' \\ V\Sigma U' & 0 \end{bmatrix}\begin{bmatrix} u_i \\ -v_i \end{bmatrix}$$

$$= \begin{bmatrix} -U\Sigma V' v_i \\ V\Sigma U' u_i \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma_i u_i \\ \sigma_i v_i \end{bmatrix}$$

$$= -\sigma_i \begin{bmatrix} u_i \\ -v_i \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} u_i \\ -v_i \end{bmatrix}$ is the eigenvector of B with the eigenvalue $-\sigma_i$

u_i, v_i are the left and right singular vectors associated with the singular value σ_i.

P9:

During this semester, I'd love to find a balance between studying and relaxing. I wish to have a healthy life-circle from the following aspects:

1. Sleep before 11 pm and wake up at 7am next morning during the weekdays. Having a good rest is a firm foundation of studying well.
2. Hit the gym four times a week. Building muscle is one of primary goals in my plans.
3. Eat healthy and hone my cooking skills.
4. Find more new friends. I need connect to more people. Go to say "Hi" and talk with them.
5. Since I want to find an internship in the next year's summer term, I need to prepare early such as doing mock interviews, practicing more Leetcode questions and reviewing about algorithm and data structure.
6. Master in Julia programing language. At the end of this term, I wish I could apply Julia for data training and solving machine learning problem.

P10.

1. Kronecker product:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

$n=3=m$ $\quad D_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

$$FXY = \begin{bmatrix} f(1,1) & f(1,2) & f(1,3) \\ f(2,1) & f(2,2) & f(2,3) \\ f(3,1) & f(3,2) & f(3,3) \end{bmatrix} N \quad DFDX \begin{bmatrix} f(2,1)-f(1,1) & f(2,2)-f(1,2) & f(2,3)-f(1,3) \\ f(3,1)-f(2,1) & f(3,2)-f(2,2) & f(3,3)-f(2,3) \\ f(1,1)-f(3,1) & f(1,2)-f(3,2) & f(1,3)-f(3,3) \end{bmatrix}$$

M

$$DFDY = \begin{bmatrix} f(1,2)-f(1,1) & f(1,3)-f(1,2) & f(1,1)-f(1,3) \\ f(2,2)-f(2,1) & f(2,3)\ \ f(2,2) & f(2,1)-f(2,3) \\ f(3,2)-f(3,1) & f(3,3)-f(3,2) & f(3,1)-f(3,3) \end{bmatrix}$$

$dfdx,\ dfdy \in R^{mn}$

$$\underset{2mn \times 1}{\begin{bmatrix} dfdx \\ dfdy \end{bmatrix}} = \underset{2mn \times mn}{A}\ \underset{mn \times 1}{fxy}$$

$$\begin{bmatrix} dfdx \\ dfdy \end{bmatrix} = \begin{bmatrix} f(2,1)-f(1,1) \\ f(3,1)-f(2,1) \\ f(1,1)-f(3,1) \\ f(2,2)-f(1,2) \\ \vdots \\ f(1,2)-f(1,1) \\ f(2,2)-f(2,1) \\ \vdots \\ f(3,1)-f(3,3) \end{bmatrix}$$

$D_n, D_m = \in R^{mn}$

$$fxy = \begin{bmatrix} f(1,1) \\ f(2,1) \\ \vdots \\ f(3,3) \end{bmatrix}$$

$$\begin{bmatrix} dfdx \\ dfdy \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \end{bmatrix} fxy = \begin{bmatrix} D_3 & & & \\ & D_3 & & \\ & & D_3 & \\ & & & D_3 \\ & & & & \ddots \end{bmatrix} fxy$$

$$A = \begin{bmatrix} I_3 \otimes D_3 \\ I_3 \otimes D_3 \end{bmatrix}$$

In Gengeral, based on example above

$$A = \begin{bmatrix} I_n \otimes D_m \\ D_n \otimes I_m \end{bmatrix}$$