EECS 455: Solutions to Problem Set 4

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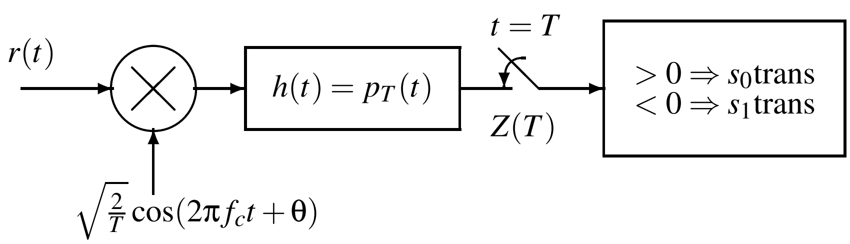
1. A binary communication system transmits one of two equally likely signals $s_0(t)$ and $s_1(t)$ of duration T given by
$$s_i(t) = A(-1)^i \cos(2\pi f_c t) p_T(t)$$
The noise in the system is white Gaussian noise with power spectral density $N_0/2$. The receiver shown below is used to demodulate the signal. However, as shown, the phase of the received signal is not know completely accurately. In fact there is a discrepancy of θ radians between the received signal (in the absence of noise) and the local reference signal. Determine the error probability at the output of the demodulator as a function of θ. (Assume $2\pi f_c T = 2\pi n$ for some integer n).



Solution: The output of the receiver due to signal alone is

$$\begin{aligned}
Z(T) &= \int_0^T A(-1)^i \cos(2\pi f_c t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t+\theta)dt \\
&= \int_0^T A(-1)^i \sqrt{\frac{2}{T}}[1/2\cos(\theta)+1/2\cos(2\pi 2 f_c t+\theta)]dt \\
&= \sqrt{A^2T/2}(-1)^i\cos(\theta)
\end{aligned}$$

The output due to noise η is a Gaussian random variable with mean 0 and variance

$$\begin{aligned}
\sigma^2 &= E[\eta^2] \\
&= E[\frac{2}{T}\int_0^T n(t)\cos(2\pi f_c t+\theta)dt \int_0^T n(s)\cos(2\pi f_c s+\theta)ds] \\
&= \frac{2}{T}\int_0^T\int_0^T E[n(t)n(s)]\cos(2\pi f_c t+\theta)\cos(2\pi f_c s+\theta)dtds] \\
&= \frac{2}{T}\int_0^T\int_0^T \frac{N_0}{2}\delta(t-s)\cos(2\pi f_c t+\theta)\cos(2\pi f_c s+\theta)dtds] \\
&= \frac{2}{T}\int_0^T \frac{N_0}{2}\cos^2(2\pi f_c t+\theta)dt \\
&= \frac{2}{T}\int_0^T \frac{N_0}{2}[1/2+1/2\cos(4\pi f_c t+2\theta)]dt \\
&= \frac{N_0}{2}
\end{aligned}$$

The probability of error given signal 0 transmitted is

$$\begin{aligned}
P_{e,0} &= P\{\sqrt{A^2T/2}\cos(\theta)+\eta<0\} \\
&= P\{\eta<-\sqrt{A^2T/2}\cos(\theta)\} \\
&= Q(\frac{\sqrt{A^2T/2}\cos(\theta)}{\sigma}) \\
&= Q(\frac{\sqrt{A^2T/2}\cos(\theta)}{\sqrt{N_0/2}}) \\
&= Q(\sqrt{\frac{(A^2T)\cos^2(\theta)}{N_0}}) \\
&= Q(\sqrt{\frac{2E\cos^2(\theta)}{N_0}})
\end{aligned}$$

The error probability given signal 1 transmitted is identical to the error probability given signal 0 transmitted.

2. A transmitter uses one of four equally likely signals to convey two bits of information. The signals are $s_0(t), s_1(t), s_2(t)$, and $s_3(t)$. The following table indicates the mapping between information bits and signals.

Information bits	Signals
00	$s_0(t)$
01	$s_1(t)$
11	$s_2(t)$
10	$s_3(t)$

The signals are received in the presence of white Gaussian noise with power spectral density $N_0/2$. The receiver consist of a filter, a sampler and a threshold device. The sampled output is denoted by $Z(T)$. The threshold device uses the following table to make a decision.

$Z(T)>2$	decide $s_0(t)$ transmitted
$0<Z(T)<2$	decide $s_1(t)$ transmitted
$-2<Z(T)<0$	decide $s_2(t)$ transmitted
$Z(T)<-2$	decide $s_3(t)$ transmitted

It is known that the output of the filter due to the signals alone at the sampling time is

$$\begin{aligned}
\hat{s}_0(T) &= +3 \\
\hat{s}_1(T) &= +1 \\
\hat{s}_2(T) &= -1 \\
\hat{s}_3(T) &= -3
\end{aligned}$$

It is also known that the variance of the output due to noise alone is $\sigma^2 = 4$.

(a) Determine the probability of error given signal $s_i(t)$ is transmitted for $i = 0, 1, 2, 3$. (Express your answers in terms of the Q function).

Solution: Consider the case of signal 0 transmitted first. Let η be the output of the filter due to noise alone (Gaussian with mean 0 and $\sigma = 2$. Then the probability of error given signal 0 transmitted is

$$\begin{aligned}
P_{e,0} &= P\{\text{error}|s_0 \text{ trans.}\} \\
&= P\{Z(T) < 2|s_0 \text{ trans.}\} \\
&= P\{3+\eta < 2\} \\
&= P\{\eta < -1\} \\
&= \Phi(\frac{-1}{2}) \\
&= Q(\frac{1}{2}).
\end{aligned}$$

For signal 1 transmitted

$$\begin{aligned}
P_{e,1} &= P\{\text{error}|s_1 \text{ trans.}\} \\
&= P\{Z(T) < 0 \text{ or } Z(T) > 2|s_1 \text{ trans.}\} \\
&= P\{1+\eta < 0\} + P\{1+\eta > 2\} \\
&= P\{\eta < -1\} + P\{\eta > 1\} \\
&= \Phi(\frac{-1}{2}) + Q(\frac{1}{2}) \\
&= 2Q(\frac{1}{2}).
\end{aligned}$$

For signal 2 transmitted

$$\begin{aligned}
P_{e,2} &= P\{\text{error}|s_2 \text{ trans.}\} \\
&= P\{Z(T) < -2 \text{ or } Z(T) > 0|s_2 \text{ trans.}\} \\
&= P\{-1+\eta < -2\} + P\{-1+\eta > 0\} \\
&= P\{\eta < -1\} + P\{\eta > 1\} \\
&= 2Q(\frac{1}{2}).
\end{aligned}$$

$$\begin{aligned}
P_{e,3} &= P\{\text{error}|s_3 \text{ trans.}\} \\
&= P\{Z(T) > -2|s_3 \text{ trans.}\} \\
&= P\{-3+\eta > -2\} \\
&= P\{\eta > 1\} \\
&= Q(\frac{1}{2}).
\end{aligned}$$

(b) Determine the probability that the receiver makes an error in the first bit given the first bit is 0. (The first bit being 0 means either signal $s_0(t)$ was transmitted or $s_1(t)$ was transmitted).

Solution: Let b_0 be the first bit. Let $\hat{b}_0$ be the decision on the first bit. Let P_e be the probabiltiy of error for the first bit. Then

$$\begin{aligned}
P_e &= P\{\hat{b}_0 = 1 | b_0 = 0\} \\
&= P\{\hat{b}_0 = 1 \cap s_0 \text{ trans.} | b_0 = 0\} + P\{\hat{b}_0 = 1 \cap s_1 \text{ trans.} | b_0 = 0\} \\
&= P\{\hat{b}_0 = 1 | s_0 \text{ trans.} \cap b_0 = 0\} P\{s_0 \text{ trans.} | b_0 = 0\} \\
&\quad + P\{\hat{b}_0 = 1 | s_1 \text{ trans.} \cap b_0 = 0\} P\{s_1 \text{ trans.} | b_0 = 0\} \\
&= P\{Z < 0 | s_0 \text{ trans.}\} P\{s_0 \text{ trans.} | b_0 = 0\} + P\{Z < 0 | s_1 \text{ trans.}\} P\{s_1 \text{ trans.} | b_0 = 0\} \\
&= P\{3 + \eta < 0\} \frac{1}{2} + P\{1 + \eta < 0\} \frac{1}{2} \\
&= P\{\eta < -3\} \frac{1}{2} + P\{\eta < -1\} \frac{1}{2} \\
&= \frac{1}{2} Q(\frac{3}{2}) + \frac{1}{2} Q(\frac{1}{2})
\end{aligned}$$

3. Show that the raised cosine pulse shape satisfies the Nyquist criteria for zero intersymbol interference.

Solution: The raised cosine pulse is

$$\begin{aligned}
H(f) &= \begin{cases} T, & |f| < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[|f| - \frac{(1-\alpha)}{2T}])\right\}, & \frac{1-\alpha}{2T} < |f| < \frac{1+\alpha}{2T} \\ 0, & |f| > \frac{1+\alpha}{2T} \end{cases} \\
&= \begin{cases} \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[-f - \frac{(1-\alpha)}{2T}])\right\}, & -\frac{1+\alpha}{2T} < f < -\frac{1-\alpha}{2T} \\ T, & -\frac{1-\alpha}{2T} < f < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[f - \frac{(1-\alpha)}{2T}])\right\}, & \frac{1-\alpha}{2T} < f < \frac{1+\alpha}{2T} \\ 0, & |f| > \frac{1+\alpha}{2T} \end{cases}
\end{aligned}$$

$$\begin{aligned}
H(f - \frac{1}{T}) &= \begin{cases} 0, & f < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[-(f - \frac{1}{T}) - \frac{(1-\alpha)}{2T}])\right\}, & \frac{1-\alpha}{2T} < f < \frac{1+\alpha}{2T} \\ \text{Don't care}, & f > \frac{1+\alpha}{2T} \end{cases} \\
&= \begin{cases} 0, & f < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[f - \frac{1}{T} + \frac{1}{2T} - \frac{\alpha}{2T}])\right\}, & \frac{1-\alpha}{2T} < f < \frac{1+\alpha}{2T} \\ \text{Don't care}, & f > \frac{1+\alpha}{2T} \end{cases} \\
&= \begin{cases} 0, & f < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[f - \frac{1}{2T} + \frac{\alpha}{2T} - \frac{\alpha}{T}])\right\}, & \frac{1-\alpha}{2T} < f < \frac{1+\alpha}{2T} \\ \text{Don't care}, & f > \frac{1+\alpha}{2T} \end{cases} \\
&= \begin{cases} 0, & f < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1 + \cos(\frac{\pi T}{\alpha}[f - \frac{1}{2T} + \frac{\alpha}{2T}] - \pi)\right\}, & \frac{1-\alpha}{2T} < f < \frac{1+\alpha}{2T} \\ \text{Don't care}, & f > \frac{1+\alpha}{2T} \end{cases}
\end{aligned}$$

$$= \begin{cases} 0, & f < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1-\cos(\frac{\pi T}{\alpha}[f-\frac{(1-\alpha)}{2T}])\right\}, & \frac{1-\alpha}{2T} < f < \frac{1+\alpha}{2T} \\ \text{Don't care}, & f > \frac{1+\alpha}{2T} \end{cases}$$

So in the interval $(1-\alpha)/(2T) < f < 1/(2T)$ the sum of $H(f)$ and $H(f-1/T)$ is T. Similarly in the interval $-1/(2T) < f < -(1-\alpha)/(2T)$ the sum of $H(f)$ and $H(f+1/T)$ is T.

4. A binary communications system operates over an AWGN channel with spectral density $N_0/2$. The transmitted signals are given by

$$s_0(t) = Ap_{T/2}(t) - Ap_{T/2}(t-T/2)$$

$$s_1(t) = 0$$

(a) Give an expression (in terms of A, T, N_0, and $Q(x)$) for the minimum average error probability when the two signals are transmitted with equal probabilities (i.e. $\pi_0 = \pi_1$).

Solution: We are interested in the minimum error probability. The signals are given but we need to optimize over the filter and threshold. The result is obtained from Step 2 of the notes and is

$$P_e = Q(\alpha)$$

where

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}$$

In this case $E_0 = A^2T$, $E_1 = 0$, so $\bar{E} = A^2T/2$. Also $r = (s_0, s_1)/\bar{E} = 0$. So

$$P_e = Q(\sqrt{\frac{A^2T}{2N_0}})$$

(b) What is the optimum filter for minimizing the error probability and the optimum threshold?

Solution:

The optimum filter is the matched filter

$$\begin{aligned} h_{opt}(t) &= s_0(T-t) - s_1(T-t) \\ &= -Ap_{T/2}(t) + Ap_{T/2}(t-T/2) \end{aligned}$$

The optimum threshold is

$$\begin{aligned} \gamma_{opt} &= \hat{s}_0(t) \\ &= \frac{A^2T}{2} \end{aligned}$$

(c) Assume that the optimum filter $h(t)$ is used for the signals above but the signal $s_0(t)$ is actually given by

$$s_0(t) = cAp_{T/2}(t) - cAp_{T/2}(t - T/2)$$

instead of the one given above while $s_1(t)$ is the same. Give an expression (in terms of c, A, T, N_0 and Φ) for the average error probability if $\pi_0 = \pi_1$.

Solution: In this case the filter is given (possibly suboptimum) and the threshold is given (possibly suboptimum). The filter is the matched filter from part b which is

$$h(t) = s_0(T-t) - s_1(T-t) = -Ap_{T/2}(t) + Ap_{T/2}(t - T/2)$$

The error probability is

$$\bar{P}_e = \frac{1}{2}Q(\frac{\hat{s}_0(T) - \gamma}{\sigma}) + \frac{1}{2}Q(\frac{\gamma - \hat{s}_1(T)}{\sigma})$$

where

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} |h(t)|^2 dt \\
&= \frac{N_0}{2}\int_0^T A^2 dt \\
&= \frac{A^2 T N_0}{2}
\end{aligned}$$

Now we need to calculate $\hat{s_0}(T)$ and $\hat{s_1}(T)$.

$$\begin{aligned}
\hat{s}_0(t) &= \int_{-\infty}^{\infty} h(t-\tau)s_0(\tau)d\tau \\
\hat{s}_0(T) &= \int_{-\infty}^{\infty} h(T-\tau)s_0(\tau)d\tau \\
\hat{s}_0(T) &= \int_{-\infty}^{\infty} [Ap_{T/2}(\tau) - Ap_{T/2}(\tau - T/2)]s_0(\tau)d\tau \\
&= \int_0^{T/2} [Ap_{T/2}(\tau)cAp_{T/2}(\tau)d\tau + \int_{T/2}^{T} [-Ap_{T/2}(\tau)][-cAp_{T/2}(\tau)]d\tau \\
&= cA^2 T \\
\hat{s}_1(T) &= 0
\end{aligned}$$

The threshold that is used (from part a) is

$$\gamma = \frac{A^2 T}{2}$$

So the error probability is

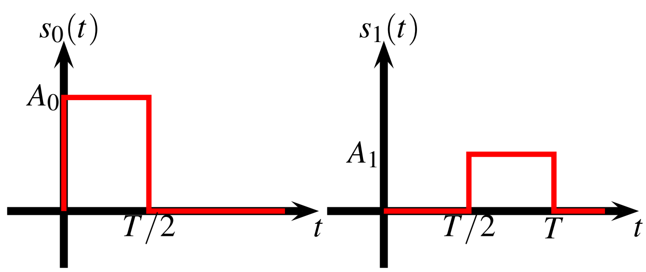
$$\bar{P}_e = \frac{1}{2}Q(\frac{\hat{s}_0(T) - \gamma}{\sigma}) + \frac{1}{2}Q(\frac{\gamma - \hat{s}_1(T)}{\sigma})$$

$$
\begin{aligned}
&= \frac{1}{2}Q\left(\frac{cA^2T - A^2T/2}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A^2T/2 - 0}{\sigma}\right) \\
&= \frac{1}{2}Q\left(\frac{A^2T(c-1/2)}{\sqrt{A^2TN_0/2}}\right) + \frac{1}{2}Q\left(\frac{A^2T/2}{\sqrt{A^2TN_0/2}}\right) \\
&= \frac{1}{2}Q\left(\sqrt{\frac{2A^2T(c-1/2)^2}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)
\end{aligned}
$$

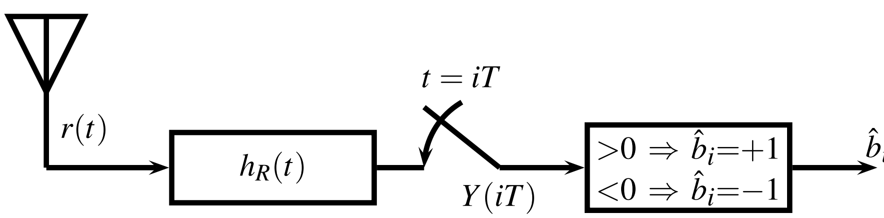
5. Consider a binary communication system that transmits one of two signal $s_0(t)$ and $s_1(t)$ over an additive white Gaussian noise channel (spectral density $N_0/2$) where

$$s_0(t) = A_0 p_{T/2}(t), \quad s_1(t) = A_1 p_{T/2}(t - T/2);$$

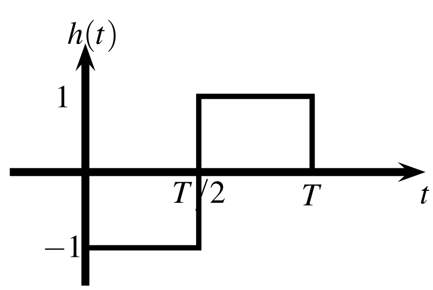
that is $s_0(t)$ is a pulse of amplitude A_0 from 0 to $T/2$ and $s_1(t)$ is a pulse of amplitude A_1 from $T/2$ to T.



The receiver shown below consist of a filter $h(t)$ which is sampled at time T and a threshold device.



(a) If $h(t) = -p_{T/2}(t) + p_{T/2}(t - T/2)$ find the output of the filter $\hat{s}_0(T)$ due to signal $s_0(t)$ at time T and the output of the filter $\hat{s}_1(T)$ due to signal $s_1(t)$ at time T



Solution: The output due to signal 0 is

$$\begin{aligned}
\hat{s}_0(T) &= \int h(T-\tau)s_0(\tau)d\tau \\
&= \int_0^{T/2} s_0(\tau)d\tau - \int_{T/2}^{T} s_0(\tau)d\tau \\
&= \int_0^{T/2} A_0 d\tau \\
&= A_0 T/2.
\end{aligned}$$

The output due to signal 1 is

$$\begin{aligned}
\hat{s}_1(T) &= \int h(T-\tau)s_1(\tau)d\tau \\
&= \int_0^{T/2} s_1(\tau)d\tau - \int_{T/2}^{T} s_1(\tau)d\tau \\
&= -\int_{T/2}^{T} A_1 d\tau \\
&= -A_1 T/2.
\end{aligned}$$

(b) Find the threshold γ that will minimize the average of the error probabilities $P_{e,0}$ and $P_{e,1}$ for the given signals and filter. Assume $\pi_0 = \pi_1$.

Solution: The threshold γ that will minimize the average error probability is

$$\begin{aligned}
\gamma_{opt} &= \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2} \\
&= (A_0 - A_1)T/4.
\end{aligned}$$

(c) Find the error average error probability $\bar{P}_e = \pi_0 P_{e,0} + \pi_1 P_{e,1}$ for the threshold found in the previous part. Assume $\pi_0 = \pi_1$.

The corresponding error probability is

$$\bar{P}_e = Q(\alpha\lambda)$$

$$\begin{aligned}
\lambda &= \frac{(h, s_T)}{||h|| ||s_T||} \\
&= \frac{(A_0 + A_1)T/2}{\sqrt{T}\sqrt{A_0^2 T/2 + A_1^2 T/2}} \\
&= \frac{(A_0 + A_1)}{\sqrt{2}\sqrt{A_0^2 + A_1^2}}
\end{aligned}$$

$$\begin{aligned} E_0 &= A_0^2 T/2 \\ E_1 &= A_1^2 T/2 \\ \bar{E} &= (A_0^2 + A_1^2)T/4 \\ r &= (s_0(t), s_1(t))/\bar{E} \\ &= 0. \end{aligned}$$

$$\begin{aligned} \alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}} \\ &= \sqrt{\frac{T(A_0^2 + A_1^2)}{4N_0}} \end{aligned}$$

$$\begin{aligned} \alpha\lambda &= \sqrt{\frac{T(A_0^2 + A_1^2)}{4N_0}} \frac{(A_0 + A_1)}{\sqrt{2}\sqrt{A_0^2 + A_1^2}} \\ &= \sqrt{\frac{(A_0 + A_1)^2 T}{8N_0}} \end{aligned}$$

So

$$\bar{P}_e = Q(\sqrt{\frac{(A_0 + A_1)^2 T}{8N_0}}).$$

(d) Find the matched filter for the same signals and find the corresponding threshold that minimizes $\bar{P}_e$. Assume $\pi_0 = \pi_1$.

Solution:

$$h_{opt} = s_0(T-t) - s_1(T-t) = -A_1 p_{T/2}(t) + A_0 p_{T/2}(t - t/2)$$

$$\gamma_{opt} = \frac{T}{4}(A_0^2 - A_1^2)$$

(e) Find $\bar{P}_e$ for the matched filter with the optimum threshold. Assume $\pi_0 = \pi_1$.

Solution: For the matched filter $\lambda = 1$ so the error probability is

$$\begin{aligned} \bar{P}_e &= Q(\alpha) \\ &= Q(\sqrt{\frac{T(A_0^2 + A_1^2)}{4N_0}}) \end{aligned}$$