1. Consider a communication system that transmits an infinite sequence of data bits $\{b_l\}_{l=-\infty}^{\infty}$ using two signals of duration T: $s_0(t) = -s_1(t) = Ap_T(t)$. Thus

$$s(t) = \sum_{l=-\infty}^{\infty} b_l A p_T(t - lT)$$

The signal $s(t)$ is transmitted over an additive white Gaussian noise channel with spectral density $N_0/2$. The receiver consists of a filter $h(t)$ the output of which is sampled at time iT and compared with a threshold of 0. If the output at time $iT > 0$ the receiver decides $b_{i-1} = +1$ otherwise the receiver decides $b_{i-1} = -1$. It is know that the filter is such that $\int_{-\infty}^{\infty} h^2(t)dt = 16$. It is also known that if the input to the filter is $p_T(t)$ ($X(t) = 0$) then the output at time iT is

$$Z(iT) = \begin{cases} 8 & i = 1 \\ 2 & i = 2 \\ 1 & i = 3 \\ 0 & i < 1,\ i > 3 \end{cases}$$

(a) Find the possible values for the output due to the desired signal (no noise) for different data bits.

Solution:

The possible values for the output in the absence of noise are 11,9,7,and 5.

(b) Find the upper and lower bounds for the $E[Z(iT)|b_{i-1} = +1, b_{i-2}, b_{i-3}, ...]$. That is, find the largest possible value for the output due to signal alone (no noise) at time iT for all possible previous data bits. Find upper and lower bounds on the probability of error for data bit b_{i-1} given that $b_{i-1} = +1$.

Solution:

If $b_{i-1} = +1$ then $E[Z(iT)|b_{i-1} = +, b_{i-2}, ...]$ can be as large as 8+2+1=11 while $E[Z(iT)]$ can be as small as 8-2-1=5.

$$5A < E[Z(iT)|b_{i-1} = +1, b_{i-2}, ...] < 11A$$
$$Q(\frac{11A}{\sigma}) < P_e < Q(\frac{5A}{\sigma})$$

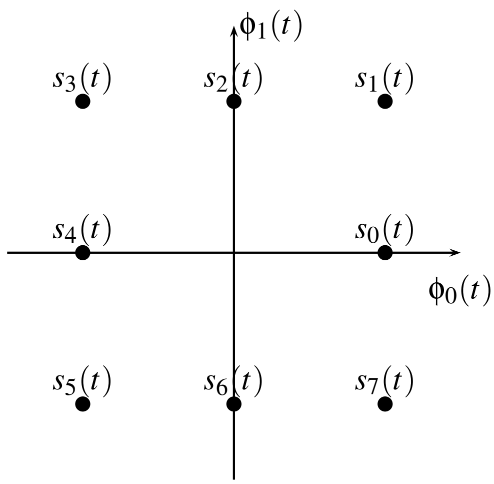
where $\sigma^2 = \frac{N_0}{2}16 = 8N_0$. (c) Give an expression for the average probability of error for the data bit b_{i-1} if each data bit is equally likely to be +1 or -1 independently of all other data bits.

Solution: There are four possible outputs given $b_{i-1}+1$ (in the absence of noise). These are 11, 9, 7, and 5. These are equally likely. The probability of error is then

$$\begin{aligned}
\bar{P}_{e,+1} = \bar{P}_{e,-1} &= \frac{1}{4}\left[Q(\frac{11A}{\sigma}) + Q(\frac{9A}{\sigma}) + Q(\frac{7A}{\sigma}) + Q(\frac{5A}{\sigma})\right] \\
\bar{P}_e &= \frac{1}{2}\bar{P}_{e,+1} + \frac{1}{2}\bar{P}_{e,-1} \\
&= \frac{1}{4}\left[Q(\frac{11A}{\sigma}) + Q(\frac{9A}{\sigma}) + Q(\frac{7A}{\sigma}) + Q(\frac{5A}{\sigma})\right]
\end{aligned}$$

where $\sigma^2 = \frac{N_0}{2}16 = 8N_0$.

2. A communication system transmits 3 bits using one of eight equally likely signals in two dimensions as shown below.



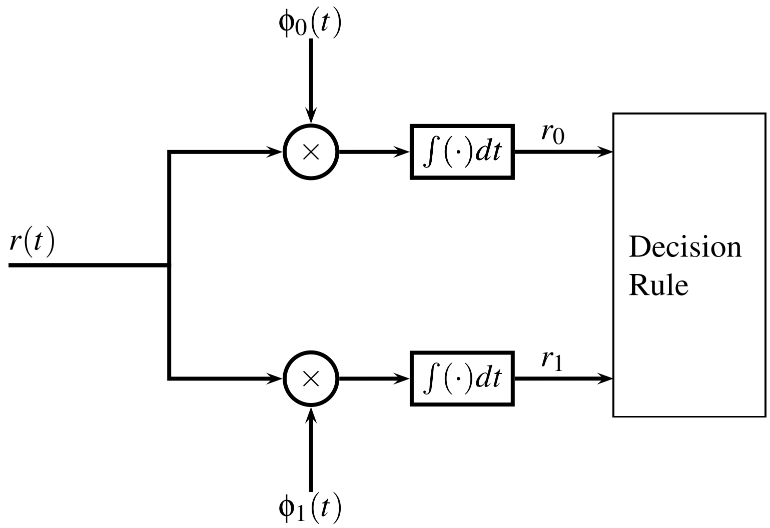
That is the signals are

$$\begin{aligned}
s_0(t) &= A\phi_0(t) \\
s_1(t) &= A\phi_0(t) + A\phi_1(t) \\
s_2(t) &= A\phi_1(t) \\
s_3(t) &= -A\phi_0(t) + A\phi_1(t) \\
s_4(t) &= -A\phi_0(t) \\
s_5(t) &= -A\phi_0(t) - A\phi_1(t) \\
s_6(t) &= -A\phi_1(t) \\
s_7(t) &= A\phi_0(t) - A\phi_1(t)
\end{aligned}$$

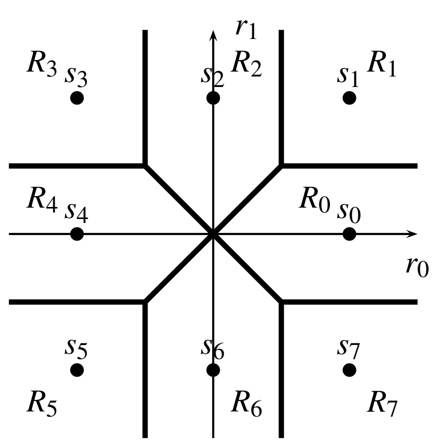
The signals $\phi_0(t)$ and $\phi_1(t)$ are orthonormal. The received signal is the transmitted signal plus white Gaussian noise.

(a) Draw a block diagram of the optimum receiver.

(b) Determine the optimal decision regions (to minimize the average error probability)



The decsion rule is shown below.



(c) Determine the probability of error for signal $s_1(t)$.

Solution: The error probability for signal 1 is

$$\begin{aligned} P_{e,1} &= 1-(1-Q\left(\frac{A/2}{\sigma}\right))^2 \\ &= 1-(1-Q\left(\sqrt{\frac{A^2/4}{N_0/2}}\right))^2 \\ &= 1-(1-Q\left(\sqrt{\frac{A^2}{2N_0}}\right))^2 \end{aligned}$$

(d) Determine a bound on the error probability for signal $s_0(t)$.

The pairwise distance between signal 0 and the other signals is given below.

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_0	0	A	$\sqrt{2}A$	$\sqrt{5}A$	$2A$	$\sqrt{5}A$	$\sqrt{2}A$	A

The union bound of the error probability given signal 0 is sent is thus

$$\begin{aligned} P_{e,0} &\leq \sum_{j=1}^{7} P_2(s_0 \to s_j) \\ &= \sum_{j=1}^{7} Q(\frac{d(s_0,s_j)}{2\sigma}) \\ &= 2Q(\frac{A}{2\sigma})+2Q(\frac{\sqrt{2}A}{2\sigma})+2Q(\frac{\sqrt{5}A}{2\sigma})+Q(\frac{2A}{2\sigma}) \end{aligned}$$

3. (a) A communication system uses BPSK in a (null-to-null) bandwidth of 20 MHz with transmitted power $P=(4\pi)^2\times 10^{-3}$ watts in the presence of white Gaussian noise with two-side power spectral density $\frac{N_0}{2}=\frac{4\times 10^{-21}}{2}$. The transmitter uses a carrier frequency of 300MHz (wavelength 1m). The transmitting antenna and receiving antenna both have gains of 2 (3dB). The data rate is 10Mbps. Assuming free space propagation what is the maximum distance for which an error probability of $Q(\sqrt{20})$ is possible.

Solution:

$$E_b/N_0 = 10$$

$$
\begin{aligned}
\frac{P_r}{RN_0} &= 10 \\
\frac{P_tG_tG_r}{(4\pi d/\lambda)^2 RN_0} &= 10 \\
\frac{P_tG_tG_r}{10(4\pi/\lambda)^2 RN_0} &= d^2 \\
\frac{(4\pi)^2 \times 10^{-3}(2)(2)}{10(4\pi)^2(10^7)4 \times 10^{-21}} &= d^2 \\
\frac{10^{-3}}{10(10^7) \times 10^{-21}} &= d^2 \\
10^{10} &= d^2 \\
10^5 &= d
\end{aligned}
$$

(b) Is it possible with the same parameters above and with BPSK to have a larger data rate if the distance is smaller than the answer to (a)? **Solution:** No, it is not possible since that would require larger bandwidth.

(c) If instead of BPSK modulation 16-ary orthogonal modulation was used with the same bandwdith can the same performance (probability of error) be achieved if the distance is larger than the answer to (a)? If so, explain how. If not, explain why not. **Solution:** Yes, with 16-ary orthogonal the required signal-to-noise ratio would be smaller. This would allow larger distance but would lower the data rate.

4. Consider a communication system that transmits one of 16 signals in 7 dimensions. The signals are

$$
\begin{aligned}
s_0 &= \sqrt{E}(+1,+1,+1,+1,+1,+1,+1) \\
s_1 &= \sqrt{E}(+1,+1,+1,-1,+1,-1,-1) \\
s_2 &= \sqrt{E}(+1,+1,-1,+1,-1,-1,+1) \\
s_3 &= \sqrt{E}(+1,+1,-1,-1,-1,+1,-1) \\
s_4 &= \sqrt{E}(+1,-1,+1,+1,-1,-1,-1) \\
s_5 &= \sqrt{E}(+1,-1,+1,-1,-1,+1,+1) \\
s_6 &= \sqrt{E}(+1,-1,-1,+1,+1,+1,-1) \\
s_7 &= \sqrt{E}(+1,-1,-1,-1,+1,-1,+1) \\
s_8 &= \sqrt{E}(-1,+1,+1,+1,-1,+1,-1)
\end{aligned}
$$

$$
\begin{aligned}
s_9 &= \sqrt{E}(-1,+1,+1,-1,-1,-1,+1)\\
s_{10} &= \sqrt{E}(-1,+1,-1,+1,+1,-1,-1)\\
s_{11} &= \sqrt{E}(-1,+1,-1,-1,+1,+1,+1)\\
s_{12} &= \sqrt{E}(-1,-1,+1,+1,+1,-1,+1)\\
s_{13} &= \sqrt{E}(-1,-1,+1,-1,+1,+1,-1)\\
s_{14} &= \sqrt{E}(-1,-1,-1,+1,-1,+1,+1)\\
s_{15} &= \sqrt{E}(-1,-1,-1,-1,-1,-1,-1)
\end{aligned}
$$

Four bits are transmitted each with energy $7E$ so the energy per bit is

$$E_b = 7E/4.$$

We will consider two receivers.

(a) The first receiver is the optimal receiver for minimizing the probability of signal error (choosing the wrong signal). For this receiver we want to determine the probability of bit error. Assume the 4 bits that are transmitted are represented in binary and the binary number determines the signal. That is the bits 0000 represent 0 and are mapped to s_0. The bits 0001 represent 1 and are mapped to s_1. The bits 0101 represent 5 and are mapped to s_5. Simulate this communication system in additive white Gaussian noise (power spectral density $N_0/2$ to determine the bit error probability. Plot P_e versus $E_b/N_0(dB)$ as $E_b/N_0(dB)$ varies from 0 dB to 7dB. Compare this to the Union Bound.

Solution: The plot and code are shown below.

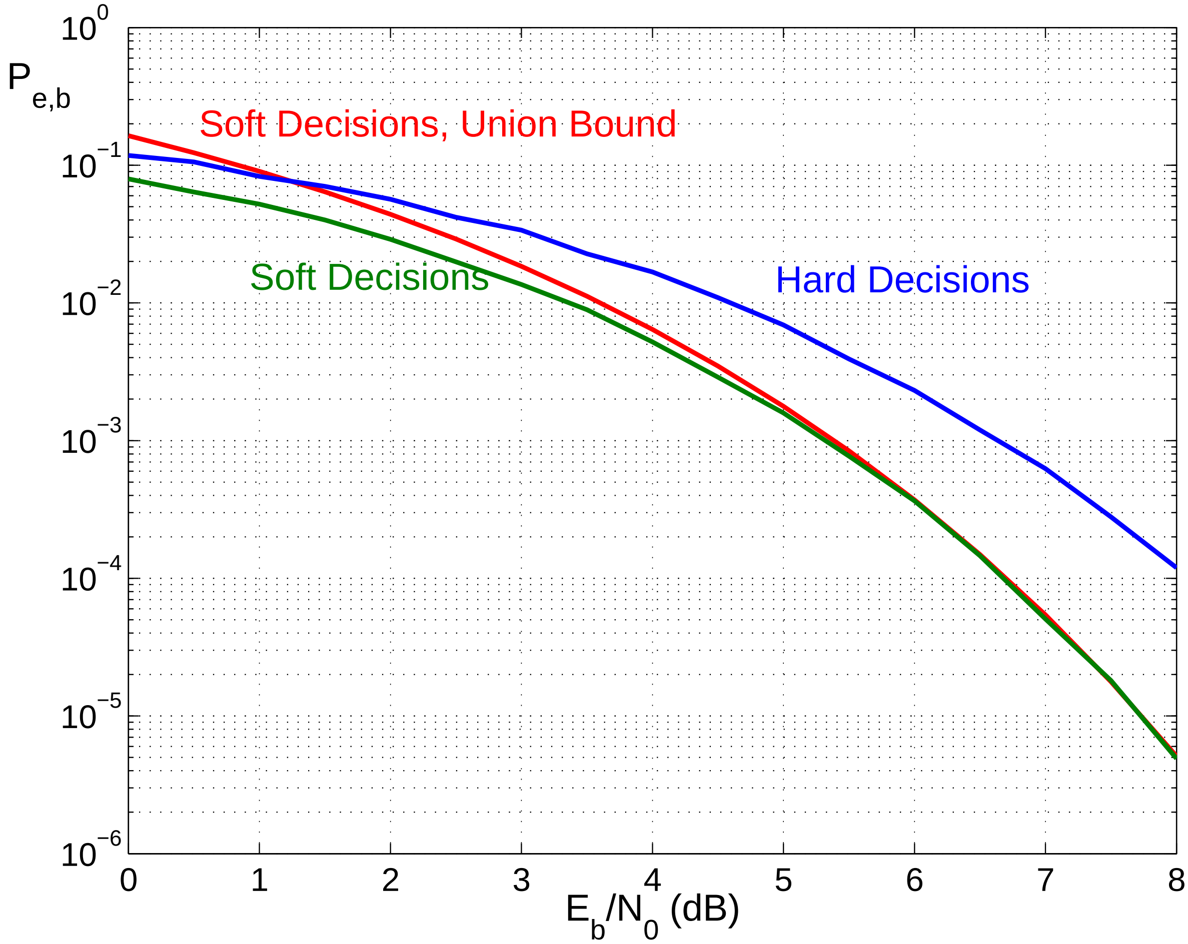
```
clear all;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                 %
%              Simulation Parameters              %
%                                                 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


ncount=input('Number of errors = ');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                 %
```

```
%                    Setup the Simulation                   %
%                                                           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for m=1:25
EbN0dB(m)=-4+(m-1)/2
EbN0(m)=10^(EbN0dB(m)/10);
Eb=1;
E=Eb*4/7;                   % Relation between energy per code symbol a
N0=Eb/EbN0(m);              % Noise power
sigma=sqrt(N0/2);

s0  =  sqrt(E)*[+1, +1, +1, +1, +1, +1, +1];
s1  =  sqrt(E)*[+1, +1, +1, -1, +1, -1, -1];
s2  =  sqrt(E)*[+1, +1, -1, +1, -1, -1, +1];
s3  =  sqrt(E)*[+1, +1, -1, -1, -1, +1, -1];
s4  =  sqrt(E)*[+1, -1, +1, +1, -1, -1, -1];
s5  =  sqrt(E)*[+1, -1, +1, -1, -1, +1, +1];
s6  =  sqrt(E)*[+1, -1, -1, +1, +1, +1, -1];
s7  =  sqrt(E)*[+1, -1, -1, -1, +1, -1, +1];
s8  =  sqrt(E)*[-1, +1, +1, +1, -1, +1, -1];
s9  =  sqrt(E)*[-1, +1, +1, -1, -1, -1, +1];
s10  =  sqrt(E)*[-1, +1, -1, +1, +1, -1, -1];
s11  =  sqrt(E)*[-1, +1, -1, -1, +1, +1, +1];
s12  =  sqrt(E)*[-1, -1, +1, +1, +1, -1, +1];
s13  =  sqrt(E)*[-1, -1, +1, -1, +1, +1, -1];
s14  =  sqrt(E)*[-1, -1, -1, +1, -1, +1, +1];
s15  =  sqrt(E)*[-1, -1, -1, -1, -1, -1, -1];

allsignals=[s0; s1; s2; s3; s4; s5; s6; s7; s8; s9; s10; s11; s12; s1
nerrors=0;
nbsim=0;
while(nerrors < ncount)


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                           %
%                 Generate data and signals                 %
```

```
%                                                  %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

b=round(rand(1,4));
index=(b(4)+2*b(3)+4*b(2)+8*b(1))+1;
s=allsignals(index,:);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                  %
%                  Add Noise                       %
%                                                  %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

noise=sigma*randn(1,7);
      rcvd=s+noise;

%FOR HARD DECISIONS DECODING ADD THIS LINE
        rcvd=sign(rcvd);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                  %
%           Decode the received signal             %
%                                                  %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

        [corr,l]=max(rcvd*allsignals');
lhat=l-1;
        bhat(4)=mod(lhat,2);
lhat=floor(lhat/2);
bhat(3)=mod(lhat,2);
lhat=floor(lhat/2);
bhat(2)=mod(lhat,2);
lhat=floor(lhat/2);
bhat(1)=mod(lhat,2);
        nerrors=nerrors+sum(abs(sign(bhat-b)));
        nbsim=nbsim+4;
if (mod(nbsim,5000)==0)
peb(m)=nerrors/nbsim;
format('long')
```

```
[EbN0dB(m),nerrors,nbsim,peb(m)]
                save s095data
end;
   end

peb(m)=nerrors/nbsim;
semilogy(EbN0dB,peb)
axis([-4 12 0.00000001 1])
grid on
hold on
xlabel('E_b/N_0 (dB)')
ylabel('P_{e,b}')
save s095data
end
```

(b) The second receiver computes the decision variable corresponding to each dimension $\tilde{r}_i = \text{sign}(r_i)$ and then finds the signal vector that has the fewest differences (number of positions where an error occurs). Simulate this system as well. Determine the bit error probability as a function of the energy per information bit. Plot P_e versu $E_b/N_0(dB)$ as $E_b/N_0(dB)$ varies from 0 dB to 8dB.



$P_{e,b}$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
Soft Decisions, Union Bound
Soft Decisions
Hard Decisions
0
1
2
3
4
5
6
7
8
E_b/N_0 (dB)