**Pr. 1.** (sol/hsj78)
We want to show that $\boldsymbol{x}'(\sigma_1^2(\boldsymbol{A})\boldsymbol{I})\boldsymbol{x} \succeq \boldsymbol{x}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{x}, \ \forall \boldsymbol{x}$.
For any $\boldsymbol{x}$: $\boldsymbol{x}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{x} = \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\|_2^2 \|\boldsymbol{x}\|^2 = \sigma_1^2(\boldsymbol{A}) \|\boldsymbol{x}\|^2 = \boldsymbol{x}'(\sigma_1^2(\boldsymbol{A})\boldsymbol{I})\boldsymbol{x}$.

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**Pr. 2.** (sol/hsj79)
Following the analysis of PGD in the notes, the convergence of $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha \boldsymbol{P}\boldsymbol{A}'(\boldsymbol{A}\boldsymbol{x}_k - \boldsymbol{y})$ is governed by the eigenvalues of

$$\boldsymbol{I} - \alpha \boldsymbol{P}^{1/2}\boldsymbol{A}'\boldsymbol{A}\boldsymbol{P}^{1/2}$$

so again following the analysis in the notes, the optimal step size is

$$\alpha_* = \frac{2}{\sigma_1^2(\boldsymbol{A}\boldsymbol{P}^{1/2}) + \sigma_N^2(\boldsymbol{A}\boldsymbol{P}^{1/2})} = \frac{2}{\sigma_1(\boldsymbol{P}^{1/2}\boldsymbol{A}'\boldsymbol{A}\boldsymbol{P}^{1/2}) + \sigma_N(\boldsymbol{P}^{1/2}\boldsymbol{A}'\boldsymbol{A}\boldsymbol{P}^{1/2})}.$$

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**Pr. 3.** (sol/hsj65)
When $\boldsymbol{Y}$ has rank $r$ and

$$\boldsymbol{Y} = \sum_{k=1}^{r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k',$$

the SVD soft-thresholding solution is

$$\hat{\boldsymbol{X}} = \sum_{k=1}^{r} [\sigma_k - \beta]_+ \boldsymbol{u}_k \boldsymbol{v}_k'.$$

Thus the approximation error is

$$\left\|\boldsymbol{Y} - \hat{\boldsymbol{X}}\right\|_{\mathrm{F}} = \left\|\sum_{k=1}^{r} \left(\sigma_k - [\sigma_k - \beta]_+\right) \boldsymbol{u}_k \boldsymbol{v}_k'\right\|_{\mathrm{F}} = \sqrt{\sum_{k=1}^{r} \left(\sigma_k - [\sigma_k - \beta]_+\right)^2}.$$

That answer is acceptable. A further simplification is:

$$\left\|\boldsymbol{Y} - \hat{\boldsymbol{X}}\right\|_{\mathrm{F}} = \sqrt{\beta \cdot |\{\sigma_k \,:\, \sigma_k > \beta\}| + \sum_{k \,:\, \sigma_k \leq \beta} \sigma_k^2},$$

where $|\cdot|$ denotes the cardinality of the set, *i.e.*, the number of singular values above $\beta$.

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**Pr. 4.** (sol/hsj32)

(a) $\{\boldsymbol{e}_1, \boldsymbol{e}_3\}$ is an orthonormal basis for the orthogonal complement of $\boldsymbol{v} = \text{span}(2\boldsymbol{e}_2)$.

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0/\sqrt{2} \end{bmatrix} \right\}$, is an orthonormal basis for the orthogonal complement of $\text{span}(\{\boldsymbol{z}\}) = \text{span}(\left\{ \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\})$.

(c) Using the general solution in the next part, the projection of $\boldsymbol{y}$ onto the orthogonal complement of $\mathcal{S} = \text{span}(\{\boldsymbol{z}\})$ is

$$\boldsymbol{y} - \frac{\boldsymbol{z}'\boldsymbol{y}}{\boldsymbol{z}'\boldsymbol{z}}\boldsymbol{z} = \boldsymbol{y} - \frac{12}{8}\boldsymbol{z} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(d) In general, the projection of $\boldsymbol{y}$ onto the orthogonal complement of $\mathcal{S} = \text{span}(\{\boldsymbol{x}\})$ is

$$\boldsymbol{P}_{\mathcal{S}^\perp}\boldsymbol{y} = \boldsymbol{P}_{\mathcal{S}}^\perp\boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{P}_{\mathcal{S}})\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{P}_{\mathcal{S}}\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{x}\boldsymbol{x}^+\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{x}(\boldsymbol{x}'\boldsymbol{x})^{-1}\boldsymbol{x}'\boldsymbol{y} = \boldsymbol{y} - \frac{\boldsymbol{x}'\boldsymbol{y}}{\boldsymbol{x}'\boldsymbol{x}}\boldsymbol{x}.$$

(e) A possible `Julia` implementation is

```
function orthcomp1(y, x)
#
# Syntax:       z = orthcomp1(y, x)
#
# Inputs:       y : vector
#               x : nonzero vector of same length, both possible very long
#
# Outputs:      z : vector of same length
#
# Description:  Projects y onto the orthogonal complement of Span({x})
#
# For full credit, your solution should be computationally efficient.

# Parse inputs
x = vec(x) # make sure it is a column
y = vec(y) # make sure it is a column

return y − ((x'y) / (x'x)) * x
end
```

This solution is likely to be the simplest possible solution and probably also the most computationally efficient solution because it uses only elementary vector arithmetic. Another solution would be to normalize $\boldsymbol{x}$ first:

```
x = x / norm(x)
return y - (x’y) * x
```

**Pr. 5.** (sol/hsx01)
Grader: full credit for any proposed exam problem that looks like a sincere attempt, iff it is submitted on time in the proper format (pdf) and location (Canvas). You do not need to check accuracy of the answer given.