EECS 455 Homework 7 YUZHAN JIANG

P1:

(a) choose $S_0(t)$ to represent the $\varphi_0(t)$

$$E_{S_0} = \int S_0^2(t)\, dt$$
$$= \left(\frac{1}{7}\right)^2 (0.5-0.1) + 1^2 \cdot (1-0.5) + \left(\frac{1}{7}\right)^2 (1.1-1)$$
$$= \frac{1}{49} \cdot 0.4 + 1 \cdot \frac{1}{2} + \frac{1}{49} \cdot 0.1$$
$$= \frac{1}{2}\left(\frac{1}{49}+1\right)$$
$$= \frac{50}{98} = \frac{25}{49}$$

Thus,
$$\varphi_0(t) = \frac{S_0(t)}{\sqrt{E_{S_0}}}$$
$$= \frac{7}{5} S_0(t)$$

$$(S_1(t), \varphi_0(t)) = \int S_1(t) \cdot \varphi_0(t) \cdot dt$$
$$= \frac{7}{5} \int S_1(t) \cdot S_0(t) \cdot dt$$
$$= \frac{7}{5}\left[\left(\frac{1}{7}\right)^2 (0.2-0.1) + \left(1 \times \frac{1}{7}\right)(0.5-0.2) + (1 \times 1)(1-0.5) + \left(\frac{1}{7}\right)^2 (1.1-1) \right]$$
$$= \frac{1}{5}\frac{1}{7} \cdot 0.1 + \frac{1}{5} \times 0.3 + \frac{7}{5} \cdot 0.5 + \frac{1}{5} \times \frac{1}{7} \times 0.1$$
$$= \frac{2}{350} + \frac{3}{50} + \frac{35}{50}$$
$$= \frac{268}{350} = \frac{134}{175}$$

$$u_1(t) = S_1(t) - \frac{134}{175}\left(\frac{7}{5} S_0(t)\right)$$
$$= S_1(t) - \frac{134}{175}\frac{7}{5} S_0(t)$$
$$= \begin{cases} 0 & 0-0.1 \\ \frac{1}{7} - \frac{134}{175}\frac{1}{7}\frac{7}{5} = -0.01 & 0.1 \le t \le 0.2 \\ 1 - \frac{134}{175} \cdot \frac{1}{7} \cdot \frac{7}{5} = 0.85 & 0.2 \le t \le 0.5 \\ 1 - \frac{134}{175} \cdot \frac{7}{5} = -0.07 & 0.5 \le t \le 1 \\ \frac{1}{7} - \frac{134}{175} \cdot \frac{1}{7} \cdot \frac{7}{5} = -0.01 & 1 \le t \le 1.1 \end{cases}$$

$$\|u_1\|_2^2 = E_{u_1}$$

$$\sqrt{E_{u_1}}$$

$$E_{u_1} = (-0.01)^2 (0.2-0.1) + (0.85)^2 (0.5-0.2) + (0.07)^2 \cdot (1-0.5) + (0.01)^2 \cdot 0.1$$
$$= 2 \times 10^{-5} + 0.217 + 0.0025 = 0.21952 \approx 0.22$$

$$\varphi_1(t) = u_1(t) / \|u_1\| = u_1(t) / 0.47$$

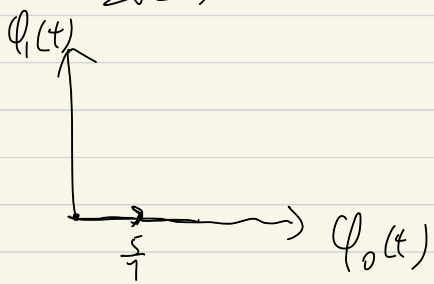
$$\varphi_1(t) = u_1(t) / \sqrt{E_{u_1}} = \begin{cases} 0 & 0 \le t \le 0.1 \\ -0.021 & 0.1 \le t \le 0.2 \\ 1.81 & 0.2 \le t \le 0.5 \\ 0.15 & 0.5 \le t \le 1 \\ -0.021 & 1 \le t \le 1.1 \end{cases}$$

$S_0(t) = \frac{5}{7}\varphi_0(t) + 0\ \varphi_1(t)$

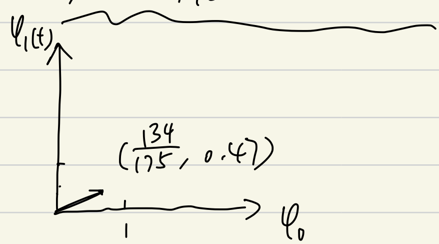
$\frac{u_1}{\sqrt{E_{u_1}}} = \frac{S_1(t)}{\sqrt{E_{u}}} - \frac{\frac{134}{175}\varphi_0}{\sqrt{E_{\varphi_1}}}$

$\varphi_1\ E_n =$

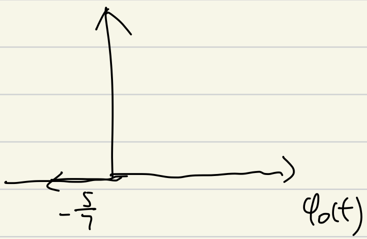
$S_0(t)$

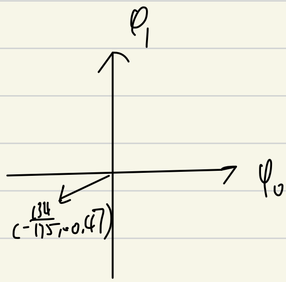


$S_1(t)$ $S_1(t) = \frac{134}{175}\varphi_0 + 0.47\varphi_1$



$S_2(t)$





(b) $E_{s_0(t)} = \int s_0^2(t) \cdot dt$

$E_{s_0(t)} = E_{s_1(t)} = \frac{25}{49} A^2$

$E_{s_1(t)} = E_{s_3(t)} = \int s_1^2(t) \cdot dt$

$= \left(\frac{1}{7}A\right)^2 \cdot (0.2 - 0.1) + (1A)^2 (1 - 0.2)$

$+ \left(\frac{1}{7}A\right)^2 (1.1 - 1)$

$= \frac{1}{49} A^2 \cdot 0.1 \times 2 + A^2 \cdot 0.8$

$= \left(\frac{2}{490} + \frac{8}{10} \cdot \frac{49}{49}\right) A^2$

$= \left(\frac{394}{490}\right) A^2$

$=$

$\therefore\ E = \frac{1}{2} \left(\frac{25}{49} A^2 + \frac{394}{490} A^2\right)$

$= 2.628\ A^2$

(c)

$$d_E^2(S_0, S_1) = \left[\left(\frac{5}{7} - \frac{134}{175}\right)^2 + 0.47^2\right] A^2 = 0.22 A^2$$

$$d_E^2(S_0, S_2) = \left[\left(\frac{5}{7} - \left(-\frac{5}{7}\right)\right)^2\right] A^2 = 2.04 A^2$$

$$d_E^2(S_0, S_3) = \left[\left(\frac{5}{7} - \left(-\frac{134}{175}\right)\right)^2 + 0.47^2\right] A^2 = 2.4 A^2$$

$$d_E^2(S_1, S_2) = \left[\left(-\frac{5}{7} - \frac{134}{175}\right)^2 + 0.47^2\right] A^2 = 2.4 A^2$$

$$d_E^2(S_1, S_3) = \left[\left(2 \times \frac{134}{175}\right)^2 + (2 \times 0.47)^2\right] A^2 = 3.23 A^2$$

$$d_E^2(S_2, S_3) = \left[\left(-\frac{5}{7} + \frac{134}{175}\right)^2 + (0.47)^2\right] A^2 = 0.22 A^2$$

(d) Optimal rec.: is to find the signal $S_i(t)$ closet to the received signal r.

(e) $$P_{e,0} \le Q\left(\frac{\sqrt{0.22A^2}}{2\sigma}\right) + Q\left(\frac{\sqrt{2.04A^2}}{2\sigma}\right) + Q\left(\frac{\sqrt{2.4A^2}}{2\sigma}\right)$$

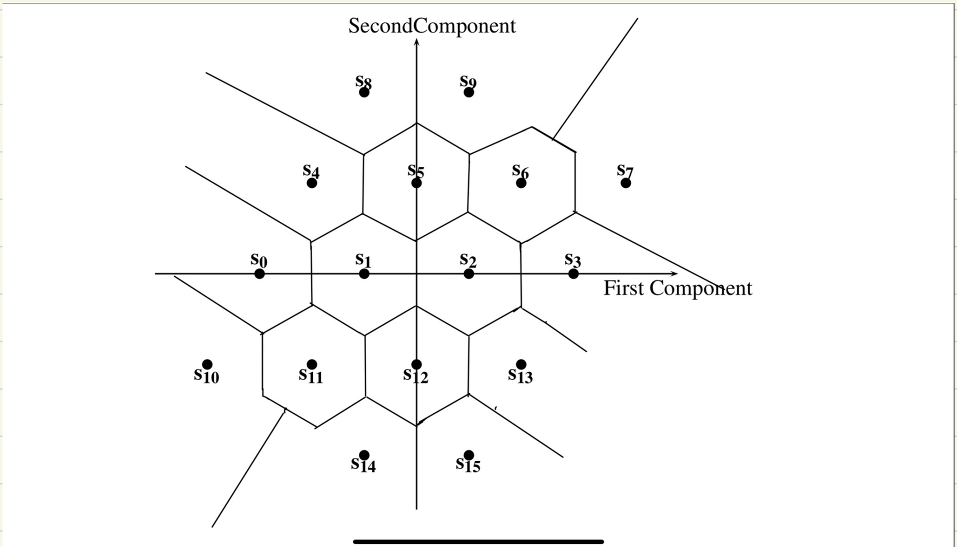
$$P_{e,1} \le Q\left(\frac{\sqrt{0.22A^2}}{2\sigma}\right) + Q\left(\frac{\sqrt{2.4A^2}}{2\sigma}\right) + Q\left(\frac{\sqrt{3.23A^2}}{2\sigma}\right)$$

$$P_{e,2} \le Q\left(\frac{\sqrt{2.04A^2}}{2\sigma}\right) + Q\left(\frac{\sqrt{2.4A^2}}{2\sigma}\right) + Q\left(\frac{\sqrt{0.22A^2}}{2\sigma}\right)$$

$$P_{e,3} \le Q\left(\frac{\sqrt{2.4A^2}}{2\sigma}\right) + Q\left(\sqrt{\frac{3.23A^2}{2\sigma}}\right) + Q\left(\frac{\sqrt{0.22A^2}}{2\sigma}\right)$$

$$\therefore \; P_e \le \frac{1}{4} P_{e,0} + \frac{1}{4} P_{e,1} + \frac{1}{4} P_{e,2} + \frac{1}{4} P_{e,3}$$

2. (a)



b)

Using Matlab, we compute the distance between pairs of signals.

0	2.00	4.00	6.00	2.00	3.4641	5.2915	7.2111	4	5.2915	2	2	3.4641	5.2915	4	5.2915
2	0	2	4	2	2	3.4641	5.2915	3.4641	4	3.4641	2	2	3.4641	3.4641	4
4	2	0	2	3.4641	2	2	3.4641	4	3.4641	5.2915	3.4641	2	2	4	3.4641
6	4	2	0	5.2915	3.4641	2	2	5.2915	4	7.2111	5.2915	3.4641	2	5.2915	4
2	2	3.4641	5.2915	0	2	4	6	2	3.4641	4	3.4641	4	5.2915	5.2915	6
3.4641	2	2	3.4641	2	0	2	4	2	2	5.2915	4	3.4641	4	5.2915	5.2915
5.2915	3.4641	2	2	4	2	0	2	3.4641	2	6.9282	5.2915	4	3.4641	6	5.2915
7.2111	5.2915	3.4641	2	6	4	2	0	5.2915	3.4641	8.7178	6.9282	5.2915	4	7.2111	6
4	3.4641	4	5.2915	2	2	3.4641	5.2915	0	2	6	5.2915	5.2915	6	6.9282	7.2111
5.2915	4	3.4641	4	3.4641	2	2	3.4641	2	0	7.2111	6	5.2915	5.2915	7.2111	6.9282
2	3.4641	5.2915	7.2111	4	5.2915	6.9282	8.7178	6	7.2111	0	2	4	6	3.4641	5.2915
2	2	3.4641	5.2915	3.4641	4	5.2915	6.9282	5.2915	6	2	0	2	4	2	3.4641
3.4641	2	2	3.4641	4	3.4641	4	5.2915	5.2915	5.2915	4	2	0	2	2	2
5.2915	3.4641	2	2	5.2915	4	3.4641	4	6	5.2915	6	4	2	0	3.4641	2
4	3.4641	4	5.2915	5.2915	5.2915	6	7.2111	6.9282	7.2111	3.4641	2	2	3.4641	0	2
5.2915	4	3.4641	4	6	5.2915	5.2915	6	7.2111	6.9282	5.2915	3.4641	2	2	2	0

Therefore, the union Bound is $P_{e,i} \leq \sum_{j \neq i}^{M-1} Q\left(\frac{d_E(s_i, s_j)}{2\sigma}\right)$

$$P_{e,0} \leq 4Q\left(\frac{2}{2\sigma}\right) + 3Q\left(\frac{4}{2\sigma}\right) + Q\left(\frac{6}{2\sigma}\right) + 2Q\left(\frac{3.4641}{2\sigma}\right) + 4Q\left(\frac{5.2915}{2\sigma}\right) + Q\left(\frac{7.2111}{2\sigma}\right)$$

$$P_{e,1} \leq 6Q\left(\frac{2}{2\sigma}\right) + 3Q\left(\frac{4}{2\sigma}\right) + 0\cdot Q\left(\frac{6}{2\sigma}\right) + 5Q\left(\frac{3.4641}{2\sigma}\right) + 1\cdot Q\left(\frac{5.2915}{2\sigma}\right) + 0Q\left(\frac{7.2111}{2\sigma}\right)$$

$$P_{e,2} \leq 6Q\left(\frac{2}{2\sigma}\right) + 3Q\left(\frac{4}{2\sigma}\right) + 0\,Q\left(\frac{6}{2\sigma}\right) + 5Q\left(\frac{3.4641}{2\sigma}\right) + 1\cdot Q\left(\frac{5.2915}{2\sigma}\right) + 0Q\left(\frac{7.2111}{2\sigma}\right)$$

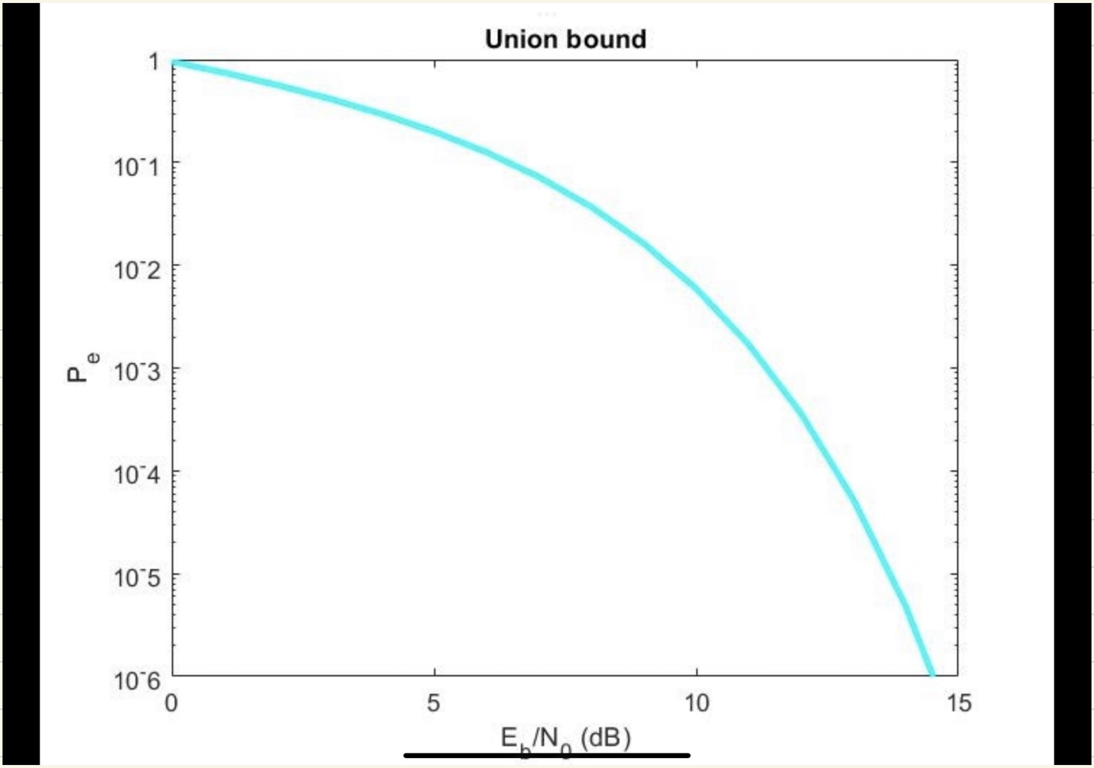
⋮

Similarly for $P_{e,3}$, $P_{e,4}$, $P_{e,5}$, $P_{e,6}$, $P_{e,7}$, $P_{e,8}$, $P_{e,9}$, $P_{e,10}$, $P_{e,11}$, $P_{e,12}$, $P_{e,13}$, $P_{e,14}$, $P_{e,15}$

Then we can compute the P_e:

$$P_e \leq \frac{1}{16} \sum_{i=0}^{15} P_{e,i}$$

The plot is looking like this if $\frac{E_b}{N_0}$ is $\in [0, 15]$



```
s0 = [-3, 0]
s1 = [-1, 0]
s2 = [+1, 0]
s3 = [3, 0]
s4 = [-2 , sqrt(3)]
s5 = [0, sqrt(3)]
s6 =[2, sqrt(3)]
s7 = [4, sqrt(3)]
s8 = [-1, 2*sqrt(3)]
s9 = [+1, 2*sqrt(3)]
s10 = [-4, -sqrt(3)]
s11 = [-2, -sqrt(3)]
s12 = [0, -sqrt(3)]
s13 = [2, -sqrt(3)]
s14 =[-1, -2*sqrt(3)]
s15 = [1, -2*sqrt(3)]


S=[s0; s1; s2; s3; s4; s5; s6; s7; s8; s9; s10; s11; s12; s13;s14;s15];
E_avg = Compute_E_average(S)
E_b = E_avg / log2(16)

x = [0:15]
y = zeros(size(x))
for m = 0:15
    Pe = 0;
    ebnodB = m;
    ebno = 10^(ebnodB/10);
    N0 = E_b / ebno
    Sigma = sqrt(N0/2)
    for j=  1:16
        p =  pij(j,S,Sigma)
        Pe =  Pe + p
    end
    Pe = Pe / 16;
    disp("Pe for dB" + m + " is :" + Pe)
    y(m+1) = Pe

end
```

```
figure
semilogy(x,y,'c','LineWidth',2.5)
% plot(x,y,'c','LineWidth',2.5)
yticks([10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 1])
yticklabels({'10^-6','10^-5','10^-4','10^-3','10^-2', '10^-1', '1'})
axis([0 15 10^-6 1])
xlabel('E_b/N_0 (dB)')
ylabel('P_{e}')
title('Union bound')
function Total_Q = pij(si, S,Sigma)
    Total_Q = 0
    for i =  1:16
        if i == si
            continue;
        else
            distance = sqrt( (S(si, 1) - S(i,1))^2 + (S(si, 2) - S(i,2))^2);
%             disp("distance in " + si +" "+ i +"  : "+ distance)
            y = qfunc(distance / (2* Sigma));
            Total_Q =  Total_Q + y;
            disp("P2 for "+ si + " " + i +" is: "  + y)
        end
    end
     disp("Pei for " + i + " is :" + Total_Q)
end

function E_avg = Compute_E_average(S)
    E_avg = 0
    for i =  1:16
      e_i = S(i,1)^2 +  S(i,2)^2 ;
      E_avg =  E_avg + e_i;
    end
    E_avg  = E_avg /16;

end
```

This is error Prob when dB = 15

```
P2 for 16 1 is: 5.309e-45
P2 for 16 2 is: 1.4335e-26
P2 for 16 3 is: 2.0963e-20
P2 for 16 4 is: 1.4335e-26
P2 for 16 5 is: 2.9059e-57
P2 for 16 6 is: 5.309e-45
P2 for 16 7 is: 5.309e-45
P2 for 16 8 is: 2.9059e-57
P2 for 16 9 is: 9.3038e-82
P2 for 16 10 is: 1.2296e-75
P2 for 16 11 is: 5.309e-45
P2 for 16 12 is: 2.0963e-20
P2 for 16 13 is: 5.7332e-08
P2 for 16 14 is: 5.7332e-08
P2 for 16 15 is: 5.7332e-08
Pei for 16 is :1.72e-07

p =

   1.7200e-07


Pe =

   3.7839e-06

Pe for dB15 is :2.3649e-07
```

This is union Bound I get:

p2_unionbound.png

y

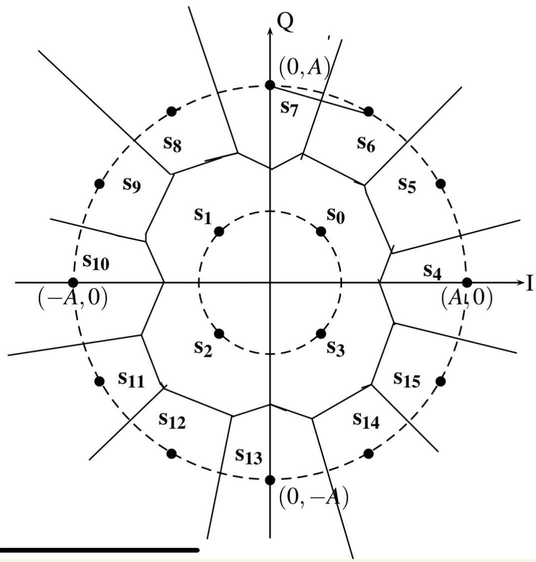
1x16 double

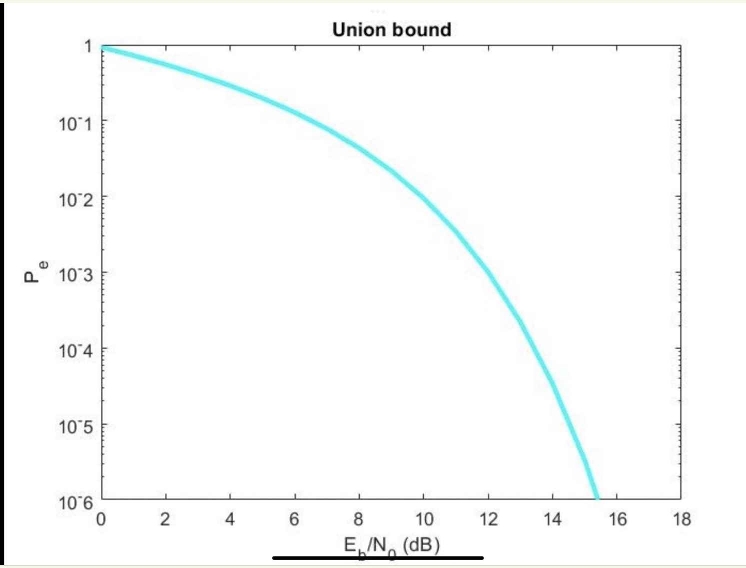
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0.9567	0.7462	0.5672	0.4180	0.2962	0.1994	0.1255	0.0722	0.0369	0.0163	0.0059	0.0017	3.5982e-04	5.2351e-05	4.7406e-06	2.3649e-07

3.

(a)

$$
\begin{aligned}
\mathbf{s_0} &= (+1,+1)\\
\mathbf{s_1} &= (-1,+1)\\
\mathbf{s_2} &= (-1,-1)\\
\mathbf{s_3} &= (+1,-1)\\
\mathbf{s_4} &= (A,0)\\
\mathbf{s_5} &= (A\cos(2\pi/12), A\sin(2\pi/12))\\
\mathbf{s_6} &= (A\cos(2\pi 2/12), A\sin(2\pi 2/12))\\
\mathbf{s_7} &= (0,A)\\
\mathbf{s_8} &= (A\cos(2\pi 4/12), A\sin(2\pi 4/12))\\
\mathbf{s_9} &= (A\cos(2\pi 5/12), A\sin(2\pi 5/12))\\
\mathbf{s_{10}} &= (-A,0)\\
\mathbf{s_{11}} &= (A\cos(2\pi 7/12), A\sin(2\pi 7/12))\\
\mathbf{s_{12}} &= (A\cos(2\pi 8/12), A\sin(2\pi 8/12))\\
\mathbf{s_{13}} &= (0,-A)\\
\mathbf{s_{14}} &= (A\cos(2\pi 10/12), A\sin(2\pi 10/12))\\
\mathbf{s_{15}} &= (A\cos(2\pi 10/12), A\sin(2\pi 10/12))
\end{aligned}
$$






```
A=sqrt(4*(2+sqrt(3)))
s0 = [+1, +1]
s1 = [-1, +1]
s2 = [-1, -1]
s3 = [+1, -1]
s4 = [A , 0]
s5 = [A*cos(2*pi/12),A*sin(2*pi/12)]
s6 =[A*cos(2*pi*2/12),A*sin(2*pi*2/12)]
s7 = [0, A]
s8 = [A*cos(2*pi*4/12),A*sin(2*pi*4/12)]
s9 = [A*cos(2*pi*5/12),A*sin(2*pi*5/12)]
s10 = [-A, 0]
s11 = [A*cos(2*pi*7/12),A*sin(2*pi*7/12)]
s12 = [A*cos(2*pi*8/12),A*sin(2*pi*8/12)]
s13 = [0, -A]
s14 =[A*cos(2*pi*10/12),A*sin(2*pi*10/12)]
s15 = [A*cos(2*pi*11/12),A*sin(2*pi*11/12)]

S=[s0; s1; s2; s3; s4; s5; s6; s7; s8; s9; s10; s11; s12; s13;s14;s15];
E_avg = Compute_E_average(S)
E_b = E_avg / log2(16)

x = [0:18]
y = zeros(size(x))
for m = 0:18
    Pe = 0;
    ebnodB = m;
    ebno = 10^(ebnodB/10);
    N0 = E_b / ebno
    Sigma = sqrt(N0/2)
    for j=  1:16
        p =  pij(j,S,Sigma)
        Pe =  Pe + p
    end
    Pe = Pe / 16;
    disp("Pe for dB" + m + " is :" + Pe)
    y(m+1) = Pe

end
```

```matlab
% yticklabel({'10^-6','10^-5','10^-4','10^-3','10^-2', '10^-1', '1'})
% yticks([10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 1])
figure
semilogy(x,y,'c','LineWidth',2.5)
% plot(x,y,'c','LineWidth',2.5)
yticks([10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 1])
yticklabels({'10^-6','10^-5','10^-4','10^-3','10^-2', '10^-1', '1'})
axis([0 18 10^-6 1])
xlabel('E_b/N_0 (dB)')
ylabel('P_{e}')
title('Union bound')
function Total_Q = pij(si, S,Sigma)
    Total_Q = 0
    for i =  1:16
        if i == si
            continue;
        else
            distance = sqrt( (S(si, 1) - S(i,1))^2 + (S(si, 2) - S(i,2))^2);
%             disp("distance in " + si +" "+ i +"  : "+ distance)
            y = qfunc(distance / (2* Sigma));
            Total_Q =  Total_Q + y;
            disp("P2 for "+ si + " " + i +" is: "  + y)
        end
    end
     disp("Pei for " + i + " is :" + Total_Q)
end

function E_avg = Compute_E_average(S)
    E_avg = 0
    for i =  1:16
      e_i = S(i,1)^2 +  S(i,2)^2 ;
      E_avg =  E_avg + e_i;
    end
    E_avg  = E_avg /16;

end
```

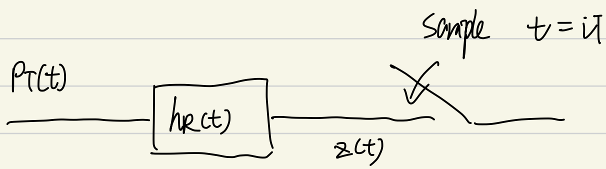
This is the Union Bound I get

19:19 Wed Nov 3 ••• 67%

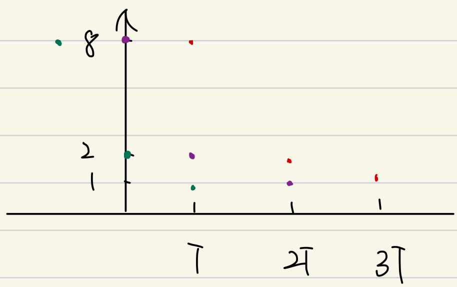
P3_unionbound2.png

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0.9268	0.7211	0.5477	0.4044	0.2887	0.1977	0.1284	0.0780	0.0435	0.0217	0.0095	0.0035	0.0010	2.2215e-04	3.4068e-05	3.3094e-06	1.8066e-07	4.7715e-09

P4:



(a)



$$E[Z(T)] = 8Ab_0 + 2Ab_{-1} + 1Ab_{-2}$$

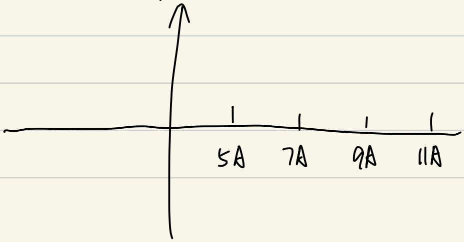
∴ The possible values for output due to the desired signal is 11, 9, 7 and 5

(b) $E[Z(iT)] = 8Ab_i + \underbrace{2Ab_{i-1} + Ab_{i-2}}$

Intersymbol Interference

$$(8A - 2A - A) \le E[Z(iT) \mid b_{i-1} = +1, b_{i-2}, b_{i-3} \dots] \le (8A + 2A + A)$$

$$5A \le E[Z(iT) \mid b_{i-1} = +1, b_{i-2}, b_{i-3} \dots] \le 11A$$



$$\Rightarrow Q\left(\frac{11A}{\sigma}\right) \le P_{e, b_{i-1}=1} \le Q\left(\frac{5A}{\sigma}\right)$$

where

$$\sigma^2 = \frac{N_0}{2} \int h^2(t) \cdot dt$$

$$= 8N_0$$

$$\sigma = \sqrt{8N_0}$$

(c) It is easy to see that $P_{e,b_{i-1}=1} = P_{e,b_{i-1}=-1}$

$$\therefore \quad P_{e,b_{i-1}=1} = \frac{1}{4}Q\left(\frac{5A}{\sigma}\right) + \frac{1}{4}Q\left(\frac{7A}{\sigma}\right) + \frac{1}{4}Q\left(\frac{9A}{\sigma}\right) + \frac{1}{4}Q\left(\frac{11A}{\sigma}\right)$$

$$\therefore \quad P_e = \frac{1}{2}P_{e,b_{i-1}=-1} + \frac{1}{2}P_{e,b_{i-1}=+1}$$

$$= \frac{1}{4}Q\left(\frac{5A}{\sigma}\right) + \frac{1}{4}Q\left(\frac{7A}{\sigma}\right) + \frac{1}{4}Q\left(\frac{9A}{\sigma}\right) + \frac{1}{4}Q\left(\frac{11A}{\sigma}\right)$$

where $\sigma = \sqrt{8N_0}$

5

(a) $\varphi_0(t) = \sqrt{\frac{2}{T}} \cdot \underline{\cos(2\pi f_c t)}\, P_T(t)$

$\varphi_1(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \cdot P_T(t)$

(b) $S_0(t) = A \cdot \sqrt{\frac{T}{2}} P_T(t) \varphi_0(t)$

$S_1(t) = \frac{A}{2} \cdot \sqrt{\frac{T}{2}} P_T(t)\, \varphi_0(t) + \frac{A}{2} \sqrt{\frac{T}{2}} P_T(t) \cdot \varphi_1(t)$

$S_2(t) = \frac{A}{2} \sqrt{\frac{T}{2}} P_T(t)\ \varphi_1(t)$

$S_3(t) = -\frac{1}{2} A \sqrt{\frac{T}{2}} P_T(t)\, \varphi_0(t) + \frac{1}{2} A \sqrt{\frac{T}{2}}\, \varphi_1(t)$

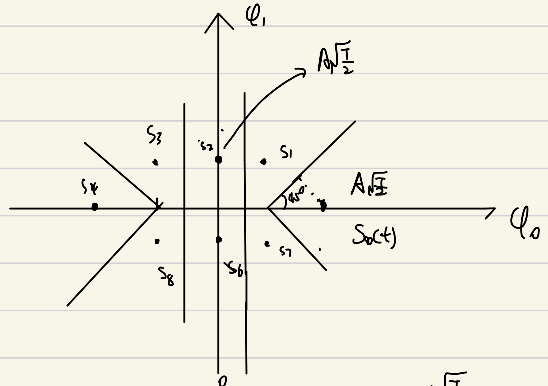
$S_4(t) = -A \sqrt{\frac{T}{2}} P_T(t)\, \varphi_0(t)$

$S_5(t) = -\frac{1}{2} A \sqrt{\frac{T}{2}} P_T(t) \cdot \varphi_0(t) - \frac{1}{2} A \sqrt{\frac{T}{2}} P_T\, \varphi_1(t)$

$S_6(t) = -\frac{1}{2} A \sqrt{\frac{T}{2}}\, \varphi_1(t)$

$S_7(t) = \frac{1}{2} A \sqrt{\frac{T}{2}}\, \varphi_0(t) - \frac{1}{2} A \sqrt{\frac{T}{2}}\, \varphi_1(t)$

(c)



(d) $P(\text{correct}, S_0(t)) = \Phi^2\left(\frac{\sin(45°) \cdot A\sqrt{\frac{T}{2}}}{\sigma}\right)$

$= \left(\Phi\left(\frac{A \frac{\sqrt{2}}{2} \sqrt{\frac{T}{2}}}{\sqrt{N_0/2}}\right)\right)^2$

(d)