1. A communication system transmits one of 8 equally likely signals. The signals (waveforms) are represented by the vectors shown below.

   (a) Determine how many information bits can be sent using these signals.

   (b) Determine the energy of each of the signals and the average energy per information bit.

   (c) Determine the squared Euclidean distance between signals s_0 and all the other signals.

   (d) Determine the rate of communication in bits/dimension for these signals.

$$
\begin{aligned}
s_0 &= (-1,-1,-1,-1,-1) \\
s_1 &= (-1,-1,+3,-3,+3) \\
s_2 &= (-1,+3,-3,+3,-1) \\
s_3 &= (-1,+3,+1,+1,+3) \\
s_4 &= (+3,-3,+3,-1,-1) \\
s_5 &= (+3,-3,-1,-3,+3) \\
s_6 &= (+3,+1,+1,+3,-1) \\
s_7 &= (+3,+1,-3,+1,+3)
\end{aligned}
$$

a. $M = 8$ signals

Number of bits per signal $= \log_2(M) = \log_2(8) = 3$

$\boxed{\text{3 information bits}}$

b. E_i = energy for signal s_i

$E_0 = (-1)^2 + (-1)^2 + (-1)^2 + (-1)^2 + (-1)^2 = 5$

$E_1 = (-1)^2 + (-1)^2 + 3^2 + (-3)^2 + 3^2 = 29$

$E_2 = (-1)^2 + 3^2 + (-3)^2 + 3^2 + (-1)^2 = 29$

$E_3 = (-1)^2 + 3^2 + 1^2 + 1^2 + 3^2 = 21$

$E_4 = 3^2 + (-3)^2 + 3^2 + (-1)^2 + (-1)^2 = 29$

$E_5 = 3^2 + (-3)^2 + (-1)^2 + (-3)^2 + 3^2 = 37$

$$E_6 = 3^2+1^2+1^2+3^2+(-1)^2 = 21$$

$$E_7 = 3^2+1^2+(-3)^2+1^2+3^2 = 29$$

$$\bar{E} = \frac{\sum_{i=0}^{7} E_i}{M} = \frac{5+29+29+21+29+37+21+29}{8}$$

$$= \frac{200}{8} = 25$$

$\bar{E}_b$ = average energy per bit $= \frac{25}{3} = 8.\bar{3}$

c. $d_E^2(s_0, s_j)$ = squared Euclidean distance between s_0 and s_j, $j \neq 0$

$$d_E^2(s_0,s_1) = (\overset{0}{\cancel{-1-(-1)}})^2 + (\overset{0}{\cancel{-1-(-1)}})^2 + (-1-3)^2 + (-1-(-3))^2 + (-1-3)^2 = (-4)^2 + 2^2 + (-4)^2 = 36$$

$$d_E^2(s_0,s_2) = (\overset{0}{\cancel{-1-(-1)}})^2 + (-1-3)^2 + (-1-(-3))^2 + (-1-3)^2 + (\overset{0}{\cancel{-1-(-1)}})^2 = (-4)^2 + 2^2 + (-4)^2 = 36$$

$$d_E^2(s_0,s_3) = (\overset{0}{\cancel{-1-(-1)}})^2 + (-1-3)^2 + (-1-1)^2 + (-1-1)^2 + (-1-3)^2 = (-4)^2 + (-2)^2 + (-2)^2 + (-4)^2 = 40$$

$$d_E^2(s_0,s_4) = (-1-3)^2 + (-1-(-3))^2 + (-1-3)^2 + (\underset{0}{\cancel{-1-(-1)}})^2 + (\underset{0}{\cancel{-1-(-1)}})^2 = (-4)^2 + 2^2 + (-4)^2 = 36$$

$$d_E^2(s_0,s_5) = (-1-3)^2 + (-1-(-3))^2 + (\overset{0}{\cancel{-1-(-1)}})^2 + (-1-(-3))^2 + (-1-3)^2 = (-4)^2 + 2^2 + 2^2 + (-4)^2 = 40$$

$$d_E^2(s_0, s_6) = (-1-3)^2 + (-1-1)^2 + (-1-1)^2 + (-1-3)^2 + (\underbrace{\cancel{-1-(-1)}}_{0})^2 = (-4)^2 + (-2)^2 + (-2)^2 + (-4)^2 = 40$$

$$d_E^2(s_0, s_7) = (-1-3)^2 + (-1-1)^2 + (-1-(-3))^2 + (-1-1)^2 + (-1-3)^2 = (-4)^2 + (-2)^2 + 2^2 + (-2)^2 + (-4)^2 = 44$$

d. Each signal can encode 3 bits

Each signal represented by a 5-dimensional vector

$$\text{Rate of communication} = \frac{3 \text{ bits}}{5 \text{ dimensions}} = \boxed{0.6 \ \text{bits/dimension}}$$

2. (a) The signal $s_0(t)$ consists of a sequence of pulses each of duration $T_c = T/7$ as shown in the figure below.

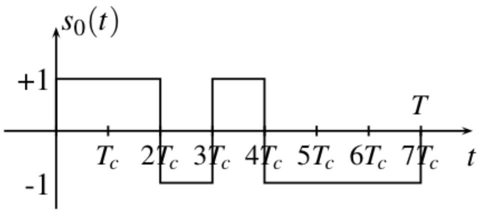
$$s_0(t) = p_{T_c}(t) + p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c) - p_{T_c}(t - 4T_c) - p_{T_c}(t - 5T_c) - p_{T_c}(t - 6T_c)$$

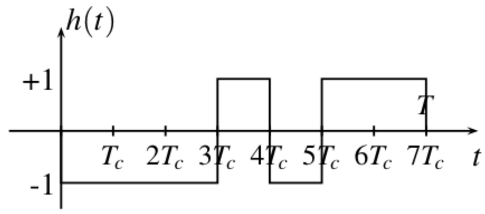
The filter is given by

$$h(t) = -p_{T_c}(t) - p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c) - p_{T_c}(t - 4T_c) + p_{T_c}(t - 5T_c) + p_{T_c}(t - 6T_c)$$

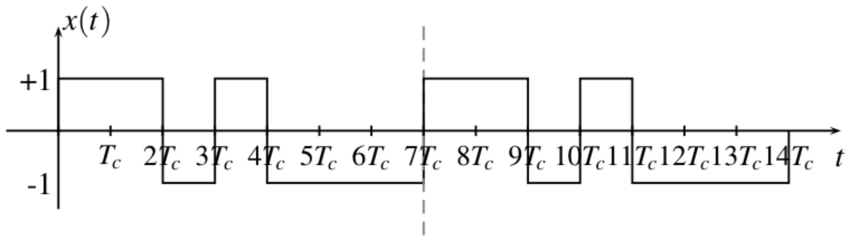
as shown below (which is the time flip of $s_0(t)$, i.e. $h(t) = s_o(T - t)$).

(a) Find (plot) the output of the filter as a function of time. The output should be a function of time beginning at time 0 and ending at time $2T = 14T_c$.

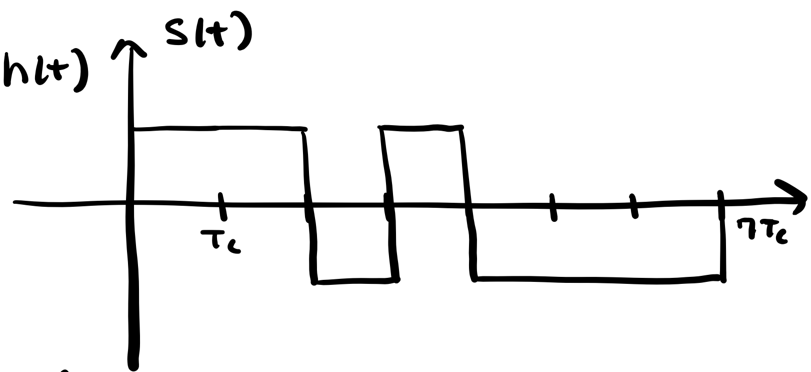


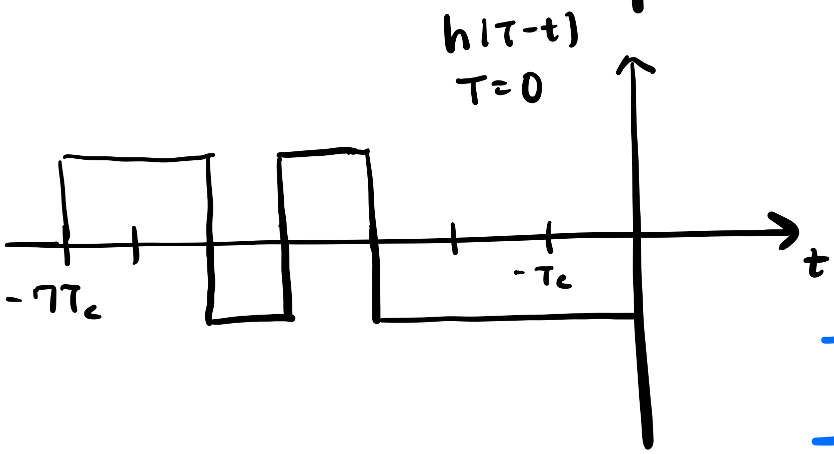


(b) Find the filter output (for the same filter) when the input is $x(t) = s(t) + s(t - T)$. The output is a function beginning at time 0 and ending at time $21T_c = 3T$.



a. Let $y(t) = s(t) * h(t)$

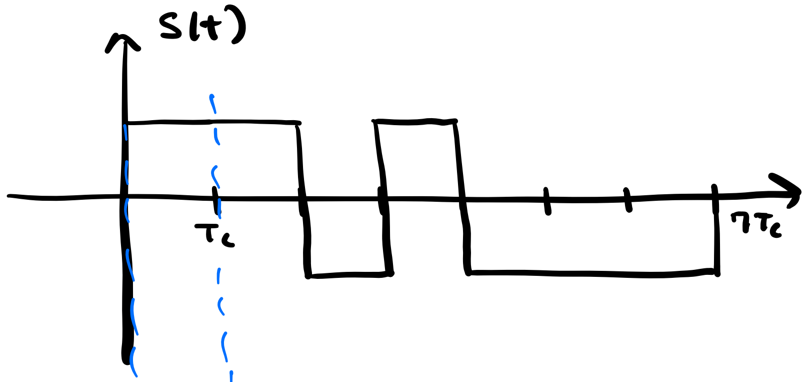


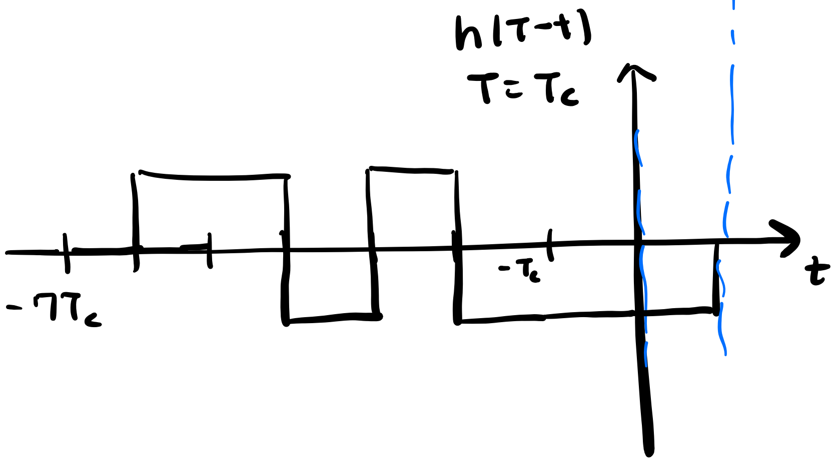


No overlap

→ Area of overlap = 0

→ $y(0) = 0$

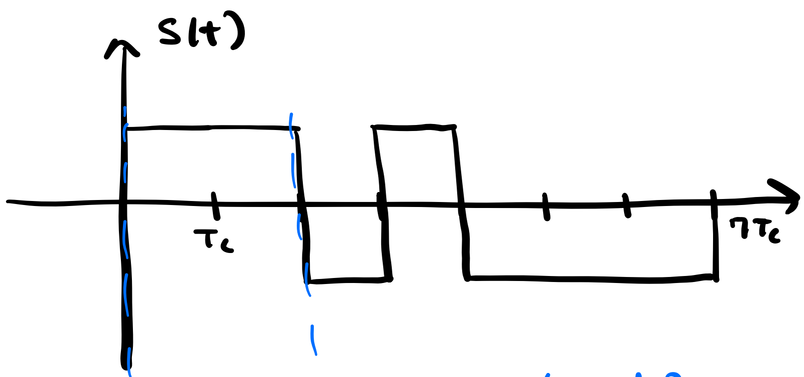


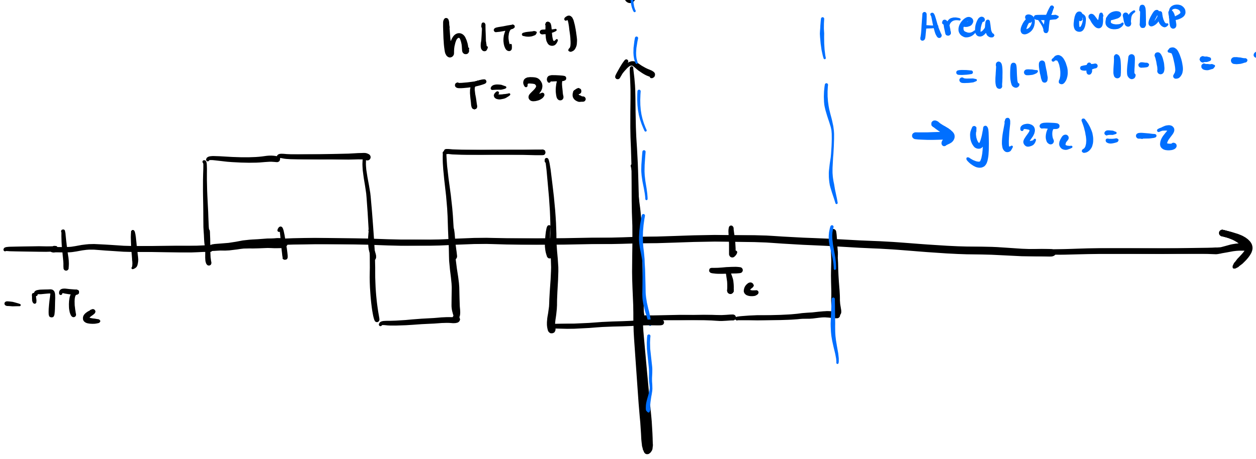


Area of overlap

$= +1(-1) = -1$

$\rightarrow y(T_c) = -1$

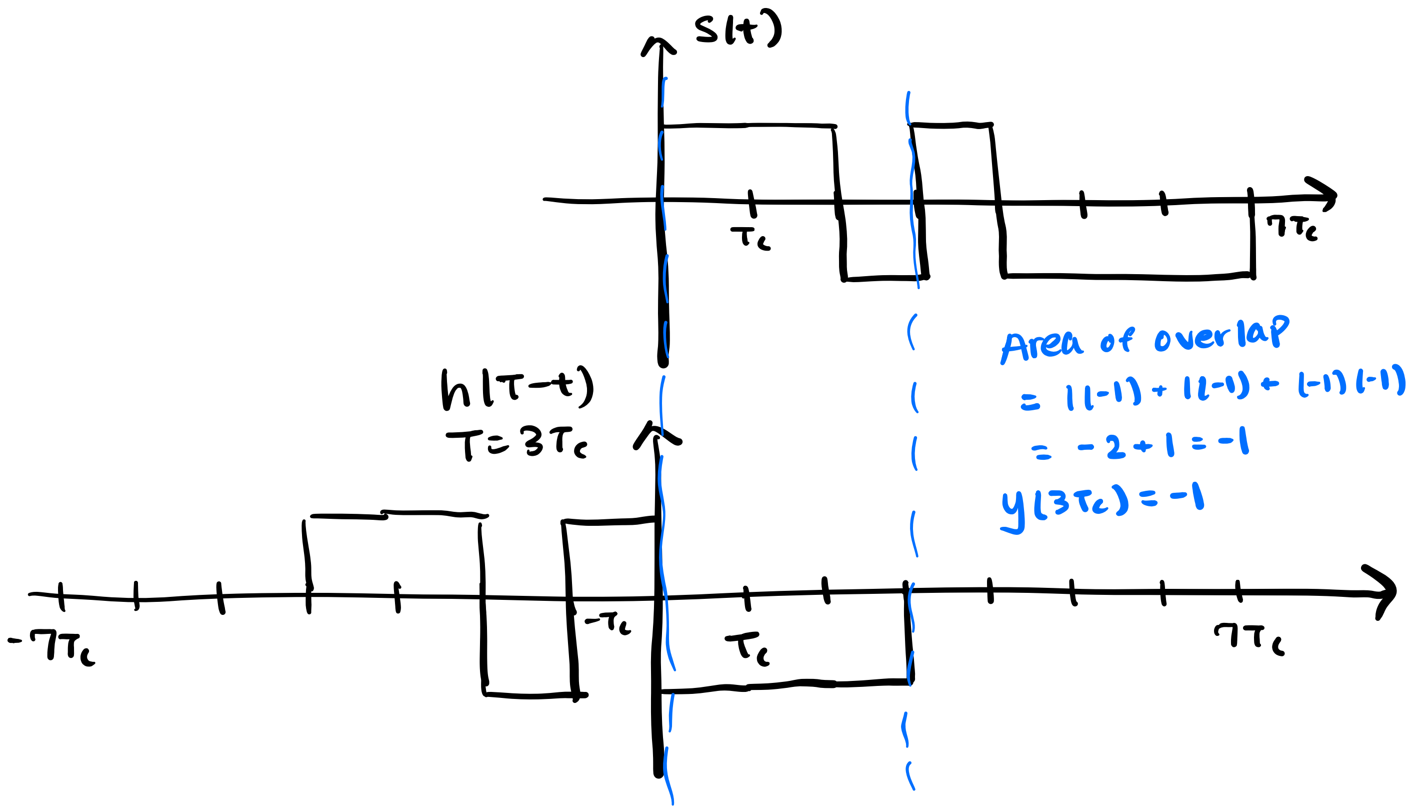




Area of overlap

$= 1(-1) + 1(-1) = -2$

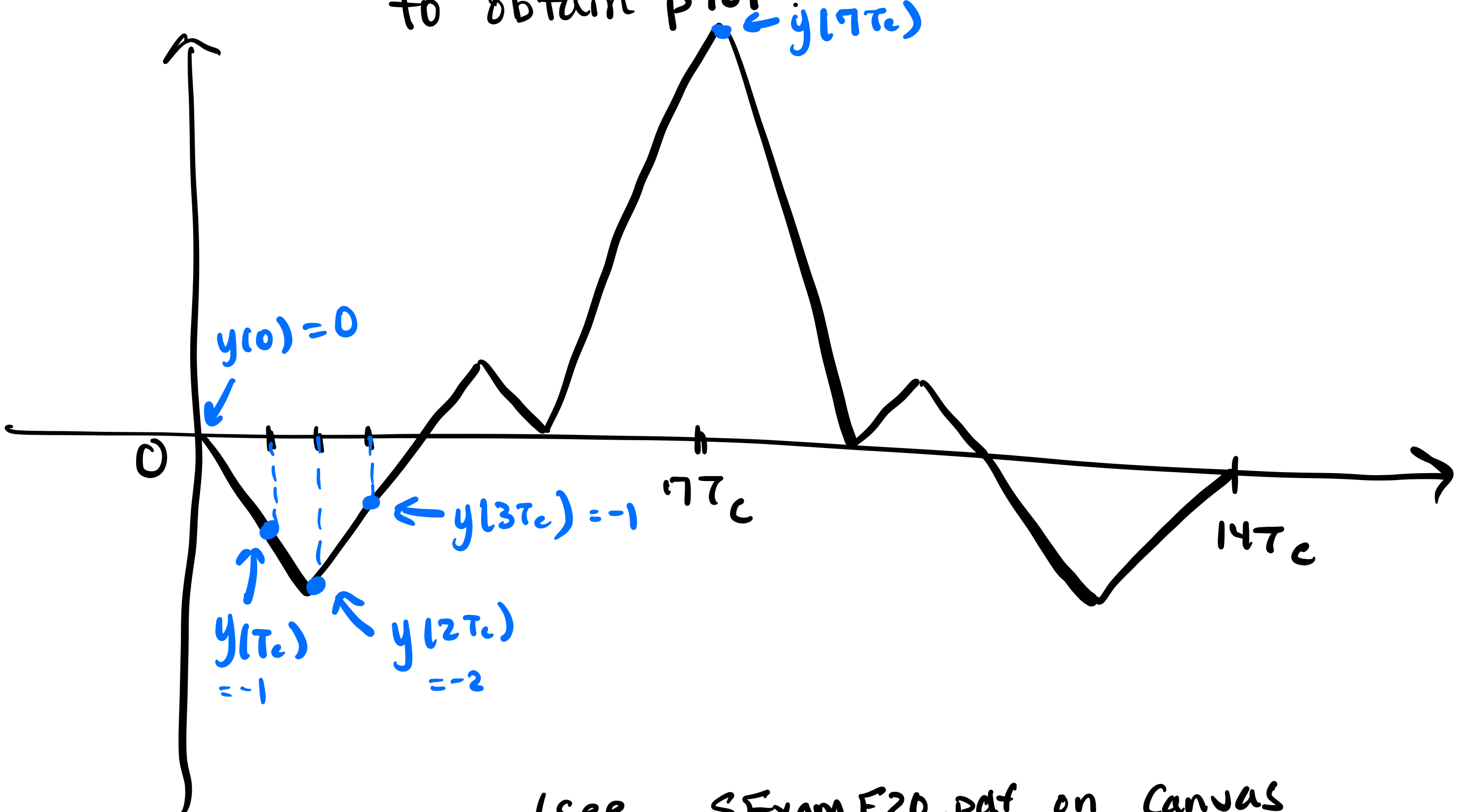
$\rightarrow y(2T_c) = -2$



Repeat for $\tau = 4T_c, \dots, 14T_c$

Use $y(0), y(T_c), y(2T_c), \dots, y(14T_c)$

to obtain plot:



(see SExamF20.pdf on Canvas
for more detailed plot)

b. $x(t) = s(t) + s(t-T)$

Let $z(t) = x(t) * h(t)$

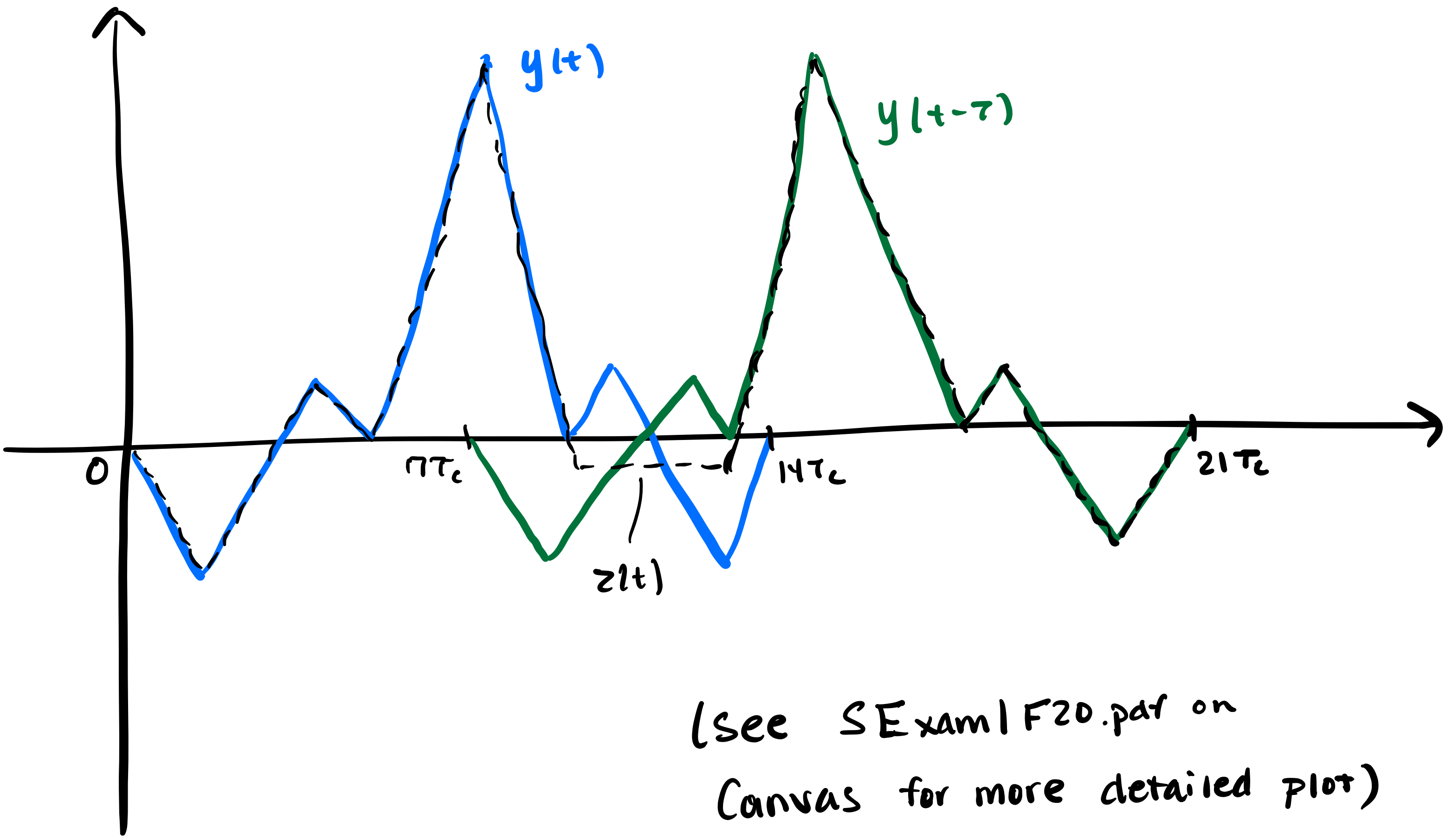
$= \left(s(t) + s(t-T)\right) * h(t)$

$= \left(s(t) * h(t)\right) + \left(s(t-T) * h(t)\right)$

$= y(t) + y(t-T)$

from part (a)

convolution is time-invariant



(see SExam1F20.pdf on Canvas for more detailed plot)

3. A communication system transmits one of three signals:

$$s_0(t) = A\cos(2\pi f_c t)p_T(t)$$

$$s_1(t) = 0$$

$$s_2(t) = -A\cos(2\pi f_c t)p_T(t)$$

over an additive white Gaussian noise channel with power spectral density $N_0/2$. Let $r(t)$ denote the received signal ($r(t) = s_i(t) + n(t)$). The receiver computes the quantity

$$Z = \int_0^T r(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt.$$

Assume $2\pi f_c T = 2\pi n$ for some large integer n (to ignore double frequency terms). The receiver output Z is compared with a threshold γ and a threshold $-\gamma$. If $Z > \gamma$, the decision is made that $s_0(t)$ was sent. If $Z < -\gamma$, the decision is made that $s_2(t)$ was sent. If $-\gamma < Z < \gamma$ the the decision is made in favor of $s_1(t)$

(a) Determine the three conditional probabilities of error: $P_{e,0}$ = probability of error given s_0 sent, $P_{e,1}$ =probability of error given s_1 sent, and $P_{e,2}$ = probability of error given s_2 sent.

(b) Determine the average error probability assuming that all three signals are equally probable of being transmitted.

a. Assume $s_0(t)$ transmitted

$$Z = \int_0^T r(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\,dt$$

$$= \int_0^T (s_0(t) + n(t))\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\,dt$$

$$= \underbrace{\int_0^T s_0(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\,dt}_{\hat{s}_0(T)\text{: output due to } s_0(t) \text{ alone}} + \underbrace{\int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\,dt}_{\eta\text{: Gaussian random variable}}$$

$$\hat{s}_0(T) = \int_0^T s_0(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\,dt$$

$$= \int_0^T A\cos(2\pi f_c t)\, P_T(t) \sqrt{\tfrac{2}{T}} \cos(2\pi f_c t)\, dt$$

$$= \sqrt{\tfrac{2}{T}} \int_0^T A\cos(2\pi f_c t)\cos(2\pi f_c t)\, dt$$

ignore double frequency term

$$= \sqrt{\tfrac{2}{T}} \int_0^T A\left[\tfrac{1}{2} + \cancel{\tfrac{1}{2}\cos(2\pi(2f_c)t)}\right] dt$$

$$= \frac{1}{2}\sqrt{\tfrac{2}{T}} \int_0^T A\, dt = \frac{1}{2}\sqrt{\tfrac{2}{T}}\, At \Big|_0^T = \frac{AT}{2}\sqrt{\tfrac{2}{T}} = \sqrt{\tfrac{A^2 T}{2}}$$

$$\mathrm{Var}\{n\} = \frac{N_0}{2}\int_0^T \left(\sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)\right)^2 dt$$

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$n = \int n(t)\, x(t)\, dt$

$\rightarrow \mathrm{Var}\{n\} = \frac{N_0}{2}\int x^2(t)\, dt$

(Assuming PSD $= N_0/2$)

$$= \frac{N_0}{2}\int_0^T \frac{2}{T}\cos^2(2\pi f_c t)\, dt$$

ignore double frequency terms

$$= \frac{N_0}{2}\,\frac{2}{T}\int_0^T \frac{1}{2} + \cancel{\tfrac{1}{2}\cos(2\pi(2f_c)t)}\, dt$$

$$= \frac{N_0}{T}\,\frac{t}{2}\Big|_0^T = \frac{N_0}{T}\,\frac{T}{2} = \frac{N_0}{2}$$

$$P_{e,0} = P\{\hat{s}_0(T) + n < \gamma\} = P\left\{n < \gamma - \sqrt{\tfrac{A^2T}{2}}\right\}$$

$$= \Phi\left(\frac{\gamma - \sqrt{\frac{A^2T}{2}}}{\sigma_N}\right) \quad \text{where } \sigma_N = \sqrt{\mathrm{Var}\{n\}} = \sqrt{\tfrac{N_0}{2}}$$

Assume $s_2(t)$ sent:

$$\hat{s}_2(T) = \int_0^T s_2(t)\sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)\, dt$$

Output due to $s_2(t)$ alone

$$= \int_0^T -A\cos(2\pi f_c t)\, P_T(t)\sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)\, dt$$

$$= \sqrt{\frac{2}{T}} \int_0^T -A \cos(2\pi f_c t) \cos(2\pi f_c t)\, dt$$

$$= \sqrt{\frac{2}{T}} \int_0^T -A \left[\frac{1}{2} + \cancel{\frac{1}{2} \cos(2\pi(2f_c)t)} \right] dt$$

$$= \sqrt{\frac{2}{T}} \int_0^T \frac{-A}{2}\, dt = \sqrt{\frac{2}{T}} \frac{-AT}{2} = -\sqrt{\frac{2}{T}} \frac{AT}{2} = -\sqrt{\frac{A^2T}{2}}$$

$$P_{e,2} = P\left\{ \hat{s}_2(T) + n > -\gamma \right\} = P\left\{ -\sqrt{\frac{A^2T}{2}} + n > -\gamma \right\}$$

$$= P\left\{ n > \sqrt{\frac{A^2T}{2}} - \gamma \right\} = Q\left(\frac{\sqrt{\frac{A^2T}{2}} - \gamma}{\sigma_N} \right) \quad \sigma_N = \sqrt{\frac{N_0}{2}}$$

Assume $s_1(t)$ sent:

$$\hat{s}_1(T) = 0$$

$$P_{e,1} = P\left\{ \hat{s}_1(T) + n < -\gamma \text{ or } \hat{s}_1(T) + n > \gamma \right\}$$

Disjoint events

$$= P\left\{ \hat{s}_1(T) + n < -\gamma \right\} + P\left\{ \hat{s}_1(T) + n > \gamma \right\}$$

$$= P\left\{ n < -\gamma - \hat{s}_1(T) \right\} + P\left\{ n > \gamma - \hat{s}_1(T) \right\}$$

$$= P\left\{ n < -\gamma \right\} + P\left\{ n > \gamma \right\}$$

$$= \Phi\left(\frac{-\gamma}{\sigma_N} \right) + Q\left(\frac{\gamma}{\sigma_N} \right) = Q\left(\frac{\gamma}{\sigma_N} \right) + Q\left(\frac{\gamma}{\sigma_N} \right)$$

$$= 2Q\left(\frac{\gamma}{\sigma_N} \right)$$

$$\boxed{\begin{aligned} P_{e,0} &= Q\left(\frac{\sqrt{\frac{A^2T}{2}} - \gamma}{\sigma_N}\right) \\ P_{e,2} &= Q\left(\frac{\sqrt{\frac{A^2T}{2}} - \gamma}{\sigma_N}\right) \qquad \sigma_N = \sqrt{\frac{N_0}{2}} \\ P_{e,1} &= 2Q\left(\frac{\gamma}{\sigma_N}\right) \end{aligned}}$$

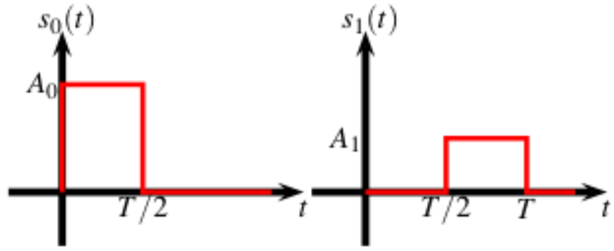
b. All 3 signals equally likely to be transmitted

$$\boxed{\begin{aligned} &\overline{P_e} = \tfrac{1}{3} P_{e,0} + \tfrac{1}{3} P_{e,1} + \tfrac{1}{3} P_{e,2} \\ &P_{e,0},\ P_{e,1} \text{ and } P_{e,2} \text{ defined in part (a)} \end{aligned}}$$

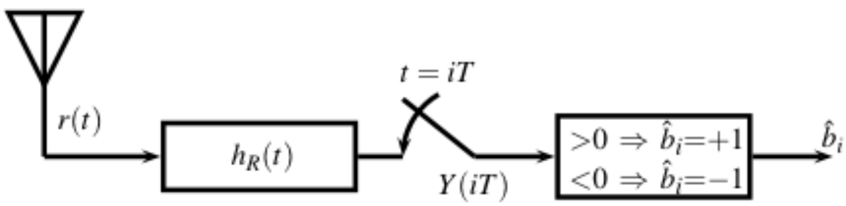
4. Consider a binary communication system that transmits one of two signal $s_0(t)$ and $s_1(t)$ over an additive white Gaussian noise channel (power spectral density $N_0/2$) where

$$s_0(t) = A_0 p_{T/2}(t), \quad s_1(t) = A_1 p_{T/2}(t - T/2);$$

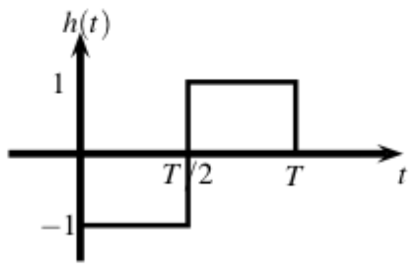
that is $s_0(t)$ is a pulse of amplitude A_0 from 0 to $T/2$ and $s_1(t)$ is a pulse of amplitude A_1 from $T/2$ to T as shown below.



The received signal, $r(t)$, is the transmitted signal with additive white Gaussian noise. The receiver shown below consist of a filter $h(t)$ which is sampled at time T and a threshold device.



(a) If $h_R(t) = -p_{T/2}(t) + p_{T/2}(t - T/2)$ shown below, find the output of the filter $\hat{s}_0(T)$ due to signal $s_0(t)$ at time T and the output of the filter $\hat{s}_1(T)$ due to signal $s_1(t)$ at time T



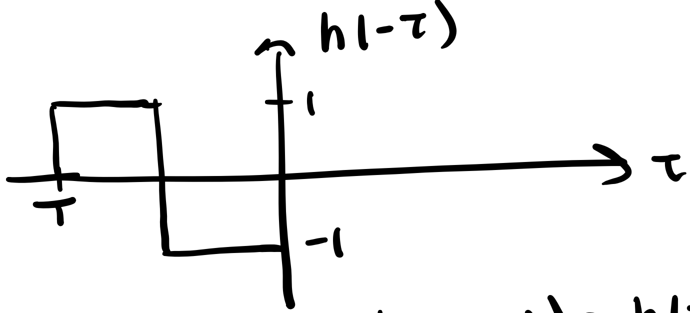
(b) Find the threshold γ that will minimize the average of the error probabilities $P_{e,0}$ and $P_{e,1}$ for the given signals and filter. Assume $\pi_0 = \pi_1$.

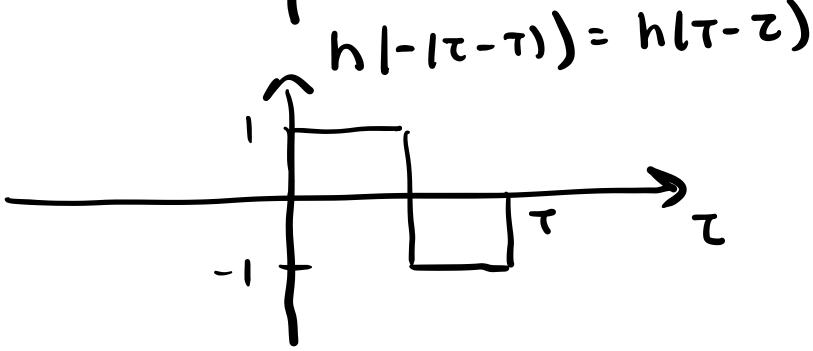
(c) Find the error average error probability $\bar{P}_e = \pi_0 P_{e,0} + \pi_1 P_{e,1}$ for the threshold found in the previous part. Assume $\pi_0 = \pi_1$.

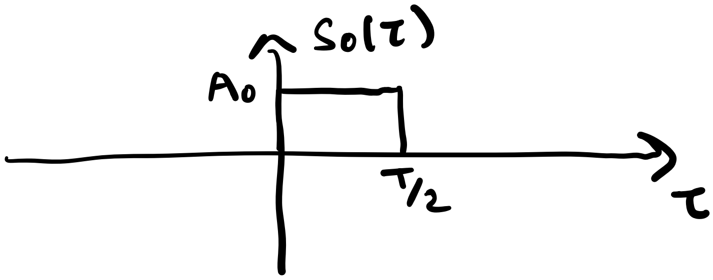
(d) Find the matched filter for the same signals and find the corresponding threshold that minimizes $\bar{P}_e$. Assume $\pi_0 = \pi_1$.

(e) Find $\bar{P}_e$ for the matched filter with the optimum threshold. Assume $\pi_0 = \pi_1$.

a. $\hat{s}_0(T) = \int_{-\infty}^{\infty} h(T-\tau)\, s_0(\tau)\, d\tau$



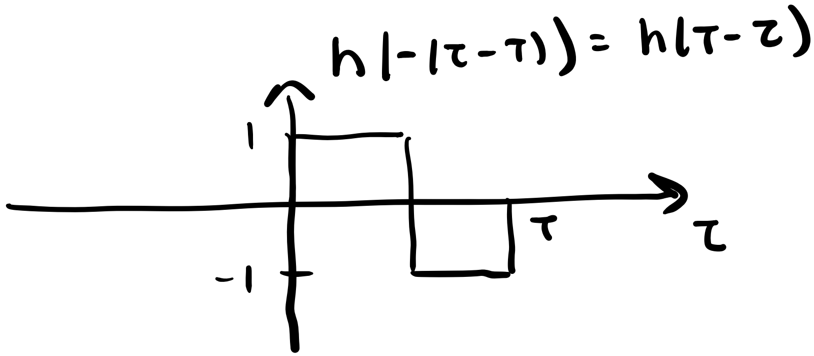


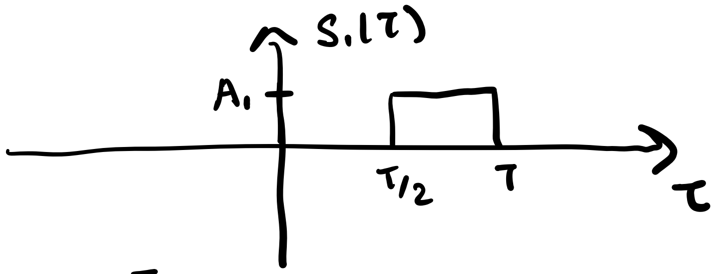


$$\hat{s}_0(T) = \int_{0}^{T/2} h(T-\tau)\, s_0(\tau)\, d\tau$$

$$= \int_{0}^{T/2} 1(A_0)\, d\tau = A_0\tau \Big|_{0}^{T/2} = \frac{A_0 T}{2}$$

$$\hat{s}_1(T) = \int_{-\infty}^{\infty} h(T-\tau)\, s_1(\tau)\, d\tau$$





$$\hat{s}_1(T) = \int_{T/2}^{T} h(T-\tau)\, s_1(\tau)\, d\tau$$

$$= \int_{T/2}^{T} -1\,(A_1)\, d\tau = -A_1\tau \Big|_{T/2}^{T} = \frac{-A_1 T}{2}$$

$$\boxed{\hat{s}_0(T) = \frac{A_0 T}{2} \qquad \hat{s}_1(T) = \frac{-A_1 T}{2}}$$

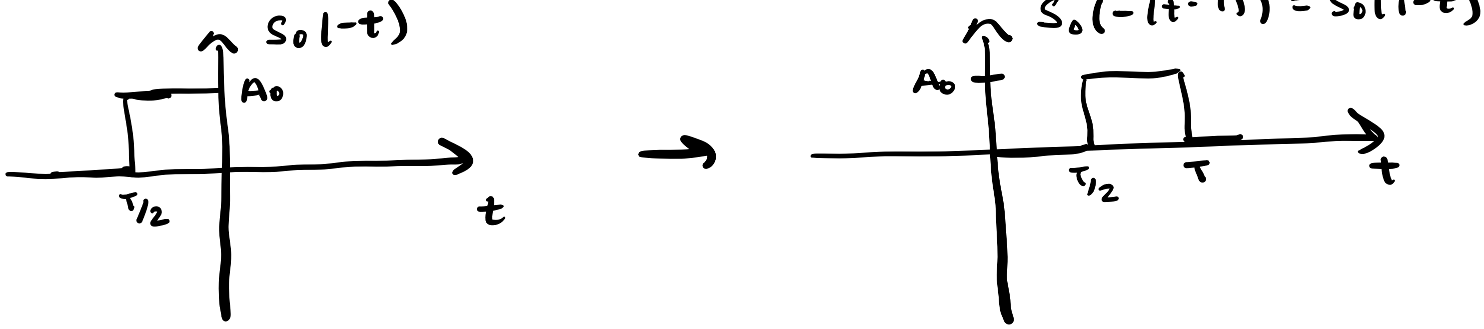
b. $\gamma_{opt} = \dfrac{\hat{s}_0(T) + \hat{s}_1(T)}{2}$ Lecture 07 Slide 21

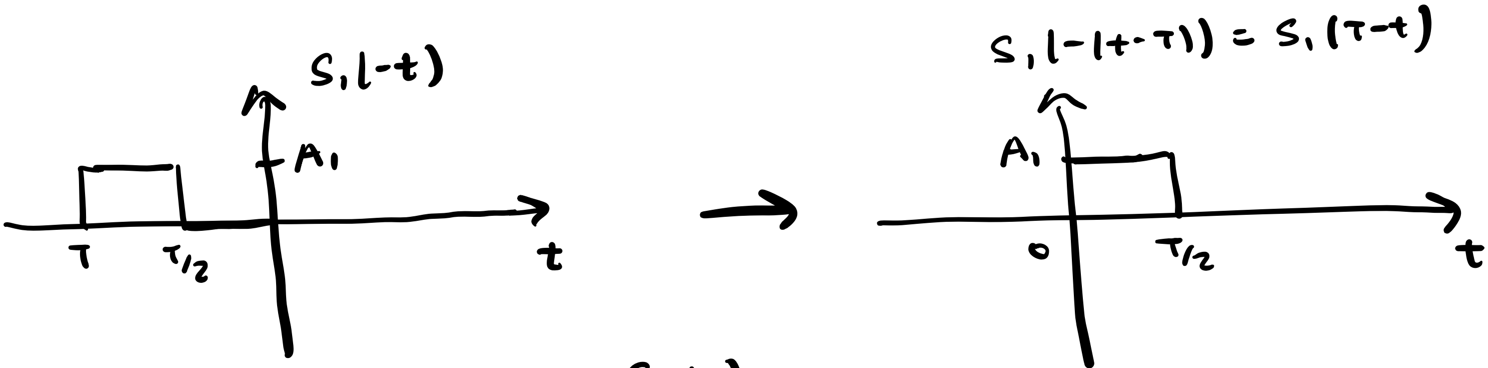
$$= \frac{\frac{A_0 T}{2} + \left(\frac{-A_1 T}{2}\right)}{2} = \frac{(A_0 - A_1)\,T}{4}$$

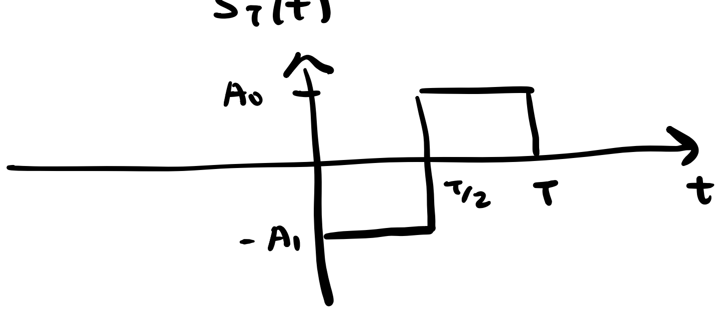
$$\boxed{\gamma_{opt} = \frac{(A_0 - A_1)\,T}{4}}$$

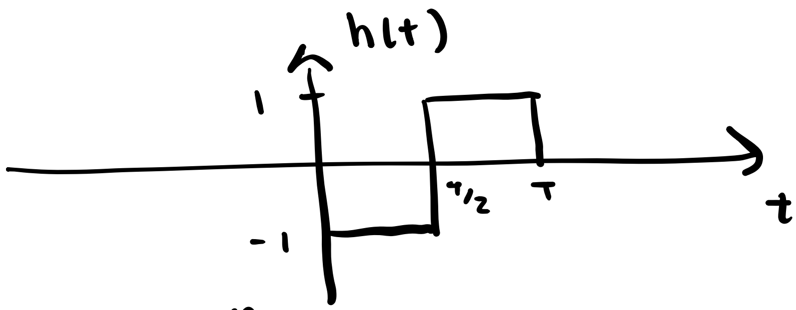
c. $\pi_0 = \pi_1 \rightarrow \bar{P}_e = Q(\alpha\lambda)$ Lecture 07 Slide 27

$$\lambda = \frac{(h(t), s_T(t))}{\|h(t)\|\ \|s_T(t)\|} \qquad s_T(t) = s_0(T-t) - s_1(T-t)$$









$$(h(t), S_T(t)) = \int_{-\infty}^{\infty} h(t)\, S_T(t)\, dt$$

$$= \int_{0}^{T} h(t)\, S_T(t)\, dt = \int_{0}^{T/2} (-1)(-A_1)\, dt + \int_{T/2}^{T} A_0\, dt$$

$$= A_1 t \Big|_{0}^{T/2} + A_0 t \Big|_{T/2}^{T} = \frac{A_1 T}{2} + \frac{A_0 T}{2}$$

$$= \frac{(A_0 + A_1) T}{2}$$

$$\|h(t)\| = \sqrt{\int_{-\infty}^{\infty} h^2(t)\,dt} = \sqrt{\int_0^T h^2(t)\,dt}$$

$$= \sqrt{\int_0^{T/2} (-1)^2\,dt + \int_{T/2}^{T} 1^2\,dt} = \sqrt{\int_0^T 1\,dt} = \sqrt{T}$$

$$\|S_T(t)\| = \sqrt{\int_{-\infty}^{\infty} S_T^2(t)\,dt} = \sqrt{\int_0^T S_T^2(t)\,dt}$$

$$= \sqrt{\int_0^{T/2} (-A_1)^2\,dt + \int_{T/2}^{T} A_0^2\,dt} = \sqrt{\frac{A_1^2 T}{2} + \frac{A_0^2 T}{2}}$$

$$= \sqrt{\frac{(A_0^2 + A_1^2)T}{2}}$$

$$\lambda = \frac{\frac{(A_0 + A_1)T}{2}}{\sqrt{T}\sqrt{\frac{(A_0^2 + A_1^2)T}{2}}}$$

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}} \qquad r = \frac{(S_0(t), S_1(t))}{\bar{E}}$$

$(S_0(t), S_1(t)) = 0$ S_0, S_1 pulses don't overlap

$$E_0 = \int_{-\infty}^{\infty} S_0^2(t)\,dt = \int_0^{T/2} S_0^2(t)\,dt = \int_0^{T/2} A_0^2\,dt = \frac{A_0^2 T}{2}$$

$$E_1 = \int_{-\infty}^{\infty} S_1^2(t)\,dt = \int_{T/2}^{T} S_1^2(t)\,dt = \int_{T/2}^{T} A_1^2\,dt = \frac{A_1^2 T}{2}$$

$$\bar{E} = \frac{E_0 + E_1}{2} = \frac{\frac{(A_0^2 T)}{2} + \frac{(A_1^2 T)}{2}}{2} = \frac{(A_0^2 + A_1^2)T}{4}$$

$$\alpha = \sqrt{\frac{\bar{E}}{N_o}} = \sqrt{\frac{(A_0^2 + A_1^2)T}{4N_o}}$$

$$\boxed{\bar{P}_e = Q(\alpha\lambda) \qquad \alpha = \sqrt{\frac{(A_0^2+A_1^2)T}{4N_o}} \qquad \lambda = \frac{\frac{(A_0+A_1)T}{2}}{\sqrt{T}\sqrt{\frac{(A_0^2+A_1^2)T}{2}}}}$$

d. $\pi_0 = \pi_1 \rightarrow \quad h_{opt} = s_0(T-t) - s_1(T-t)$ Lecture 07 Slide 32

$$\gamma_{opt} = \frac{1}{2}(E_0 - E_1) + \cancel{\frac{1}{2}\ln\frac{\pi_0}{\pi_1}} \quad \pi_0 = \pi_1$$

$$= \frac{1}{2}\left(\frac{A_0^2 T}{2} - \frac{A_1^2 T}{2}\right) = \frac{(A_0^2 - A_1^2)T}{4}$$

$$\boxed{h_{opt} = s_0(T-t) - s_1(T-t) \qquad \gamma_{opt} = \frac{(A_0^2 - A_1^2)T}{4}}$$

e. $\pi_0 = \pi_1$, using $h_{opt}(t)$ and corresponding γ_{opt}

$\bar{P}_e = Q(\alpha)$ Lecture 07 slide 33

$\alpha = \sqrt{\frac{T(A_0^2+A_1^2)}{4N_o}}$ from part (c)

$$\boxed{\bar{P}_e = Q\left(\sqrt{\frac{T(A_0^2+A_1^2)}{4N_o}}\right)}$$