University of Michigan FALL 2020

EECS501: Solutions to Practice Problems for Mid-term Exam 1

1. Rooks

There are two ways based on counting to approach the problem, depending on whether we view the rooks as ordered/distinguishable or unordered/indistinguishable. Either approach is fine, since the outcomes are equally likely in either case and we can using counting principles, but it is essential to be consistent in how one counts.

First, let's consider the unordered approach. We are choosing 8 squares without replacement from 64 possible, so there are $\binom{64}{8}$ ways to place the rooks. For the rooks to be safe, we can count as follows. Every safe arrangement has one rook per column, so each safe arrangement is defined by which row is associated with which column. For example, we could use the notation $(2, 5, 6, 3, 1, 7, 4, 8)$ to denote the outcome where the rook in the first column is on the 2nd row, the rook in the 2nd column is in the 5th row, etc. Then the distinct permutations of the numbers 1 through 8 correspond to the unique outcomes with safe rooks, when order is not considered. There are 8! such permutations, so the probability of a safe arrangement is

$$\frac{8!}{\binom{64}{8}}.$$

If we view the outcomes as ordered, then we can use the multiplication principle. The number of possible outcomes of the experiment is 64!/56!, using the formula for counting ordered outcomes obtained by drawing 8 squares from 64 without replacement. To count the number of safe outcomes, there are 64 ways to place the first rook. Once this rook is placed, 15 squares are eliminated, leaving 49 safe squares for the second rook. Once the first and second rooks are safely placed, there are 36 = 49-13 safe squares for the third rook, and so on. By the multiplication principle, there are

$$64 \cdot 49 \cdot 36 \cdots 4 \cdot 1 = (8!)^2$$

safe, ordered outcomes, so the probability of a safe outcome is

$$\frac{(8!)^2}{64!/56!}.$$

A third approach is to use conditional probabilities. Thus assume the rooks are placed one at a time, and let A_k be the event that the kth rook is safe. Then

$$\begin{aligned} P(\cap_{k=1}^{8} A_k) &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_8|A_1 \cap \cdots \cap A_7) \\ &= 1 \cdot \frac{49}{63} \cdot \frac{36}{62} \cdots \frac{1}{57}, \end{aligned}$$

where we have used simple counting principles to fill in the conditional probabilities.

All three solutions give the same numerical answer.

2. Badminton

Let B denote the event that Ann wins the next point. Let A_1 denote the event that Ann wins the current serve. Let A_2 denote the event that Ann loses the current serve and Bob wins the next serve. Let A_3 denote the event that Ann loses the current serve and Ann wins the next serve. Note that $P(B|A_1) = 1$, $P(B|A_2) = 0$. Further, $P(A_1) = p$ and $P(A_2) = (1-p)^2$. Moreover, $P(A_3) = (1-p)p$, and $P(B|A_3) = P(B)$. Hence we can solve for $P(B)$ as follows:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) = p + 0 + p(1-p)P(B).$$

We get

$$P(B) = \frac{p}{1 - p(1-p)}.$$

3. Repetition Code over Asymmetric Channel

(a) Let D denote the event that the true pixel color is B. Let E denote the event that the received block is $WBBWB$. We want to find $P(D|E)$. We know that $P(D) = P(D^c) = 0.5$. Let us calculate $P(D|E)$. Since the pixels are corrupted independently, we have

$$P(E|D) = (0.3)^2(0.7)^3 = 0.03087.$$

Similarly we have

$$P(E|D^c) = (0.1)^3(0.9)^2 = 0.00081.$$

Since D and D^c induce a partition of the sample space, using Bayes rule, we have

$$P(D|E) = \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} = 0.9744.$$

(b) Given that Black pixels are 5 times as prevalent as the white pixels, we have $P(D) = 5/6$ and $P(D^c) = 1/6$. Using Bayes rule, we have

$$P(D|E) = \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} = 0.9948.$$

4. State TRUE or FALSE by giving reasons

(a) FALSE. Using the law of the complements, DeMorgan's laws and the inclusion-exclusion principle, we have

$$P(A^c \cap B^c) = P[(A \cup B)^c] = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - \frac{8}{5} + \frac{16}{25} = \frac{25 - 40 + 16}{25} = \frac{1}{25}.$$

(b) FALSE. Let A, B and C be the events that 4, 5 and 6 oranges are picked, respectively. Let E be the event of interest. Then

$$P(E) = P(A) + P(B) + P(C) = \frac{\binom{10}{4}\binom{8}{2} + \binom{10}{5}\binom{8}{1} + \binom{10}{6}}{\binom{18}{6}}.$$

(c) TRUE. Consider the event $\{X = k\}$. In this event, at the kth toss, we have a Heads. The other Heads can take any of the $(k-1)$ positions. The rest are filled with Tails.

(d) TRUE. Let A_3 denote the event that component 3 works, and E be the event that the system works. We have using Bayes' rule

$$P(A_3|E) = \frac{P(E|A_3)P(A_3)}{P(E)} = \frac{p(2p - p^2)}{p + p^2 - p^3}.$$

(e) FALSE. Note that $\mathcal{F} = \{\phi, \Omega, A, A^c\}$. We cannot compute $P(X = 1) = P\{w : w \text{ is even}\}$ because the event $\{2, 4, 6\}$ is not in $\mathcal{F}$.

5. State TRUE or FALSE by giving reasons

(a) TRUE. $\mathcal{F}$ is closed under union and complementation. Hence it is a sigma-algebra. $\mathcal{F}$ contains both A and B. In the Venn diagram for A, B and Ω, we see that there are three atoms, and hence the smallest sigma algebra that contains both A and B must be of size 8.

(b) FALSE. Using the Law of complements and the inclusion-exclusion principle, we have $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) = 1 - \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$.

(c) FALSE. X is the sum of 5 independent geometric random variables with parameter 6. Hence, $E(X) = 30$.

(d) FALSE. Let A_i denote the event that ith component works. Let S denote the event that the system works. We have

$$P(A_1|S) = \frac{P(S|A_1)P(A_1)}{P(S)} = \frac{p}{p + p^2 - p^3}.$$

(e) FALSE. P satisfies the first two axioms of probability. We show that it does not satisfy the third axiom. Consider two disjoint events: $A_1 = \{\frac{1}{\pi}\}$ and $A_2 = \{\frac{1}{e}\}$. Let $A = A_1 \cup A_2$. Note that $P(A_1) = P(A_2) = 0$. But A contains both $\frac{1}{\pi}$ and $\frac{1}{e}$. Hence $P(A) = 1 \neq P(A_1) + P(A_2)$.

6. Dice Game

Let A_1 denote the event that you get 3 on first roll. Let A_2 denote the event that you get 1 on first roll. Let A_3 denote the event that you get $2, 4, 5, 6$ on first roll. Let B_1 denote the event that you get 3 on second roll. Let B_2 denote the event that you get 1 on second roll. Let B_3 denote the event that you get $2, 4, 5, 6$ on second roll.

Using the Law of total probability, we have

$$\begin{aligned} E(X) &= E(X|A_1)P(A_1) + E(X|A_2)P(A_2) + E(X|A_3)P(A_3) && (1) \\ &= \frac{1}{6}E(X|A_1) + \frac{1}{6}E(X|A_2) + \frac{2}{3}E(X+1) && (2) \end{aligned}$$

Using the Law of total probability, we have

$$\begin{aligned} E(X|A_1) &= E(X|A_1B_1)P(B_1) + E(X|A_1B_2)P(B_2) + E(X|A_1B_3)P(B_3) && (3) \\ &= \frac{1}{6}2 + \frac{1}{6}E(X+1|A_2) + \frac{2}{3}E(X+2) && (4) \end{aligned}$$

Using the Law of total probability, we have

$$\begin{aligned} E(X|A_2) &= E(X|A_2B_1)P(B_1) + E(X|A_2B_2)P(B_2) + E(X|A_2B_3)P(B_3) && (5) \\ &= \frac{1}{6}E(X+1|A_1) + \frac{1}{6}2 + \frac{2}{3}E(X+2) && (6) \end{aligned}$$

By symmetry we must have $E(X|A_1) = E(X|A_2)$. The above equations can be simplified as $E(X) = E(X|A_1) + 2$ and $5E(X|A_1) = 4E(X) + 11$. Solving we get, $E(X) = 21$.

7. Chess Problem

Let A_i denote the event that ith rank has exactly 3 black and 3 white pawns. Note that

$$P(A_i) = \frac{|A_i|}{|\Omega|}$$

where

$$|A_i| = \binom{8}{3}\binom{8}{3}\frac{8!}{(8-6)!}\frac{56!}{(56-10)!}$$

and

$$|\Omega| = \frac{64!}{(64-16)!}$$

The argument is: we need to first choose 3 (without order) out of 8 black pawns, then choose 3 (without order) out of 8 white pawns, and then choose 6 spaces on the ith row to place them (with order). Further, we need to place the rest 10 pieces on the rest of 56 spaces.

Next,

$$P(A_i \cap A_j) = \frac{|A_i \cap A_j|}{|\Omega|},$$

where

$$|A_i \cap A_j| = \binom{8}{3}\binom{8}{3}\binom{5}{3}\binom{5}{3}\frac{8!}{(8-6)!}\frac{8!}{(8-6)!}\frac{48!}{(48-4)!}.$$

The argument is similar to one used above: we need to first choose 3 (without order) out of 8 black pawns, then choose 3 (without order) out of 8 white pawns, and then choose 6 spaces on the ith row to place them (with order). Similarly for jth row. Further, we need to place the rest 4 pieces on the rest of 48 spaces. Let E denote the event of interest. Then using inclusion-exclusion principle, we have

$$P(E) = 8P(A_1) - \binom{8}{2}P(A_1 \cap A_2).$$

8. Dice Game

1. Let A_i, B_j denote the event that the first and second die outcomes are i, j, respectively. Then by the Law of Total Probability, for any $k > 2$,

$$P(X = k) = \frac{1}{6}P(X = k|A_6) + \frac{5}{6}P(X = k|A_{1,2,3,4,5}). \quad (7)$$

We have,

$$P(X = k|A_{1,2,3,4,5}) = P(X = k-1), \quad (8)$$

$$P(X = k|A_6) = \frac{1}{6}P(X = k|A_6, B_6) + \frac{5}{6}P(X = k|A_6, B_{1,2,3,4,5}). \quad (9)$$

Then,

$$P(X = k|A_6, B_6) = 0, \quad (10)$$

$$P(X = k|A_6, B_{1,2,3,4,5}) = P(X = k-1|A_6). \quad (11)$$

Combining all this gives, for all $k > 2$,

$$P(X = k) = \frac{1}{6} P(X = k|A_6) + \frac{5}{6} P(X = k-1), \tag{12}$$

$$P(X = k|A_6) = \frac{5}{6} P(X = k-1|A_6). \tag{13}$$

Initial conditions are,

$$P(X = 1) = P(X = 1|A_6) = 0, \qquad P(X = 2) = \frac{1}{36}, \qquad P(X = 2|A_6) = \frac{1}{6}. \tag{14}$$

2. Let A_i, B_j denote the event that the first and second outcomes are i, j, respectively. Then by the Law of Total Expectation,

$$\mathbb{E}[X] = \frac{1}{6} \sum_{i=1}^{6} \mathbb{E}[X|A_i]. \tag{15}$$

Now for any $i \in \{1, 2, 3, 4, 5, 6\}$, we have

$$\mathbb{E}[X|A_i] = \frac{1}{6} \sum_{j=1}^{6} \mathbb{E}[X|A_i B_j]. \tag{16}$$

Now for $(i, j) \in A \triangleq \{(1, 1), (1, 2), (2, 1), (5, 5), (5, 6), (6, 5)\}$, we have $i + j$ either smaller than 4 or bigger than 10, hence

$$\mathbb{E}[X|A_i B_j] = \begin{cases} 2 & \text{if } (i, j) \in A, \\ 1 + \mathbb{E}[X|A_j] & \text{if } (i, j) \notin A. \end{cases} \tag{17}$$

There are 43 equations in (15), (16) and (17), they involve exactly 43 variables:

$$\mathbb{E}[X] \quad \text{and} \quad \mathbb{E}[X|A_i] \text{ for } 1 \leq i \leq 6 \quad \text{and} \quad \mathbb{E}[X|A_i B_j] \text{ for } 1 \leq i, j \leq 6. \tag{18}$$

Hence this linear system can be solved.

9. Differential Entropy

$$h(X)=\int_0^2 (1/2)\log 2\,dx=\log 2 \quad \text{and} \quad h(X)=\int_0^{1/2} 2\log(1/2)\,dx=\log(1/2).$$

For the third calculation, note that $-\ln f(x)=\frac{1}{2}[(x-m)/\sigma]^2+\frac{1}{2}\ln 2\pi\sigma^2$. Then

$$\begin{aligned}
h(X) &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\Big([(x-m)/\sigma]^2+\ln 2\pi\sigma^2\Big)dx \\
&= \frac{1}{2}\left\{\mathsf{E}\left[\left(\frac{X-m}{\sigma}\right)^2\right]+\ln 2\pi\sigma^2\right\} \;=\; \tfrac{1}{2}\{1+\ln 2\pi\sigma^2\} \;=\; \tfrac{1}{2}\ln 2\pi\sigma^2 e.
\end{aligned}$$

10. RLC circuit

(a) $R \sim \text{uniform}[0,\sqrt{2}]$ and $Y=\sqrt{1-R^2/2}$. For $0 \le y \le 1$,

$$\begin{aligned}
F_Y(y) &= P(Y \le y) = P(\sqrt{1-R^2/2} \le y) \\
&= P(1-R^2/2 \le y^2) = P(2-2y^2 \le R^2) \\
&= P(\sqrt{2-2y^2} \le R) = 1-\frac{\sqrt{2-2y^2}}{\sqrt{2}} \\
&= 1-\sqrt{1-y^2}
\end{aligned}$$

We then differentiate to get the pdf of Y

$$f_Y(y)=\begin{cases} y/\sqrt{1-y^2}, & \text{for } 0 \le y \le 1 \\ 0, & \text{otherwise.}\end{cases}$$

(b) When $L=C=1$, $\omega_0=\sqrt{1-\frac{1}{2}R^2}$. Hence

$$\begin{aligned}
Z &\triangleq |H(\omega_0)|^2 \\
&= \frac{1}{(1-\omega_0^2)^2+(R\omega_0)^2} \\
&= \frac{1}{(1-(1-\frac{1}{2}R^2))^2+R^2(1-\frac{1}{2}R^2)} \\
&= \frac{1}{\frac{1}{4}R^4+R^2-\frac{1}{2}R^4} \\
&= \frac{1}{R^2-\frac{1}{4}R^4} \\
&= \frac{1}{R^2(1-R^2/4)}.
\end{aligned}$$

The first thing to note is that for $0 \le R \le \sqrt{2}$, $0 \le R^2 \le 2$. It is then easy to see that the minimum value of $Z = [R^2(1-R^2/4)]^{-1}$ occurs when $R^2 = 2$ or $R = \sqrt{2}$. Hence, the random variable Z takes values in the range $[1,\infty)$. So, for $z \ge 1$, we write

$$\begin{aligned} F_Z(z) &= \mathsf{P}\left(\frac{1}{R^2(1-R^2/4)} \le z\right) = \mathsf{P}\left(R^2(1-R^2/4) \ge 1/z\right) \\ &= \mathsf{P}\left((R^2/4)(1-R^2/4) \ge 1/(4z)\right). \end{aligned}$$

Put $y := 1/(4z)$ and observe that $x(1-x) \ge y$ if and only if $0 \ge x^2 - x + y$, with equality if and only if

$$x = \frac{1 \pm \sqrt{1-4y}}{2}.$$

Since we will have $x = R^2/4 \le 1/2$, we need the negative root. Thus,

$$\begin{aligned} F_Z(z) &= \mathsf{P}\left((R^2/4) \ge \frac{1-\sqrt{1-4y}}{2}\right) = \mathsf{P}\left(R^2 \ge 2[1-\sqrt{1-1/z}]\right) \\ &= \mathsf{P}\left(R \ge \sqrt{2[1-\sqrt{1-1/z}]}\right) = \frac{\sqrt{2}-\sqrt{2[1-\sqrt{1-1/z}]}}{\sqrt{2}} \\ &= 1-\sqrt{1-\sqrt{1-1/z}}. \end{aligned}$$

Differentiating, we obtain

$$\begin{aligned} f_Z(z) &= -\frac{1}{2}\left(1-\sqrt{1-1/z}\right)^{-1/2}\frac{d}{dz}[1-\sqrt{1-1/z}] \\ &= -\frac{1}{2}\left(1-\sqrt{1-1/z}\right)^{-1/2} \cdot -\frac{1}{2}(1-1/z)^{-1/2} \cdot \frac{1}{z^2} \\ &= \frac{1}{4z^2}[(1-\sqrt{1-1/z})(1-1/z)]^{-1/2}. \end{aligned}$$

11. Coin Game

(a) We condition on the result of the first two outcomes. According to law of total expectations,

$$E(X) = E(X|HH)P(HH) + E(X|HT)P(HT) + E(X|TH)P(TH) + E(X|TT)P(TT)$$

and we have

$$E(X|HT) = E(X|HTT)P(T) + E(X|HTH)P(H) = 3 \times \frac{1}{2} + \frac{1}{2}(1 + E(X|TH))$$
$$E(X|TH) = E(X|THT)P(T) + E(X|THH)P(H) = \frac{1}{2}(1 + E(X|HT)) + 3 \times \frac{1}{2}$$
$$E(X|HH) = E(X|HHT)P(T) + E(X|HHH)P(H) = \frac{1}{2}(1 + E(X|HT)) + \frac{1}{2}(1 + E(X|HH))$$
$$E(X|TT) = E(X|TTT)P(T) + E(X|TTH)P(H) = \frac{1}{2}(1 + E(X|TT)) + \frac{1}{2}(1 + E(X|TH))$$

By solving the above system of equations, we get

$$E(X|HT) = E(X|TH) = 4, \quad E(X|HH) = E(X|TT) = 6 \;\Rightarrow E(X) = 5$$

(b) Using LOTP,

$$P_X(k) = \frac{1}{4}P(X = k|HH) + \frac{1}{4}P(X = k|HT) + \frac{1}{4}P(X = k|TH) + \frac{1}{4}P(X = k|TT)$$

and for $k > 3$, we have

$$P(X = k|HT) = \frac{1}{2}P(X = k|HTT) + \frac{1}{2}P(X = k|HTH) = \frac{1}{2}P(X = k-1|TH)$$
$$P(X = k|TH) = \frac{1}{2}P(X = k|THT) + \frac{1}{2}P(X = k|THH) = \frac{1}{2}P(X = k-1|HT)$$
$$P(X = k|HH) = \frac{1}{2}P(X = k|HHT) + \frac{1}{2}P(X = k|HHH)$$
$$= \frac{1}{2}P(X = k-1|HT) + \frac{1}{2}P(X = k-1|HH)$$
$$P(X = k|TT) = \frac{1}{2}P(X = k|TTT) + \frac{1}{2}P(X = k|TTH)$$
$$= \frac{1}{2}P(X = k-1|TT) + \frac{1}{2}P(X = k-1|TH)$$

with boundary conditions being

$$P(X < 3) = 0$$
$$P(X = 3) = \frac{1}{4}$$
$$P(X < 3|HT) = P(X < 3|TH) = P(X < 3|HH) = P(X < 3|TT) = 0$$
$$P(X = 3|HT) = P(X = 3|TH) = \frac{1}{2}$$
$$P(X = 3|HH) = P(X = 3|TT) = 0$$