EECS 455: Solutions to Problem Set 10
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1. A (7,5) Reed Solomon code is used on a channel. The code operates on the finite field $GF(2^3)$ shown below using the polynominal $\alpha^3 = \alpha + 1$. The code has minimum distance 3 and can correct 1 error.

$-$	0	$(0,0,0)$
1	1	$(0,0,1)$
α	α	$(0,1,0)$
α^2	α^2	$(1,0,0)$
α^3	$\alpha+1$	$(0,1,1)$
α^4	$\alpha^2+\alpha$	$(1,1,0)$
α^5	$\alpha^2+\alpha+1$	$(1,1,1)$
α^6	α^2+1	$(1,0,1)$

The code has generator polynomial

$$\begin{aligned} g(x) &= (x-\alpha)(x-\alpha^2) \\ &= x^2 - (\alpha+\alpha^2)x + \alpha^3 \\ &= x^2 + \alpha^4 x + \alpha^3 \end{aligned}$$

A codeword is generated from an information polynomial $i(x)$ by

$$c(x) = g(x)i(x)$$

If $r(x) = x^6 + \alpha^4 x^5$ determine the codeword transmitted.

Solution: Let

$$r(x) = c(x) + e(x)$$

where $e(x)$ is an error vector. If we assume just one error then

$$e(x) = e_j x^j.$$

Since $c(x) = g(x)i(x)$ we have

$$\begin{aligned} r(x) &= g(x)i(x) + e(x) \\ r(\alpha) &= g(\alpha)i(\alpha) + e(\alpha) \\ &= 0 + e(\alpha) \\ &= e_j \alpha^j. \end{aligned}$$

Similarly

$$\begin{aligned} r(\alpha^2) &= g(\alpha^2)i(\alpha^2)+e(\alpha^2) \\ &= g(\alpha^2)i(\alpha^2)+e(\alpha^2) \\ &= 0+e_j\alpha^{2j} \\ &= e_j\alpha^{2j} \end{aligned}$$

Now notice that

$$\frac{r(\alpha^2)}{r(\alpha)} = \alpha^j$$

and

$$\frac{r^2(\alpha)}{r(\alpha^2)} = e_j$$

For the received signal above $r(\alpha) = 1$ and $r(\alpha^2) = \alpha^4$. Thus

$$\alpha^j = \frac{r(\alpha^2)}{r(\alpha)} = \alpha^4$$

So $j = 4$. That is the error occurred in the x^4 position of the received word. Similarly

$$\begin{aligned} e_j &= \frac{r^2(\alpha)}{r(\alpha^2)} \\ &= \frac{1}{\alpha^4} \\ &= \frac{\alpha^7}{\alpha^4} \\ &= \alpha^3 \end{aligned}$$

Thus the magnitude of the error is α^3. Thus

$$e(x) = \alpha^3x^4$$

so

$$\begin{aligned} \hat{c}(x) &= r(x)-e(x) \\ &= x^6+\alpha^4x^5-\alpha^3x^4 \\ &= x^6+\alpha^4x^5+\alpha^3x^4 \end{aligned}$$

2. A (7,5) Reed Solomon code is used on a channel. The code operates on the finite field $GF(2^3)$ shown below using the polynominal $\alpha^3 = \alpha + 1$. The code has minimum distance 3 and can correct 1 error.

$-$	0	$(0,0,0)$
1	1	$(0,0,1)$
α	α	$(0,1,0)$
α^2	α^2	$(1,0,0)$
α^3	$\alpha+1$	$(0,1,1)$
α^4	$\alpha^2+\alpha$	$(1,1,0)$
α^5	$\alpha^2+\alpha+1$	$(1,1,1)$
α^6	α^2+1	$(1,0,1)$

The code has generator polynomial

$$\begin{aligned} g(x) &= (x-\alpha)(x-\alpha^2) \\ &= x^2 - (\alpha+\alpha^2)x + \alpha^3 \\ &= x^2 + \alpha^4 x + \alpha^3 \end{aligned}$$

A codeword is generated from an information polynomial $i(x)$ by

$$c(x) = g(x)i(x)$$

The channel drops two symbols (erases them) but receives the vector $r(x)$. If $r(x) = ?x^4 + ?x^2 + \alpha^5 x + \alpha^3$ determine the codeword transmitted.

Solution: There are two unknows corresponding to the received signal in position 4 and 2. Let these be a and b. Then

$$r(x) = ax^4 + bx^2 + \alpha^5 x + \alpha^3.$$

Since $r(\alpha) = 0$ and $r(\alpha^2) = 0$ we get two equations in two unknows.

$$\begin{aligned} r(\alpha) &= a\alpha^4 + b\alpha^2 + \alpha^6 + \alpha^3 = 0 \\ r(\alpha^2) &= a\alpha^8 + b\alpha^4 + \alpha^7 + \alpha^3 = 0 \end{aligned}$$

Solving these two equations yields $a = 1$ and $b = 0$. Thus the transmitted codeword is

$$c(x) = x^4 + \alpha^5 x + \alpha^3.$$

3. Consider a Reed Solomon code with 24 information symbols and 28 coded symbols. So there are 4 redundant symbols. The distance of the code is $d = n - k + 1 = 28 - 24 + 1 = 5$. So the code can correct up to 2 errors. Each symbol is an eight bit byte. Each 8 bit symbol is transmitted over a binary symmetric channel with crossover probability p. Errors occur independently from one bit to the next. Assume that if more than 2 symbol errors occurs the

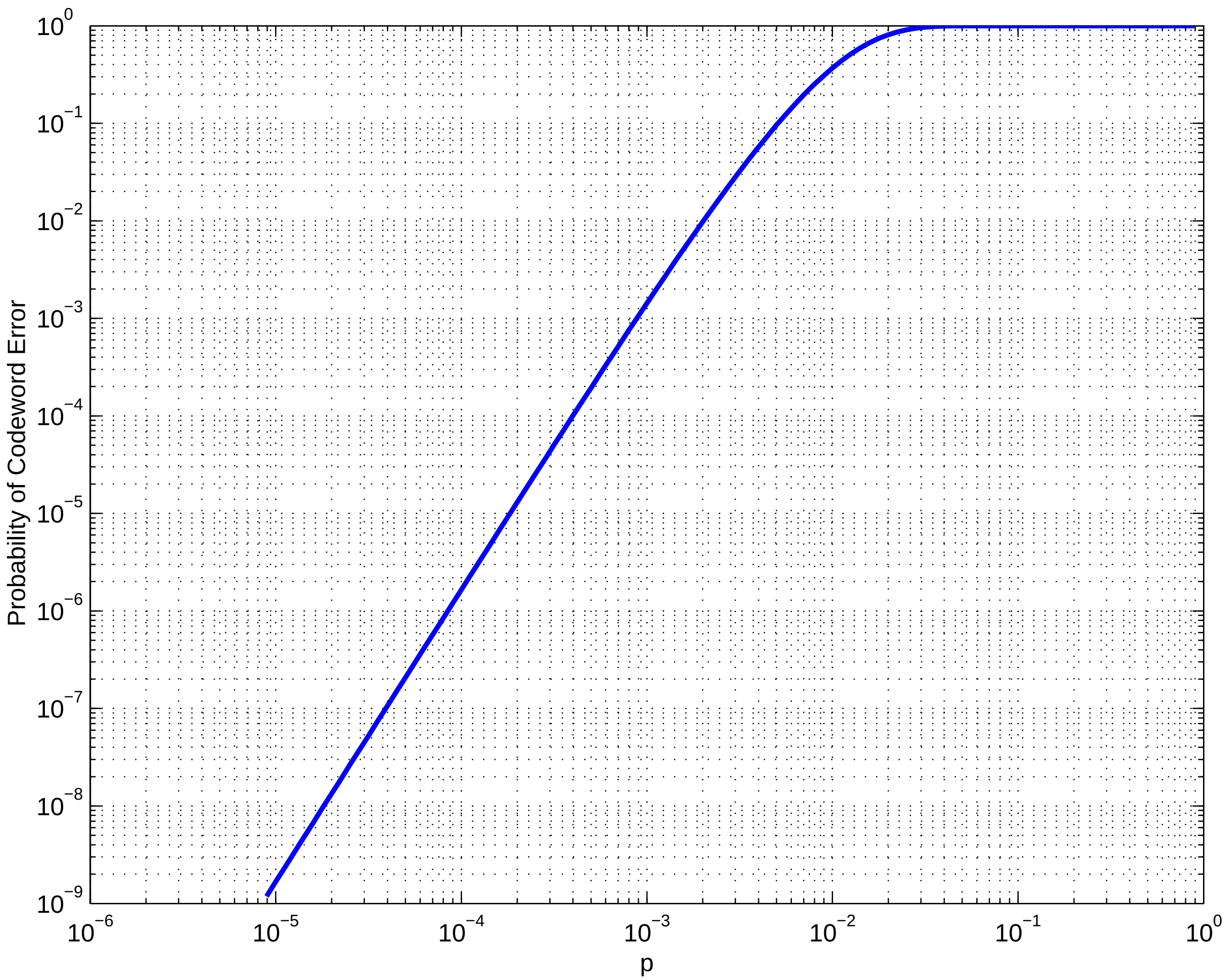
decder fails. Determine the probability that the decoder fails. Plot this on a log-log scale as a function of p for p=0.00001 to 1.

Solution: First the probability that an eight bit byte is received incorrectly, the symbol error probability, is

$$P_s = 1 - (1-p)^8.$$

The probability of failure is the probability that in 28 symbols two or more symbol errors are made. Since symbol errors occur independently the probability of two or more symbol errors is

$$P_s = \sum_{l=3}^{28} \binom{28}{l} P_s^l (1-P_s)^{28-l}$$




```
for m=1:101
p(m)=10^(-5*m/100);
```

```
n=28;
l=3;
ncl=14*9*26;
ps=1-(1-p(m))^8;
bin=ps^l*(1-ps)^(n-l);
pce(m)=ncl*bin;
for l=4:28
ncl=ncl*(n-l+1)/l;
bin=bin*ps/(1-ps);
pce(m)=pce(m)+ncl*bin;
end
end
loglog(p,pce,'LineWidth',2)
xlabel('p')
ylabel('Probability of Codeword Error')
grid on
```

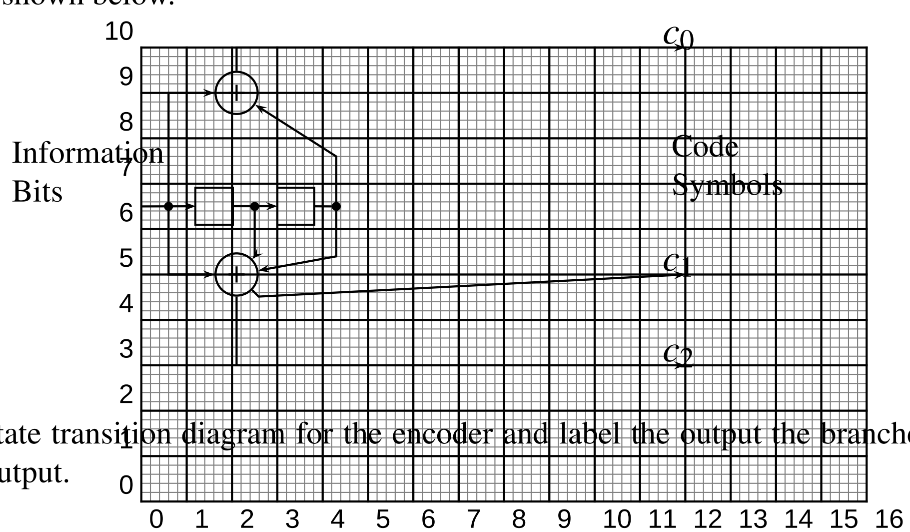
4. Consider the rate 1/3 convolutional code with memory 2 and generator polynomials

$$\begin{aligned} g_0 &= [101] \\ g_1 &= [111] \\ g_2 &= [111] \end{aligned}$$

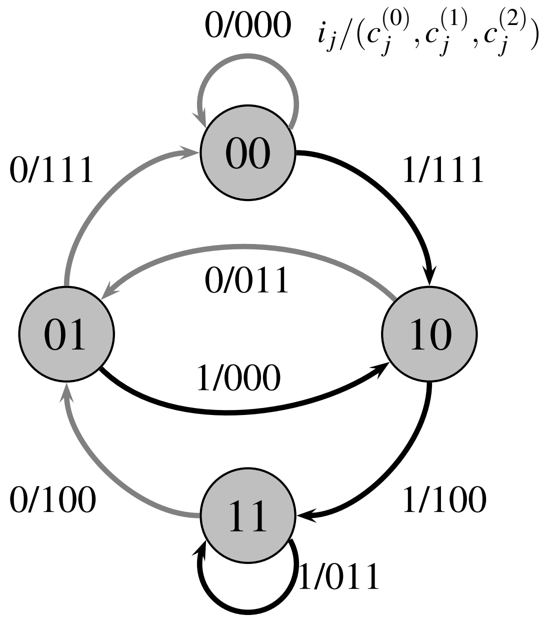
(a) Suppose the information bits are $u = [01101]$. Find the output of the encoder. The output should be a vector of length $3(5+2) = 21$.

Solution: The output codeword is (000 111 100 100 000 011 111)

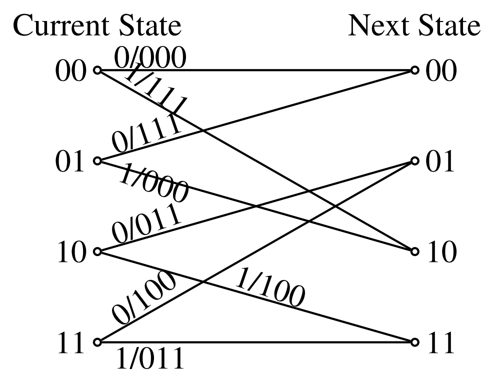
The encoder is shown below.



(b) Draw the state transition diagram for the encoder and label the output the branches by the input and output.



(c) Draw the trellis diagram from one state to the next labeling the transition with the input bit and the output bits.



(d) The transmitted signal is found by encoding the information, mapping the bits to levels ± 1 ($0 \to +1$, $1 \to -1$). Suppose the received signal is

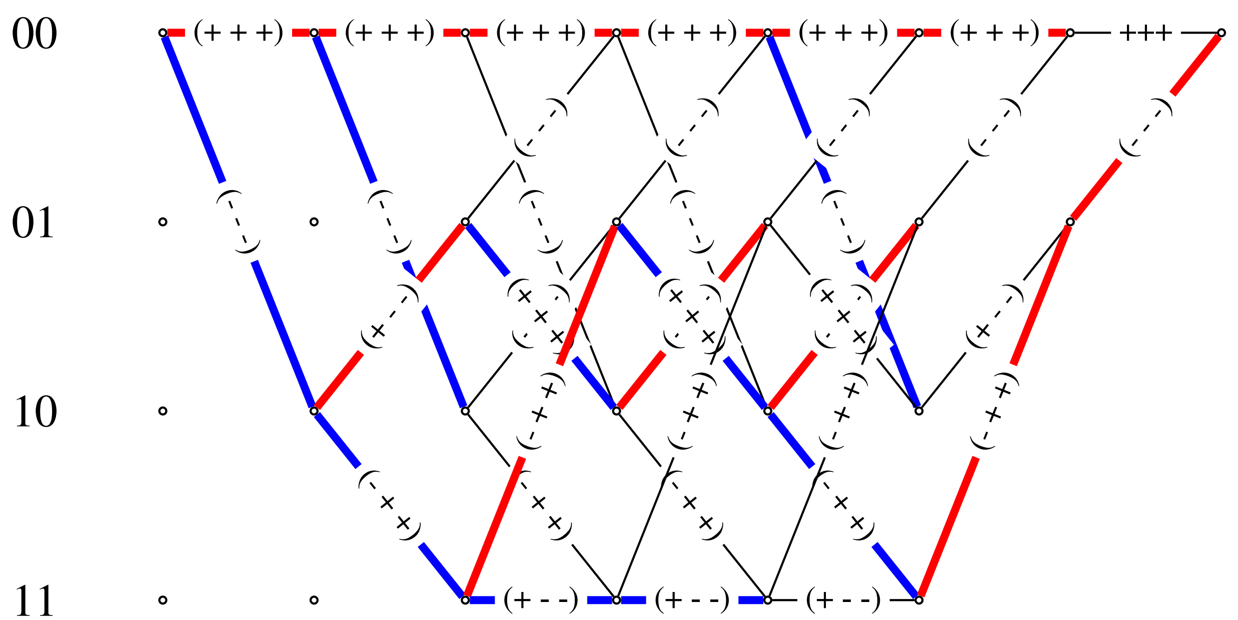
$$
\begin{aligned}
rcvd[0] &= -2.56\\
rcvd[1] &= 1.69\\
rcvd[2] &= -1.56\\
rcvd[3] &= 0.37\\
rcvd[4] &= -0.14\\
rcvd[5] &= 2.19\\
rcvd[6] &= 1.13\\
rcvd[7] &= 1.05\\
rcvd[8] &= 1.78
\end{aligned}
$$

$$
\begin{aligned}
rcvd[9] &= 4.51 \\
rcvd[10] &= 2.17 \\
rcvd[11] &= -0.41 \\
rcvd[12] &= -0.46 \\
rcvd[13] &= -0.92 \\
rcvd[14] &= 0.68 \\
rcvd[15] &= 0.93 \\
rcvd[16] &= 1.54 \\
rcvd[17] &= 2.34 \\
rcvd[18] &= 0.13 \\
rcvd[19] &= -1.66 \\
rcvd[20] &= -0.82
\end{aligned}
$$

Find the most likely information sequence at the input to the encoder.

Solution: The most likely information sequence is [11011];

-2.56	0.37	1.13	4.51	-0.46	0.93	0.13
1.69	-0.14	1.05	2.17	-0.92	1.54	-1.66
-1.56	2.19	1.78	-0.41	0.68	2.34	-0.82

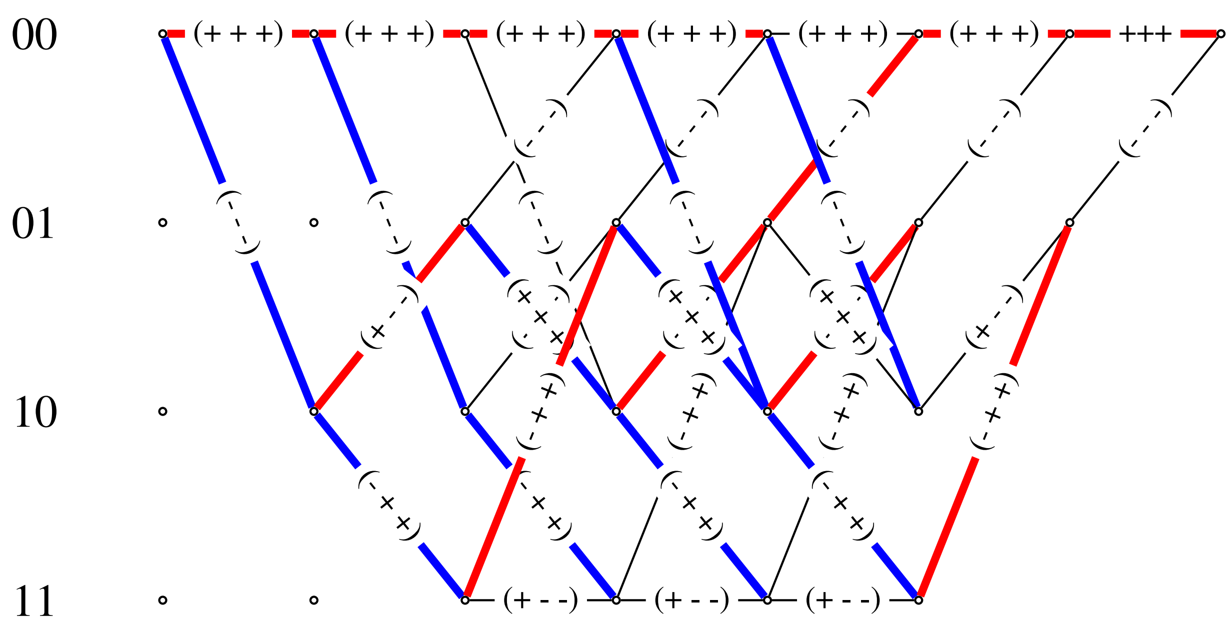


-2.43	-0.01	3.95	10.22	9.52	14.33	17.6
NA	0.75	5.81	7.46	11.86	15.25	
2.43	-4.85	4.71	12.08	10.92		
NA	4.11	2.41	5.16	12.30		

(e) If the demodulator made a hard decision on each bit find the most likely encoder input (for the same sequence received out of the demodulator except that a hard decision is made on each of the signals received whether it corresponds to a +1 or -1.

Solution: The most likely information sequence is [10100];

RCVD SIGNAL	-1	1	1	1	-1	1	1
	1	-1	1	1	-1	1	-1
	-1	1	1	-1	1	1	-1



METRICS	-1	0	3	4	7	10	9
	NA	2	1	6	5	4	
	1	-2	5	2	5		
	NA	0	-1	4	3		