Example: PageRank

Ranking the importance of webpages

- Idea 1: View in-links as votes
- Idea 2: Link from an important page counts more
- Idea 3: Each page's vote is evenly split among its out-links.
- Idea 4: Each link's vote is proportional to the importance of the page.

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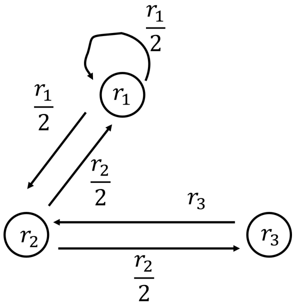
Consider a network with three webpages, where r_i is the importance score of webpage i and the direct edges are hyperlinks.

$$
\begin{aligned}
r_1 &= \frac{r_1}{2} + \frac{r_2}{2} \\
r_2 &= \frac{r_1}{2} + r_3 \\
r_3 &= \frac{r_2}{2}
\end{aligned}
$$



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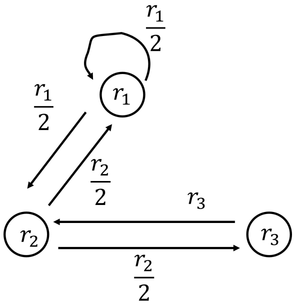
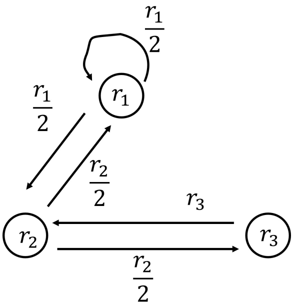


- There is no unique solution.
- Add an additional constraint $r_1 + r_2 + r_3 = 1$.
- Unique solution: $r_1 = \frac{2}{5}$, $r_2 = \frac{2}{5}$, and $r_3 = \frac{1}{5}$.

Example: PageRank

Markov chain perspective

- Imagine a random Web surfer.
- At any time t, surfer is on web i. At time $t+1$ the surfer picks an out-link uniformly at random and goes to the next web.
- $p_i(t)$: probability that the surfer is at web i at time t.



$$p_i(t+1) = \sum_{j:j\to i} \frac{1}{d_j} p_j(t) \Rightarrow p(t+1) = p(t) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$\pi = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$ is the stationary distribution of this Markov chain.

Random Walks and Gambler's Ruin

Consider a random process

$$X_n = X_0 + W_1 + \cdots + W_n,$$

where $W_1, W_2, \cdots$ are independent random variables with $\mathbb{P}(W_i = 1) = p$ and $\mathbb{P}(W_i = -1) = 1 - p$.

Gambler's ruin problem

Assume $X_0 = k$. The random process terminates when $X_n = 0$ (the gambler is ruined) or $X_n = b$ (the gambler is successful). Define S_b to be the event that the gambler is successful without being ruined first. What is $\mathbb{P}(S_b)$?

Define $s_k = \mathbb{P}(S_b | X_0 = k)$. Clearly $s_0 = 0$ and $s_b = 1$.

Random Walks and Gambler's Ruin

It is a discrete-time Markov chain with two absorbing states (i.e. not irreducible).

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Key idea

Conditioning on the value of the first step W_1, the conditional probability of success, i.e. $\mathbb{P}(S_b|X_0 = k, W_1 = w)$ is the same as the unconditional probability of success from initial condition $k + w$, i.e., $\mathbb{P}(S_b|W_0 = k + w)$.

$$s_k = ps_{k+1} + (1 - p)s_{k-1}$$

Boundary conditions: $s_0 = 0$ and $s_b = 1$.

Random Walks and Gambler's Ruin

$$s_k = ps_{k+1} + (1-p)s_{k-1} \quad \text{with} \quad s_0 = 0,_b = 1.$$

Case 1: $p = \frac{1}{2}$

$$s_k = \frac{1}{2}\left(s_{k+1} + s_{k-1}\right) \Rightarrow s_k = \frac{k}{b}.$$

Random Walks and Gambler's Ruin

$$s_k = p s_{k+1} + (1-p) s_{k-1} \quad \text{with} \quad s_0 = 0,_b = 1.$$

Case 2: $p \neq \frac{1}{2}$

$$s_k = \frac{1 - \left(\frac{1-p}{p}\right)^k}{1 - \left(\frac{1-p}{p}\right)^b}.$$

For $p > \frac{1}{2}$,

$$\lim_{b \to \infty} s_k = 1 - \left(\frac{1-p}{p}\right)^k.$$

The probability of ruin decreases geometrically with the initial wealth k.

Kelly's Formula

Assume the gambler wins with probability 0.6 and loses with probability 0.4. We assume the gambler can bit any nonzero value (i.e. any positive real number).

How to bet to maximize the wealth?

Question: Bet a fixed amount or a fixed fraction?

Kelly's Formula

Assume the gambler wins with probability 0.6 and loses with probability 0.4. We assume the gambler can bit any nonzero value (i.e. any positive real number).

The gambler bets α fraction of the wealth each time. Define Z_n such that

$$\mathbb{P}(Z_n = 1+\alpha) = 0.6 \quad \text{and} \quad \mathbb{P}(Z_n = 1-\alpha) = 0.4.$$

Then

$$W_T = W_0 \prod_n Z_n \quad \text{or} \quad \log W_T = \log W_0 + \sum_n \log Z_n.$$

From LLN, we use the following approximation:

$$\frac{\log W_T}{T} \to 0.6 \log(1+\alpha) + 0.4 \log(1-\alpha) \quad (a.s.)$$

Kelly's Formula

$$\max_{\alpha} 0.6 \log (1+\alpha) + 0.4 \log (1-\alpha)$$

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$$\max_{\alpha} 0.6 \log\left(1+\alpha\right) + 0.4 \log\left(1-\alpha\right)$$

$$\frac{0.6}{1+\alpha} = \frac{0.4}{1-\alpha} \Rightarrow \alpha = 0.2.$$

Kelly's Formula

When betting x dollars, the gambler wins with probability p and gets Ax dollars and loses with probability $1 - p$ and gets 0 dollars.

$$\max_{\alpha} p \log(1 - \alpha + A\alpha) + (1 - p) \log(1 - \alpha)$$

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$$\frac{p(A-1)}{1-\alpha+A\alpha}-\frac{1-p}{1-\alpha}=0\Rightarrow\frac{p(A-1)-(1-p)}{A-1}$$

Kelly's Formula

$$\text{Fraction} = \frac{\text{Edge}}{\text{Odd}}.$$

- Edge: the fraction of money you win on average when betting a unit amount of money.
- Odd: when you win, the profit you make.