Chapter 2

Modulation and Demodulation

In this chapter we review the basic concepts of digital modulation and demodulation in the absence of noise. Digital modulation is the process of mapping information bits into transmitted waveforms. Demodulation is the process of mapping received signals back into bits. Concepts from signals and systems that are needed in understanding modulation and demodulation are reviewed.

Because sinusoidal signals are fundamental in communication systems we review of representations of sinusoidal signals. Next we show how vectors can be mapped into waveforms, as described in Chapter 1. This is the essence of modulation: for binary modulation a bit of information (a 0 or a 1) is mapped into one of two vectors. The vectors along with a set of orthonormal waveforms (described below) is mapped into two signals that are transmitted. For M-ary modulation a set of $\log_2(M)$ bits are mapped into one of M vectors that are then converted to signals for transmission. This describes the operation of a transmitter in a communication system. We also describe how to reverse this and generate vectors from a received waveform. This is the essence of demodulation. This describes the operation of a receiver in a communication system.

One particular type of signal, which we call time-shifted orthogonal signals, can be generated using linear time invariant systems. That is, modulation can be achieved with a single linear time-invariant system. To this end we show how to calculate the output of a linear time-invariant system (convolution) when the input is a specified function of time (or frequency). Because of the importance of linear time-invariant systems we review time domain and frequency domain signal representations (Fourier transforms) It is important in understanding digital communication to know how to work in both time domain and frequency domain. Finally, we show how to go from baseband signals to passband signals and back (up conversion and down conversion).

Objectives of this Chapter

- Understand the composition of signal waveforms from signal vectors and orthonormal waveforms.
- Understand the decomposition of signal waveforms to signal vectors.
- Understand the relation between of signal waveforms and signal vectors.
- Understand the time-domain and frequency domain processing of signal waveforms.
- Understand intersymbol interference and the design of signal waveforms without intersymbol interference.
- Understand the generation of passband signals from baseband signals and the reverse.

2.1 Sinusoidal Signals

A sinusoidal signal with frequency f_c can be represented mathematically in various ways. One way is in terms of what is known as quadrature components. Another way is in terms of amplitude and phase. A third way is in terms of a complex number. Starting with a sinusoidal signal and using the trigonometric identity (see Appendix B for such identities) we can express a sinusoid as

$$\begin{aligned} s(t) &= c\cos(2\pi f_c t+\theta) \\ &= c\cos(\theta)\cos(2\pi f_c t) - c\sin(\theta)\sin(2\pi f_c t) \\ &= a\cos(2\pi f_c t) - b\sin(2\pi f_c t) \end{aligned}$$

where $a = c\cos(\theta)$ and $b = c\sin(\theta)$. The frequency, f_c, is sometimes called the "carrier" frequency. The amplitude is c and the phase is θ. The components a and b are called the "in-phase" component and the "quadrature-phase" component or often just the I and Q components. There is a relation between a, b, c and θ that can be visualized with a triangle, as shown in Figure 2.1. Using this triangle we can also start with a signal of the form $s(t) = a\cos(2\pi f_c t) - b\sin(2\pi f_c t)$ and then determine c and θ. Namely, $c = \sqrt{a^2+b^2}$ and $\theta = \tan^{(-1)}(b/a)$.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ \theta &= \tan^{(-1)}\left(\frac{b}{a}\right) \\ a &= c\cos(\theta) \\ b &= c\sin(\theta) \end{aligned}$$

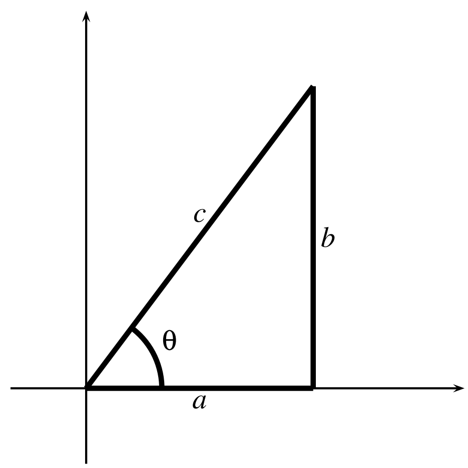


Figure 2.1: Relation between sinusoidal parameters.

$\cos(2\pi t) - \sin(2\pi t)$
$= \sqrt{2}\cos(2\pi t + \frac{\pi}{4})$

As an example consider the following sinusoidal signal

$$\begin{aligned} s(t) &= 2\cos(2\pi t+\pi/4) \\ &= 2\cos(\pi/4)\cos(2\pi t) - 2\sin(\pi/4)\sin(2\pi t) \\ &= \sqrt{2}\cos(2\pi t) - \sqrt{2}\sin(2\pi t). \end{aligned}$$

In this example $f_c = 1$, $c = 2$ and $\theta = \pi/4$ so that $a = b = \sqrt{2}$. Figure 2.2 shows these signals from time $t = 0$ to time $t = 2$ seconds.

Note that by using the representation of an arbitrary sinusoid by an in-phase and quadrature representation we can see that the sum of two sinusoids of the same frequency is another sinusoid of the same

$a\cos x + b\sin x = \sqrt{a^2+b^2}\ \cos(x - \text{arctans}(b/a))$

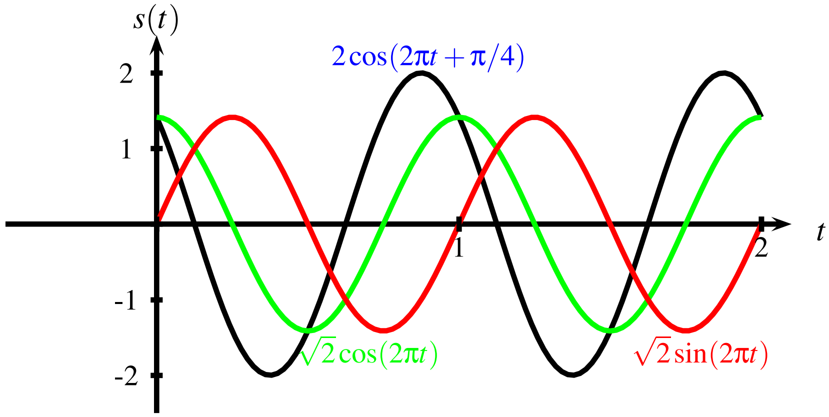


Figure 2.2: Sinusoidal signals.

frequency. That is,

$$\begin{aligned} s_1(t) &= a_1\cos(2\pi f_c t) - b_1\sin(2\pi f_c t) \\ s_2(t) &= a_2\cos(2\pi f_c t) - b_2\sin(2\pi f_c t) \\ s(t) &= s_1(t)+s_2(t) \\ &= (a_1+a_2)\cos(2\pi f_c t) - (b_1+b_2)\sin(2\pi f_c t). \end{aligned}$$

So the representation of the sum of sinusoidal signals of the same frequency in terms of the quadrature components is just the sum of the quadrature components of the individual sinusoidal signals. A third representation of a sinusoidal signal is in terms of complex exponentials. Using Euler's formula

$$\begin{aligned} s(t) &= c\cos(2\pi f_c t+\theta) \\ &= c\left[\frac{e^{j(2\pi f_c t+\theta)}+e^{-j(2\pi f_c t+\theta)}}{2}\right] \qquad \frac{e^{j\theta}+e^{-j\theta}}{2} = \cos(\theta) \\ &= \frac{ce^{j\theta}}{2}e^{j2\pi f_c t} + \frac{ce^{-j\theta}}{2}e^{-j2\pi f_c t} \\ &= de^{j2\pi f_c t} + d^* e^{-j2\pi f_c t} \\ &= \Re\left[2de^{j2\pi f_c t}\right] \end{aligned}$$

where $d = ce^{j\theta}/2 = (a+jb)/2$ and $\Re(x)$ denotes the real part of x. The constant d is the amplitude of the complex exponential at frequency f_c and d^* is the amplitude of the complex exponential at frequency $-f_c$. In the above example $d = \sqrt{2}/2 + j\sqrt{2}/2$. One way to think about the signal $s(t)$ generated from d and d^* is two vectors rotating around the circle in opposite directions as seen in Figure 2.3. The rate of rotation is f_c cycles per second. One vector starts at the point d in the plane while the other vector starts at d^*. The signal $s(t)$ is the sum of the two vectors. Since the vectors are always on opposite sides of the horizontal axis, the vertical components (imaginary component) are cancelled when summing the two vectors and all that is left is the horizontal component (real component). In summary, a pair or real numbers, either (a,b) or (c,θ) or the complex number d, and knowledge of the frequency f_c characterizes a sinusoidal signal.

A signal can be limited to a given time interval by multiplying the signal by a pulse function. To limit a signal to the interval $[0,\ T]$ we multiply the signal by a simple unit amplitude pulse of duration T

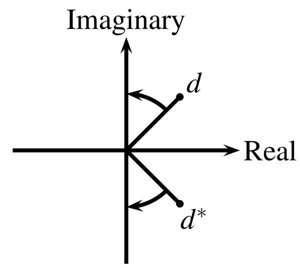


Figure 2.3: Complex exponentials.

starting at time 0 and ending at time T. This pulse will be denoted by $p_T(t)$. That is,

$$p_T(t) = \begin{cases} 1 & 0 \leq t \leq T, \\ 0 & \text{otherwise.} \end{cases}$$

For example a sinusoidal signal with frequency f_c of duration T starting at time $t = 0$ is $\cos(2\pi f_c t)p_T(t)$. Figure 2.2 shows sinusoidal signals of duration 2 seconds with frequency 1.

2.2 Signals and Vectors

In this section we describe the mapping of signal vectors to signal waveforms (functions of time), which is part of the process of modulating, and the reverse of translating a signal waveform back into a signal vector, which is part of the process of demodulating. Before we show how signals and vectors are related a few definitions are in order. We will sometimes use complex signals or waveforms. For real waveforms, the imaginary part is zero and complex conjugates are not needed. The energy of a signal $x(t)$ is

$$E = ||x(t)||^2 = \int |x(t)|^2 dt.$$

Note that $||x(t)||$ is called the norm of $x(t)$. A signal $x(t)$ is said to be normalized if the energy (and the norm) is 1. The energy of a vector $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$ is

$$E = ||\mathbf{x}||^2 = \sum_{i=0}^{N-1} |x_i|^2.$$

A vector $\mathbf{x}$ is said to be normalized if the energy is 1. The inner product of two signals $x(t)$ and $y(t)$ is

$$(x(t), y(t)) = \int x(t)y^*(t)dt.$$

Two signals $x(t)$ and $y(t)$ are said to be orthogonal if the inner product is 0. The inner product of two vectors $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{N-1})$ is

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{N-1} x_i y_i^*.$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal if the inner product is 0. Two signals or vectors are said to be *orthonormal* if they are orthogonal and normalized. The squared Euclidean distance between two possibly complex signals $x(t)$ and $y(t)$ is defined as

$$d_E^2(x(t), y(t)) = \int |x(t) - y(t)|^2 dt.$$

The squared Euclidean distance between two possibly complex vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$d_E^2(\mathbf{x},\mathbf{y}) = \sum_{n=0}^{N-1} |x_n - y_n|^2.$$

Note that

$$\begin{aligned}
d_E^2(x(t),y(t)) &= \int |x(t)-y(t)|^2 dt \\
&= \int |x(t)|^2 - 2\Re[x(t)y^*(t)] - |y(t)|^2 dt \\
&= \int |x(t)|^2 dt - 2\Re[\int x(t)y^*(t)dt] - \int |y(t)|^2 dt \\
&= ||x(t)||^2 - 2\Re[(x(t),y^(t))] - ||y(t)||^2 \\
&= E_x - 2\Re[(x(t),y(t))] + E_y \qquad (2.1)
\end{aligned}$$

where E_x is the energy of signal $x(t)$ and E_y is the energy of signal $y(t)$ and $\Re[a]$ is the real part of the complex number a. The same property holds for vectors as well. The *average* power of a signal $x(t)$ is

$$P_{ave} = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} |x(t)|^2 dt.$$

For a signal $x(t)$ of duration T beginning at time 0 the average power is

$$P = \frac{1}{T}\int_0^T |x(t)|^2 dt.$$

The peak power of a signal $x(t)$ is

$$P_{max} = \max_t |x(t)|^2.$$

2.2.1 Modulation: Composition from vectors to signals

vectors → signals

In Chapter 1 we discussed the mapping of bits to vectors for the purpose of modulation. In this subsection we describe how to form signal waveforms from signal vectors. The combination of the two operations is called modulation. The operation of mapping a vector to a waveform occurs at the transmitter in a communication system. A set of M vectors of length N with a set of N orthonormal waveforms is mapped into M signal waveforms that can be used for transmitting data.

☆ 很重要

Definition.: A set of N (possibly complex) waveforms $\varphi_n(t), n = 0,1,\ldots,N-1$ is said to be orthonormal over a time interval I if

orthonormal waveform

$$(\varphi_n(t),\varphi_l(t)) = \int_I \varphi_n(t)\varphi_l^*(t)dt = \begin{cases} 1 & n = l \\ 0 & n \neq l. \end{cases}$$

Given a set of N orthonormal waveforms $\{\varphi_n(t), n = 0,1,\ldots,N-1\}$ over time interval I and a set of M vectors of length N, $\mathbf{s_m} = (s_{m,0},\ldots,s_{m,N-1})$, $m = 0,1,\ldots,M-1$, we can construct a set of M signals over the time interval I as

vector waveforms

$$s_m(t) = \sum_{n=0}^{N-1} s_{m,n}\varphi_n(t), \; t \in I, m = 0,1,\ldots,M-1.$$

A block diagram showing the generation of a signal waveform from N orthonormal waveforms and one of M vectors of length N is shown in Figure 2.4. The input is a vector $\mathbf{s_m}$. Then the components of the vector $s_{m,n}$, $n = 0,1,\ldots,N-1$ are multiplied by $\varphi_n(t)$ and summed from $n = 0$ to $N-1$. We will abuse notation by using the same label for the signal vectors and the signal waveforms. It will be clear that any use of $s_m(t)$ means the waveform and just $\mathbf{s_m}$ means the vector.

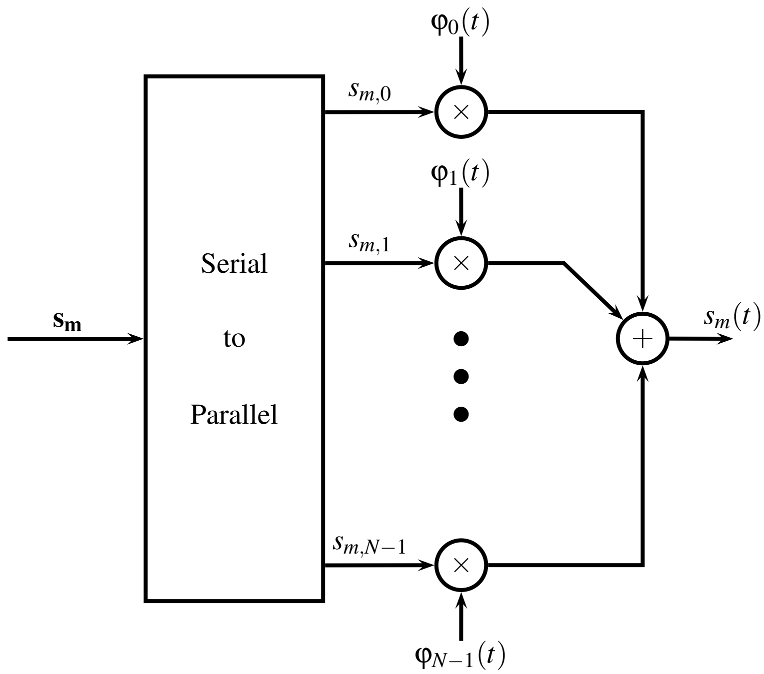


Figure 2.4: Modulation: generating signals from vectors and orthogonal waveforms.

2.2.2 Relations between operations on vectors and operations on signals

There is a correspondence between certain mathematical functions when applied to the representation of signals as vectors or waveforms. Using the notation above these relations between the waveforms and the vectors is as follows.

Properties

signals

1. $||s_m(t)||^2 = \int |s_m(t)|^2 dt = \sum_{n=0}^{N-1} |s_{m,n}|^2 = ||\mathbf{s_m}||^{\mathbf{2}}$

2. $(s_m(t), s_l(t)) = \int s_m(t) s_l^*(t) dt = \sum_{n=0}^{N-1} s_{m,n} s_{l,n}^* = (\mathbf{s_m}, \mathbf{s_l})$

3. $d_E^2(s_m(t), s_l(t)) = ||s_m(t) - s_l(t)||^2 = \int |s_m(t) - s_l(t)|^2 dt = \sum_{n=0}^{N-1} |s_{m,n} - s_{l,n}|^2 = ||\mathbf{s_m} - \mathbf{s_l}||^2 = d_E^2(\mathbf{s_m}, \mathbf{s_l})$.

These relations indicate that the geometry of signal vectors and signal waveforms is identical. It is instructive to prove one of these relationships. The others are proved in a similar manner. The proof technique involves changing the order of summation and integration.

Proof of 2:

PAPR

energy of signal = energy of vector

$$
\begin{aligned}
(s_m(t), s_l(t)) &= \int s_m(t) s_l^*(t) dt \\
&= \int \sum_{n=0}^{N-1} s_{m,n} \varphi_n(t) \sum_{p=0}^{N-1} s_{l,p}^* \varphi_p^*(t) dt \\
&= \sum_{n=0}^{N-1} s_{m,n} \sum_{p=0}^{N-1} s_{l,p}^* \int \varphi_n(t) \varphi_p^*(t) dt \\
&= \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} s_{m,n} s_{l,n}^* \delta_{n,p} \\
&= \sum_{n=0}^{N-1} s_{m,n} s_{l,n}^* \\
&= (\mathbf{s_m}, \mathbf{s_l})
\end{aligned}
$$

where $\delta_{n,p} = 1$ if $n = p$ and is zero otherwise. This is called the Kronecker delta function. From property 2 the energy of the vector is the same as the energy of the signal. This is a consequence of property 2 (set $l = m$ in property 2). Note using (2.1) and property 2 we can show property 3. Property 3 states that the distance between two vectors is the same as the distance between the signal waveforms. In a digital communication system where the performance depends on the distance between signals and the energy of the signals, it is sufficient to know the distance between the signal vectors and the energy of the vectors.

PAPR :

An important measure for a signal is the peak-to-average power ratio (PAPR) of a set of signal waveforms. The PAPR for a particular signal is defined as

$$
\begin{aligned}
\text{PAPR}(m) = \Gamma_w(m) &= \frac{P_{max}(m)}{P_{ave}(m)} \\
&= \frac{\max_m |s_m(t)|^2}{P_m}.
\end{aligned}
$$

$\|s_m\|^2 / T$

$P_m = \left(\int |s_m(t)|^2 dt \right) / T$

where P_m is the average power of signal $s_m(t)$. The *peak-to-average power ratio* (PAPR) Γ of a signal set is the maximum over all possible signals of the peak power of a signal normalized by the average power of the signals. That is,

$$\Gamma_w = \max_m \Gamma_w(m) = \frac{\max_{t,m} |s_m(t)|^2}{\sum_{m=0}^{M-1} P_m / M}.$$

A low PAPR allows for an amplifier to efficiently convert DC energy to RF energy without distorting the signal due to the nonlinear nature of the amplifier. A high PAPR can result in distortion when amplifying the signal unless the input amplitude is reduced (backed-off) so that the signal does not cause the amplifier to operate near saturation. But this usually means the amplifier is not efficiently converting DC energy to radio frequency (RF) energy or transmitted energy. The peak-to-average power ratio is important in determining the effect of a nonlinear amplifier on the distortion incurred and in the power consumed by an amplifier. Sometimes the peak-to-average power ratio is defined for a set of vectors as opposed to a set of waveforms. For a set of vectors the peak-to-average power ratio is

$$\Gamma_v = \frac{\max_m |\mathbf{s_m}|^2}{\sum_{m=0}^{M-1} |\mathbf{s_m}|^2 / M}.$$

Typically there is a factor of 2 (3 dB) between the waveform PAPR and the vector PAPR due to the use of sinusoidal signals.

2.2.3 Demodulation: decomposition from signals to vectors

At the receiver in a communication system the received signal is often mapped back to a vector. From a set of M signals and a set of N orthogonal signals we can find the vector representation.

vectors

$$s_{m,n} = \int s_m(t)\varphi_n^*(t)dt, \; m = 0, 1, \ldots, M-1, \; n = 0, 1, \ldots, N-1$$

Signals → vectors

signals

This is shown in Figure 2.5.

To show the validity of the generation of the coefficient from the waveforms consider

$$\begin{aligned}
(s_m(t), \varphi_n(t)) &= \int s_m(t)\varphi_n^*(t)dt \\
&= \int \left(\sum_{l=0}^{N-1} s_{m,l}\varphi_l(t) \right) \varphi_n^*(t)dt \\
&= \sum_{l=0}^{N-1} s_{m,l} \int \varphi_l(t)\varphi_n^*(t)dt \\
&= \sum_{l=0}^{N-1} s_{m,l}\delta_{l,n} \\
&= s_{m,n}.
\end{aligned}$$

Thus the inner product of $s_m(t)$ with $\varphi_n(t)$ produces the n-th coefficient in the representation of $s_m(t)$ with orthonormal functions $\varphi_n(t)$.

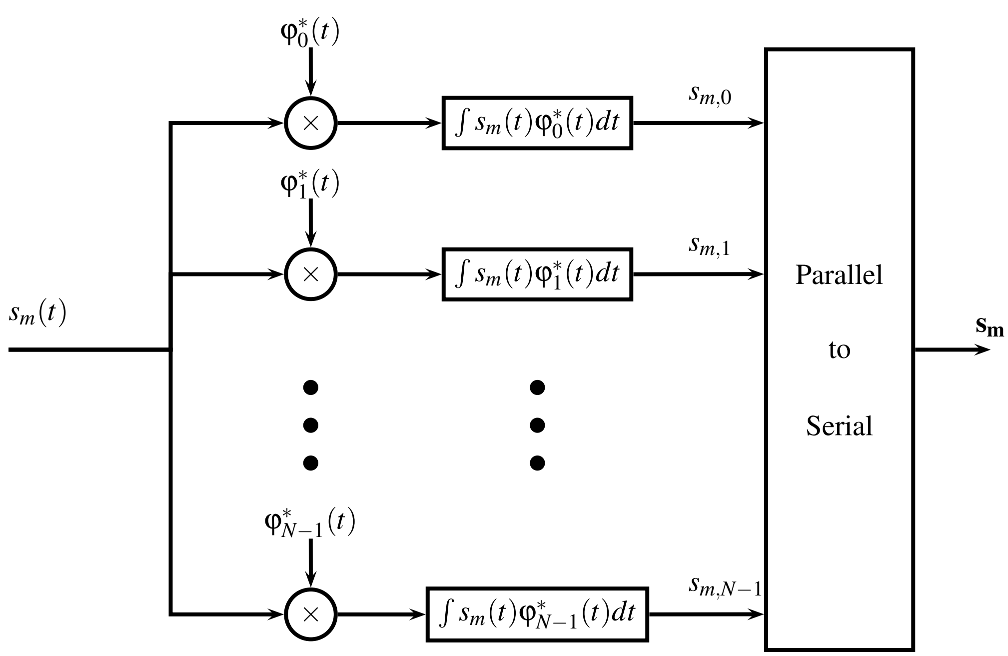


Figure 2.5: Demodulation: determining vectors from a signal.

2.3 Modulation/Demodulation Examples

In this section we illustrate many different types of modulation and demodulation. In essence modulation and demodulation is the process of mapping signal vectors to signal waveforms and back. The first few examples are in the form of baseband signals (no carrier frequency). Examples showing how signal vectors can be mapped to bandpass or passband signals are presented as well.

2.3.1 Example 1:

The first example of modulation is mapping one bit of information into one of two signals. Consider the one normalized[1] waveform $\varphi_0(t)$ over the interval $I = [0,T]$ described below and shown in Figure 2.6.

$$\begin{aligned} \varphi_0(t) &= \sqrt{\frac{1}{T}} p_T(t) \\ &= \begin{cases} \sqrt{\frac{1}{T}}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where T is the duration of the waveform. This waveform is shown in Figure 2.6. Clearly this waveform has energy 1. That is,

$$\int |\varphi_0(t)|^2 dt = 1.$$

The signal vectors (actually scalars since $N = 1$) are

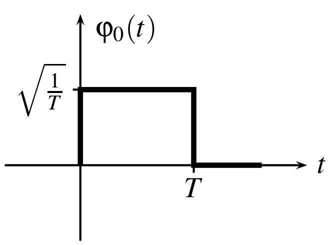


Figure 2.6: Normalized waveform $\varphi_0(t)$ for Example 1.

$$\begin{aligned} \mathbf{s_0} &= (s_{0,0}) = +\sqrt{E} \\ \mathbf{s_1} &= (s_{1,0}) = -\sqrt{E}. \end{aligned}$$

The constellation is shown in Figure 2.7. The signals are

$$\begin{aligned} s_0(t) &= s_{0,0}\varphi_0(t) \\ &= +\sqrt{E}\sqrt{1/T}\, p_T(t) \\ &= +\sqrt{P} p_T(t) \\ s_1(t) &= s_{1,0}\varphi_1(t) \\ &= -\sqrt{P} p_T(t) \end{aligned}$$

since $P = E/T$. The signals corresponding to the one normalized waveform and the signal vectors are shown in Figure 2.8. These signals are known as antipodal signals as $s_1(t) = -s_0(t)$. The square distance

[1] There is no other waveform to be orthogonal to so this is just a normalized waveform as opposed to a set of orthonormal waveforms.

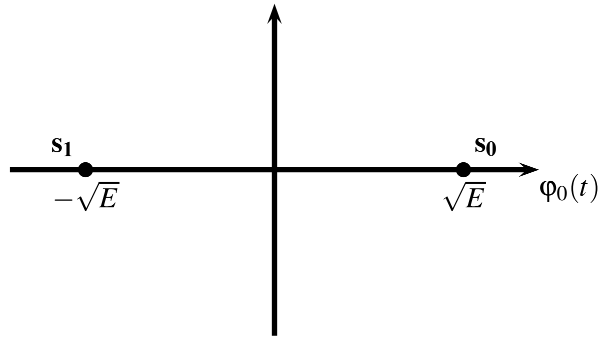


Figure 2.7: Constellation for Example 1.

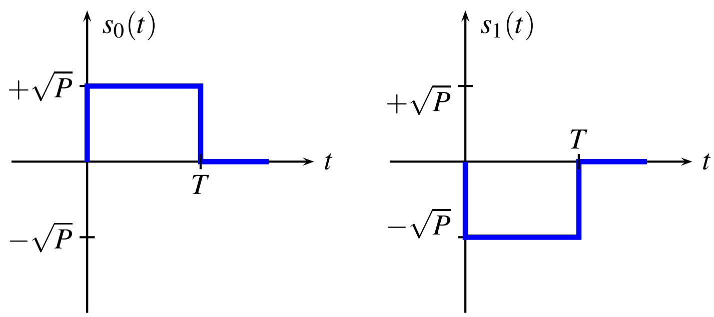


Figure 2.8: Signals $s_0(t)$ and $s_1(t)$ for Example 1.

between the two signals is

$$\begin{aligned} d_E^2(s_0, s_1) &= \int (s_0(t) - s_1(t))^2 dt \\ &= (\sqrt{E} - (-\sqrt{E})^2 \\ &= 4E. \end{aligned}$$

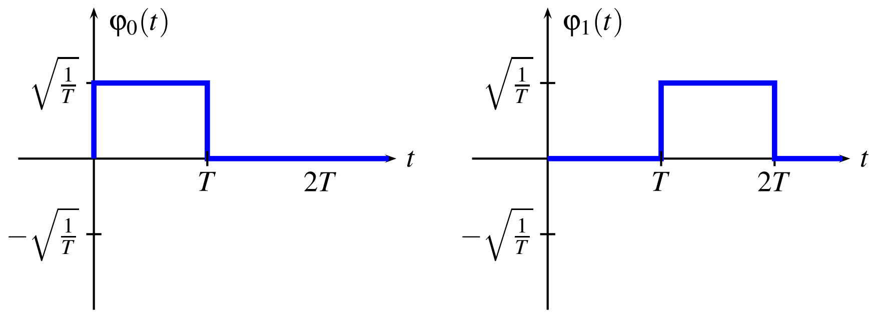
The energy of each signal is E so the average energy is E and the energy per bit E_b is E. The rate of communication is $R = 1/T$ bits/second. The signals are baseband signals since their frequency content is concentrated around DC (as seen in Section XXX).

2.3.2 Example 2:

Consider the two orthogonal waveforms described below and shown in Figure 2.9 that are orthogonal over the interval $I = [0,\ 2T]$. This is also a set of baseband signals.

$$\begin{aligned} \varphi_0(t) &= \sqrt{\frac{1}{T}} p_T(t), \\ \varphi_1(t) &= \sqrt{\frac{1}{T}} p_T(t - T). \end{aligned}$$

These waveforms $\varphi_0(t)$ and $\varphi_1(t)$ are said to be orthogonal in time. Because signal $\varphi_1(t)$ is a time shift of waveform $\varphi_0(t)$ we call this type of signal set as time-shifted orthonormal. As we will see in Section



$P = \frac{E}{T}$

Figure 2.9: Orthonormal waveforms $\varphi_0(t)$ with $\varphi_1(t)$ for Example 2.

2.5 there are implementation advantages of using time-shifted orthonormal signals. The signal vectors in this example are of length $N = 2$ and are given by

$$
\begin{aligned}
\mathbf{s}_0 &= \sqrt{E/2}(+1,+1) \\
\mathbf{s}_1 &= \sqrt{E/2}(+1,-1) \\
\mathbf{s}_2 &= \sqrt{E/2}(-1,+1) \\
\mathbf{s}_3 &= \sqrt{E/2}(-1,-1)
\end{aligned}
$$

$$
\begin{aligned}
S_0(t) &= \sqrt{\tfrac{E}{2}}\sqrt{\tfrac{1}{T}}\,P_T(t) + \sqrt{\tfrac{E}{2}}\sqrt{\tfrac{1}{T}}\,P_T(t-T) \\
&= 2\sqrt{\tfrac{E}{2T}}\left(P_T(t) + P_T(t-T)\right) \\
&= \sqrt{2P}\left(P_T(t) + P_T(t-T)\right)
\end{aligned}
$$

where E represents the energy per vector. These vectors are shown in Figure 2.10.

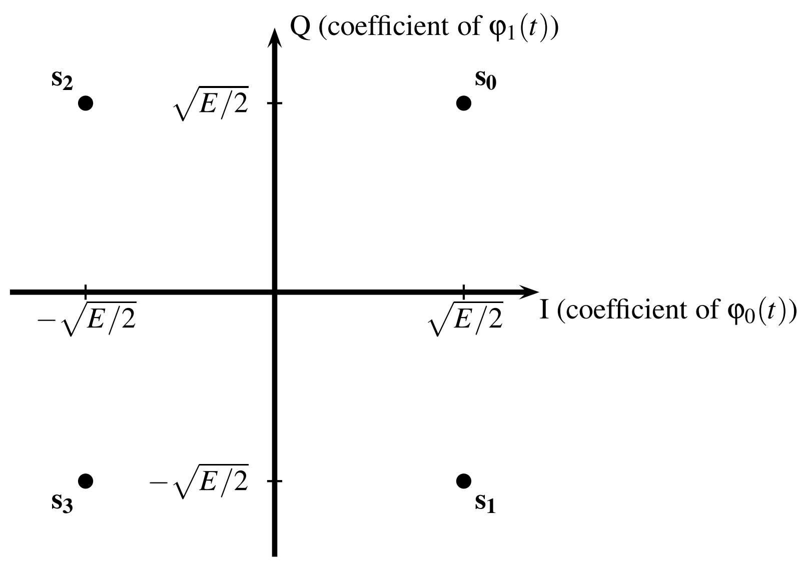


Figure 2.10: Constellation for Example 2.

The signals generated from these orthonormal waveforms and signal vectors are shown in Figure 2.11. The energy per signal is E, the average energy is E and the energy per bit is $2E/2 = E$. The rate of communications is $R = 2/(2T) = 1/T$ bits/second. We can generate more orthogonal waveforms by having more pulses that do not overlap. A pulse from time $2T$ to $3T$ would be the next pulse, then time $3T$ to $4T$ and so on. The energy per bit and the rate remain the same. These signals are known as

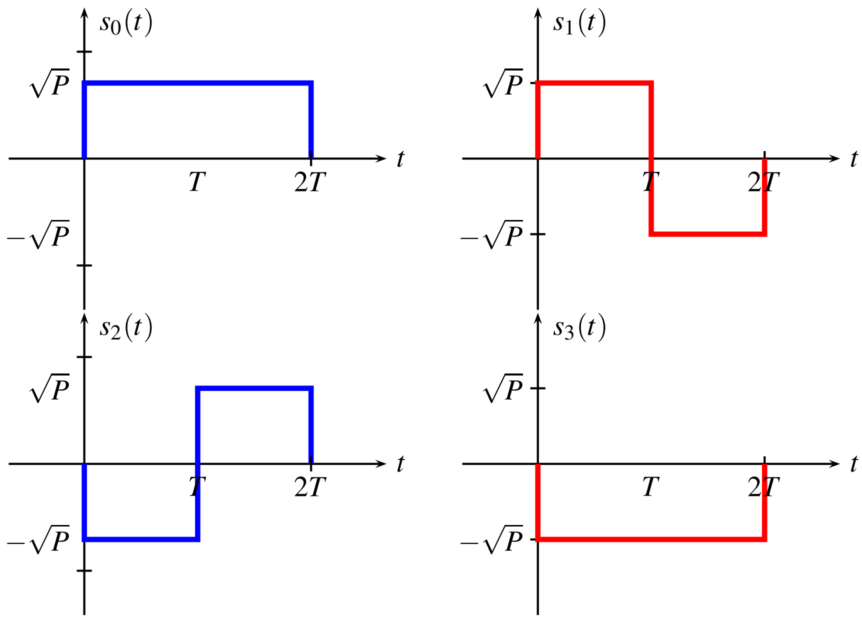


Figure 2.11: Signals $s_0(t), s_1(t), s_2(t)$, and $s_3(t)$ for Example 2.

baseband signals because the frequency content is centered at DC (0 frequency). The next few examples have signals at a carrier frequency f_c.

2.3.3 Example 3:

Consider a set of $M = 2$ signals with $N = 1$, that is just one dimension. Suppose the single orthonormal waveform (just a normalized waveform since there is only one waveform) is

$$\varphi_0(t) = +\sqrt{\frac{2}{T}}\cos(2\pi f_c t) p_T(t)$$

where $2f_cT$ is an integer. Here f_c represents a carrier frequency. The energy of $\varphi_0(t)$ is

$$\begin{aligned} E = \int_0^T |\varphi_0(t)|^2 dt &= \int_0^T \frac{2}{T}\cos^2(2\pi f_c t) dt \\ &= \frac{2}{T}\int_0^T \frac{1}{2}(1 + \cos(2\pi 2 f_c t)) dt \\ &= \frac{1}{T}[t + \frac{\sin(2\pi 2 f_c t)}{2\pi f_c}]_0^T \\ &= 1 + \frac{\sin(2\pi 2 f_c T)}{2\pi 2 f_c T} \\ &= 1. \end{aligned}$$

The last step follows since $2f_cT$ is an integer so that $\sin(2\pi 2 f_c T) = 0$. The energy is also approximately 1 if $2f_cT \gg 1$. The term $\sin(2\pi 2 f_c t)/(2\pi f_c T)$, is often called the double frequency term. In most communication systems this term can be neglected because f_cT is much greater than 1 so that the second term becomes negligible compared to the first term. The single orthonormal waveform $\varphi_0(t)$ is shown in Figure 2.12.

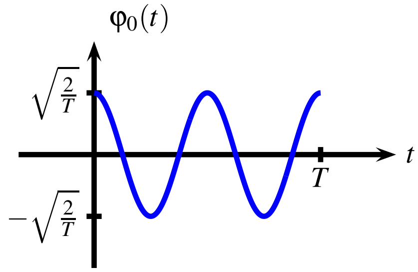


Figure 2.12: Orthonormal waveform $\varphi_0(t)$ of Example 3.

The signal vectors (actually scalars since $N = 1$) are the same as Example 1.

$$\begin{aligned} \mathbf{s_0} &= +\sqrt{E} \\ \mathbf{s_1} &= -\sqrt{E}. \end{aligned}$$

The two signal vectors are shown in Figure 2.13. These signal vectors are plotted in the two-dimensional plane even though there is just one dimension. The two signal waveforms are

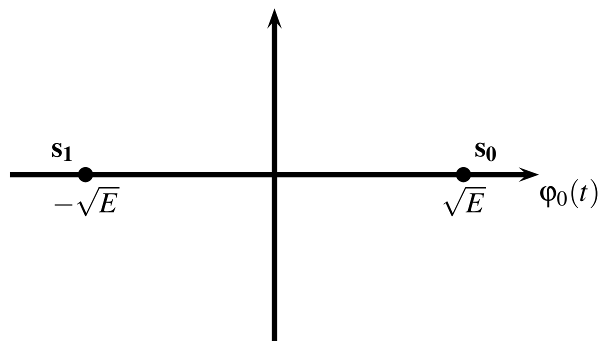


Figure 2.13: Constellation for Example 3.

$$\begin{aligned} s_0(t) &= \sqrt{E}\varphi_0(t) \\ &= +\sqrt{2E/T}\cos(2\pi f_c t)p_T(t) \\ s_1(t) &= -\sqrt{E}\varphi_0(t) \\ &= -\sqrt{2E/T}\cos(2\pi f_c t)p_T(t). \end{aligned}$$

Since E/T is power P and $-\cos(x) = \cos(x+\pi)$ an equivalent form for the signals is

$$\begin{aligned} s_0(t) &= \sqrt{2P}\cos(2\pi f_c t)p_T(t) \\ s_1(t) &= \sqrt{2P}\cos(2\pi f_c t + \pi)p_T(t). \end{aligned}$$

As can be seen the two signal waveforms are distinguished by the phase of a sinusoid. This modulation technique is called binary phase shift keying or BPSK. The two signals waveforms are also opposite amplitude or polarity. A pair of signals is called an antipodal signal set if $s_0(t) = -s_1(t)$. The two signals are shown in Figure 2.14. The squared Euclidean distance between the signal vectors or wave-

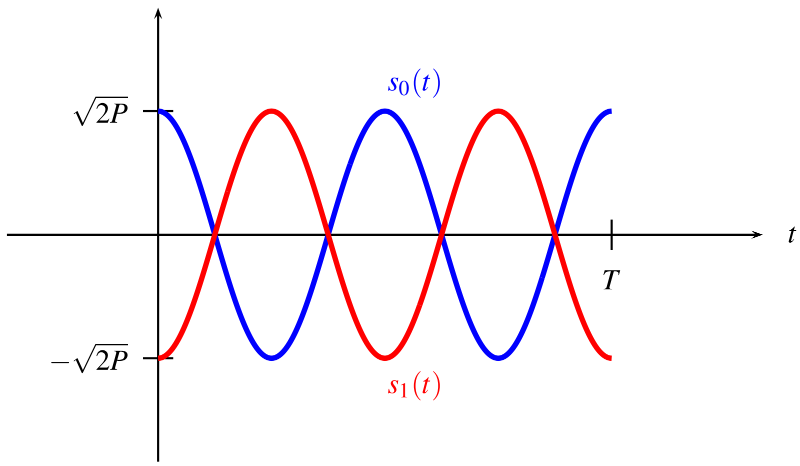


Figure 2.14: Signals $s_0(t)$ with $s_1(t)$ of Example 3.

forms is $d_E^2(\mathbf{s_0}, \mathbf{s_1}) = 4E$. While this is a one dimensional signal set, there is an unused dimension (using $\sin(2\pi f_c t)$) that does not take up any extra bandwidth as will be seen in Example 4. So the number of dimensions is really 2 but we do not use the second dimension. So the rate is actually $r = 1/2$ bits/dimension. The bandwidth efficiency in this case is $R/W = 1$ bps/Hz. We can double the bandwidth efficiency of BPSK by adding a nonzero signal component in the unused dimension (see the next Example). The peak-to-average power ratio Γ_w of this set of signal waveforms is 2 (3dB) while the peak-to-average power ratio Γ_v of the constellation is 1 (0dB). This is a passband or bandpass signal set since the frequency content is centered around a carrier frequency f_c.

2.3.4 Example 4:

Consider a set of $M = 4$ signals with $N = 2$, that is using two dimensions. The two (real) orthonormal waveforms are

$$\varphi_0(t) = +\sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t)$$
$$\varphi_1(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t)$$

where the waveforms $\varphi_0(t)$ and $\varphi_1(t)$ are orthogonal over the interval $I = [0,\ T]$. It will become clear shortly why the signal $\varphi_1(t)$ has a negative sign. These waveforms are shown in Figure 2.15. The duration of the orthonormal waveforms is T and f_c is called the carrier frequency. Typically we set $f_c T$ to be an integer or at least $f_c T \gg 1$. Often these two waveforms are called the "in-phase" and "quadrature-phase" signals or just I and Q signals. These signals are said to be orthogonal in phase. That is, they use different phases of a sinusoid of frequency f_c.

The two signals $\varphi_0(t)$ and $\varphi_1(t)$ are orthogonal and each has unit energy. We have already shown in the previous example that the orthonormal waveform $\varphi_0(t)$ is normalized (has energy 1). A similar analysis shows that

$$\int_0^T |\varphi_1(t)|^2 dt \quad = \quad 1$$

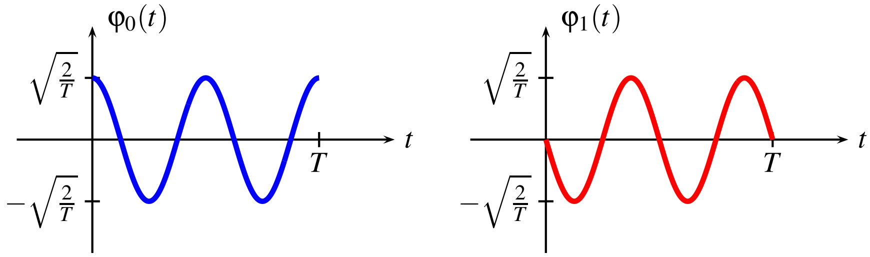


Figure 2.15: Orthogonal waveforms $\varphi_0(t)$ with $\varphi_1(t)$ of Example 4.

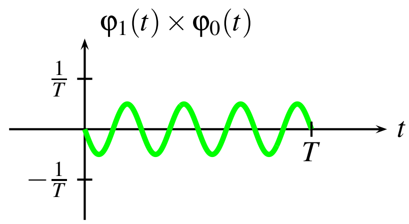


Figure 2.16: Product of $\varphi_0(t)$ with $\varphi_1(t)$ of Example 4.

under the same conditions (e.g. $2f_cT$ is an integer or is very large). Second consider the orthogonality. The product of the two signals is shown in Figure 2.16. It is apparent that the product of these two signals is a sinusoid and has twice the frequency of each. As long as there is an integer number of cycles of this sinusoid between 0 and T the integral of the product will be zero. Mathematically,

$$\begin{aligned}
\int_0^T \varphi_0(t)\varphi_1^*(t)dt &= \int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_c t)[-\sqrt{\frac{2}{T}}\sin(2\pi f_c t)]dt \\
&= -\frac{2}{T}\int_0^T \cos(2\pi f_c t)\sin(2\pi f_c t)dt \\
&= -\frac{2}{T}\int_0^T \frac{1}{2}\sin(2\pi(2f_c)t)dt \\
&= -\left[\frac{\cos(2\pi(2f_c)t)}{(2\pi 2f_c T)}\right]_0^T \\
&= \frac{1-\cos(2\pi(2f_cT))}{2\pi(2f_cT)} \\
&= 0.
\end{aligned}$$

The last step follows if $2f_cT$ is an integer in which case $\cos(2\pi 2f_cT) = 1$. Note that if $2f_cT \gg 1$ then the signals are approximately orthogonal. That is, the integral of the product will be much less than 1, the energy of each signal. For example if $f_c = 1$ GHz and $T = 1\mu s$ then $f_cT = 10^3$ and the integral of the product of $\varphi_0(t)$ and $\varphi_1(t)$ will be no more than 2×10^{-3}.

The signal vectors in this example are given by

$$\begin{aligned}
\mathbf{s}_0 &= \sqrt{E/2}(+1,+1)\\
\mathbf{s}_1 &= \sqrt{E/2}(+1,-1)\\
\mathbf{s}_2 &= \sqrt{E/2}(-1,+1)\\
\mathbf{s}_3 &= \sqrt{E/2}(-1,-1)
\end{aligned}$$

where E represents the energy per vector. These vectors are shown in Figure 2.17. We label the horizontal axis representing the first component as I, short for "in-phase" component and the vertical axis representing the second component as Q, short for "quadrature-phase" component. The in-phase component is the term that is multiplied by $\varphi_0(t) = \sqrt{2/T}\cos(2\pi f_c t)$ and the quadrature-phase component is the term that multiplies $\varphi_1(t) = -\sqrt{2/T}\sin(2\pi f_c t)$.

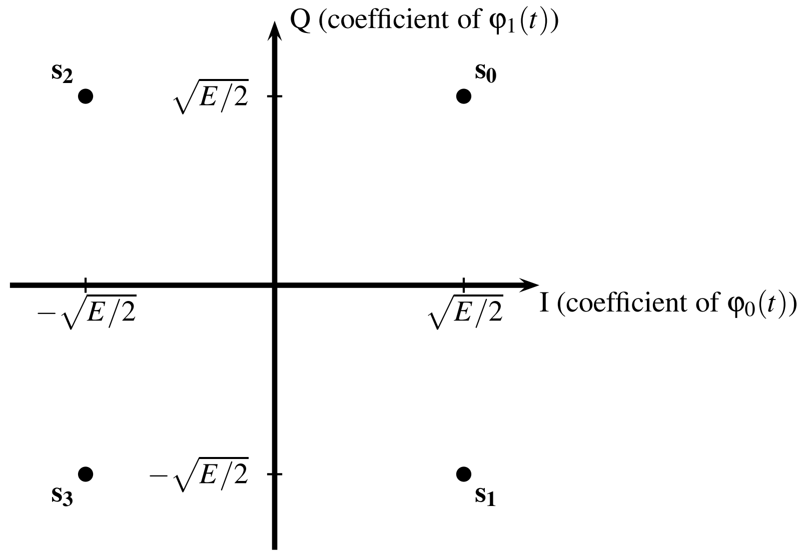


Figure 2.17: Constellation for Example 4.

The four vectors $\mathbf{s_0}, \mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}$ of length $\sqrt{E}$ each have energy E. The vectors and two orthonormal waveforms, $\varphi_0(t)$ and $\varphi_1(t)$, can generate four signals $s_0(t), s_1(t), s_2(t), s_3(t)$. These are shown in Figure 2.18. Consider first the signal $s_0(t)$.

$$\begin{aligned}
s_0(t) &= s_{0,0}\varphi_0(t) + s_{0,1}\varphi_1(t)\\
&= \sqrt{E/2}\varphi_0(t) + \sqrt{E/2}\varphi_1(t)\\
&= \sqrt{E/2}\sqrt{2/T}\cos(2\pi f_c t)p_T(t) - \sqrt{E/2}\sqrt{2/T}\sin(2\pi f_c t)p_T(t)\\
&= \sqrt{E/T}\cos(2\pi f_c t)p_T(t) - \sqrt{E/T}\sin(2\pi f_c t)p_T(t)\\
&= \sqrt{P}\cos(2\pi f_c t)p_T(t) - \sqrt{P}\sin(2\pi f_c t)p_T(t)\\
&= \sqrt{2P}\cos(2\pi f_c t + \pi/4)p_T(t).
\end{aligned}$$

$\sqrt{P}(\cos(2\pi f_c t) - \sin(2\pi f_c t))P_T(t)$

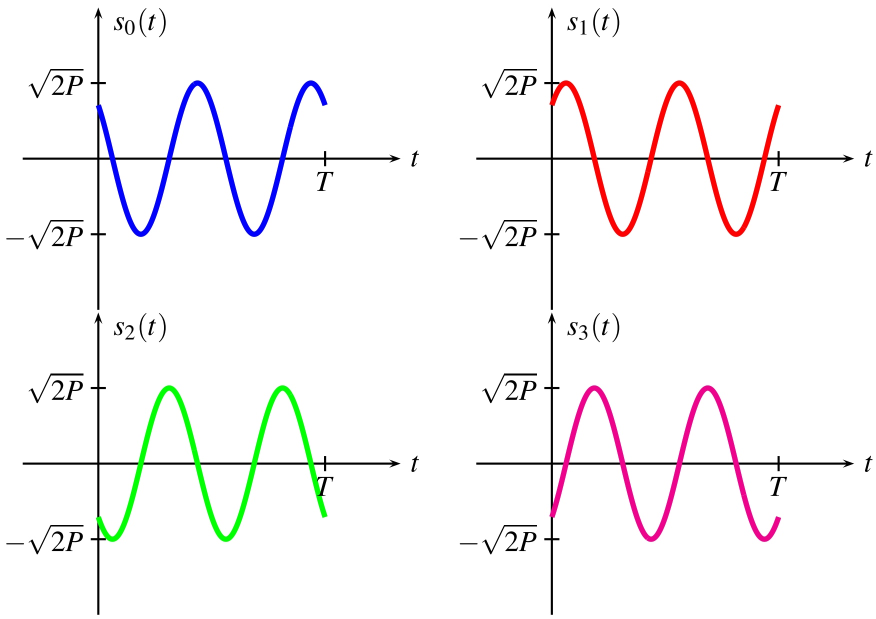
$= \sqrt{2P}(\cos(2\pi f_c t + \frac{\pi}{4}))P_T(t)$

Here $P = E/T$ is the power of the transmitted signal. Similarly the other signals can be expressed as sinusoids with a certain phase. All four signals are sinusoids of duration T but with four different phases.

$$\begin{aligned}
s_0(t) &= \sqrt{2P}\cos(2\pi f_c t + \pi/4)p_T(t)\\
s_1(t) &= \sqrt{2P}\cos(2\pi f_c t + 7\pi/4)p_T(t)\\
s_2(t) &= \sqrt{2P}\cos(2\pi f_c t + 3\pi/4)p_T(t)\\
s_3(t) &= \sqrt{2P}\cos(2\pi f_c t + 5\pi/4)p_T(t).
\end{aligned}$$

$P_{max} = 2P \quad \int |S_0|^2 dt / T$

These signals have the same energy as the energy of the vectors. This modulation is known as quadrature



$P_{avg} = \int |S_m(t)|^2 dt / T M$
$= \|S_m\|^2 / TM$
$= 4E / T4$
$= E/T = P$

Figure 2.18: Signals $s_0(t), s_1(t), s_2(t)$, and $s_3(t)$ for Example 4.

phase shift keying (QPSK) and can transmits two bits of information using one of four signals. These signals/vectors can be represented as points in a 2 dimensional plane. Note that the phase of the signal point in the 2-dimensional plane corresponds to the phase of the sinusoidal signal. This is because of the choice of the orthonormal waveforms, in particular the sign of $\varphi_1(t)$. The rate is 1 bit/dimension or 2 bits/second/Hz. The minimum squared Euclidean distance is $2E$. The energy per bit is $E_b = E/2$. So the minimum squared Euclidean distance normalized by the energy per bit is 4. The peak power of these signals is $2P$ while the average power is P. So the peak-to-average power ratio in Γ_w this case is 2 or 3dB.

? How to compute P_{max} P_{avg}

2.3.5 Example 5:

In this example $N = 1$ and $M = 4$. The single orthonormal waveform is complex and the signal vectors (scalars) are complex.

$$\varphi_0(t) = \sqrt{\frac{1}{T}} e^{j2\pi f_c t} p_T(t).$$

For complex orthonormal waveforms, besides the requirement that

$$\int |\varphi_0(t)|^2 dt = 0$$

we also require that

$$\int \varphi_0^2(t)dt = 0$$

where the square in the above equation is not the magnitude squared but the ordinary square. For the waveform above the first condition is met as follows.

$$\begin{aligned}
\int |\varphi_0(t)|^2 dt &= \int |\sqrt{\frac{1}{T}} e^{j2\pi f_c t} p_T(t)|^2 dt \\
&= \frac{1}{T}\int_0^T |e^{j2\pi f_c t}|^2 dt \\
&= \frac{1}{T}\int_0^T 1 dt = 1.
\end{aligned}$$

The second condition is

$$\begin{aligned}
\int \varphi_0^2(t)dt = &= \int (\sqrt{\frac{1}{T}} e^{j2\pi f_c t} p_T(t))^2 dt \\
&= \int_0^T \frac{1}{T} e^{j2\pi 2 f_c t} dt \\
&= \frac{1}{T}\int_0^T e^{j2\pi 2 f_c t} dt \\
&= \frac{1}{T}\left[\frac{e^{j2\pi 2 f_c T} - 1}{j2\pi 2 f_c}\right].
\end{aligned}$$

As in the previous example, if $2f_cT$ is an integer then $\int \varphi^2(t)dt = 0$. Alternatively if $f_cT \gg 1$ then $\int \varphi^2(t)dt \approx 0$. Similarly, $\int(\varphi_0^*(t))^2 dt = 0$ if $2f_cT$ is an integer or $\int(\varphi_0^*(t))^2 dt \approx 0$ if $f_cT \gg 1$. Suppose that the complex signal waveforms are

$$s_m(t) = s_{m,0}\varphi_0(t), \ \ m = 0,1,2,3;$$

where the complex vectors are

$$\begin{aligned}
\mathbf{s}_{0,0} &= \sqrt{E/2}(+1+j) \\
\mathbf{s}_{1,0} &= \sqrt{E/2}(+1-j) \\
\mathbf{s}_{2,0} &= \sqrt{E/2}(-1+j) \\
\mathbf{s}_{3,0} &= \sqrt{E/2}(-1-j).
\end{aligned}$$

Note that this is the same set of signals as Example 3 but uses complex numbers rather than two dimensional vectors. Since communication systems do not transmit complex signals, there must be a conversion from complex signals to real signals. The way complex signals are mapped to real signals is as follows.

$$\begin{aligned}
u_m(t) &= \Re\left(\sqrt{2} s_m(t)\right) \\
&= \Re\left[\frac{\sqrt{2}}{2}(s_m(t) + s_m^*(t))\right]
\end{aligned}$$

where $\Re(x)$ is the real part of the complex number x. The energy of the real signal can be determined as follows.

$$\begin{aligned}
E_m &= \int u_m^2(t)dt \\
&= \int \left[\Re\left(\sqrt{2}s_m(t)\right)\right]^2 dt. \\
&= \frac{1}{2}\int \left[s_m(t)+s_m^*(t)\right]^2 dt. \\
&= \frac{1}{2}\int s_m^2(t)+2s_m(t)s_m^*(t)+(s_m^*(t))^2 dt. \\
&= \frac{1}{2}\int s_{m,0}^2\varphi_0^2(t)+2|s_{m,0}|^2\varphi_0(t)\varphi_0^*(t)+(s_{m,0}^*)^2(\varphi_0^*(t))^2 dt \\
&= \frac{1}{2}\int 2|s_{m,0}|^2\varphi_0(t)\varphi_0^*(t)dt. \\
&= |s_{m,0}|^2\int|\varphi_0(t)|^2 dt. \\
&= |s_{m,0}|^2.
\end{aligned}$$

Thus again, the energy of the real signal $u_m(t)$ is the same as the energy of the vector (a complex scalar in this example). At a receiver, the signal $u_m(t)$ is processed as follows:

$$\begin{aligned}
\int \sqrt{2}u_m(t)\varphi_0^*(t)dt &= \sqrt{2}\int \Re[\sqrt{2}s_m(t)]\varphi_0^*(t)dt \\
&= 2\int \frac{s_m(t)+s_m^*(t)}{2}\varphi_0^*(t)dt \\
&= \int (s_{m,0}\varphi_0(t)+s_{m,0}^*\varphi_0^*(t))\varphi_0^*(t)dt \\
&= s_{m,0}\int \varphi_0(t)\varphi_0^*(t)dt + s_{m,0}^*\int \varphi_0^*(t)\varphi_0^*(t)dt \\
&= s_{m,0}+s_{m,0}^*\int \varphi_0^*(t)\varphi_0^*(t)dt \\
&= s_{m,0}.
\end{aligned}$$

This example is really the same as Example 4 but uses complex notation rather than two dimensional vectors.

2.3.6 Example 6:

In this example the same orthonormal waveforms are used as in Example 4 but there are eight signal vectors in two dimensions generating eight signals. The signal vectors are

$$\begin{aligned}
s_0 &= (\sqrt{E},0) \\
s_1 &= (\sqrt{E/2},\sqrt{E/2}) \\
s_2 &= (0,\sqrt{E}) \\
s_3 &= (-\sqrt{E/2},\sqrt{E/2}) \\
s_4 &= (-\sqrt{E},0) \\
s_5 &= (-\sqrt{E/2},-\sqrt{E/2}) \\
s_6 &= (0,-\sqrt{E}) \\
s_7 &= (+\sqrt{E/2},-\sqrt{E/2})
\end{aligned}$$

These signal vectors are shown in Figure 2.19. The eight signals are

$$s_m(t) = \sqrt{2P}\cos(2\pi f_c t + m\pi/4)p_T(t), \quad m = 0,1,\ldots,7.$$

This signal set is known as 8-ary phase shift keying or 8PSK. The minimum squared Euclidean distance

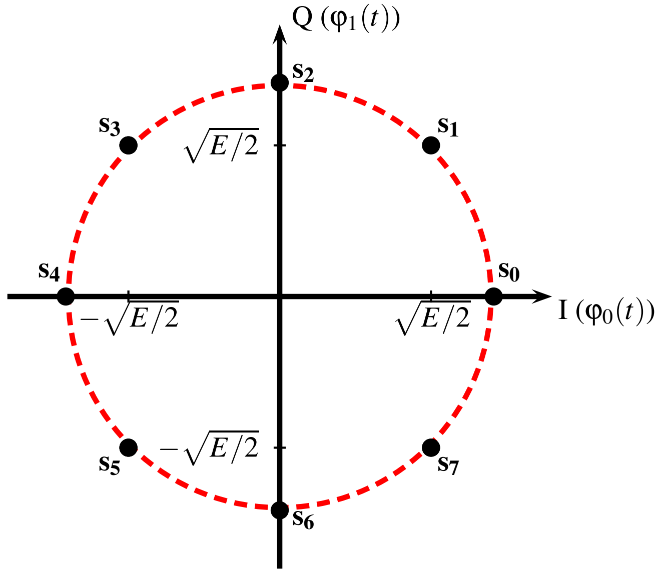


Figure 2.19: Constellation for Example 4.

between signal vectors or waveforms is the distance between neighboring signals which is $d_E^2(\mathbf{s_0},\mathbf{s_1}) = (\sqrt{E}-\sqrt{E/2})^2 + (\sqrt{E/2})^2 = [(1-\sqrt{1/2})^2 + (1/\sqrt{2})^2]E = (2-\sqrt{2}) = .5857E$. The energy per bit is $E_b = E/3$. The minimum normalized squared Euclidean distance is

$$\min_{m\neq l}\frac{d_E^2(\mathbf{s_m},\mathbf{s_l})}{E_b} = \frac{(2-\sqrt{2})E}{E/3} = 3(2-\sqrt{2}) = 1.7474.$$

The rate is $r = 3/2$ (bits/dimension) or $R = 3$ bits/second/Hz. This signal set has higher rate than Example 3 but the distance between signals is smaller for the same energy per bit.

2.3.7 Example 7:

In this example there are $M = 16$ signals in $N = 2$ dimensions. The orthogonal waveforms are the same as in Example 4:

$$\begin{aligned}\varphi_0(t) &= +\sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t)\\ \varphi_1(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t).\end{aligned}$$

The signal vectors are shown in Table 2.1 and in Figure 2.20. This signal set is called 16-ary quadrature amplitude modulation or 16QAM. The signal $s_0(t)$ can be written in different forms.

$$\begin{aligned}s_0(t) &= -3A\sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t) - 3A\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t)\\ &= 6\sqrt{\frac{A^2}{T}}\cos(2\pi f_c t + 3\pi/4)p_T(t).\end{aligned}$$

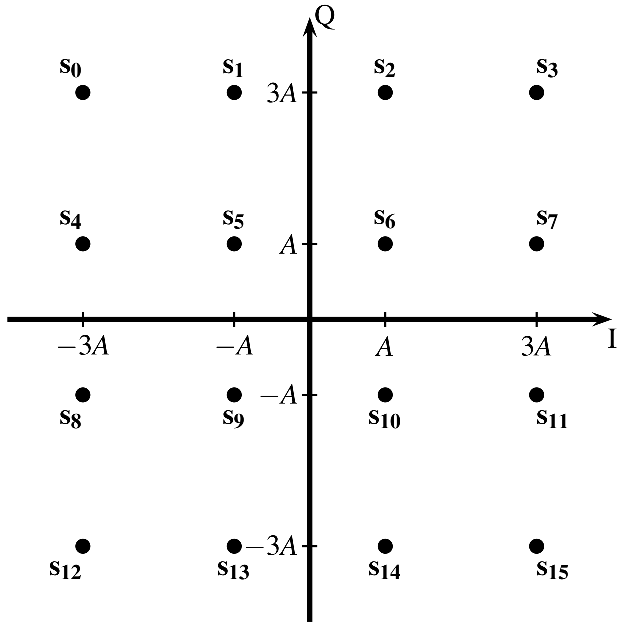
$= -3A\sqrt{\frac{2}{T}}\left(\cos(2\pi f_c t) + \sin(2\pi f_c t)\right) p_T(t)$

$= -3A\sqrt{\frac{2}{T}}\cdot\sqrt{2}\cos(2\pi f_c t - \frac{\pi}{4}))\, p_T(t)$

$= 6\sqrt{\frac{A^2}{T}}\cos(2\pi f_c t + \frac{3\pi}{4})\, p_T(t)$

$\mathbf{s_0} = A(-3,+3)$	$\mathbf{s_1} = A(-1,+3)$	$\mathbf{s_2} = A(+1,+3)$	$\mathbf{s_3} = A(+3,+3)$
$\mathbf{s_4} = A(-3,+1)$	$\mathbf{s_5} = A(-1,+1)$	$\mathbf{s_6} = A(+1,+1)$	$\mathbf{s_7} = A(+3,+1)$
$\mathbf{s_8} = A(-3,-1)$	$\mathbf{s_9} = A(-1,-1)$	$\mathbf{s_{10}} = A(+1,-1)$	$\mathbf{s_{11}} = A(+3,-1)$
$\mathbf{s_{12}} = A(-3,-3)$	$\mathbf{s_{13}} = A(-1,-3)$	$\mathbf{s_{14}} = A(+1,-3)$	$\mathbf{s_{15}} = A(+3,-3)$

Table 2.1: Signal vectors for 16QAM



18 20

Figure 2.20: Constellation for Example 7 (16QAM).

Similarly,

$$
\begin{aligned}
s_{10}(t) &= A\sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t) + A\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t) \\
&= 2\sqrt{\frac{A^2}{T}}\cos(2\pi f_c t - \pi/4)p_T(t).
\end{aligned}
$$

Each signal can be written as a single sinusoid with some amplitude and phase or as a linear combination of the quadrature signals $\cos(2\pi f_c t)$ and $-\sin(2\pi f_c t)$. The rate of this signal set is $r = \log_2(16)/2 = 2$ bits/dimension or $R = 4$ bits/second/Hz. Signals s_5, s_6, s_9 and s_{10} have energy $2A^2$. Signals $s_1, s_2, s_4, s_7, s_8, s_{11}, s_{13}, s_{14}$ have energy $10A^2$. Signals s_0, s_3, s_{12}, s_{15} have energy $18A^2$. The average energy of this signal set (assuming each signal is used with the same probability) is $E_{ave} = 10A^2$. The average energy per bit is $E_b = E_{ave}/4 = 5A^2/2$. The minimum squared Euclidean distance between signals is $d^2_{E,min} = 4A^2$. The normalized minimum squared Euclidean distance is

$$
\frac{\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{4A^2}{5A^2/2} = 8/5 = 1.6.
$$

The peak power of the signals is $P_{max} = 36A^2/T$. The average power of the signals is $P_{ave} = 10A^2/T$. The peak-to-average power ratio, Γ_w is $PAPR = 36/10 = 3.6$ which is 5.56dB. The peak-to-average power ratio of the constellation is $\Gamma_v = 2.56$dB. This modulation technique has worse PAPR compared to 8PSK (Example 4).

2.3.8 Example 8:

Consider the following $N = 2$ orthogonal waveforms to generate $M = 4$ signals using the same signal vectors as Example 2 and 4. These orthogonal waveforms use sinusoids of different frequencies.

$$\begin{aligned}\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)\\ \varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t).\end{aligned}$$

These waveforms are orthogonal over the interval $I = [0,\ T]$ for certain choices of the frequencies f_0 and f_1. These two waveforms are shown in Figure 2.21 where $f_0 = 7/(2T)$ and $f_1 = 8/(2T)$. The product

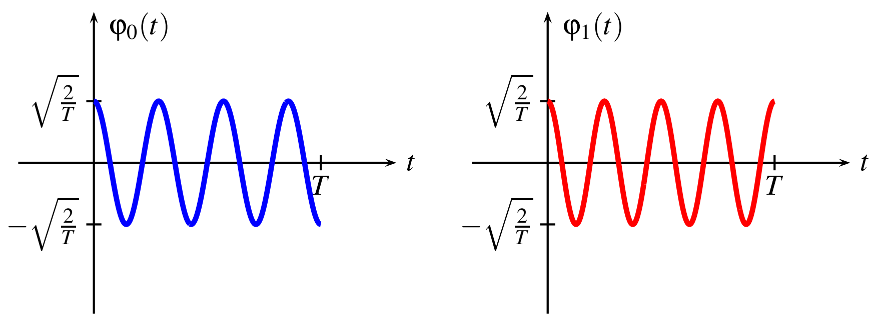


Figure 2.21: Orthonormal waveforms $\varphi_0(t)$ with $\varphi_1(t)$ for Example 8.

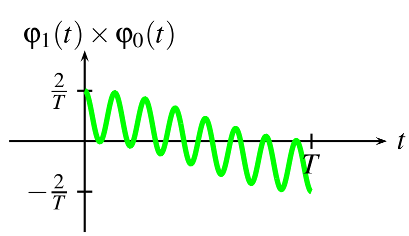


Figure 2.22: Product of $\varphi_0(t)$ with $\varphi_1(t)$ of Example 8

of these two waveforms is shown in Figure 2.22. These waveforms are orthogonal provided there is a certain separation between f_0 and f_1. For example if $f_1 - f_0 = m/(2T)$, for m a non-negative integer then $\varphi_0(t)$ and $\varphi_1(t)$ will be orthogonal. We could add more signals at different frequencies as long as the separation between frequencies was maintained. For example $f_0 = 1/(2T), f_1 = 2/(2T), f_2 = 3/(2T), \ldots$

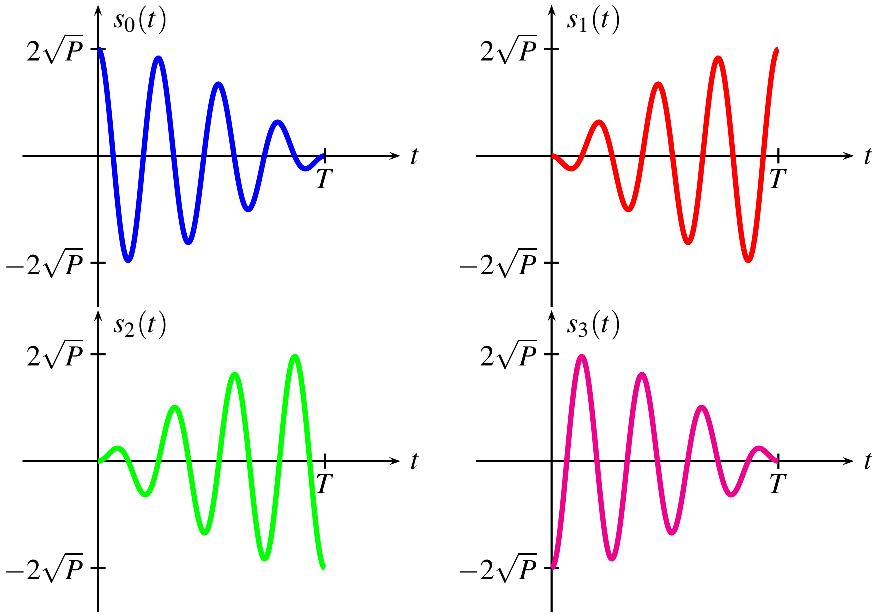


Figure 2.23: Signals $s_0(t), s_1(t), s_2(t)$, and $s_3(t)$ for Example 8.

The signal vectors in this example are the same as in Example 2 and 4 above.

$$s_{0,0} = \sqrt{E/2}(+1,+1)$$
$$s_{1,0} = \sqrt{E/2}(+1,-1)$$
$$s_{1,0} = \sqrt{E/2}(-1,+1)$$
$$s_{1,0} = \sqrt{E/2}(-1,-1).$$

The signals generated from the signal vectors and the orthonormal waveforms above (see Figure 2.21) are shown in Figure 2.23. Notice that the signals $s_0(t), s_1(t), s_2(t), s_3(t)$ are not pure sinusoidal signals but are the sum of two sinusoidal signals of different frequencies.

$$\begin{aligned}
s_0(t) &= +\sqrt{P}\cos(2\pi f_0 t)p_T(t) + \sqrt{P}\cos(2\pi f_1 t)p_T(t) \\
s_1(t) &= +\sqrt{P}\cos(2\pi f_0 t)p_T(t) - \sqrt{P}\cos(2\pi f_1 t)p_T(t) \\
s_2(t) &= -\sqrt{P}\cos(2\pi f_0 t)p_T(t) + \sqrt{P}\cos(2\pi f_1 t)p_T(t) \\
s_3(t) &= -\sqrt{P}\cos(2\pi f_0 t)p_T(t) - \sqrt{P}\cos(2\pi f_1 t)p_T(t).
\end{aligned}$$

This is one form of transmission technique known as *orthogonal frequency division multiplexing* (OFDM) because we are transmitting two bits on two different frequency carriers (f_0 and f_1). The minimum squared Euclidean distance between signals is $2E$ while the energy per bit is $E_b = E/2$. So the normalized minimum squared Euclidean distance is

$$\min_{m \neq l} \frac{d_E^2(\mathbf{s_m}, \mathbf{s_l})}{E_b} = \frac{2E}{E/2} = 4.$$

The peak power of these signals is $(2\sqrt{P})^2 = 4P$ while the average power is P. As a result the peak-to-average power ratio Γ_w is PAPR$= 4 = 6.02$dB. The signals do not have a constant "envelope" in that the amplitude of the signals varies with time compared to Example 1. The envelope of a signal, which we define more precisely later, is essentially the waveform generated by connecting the peaks of the signal. This variation of the envelope is important because the effect of the amplifier on the signals. Generally signals with constant envelope, like Example 1, 2 and 3, can be amplified in a more energy efficient way than a non-constant envelope signal, such as this example. However, signals with a nonconstant envelope can have better spectral characteristics.

2.3.9 Example 9:

As another example of modulation consider the same $N = 2$ orthogonal waveforms as in Example 8.

$$\begin{aligned} \varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t) \\ \varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t). \end{aligned}$$

The $M = 2$ signal vectors of length $N = 2$ for this example are

$$\begin{aligned} \mathbf{s_0} &= \sqrt{E}(1,0) \\ \mathbf{s_1} &= \sqrt{E}(0,1). \end{aligned}$$

The signals is this example are

$$\begin{aligned} s_0(t) &= \sqrt{E}\varphi_0(t) + 0\varphi_1(t) \\ &= \sqrt{2P}\cos(2\pi f_0 t)p_T(t) \\ s_1(t) &= 0\varphi_0(t) + \sqrt{E}\varphi_1(t) \\ &= \sqrt{2P}\cos(2\pi f_1 t)p_T(t). \end{aligned}$$

The frequency separation $f_1 - f_0$ is the same as in Example 8. In this example a single information bit is sent by transmitting a sinusoid at one of two frequencies. This is known as binary frequency shift keying (BFSK). The signals are just the orthonormal waveforms scaled by $\sqrt{E}$. The energy per bit of information is $E_b = E$.The squared Euclidean distance between signals is $d_E^2(\mathbf{s}_0, \mathbf{s}_1) = 2E$. The energy per bit is $E_b = E$. So the normalized minimum squared Euclidean distance is $d_E^2(\mathbf{s}_0, \mathbf{s}_1)/E_b = 2E/E = 2$. The rate is $r = 1/2$ bits/dimension or $R = 1$ bits/second/Hz. This is a signal set that is worse than Example 4 in both normalized distance and rate. However, there are implementation advantages of BFSK. In particular the data can be recovered without knowing the phase of the received signal.

More orthonormal waveforms can be added with different frequencies to generate more orthonormal waveforms. The case of M different orthonormal waveforms at M different frequencies using signal vectors that are nonzero in only one component is called M-ary FSK. As in Example 8, a minimum spacing between frequencies is required to ensure the orthogonality of the waveforms.

2.3.10 Example 10:

Consider the case of M signals that are scaled versions of a set of M orthonormal waveforms. Any set of M orthonormal waveforms can be used. The waveforms of example 2 extended to have M time-shifted orthonormal signals can be used. Here the number of signals M is the same as the number of dimensions

N. So $N = M$. The signal vectors are

$$\begin{aligned}
\mathbf{s}_0 &= \sqrt{E}(1,0,0,0,\cdots,0) \\
\mathbf{s}_1 &= \sqrt{E}(0,1,0,0,\cdots,0) \\
\mathbf{s}_2 &= \sqrt{E}(0,0,1,0,\cdots,0) \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
\mathbf{s}_{M-1} &= \sqrt{E}(0,0,0,0,\cdots,1).
\end{aligned}$$

The signals are

$$\begin{aligned}
s_0(t) &= \sqrt{E}\varphi_0(t) \\
s_1(t) &= \sqrt{E}\varphi_1(t) \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
s_{M-1}(t) &= \sqrt{E}\varphi_{M-1}(t).
\end{aligned}$$

Any set of orthonormal waveforms can be used and we will get the same distance between signals, the same rate and the same energy. The minimum squared Euclidean distance of these signals is the same as Example 6, $\min_{m\neq l} d_E^2(\mathbf{s_m},\mathbf{s_l}) = 2E$. The rate is $r = \log_2(M)/M$ since we can communicate $\log_2(M)$ bits using M signals and there are M dimensions. The energy per bit is $E_b = E/\log_2(M)$ since each signal has energy E. From this we see that the squared Euclidean distance normalized by the energy per bit is

$$\frac{\min_{m\neq l} d_E^2(\mathbf{s}_m,\mathbf{s}_l)}{E_b} = \frac{2E}{E/\log_2(M)} = 2\log_2(M).$$

So a set of M orthogonal signals has squared Euclidean distance normalized by the energy per bit that grows with M but the rate is going to 0 as M gets larger. So there is a larger and larger distance between signals (for a given amount of energy per bit) but at a smaller and smaller rate. This implies that the probability of error is decreasing as M grows (as will be shown in Chapter6) but also that the rate is going to 0. This is in contrast to what Shannon showed that the error probability can be made arbitrarily small as long as the rate is smaller than the capacity. That is, the rate does not have to go to zero for the error probability to become small.

In each of Example 2, 4, and 8 we have two orthogonal waveforms. In these examples we mapped 4 signal vectors into 4 signal waveforms using two orthogonal waveforms. In Example 2 the two waveforms $\varphi_0(t)$ and $\varphi_1(t)$ are orthogonal in **time**. Example 4 demonstrates orthogonality in **phase** of the transmitted signals. In that example the two orthogonal waveforms have a phase difference of $\pi/2$. From this we generated four signals that are known as quadrature phase shift keying (QPSK). Example 8 has two signals that are orthogonal in **frequency**. We can also construct orthogonal waveforms that are combinations of being orthogonal in time, frequency and phase as shown below.

2.3.11 Example 11:

Consider the eight orthonormal waveforms below. The first two are exactly the same as Example 2 and are orthogonal in phase to each other. The first two are orthogonal in time to the remaining 6 waveforms.

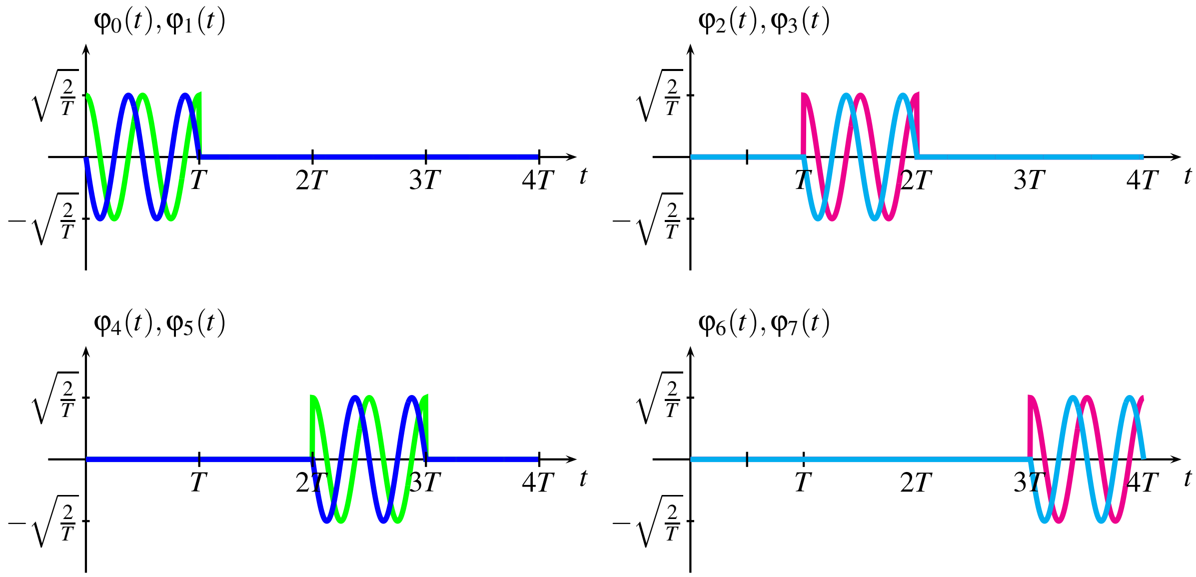


Figure 2.24: Orthonormal waveforms $\varphi_0(t)$,..., $\varphi_7(t)$ for Example 10.

These waveforms are orthogonal over the time interval $I = [0,\ 4T]$ provided $2f_cT$ is an integer. These signals are shown in Figure 2.24.

$$
\begin{aligned}
\varphi_0(t) &= \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t), & \varphi_4(t) &= \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t-2T)\\
\varphi_1(t) &= -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t), & \varphi_5(t) &= -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t-2T)\\
\varphi_2(t) &= \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t-T), & \varphi_6(t) &= \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)p_T(t-3T)\\
\varphi_3(t) &= -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t-T), & \varphi_7(t) &= -\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)p_T(t-3T).
\end{aligned}
$$

Since there are $N = 8$ orthogonal waveforms we can map signal vectors of length 8 to signal waveforms. Consider the $M = 16$ vectors shown in Table 2.2. From these 16 vectors of length 8 and the eight orthogonal waveforms of time duration $4T$ we can generate 16 signals of duration $4T$.

$$s_m(t) = \sum_{n=0}^{7} s_{m,n}\varphi_n(t),\ m = 0, 1, \ldots, 15$$

These signals are shown in Figure 2.25. The energy of each signal/vector is E. The duration of each signal is $4T$. The rate of the signal set is $r = 4$ bits/8 dimensions or 1/2 bit/dimension. The bandwidth efficiency of communication is $R/W = 2r = 1$ bits/second/Hz. The energy per bit is $E_b = E/4$. The squared Euclidean distance between signals $s_0(t)$ and $s_1(t)$ is the same as the squared Euclidean distance between the corresponding vectors. That is,

$$d_E^2(\mathbf{s_0}, \mathbf{s_1}) = 4(\frac{E}{8}4) = 2E.$$

This is because the two vectors different in 4 components and in each component where the two vectors differ the squared Euclidean distance is $\frac{E}{8}4$. The squared distance between vector $\mathbf{s_0}$ and vectors $\mathbf{s_i}$ for

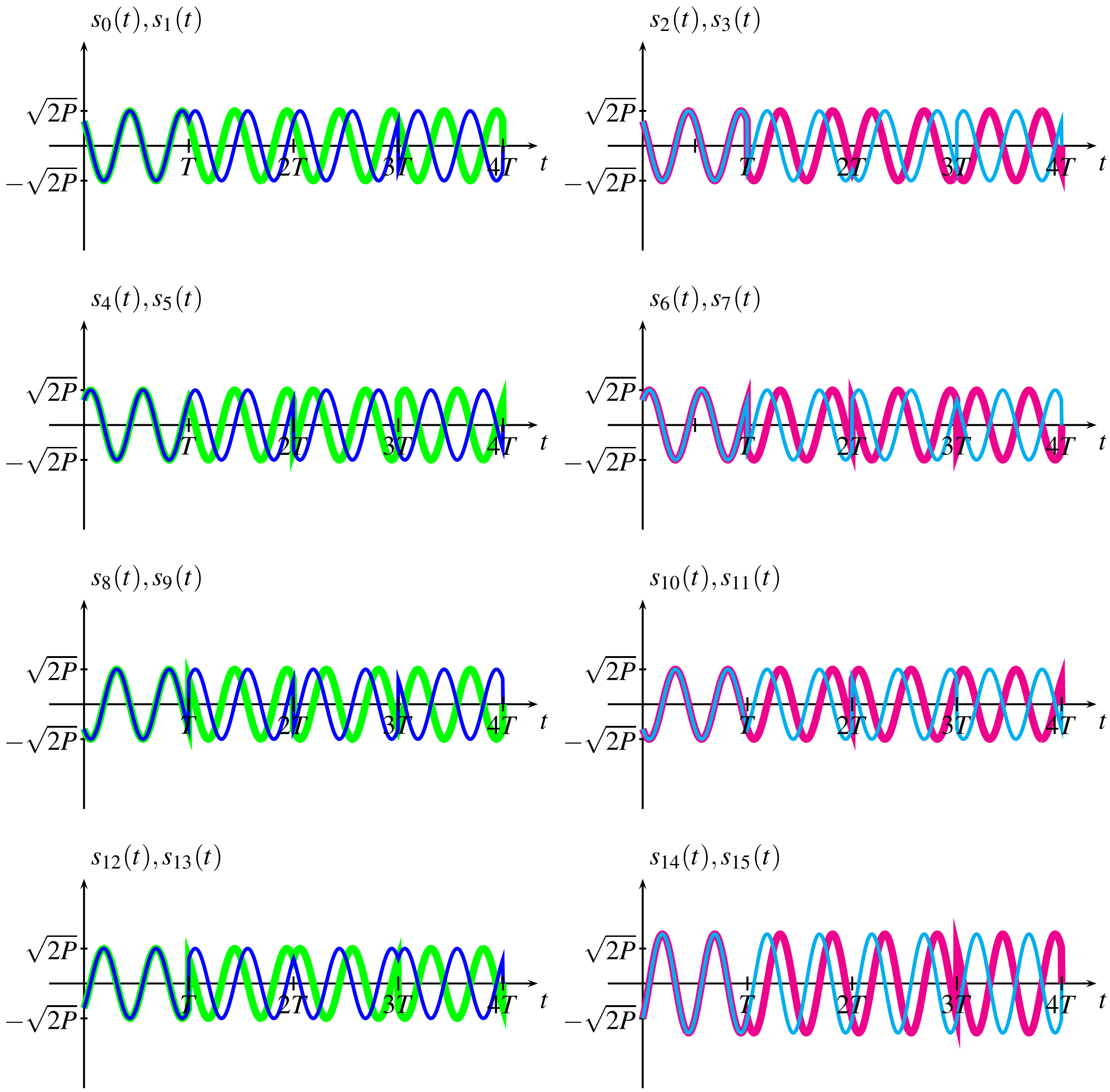


Figure 2.25: Waveforms $s_0(t)$,..., $s_{15}(t)$ for Example 10.

$\mathbf{s_0} = \sqrt{\frac{E}{8}}(+1,+1,+1,+1,+1,+1,+1,+1)$	$\mathbf{s_1} = \sqrt{\frac{E}{8}}(+1,+1,+1,-1,+1,-1,-1,-1)$
$\mathbf{s_2} = \sqrt{\frac{E}{8}}(+1,+1,-1,+1,-1,-1,+1,-1)$	$\mathbf{s_3} = \sqrt{\frac{E}{8}}(+1,+1,-1,-1,-1,+1,-1,+1)$
$\mathbf{s_4} = \sqrt{\frac{E}{8}}(+1,-1,+1,+1,-1,-1,-1,+1)$	$\mathbf{s_5} = \sqrt{\frac{E}{8}}(+1,-1,+1,-1,-1,+1,+1,-1)$
$\mathbf{s_6} = \sqrt{\frac{E}{8}}(+1,-1,-1,+1,+1,+1,-1,-1)$	$\mathbf{s_7} = \sqrt{\frac{E}{8}}(+1,-1,-1,-1,+1,-1,+1,+1)$
$\mathbf{s_8} = \sqrt{\frac{E}{8}}(-1,+1,+1,+1,-1,+1,-1,-1)$	$\mathbf{s_9} = \sqrt{\frac{E}{8}}(-1,+1,+1,-1,-1,-1,+1,+1)$
$\mathbf{s_{10}} = \sqrt{\frac{E}{8}}(-1,+1,-1,+1,+1,-1,-1,+1)$	$\mathbf{s_{11}} = \sqrt{\frac{E}{8}}(-1,+1,-1,-1,+1,+1,+1,-1)$
$\mathbf{s_{12}} = \sqrt{\frac{E}{8}}(-1,-1,+1,+1,+1,-1,+1,-1)$	$\mathbf{s_{13}} = \sqrt{\frac{E}{8}}(-1,-1,+1,-1,+1,+1,-1,+1)$
$\mathbf{s_{14}} = \sqrt{\frac{E}{8}}(-1,-1,-1,+1,-1,+1,+1,+1)$	$\mathbf{s_{15}} = \sqrt{\frac{E}{8}}(-1,-1,-1,-1,-1,-1,-1,-1)$

Table 2.2: Signal vectors for Example 10.

$i = 1,2,...,14$ are also $2E$. The squared distance between vector $\mathbf{s_0}$ and vector $\mathbf{s_{15}}$ is $4E$. The set of squared distances from vector $\mathbf{s_1}$ is the same as from $\mathbf{s_0}$. That is, there are also 14 signals with squared distance $2E$ from $\mathbf{s_1}$ and one signal with distance $4E$ from $\mathbf{s_1}$. In fact every signal in the constellation has the same set of distances. Because of this, the signal set is said to be *geometrically uniform*. This example is something called the extended Hamming code that will be revisited in Chapter 8 (see 8.8.5. The energy per bit is $E_b = E/4$. So the normalized minimum squared Euclidean distance is

$$\frac{\min_{m \neq l} d_E^2(\mathbf{s}_m, \mathbf{s}_l)}{E_b} = \frac{2E}{E/4} = 8.$$

The rate is $r = 3/8$ bits/dimension or $R = 3/4$ bits/second/Hz. This example has better distance properties for the same average energy per bit but has lower rate .

2.3.12 Example 12:

This is an important example for practical systems since these pulses are widely used. In this example we have N orthonormal waveforms defined over the entire real line, $I = (-\infty, +\infty)$, as follows.

$$\begin{aligned}
\varphi_0(t) &= \frac{1}{\sqrt{T}}\left(\frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T\cos(\pi(1+\alpha)t/T)}{[1-(4\alpha t/T)^2]\pi t/T}\right), \; t \neq 0, \; t \neq \pm T/(4\alpha) \quad (2.2)\\
\varphi_0(0) &= \frac{1}{\sqrt{T}}\left[\frac{\pi(1-\alpha)+4\alpha}{\pi}\right]\\
\varphi_0(\pm\frac{T}{4\alpha}) &= \frac{1}{\sqrt{2T}}\alpha[C(1-2/\pi)+S(1+2/\pi)]\\
\varphi_n(t) &= \varphi_0(t-nT), \; n = 1,2,...,N-1. \quad (2.3)
\end{aligned}$$

Here α is a parameter between 0 and 1 called the rolloff parameter. Also, $C = \cos(\pi/(4\alpha)$ and $S = \sin(\pi/(4\alpha))$. Note that for when $t = 0, \pm T/(4\alpha)$ the numerator and denominator in (2.2) both evaluate to 0. The limit as t approaches these values is derived in Appendix 2A. These pulses are called square-root raised cosine pulses because in the frequency domain the spectrum is the square-root of a raised cosine shape. We will show this in Appendix 2B and 2C. If $\alpha = 0$ then

$$\varphi_0(t) = \frac{1}{\sqrt{T}}\left(\frac{\sin(\pi t/T)}{\pi t/T}\right) = \frac{1}{\sqrt{T}}\text{sinc}(t/T), \quad -\infty < t < \infty \tag{2.4}$$

where $\text{sinc}(x) = \sin(\pi x)/(\pi x)$. This function has a flat frequency spectrum over the frequency interval $[-1/(2T), 1/2T]$ as will be seen in Section 2.6 . The orthonormality of these functions can be shown using frequency domain techniques, discussed also in Section 2.6 . In this example different orthonormal waveforms are *time-shifted* versions of one of the orthonormal waveform as seen in (2.3). These waveforms are shown in Figure 2.26 for $N = 3$. These are baseband signals centered in the frequency domain at DC (0 Hz). If $\alpha > 0$ these signals decay in time as $1/(4\pi\alpha(t/T)^2)$. That is, for large t

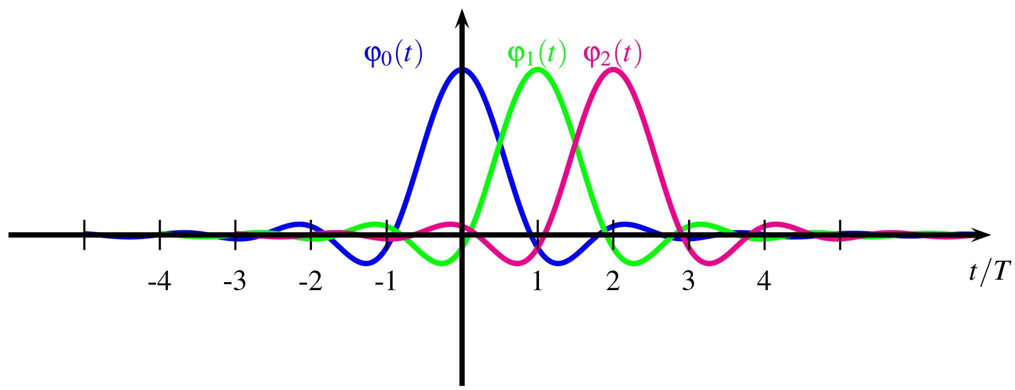


Figure 2.26: Example 12 orthonormal waveforms ($\alpha = 0.35$).

$$|\varphi_0(t)| \approx \frac{1}{\sqrt{T}4\pi\alpha(t/T)^2}, \quad \alpha > 0$$
$$|\varphi_0(t)| \approx \frac{1}{\sqrt{T}\pi(t/T)}, \quad \alpha = 0.$$

1 2

If $\alpha = 0$ then the decay with time is inverse with t but if $\alpha > 0$ the decay with time is inverse with t^2. The faster rolloff for $\alpha > 0$ corresponds to a larger bandwidth, as will be seen later in this chapter. Note that the zero crossings occur at different times for different values of α. Figure 2.27 shows the time domain signals $\varphi_0(t)$ for different values of α. Also for larger values of α the signal decays faster in time. This can be seen in Figure **??** where the signal is plotted in dB scale on a log plot for time. The asymptotic decay, shown as the dashed line in Figure 2.28, has slope that is either 2 (for $\alpha > 0$) or 1 (for $\alpha = 0$). Using the asymptotic decay we can roughly determine the time duration for a truncated pulse so that the part that is truncated is below some threshold. For example, suppose we desire the truncated part to be more than 40dB below the peak value and $\alpha > 0$. For large t

$$
\begin{aligned}
\frac{|\varphi_0(t)|}{\varphi_0(0)} &\approx \frac{\frac{1}{\sqrt{T}4\pi\alpha(t/T)^2}}{\frac{\pi(1-\alpha)+4\alpha}{\sqrt{T}\pi}} \\
&= \frac{1}{4\alpha(t/T)^2[\pi(1-\alpha)+4\alpha]} \\
20\log_{10}\left(\frac{|\varphi_0(t)|}{\varphi_0(0)}\right) &< -40 \\
\log_{10}(4\alpha(t/T)^2[\pi(1-\alpha)+4\alpha]) &> 2 \\
4\alpha(t/T)^2[\pi(1-\alpha)+4\alpha]) &> 100 \\
(t/T)^2 &> \frac{100}{4\alpha[\pi(1-\alpha)+4\alpha]} \\
(t/T) &> \frac{10}{\sqrt{4\alpha[\pi(1-\alpha)+4\alpha]}}
\end{aligned}
$$

$\alpha = 0.5$, $\frac{10}{\sqrt{2(\pi\frac{1}{2}+2)}}$ $= 3.2821$

For $\alpha = 0.25$ this means that truncation can happen after about a total 11 symbol durations . For $\alpha = 0.50$ truncation can happen after 7.5 symbols total. In order to minimize the sidelobes in the frequency domain created by truncation, the pulse should be truncated as close to a zero crossing as possible. For $\alpha = 0.25$ a zero crossing at about $5.18T$ on either side of the pulse peak value will be such that the part of the signal that is truncated is smaller than 40dB below the peak value. The power spectral density for this truncation is shown in Section 3.5. For larger α the time till the pulse is smaller than some value decreases but the bandwidth of the signal increases. Of course, in truncating these pulses, the waveform are no longer orthogonal. As such there will be intersymbol interference from one symbol to another.

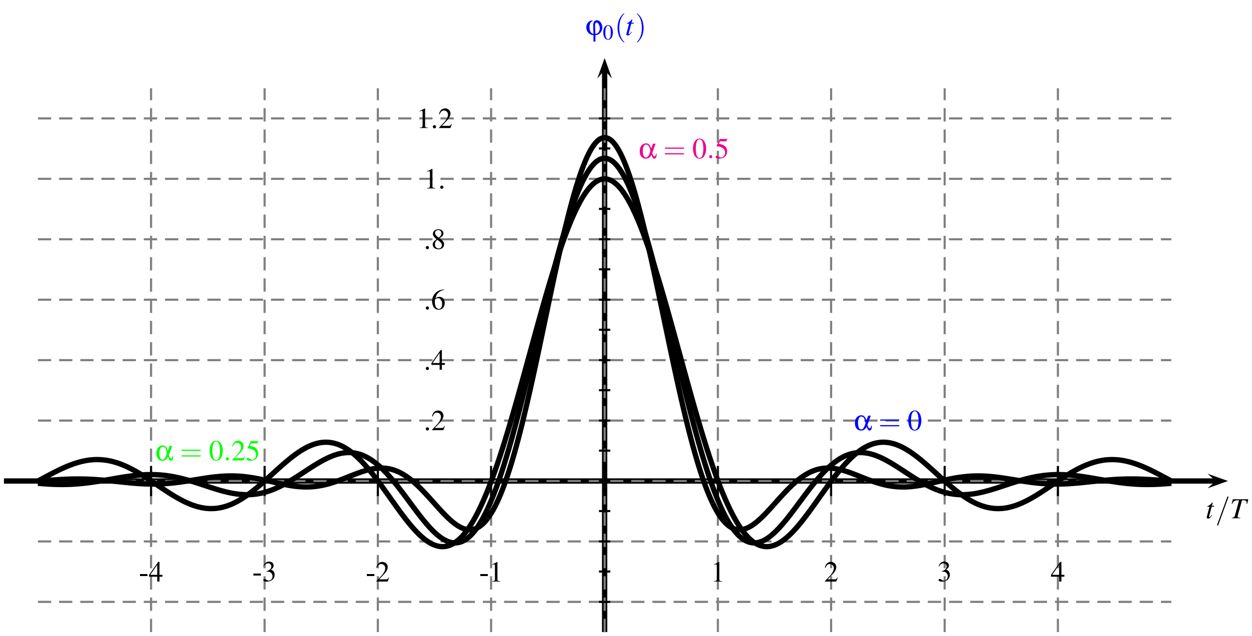


Figure 2.27: Square-root raised cosine pulses for Example 12

The signal vectors in this example are all vectors of length N with components being either $+\sqrt{E/N}$ or $-\sqrt{E/N}$ so that the energy of the signals is E. There are $M = 2^N$ such vectors. In the case that $N = 3$

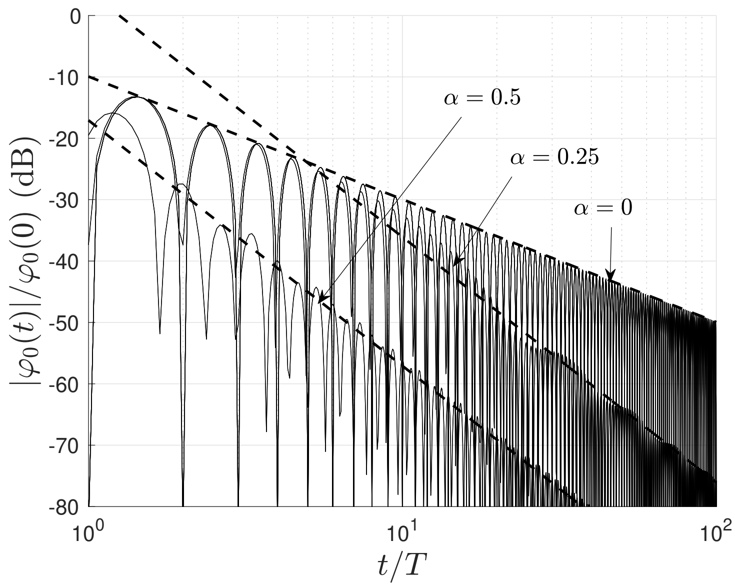


Figure 2.28: Square-root raised cosine pulses for Example 12.

the $M = 8$ signal vectors are

$$\begin{aligned}
s_0 &= \sqrt{E/3}(+1,+1,+1), \\
s_1 &= \sqrt{E/3}(+1,+1,-1), \\
s_2 &= \sqrt{E/3}(+1,-1,+1), \\
s_3 &= \sqrt{E/3}(+1,-1,-1), \\
s_4 &= \sqrt{E/3}(-1,+1,+1), \\
s_5 &= \sqrt{E/3}(-1,+1,-1), \\
s_6 &= \sqrt{E/3}(-1,-1,+1), \\
s_7 &= \sqrt{E/3}(-1,-1,-1).
\end{aligned}$$

The signals are

$$s_m(t) = \sum_{n=0}^{2} s_{m,n}\phi_n(t), \quad m = 0,1,...,7.$$

The eight signals are shown in Figure 2.29.

One advantage of these signals is the continuity of the signals. There are no abrupt (instantaneous) changes in the signals. This reduces the bandwidth needed compared to signals that are discontinuous, such as Example 3. One disadvantage is that the signals do not have constant envelope which makes it more difficult to amplify in an energy efficient manner. The peak-to-average power ratio of these signals when N is large is shown in Figure 2.30. Small values of α have larger peak-to-average power ratio but small bandwidth. For values of $\alpha < 0.1$ the data bits yielding the largest PAPR are alternating bits (+1, -1) with the middle two bits being the same sign. For example, the signal vector

$$s_0 = \sqrt{E/N}(+1,-1,+1,-1,....,+1,-1,+1,+1,-1,+1,...,-1,+1,-1,+1)$$

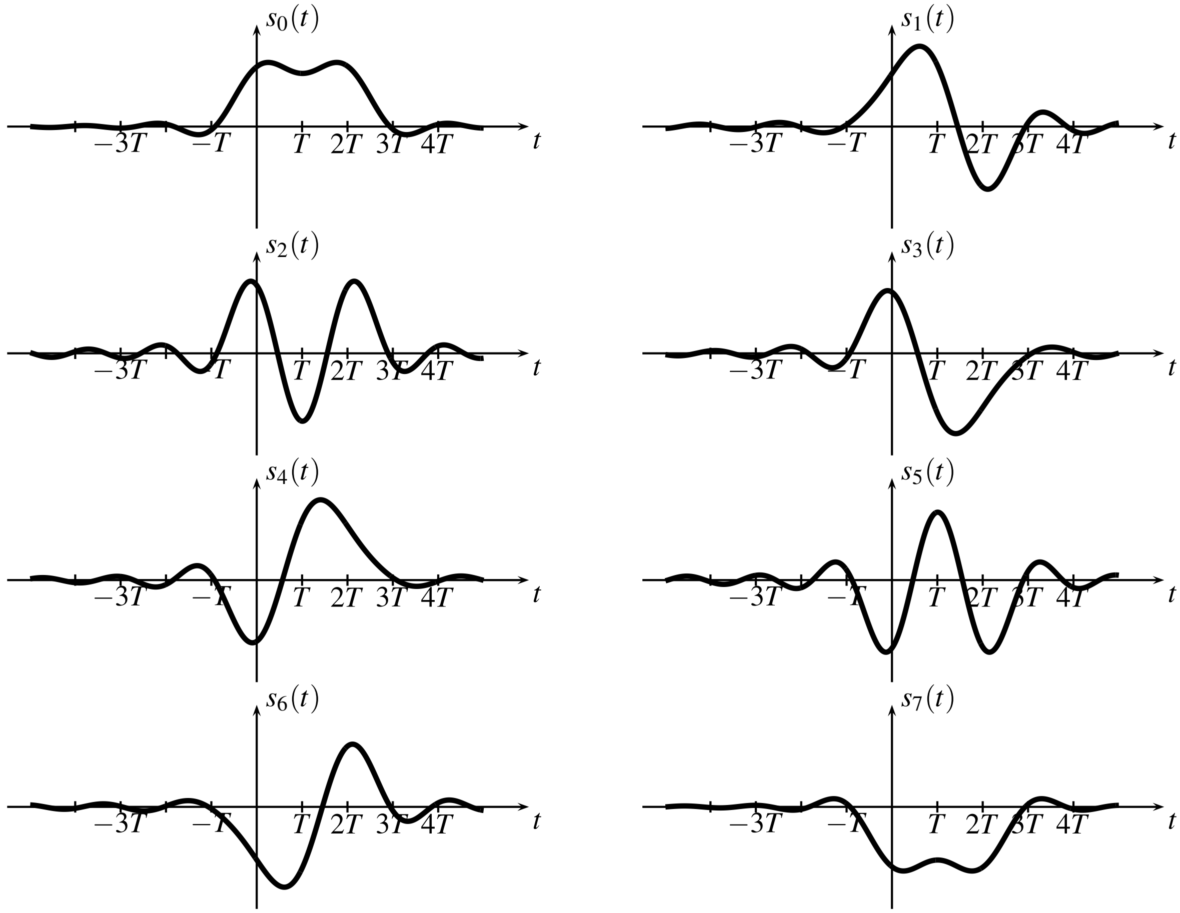


Figure 2.29: Waveforms $s_0(t)$,..., $s_7(t)$ for Example 12 ($\alpha = 0.35$).

with the square-root raised cosine pulses for small α would yield a large PAPR. In calculating the PAPR the peak power is found by looking at all time and over all possible transmitted data bits. In reality the data sequence described above which achieves the peak power has extremely small probability when the number of bits is large (e.g. > 100). For α near 0 and for most data sequences the peak over time of the power is within a dB of the average over all sequences of the peak over time of about 4dB. That is, with high probability the peak over time of the power is between 3dB and 5dB depending on the data sequence. The smaller the length of the data sequences the smaller the peak power over time and over the data sequence.

2.4 Orthogonal Signals from Arbitrary Signals: Gram-Schmidt procedure

We have shown how to construct a set of M signals from a set of M vectors and a set of N orthonormal waveforms. It is also possible to start with a set of M signals and generate a set of M vectors and a set of N orthonormal waveforms. This procedure is called the Gram-Schmidt procedure.

Given a set of signals $s_0(t),...,s_{M-1}(t)$ there exists a set of orthonormal waveforms $\varphi_0(t),\varphi_1(t),...,\varphi_{N-1}(t)$

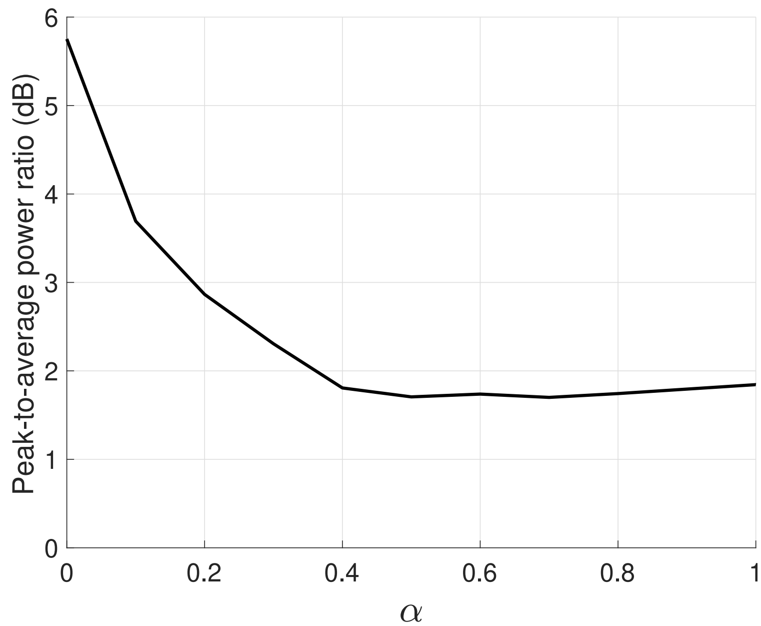


Figure 2.30: Peak-to-average power ratio for square-root raised cosine pulses.

with $N \leq M$ such that

$$s_m(t) = \sum_{n=0}^{N-1} s_{m,n}\varphi_n(t).$$

The coefficients are determined as

$$s_{m,n} = \int s_m(t)\varphi_n^*(t)dt.$$

These orthogonal signals can be determined using a procedure known as the Gram-Schmidt orthogonalization procedure. This procedure is described by the following steps.

Gram-Schmidt: Step 1 The first step of the Gram-Schmidt procedure is to normalize the first signal to find the first orthonormal waveform.

$$\begin{aligned} u_0(t) &= s_0(t) \\ \varphi_0(t) &= u_0(t)/||u_0(t)||. \end{aligned}$$

The first coefficient is

$$s_{0,0} = (s_0(t), \phi_0(t)) = \int s_0(t)\varphi_0^*(t)dt = \left(\int |s_0(t)|^2 dt\right)^{1/2}.$$

Gram-Schmidt: Step 2 To find the second orthonormal waveform we need to find the part of the second signal $s_1(t)$ in the direction of the first orthonormal waveform $\varphi_0(t)$. This is the correlation $s_{1,0}$ of $s_1(t)$ with $\varphi_0(t)$. This component is then subtracted from the second signal to find the part of $s_1(t)$

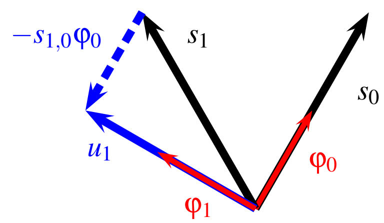


Figure 2.31: Gram-Schmidt orthogonalization.

that is orthogonal to the first orthonormal waveform $\varphi_0(t)$.

$$\begin{aligned} s_{1,0} &= (s_1(t), \varphi_0(t)) \\ u_1(t) &= s_1(t) - s_{1,0}\varphi_0(t) \\ \varphi_1(t) &= u_1(t)/||u_1(t)|| \\ s_{1,1} &= (s_1(t), \varphi_1(t)). \end{aligned}$$

The waveform $s_{1,0}\varphi_0(t)$ is the component of $s_1(t)$ in direction of $\varphi_0(t)$ as shown in Figure 2.31

Gram-Schmidt: Step 3

$$\begin{aligned} s_{2,0} &= (s_2(t), \varphi_0(t)) \\ s_{2,1} &= (s_2(t), \varphi_1(t)) \\ u_2(t) &= s_2(t) - s_{2,1}\varphi_1(t) - s_{2,0}\varphi_0(t) \\ \varphi_2(t) &= u_2(t)/||u_2(t)|| \\ s_{2,2} &= (s_2(t), \varphi_2(t)) \end{aligned}$$

Gram-Schmidt: Step 4

$$\begin{aligned} s_{3,0} &= (s_3(t), \varphi_0(t)) \\ s_{3,1} &= (s_3(t), \varphi_1(t)) \\ s_{3,2} &= (s_3(t), \varphi_2(t)) \\ u_3(t) &= s_3(t) - s_{3,2}\varphi_2(t) - s_{3,1}\varphi_1(t) - s_{3,0}\varphi_0(t) \\ \varphi_3(t) &= u_3(t)/||u_3(t)|| \\ s_{3,3} &= (s_3(t), \varphi_3(t)) \end{aligned}$$

This process continues until every signal can be represented by a linear combination of the orthogonal signals found. It may be the case that fewer orthogonal signals are needed then the number of original signals. For example if $s_0(t) = Ks_1(t)$ where K is a constant then only one orthogonal signal is needed to represent both $s_0(t)$ and $s_1(t)$.

Example 13:

Consider the following set of four signals shown in Figure 2.32. The unnormalized signals used in the

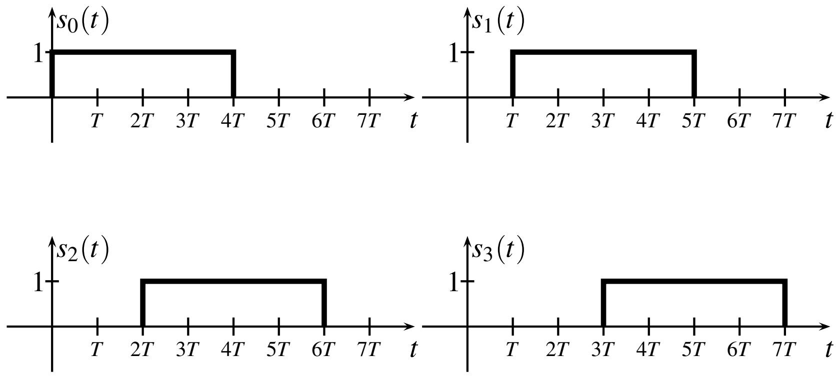


Figure 2.32: Example 13 signals.

Gram-Schmidt procedure are show in Figure 2.33. The normalized orthonormal waveforms from the Gram-Schmidt procedure are show in Figure 2.34.

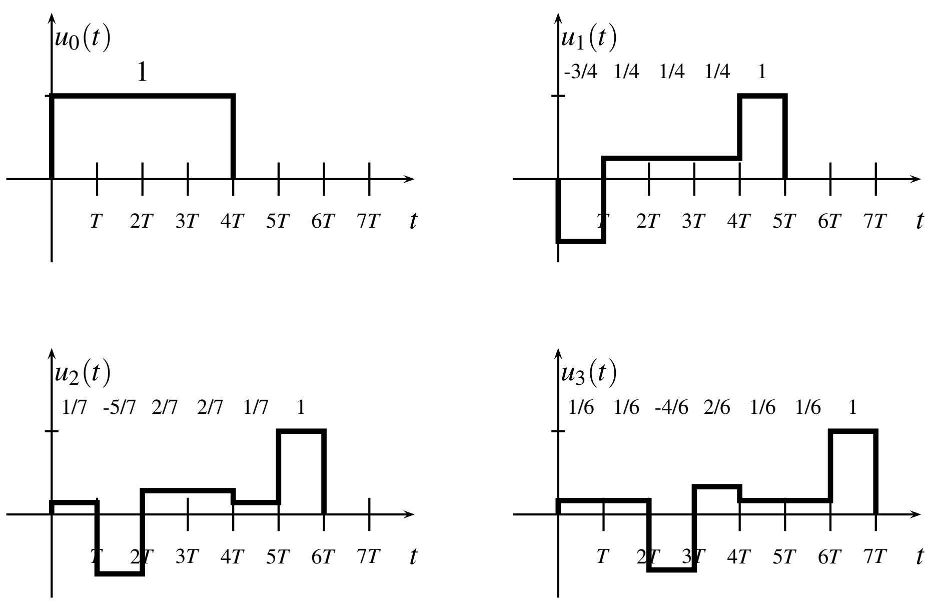


Figure 2.33: Orthogonal basis functions for Example 13.

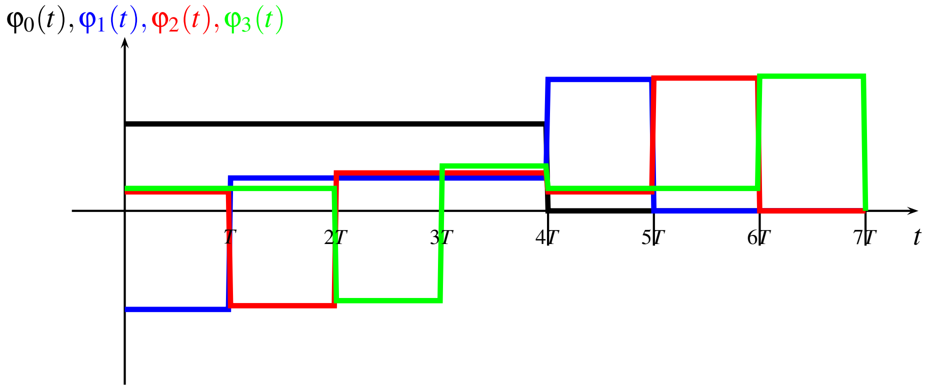


Figure 2.34: Orthonormal waveforms for Example 13.

The orthonormal representation of the signals is as follows.

$$\begin{aligned}
s_0(t) &= 2.00\varphi_0(t)+0.00\varphi_1(t)+0.00\varphi_2(t)+0.00\varphi_3(t)\\
s_1(t) &= 1.50\varphi_0(t)+1.32\varphi_1(t)+0.00\varphi_2(t)+0.00\varphi_3(t)\\
s_2(t) &= 1.00\varphi_0(t)+1.13\varphi_1(t)+1.31\varphi_2(t)+0.00\varphi_3(t)\\
s_3(t) &= 0.50\varphi_0(t)+0.94\varphi_1(t)+1.09\varphi_2(t)+1.29\varphi_3(t)
\end{aligned}$$

$S_n(t) = \sum S_{m,n} \varphi_n(t)$

2.5 Filtering: Time-Domain Analysis

Often a large number of signal vectors are needed with a large number of orthonormal waveforms. Shannon's theorem gives a hint that large dimensions are a way of getting the error probability to become small. Figure 2.4 shows one way of generating a signal waveform from a signal vector and a set of orthonormal waveforms. Figure 2.5 shows one way of determining the coefficients of the orthonormal expansion of a signal. As shown there, for each orthonormal waveform there is a correlator (multiply and integrate). We would like a very large number of orthogonal waveforms but having a correlator for each waveform makes the complexity large. For example suppose we have 200 orthogonal signals that are just time shifts of a rectangular pulse. This would require 200 correlators. Figure 2.35 shows nine such pulses.

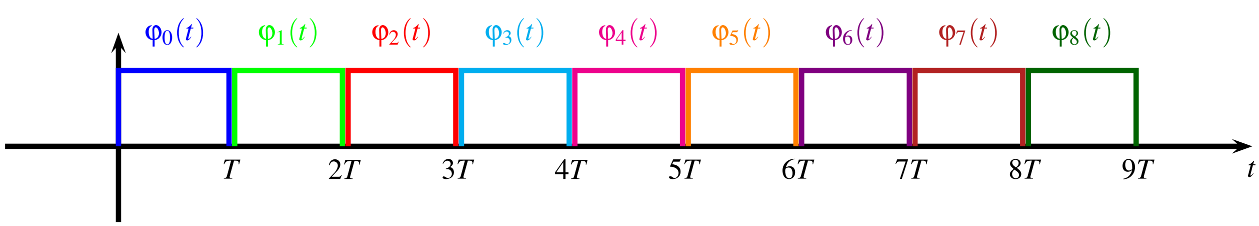


Figure 2.35: Time-orthogonal pulses.

filter 是干啥的

①

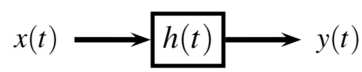
We need to find a way of having a large number of orthogonal waveforms but reduced complexity (fewer correlators). This is where filtering and sampling combined with time-shifted orthogonal wave-

(2)

forms is useful. Also, filtering is a way to remove as much of the noise as possible without removing much of the desired signal. At the transmitter we need to generate a large number of possible signals. For time-shifted orthonormal waveforms there is a simple way to generate the signals from the signal vector using a single filter which we describe below. In most receivers in a digital communication system the received signal is filtered before a decision is made as to the data bit that is transmitted. The purpose of filtering is to remove as much of the noise as possible while removing as little of the signal as possible. So it is important to understand how a signal is affected or changed by a filter and how noise is affected by a filter. The filters often used in practical systems are most conveniently modeled as linear, time-invariant filters. Linear systems have the property that, if when the input is $x_1(t)$ the output is $y_1(t)$ and when the input is $x_2(t)$ the output is $y_2(t)$, then the output when the input is $\alpha x_1(t) + \beta x_2(t)$ is $\alpha y_1(t) + \beta y_2(t)$. Time-invariant systems have the property that if $x(t)$ is the input the output is $y(t)$, then the output when the input is $x(t-\tau)$ is $y(t-\tau)$.

什么是 Time-invariant?
Time-invariant 的性质

The operation of filtering is a modification of the input signal to produce an output signal. A diagram representing a filter is typically drawn as shown in Figure 2.36. The input shown is $x(t)$, the output is $y(t)$ and $h(t)$ is the impulse response of the filter. Mathematically, the output of a linear time-invariant



impulse response

Figure 2.36: Filtering.

Convolution of input signals x(t) with impulse response h(t)

filter or system is the convolution of the input signal with the impulse response of the filter. That is, if the input to the filter is the signal $x(t)$ and the impulse response of the filter is $h(t)$ then the output of the filter $y(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} h(t-\alpha)x(\alpha)d\alpha$$

The above mathematical operation on $x(t)$ and $h(t)$ is called the convolution of h with x written as $h * x$. For each value of t the output, $y(t)$, is the integral of $x(\alpha)h(t-\alpha)$ with respect to α. Thus convolution is just doing an integral to determine the output for each value of time.

2.5.1 Filtering applied to time-shifted orthonormal waveforms

Consider a set of signals constructed from time-shifted orthonormal waveforms. The signals in the signal set are linear combinations of the orthonormal waveforms.

$$s_m(t) = \sum_{n=0}^{N-1} s_{m,n}\varphi_n(t).$$

Assume that the orthonormal waveforms used are time shifts of each other, such as Example 9 or 11 earlier. That is,

$$\varphi_n(t) = \varphi_0(t-nT), n = 1,2,\ldots,N-1$$

where $\varphi_0(t)$ is an energy 1 signal. Because $\{\varphi_n(t), n = 0,1,\ldots,N-1\}$ is an orthonormal waveform set we have that

$$\int \varphi_n(t)\varphi_m^*(t)dt = \delta_{n,m}.$$

$$(\varphi_n(t), \varphi_l(t)) = \int_I \varphi_n(t)\varphi_l^*(t)\,dt = \begin{cases} 1, & n=l \\ 0, & n\neq l \end{cases}$$

I = time interval

h x

Fist consider a filter with impulse response $h_T(t) = \varphi_0(t)$. If the input to that filter is a sequence of impulses with amplitudes corresponding to the coefficients of the signal vector then the output is the desired signal. That is, let

$$d_m(t) = \sum_{n=0}^{N-1} s_{m,n}\delta(t - nT).$$

Then when $d_m(t)$ is filtered by $h_T(t)$ the result is

$$\begin{aligned}
d_m * h(t) &= \int_\tau d_m(\tau)h(t-\tau)d\tau \\
&= \int_\tau \sum_{n=0}^{N-1} s_{m,n}\delta(\tau - nT)\varphi_0(t-\tau)d\tau \\
&= \sum_{n=0}^{N-1} s_{m,n} \int_\tau \delta(\tau - nT)\varphi_0(t-\tau)d\tau \\
&= \sum_{n=0}^{N-1} s_{m,n}\varphi_0(t - nT) \\
&= \sum_{n=0}^{N-1} s_{m,n}\varphi_n(t) \\
&= s_m(t).
\end{aligned}$$

So for time-shifted orthonormal waveforms a transmitter can generate the signal using a single filter with impulse response $h_T(t) = \varphi_0(t)$ as shown in Figure 2.37.

$$d_m(t) = \textstyle\sum_n s_{m,n}\delta(t-nT) \longrightarrow \boxed{h_T(t) = \varphi_0(t)} \longrightarrow s_m(t)$$

$d_m(t) * h_T(t)$

Figure 2.37: Generating signals for time-shifted orthonormal waveforms.

Now consider a filter "matched" to $\varphi_0(t)$. That is, $h_R(t) = \varphi_0^*(-t)$. The filter impulse response is the time flip of the signal $\varphi_0(t)$. The output of the filter is

$$\begin{aligned}
y(t) &= \int h_R(t-\tau)s_m(\tau)d\tau \\
&= \int \varphi_0^*(-(t-\tau))s_m(\tau)d\tau \\
&= \int \varphi_0^*(\tau - t)s_m(\tau)d\tau.
\end{aligned}$$

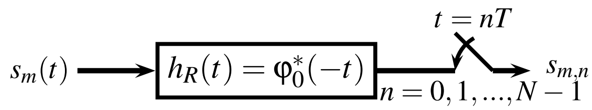


Figure 2.38: Generating signal vector components for time-shifted orthonormal waveforms.

Consider sampling the output $y(t)$ at time $t = 0, T, 2T, \ldots, (N-1)T$. The sampled output is then

$$\begin{aligned}
y(nT) &= \int \varphi_0^*(\tau - nT) s_m(\tau) d\tau \\
&= \int \varphi_0^*(\tau - nT) \sum_{k=0}^{N-1} s_{m,k} \varphi_k(\tau) d\tau \\
&= \sum_{k=0}^{N-1} s_{m,k} \int \varphi_0^*(\tau - nT) \varphi_k(\tau) d\tau \\
&= \sum_{k=0}^{N-1} s_{m,k} \int \varphi_n^*(\tau) \varphi_k(\tau) d\tau \\
&= \sum_{k=0}^{N-1} s_{m,k} \delta_{n,k} \\
&= s_{m,n-1}.
\end{aligned}$$

So the samples at times $0, T, 2T, \ldots, (N-1)T$ correspond to the coefficients of the signal composition $s_{m,0}, \ldots, s_{m,N-1}$. The samples of the filter output are exactly the coefficients in the signal. Note that the filter with impulse response $h_R(t) = \varphi^*(-t)$ is not a causal filter for the case that $\varphi_0(t) \neq 0$ for $t > 0$ because then $h_R(t) = \varphi(-t) \neq 0$ for $t < 0$. As such typically a delay is added to the filter. For example if the signal $\varphi_0(t)$ is only nonzero in the interval $0 \leq t \leq T$ then a delay of T will make the filter $h_R(t) = \varphi^*(T-t)$ causal. For example, if $h_R(t) = \varphi_0^*(T-t)$ then the outputs at times $T, 2T, \ldots, NT$ would have corresponded to the coefficients $s_{m,0}, \ldots, s_{m,N-1}$. The impulse response will be causal if $\varphi(t) = 0$ for $t > T$. So matched filtering a signal composed of a linear combination time-shifted orthonormal waveforms will recover the coefficients of the signal. Using signals constructed from time-shifted orthonormal waveforms allows us to recover the coefficients of the signals from samples of a single filter as shown in Figure 2.38.

2.5.2 Properties of LTI Systems Linear Time Invariant

The above property shows we can recover the signal vector from a set of signal waveforms generated from signal vectors and time-shifted orthonormal waveforms by a set of samples from a single filter matched to the orthonormal waveforms. However, to understand synchronization and to analyze the case of non-ideal channels it is important to understand the continuous output of a filter. To this end we examine several properties of LTI systems that are useful in calculating the output of a linear time-invariant system.

We consider a filter with impulse response $h(t)$ with input $x(t)$ and output $y(t)$ as shown in Figure 2.36.

Properties

1. Linearity-Part A: If an input $x_1(t)$ to a filter with impulse response $h(t)$ results in an output of $y_1(t)$ and an input $x_2(t)$to the same filter results in an output $y_2(t)$ then the input $\alpha_1 x_1(t)+\alpha_2 x_2(t)$ to the same filter results in an output $\alpha_1 y_1(t)+\alpha_2 y_2(t)$.

2. Linearity-Part B: If an input $x(t)$ into a filter with impulse response $h_1(t)$ results in an output of $y_1(t)$ and an input $x(t)$ into a filter with impulse response $h_2(t)$ results in an output $y_2(t)$ then the input $x(t)$ into a filter with impulse response $\alpha_1 h_1(t)+\alpha_2 h_2(t)$ results in an output $\alpha_1 y_1(t)+\alpha_2 y_2(t)$.

3. Time Invariance: If $x(t)$ results in an output of $y(t)$ then $x(t-\tau)$ results in an output of $y(t-\tau)$.

4. If the input to a LTI system is a rectangular pulse and the impulse response is a rectangular pulse then the output is a piecewise linear function. In other words, the convolution of two rectangular pulses results in a piece-wise linear output. If the two rectangular pulses have identical durations then the output is a triangular pulse.

5. If the impulse response is a rectangular pulse $h(t) = p_T(t)$ then the output of the filter at time t is the integral of the input over the last T seconds. That is,

$$\begin{aligned} h(t) &= p_T(t) \\ y(t) &= \int_{t-T}^{t} x(\tau)d\tau. \end{aligned}$$

   This is called a sliding integrator.

6. If $h(t) = \varphi^*(T-t)$ where $\varphi(t)$ is zero outside the interval $[0,T]$ then the output of the LTI system at time T is a correlation of the input with $\varphi^*(t)$. That is

$$y(T) = \int_0^T x(\tau)\varphi^*(\tau)d\tau.$$

7. When the impulse response of the filter is matched to the input signal $h(t) = x^*(-t)$, the output is the autocorrelation of the input.

$$\begin{aligned} h(t) &= x(-t) \\ y(t) &= \int x(\tau)h(t-\tau)d\tau \\ &= \int x(\tau)x^*(\tau-t)d\tau \end{aligned}$$

Example 14:

In this example $x(t)$ is a rectangular pulse of duration T and amplitude $\sqrt{P}$ and $h(t)$ is a rectangular pulses of amplitude $\sqrt{\frac{1}{T}}$ and duration T beginning at $t = 0$. Here P represents the power of the signal. The filter is normalized to have energy of the impulse response equal to 1. The signal and filter impulse response are shown in Figure 2.39.

$$x(t) = \sqrt{P}p_T(t) = \begin{cases} \sqrt{P}, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$$
$$h(t) = \sqrt{\frac{1}{T}}p_T(t) = \begin{cases} \sqrt{\frac{1}{T}}, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$$

The output of the filter with rectangular impulse response of duration T to an input that is also rectangular

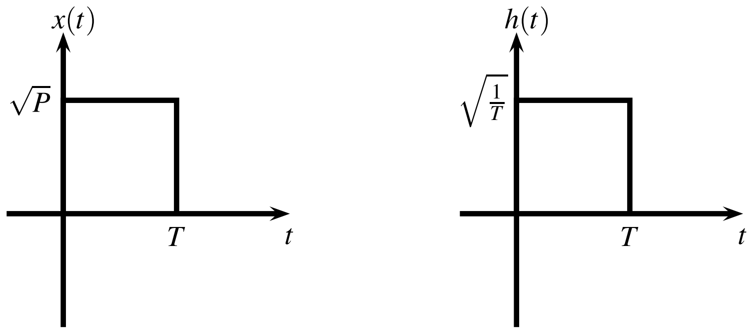


Figure 2.39: Rectangular pulses.

of duration T is a triangular pulse of duration $2T$. This is shown in 2.40. Convolution of a rectangular

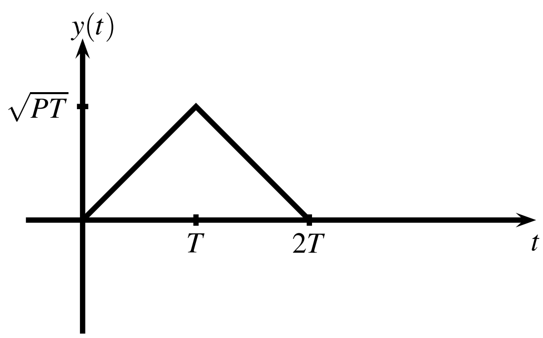


Figure 2.40: Triangular pulse: Convolution of a rectangular pulse with a rectangular pulse.

pulse with a rectangular pulse yields a triangular pulse. Note that the duration of the output that is not zero is the sum of the durations of the input and the impulse response. In this example those two durations are the same. The peak output amplitude is $\sqrt{PT} = \sqrt{E}$. The use of a normalized filter impulse response (energy 1) is a convenient normalization as will be seen later.

Example 15: Input of two time-shifted pulses

Consider a filter with impulse response that is a rectangular pulse of duration T as shown in Figure 2.41. Find the output due to a sequence of rectangular pulses. If the input is shifted in time, the output is shifted in time. The output due to the sum of two inputs is the sum of the two outputs. This is shown in Figure 2.42.

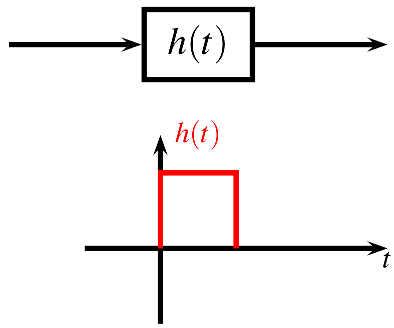


Figure 2.41: Filter with rectangular impulse response.

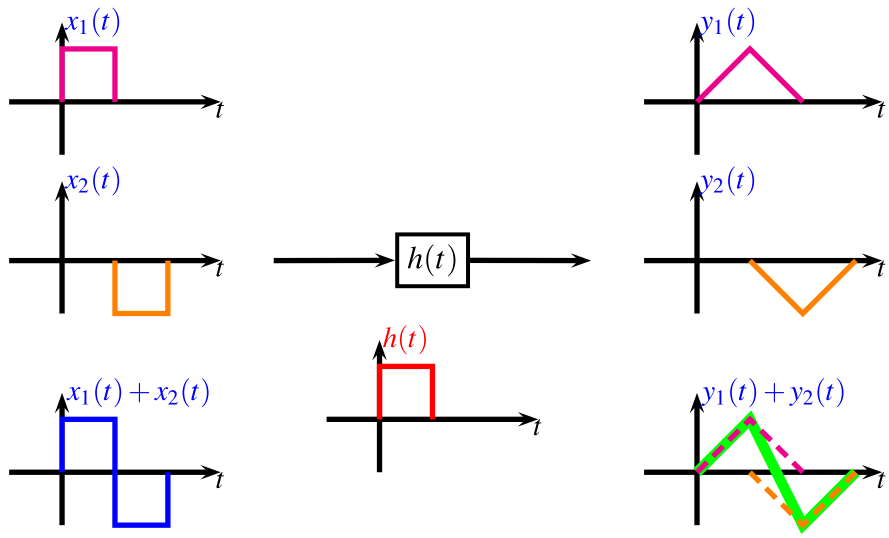


Figure 2.42: Filter with rectangular impulse response and two input pulses.

Thus this filter applied to the signals of Example 2 above when sampled at the appropriate time produces the signal vector.

Example 16: Input of three pulses

Consider a filter with impulse response that is rectangular pulse of duration T. Find the output due to a sequence of rectangular pulses. If the input is shifted in time, the output is shifted in time. The output due to the sum of two inputs is the sum of the two outputs. This is shown in Figure 2.43.

Notice that the output at time T is the correlation of the input with a rectangular pulse. That is, from Example 3 in Section 2.2, the filter output at time T is the same as performing the correlation with $\varphi_0(t)$ from example 1. Similarly, the output at time $2T$ is the same as the correlation with $\varphi_1(t)$ from example 1. Thus filtering with a rectangular pulse and sampling can provide the correlations needed to go from a signal to a vector.

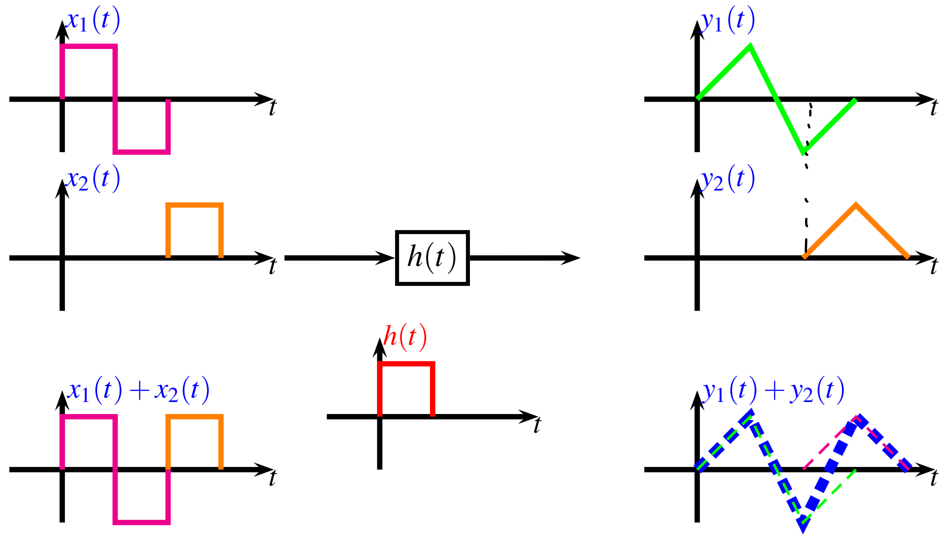


Figure 2.43: Filter with rectangular impulse response and three input pulses.

Example 17: Input of 5 pulses into a matched filter

Consider a signal consisting of a sequence of pulses.

$h(t) = P_T(t)$

$y(t) = \int_{t-T}^{t} x(\tau)\, d\tau$

$$x(t) = \sum_{n=0}^{4} a_n p_T(t)$$

where $\mathbf{a} = (a_0, a_1, ..., a_4) = (+1, +1, +1, -1, +1)$. Suppose the filter is matched to $x(t)$. In that case

$$h(t) = x(5T - t) = \sum_{n=0}^{4} h_n p_T(t)$$

where $\mathbf{h} = (h_0, h_1, ..., h_4) = (+1, -1, +1, +1, +1)$. In this case $h_n = a_{4-n}$ and $\mathbf{h}$ is the reverse of $\mathbf{a}$ The output of the filter can be derived using several of the properties of linearity (both Parts A and B), time-invariance and the Property 5. The output of the first input pulse with a single pulse of the filter will yield a triangular pulse. By combining multiple delayed pulses with the sign inverted we end up with an output that is still piecewise linear. Figure 2.44 shows the signals $x(t)$, $h(t)$ and $y(t)$. The output has a peak value of $5T$ at time $5T$. This is because at time $5T$ $h(5T - \tau) = x(t)$ and thus the product is just 1 which we integrate from 0 to $5T$ to determine the output. The part of the output between $4T$ and $6T$ is called the main lobe while the other parts are called sidelobes. The sequence used to generate $x(t)$ is called a Barker sequence. It has the property that the sidelobes have absolute value of the amplitude less than $1T$ or, if we normalize the output by T the magnitude of the output is limited to 1. A signal with very narrow and high main lobe with low sidelobes is desired for various purposes. One purpose is timing. By transmitting such a signal the receiver will be able to tell where a particular symbol starts or ends. A few sequences with good autocorrelation properties are discussed in Appendix A. As discussed there the autocorrelation above is the aperiodic autocorrelation. The other autocorrelation is the periodic autocorrelation. In the periodic autocorrelation the signal $x(t)$ is repeated (e.g. every $5T$) to create an infinite length signal. The periodic autocorrelation is the result of filtering the repeated (periodic) signal with the same filter. See Appendix A.

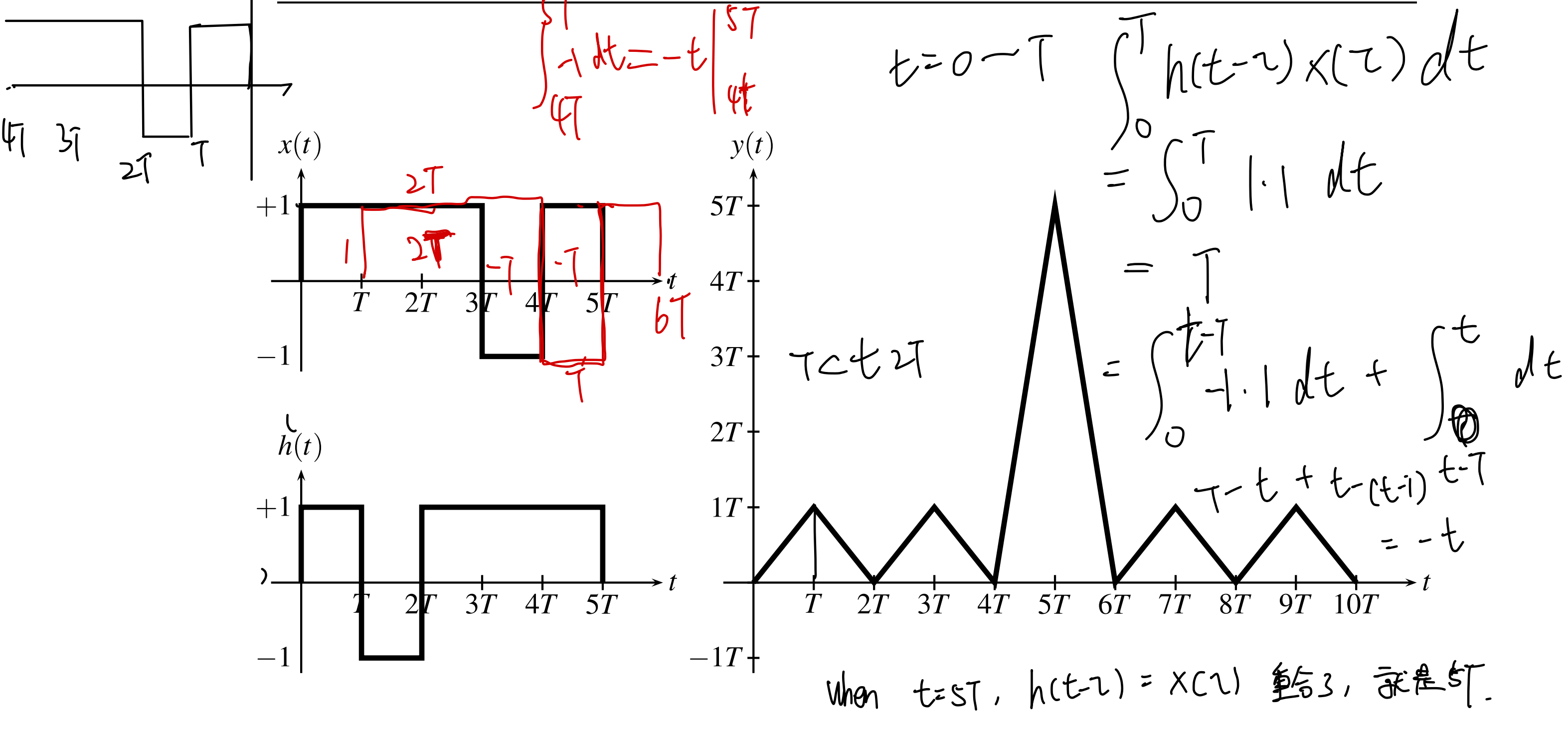


Figure 2.44: Input $x(t)$, impulse response $h(t)$, filter output $y(t)$ for Example 17

Notice that the output duration of the signal is the sum of the input duration and the impulse response duration. For the case where the input is a sequence of rectangular pulses and the impulse response is also a sequence of rectangular pulses the output is a piece-wise linear function. For a single rectangular pulse of the input and the impulse response the output is a triangular wave.

Example 18: Filtering square-root raised cosine pulses

Now suppose that the input to a filter is a single square-root raised cosine pulse. The filter impulse response is also a square-root raised cosine pulse. Then the output is a raised cosine pulse.

$$
\begin{aligned}
x(t) = h(t) &= \frac{1}{\sqrt{T}} \frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T \cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T} \\
y(t) &= \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\alpha\pi t/T)}{1-4\alpha^2 t^2/T^2}.
\end{aligned}
$$

This is shown in the next section using results from Appendix 2B and 2C. If the input to a filter is a sum of delayed square-root raised cosine pulses, like the signal in Example 11 then the output is the sum of delayed raised cosine pulses. In Figure 2.45 the input of a square-root raised cosine signal, with the impulse response being a square-root raised cosine pulse and output being a raised cosine pulse are shown. In Figure 2.46 the input of a pulse and a delayed pulse are shown along with the corresponding outputs. In Figure 2.47 the combination of the two pulses at the input and the two pulses at the output are shown. In Figure 2.48 the response to five different inputs is shown. The inputs are shifted by T seconds from one another. Notice that when any one of these outputs reaches its peak the other outputs are zero. So if the total output was sampled every T there would only be a nonzero output for one of the pulses at the sampling time.

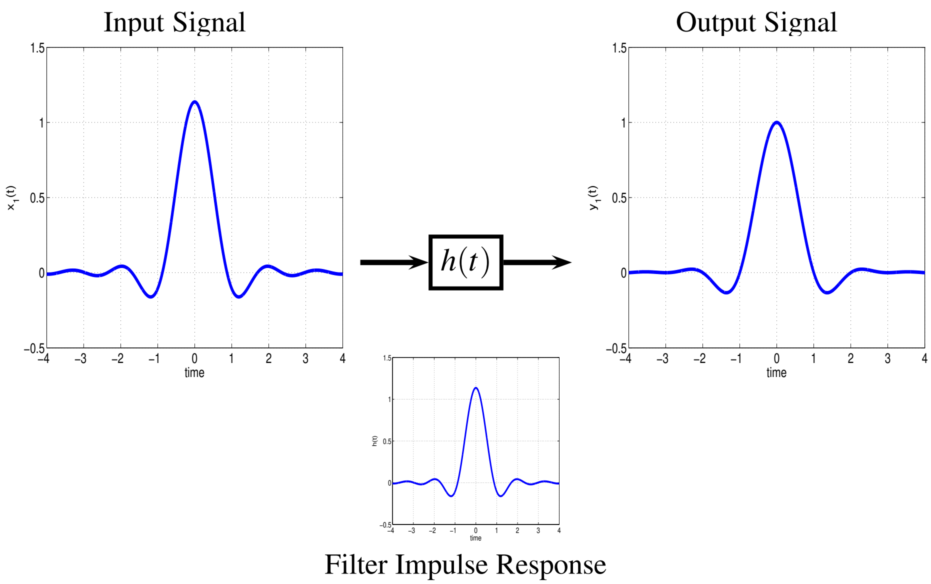


Figure 2.45: Root-raised cosine filter of root-raised cosine signals for one input pulse

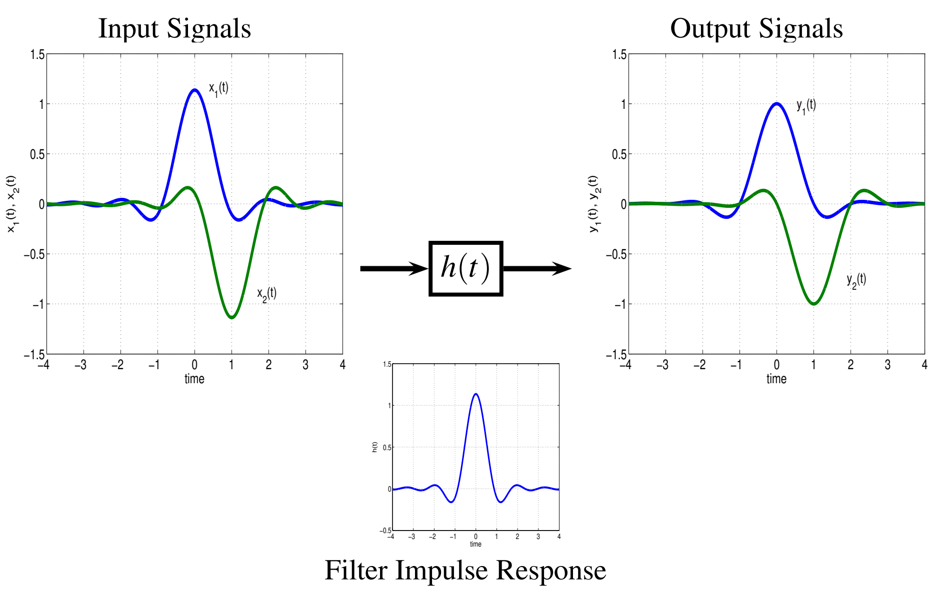


Figure 2.46: Root-raised cosine filter of root-raised cosine signals for two pulses.

2.6 Filtering: Frequency Domain Analysis

Signals and filtering can also be described in the frequency domain. The frequency content of a signal is obtained via the Fourier Transform.

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt.$$

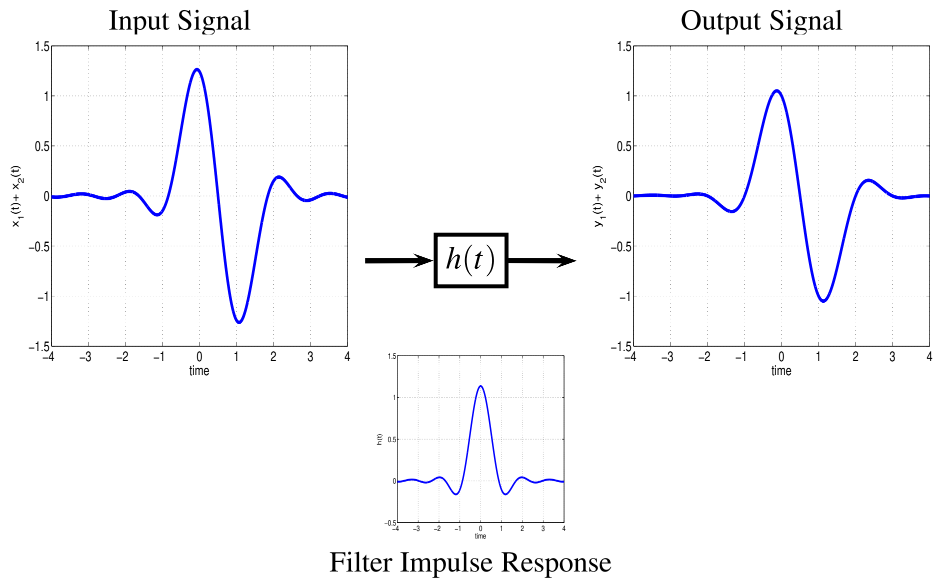


Figure 2.47: Root-raised cosine filter of root-raised cosine signals for the sum of two pulses

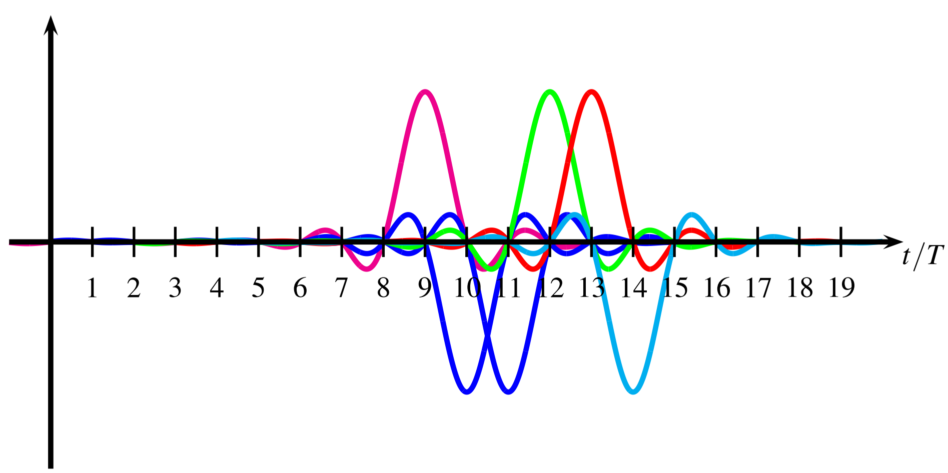


Figure 2.48: Response to five delayed inputs.

Convolution in the time domain corresponds to multiplication in the frequency domain and thus

$$y(t) = x(t) * h(t) \Leftrightarrow Y(f) = H(f)X(f).$$

One useful relation between the frequency domain and time domain is Parseval's Theorem (see Appendix 2E)

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)df.$$

As a special case when $x_1(t) = x_2(t) = x(t)$ and $X_1(f) = X_2(f) = X(f)$ we get

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Example 19: Rectangular Filtering of Rectangular Pulses

In this example the signal consists of a single pulse of duration T. A single data bit is transmitted by sending either a positive pulse to represent a 0 or a negative pulse of to represent a 1. The receiver decides which bit was transmitted by filtering the received signal with a filter matched to the transmitted signal and sampling the filter output. The signal at the filter input is

$$x(t) = p_T(t)$$

The filter has an impulse response that is also a pulse of duration T. That is

$$h(t) = x(T-t) = p_T(T-t) = p_T(t).$$

The output of the filter is then a triangular shape.

$$y(t) = h(t) * x(t) = \Lambda_T(t) = \begin{cases} t, & 0 \le t \le T \\ (2 - \frac{t}{T})T, & T \le t \le 2T \\ 0, & \text{otherwise} \end{cases}.$$

The output at time T is $y(T) = T$. In the frequency domain we have

$$\begin{aligned} X(f) &= T\operatorname{sinc}(fT)e^{-j\pi fT} \\ H(f) &= T\operatorname{sinc}(fT)e^{-j\pi fT} \\ &= T\frac{\sin(\pi fT)}{\pi fT}e^{-j\pi fT}. \\ Y(f) &= H(f)X(f) = T^2\operatorname{sinc}^2(fT)e^{-j2\pi fT}. \end{aligned}$$

Figure 2.49 shows the time and frequency domain for the transmitted pulse, the filter and the filter output.

Example 20: Square Root Raised Cosine Filtering of Square-Root Raised Cosine Pulses

Consider the orthonormal pulse of Example 11 for the signal and the filter. That is $x(t) = h(t) = h(-t)$ where

$$\begin{aligned} x(t) = h(t) &= \frac{1}{\sqrt{T}}\frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T\cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T} \\ y(t) &= \frac{\sin(\pi t/T)}{\pi t/T}\frac{\cos(\alpha\pi t/T)}{1-4\alpha^2t^2/T^2}. \end{aligned}$$

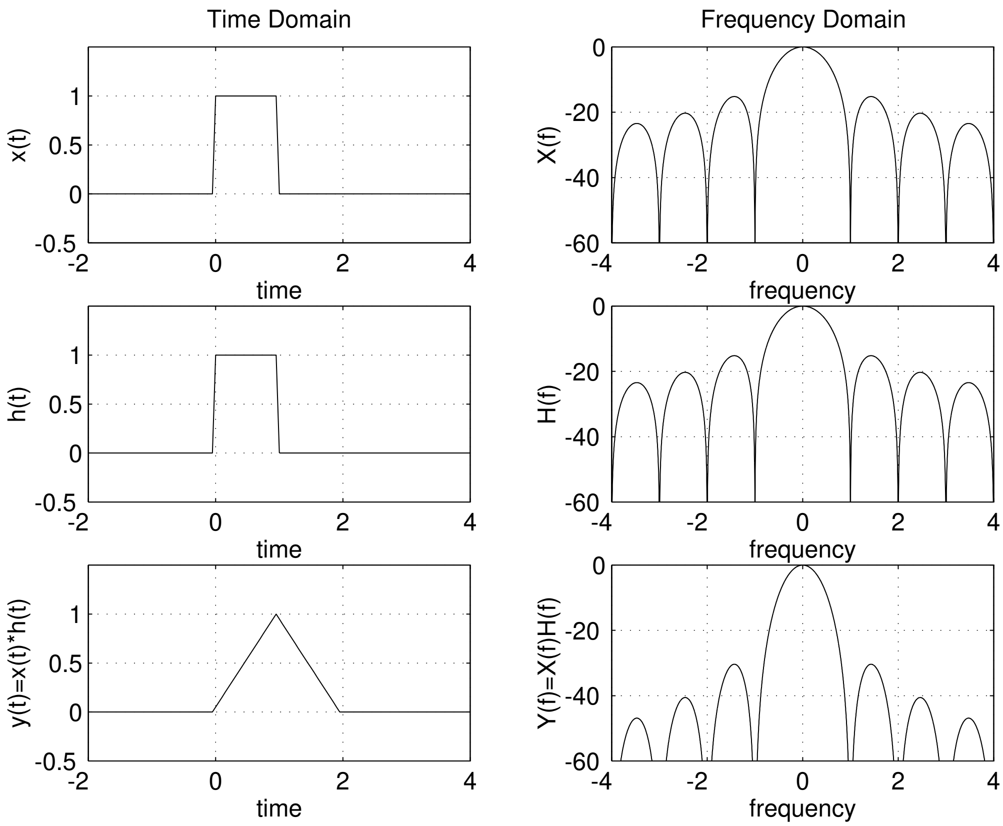


Figure 2.49: Time and frequency domain for rectangular pulses.

The frequency content of these signals can be expressed in several equivalent forms. In Appendix 2B we start with one of the forms given below and then derive the inverse Fourier transform that gives the time domain signal. The different forms for the frequency content are stated here. These can be derived from one another by simple trigonometric identities: namely $\sin(x) = \cos(x - \pi/2) = -\cos(x + \pi/2)$.

$$
\begin{aligned}
X(f) = H(f) &= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}[1 - \sin(\pi T(|f| - \frac{1}{2T})/\alpha)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}}\sqrt{[1 - \cos(\gamma|f| - \delta)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}}\sqrt{[1 + \cos(\frac{\pi T}{\alpha}(|f| - \frac{1-\alpha}{2T})]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Here $\gamma = \pi T/\alpha$ and $\delta = \pi(1+\alpha)/(2\alpha)$.

The frequency content of these signals is the square-root of a half cycle of a cosine shape that is raised (shifted vertically) to start at 1 at frequency $(1-\alpha)/(2T)$ Hz and goes to zero at frequency $(1+\alpha)/(2T)$. For this reason it is called a square-root raised cosine pulse. If $\alpha = 0$ then the frequency content is constant over the interval $[-1/(2T), 1/(2T)]$. Next we use the fact that $\cos(x) = 1 - 2\sin^2(x/2)$.

$$
\begin{aligned}
X(f) &= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}}\sqrt{[1 - (1 - 2\sin^2(\gamma/2|f| - \delta/2))]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{T}\sqrt{[\sin^2(\gamma/2|f| - \delta/2)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{T}[|\sin(\gamma/2|f| - \delta/2)|], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{T}[|\sin(\frac{\pi}{2\alpha}(T|f| - \frac{(1+\alpha)}{2}))|], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

The parameter α is called the roll-off factor and is between 0 and 1. The (absolute) bandwidth of the baseband signal is $W = (1+\alpha)/2T$. This last form is the starting point for deriving the time domain representation in Appendix 2B. Namely,

$$
x(t) = \frac{1}{\sqrt{T}}\left[\frac{\sin(\pi(1-\alpha)t/T) + (4\alpha t/T)\cos(\pi(1+\alpha)t/T)}{(1 - (4\alpha t/T)^2)\pi t/T}\right].
$$

Notice that the pulse in the time domain has infinite duration. Any implementation would naturally truncate the pulse shape. Truncating the pulse shape will change the frequency content of the signal. For example, if the pulse is truncated in time to contain the center lobe and three sidelobes on each side then the spectrum will not be limited in frequency but contain sidelobes as shown in Figure 2.50. In this example (with $\alpha = 0.25$) the newly created sidelobes in the frequency domain are about 30dB down compared to the main lobe. In addition, there is some ripple in the portion of the spectrum that is originally a constant.

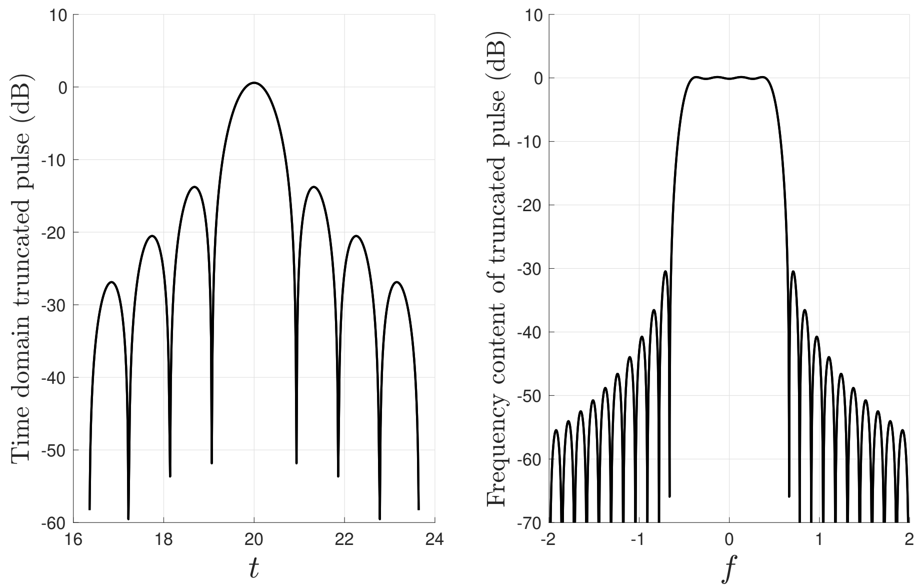


Figure 2.50: Time and frequency domain of truncated square-root raised cosine pulse

In the frequency domain $Y(f) = H(f)X(f)$. As a result

$$Y(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}[1 - \sin(\pi T(|f| - \frac{1}{2T})/\alpha)], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

The frequency content, $Y(f)$, is shown in Figure 2.51 for $\alpha = 0.05, 0.25, 0.45, 0.65, 0.85$ (in non-dB units (left) and in dB units (right)). In the time domain when $x(t)$ is filtered by $h(t)$ the result is $y(t)$ (see

Appendix 2B).

$$y(t) = \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\alpha\pi t/T)}{1 - 4\alpha^2 t^2/T^2}.$$

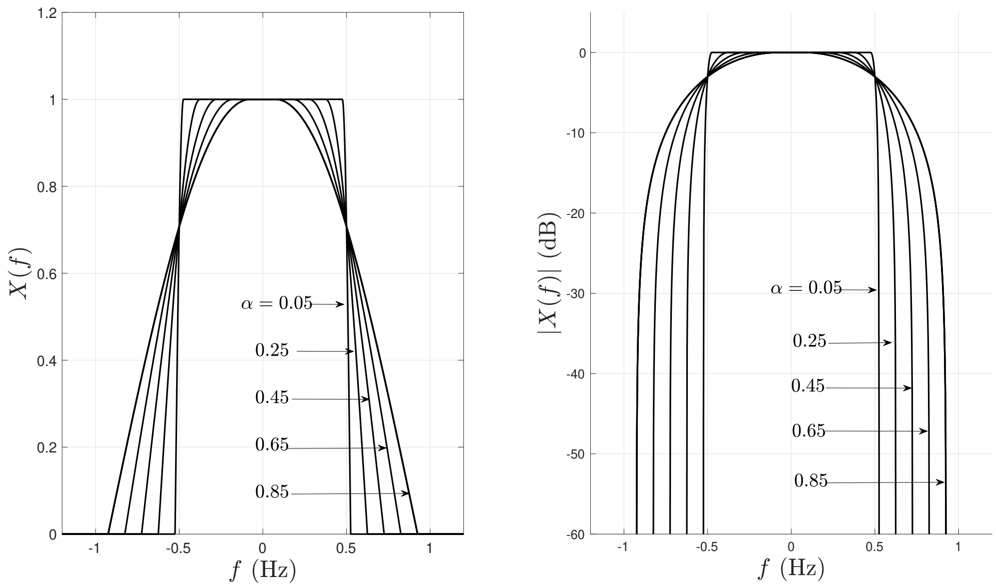


Figure 2.51: Spectrum of square-root raised cosine signal in non-dB and dB units

Notice that the output $y(t)$ is zero at integer multiples of T except at $t = 0T$. Note also that

$$\begin{aligned}
\int_{-\frac{1+\alpha}{2T}}^{\frac{1+\alpha}{2T}} Y(f)df &= 2\int_0^{(1-\alpha)/(2T)} T df + 2\int_{(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} T/2[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)]df \\
&= (1-\alpha)+[(1+\alpha)/(2T)-(1-\alpha)/(2T)](T/2)-T/2\int_{(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \sin(\pi T(f-\frac{1}{2T})/\alpha)]df \\
&= (1-\alpha)+\alpha+T/2\frac{1}{\pi T/\alpha}\cos(\pi T/\alpha(f-\frac{1}{2T})|_{(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \\
&= 1+\frac{\alpha}{2\pi}\cos(\pi T/\alpha(f-\frac{1}{2T})|_{(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \\
&= 1.
\end{aligned}$$

This means, because of Parseval (see Appendix 2E) that the energy of the signal $x(t)$ is 1, i.e. the signal is normalized. The relevant pulses in the time domain and frequency domain are shown in Figure 2.52.

Example 21: Matched Filters:

In a digital communication system it is usual for the received filter to be matched to the transmitted signal. In this case, if $s(t)$ is the transmitted signal and is of duration T beginning at 0, we sample the filter output at time T then $h(t) = s^*(T-t)$ is called the *matched filter*. In the frequency domain the

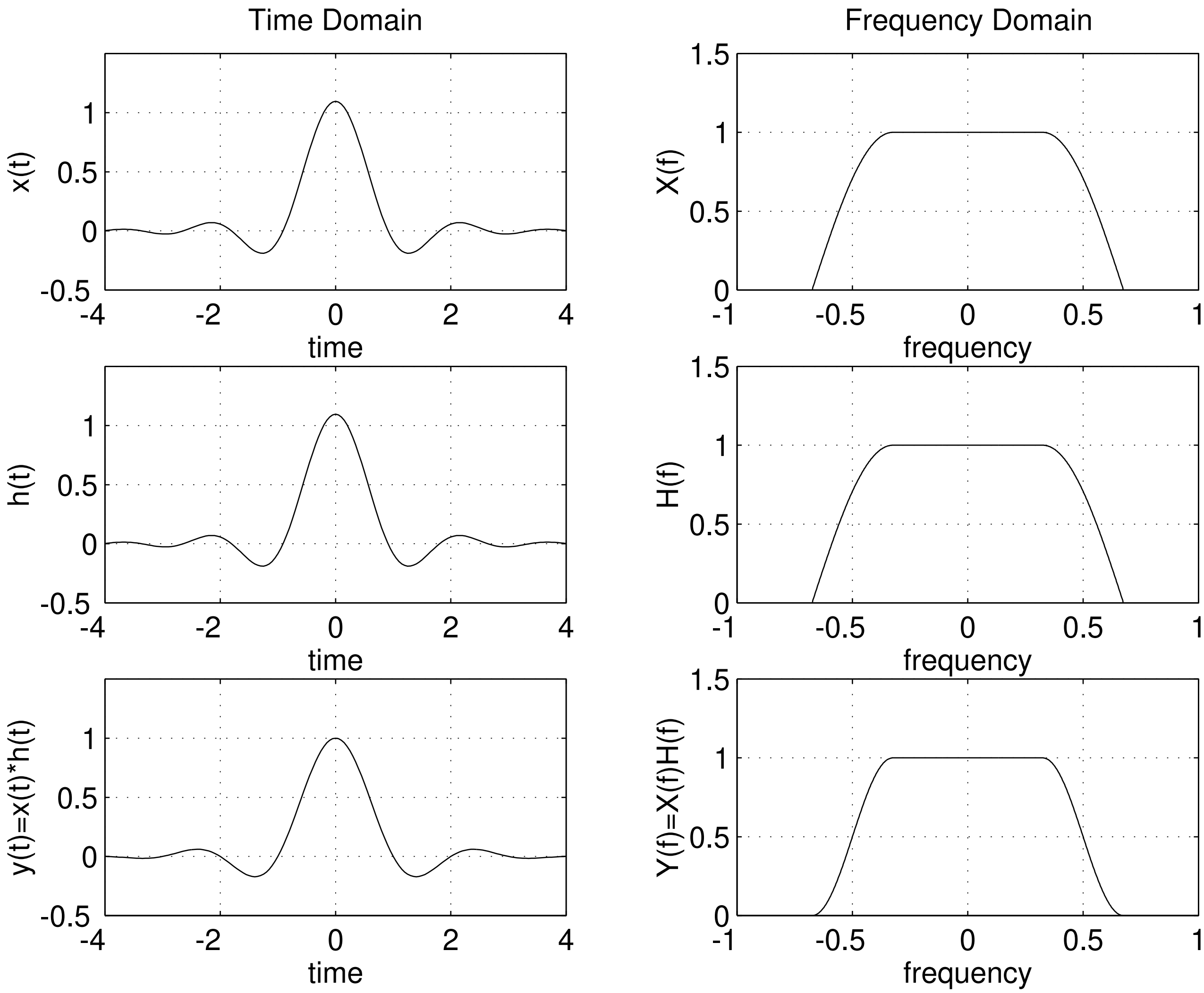


Figure 2.52: Filtering of square-root raised cosine pulses: time and frequency domain.

matched filter is $H(f) = S^*(f)e^{-j2\pi fT}$. The matched filter output is given by

$$
\begin{aligned}
y(t) &= \left[\int_{-\infty}^{\infty} h(t-\tau)s(\tau)d\tau\right] \\
&= \left[\int_{t-T}^{t} h(t-\tau)s(\tau)d\tau\right] \\
&= \left[\int_{t-T}^{t} s^*(T-(t-\tau))s(\tau)d\tau\right] \\
&= \left[\int_{t-T}^{t} s^*(\tau-(t-T))s(\tau)d\tau\right].
\end{aligned}
$$

This is the autocorrelation of the signal $s(t)$. The desired filter output due to the transmitted signal is the output at sample at time T.

$$\begin{aligned} y(T) = \int_0^T h(T-\tau)s(\tau)d\tau &= \int_0^T s^*(\tau)s(\tau)d\tau \\ &= \int_0^T |s(\tau)|^2 d\tau. \end{aligned}$$

Note that since $y(T)$ is the energy of the signal $s(t)$, it is only the real part of the matched filter output that is needed when processing the signal. So the output of the matched filter at the sample time T is the energy of the signal.

2.7 Intersymbol Interference Free Pulses

If pulses used to transmit data are orthogonal then at the receiver there will be no interference from one symbol (vector component) to another symbol. In this case we say the pulses are free of intersymbol interference. In this section we derive a condition for the pulses to be intersymbol interference free (i.e. orthogonal) in the frequency domain. 码间干扰

Recall that time-shifted orthonormal pulses satisfied

$$\varphi_n(t) = \varphi_0(t-nT).$$

The condition for the signals to be orthonormal in the time domain is

$$\int_t \varphi_0(t)\varphi_n^*(t)dt = \int_t \varphi_0(t)\varphi_0^*(t-nT)dt = \begin{cases} 1, & n=0 \\ 0, & n \neq 0. \end{cases}$$

Let $\Phi_0(f)$ be the Fourier transform of $\varphi_0(t)$. The Fourier transform of $\varphi_0(t-nT)$ is $\Phi_0(f)e^{-j2\pi fnT}$. Parseval's identity (Appendix 2E) then indicates that

$$\begin{aligned} \int_t \varphi_0(t)\varphi_0^*(t-nT)dt &= \int_{f=-\infty}^{\infty} \Phi_0(f)[\Phi_0(f)e^{-j2\pi fnT}]^* df \\ &= \int_{f=-\infty}^{\infty} \Phi_0(f)\Phi_0^*(f)e^{+j2\pi fnT} df. \end{aligned}$$

Now break the integral into nonoverlapping segments of size $1/T$ and sum the different integrals.

$$\begin{aligned} \int_t \varphi_0(t)\varphi_0^*(t-nT)dt &= \sum_{m=-\infty}^{\infty} \int_{f=(2m-1)/2T}^{2m+1/2T} |\Phi_0(f)|^2 e^{+j2\pi fnT} df \\ &= \sum_{m=-\infty}^{\infty} \int_{u=-1/2T}^{1/2T} |\Phi_0(u+m/T)|^2 e^{+j2\pi(u+m/T)nT} df \\ &= \int_{u=-1/2T}^{1/2T} [\sum_{m=-\infty}^{\infty} |\Phi_0(u+m/T)|^2] e^{+j2\pi unT} df \\ &= \int_{u=-1/2T}^{1/2T} \tilde{\Phi}(f) e^{+j2\pi unT} df \end{aligned}$$

where

$$\tilde{\Phi}(f) = \sum_{m=-\infty}^{\infty} |\Phi_0(f+m/T)|^2], \quad -1/(2T) < f < 1/(2T).$$

Let $\tilde{\varphi}(t)$ be the inverse Fourier transform of $\tilde{\Phi}(f)$. Then

$$\begin{aligned}
\tilde{\varphi}(t) &= \int_{f=-1/(2T)}^{1/(2T)} \tilde{\Phi}(f)e^{+j2\pi ft}df \\
\tilde{\varphi}(nT) &= \int_{f=-1/(2T)}^{1/(2T)} \tilde{\Phi}(f)e^{+j2\pi fnT}df \\
&= \int_t \varphi_0(t)\varphi_0^*(t-nT)dt.
\end{aligned}$$

If $\tilde{\varphi}(nT)$ is 0 for $n \neq 0$ and 1 for $n = 0$ then $\varphi_0(t)$ and $\varphi_n(t)$ will be orthonormal. If $\tilde{\Phi}(f)$ satisfies

$$\tilde{\Phi}(f) = \sum_{m=-\infty}^{\infty} |\Phi_0(f+m/T)|^2] = \begin{cases} T, & -1/(2T) \leq f \leq 1/(2T) \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

then we have that $\varphi_0(t)$ and $\varphi_n(t)$ will be orthonormal. This is because

$$\begin{aligned}
\int_{f=-1/(2T)}^{1/(2T)} Te^{+j2\pi fnT}df &= T\frac{e^{+j2\pi fnT}}{j2\pi nT}\Big|_{f=-1/(2T)}^{1/(2T)} \\
&= T\frac{e^{+j\pi n} - e^{-j\pi n}}{j2\pi nT} \\
&= \frac{e^{j\pi n} - e^{-j\pi n}}{2j\pi n} \\
&= \frac{\sin(\pi n)}{\pi n} \\
&= \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}
\end{aligned}$$

The condition that the frequency response $\tilde{\Phi}(f)$ is flat for $f \in [-1/(2T), 1/(2T)]$ is the same as the condition that when the frequency content of $|\Phi_0(f)|^2$ in bands outside $[-1/(2T), +1/(2T)]$ are shifted to occupy the frequency range $[-1/(2T), +1/(2T)]$ and summed together the result is a constant 1. That is, the signal in frequency range $[1/(2T), 3/(2T)]$ is shifted left by $1/T$ so as to be in the range $[-1/(2T), 1/(2T)]$. Similarly the signal in frequency range $[3/(2T), 5/(2T)]$ is shifted to the range $[-1/(2T), 1/(2T)]$. Similarly signals with frequencies less than $-1/(2T)$ are shifted to the right. Figure 2.53 shows this graphically for one possible frequency pulse shape that has bandwidth that extends past $f = 1/(2T)$ but not past $f = 2/(2T)$. In this case the spectrum from $[1/(2T), 3/(2T)]$ is shifted to $[-1/(2T), 1/(2T)]$ and similarly for negative frequencies.

One pulse shape that satisfy this condition is the "box car" shape.

$$\Phi_0(f) = \begin{cases} \sqrt{T}, & -1/(2T) < f < 1/(2T) \\ 0, & \text{elsewhere}. \end{cases}$$

This corresponds to $\varphi_0(t)$ being a sinc pulse shape. That is

$$\varphi_0(t) = \frac{1}{\sqrt{T}}\frac{\sin(\pi t/T)}{\pi t/T}.$$

The sinc pulse occupies a bandwidth of $W = 1/(2T)$ Hz. Different pulses can be separated by T seconds and not have any intersymbol interference at the receiver. Thus each symbol (one dimension) mapped into time-orthogonal pulses separated by T seconds occupies a bandwidth of $W = 1/(2T)$. Each pulse

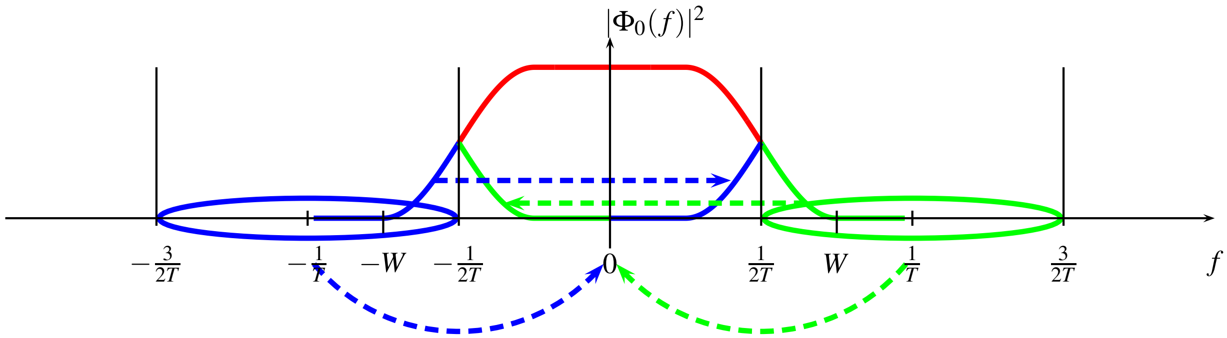


Figure 2.53: Frequency content of time-shifted orthonormal pulses.

can have an arbitrary amplitude and there is no intersymbol interference at the output of the receiver. A sequence of N pulses with each pulse separated in time by T seconds will also occupy bandwidth $1/2T$. Now there are N dimensions in time (roughly) NT with bandwidth $W = 1/(2T)$. So $N/(NT)$ (the number of dimensions per second) is $1/T = 2W$. This is an example of the relation $N = 2WT$ of Chapter 1 (see (**??**)) which states that there are $2W$ available (orthogonal) dimensions per second in a bandwidth W.

Another pulse that satisfies the condition (2.5) for no intersymbol interference is the square root raised cosine pulse. A special case of square-root raised cosine pulse is when $\alpha = 0$ and is the sinc pulse shape discussed above. Any pulse that satisfies (2.5) condition is called a Nyquist pulse.

2.8 Eye Diagrams

One way to visualize the lack of intersymbol interference or the amount of intersymbol interference is though an eye diagram. The eye diagram is obtained by repeatedly plotting a two symbol interval of the signal at the output of a receiver filter. Consider the system shown in Figure 2.54. A sequence of data symbols d_n is the input to the transmitter filter $h_T(t)$ and the output of the receiver filter $h_R(t)$ is $e(t)$. If the pulses $\varphi_0(t-nT)$ are orthonormal for different n over the time interval of the pulses then at the appropriate sampling time there will be no intersymbol interference at the output of the receiver filter. At other times the output of the filter will depend on other data symbols. The eye diagram is a way to visualize the filter output for many different possible data sequences.

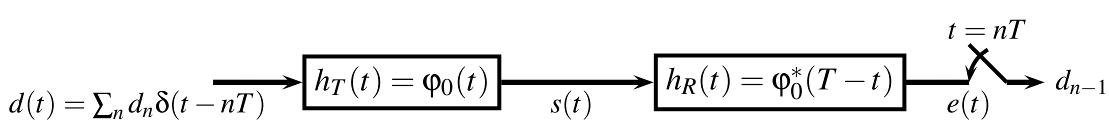


Figure 2.54: Generating signals for time-shifted orthonormal waveforms.

The eye diagram for a square-root raised cosine pulse used at the transmitter and then filtered by another square-root raised cosine pulse at the receiver is shown in Figure 2.55. In this case the data is either +1 or -1, i.e. $d_n \in \{+1, -1\}$ and the symbol duration is 1 second. Different sequences of data bits are considered which changes the signal at the output of the receiver. Each distinct signal plotted in the eye diagram corresponds to a distinct sequence of data bits. Similarly the eye diagram for the square-root raised cosine pulse with $\alpha = 0.25, 0.5$ and 0.75 are shown in Figure 2.56, 2.57, and 2.58. The eye diagram illustrates the effect of imperfect sampling at the receiver. As can be seen in these figures the output for $\alpha = 0.05$ is much more sensitive to not sampling at the correct time compared to $\alpha = 0.5$ or 0.75. An eye diagram for the case $d_n \in \{-3, -1, +1, +3\}$ is shown in Figure 2.59 with $\alpha = 0.5$.

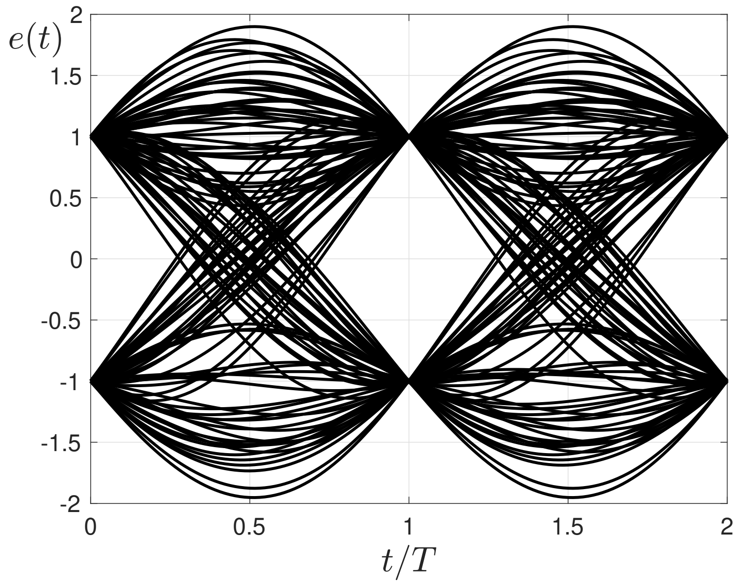


Figure 2.55: Eye diagram for square-root raised cosine pulses, $\alpha = 0.05$

Draw eye diagram for truncated pulse

2.9 Baseband-to-Passband, Passband-to-Baseband

In this section we show how two baseband signals (signals with frequency content centered around 0 frequency) can be mixed onto a carrier frequency and combined and then the original signals can be recovered. Alternatively, the two baseband signals can be viewed as a single complex signal. One motivation of using complex signals is as a representation of a bandpass signal. Equivalently we can generate a bandpass signal from two baseband (lowpass) signals.

Start with two signals, $x_I(t)$ and $x_Q(t)$ with frequency content $X_I(f)$ and $X_Q(f)$, respectively. These signals are limited in frequency content to frequencies between $-W$ and W Hz. That is, $|X_I(f)| = |X_Q(f)| = 0$ for $f \notin [-W, W]$. We call these baseband signals relative to f_c if $W \ll f_c$. That is the largest frequency in $x_I(t)$ and $x_Q(t)$ is much smaller than f_c. As an example of the parameters representative values might be $f_c = 10^9$ while $W = 10^6$ in which case $W \ll f_c$. These baseband signals are called the in-phase and quadrature-phase signal, or I and Q signals. These signals are then mixed onto a carrier and combined as follows as shown in Figure 2.60

$$x(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t).$$

The lowpass complex representation of $x(t)$ is

$$\tilde{x}(t) = x_I(t) + jx_Q(t).$$

An alternative way of expressing $x(t)$ is

$$\begin{aligned} x(t) &= \Re\{\tilde{x}(t)\sqrt{2}e^{j2\pi f_c t}\} \\ &= \Re\{(x_I(t) + jx_Q(t))\sqrt{2}e^{j2\pi f_c t}\}. \end{aligned}$$

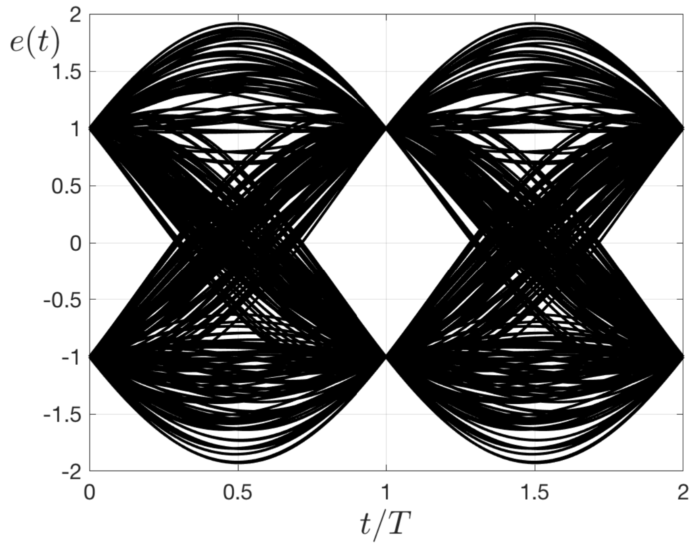


Figure 2.56: Eye diagram for square-root raised cosine pulses, $\alpha = 0.25$

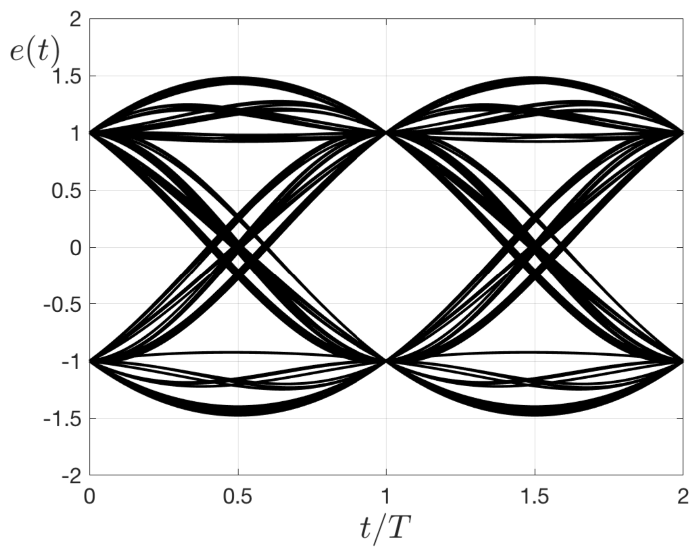


Figure 2.57: Eye diagram for square-root raised cosine pulses, $\alpha = 0.5$

Finally another expression for $x(t)$ is

$$x(t) = x_e(t)\cos(2\pi f_c t + \theta(t))$$

where

$$\begin{aligned} x_e(t) &= \sqrt{2(x_I^2(t) + x_Q^2(t))} \\ \theta(t) &= \tan^{-1}\left(\frac{x_Q(t)}{x_I(t)}\right). \end{aligned}$$

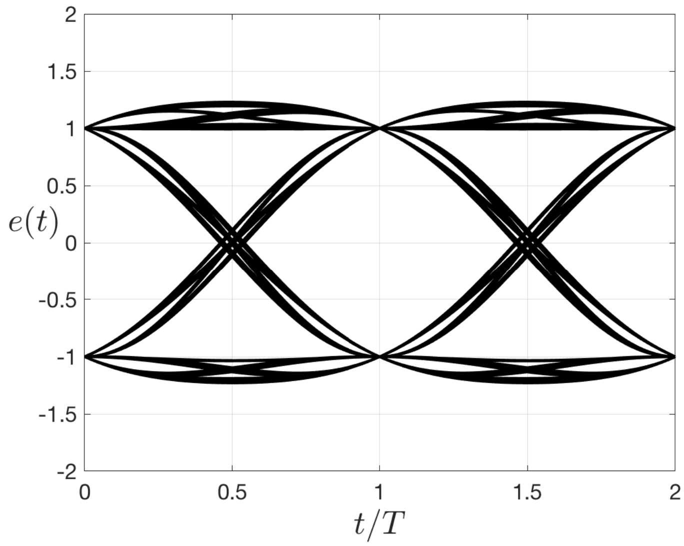


Figure 2.58: Eye diagram for square-root raised cosine pulses, $\alpha = 0.75$

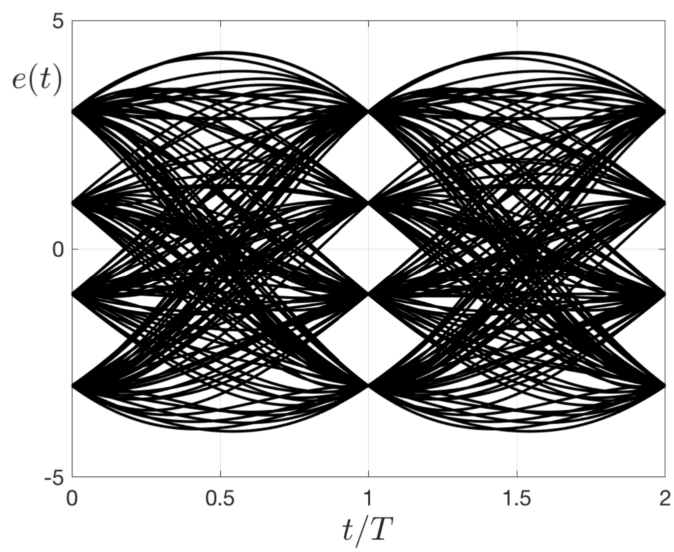


Figure 2.59: Eye diagram for square-root raised cosine pulses with 4-ary amplitude, $\alpha = 0.5$

The signal $x_e(t)$ is known as the envelope of the signal and $\theta(t)$ is the phase. The peak-to-mean envelope power ratio is the peak-to-mean power ratio of the envelope $x_e(t)$. This is a factor 2 larger than the peak-to-mean power ratio of the actual signal $x(t)$.

Suppose the frequency content of the signals $x_I(t)$ and $x_Q(t)$ are $X_I(f)$ and $X_Q(f)$. Then frequency content of the signal $x(t)$ is

$$X(f) = \frac{\sqrt{2}}{2}\left[X_I(f+f_c) + X_I(f-f_c) - j(X_Q(f+f_c) - X_Q(f-f_c))\right]$$

The signal $x(t)$ is called a *bandpass* signal since the frequency content of $x(t)$ is limited to frequencies

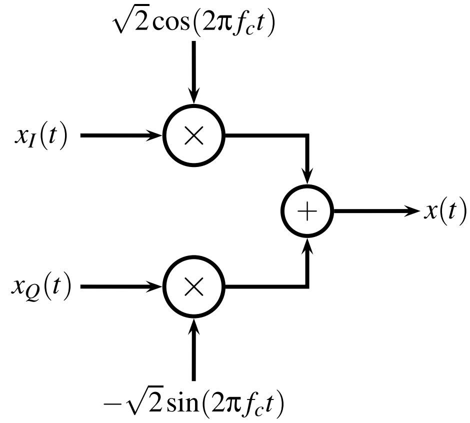


Figure 2.60: Mixing two baseband signals onto a carrier and combining.

within W of $\pm f_c$. That is, $|X(f)| = 0$ for $f \notin [-f_c - W, -f_c + W]$ or $f \notin [f_c - W, f_c + W]$. The complex signal $\tilde{x}(t) = x_I(t) + jx_Q(t)$ is called the *complex baseband representation* or *lowpass complex representation* of the bandpass signal $x(t)$. Note that the energy of $x(t)$ is the same as the energy of the complex baseband signal $x_I(t) + jx_Q(t)$.

$$\int |x(t)|^2 dt = \int |x_I(t) + jx_Q(t)|^2 dt.$$

Also the Euclidean distance between two bandpass signals $x_1(t)$ and $x_2(t)$ is the same as the distance between the corresponding complex baseband signals. That is, consider two pairs of baseband signals $(x_{I,1}(t), x_{Q,1}(t))$ and $(x_{I,2}(t), x_{Q,2}(t))$ with corresponding real bandpass signal $x_1(t)$ and $x_2(t)$. Let $\tilde{x}_1(t) = x_{I,1}(t) + jx_{Q,1}(t)$ and $\tilde{x}_2(t) = x_{I,2}(t) + jx_{Q,2}(t)$ be the complex representation of these signals. Then

$$\int |x_1(t) - x_2(t)|^2 dt = \int |\tilde{x}_1(t) - \tilde{x}_2(t)|^2 dt.$$

The operation of generating a bandpass signal is what typically happens at the transmitter in a communication system.

At a receiver we can recover the two baseband signals from the single bandpass signal as follows.

$$\begin{aligned} x_I(t) &= \text{LPF}\left[x(t)\sqrt{2}[\cos(2\pi f_c t)]\right] \\ x_Q(t) &= \text{LPF}\left[x(t)\sqrt{2}[-\sin(2\pi f_c t)]\right] \end{aligned}$$

where LPF is an ideal low pass filter which passes frequencies between $-W$ and W. This operation is shown in Figure 2.61. The frequency representation of $x_I(t)$ and $x_Q(t)$ in terms of $X(f)$ is

$$\begin{aligned} X_I(f) &= \frac{\sqrt{2}}{2}\text{LPF}[X(f - f_c) + X(f + f_c)] \\ X_Q(f) &= \frac{\sqrt{2}j}{2}\text{LPF}[X(f - f_c) - X(f + f_c)] \end{aligned}$$

One way to show that each of the lowpass or baseband signals can be reconstructed from the passband signals is in the frequency domain. Let

$$w_I(t) = x(t)\sqrt{2}\cos(2\pi f_c t).$$

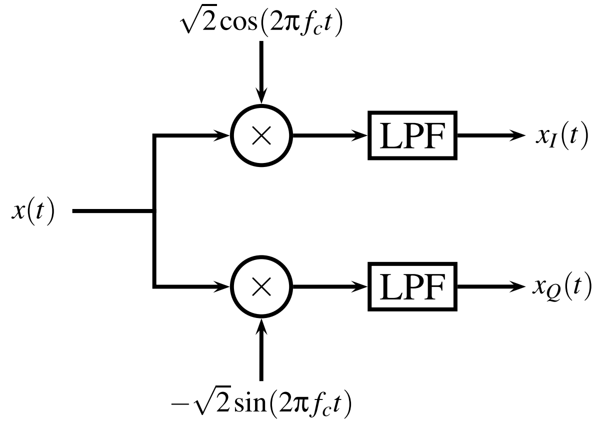


Figure 2.61: Recovering a lowpass complex signal from a bandpass signal.

Then

$$\begin{aligned}
W_I(f) &= \frac{\sqrt{2}}{2}[X(f-f_c)+X(f+f_c)] \\
&= \frac{1}{2}[X_I(f)+X_I(f-2f_c)-j(X_Q(f)-X_Q(f-2f_c)) \\
&\quad +X_I(f+2f_c)+X_I(f)-j(X_Q(f+2f_c)-X_Q(f))] \\
&= X_I(f)+\frac{1}{2}[X_I(f-2f_c)+X_I(f+2f_c)+j(X_Q(f-2f_c)-X_Q(f+2f_c))].
\end{aligned}$$

Besides the first term, all other terms in $W_I(f)$ are double frequency terms. The result of lowpass filter on $W_I(f)$ is then to remove all double frequency signals and the end result is a regeneration of $X_I(f)$. Similarly for $X_Q(f)$.

Note that the real part of the integral of the regular square (not magnitude square) of the lowpass complex signal generated from a bandpass signal is 0. That is,

$$\Re\left[\int (x_I(t)+jx_Q(t))^2\,dt\right] = 0.$$

Example 22:

Consider two signals each like the signals in Example 11.

$$\begin{aligned}
x_I(t) &= \sum_{n=0}^{N} b_{I,n}\varphi(t-nT) \\
x_Q(t) &= \sum_{n=0}^{N} b_{Q,n}\varphi(t-nT)
\end{aligned}$$

Here $b_{I,n}$ and $b_{Q,n}$, $n=0,1,\ldots,N-1$ are binary (i.e. +1 or -1) representing data and $\varphi(t)$ is the square-root raised cosine pulses in Example 11 with $\alpha = 0.5$. The low pass signals $x_I(t)$ and $x_Q(t)$ are shown in Figure 2.62 in the time domain and frequency domain. The bandpass signal $x(t)$ and the corresponding frequency representation $X(f)$ is shown in Figure 2.63. In addition, the envelope $x_e(t)$ of $x(t)$ is shown. In this example the bit duration is $T = 1$ and the carrier frequency is $f_c = 6$.

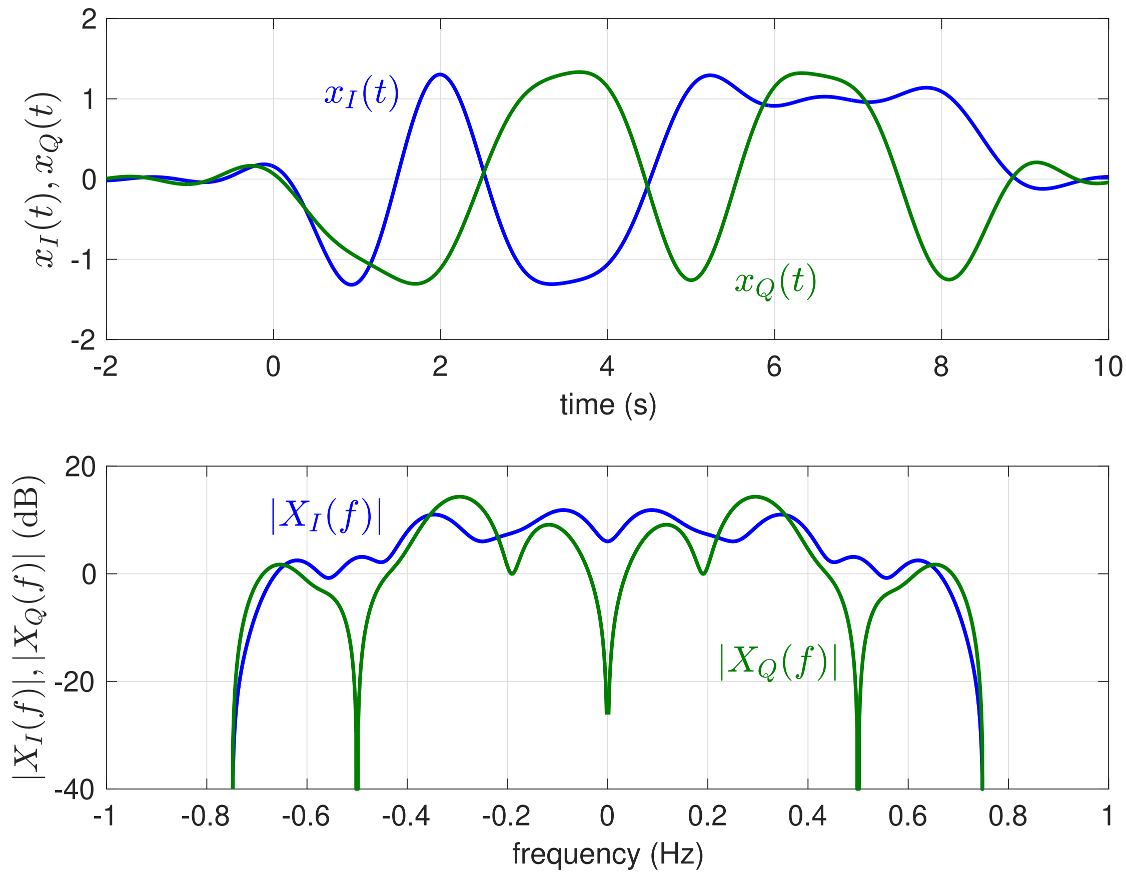


Figure 2.62: Lowpass signals.

The signals $x_I(t)$ and $x_Q(t)$ can be recovered by mixing and performing a lowpass filtering such that frequencies below W are passed through the filter unaltered while the double frequency signals in the frequency range $[2f_c - W, 2f_c + W]$ and $[-2f_c - W, -2f_c + W]$ are removed. If we are interested in recovering the data, $b_{I,n}$ and $b_{Q,n}$ then additional filtering with a filter matched to $\varphi(t)$ should be used. However, note that the filter matched to $\varphi(t)$ also will remove the double frequency term after multiplying the signal $x(t)$ by $\sqrt{2}\cos(2\pi f_c t)$ and $\sqrt{2}\sin(2\pi f_c t)$. As a result the filter in Figure 2.61 can be a filter matched to $\varphi(t)$ and the result sampled to reproduce the data.

Example 23:

Consider now the signals of Example 22 but the orthonormal waveforms $\varphi_n(t)$ are time-shifted rectangular pulses of Figure 2.35 instead of time-shifted square-root raised cosine pulses. These baseband signals are shown in Figure 2.64. The spectrum of these signals is shown in this Figure. Note that the spectrum of the baseband signals decays toward zero but only approaches zero at large frequencies. So these signals are not technically baseband signals but they are approximately baseband signals. The approximation gets better the larger f_c is compared to $1/T$.

The bandpass signal is shown in Figure 2.65 along with the envelope of the signal. This signal has a carrier frequency of 64. For these baseband signals the envelope is constant. A constant envelop corresponds to a better peak-to-average power ratio than a nonconstant envelope signal like Example 6

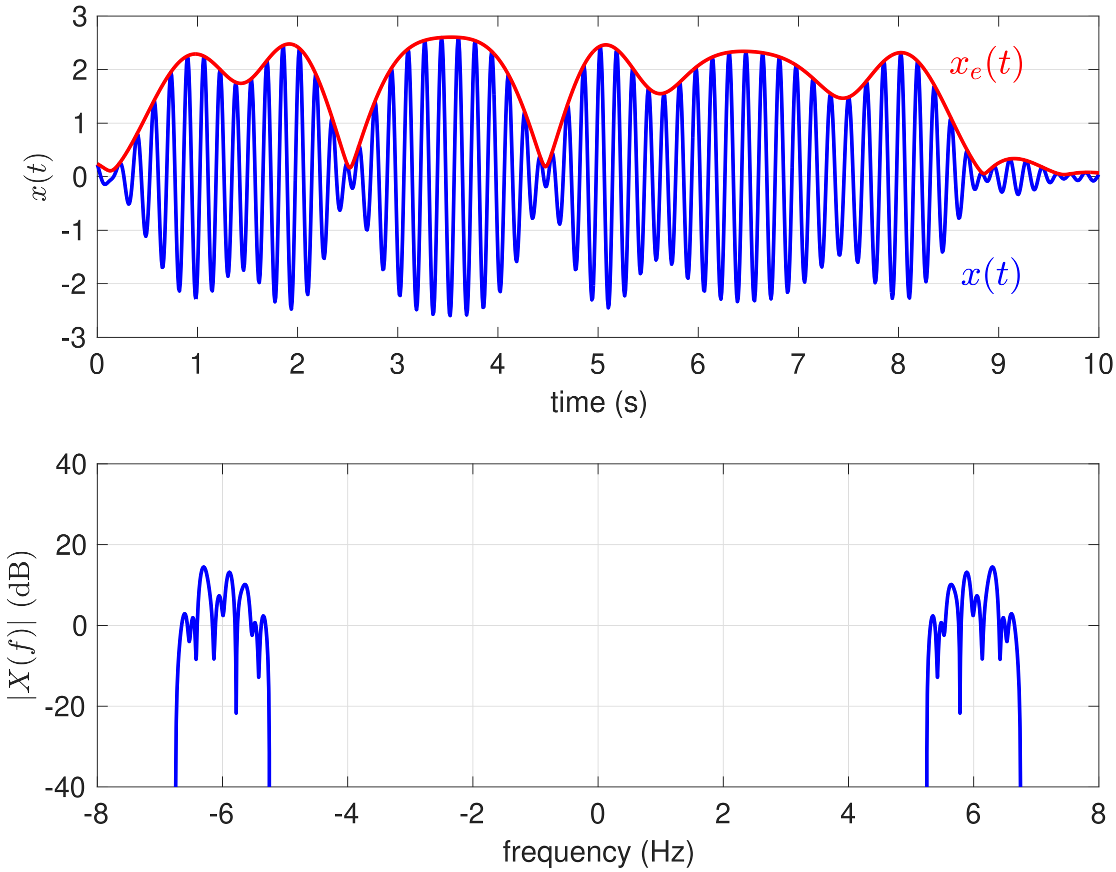


Figure 2.63: Bandpass signal.

and thus an amplifier can more efficiently convert DC power to RF power. However, the bandwidth of the constant envelope signal is larger. This is another example of how bandwidth efficiency and energy efficiency are hard to achieve simultaneously. The bandpass signal makes an abrupt change at time 1s because the data bit for the quadrature-phase signal has changed from -1 to +1. It is this instantaneous change in the signals that cause the spectrum to only go to zero as the frequency gets very large. This modulation is called quadrature-phase shift keying (QPSK). The regeneration of the baseband signals in this example will not be perfect due to the violation of the requirement that the maximum bandwidth W be less than the carrier frequency f_c. In this example the carrier frequency f_c is 64 times the bit rate $1/T$. In practice the carrier frequency might be on the order of 1000 times the bit rate or even more.

Now consider combining the up conversion at the transmitter with a time-shifted orthonormal set of waveforms. The block diagram of the transmitter is shown in Figure 2.66. The filters in Figure 2.66 have frequency response limited to $[-W,W]$ such that $W \ll f_c$. So the inputs to the mixers, $x_I(t)$,$x_Q(t)$ have frequency content limited to $[-W,W]$. We can recover the two signals $x_I(t)$ and $x_Q(t)$ by mixing down to baseband and low pass filtering to remove the double frequency component as shown in Figure 2.61. However, the goal is to recover the data $s_{I,n}$ and $s_{Q,n}$. To recover the data a matched filter, $h_R(t) = h_T(-t)$, can be appended to the low pass filter in Figure 2.61. However, since $h_T(t)$ is a filter with frequency response limited to $[-W,W]$ this filter also can function to remove the double frequency component present after mixing. So only a filter with impulse response $h_R(t)$ is needed. This is shown in Figure 2.67. This system will recover the data exactly provided a matched filter is used $h_R(t) = h_T(-t)$ and that

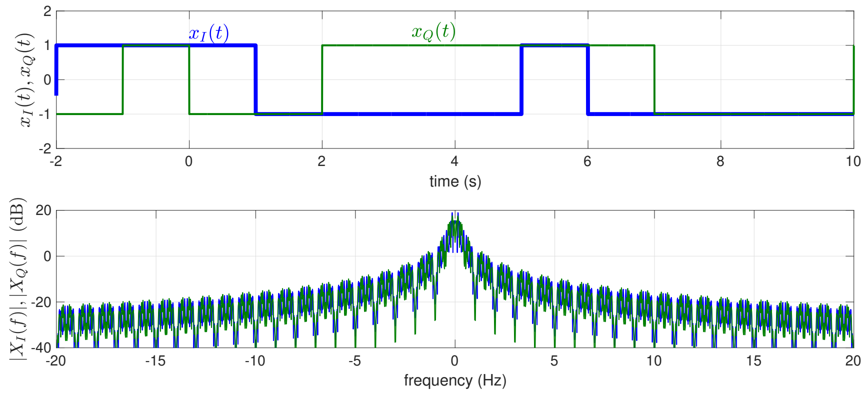


Figure 2.64: Lowpass signals for Example 22.

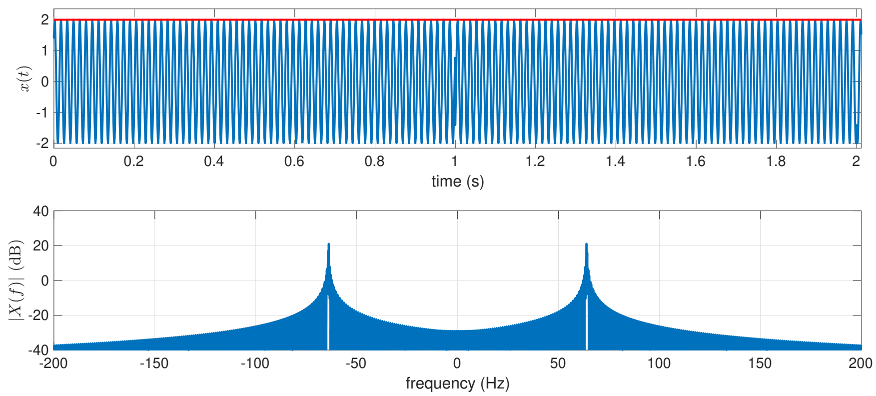


Figure 2.65: Bandpass signal for Example 22

the filter $h_T(t)$ corresponds to a time-shifted orthogonal signal. That is,

$$\int h_T(t)h_T(t-nT)dt = \begin{cases} A, & n=0 \\ 0, & n \neq 0 \end{cases}$$

where A is some constant.

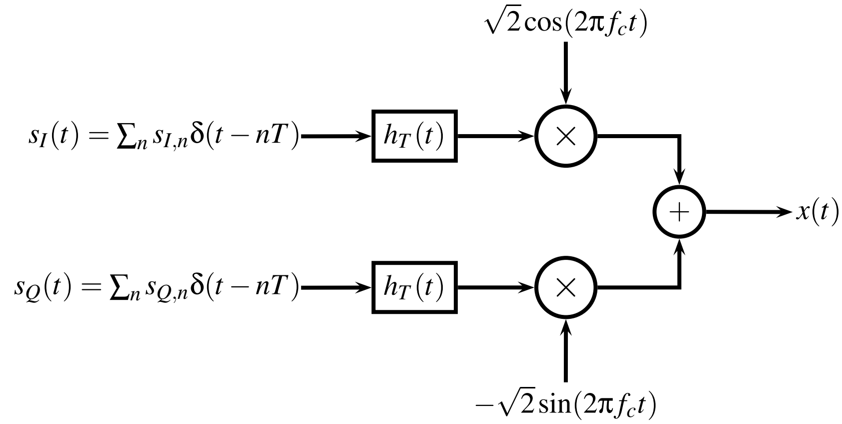


Figure 2.66: Mixing two baseband signals onto a carrier and combining.

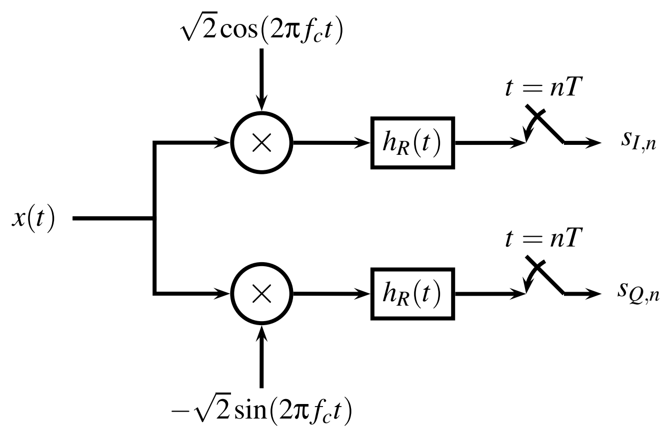


Figure 2.67: Recovering a data from a bandpass signal.

Summary of Chapter 2 Concepts:

- A sinusoidal signal of known frequency is determined by either an amplitude and phase, the in-phase and quadrature-phase components or a complex number.
- Signal composition: A set of M vectors of length N and a set of of N orthonormal waveforms can generate a set of M signals where $N \leq M$.
- Signal decomposition: The vectors associated with a set of M signals can be recovered by correlating with the orthonormal waveforms.
- A signal that is the composition of time-shifted orthogonal waveforms can be generated from a vector by generating a sequence of impulses with the amplitude being the coefficients of the vector and a filter that is one of the orthogonal waveforms.
- A vector used with a set of time-shifted orthogonal waveforms $\varphi_i(t) = \varphi_0(t - iT)$ to generate a signal can be recovered from the signal by sampling a single filter matched to the orthonormal signal $\varphi_0(t)$.
- A pulse shape can generate time-shifted (shifted by T) orthonormal waveforms if the spectrum when all frequency content outside the frequency range $[-1/(2T), 1/(2T)]$ when shifted to $[-1/(2T), 1/(2T)]$ and summed is a constant.
- Two baseband signals (frequency content limited to $[-W, W]$ Hz can be mixed onto carriers using $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ and combined to form a single bandpass signal. From the bandpass signal the two baseband signals can be recovered.

Key Relations for Chapter 2:

- $s(t) = c\cos(2\pi f_c t + \theta) = a\cos(2\pi f_c t) - b\sin(2\pi f_c t) = \Re[2de^{j2\pi f_c t}]$
- $a = c\cos(\theta)$, $b = c\sin(\theta)$, $c = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}(b/a)$, $d = (a + jb)/2$.
- Modulation: Signals from vectors $s_m(t) = \sum_{n=0}^{N-1} s_{m,n}\varphi_n(t)$
- Demodulation: Vectors from signals $s_{m,n} = \int s_m(t)\varphi_n^*(t)dt$
- Equivalency between signals and vectors:
  - $(s_m(t), s_l(t)) = \int s_m(t)s_l^*(t)dt = \sum_{n=0}^{N-1} s_{m,n}s_{l,n}^*$
  - $||s_m(t)||^2 = \int |s_m(t)|^2 dt = \sum_{n=0}^{N-1} |s_{m,n}|^2$
  - $d_E^2(s_m(t), s_l(t)) = ||s_m(t) - s_l(t)||^2 = \int |s_m(t) - s_l(t)|^2 dt = \sum_{n=0}^{N-1} |s_{m,n} - s_{l,n}|^2$
- A set of time-shifted pulses ($\varphi_n(t) = \varphi_0(t - nT)$) will be orthonormal provided that

$$\sum_{m=-\infty}^{\infty} |\Phi_0(f + \frac{m}{T})|^2 = T, \quad -\frac{1}{2T} < f < \frac{1}{2T}$$

  where $\Phi_0(f)$ is the Fourier transform of the pulse shape $\varphi_0(t)$ that generates the time-shifted (by T) orthonormal pulses.

- Passband signals can be represented by two lowpass signals or a single complex lowpass signal.

$$\begin{aligned} x(t) &= x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t))\sqrt{2}\cos(2\pi f_c t) \\ &= \Re\{(x_I(t) + jx_Q(t))\sqrt{2}e^{j2\pi f_c t}\} \\ &= x_e(t)\cos(2\pi f_c t + \varphi(t)). \end{aligned}$$

The baseband signals can be recovered from the passband signal.

$$\begin{aligned} x_I(t) &= \text{LPF}\left[x(t)\sqrt{2}[\cos(2\pi f_c t)]\right] \\ x_Q(t) &= \text{LPF}\left[x(t)\sqrt{2}[-\sin(2\pi f_c t)]\right] \\ x_e(t) &= \sqrt{2(x_I^2(t) + x_Q^2(t))} \\ \varphi(t) &= \tan^{(} x_Q(t)/x_I(t)) \\ \int |x(t)|^2 dt &= \int |x_I(t) + jx_Q(t)|^2 dt \end{aligned}$$

2.10 Appendix 2A: Evaluation of Square Root Raised Cosine Pulses

The square root raised cosine pulse shape $x(t)$ can be evaluated at $t = -T/(4\alpha), 0, T/(4\alpha)$ using L'Hospitals rule. The square root raised cosine pulse shape is

$$x(t) = \frac{1}{\sqrt{T}}\left[\frac{\sin(\pi(1-\alpha)t/T) + (4\alpha t/T)\cos(\pi(1+\alpha)t/T)}{(1-(4\alpha t/T)^2)\pi t/T}\right].$$

First note that both the numerator and denominator are 0 at $t = 0$. Applying L'Hospitals rule at $t = 0$ we get

$$\begin{aligned}
\lim_{t\to 0} x(t) &= \lim_{t\to 0}\frac{1}{\sqrt{T}}\frac{\cos[\pi(1-\alpha)t/T]\pi(1-\alpha)/T + 4\alpha/T\cos[\pi(1+\alpha)t/T)] - 4\alpha t/T\sin[\pi(1+\alpha)t/T]\pi(1+\alpha)/T}{\pi/T[1-(4\alpha t/T)^2] + \pi t/T[-2(4\alpha/T)^2 t]} \\
&= \frac{1}{\sqrt{T}}\left[\frac{\pi(1-\alpha)+4\alpha}{\pi}\right] \\
&= \frac{1}{\sqrt{T}}\left[(1-\alpha)+\frac{4\alpha}{\pi}\right].
\end{aligned}$$

Consider now, the case where $t = \pm T/(4\alpha)$. Applying L'Hospitals rule again we get

$$\begin{aligned}
\lim_{t\to\pm T/(4\alpha)} x(t) &= \frac{1}{\sqrt{T}}\lim_{t\to\pm T/(4\alpha)}\frac{\cos[\pi(1-\alpha)t/T]\pi(1-\alpha)/T + 4\alpha/T\cos[\pi(1+\alpha)t/T)] - 4\alpha t/T\sin[\pi(1+\alpha)t/T]\pi(1+\alpha)/T}{\pi/T[1-(4\alpha t/T)^2] + \pi t/T[-2(4\alpha/T)^2 t]} \\
&= \frac{1}{\sqrt{T}}\frac{\cos[\pi(1-\alpha)/(4\alpha)]\pi(1-\alpha)/T + 4\alpha/T\cos[\pi(1+\alpha)/(4\alpha)] - \sin[\pi(1+\alpha)/(4\alpha)]\pi(1+\alpha)/T}{\pi/(4\alpha)[-2(16\alpha^2/T^2(T/(4\alpha))]} \\
&= \frac{1}{\sqrt{T}}\frac{\cos[\frac{\pi}{4\alpha}-\frac{\pi}{4}]\pi(1-\alpha) + 4\alpha\cos[\frac{\pi}{4\alpha}+\frac{\pi}{4}] - \sin[\frac{\pi}{4\alpha}+\frac{\pi}{4}]\pi(1+\alpha)}{-2\pi} \\
&= \frac{1}{\sqrt{T}}\frac{\cos[\frac{\pi}{4\alpha}-\frac{\pi}{4}]\pi(1-\alpha) + 4\alpha\cos[\frac{\pi}{4\alpha}+\frac{\pi}{4}] - \sin[\frac{\pi}{4\alpha}+\frac{\pi}{4}]\pi(1+\alpha)}{-2\pi}
\end{aligned}$$

Let $C = \cos[\pi/(4\alpha)]$ and $S = \sin[\pi/(4\alpha)]$. We will use the following facts

$$\begin{aligned}
\cos(a+\frac{\pi}{4}) &= \cos(a)\cos(\pi/4) - \sin(a)\sin(\pi/4) \\
&= K[\cos(a) - \sin(a)] \\
&= K[C-S] \\
\cos(a-\frac{\pi}{4}) &= \cos(a)\cos(\pi/4) + \sin(a)\sin(\pi/4) \\
&= K[\cos(a) + \sin(a)] \\
&= K[C+S] \\
\sin(a+\frac{\pi}{4}) &= \sin(a)\cos(\pi/4) + \cos(a)\sin(\pi/4) \\
&= K[\cos(a) + \sin(a)] \\
&= K[C+S]
\end{aligned}$$

where $K = \cos(\pi/4)$. Consider the numerator only.

$$
\begin{aligned}
N &= \cos[\frac{\pi}{4\alpha}-\frac{\pi}{4}]\pi(1-\alpha)+4\alpha\cos[\frac{\pi}{4\alpha}+\frac{\pi}{4}]-\sin[\frac{\pi}{4\alpha}+\frac{\pi}{4}]\pi(1+\alpha) \\
&= K\{[C+S]\pi(1-\alpha)+4\alpha[C-S]-[C+S]\pi(1+\alpha)\} \\
&= K\{[C+S][\pi(1-\alpha)-\pi(1+\alpha)]+4\alpha[C-S]\} \\
&= K\{[C+S][-2\alpha\pi]+4\alpha[C-S]\} \\
&= 2K\alpha\{C(2-\pi)+S(2+\pi)\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{t\to T/(4\alpha)} h(t) &= \frac{1}{\sqrt{T}}\frac{2K\alpha\{C(2-\pi)-S(2+\pi)\}}{-2\pi} \\
&= \frac{1}{\sqrt{T}}\frac{K\alpha\{C(2-\pi)-S(2+\pi)\}]\}}{-\pi} \\
&= \frac{1}{\sqrt{T}}K\alpha\{C(1-\frac{2}{\pi})+S(1+\frac{2}{\pi})\}]\}.
\end{aligned}
$$

2.11 Appendix 2B: Square Root Raised Cosine Pulses

In this appendix we derive the time-domain pulse shape of a square root raised cosine pulse. Consider the signal with frequency representation

$$
\begin{aligned}
X(f) &= \begin{cases} \sqrt{T}, & 0\le|f|\le\frac{1-\alpha}{2T} \\ \sqrt{T}|\sin(\frac{\pi}{2\alpha}(T|f|-\frac{(1+\alpha)}{2}))|, & \frac{1-\alpha}{2T}\le|f|\le\frac{1+\alpha}{2T} \\ 0, & \text{otherwise}. \end{cases} \\
&= \begin{cases} \sqrt{T}, & 0\le|f|\le\frac{1-\alpha}{2T} \\ \sqrt{T}\sin(\frac{\pi}{2\alpha}(\frac{(1+\alpha)}{2}-T|f|)), & \frac{1-\alpha}{2T}\le|f|\le\frac{1+\alpha}{2T} \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}
$$

The inverse Fourier transform is found as follows.

$$
\begin{aligned}
x(t) &= \int X(f)e^{j2\pi ft}df \\
&= \int_{-\infty}^{0} X(f)e^{j2\pi ft}df+\int_{0}^{\infty} X(f)e^{j2\pi ft}df \\
&= \int_{0}^{\infty} X(-f)e^{-j2\pi ft}df+\int_{0}^{\infty} X(f)e^{j2\pi ft}df \\
&= \int_{0}^{\infty} X(f)[e^{-j2\pi ft}+e^{j2\pi ft}]df \\
&= \int_{0}^{\infty} 2X(f)\cos(2\pi ft)df \\
&= \int_{0}^{\frac{1-\alpha}{2T}} 2X(f)\cos(2\pi ft)df+\int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} 2X(f)\cos(2\pi ft)df \\
&= \sqrt{T}\left[\int_{0}^{\frac{1-\alpha}{2T}} 2\cos(2\pi ft)df-\int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} 2\sin(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2}))\cos(2\pi ft)df\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{T}\left[\int_0^{\frac{1-\alpha}{2T}} 2\cos(2\pi f t)df - 2\int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \sin(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2}))\cos(2\pi f t)df\right] \\
&= \sqrt{T}\left[\frac{2\sin(2\pi f t)}{2\pi t}\Big|_{f=0}^{\frac{1-\alpha}{2T}} - 2\int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \sin(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2}))\cos(2\pi f t)df\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} - 2\int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \sin(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2}))\cos(2\pi f t)df\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} - \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} [\sin(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2})+2\pi f t)+\sin(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2T})-2\pi f t)]df\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + [\frac{\cos(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2})+2\pi f t)}{\pi T/(2\alpha)+2\pi t} + \frac{\cos(\frac{\pi}{2\alpha}(Tf-\frac{1+\alpha}{2})-2\pi f t)}{\pi T/(2\alpha)-2\pi t}]\Big|_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}}\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + [\frac{\cos(\pi(1+\alpha)t/T)}{\pi T/(2\alpha)+2\pi t} + \frac{\cos(\pi(1+\alpha)t/T)}{\pi T/(2\alpha)-2\pi t}]\right. \\
&\quad \left. -[\frac{\cos(\frac{\pi}{2\alpha}(\frac{1-\alpha}{2}-\frac{1+\alpha}{2})+\pi(1-\alpha)t/T)}{\pi T/(2\alpha)+2\pi t} + \frac{\cos(\frac{\pi}{2\alpha}(\frac{1-\alpha}{2}-\frac{1+\alpha}{2})-\pi(1-\alpha)t/T)}{\pi T/(2\alpha)-2\pi t}]\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + \cos(\pi(1+\alpha)t/T)[\frac{1}{\pi T/(2\alpha)+2\pi t} + \frac{1}{\pi T/(2\alpha)-2\pi t}]\right. \\
&\quad \left. -[\frac{\cos(\pi(1-\alpha)t/T-\frac{\pi}{2})}{\pi T/(2\alpha)+2\pi t} + \frac{\cos(\pi(1-\alpha)t/T+\frac{\pi}{2})}{\pi T/(2\alpha)-2\pi t}]\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + \cos(\pi(1+\alpha)t/T)[\frac{\pi T/\alpha}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right. \\
&\quad \left. -[\frac{\sin(\pi(1-\alpha)t/T)}{\pi T/(2\alpha)+2\pi t} - \frac{\sin(\pi(1-\alpha)t/T)}{\pi T/(2\alpha)-2\pi t}]\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + \cos(\pi(1+\alpha)t/T)[\frac{\pi T/\alpha}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right. \\
&\quad \left. -\sin(\pi(1-\alpha)t/T)[\frac{1}{\pi T/(2\alpha)+2\pi t} - \frac{1}{\pi T/(2\alpha)-2\pi t}]\right] \\
&= \sqrt{T}\left[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + \cos(\pi(1+\alpha)t/T)[\frac{\pi T/\alpha}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right. \\
&\quad \left. +\sin(\pi(1-\alpha)t/T)[\frac{4\pi t}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right] \\
&= \sqrt{T}\left[\sin(\pi(1-\alpha)t/T)[\frac{1}{\pi t} + \frac{4\pi t}{(\pi T/(2\alpha))^2-(2\pi t)^2}] + \cos(\pi(1+\alpha)t/T)[\frac{\pi T/\alpha}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right] \\
&= \sqrt{T}\left[\sin(\pi(1-\alpha)t/T)[\frac{(\pi T/(2\alpha))^2-(2\pi t)^2+4(\pi t)^2}{(\pi t)(\pi T/(2\alpha))^2-(2\pi t)^2)} + \cos(\pi(1+\alpha)t/T)[\frac{\pi T/\alpha}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right] \\
&= \sqrt{T}\left[\sin(\pi(1-\alpha)t/T)[\frac{(\pi T/(2\alpha))^2}{(\pi t)(\pi T/(2\alpha))^2-(2\pi t)^2)} + \cos(\pi(1+\alpha)t/T)[\frac{\pi T/\alpha}{(\pi T/(2\alpha))^2-(2\pi t)^2}]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{T}\left[\sin(\pi(1-\alpha)t/T)[\frac{1}{(\pi t)(1-(4\alpha t/T)^2)} + \cos(\pi(1+\alpha)t/T)[\frac{4\alpha/T}{\pi(1-(4\alpha t/T)^2)}]\right] \\
&= \frac{1}{\sqrt{T}}\left[\frac{\sin(\pi(1-\alpha)t/T) + (4\alpha t/T)\cos(\pi(1+\alpha)t/T)}{(1-(4\alpha t/T)^2)\pi t/T}\right] \\
&= \frac{1}{\sqrt{T}}\left[\frac{\sin(\pi(1-\alpha)t/T) + (4\alpha t/T)\cos(\pi(1+\alpha)t/T)}{(1-(4\alpha t/T)^2)\pi t/T}\right]
\end{aligned}
$$

2.12 Appendix 2C: Raised Cosine Pulses

In this appendix we derive the time-domain pulse shape of a raised cosine pulse. The frequency domain representation of raised cosine pulse is given by

$$
Y(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}[1-\sin(\pi T(|f| - \frac{1}{2T})/\alpha)], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
y(t) &= \int_f Y(f)e^{j2\pi ft}df \\
&= \int_{f=-\infty}^{0} Y(f)e^{j2\pi ft}df + \int_{f=0}^{\infty} Y(f)e^{j2\pi ft}df \\
&= \int_{f=0}^{\infty} Y(-f)e^{-j2\pi ft}df + \int_{f=0}^{\infty} Y(f)e^{j2\pi ft}df \\
&= \int_{f=0}^{\infty} Y(f)e^{-j2\pi ft}df + \int_{f=0}^{\infty} Y(f)e^{j2\pi ft}df \\
&= \int_{f=0}^{\infty} Y(f)[e^{-j2\pi ft} + e^{j2\pi ft}]df \\
&= 2\int_{f=0}^{f=\infty} Y(f)\cos(2\pi ft)df \\
&= 2\int_{f=0}^{(1-\alpha)/(2T)} Y(f)\cos(2\pi ft)df + 2\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} Y(f)\cos(2\pi ft)df \\
&= 2\int_{f=0}^{(1-\alpha)/(2T)} T\cos(2\pi ft)df + 2\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \frac{T}{2}[1-\sin(\pi T(f-\frac{1}{2T})/\alpha)]\cos(2\pi ft)df \\
&= 2T\frac{\sin(2\pi ft)}{2\pi t}\Big|_{f=0}^{(1-\alpha)/(2T)} + 2\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \frac{T}{2}[1-\sin(\pi T(f-\frac{1}{2T})/\alpha)]\cos(2\pi ft)df \\
&= T\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + T\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \cos(2\pi ft) - \sin(\pi T(f-\frac{1}{2T})/\alpha)\cos(2\pi ft)df
\end{aligned}
$$

$$
\begin{aligned}
&= T\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + T\frac{\sin(2\pi f t)}{2\pi t}\Big|_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} - T\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \sin(\pi T(f-\frac{1}{2T})/\alpha)\cos(2\pi f t)df \\
&= T\frac{\sin(\pi(1-\alpha)t/T)}{\pi t} + T\frac{\sin(\pi(1+\alpha)t/T)}{2\pi t} - T\frac{\sin(\pi(1-\alpha)t/T)}{2\pi t} \\
&\quad -T\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \sin(\pi T(f-\frac{1}{2T})/\alpha)\cos(2\pi f t)df \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad -T/2\int_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} [\sin(\pi T(f-\frac{1}{2T})/\alpha + 2\pi f t) + \sin(\pi T(f-\frac{1}{2T})/\alpha - 2\pi f t)]df \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad +T/2[\frac{\cos(\pi T(f-\frac{1}{2T})/\alpha + 2\pi f t)}{\pi T/\alpha + 2\pi t} + \frac{\cos(\pi T(f-\frac{1}{2T})/\alpha - 2\pi f t)}{\pi T/\alpha - 2\pi t}]\Big|_{f=(1-\alpha)/(2T)}^{(1+\alpha)/(2T)} \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad +T/2\left(\frac{\cos(\pi T((1+\alpha)/(2T)-\frac{1}{2T})/\alpha + 2\pi(1+\alpha)/(2T)t)}{\pi T/\alpha + 2\pi t} + \frac{\cos(\pi T((1+\alpha)/(2T)-\frac{1}{2T})/\alpha - 2\pi(1+\alpha)/(2T)t)}{\pi T/\alpha - 2\pi t}\right. \\
&\quad \left. -[\frac{\cos(\pi T((1-\alpha)/(2T)-\frac{1}{2T})/\alpha + 2\pi(1-\alpha)/(2T)t)}{\pi T/\alpha + 2\pi t} + \frac{\cos(\pi T((1-\alpha)/(2T)-\frac{1}{2T})/\alpha - 2\pi(1-\alpha)/(2T)t)}{\pi T/\alpha - 2\pi t}]\right) \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad +T/2\left(\frac{\cos(2\pi(1+\alpha)/(2T)t + \pi/2)}{\pi T/\alpha + 2\pi t} + \frac{\cos(2\pi(1+\alpha)/(2T)t - \pi/2)}{\pi T/\alpha - 2\pi t}\right. \\
&\quad \left. -[\frac{\cos(2\pi(1-\alpha)/(2T)t - \pi/2)}{\pi T/\alpha + 2\pi t} + \frac{\cos(2\pi(1-\alpha)/(2T)t + \pi/2)}{\pi T/\alpha - 2\pi t}]\right) \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad -T/2\left(\frac{\sin(2\pi(1+\alpha)/(2T)t)}{\pi T/\alpha + 2\pi t} - \frac{\sin(2\pi(1+\alpha)/(2T)t)}{\pi T/\alpha - 2\pi t}\right. \\
&\quad \left. +[\frac{\sin(2\pi(1-\alpha)/(2T)t)}{\pi T/\alpha + 2\pi t} - \frac{\sin(2\pi(1-\alpha)/(2T)t)}{\pi T/\alpha - 2\pi t}]\right) \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad -\left(\frac{\sin(2\pi(1+\alpha)/(2T)t)}{2\pi/\alpha + 4\pi t/T} - \frac{\sin(2\pi(1+\alpha)/(2T)t)}{2\pi/\alpha - 4\pi t/T}\right. \\
&\quad \left. +[\frac{\sin(2\pi(1-\alpha)/(2T)t)}{2\pi/\alpha + 4\pi t/T} - \frac{\sin(2\pi(1-\alpha)/(2T)t)}{2\pi/\alpha - 4\pi t/T}]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad -\frac{\alpha}{2\pi}\left(\frac{\sin(2\pi(1+\alpha)/(2T)t)}{1+2\alpha t/T} - \frac{\sin(2\pi(1+\alpha)/(2T)t)}{1-2\alpha t/T}\right. \\
&\quad \left. +[\frac{\sin(2\pi(1-\alpha)/(2T)t)}{1+2\alpha t/T} - \frac{\sin(2\pi(1-\alpha)/(2T)t)}{1-2\alpha t/T}]\right) \\
&= \frac{\sin(\pi(1-\alpha)t/T)}{2\pi t/T} + \frac{\sin(\pi(1+\alpha)t/T)}{2\pi t/T} \\
&\quad +\frac{\alpha}{2\pi}\left(\sin(2\pi(1+\alpha)/(2T)t)\frac{4\alpha t/T}{1-(2\alpha t/T)^2}\right. \\
&\quad \left. +\sin(2\pi(1-\alpha)/(2T)t)\frac{4\alpha t/T}{1-(2\alpha t/T)^2}\right) \\
&= \frac{\sin(\pi t/T)}{\pi t/T}\cos(\pi\alpha t/T) \\
&\quad +\frac{4\alpha^2 t/T}{2\pi(1-(2\alpha t/T)^2)}\left(\sin(\pi(1+\alpha)t/T) + \sin(\pi(1-\alpha)t/T)\right) \\
&= \frac{\sin(\pi t/T)}{\pi t/T}\cos(\pi\alpha t/T) + \frac{4\alpha^2 t/T}{\pi(1-(2\alpha t/T)^2)}\left(\sin(\pi t/T)\cos(\pi\alpha t/T)\right) \\
&= \sin(\pi t/T)\cos(\pi\alpha t/T)\left(\frac{1}{\pi t/T} + \frac{4\alpha^2 t/T}{\pi(1-(2\alpha t/T)^2)}\right) \\
&= \sin(\pi t/T)\cos(\pi\alpha t/T)\left(\frac{1-(2\alpha t/T)^2+4(\alpha t/T)^2}{\pi t/T(1-(2\alpha t/T)^2)}\right) \\
&= \frac{\sin(\pi t/T)}{\pi t/T}\frac{\cos(\pi\alpha t/T)}{(1-4(\alpha t/T)^2)}.
\end{aligned}
$$

2.13 Appendix 2D: Orthogonality of Square Root Raised Cosine Pulses

In this appendix we show that the square root raised cosine pulses are orthogonal. Consider a square-root raised cosine pulse $x(t)$ given by

$$
x(t) = \frac{1}{\sqrt{T}}\frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T\cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T}.
$$

Claim:

$$
\int x(t)x(t-mT)dt = 0.
$$

Proof: Let $X(f)$ be the Fourier transform of $x(t)$. Then

$$
X(f) = \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}
$$

Let $z(t) = x(t-mT)$. Then

$$
\begin{aligned}
Z(f) &= X(f)e^{-j2\pi fmT} \\
&= \begin{cases} \sqrt{T}e^{-j2\pi fmT}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ e^{-j2\pi fmT}\sqrt{\frac{T}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T}. \end{cases}
\end{aligned}
$$

From Parseval's, (see Appendix 2E below), identity we know

$$\int x(t)z^*(t)dt = \int X(f)Z^*(f)df.$$

Thus

$$\begin{aligned}
\int x(t)z^*(t)dt &= \int X(f)Z^*(f)df \\
&= \int e^{j2\pi fmT}|X(f)|^2 df \\
&= \int e^{j2\pi fmT}|X(f)|^2 df \\
&= y(mT).
\end{aligned}$$

where $y(t)$ is the raised cosine pulse shape. Let $Y(f) = |X(f)|^2$. The above integral is the inverse Fourier transform of $Y(f)$ evaluated at $t = mT$. The time domain raised cosine pulse is (see previous Appendix 2C)

$$y(t) = \frac{\sin(\pi t/T)}{\pi t/T}\frac{\cos(\pi\alpha t/T)}{(1-4(\alpha t/T)^2)}.$$

Clearly $y(mT)$ is zero except when $m = 0$ in which case it is 1.

2.14 Appendix 2E: Parseval's Theorem

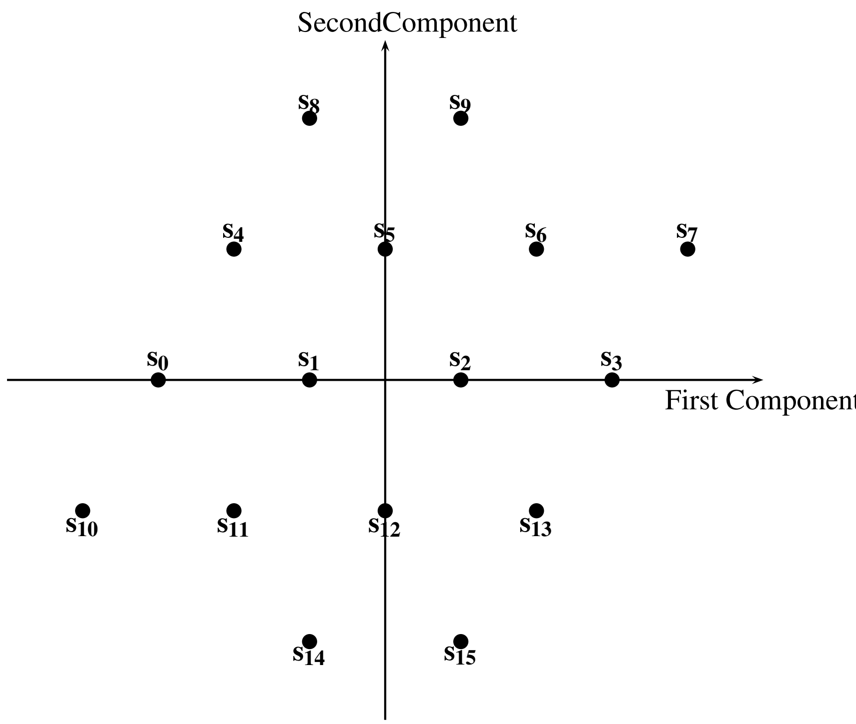
Parseval's theorem states that the inner product of two time domain signals is the same as the inner product of the corresponding two frequency domain signals.

$$\begin{aligned}
\int_t x(t)y^*(t)dt &= \int_t x(t)\left[\int_f Y(f)e^{j2\pi f_1 t}df\right]^* dt \\
&= \int_f \left[\int_t x(t)e^{-j2\pi f_1 t}dt\right] Y^*(f)df \\
&= \int_f X(f)Y^*(f)df.
\end{aligned}$$

2.15 Problems

1. Consider the following 16 signal vectors.

$$\begin{array}{ll}
\mathbf{s_0} = (-3,0) & \mathbf{s_8} = (-1,2\sqrt{3}) \\
\mathbf{s_1} = (-1,0) & \mathbf{s_9} = (+1,2\sqrt{3}) \\
\mathbf{s_2} = (+1,0) & \mathbf{s_{10}} = (-4,-\sqrt{3}) \\
\mathbf{s_3} = (+3,0) & \mathbf{s_{11}} = (-2,-\sqrt{3}) \\
\mathbf{s_4} = (-2,\sqrt{3}) & \mathbf{s_{12}} = (0,-\sqrt{3}) \\
\mathbf{s_5} = (0,\sqrt{3}) & \mathbf{s_{13}} = (+2,-\sqrt{3}) \\
\mathbf{s_6} = (+2,\sqrt{3}) & \mathbf{s_{14}} = (-1,-2\sqrt{3}) \\
\mathbf{s_7} = (+4,\sqrt{3}) & \mathbf{s_{15}} = (+1,-2\sqrt{3})
\end{array}$$



(a) Calculate the Euclidean distance $d_E(i,k)$ between every pair $(\mathbf{s_i},\mathbf{s_k})$ of distinct signals and the minimum Euclidean distance $d_{E,min}$ between distinct signals.

(b) Calculate the average energy per information bit, E_b.

(c) Calculate the normalized squared Euclidean distance $(d^2_{E,min}/E_b)$.

(d) Calculate the peak-to-average power ratio Γ_v for this constellation.

(e) If $\varphi_0(t) = \sqrt{2/T}\cos(2\pi f_c t)p_T(t)$ and $\varphi_1(t) = -\sqrt{2/T}\sin(2\pi f_c t)p_T(t)$ calculated the peak-to-average power ratio Γ_w for the set of 16 signal waveforms.

2. Consider the following 16 signals in 2 dimensions.

$$\begin{aligned}
\mathbf{s_0} &= (1.2028, -3.4096) \\
\mathbf{s_1} &= (2.8312, -2.2484) \\
\mathbf{s_2} &= (3.5934, -0.3994) \\
\mathbf{s_3} &= (3.2558, +1.5720) \\
\mathbf{s_4} &= (1.9220, +3.0622) \\
\mathbf{s_5} &= (0.0000, +3.6154) \\
\mathbf{s_6} &= (-1.9220, +3.0622) \\
\mathbf{s_7} &= (-3.2558, +1.5720) \\
\mathbf{s_8} &= (-3.5934, -0.3994) \\
\mathbf{s_9} &= (-2.8312, -2.2484) \\
\mathbf{s_{10}} &= (-1.2028, -3.4096) \\
\mathbf{s_{11}} &= (0.0000, -1.8116) \\
\mathbf{s_{12}} &= (1.0000, +1.2874) \\
\mathbf{s_{13}} &= (1.6054, -0.6188) \\
\mathbf{s_{14}} &= (-1.0000, +1.2874) \\
\mathbf{s_{15}} &= (-1.6054, -0.6188)
\end{aligned}$$

(a) Calculate the Euclidean distance $d_E(i,k)$ between every pair (s_i, s_k) of distinct signals and the minimum Eucidean distance $d_{E,min}$ between distinct signals.

(b) Plot the signals in the plane. On the sample plot draw circles around each signal point with radius being half the minimum distance $d_{E,min}$ calculated in part (a).

(c) Calculate the average energy per information bit, E_b.

(d) Calculate the normalized squared Euclidean distance $(d_{E,min}^2/E_b)$.

(e) Calculate the peak-to-average power ratio for this constellation.

(f) If $\varphi_0 = \sqrt{2/T}\cos(2\pi f_c t)p_T(t)$ and $\varphi_1 = -\sqrt{2/T}\sin(2\pi f_c t)p_T(t)$ calculated the peak-to-average power ratio for the set of 16 signal waveforms.

3. A data signal consists of an infinite sequence of rectangular pulses of duration T. That is

$$s(t) = \sum_{l=-\infty}^{\infty} b_l p_T(t - lT)$$

where $p_T(t)$ is 1 for $0 \leq t \leq T$ and zero elsewhere. The data is represented by b_l and is either +1 or -1. The signal is filtered by a low pass RC filter with impulse response

$$h(t) = \alpha e^{-\alpha t} u(t)$$

where $u(t)$ is one for $t > 0$ and is 0 otherwise. The filter output is sampled every T seconds.

(a) Find an expression for the output of the filter at time T in terms of $b_0, b_{-1}, b_{-2}, \ldots$.

(b) Suppose that $b_0 = +1$. Find the largest and smallest possible value (over all possible data sequences except b_0) of the sampled output.

4. CONSIDER REWORDING THIS FOR CLARITY (PART D). (a) Consider two signals of duration T seconds.

$$\begin{aligned}\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)\\ \varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t).\end{aligned}$$

Determine the minimum separation between f_0 and f_1 so that $\varphi_0(t)$ and $\varphi_1(t)$ are orthogonal. You may assume that $(f_0+f_1)T \gg 1$.

(b) Consider two signals of duration T seconds.

$$\begin{aligned}\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)\\ \varphi_1(t) &= \sqrt{\frac{2}{T}}\sin(2\pi f_1 t)p_T(t).\end{aligned}$$

Determine the minimum (non-zero) separation between f_0 and f_1 so that $\phi_0(t)$ and $\phi_1(t)$ are orthogonal. You may assume that $(f_0+f_1)T \gg 1$.

(c) Consider two sets of orthogonal signals of duration T seconds. Signal set one consists of signals $\varphi_0(t)$ and $\varphi_1(t)$.

$$\begin{aligned}\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)\\ \varphi_1(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_0 t)p_T(t).\end{aligned}$$

Signal set two consists of signals $\psi_0(t)$ and $\psi_1(t)$.

$$\begin{aligned}\psi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t)\\ \psi_1(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_1 t)p_T(t).\end{aligned}$$

Determine the minimum separation between f_0 and f_1 so that either signal in signal set 1 is orthogonal to any signal in signal set 2. You may assume that $(f_0+f_1)T \gg 1$.

(d) Define the bandwidth of a set of orthonormal signals as follows. Consider two different signal sets with different frequencies (like part c). Define the bandwidth to be the minimum separation in frequency between signals sets so that any signal from the first signal set is orthogonal to any signal in the second signal set. Determine the bandwidth as a function of T for the signal sets defined in part (a).

To be clear, consider that you have just one orthogonal signal, $\varphi_0(t)$ for one user and a second orthogonal signal $\psi_0(t)$ for a second user. So each user just gets one orthogonal signal.

$$\begin{aligned}\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)\\ \psi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t)\end{aligned}$$

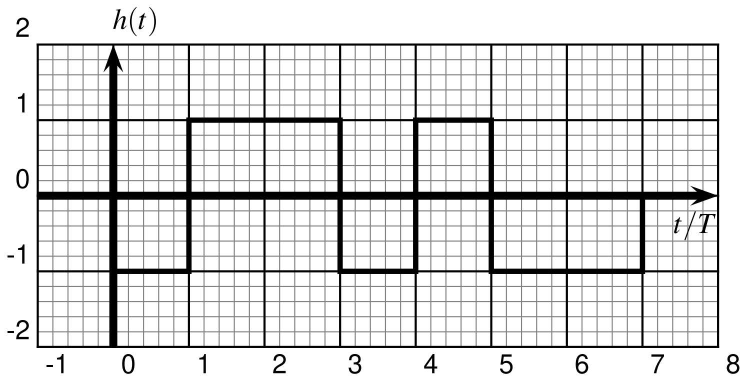
What is the bandwidth for this situation, i.e. the minimum separation between f_1 and f_0 so that $\varphi_0(t)$ is orthogonal to $\psi_0(t)$. This is really just a repeat of part (a). This is the same question as (c) except applied to just using the cosine without the sine function.

(e) From part (a) or (d) and (c) compute the time-bandwidth product.

Signals per frequency	WT
One orthogonal signal per frequency (part a)	
Two orthogonal signals per frequency (part c)	

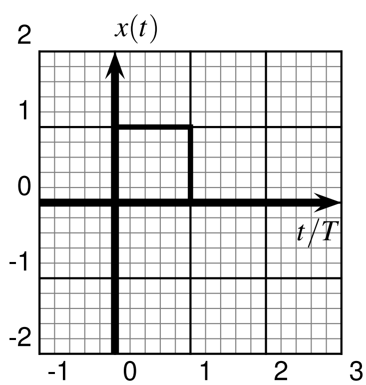
5. A filter has impulse response $h(t)$ shown in the figure below.

$$h(t) = -p_T(t) + p_T(t-T) + p_T(t-2T) - p_T(t-3T) + p_T(t-4T) - p_T(t-5T) - p_T(t-6T)$$



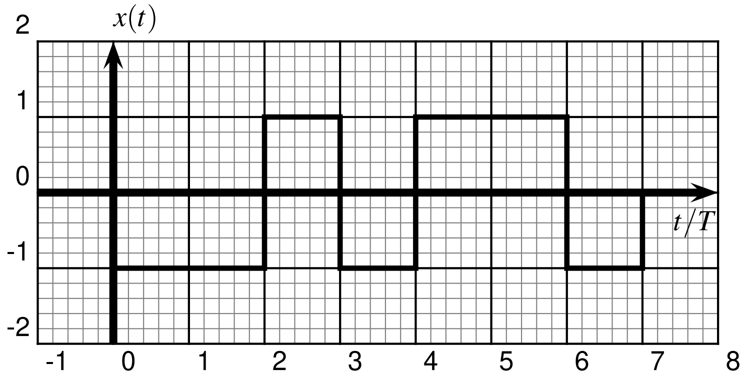
Note that each pulse lasts T seconds since the plot has a scale of t/T.

(a) The input $x(t)$ to the filter is a single rectangular pulse of duration T.



Find the output of the filter. Plot this from time $-T$ to time $9T$.

(b) Using the result above and superposition (linearity principle) find the output due to a sequence of pulses shown below. Plot the output from time -1 to time $15T$.



6. (a) The signal $x(t)$ consists of a sequence of pulses each of duration $T_c = T/7$ as shown in Figure 2.68.

$$s(t) = p_{T_c}(t) + p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c) - p_{T_c}(t - 4T_c) - p_{T_c}(t - 5T_c) - p_{T_c}(t - 6T_c)$$

Find the (matched) filter is given by.

$$h(t) = -p_{T_c}(t) - p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c) - p_{T_c}(t - 4T_c) + p_{T_c}(t - 5T_c) + p_{T_c}(t - 6T_c)$$

as shown below. Find the output of the matched filter and plot. The output should be a function of time beginning at time 0 and ending at time $2T = 14T_c$.

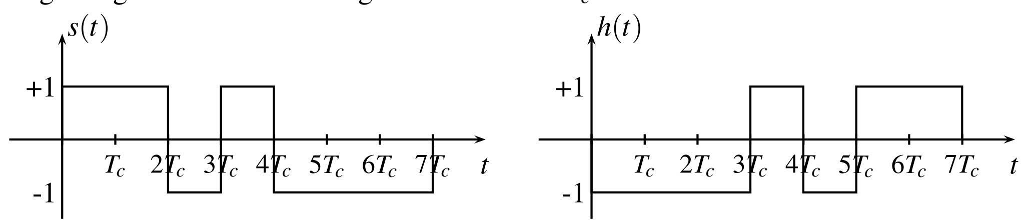
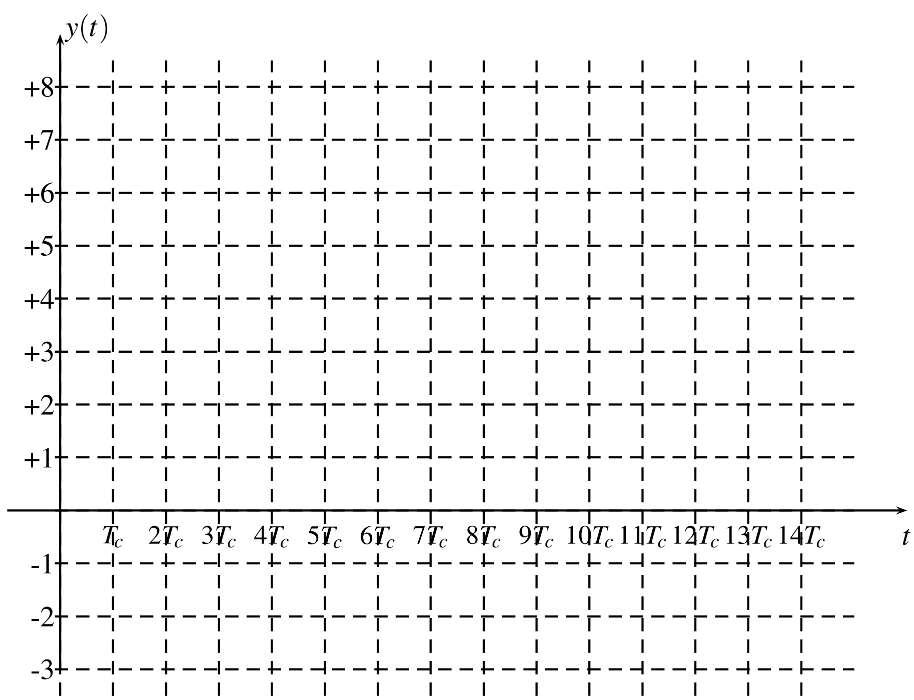
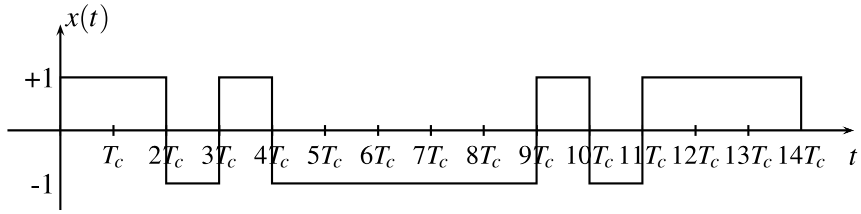


Figure 2.68: Signal $s(t)$ and filter $h(t)$.



(b) Find the filter output (for the same filter) when the input is $x(t) = s(t) - s(t-T)$. The output is a function beginning at time 0 and ending at time $21T_c = 3T$.



7. (a) The signal $x(t)$ is a complex signal with a real part and an imaginary part.

$$x(t) = x_I(t) + jx_Q(t).$$

The (matched) filter has a real and imaginary part and is given by

$$h(t) = h_I(t) + jh_Q(t).$$

Only the real part of the output of the matched filter is of interest.

(a) Find the real part of the output of the matched filter in terms of the real and imaginary part of the input and the real and imaginary parts of the impulse response.

$$y(t) = \Re\left\{\int h(t-\tau)x(\tau)d\tau\right\}.$$

That is, express $y(t)$ in terms of $h_I(t), h_Q(t), x_I(t), x_Q(t)$. You just need to substitute in for $h(t)$ and $x(t)$ and find the real part. The output consists of filtering by $h_I(t)$ and $h_Q(t)$ with $x_I(t)$ and $x_Q(t)$. You just need to figure out which of the two filters filters has which of the two inputs and how the overall result is determined.

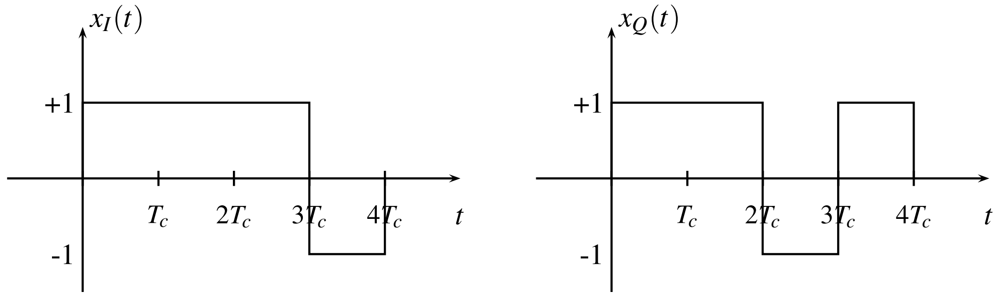
(b) Suppose that

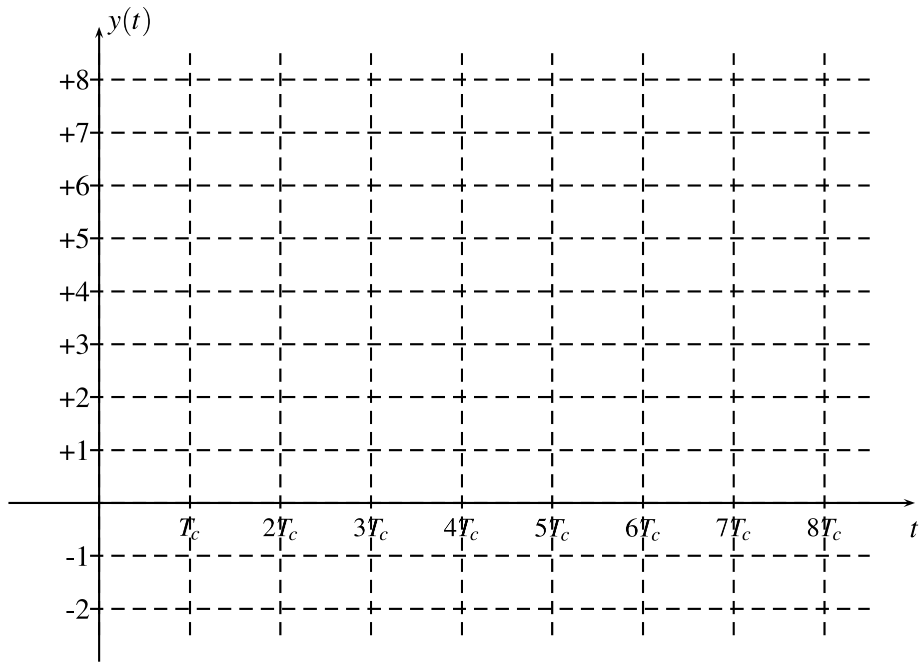
$$\begin{aligned} x_I(t) &= p_{T_c}(t) + p_{T_c}(t-T_c) + p_{T_c}(t-2T_c) - p_{T_c}(t-3T_c) \\ x_Q(t) &= p_{T_c}(t) + p_{T_c}(t-T_c) - p_{T_c}(t-2T_c) + p_{T_c}(t-3T_c) \end{aligned}$$

which are sequences of pulses each of duration $T_c = T/4$ shown below. The matched filter is given by

$$\begin{aligned} h(t) &= x^*(T-t) \\ &= x_I(T-t) - jx_Q(T-t) \\ h_I(t) &= x_I(T-t) \\ h_Q(t) &= -x_Q(T-t) \end{aligned}$$

Find the output of the matched filter and plot. Hint: Plot each term individually then plot the total. The output should be a function of time beginning at time 0 and ending at time $T = 8T_c$.





8. (a) Consider a signal $f(t)$ with unit energy defined over the time interval $[0,T]$ that is zero outside that time interval. For example $f(t)=\sqrt{1/T}p_T(t)$. Consider two signals of duration NT.

$$
\begin{aligned}
s_0(t) &= \sum_{i=0}^{N-1} s_{0,i} f(t-iT) \\
s_1(t) &= \sum_{i=0}^{N-1} s_{1,i} f(t-iT)
\end{aligned}
$$

Determine $(s_0(t),s_1(t))$ in terms of the sequence $s_{0,i}, i=0,1,...,N-1$ and $s_{1,i}, i=0,1,...,N-1$.

(b) Consider the two signals of duration $2T$ generated from $f(t)$ and the two vectors $s_0=(+1,+1)$ and $s_1=(+1,-1)$. We will form a matrix of vectors. In this case $N=2$.

$$
\begin{aligned}
H_2 &= \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \\
&= \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}.
\end{aligned}
$$

where the first row can be used to generate a first signal and the second row used to generate a second signal. Using part (a) show that the two signals

$$s_0(t) = s_{0,0}f(t)p_T(t) + s_{0,1}f(t-T)p_T(t-T)$$
$$s_1(t) = s_{1,0}f(t)p_T(t) + s_{1,1}f(t-T)p_T(t-T)$$

are orthogonal.

(c) Now consider four signals $s_0(t)$, $s_1(t)$, $s_2(t)$, $s_3(t)$ generated with a matrix H_4 where

$$H_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}.$$

Each row of the matrix represents one of four signals. The components of a row are the coefficients of $f(t)$ in different time intervals. Show that using part (a) that any distinct pair of signals is orthogonal.

(d) Consider H_8

$$\begin{aligned} H_8 &= \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} \\ &= \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{bmatrix} \end{aligned}$$

Show that the signals in the set generated by H_8 are orthogonal using the fact that the signals in set generated by H_4 are orthogonal.

9. (a) Consider a complex sequence of length $N = 10$;

$$x = (x_0, \ldots, x_9) = (-1-j, 1+j, 1+j, -1+j, 1+j, -1+j, 1+j, 1-j, 1-j, -1+j)$$

Now consider a matched filter $h(n) = x^*((N-1)-n) = x^*(9-n)$ Determine the output of the matched filter when the input is x. That is determine y where

$$y(n) = \Re\left[\sum_{l=0}^{N-1} h(n-l)x(l)\right]$$

and $\Re(x)$ is the real part of x. That is, plot the result.

(b) Consider the sequence of length 20

$$\begin{aligned} \Re(u) &= [\Re(x), \quad +\Im(x)] \\ \Im(u) &= [\Re(x), \quad -\Im(x)] \\ u &= \Re(u) + j\Im(u) \end{aligned}$$

That is, the first half of the real part of u is the real part of x. The second half of the real part of u is the imaginary part of x. Similarly for the imaginary part of u. Determine the output of a filter matched to the sequence u. Plot the result.

10. A first signal set with $M = 16$ signals in two dimensions that can transmit 4 bits of information has the following signals.

$$
\begin{aligned}
s_0 &= A(-3,-3)\\
s_1 &= A(-3,-1)\\
s_2 &= A(-3,+3)\\
s_3 &= A(-3,+1)\\
s_4 &= A(-1,-3)\\
s_5 &= A(-1,-1)\\
s_6 &= A(-1,+3)\\
s_7 &= A(-1,+1)\\
s_8 &= A(+1,-3)\\
s_9 &= A(+1,-1)\\
s_{10} &= A(+1,+3)\\
s_{11} &= A(+1,+1)\\
s_{12} &= A(+3,-3)\\
s_{13} &= A(+3,-1)\\
s_{14} &= A(+3,+3)\\
s_{15} &= A(+3,+1)
\end{aligned}
$$

(a) Simulate the probability of error for this signal set as a function of the signal-to-noise ratio (E_b/N_0). That is, plot the probability of choosing the wrong transmitted signal at the receiver as a function of the signal-to-noise ratio. Consider signal-to-noise ratios that yield an error probability between 0.0001 and 1.

(b) Repeat part a for the signal set below.

$$
\begin{aligned}
s_0 &= A(+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1)\\
s_1 &= A(+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1)\\
s_2 &= A(+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1)\\
s_3 &= A(+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,+1)\\
s_4 &= A(+1,+1,+1,+1,-1,-1,-1,-1,+1,+1,+1,+1,-1,-1,-1,-1)\\
s_5 &= A(+1,-1,+1,-1,-1,+1,-1,+1,+1,-1,+1,-1,-1,+1,-1,+1)\\
s_6 &= A(+1,+1,-1,-1,-1,-1,+1,+1,+1,+1,-1,-1,-1,-1,+1,+1)\\
s_7 &= A(+1,-1,-1,+1,-1,+1,+1,-1,+1,-1,-1,+1,-1,+1,+1,-1)\\
s_8 &= A(+1,+1,+1,+1,+1,+1,+1,+1,-1,-1,-1,-1,-1,-1,-1,-1)\\
s_9 &= A(+1,-1,+1,-1,+1,-1,+1,-1,-1,+1,-1,+1,-1,+1,-1,+1)\\
s_{10} &= A(+1,+1,-1,-1,+1,+1,-1,-1,-1,-1,+1,+1,-1,-1,+1,+1)\\
s_{11} &= A(+1,-1,-1,+1,+1,-1,-1,+1,-1,+1,+1,-1,-1,+1,+1,-1)\\
s_{12} &= A(+1,+1,+1,+1,-1,-1,-1,-1,-1,-1,-1,-1,+1,+1,+1,+1)\\
s_{13} &= A(+1,-1,+1,-1,-1,+1,-1,+1,-1,+1,-1,+1,+1,-1,+1,-1)\\
s_{14} &= A(+1,+1,-1,-1,-1,-1,+1,+1,-1,-1,+1,+1,+1,+1,-1,-1)\\
s_{15} &= A(+1,-1,-1,+1,-1,+1,+1,-1,-1,+1,+1,-1,+1,-1,-1,+1)
\end{aligned}
$$

(c) Determine the improvement in signal-to-noise ratio required for an error probability of 0.0001 by using the smaller rate (bits/dimension) of the signal set in part (b) compared to part (a).

Use the code below as the starting point.

```
clear all
A=1;
signal_set1
N=length(s(1,:));
E=sum(sum(abs(s).^2))/16;
Eb=E/4;
for j1=1:29
    EbN0dB(j1)=(j1-1)/2;                % Change values of EbN0 (dB)
    EbN0=10.^(EbN0dB(j1)/10);           % Convert to non-dB units
    N0=Eb/EbN0;                         % Determine N0 from Eb and Eb/N0
    sigma=sqrt(N0/2);                   % Sqrt of variance
    biterror=0;
    n=0;                                % Number of iterations
    while (biterror < 100)              % Do this loop till you count a certain number
        data=randi(16,1);               % Choose a signal at random
        r=s(data,:)+sigma*randn(1,N);   % Generate received signal


%==========================================================================
%                    YOU FILL IN THIS PART.
%==========================================================================
%  a)  FIND THE SIGNAL CLOSEST TO THE RECEIVED SIGNAL
```

```
%  b)  FIND THE BITS (called this bhat) ASSOCIATED WITH THAT SIGNAL
%  c)  FIND THE BITS (called this b) ASSOCIATED WITH THE ACTUAL TRANSMITTED SIGNAL
%  Try to write the code for this without using a for loop to keep speed up.
%=========================================================================

        biterror =biterror+sum(abs(b-bhat));  % Count the number of errors
        n=n+1;
    end

    Pe(j1)=biterror/(4*n);

end
semilogy(EbN0dB,Pe,'LineWidth',2)
grid on
xlabel('$E_b/N_0$ (dB)','FontSize',16,'Interpreter','Latex')
ylabel('$P_{e,b}$', 'FontSize',16,'Interpreter','Latex','Rotation',0)
set(gca,'FontSize',16)
axis([0 14 1e-6 1])
```

```
signal_set1.m

%  First Signal Set
s( 1,:) = A *[-3, -3];
s( 2,:) = A *[-3, -1];
s( 3,:) = A *[-3, +3];
s( 4,:) = A *[-3, +1];
%=====================================================================
s( 5,:) = A *[-1, -3];
s( 6,:) = A *[-1, -1];
s( 7,:) = A *[-1, +3];
s( 8,:) = A *[-1, +1];
%=====================================================================
s( 9,:) = A *[+1, -3];
s(10,:) = A *[+1, -1];
s(11,:) = A *[+1, +3];
s(12,:) = A *[+1, +1];
%=====================================================================
s(13,:) = A *[+3, -3];
s(14,:) = A *[+3, -1];
s(15,:) = A *[+3, +3];
s(16,:) = A *[+3, +1];
%=====================================================================
```

11. Consider a bandpass signals $x(t)$ with frequency content $X(f)$ generated from two signals $x_I(t)$ and $x_Q(t)$ with frequency content $X_I(f)$ and $X_Q(f)$ as in the notes. The point of this homework is to show you can recover $x_I(t)$ from $x(t)$.

    (i) Find the frequency content of the signal $y_I(t) = x(t)\sqrt{2}\cos(2\pi f_c t)$. in terms of $X_I(f)$ and $X_Q(f)$.

    (ii) Find the frequency content of $y_I(t) * g_{LP}(t)$ where $g_{LP}(t)$ is an ideal low pass filter.

12. A communication system uses BPSK modulation to transmit data bits $b_l, l = 0,1,2,\ldots$. In the transmitter a sequence of rectangular pulses is mixed to a carrier frequency by multiplying the rectangular pulses by $\sqrt{2P}\cos(2\pi f_1 t)$.

$$s(t) = \sqrt{2P} \sum_{l=0,1,\ldots} b_l p_T(t - lT)\cos(2\pi f_1 t)$$

At the receiver the received signal is first mixed down to baseband by multiplying by $\sqrt{2/T}\cos(2\pi f_2 t)$ where $f_2 - f_1 = \Delta f$ is the offset of the two oscillators. After the signal is mixed down it is filtered with a matched filter (that is $h(t) = p_T(t)$). The filter is sampled at time $t = iT$ for $i = 1,2,\ldots$. In addition the signal is mixed down by multiplying by $-\sqrt{2/T}\sin(2\pi f_2 t)$. Let $y_c(iT)$ denote the first output and $y_s(iT)$ denote the second output. Then

$$\begin{aligned} y_c(iT) &= \int_{-\infty}^{\infty} s(\tau)\sqrt{\frac{2}{T}}\cos(2\pi f_2 \tau)h(iT-\tau)d\tau \\ y_s(iT) &= -\int_{-\infty}^{\infty} s(\tau)\sqrt{\frac{2}{T}}\sin(2\pi f_2 \tau)h(iT-\tau)d\tau \end{aligned}$$

   (a) In the absence of noise evaluate the outputs $y_c(iT)$ and $y_s(iT)$ in terms of b_{i-1}, $E = PT$, $\Delta f T$ and i. Ignore double frequency terms in evaluating the output. That is, derive an expression for $y_c(iT)$ and $y_s(iT)$. (Useful trig identity $\sin(u) - \sin(v) = 2\cos(\frac{u+v}{2})\sin(\frac{u-v}{2})$).

(b) Assume you buy two crystal oscillators at a 10MHz nominal frequency that have $\pm$ 10 PPM accuracy. That is, $f_{\text{actual}} = f_{\text{nominal}}(1 \pm 10/10^6)$. Assume that the data rate is 100kbps ($T = 10^{-5}$), that the data bits are all positive ($b_i = 1, i = 0, 1, 2, ..., 500$) and that $E = 1$.

i. Are the double frequency terms negligible?

ii. Plot the output of the filters $y_c(iT)$ and $y_s(iT)$ as a function of i for $1 \le i \le 500$. Assume all the data bits are +1 and $f_2 - f_1 = \Delta f = 200$ Hz and $T = 10^{-5}$.

iii. Plot $y_s(iT)$ versus $y_c(iT)$ for the case where $\Delta f T = 0.01$, the data bits are all 1 for $i = 0, ..., 39$. Plot each point with a "marker". Use the Matlab command `plot(yc,ys,'+')` where yc is a vector of samples representing $y_c(iT)$ and `ys` is a vector of samples representing $y_s(iT)$ for $i = 0, 1, ..., 39$.

13. Consider a bandpass signal $x(t)$ with lowpass complex representation $\tilde{x}(t) = x_I(t) + jx_Q(t)$. Show that

$$\int_t |\tilde{x}(t)|^2 dt = \int_t |x_I(t) + jx_Q(t)|^2 dt \quad = \quad \int |x(t)|^2 dt.$$

14. Consider two bandpass signals $x(t)$ and $y(t)$ with lowpass complex representation $\tilde{x}(t) = x_I(t) + jx_Q(t)$ and $\tilde{y}(t) = y_I(t) + jy_Q(t)$. Show that

$$\Re[\int_t \tilde{x}(t)\tilde{y}^*(t)dt] = \quad \int_t x(t)y(t)dt.$$

15. Suppose you want to implement a square-root raised cosine pulse shape. The pulse shape theoretically last forever. You truncate the pulse to some number of samples. Suppose you generated 8 samples per T seconds where T is the time between pulses. You can assume $T = 1$. You can use Matlab. Do the following for $\alpha = 0.05, 0.15, 0.25$.

(a) Determine how many samples you need for the maximum sample you ignore is 40dB down relative to the peak sample. That is the amplitude of the any sample you ignore is 0.01 times as small as the peak sample.

(b) Determine the frequency content of the signal (plot the frequency content in dB) versus f.

(c) Suppose you add samples to the truncated pulse. You want to add samples so that the last sample is as close as possible to 0. What is your new pulse length in samples. Compare the frequency response to that in part (b).

16. Consider a baseband signal

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

where $x_I(t)$ and $x_Q(t)$ are baseband signals with frequency content limited to $[-W, +W]$. Let $X_I(f)$ and $X_Q(f)$ be the frequency content of the signals. So

$$X_I(f) = X_Q(f) = 0 \text{ for } f \notin [-W, W]$$

The energy of the lowpass complex signal is

$$E_l = \int |\tilde{x}(t)|^2 dt$$

The passband signal is

$$x(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t)$$

where $f_c > W$. The energy of the passband signals is

$$E_p = \int |x(t)|^2 dt$$

Show that $E_l = E_p$.

Hint: Derive expressions for the energy in the frequency domain for $x(t)$. Use Parseval's Theorem

$$\int u(t)v^*(t)dt = \int U(f)V^*(f)df.$$

where $u(t) = x_I(t)$ and $v(t) = x_I(t)\cos(2\pi(2f_c)t)$.

17. Consider a baseband signal

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

where $x_I(t)$ and $x_Q(t)$ are baseband signals with frequency content limited to $[-W,+W]$. Let $X_I(f)$ and $X_Q(f)$ be the frequency content of the signals. So

$$X_I(f) = X_Q(f) = 0 \text{ for } f \notin [-W,W].$$

The energy of the lowpass complex signal is

$$E_l = \int |\tilde{x}(t)|^2 dt.$$

The passband signal is

$$x(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t)$$

where $f_c > W$. The energy of the passband signals is

$$E_p = \int |x(t)|^2 dt.$$

Show that $E_l = E_p$.

Hint: Derive expressions for the energy E_p for $x(t)$ and convert to the frequency domain. Use Parseval's Theorem

$$\int u(t)v^*(t)dt = \int U(f)V^*(f)df.$$

where, for example, $u(t) = x_I(t)$ and $v(t) = x_I(t)\cos(2\pi(2f_c)t)$.

18. Consider a baseband signal

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

where $x_I(t)$ and $x_Q(t)$ are baseband signals with frequency content limited to $[-W,+W]$. Let $X_I(f)$ and $X_Q(f)$ be the frequency content of the signals. So

$$X_I(f) = X_Q(f) = 0 \text{ for } f \notin [-W,W]$$

The passband signal is

$$x(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t)$$

where $f_c > W$.

Suppose that $x(t)$ is the transmitted signal. The received signal is

$$r(t) = x(t-\tau)$$

where τ is a delay. The receiver processes the received signal by mixing down to baseband and filtering out the double frequency terms. That is,

$$\begin{aligned} y_I(t) &= +\int h(t-s)r(s)\sqrt{2}\cos(2\pi f_c s)ds \\ y_Q(t) &= -\int h(t-s)r(s)\sqrt{2}\sin(2\pi f_c s)ds \end{aligned}$$

where $h(t)$ is an ideal low pass filter passing frequencies between $-W$ and W and removing other frequencies. That is,

$$\begin{aligned} \int h(t-s)x_I(s)ds &= x_I(t) \\ \int h(t-s)x_Q(s)ds &= x_Q(t). \end{aligned}$$

Derive an expression relating $y_I(t), y_Q(t)$ to $x_I(t), x_Q(t)$ and τ.

19. Consider a baseband signal

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

where $x_I(t)$ and $x_Q(t)$ are baseband signals with frequency content limited to $[-W,+W]$. Let $X_I(f)$ and $X_Q(f)$ be the frequency content of the signals. So

$$X_I(f) = X_Q(f) = 0 \text{ for } f \notin [-W,W]$$

The passband signal is

$$\begin{aligned} x(t) &= x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t) \\ X(f) &= \frac{\sqrt{2}}{2}[X_I(f-f_c)+X_I(f+f_c)] - j\frac{\sqrt{2}}{2}[X_Q(f-f_c)-X_Q(f+f_c)] \end{aligned}$$

where $f_c > W$. Suppose that the signal is mixed down to baseband using frequency f_d rather than f_c. However, $|f_d - f_c| \ll W$ so that a low pass filter of a frequency shift of the baseband signal by $\Delta f = f_d - f_c$ is still in the passband of the low pass filter.

Let $w_I(t), w_Q(t), W_I(f), W_Q(f)$ be the result (time domain and frequency domain) of mixing down (without filtering) the signal $x(t)$ by $\sqrt{2}\cos(2\pi f_d t)$ and $-\sqrt{2}\sin(2\pi f_d t)$. That is,

$$\begin{aligned} w_I(t) &= +x(t)\sqrt{2}\cos(2\pi f_d t) \\ w_Q(t) &= -x(t)\sqrt{2}\sin(2\pi f_d t). \end{aligned}$$

Let $\hat{x}_I(t)$ and $\hat{x}_Q(t)$ be the result after filtering $w_I(t)$ and $w_Q(t)$ to remove the double frequency terms.

Show that

$$\begin{aligned} \hat{x}_I(t) + j\hat{x}_Q(t) &= [x_I(t)+jx_Q(t)]e^{-j2\pi\Delta f t} \\ &= [x_I(t)+jx_Q(t)](\cos(2\pi\Delta f t) - j\sin(2\pi\Delta f t)) \\ &= [x_I(t)\cos(2\pi\Delta f t) + x_Q(t)\sin(2\pi\Delta f t)] \\ &\quad + j[x_Q(t)\cos(2\pi\Delta f t) - x_I(t)\sin(2\pi\Delta f t)] \end{aligned}$$

Hint: See Lecture Notes 2 and work in the frequency domain.

20. Consider a baseband signal

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

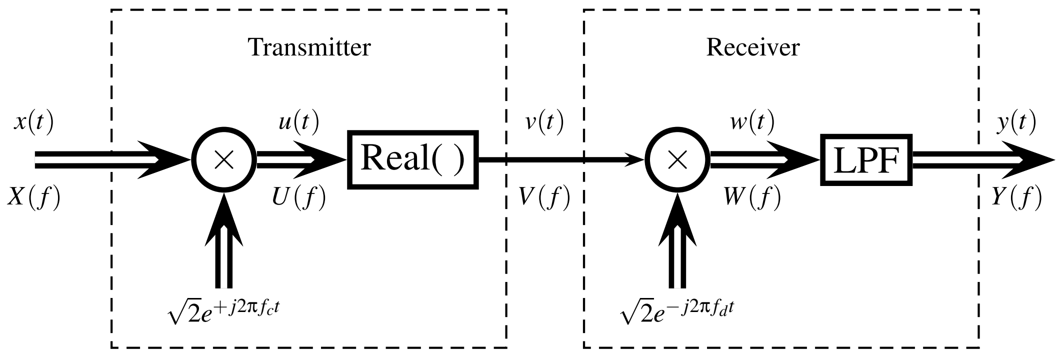
where $x_I(t)$ and $x_Q(t)$ are baseband signals with frequency content limited to $[-W, +W]$. Let $X_I(f)$ and $X_Q(f)$ be the frequency content of the signals. So

$$X_I(f) = X_Q(f) = 0 \text{ for } f \notin [-W, W]$$

The passband signal is

$$\begin{aligned} x(t) &= x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t) \\ X(f) &= \frac{\sqrt{2}}{2}[X_I(f-f_c) + X_I(f+f_c)] - j\frac{\sqrt{2}}{2}[X_Q(f-f_c) - X_Q(f+f_c)] \end{aligned}$$

where $f_c > W$.



Let $y(t) = y_I(t) + jy_Q(t)$.

(a) Derive the relation between $x(t) = x_I(t) + jx_Q(t)$ and $y(t) = y_I(t) + jy_Q(t)$ in the time domain.

(b) Derive the relation between $X(f)$ and $Y(f)$ in the frequency domain.

21. (a) Consider a narrowband filter with

$$h(t) = 2g(t)\cos(\omega_c t)$$

and a narrowband signal with

$$s(t) = v(t)\sin(\omega_c t)$$

Show that the output of the filter $h(t)$ when $s(t)$ is the input is

$$\hat{s}(t) = \hat{v}(t)\sin(\omega_c t)$$

where

$$\hat{v}(t) = \int_{-\infty}^{\infty} g(t-\tau)v(\tau)d\tau.$$

(b) For the same filter but with a signal

$$s(t) = v(t)\cos(\omega_c t + \theta)$$

show that

$$\hat{s}(t) = \hat{v}(t)\cos(\omega_c t + \theta)$$

22. A signal $s(t)$ of duation T consists of 15 consectuive pusles (of duration $T/15$) of amplitude ± 1. The sequence of amplitudes is (-1 -1 -1 -1 +1 -1 +1 -1 -1 +1 +1 -1 +1 +1 +1). Assume that this signal is input to a linear time-invariant system (filter) with impulse response $h(t) = s(T-t)$. Find (plot) the output of the filter.

23. A signal $s(t)$ of duration T consists of 15 consecutive pulses (of duration $T/15$) of amplitude ± 1. The sequence of amplitudes is (-1 -1 -1 -1 +1 -1 +1 -1 -1 +1 +1 -1 +1 +1 +1). Data is transmitted using this signal. The data sequence 011010 is transmitted in 6T seconds by transmitting the signal $s(t)$ for the first T seconds, $-s(t-T)$ during the interval $[T, 2T]$, $-s(t-2T)$ during the interval $[2T, 3T]$ and so on. The total signal transmitted is thus

$$s(t) - s(t-T) - s(t-2T) + s(t-3T) - s(t-4T) + s(t-5T)$$

Assume that this signal is input to a linear time-invariant system (filter) with impulse response $h(t) = s(T-t)$. Find (plot) the output of the filter.

24. (a) Data is transmitted using a rectangular pulse with amplitude A or $-A$ and duration T. The receiver filters the signal with a filter impulse resonse

$$h(t) = e^{-\alpha t} u(t).$$

The filter output is sampled at time T. Find $y(T)$, the filter output at time T if the input is a pulse with amplitude $+A$.

(b) White Gaussian noise with power spectral density $N_0/2$ is the input to an RC filter with impulse response

$$h(t) = e^{-\alpha t} u(t)$$

where $u(t)$ is one for $t > 0$ and is 0 otherwise. Find σ^2, the variance of the noise at the output of the filter.

(c) Let $\beta = \alpha T$. Find the value of β that maximizes $y(T)/\sigma$, the signal-to-noise ratio. (You probably need to do this numerically, e.g. Matlab). Find the resulting signal-to-noise ratio for the best choise of β. Express your answer in terms of the energy of the signal and the noise power spectral density (E/N_0).

25. Consider the systems shown below. Assume $G_{LP}(f)$ is an ideal low pass filter, namely

$$G_{LP}(f) = \begin{cases} 1, & |f| < W \\ 0, & |f| \geq W. \end{cases}$$

Also assume $f_c > 2W$. Show that the system is an ideal bandpass filter; that is show

$$H(f) = \begin{cases} 1, & |f - f_c| < W \\ 1, & |f + f_c| < W \\ 0, & \text{otherwise} \end{cases}$$

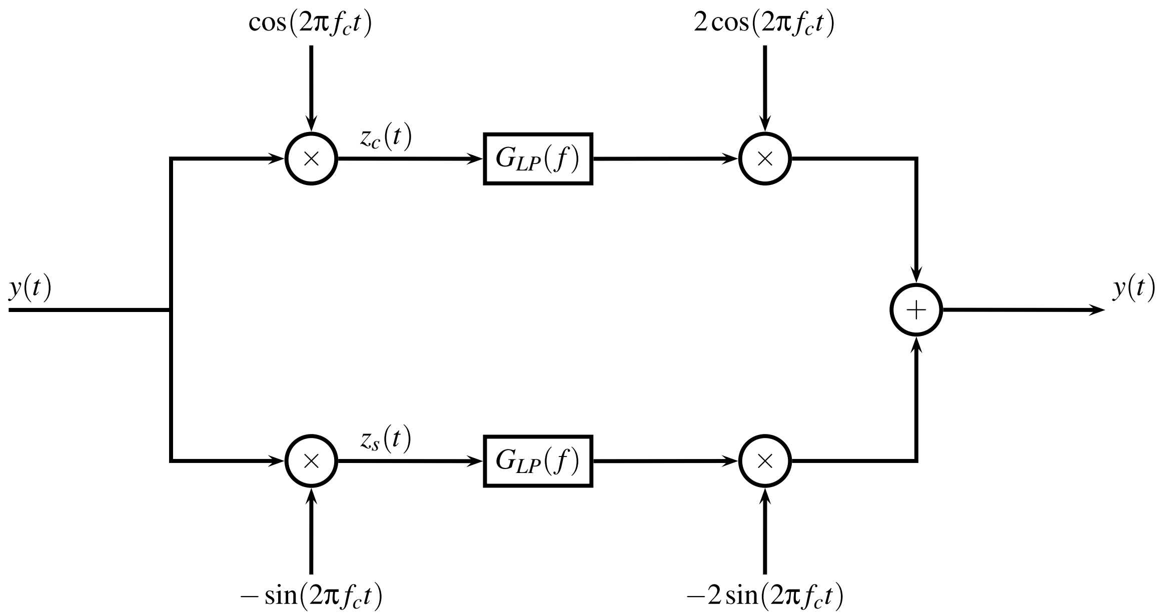
where $H(f)$ is the transfer function from the input to the output.

A communication system uses a combination of transmit, channel and receiver filtering that satisfies

$$P_r(f) = j2T_b \sin(2\pi f T_b), |f| < \frac{1}{2T_b}.$$

(a) Determine the impulse response of the system. (b) Determine the precoding that should be used to remove the intersymbol interference.

26. Consider a time-shifted set of orthonormal rectangular pulses with amplitudes given by



s(0)= 0.0460 - 0.0460j
s(1)= -0.1320 - 0.0020j
s(2)= -0.0130 + 0.0790j
s(3)= 0.1430 + 0.0130j
s(4)= 0.0920 + 0.0000j
s(5)= 0.1430 + 0.0130j
s(6)= -0.0130 + 0.0790j
s(7)= -0.1320 - 0.0020j
s(8)= 0.0460 - 0.0460j
s(9)= 0.0020 + 0.1320j
s(10)= -0.0790 + 0.0130j
s(11)= -0.0130 - 0.1430j
s(12)= 0.0000 - 0.0920j
s(13)= -0.0130 - 0.1430j
s(14)= -0.0790 + 0.0130j
s(15)= 0.0020 + 0.1320j

The signal is then

$$s(t) = \sum_{n=0}^{15} s(n) p_T(t - nT).$$

This signal is used in the preamble of the IEEE 802.11 system.The signal is filtered with a filter with impulse response $h(t) = s^*(16T - t)$.

(a) Find and plot the magnitude of the output of the filter.

(b) If the signal is repeated 8 times plot the real part, imaginary part and magnitude of the output of the same filter.

(c) If there is a frequency shift so that $\Delta f T = 0.01$ find the reduction in the magnitude of the output of the filter for the signal part (b).

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