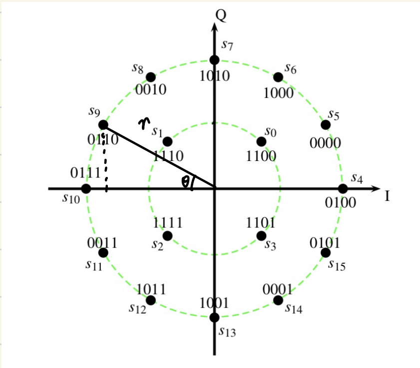
Homework 2 YUZHAN JIANG

1、(a) we have

$\min_{m \neq 1} d_E^2(S_m, S_1) = 4$ between signals on the Outer circle



Let's consider S_{10} and S_9, the angle between S_{10} and S_9 is $\frac{360°}{12} = 30°$

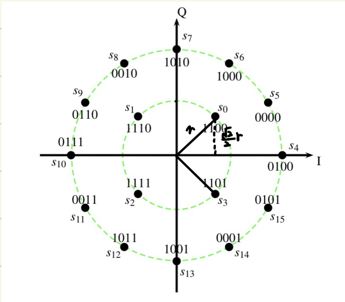
Let $S_{10} = (-r, 0)$ So that $S_9 = (-\frac{\sqrt{3}}{2}r, \frac{1}{2}r)$

$$d_E^2(S_{10}, S_9) = (-r + \tfrac{\sqrt{3}}{2}r)^2 + (\tfrac{1}{2}r)^2$$
$$= \left(\frac{\sqrt{3}-2}{2}\right)^2 r^2 + \frac{1}{4}r^2$$
$$= \left(\frac{3+4-4\sqrt{3}}{4}\right) r^2 + \frac{1}{4}r^2$$
$$= \left(\frac{8-4\sqrt{3}}{4}\right) r^2$$
$$= (2-\sqrt{3})\, r^2$$

Since $\min_{m \neq i} d_E^2(S_m, S_i) = 4 \Rightarrow d_E^2(S_{10}, S_9) = 4$

$$\therefore \quad (2-\sqrt{3})\, r^2 = 4$$
$$r^2 = \frac{4}{2-\sqrt{3}}$$
$$r = \sqrt{4(2+\sqrt{3})}$$
$$r = 2\sqrt{2+\sqrt{3}}$$

1. (b)



Let's consider S_0, S_3 on the inner circle, assume the radius of inner circle is r

the angle between S_0 and S_3 is $\frac{360^\circ}{4} = 90^\circ$

$\therefore \quad S_0 = \left(\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r\right) \qquad S_3 = \left(\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r\right)$

$$d_E^2(S_0, S_3) = \left(\frac{\sqrt{2}}{2}r - \frac{\sqrt{2}}{2}r\right)^2 + \left(\frac{\sqrt{2}}{2}r - \left(-\frac{\sqrt{2}}{2}\right)r\right)^2$$

$$= 2r^2$$

Since $\min\limits_{m \neq i} d_E^2(S_m, S_i) = 4$ on the inner circle

$\therefore \quad d_E^2(S_0, S_3) = 4 \implies 2r^2 = 4$

$r^2 = 2$

$r = \sqrt{2}$

$\therefore$ the radius of inner circle is $\sqrt{2}$

(C). Yes,

(d) Let r_1 be the radius of outer circle, r_2 be the radius of inner circle

$E_{0,1,2,3} = r_2^2$ for signal S_0, S_1, S_2, S_3

$E_{4,5,6,7,8,9,10,11,12,13,14,15} = r_1^2$ for signal S_i, $i \in [4,15]$

$$\therefore \quad \bar{E} = \frac{1}{M} \sum_{i=0}^{M-1} E_i$$

$$= \frac{1}{16}\left(4r_2^2 + 12r_1^2\right)$$

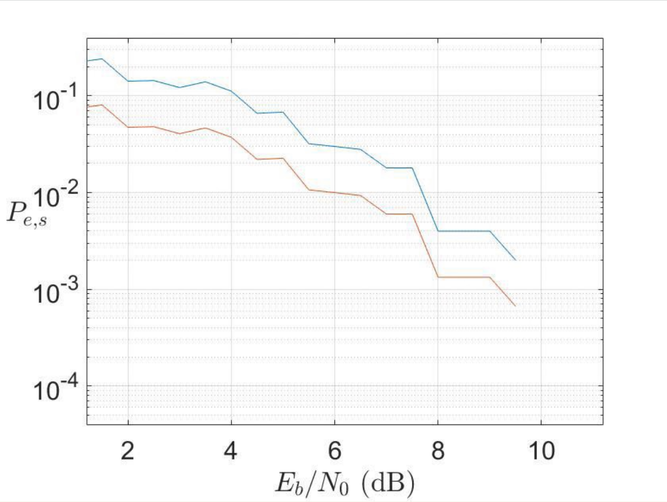
$$= \frac{1}{4}r_2^2 + \frac{3}{4}r_1^2$$

$$\therefore \quad \bar{E}_b = \bar{E} / \log_2(M)$$

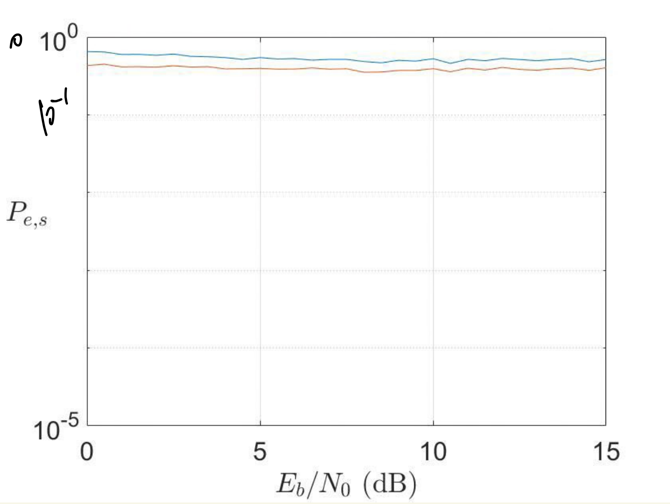
$$= \bar{E} / 4$$

$$= \frac{1}{16}r_2^2 + \frac{3}{16}r_1^2$$

1.(e)(f)



8PSK



16APSK

The bit error probability of 16 APSK stays steady which is between 0~0.1 and the bit error probability of 8PSK decrease sharply.

2 (a) $d_E(S_i, S_k)$ all results are computed by Julia program

$d_E(S_0, S_1) = 2$
$d_E(S_0, S_2) = 3.84$
$d_E(S_0, S_3) = 5.39$
$d_E(S_0, S_4) = 6.51$
$d_E(S_0, S_5) = 7.21$
$d_E(S_0, S_6) = 7.19$
$d_E(S_0, S_7) = 6.69$
$d_E(S_0, S_8) = 5.66$
$d_E(S_0, S_9) = 4.20$
$d_E(S_0, S_{10}) = 2.41$
$d_E(S_0, S_{11}) = 2$
$d_E(S_0, S_{12}) = 4.70$
$d_E(S_0, S_{13}) = 2.82$
$d_E(S_0, S_{14}) = 5.19$
$d_E(S_0, S_{15}) = 4.00$

$d_E(S_1, S_2) = 2.00$
$d_E(S_1, S_3) = 3.84$
$d_E(S_1, S_4) = 5.39$
$d_E(S_1, S_5) = 6.51$
$d_E(S_1, S_6) = 7.13$
$d_E(S_1, S_7) = 7.19$
$d_E(S_1, S_8) = 6.69$
$d_E(S_1, S_9) = 5.66$
$d_E(S_1, S_{10}) = 4.20$
$d_E(S_1, S_{11}) = 2.86$
$d_E(S_1, S_{12}) = 3.98$
$d_E(S_1, S_{13}) = 2.04$
$d_E(S_1, S_{14}) = 5.21$
$d_E(S_1, S_{15}) = 4.73$

$d_E(S_2, S_3) = 2.00$
$d_E(S_2, S_4) = 3.84$
$d_E(S_2, S_5) = 5.39$
$d_E(S_2, S_6) = 6.51$
$d_E(S_2, S_7) = 7.12$
$d_E(S_2, S_8) = 7.19$
$d_E(S_2, S_9) = 6.69$
$d_E(S_2, S_{10}) = 5.66$
$d_E(S_2, S_{11}) = 3.86$
$d_E(S_2, S_{12}) = 3.09$
$d_E(S_2, S_{13}) = 2.00$
$d_E(S_2, S_{14}) = 4.89$
$d_E(S_2, S_{15}) = 5.20$

$d_E(S_3, S_4) = 2.00$
$d_E(S_3, S_5) = 3.84$
$d_E(S_3, S_6) = 5.39$
$d_E(S_3, S_7) = 6.51$
$d_E(S_3, S_8) = 7.13$
$d_E(S_3, S_9) = 7.19$
$d_E(S_3, S_{10}) = 6.69$
$d_E(S_3, S_{11}) = 4.70$
$d_E(S_3, S_{12}) = 2.27$
$d_E(S_3, S_{13}) = 2.74$
$d_E(S_3, S_{14}) = 4.27$
$d_E(S_3, S_{15}) = 5.33$

2(a)

$d_E(S_4, S_5) = 2.00$
$d_E(S_4, S_6) = 3.84$
$d_E(S_4, S_7) = 5.39$
$d_E(S_4, S_8) = 6.51$
$d_E(S_4, S_9) = 7.13$
$d_E(S_4, S_{10}) = 7.19$
$d_E(S_4, S_{11}) = 5.24$
$d_E(S_4, S_{12}) = 2.00$
$d_E(S_4, S_{13}) = 3.69$
$d_E(S_4, S_{14}) = 3.42$
$d_E(S_4, S_{15}) = 5.10$

$d_E(S_5, S_6) = 2.00$
$d_E(S_5, S_7) = 3.84$
$d_E(S_5, S_8) = 5.39$
$d_E(S_5, S_9) = 6.51$
$d_E(S_5, S_{10}) = 7.13$
$d_E(S_5, S_{11}) = 5.43$
$d_E(S_5, S_{12}) = 2.53$
$d_E(S_5, S_{13}) = 4.53$
$d_E(S_5, S_{14}) = 2.53$
$d_E(S_5, S_{15}) = 4.53$

$d_E(S_6, S_7) = 2.00$
$d_E(S_6, S_8) = 3.84$
$d_E(S_6, S_9) = 5.39$
$d_E(S_6, S_{10}) = 6.51$
$d_E(S_6, S_{11}) = 5.24$
$d_E(S_6, S_{12}) = 3.42$
$d_E(S_6, S_{13}) = 5.10$
$d_E(S_6, S_{14}) = 2.00$
$d_E(S_6, S_{15}) = 3.69$

$d_E(S_7, S_8) = 2.00$
$d_E(S_7, S_9) = 3.84$
$d_E(S_7, S_{10}) = 5.39$
$d_E(S_7, S_{11}) = 4.70$
$d_E(S_7, S_{12}) = 4.27$
$d_E(S_7, S_{13}) = 5.33$
$d_E(S_7, S_{14}) = 2.27$
$d_E(S_7, S_{15}) = 2.74$

$d_E(S_8, S_9) = 2.00$
$d_E(S_8, S_{10}) = 3.84$
$d_E(S_8, S_{11}) = 3.86$
$d_E(S_8, S_{12}) = 4.89$
$d_E(S_8, S_{13}) = 5.20$
$d_E(S_8, S_{14}) = 3.09$
$d_E(S_8, S_{15}) = 2.00$

$d_E(S_9, S_{10}) = 2.00$
$d_E(S_9, S_{11}) = 2.86$
$d_E(S_9, S_{12}) = 5.21$
$d_E(S_9, S_{13}) = 4.73$
$d_E(S_9, S_{14}) = 3.98$
$d_E(S_9, S_{15}) = 2.04$

2(a)

$d_E(S_{10}, S_{11}) = 2.00$

$d_E(S_{10}, S_{12}) = 5.19$

$d_E(S_{10}, S_{13}) = 3.96$

$d_E(S_{10}, S_{14}) = 4.70$

$d_E(S_{10}, S_{15}) = 2.82$

$d_E(S_{11}, S_{12}) = 3.26$

$d_E(S_{11}, S_{13}) = 2.00$

$d_E(S_{11}, S_{14}) = 3.26$

$d_E(S_{11}, S_{15}) = 2.00$

$d_E(S_{12}, S_{13}) = 2.00$

$d_E(S_{12}, S_{14}) = 2.00$

$d_E(S_{12}, S_{15}) = 3.23$

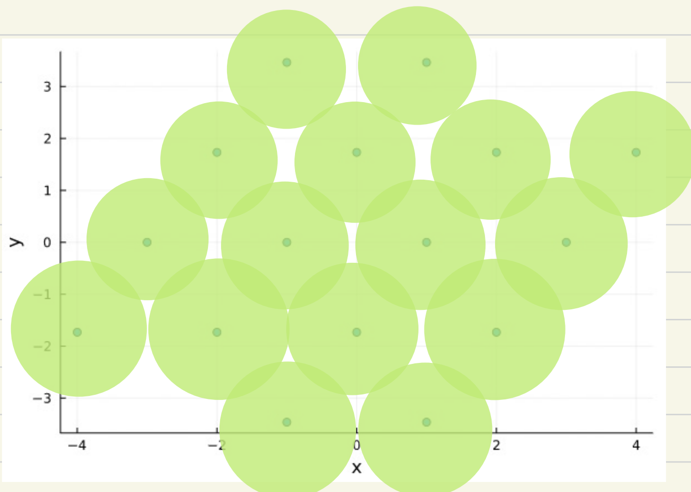
$d_E(S_{13}, S_{14}) = 3.23$

$d_E(S_{13}, S_{15}) = 3.21$

$d_E(S_{14}, S_{15}) = 2.00$

the minimum Euclidean distance between distinct signals is 2.00

2(b)



2(c) By the formula,

$$\bar{E_b} = \bar{E} / \log_2(M)$$, first we need compute $\bar{E}$

$$\bar{E} = \frac{1}{M} \sum_{i=0}^{M-1} E_i$$

$E_0 = 13.07$ $\quad$ $E_8 = 13.07$

$E_1 = 13.07$ $\quad$ $E_9 = 13.07$

$E_2 = 13.07$ $\quad$ $E_{10} = 13.07$

$E_3 = 13.07$ $\quad$ $E_{11} = 3.28$

$E_4 = 13.07$ $\quad$ $E_{12} = 2.66$

$E_5 = 13.07$ $\quad$ $E_{13} = 2.96$

$E_6 = 13.07$ $\quad$ $E_{14} = 2.66$

$E_7 = 13.07$ $\quad$ $E_{15} = 2.96$

$\therefore \bar{E} = 158.3/16 = 9.89$

$$\therefore \overline{E_b} = \bar{E} / \log_2(M)$$

$$= 9.89 / \log_2(16)$$

$$= 2.4725$$

(d) The normalized squared Euclidean distance

$$d_{E,min}^2 / E_b = \frac{\min_{m \neq i}^2 d_E^2(s_m, s_i)}{\bar{E_b}}$$

$$= \frac{4}{2.4725}$$

$$= 1.62$$

2(e) PAPR of constellation of vectors:

$$\Gamma_V = \frac{\max_m |S_m|^2}{\sum_{m=0}^{M-1} |S_m|^2 / M}$$

$$= \frac{13.07}{158.3/16}$$

$$= \frac{13.07}{9.89}$$

$$= 1.32$$

2(f) The orthogonal waveforms are:

$$\varphi_0(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t) \quad \text{and} \quad \varphi_1(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t) P_T(t)$$

$\therefore$ $S_0(t)$ can be written in different form:

$$S_0(t) = 1.2028 \sqrt{\frac{2}{T}} \cos(2\pi f_c(t)) P_T(t) + 3.4096 \sqrt{\frac{2}{T}} \sin(2\pi f_c t) P_T(t)$$

$$= \sqrt{\frac{2}{T}} \cdot 3.62 \cos(2\pi f_c t - 0.392\pi)$$

$\vdots$

The peak power of these signals is $\frac{26.1442}{T}$, and the avergy power P_{avg} is $\frac{9.8940}{T}$.

Therefore, the PAPR is $\frac{P_{max}}{P_{avg}} = 2.6424$

3. (a) $d_E(S_i, S_k)$

$d_E(S_0, S_1) = \sqrt{(-3-(-1))^2} = 2$

$d_E(S_0, S_2) = \sqrt{(-3-1)^2} = 4$

$d_E(S_0, S_3) = \sqrt{(-3-3)^2} = 6$

$d_E(S_0, S_4) = \sqrt{(-3-(-2))^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$d_E(S_0, S_5) = \sqrt{(-3)^2 + (0-\sqrt{3})^2} = \sqrt{9} = 2\sqrt{3}$

$d_E(S_0, S_6) = \sqrt{(-3-2)^2 + (0+\sqrt{3})^2} = \sqrt{25+3} = 2\sqrt{7}$

$d_E(S_0, S_7) = \sqrt{(-3-4)^2 + (0-\sqrt{3})^2} = \sqrt{49+3} = \sqrt{52}$

$d_E(S_0, S_8) = \sqrt{(-3-(-1))^2 + (0-2\sqrt{3})^2} = \sqrt{4+12} = 4$

$d_E(S_0, S_9) = \sqrt{(-3-1)^2 + (0-2\sqrt{3})^2} = \sqrt{16+12} = \sqrt{28} = 2\sqrt{7}$

$d_E(S_0, S_{10}) = \sqrt{(-3-(-4))^2 + (0+\sqrt{3})^2} = \sqrt{1+3} = 2$

$d_E(S_0, S_{11}) = \sqrt{(-3-(-2))^2 + (0+\sqrt{3})^2} = \sqrt{1+3} = 2$

$d_E(S_0, S_{12}) = \sqrt{(-3-0)^2 + (0+\sqrt{3})^2} = \sqrt{9+3} = 2\sqrt{3}$

$d_E(S_0, S_{13}) = \sqrt{25+3} = 2\sqrt{7}$

$d_E(S_0, S_{14}) = \sqrt{4+12} = 4$

$d_E(S_0, S_{15}) = \sqrt{16+12} = \sqrt{28} = 2\sqrt{7}$

$d_E(S_1, S_2) = \sqrt{(1+1)^2} = 2$

$d_E(S_1, S_3) = \sqrt{(-1-3)^2} = 4$

$d_E(S_1, S_4) = \sqrt{1+3} = 2$

$d_E(S_1, S_5) = 2$

$d_E(S_1, S_6) = 2\sqrt{3}$

$d_E(S_1, S_7) = 2\sqrt{7}$

$d_E(S_1, S_8) = 2\sqrt{3}$

$d_E(S_1, S_9) = 4$

$d_E(S_1, S_{10}) = 2\sqrt{3}$

$d_E(S_1, S_{11}) = 2$

$d_E(S_1, S_{12}) = 2$

$d_E(S_1, S_{13}) = 2\sqrt{3}$

$d_E(S_1, S_{14}) = 2\sqrt{3}$

$d_E(S_1, S_{15}) = 4$

$d_E(S_2, S_3) = 2$

$d_E(S_2, S_4) = 2\sqrt{3}$

$d_E(S_2, S_5) = 2$

$d_E(S_2, S_6) = 2$

$d_E(S_2, S_7) = 2\sqrt{3}$

$d_E(S_2, S_8) = 4$

$d_E(S_2, S_9) = 2\sqrt{3}$

$d_E(S_2, S_{10}) = 2\sqrt{7}$

$d_E(S_2, S_{11}) = 2\sqrt{3}$

$d_E(S_2, S_{12}) = 2$

$d_E(S_2, S_{13}) = 2$

$d_E(S_2, S_{14}) = 2$

$d_E(S_2, S_{15}) = 4$

$d_E(S_3, S_4) = 2\sqrt{7}$

$d_E(S_3, S_5) = 2\sqrt{3}$

$d_E(S_3, S_6) = 2$

$d_E(S_3, S_7) = 2$

$d_E(S_3, S_8) = 2\sqrt{7}$

$d_E(S_3, S_9) = 4$

$d_E(S_3, S_{10}) = \sqrt{52} = 2\sqrt{13}$

$d_E(S_3, S_{11}) = 2\sqrt{7}$

$d_E(S_3, S_{12}) = 2\sqrt{3}$

$d_E(S_3, S_{13}) = 2$

$d_E(S_3, S_{14}) = 2\sqrt{7}$

$d_E(S_3, S_{15}) = 4$

$d_E(S_4, S_5) = 2$
$d_E(S_4, S_6) = 4$
$d_E(S_4, S_7) = 6$
$d_E(S_4, S_8) = 2$
$d_E(S_4, S_9) = 2\sqrt{3}$
$d_E(S_4, S_{10}) = 4$
$d_E(S_4, S_{11}) = 2\sqrt{3}$
$d_E(S_4, S_{12}) = 4$
$d_E(S_4, S_{13}) = 2\sqrt{7}$
$d_E(S_4, S_{14}) = 2\sqrt{7}$
$d_E(S_4, S_{15}) = 6$

$d_E(S_5, S_6) = 2$
$d_E(S_5, S_7) = 4$
$d_E(S_5, S_8) = 2$
$d_E(S_5, S_9) = 2$
$d_E(S_5, S_{10}) = 2\sqrt{7}$
$d_E(S_5, S_{11}) = 4$
$d_E(S_5, S_{12}) = 2\sqrt{3}$
$d_E(S_5, S_{13}) = 4$
$d_E(S_5, S_{14}) = 2\sqrt{7}$
$d_E(S_5, S_{15}) = 2\sqrt{7}$

$d_E(S_6, S_7) = 2$
$d_E(S_6, S_8) = 2\sqrt{3}$
$d_E(S_6, S_9) = 2$
$d_E(S_6, S_{10}) = \sqrt{6^2+(2\sqrt{3})^2} = \sqrt{36+12} = \sqrt{48} = 4\sqrt{3}$
$d_E(S_6, S_{11}) = 2\sqrt{7}$
$d_E(S_6, S_{12}) = 4$
$d_E(S_6, S_{13}) = 2\sqrt{3}$
$d_E(S_6, S_{14}) = 6$
$d_E(S_6, S_{15}) = 2\sqrt{7}$

$d_E(S_7, S_8) = 2\sqrt{7}$
$d_E(S_7, S_9) = 2\sqrt{3}$
$d_E(S_7, S_{10}) = \sqrt{8^2+(2\sqrt{3})^2} = \sqrt{64+12} = \sqrt{76}$
$d_E(S_7, S_{11}) = 4\sqrt{3}$
$d_E(S_7, S_{12}) = 2\sqrt{7}$
$d_E(S_7, S_{13}) = 4$
$d_E(S_7, S_{14}) = 2\sqrt{13}$
$d_E(S_7, S_{15}) = 6$

$d_E(S_8, S_9) = 2$
$d_E(S_8, S_{10}) = 6$
$d_E(S_8, S_{11}) = 2\sqrt{7}$
$d_E(S_8, S_{12}) = 2\sqrt{7}$
$d_E(S_8, S_{13}) = 6$
$d_E(S_8, S_{14}) = 4\sqrt{3}$
$d_E(S_8, S_{15}) = 2\sqrt{13}$

$d_E(S_9, S_{10}) = 2\sqrt{13}$
$d_E(S_9, S_{11}) = 6$
$d_E(S_9, S_{12}) = 2\sqrt{7}$
$d_E(S_9, S_{13}) = 2\sqrt{7}$
$d_E(S_9, S_{14}) = 2\sqrt{13}$
$d_E(S_9, S_{15}) = 4\sqrt{3}$

$d_E(S_{10}, S_{11}) = 2$

$d_E(S_{10}, S_{12}) = 4$

$d_E(S_{10}, S_{13}) = 6$

$d_E(S_{10}, S_{14}) = 2\sqrt{3}$

$d_E(S_{10}, S_{15}) = 2\sqrt{7}$

$d_E(S_{11}, S_{12}) = 2$

$d_E(S_{11}, S_{13}) = 4$

$d_E(S_{11}, S_{14}) = 2$

$d_E(S_{11}, S_{15}) = 2\sqrt{3}$

$d_E(S_{12}, S_{13}) = 2$

$d_E(S_{12}, S_{14}) = 2$

$d_E(S_{12}, S_{15}) = 2$

$d_E(S_{13}, S_{14}) = 2\sqrt{3}$

$d_E(S_{13}, S_{15}) = 2$

$d_E(S_{14}, S_{15}) = 2$

the minimum Euclidean distance between distinct signals is 2.

3.(b)

$\bar{E}_b = \bar{E} / \log_2(M)$ where $\bar{E} = \frac{1}{M}\sum_{i=0}^{M-1} E_i$, So Let's compute $\bar{E}$ first

$E_0 = 9$ $\quad E_7 = 19$ $\quad E_{14} = 13$

$E_1 = 1$ $\quad E_8 = 13$ $\quad E_{15} = 13$

$E_2 = 1$ $\quad E_9 = 13$

$E_3 = 9$ $\quad E_{10} = 19$

$E_4 = 7$ $\quad E_{11} = 7$

$E_5 = 3$ $\quad E_{12} = 3$

$E_6 = 7$ $\quad E_{13} = 7$

$\therefore \bar{E} = \frac{1}{16}(E_0 + \cdots + E_{15})$

$= 9$

$\therefore \bar{E}_b = \bar{E} / \log_2(M) = 9 / \log_2(16)$

$= \frac{9}{4}$

3.(c) $\quad d_{E,min}^2 = 2^2 = 4$ by part (a)

$d_{E,min}^2 / E_b = \frac{4}{\frac{9}{4}} = \frac{16}{9}$

$\therefore$ The normalized squared Euclidean distance is $\frac{16}{9}$

3(d) PAPR of constellation of vectors:

$$\Gamma_V = \frac{\max_m |S_m|^2}{\sum_{m=0}^{M-1} |S_m|^2 / M}$$

$$= \frac{19}{144/16} \quad \text{(by part (b))}$$

$$= \frac{19}{9}$$

$$= 2.11$$

3(e) The orthogonal waveforms are:

$$\varphi_0(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t) \quad \text{and} \quad \varphi_1(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t) P_T(t)$$

$\therefore$ $S_0(t)$ can be written in different form:

$$S_0(t) = -3\sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t)$$

$$S_1(t) = -1\sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t)$$

$$S_2(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t)$$

$$S_3(t) = 3 \cdot \sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t)$$

$$S_4(t) = -2\sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t) - \sqrt{3}\sqrt{\frac{2}{T}} \sin(2\pi f_c t) P_T(t)$$

$$= -\sqrt{\frac{2}{T}} \cdot \sqrt{2^2+(\sqrt{3})^2} \left(\cos(2\pi f_c t - 40.89^\circ) \right) P_T(t)$$

$$= \sqrt{\frac{14}{T}} \left(\cos(2\pi f_c t + 139.1^\circ) \right) P_T(t)$$

$$S_5(t) = -\sqrt{3} \cdot \sqrt{\frac{2}{T}} \sin(2\pi f_c t) P_T(t)$$

$$\vdots$$

$$S_7(t) = 4\sqrt{\frac{2}{T}} \cos(2\pi f_c t) P_T(t) - \sqrt{3}\sqrt{\frac{2}{T}} \cdot \sin(2\pi f_c t) \cdot P_T(t)$$

$$= \sqrt{\frac{2}{T}} \cdot \sqrt{4^2+(-\sqrt{3})^2} \cdot \cos(2\pi f_c t + 23.41^\circ) P_T(t)$$

$$= \sqrt{\frac{38}{T}} \cos(2\pi f_c t + 23.41^\circ) P_T(t)$$

$\cdots\cdots$

The peak power of these signals is $P_{max} = \frac{38}{T}$, the average peak power is $P_{avg} = \frac{38}{9}$

$\therefore$ The PAPR for the 16 signals is $\frac{38}{9}$