EECS 551 Homework 10 YUZHAN JIANG

$a \in R^{n+1}$

P1: $p(x) = a[1] + a[2]x + a[3]x^2 + \dots + a[n+1]x^n = 0$

(a) Given $p(x)$

Then companion matrix $A = \begin{bmatrix} -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & \cdots & -\frac{a_0}{a_n} \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}$

using Power iteration. We can find largest root

which is try to find its eigenvector by $x_{k+1} = \frac{A x_k}{\|A x_k\|_2}$

Then its eigenvalue is $x_{k+1}' A x_{k+1}$

(b) $B = A - \lambda_1 I$, the largest in magnitude of the matrix B

Corresponds to the smallest eigenvalue of A, then apply the power method

to B to obtain the v_N with the smallest eigenvalue λ_N

In HW9, if vector b is in reverse order of a, then

eigenvalues of A = $\frac{1}{\text{eigenvalues of } B}$

$b = [\, a_n \quad a_{n-1} \quad \dots \quad a_0 \,]$

companion matrix B is $\begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & \cdots & -\frac{a_n}{a_0} \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}$

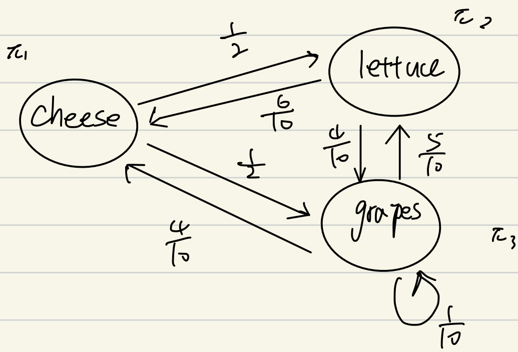
Same in part (a)

$$x_{k+1} = \frac{B x_k}{\|B x_k\|_2}$$

and compute corresponding eigenvalue

min eigvalue is $\frac{1}{x_{k+1}' B x_{k+1}}$

P2.



(a) $\pi P = \pi$

$$P = \begin{bmatrix} 0 & \frac{6}{10} & \frac{4}{10} \\ \frac{1}{2} & 0 & \frac{5}{10} \\ \frac{1}{2} & \frac{4}{10} & \frac{1}{10} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{6}{10}\pi_2 + \frac{4}{10}\pi_3 \\ \pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \\ \pi_3 = \frac{1}{10}\pi_3 + \frac{1}{2}\pi_1 + \frac{4}{10}\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{3}{5}\pi_2 + \frac{2}{5}\pi_3 \\ \pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \\ \pi_3 = \frac{1}{2}\pi_1 + \frac{2}{5}\pi_2 + \frac{1}{10}\pi_3 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{1}{3} \\ \pi_2 = \frac{1}{3} \\ \pi_3 = \frac{1}{3} \end{cases}$$

$\therefore$ the long term average probability is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of eating cheese grapes or lettuce

(b) This answer $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the unique solution.
Because P is square non-negative and strongly connected graph, so it is irreducible, then the equilibrium distribution is unique.

Problem 3:

(a) $\pi = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]^T$

rank-1 transition matrix $P \in R^{4\times 4}$

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad \pi P = \pi \Rightarrow \pi = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

(b) This example has unique Solution,
Because P is square non-negative and strongly connected graph, so it is irreducible, then the equilibrium distribution is unique.

P4:

Given that A has full column rank and P is symmetric positive definite.

$$-1 < \text{eig}(I - P^{\frac{1}{2}} A'A P^{\frac{1}{2}}) < 1$$

LHS :

$$-1 < \text{eig}(I - P^{\frac{1}{2}} A'A P^{\frac{1}{2}})$$

$$-1 < 1 - \text{eig}(P^{\frac{1}{2}} A'A P^{\frac{1}{2}})$$

$$\text{eig}(P^{\frac{1}{2}} A'A P^{\frac{1}{2}}) < 2$$

$$2 - \text{eig}(P^{\frac{1}{2}} A'A P^{\frac{1}{2}}) > 0$$

$$\text{eig}(2I - P^{\frac{1}{2}} A'A P^{\frac{1}{2}}) > 0$$

$$2I - P^{\frac{1}{2}} A'A P^{\frac{1}{2}} \succ 0$$

$$P^{-\frac{1}{2}}(2I - P^{\frac{1}{2}} A'A P^{\frac{1}{2}}) P^{-\frac{1}{2}} \succ 0$$

$$2P^{-1} - A'A \succ 0$$

$$\therefore \quad 2P^{-1} \succ A'A$$

Problem 5:

(a) If H is a (Hermitian) symmetric matrix that is diagonally dominant, then $|h_{ii}| \geq \sum_{j \neq i} |h_{ij}|$ for $i = 1 \cdots n$

Since $h_{ii} > 0$ and by Gershgorin disk theorem, the disk D doesn't contain the negative value since $\underbrace{h_{ii} - \sum_{j \neq i} |h_{ij}|}_{R:\ \text{radius of disk.}} \geq 0$

$\therefore$ the eigenvalue of H are nonegative and symmetric matrix

$\therefore$ $H \succeq 0$

(b) $D \triangleq \mathrm{Diag}(|B| \mathbb{1}_n)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \mathrm{Diag}\left(\begin{bmatrix} \Sigma |B_{1,:}| \\ \Sigma |B_{2,:}| \\ \vdots \\ \Sigma |B_{n,:}| \end{bmatrix} \right)$$

and $D - B \Rightarrow$

$$\begin{bmatrix} \Sigma |B_{1,:}| - B_{11} & & & \\ & \Sigma |B_{2,:}| - B_{22} & & \\ & & \ddots & \\ & & & \Sigma |B_{n,:}| - B_{nn} \end{bmatrix}$$

rest are entries of B

We notice that the matrix $[D-B]_{ii} = \sum_j |B_{i,j}| - B_{ii} > \sum_{j \neq i} B_{i,j}$

$\left(\text{Since } \sum_j |B_{i,j}| > \sum_j B_{i,j} \right)$

So $D - B$ is diagonally dominant

Therefore, based on part (a) $D-B$ is $n \times n$ Hermitian symmetric matrix and is diagonally dominant, then $D - B \succeq 0$

$\therefore$ $D \triangleq \mathrm{Diag}(|B| \mathbb{1}_n) \succeq B$

P6:

$$f(x) = \frac{1}{2}\|Ax-y\|_2^2 + \delta^2 \frac{1}{2}\|x\|_2^2$$

(a) gradient of this cost function:

$$\nabla f(x) = A'(Ax-y) + \delta^2 x$$

(b) $\forall\, x, z \in R^N$

$$\begin{aligned}
\|\nabla f(x) - \nabla f(z)\|_2 &= \| A'(Ax-y) + \delta^2 x - (A'(Az-y) + \delta^2 z)\|_2 \\
&= \| A'(Ax-y) - A'(Az-y) + \delta^2 (x-z) \\
&= \| A'A(x-z) + \delta^2 (x-z)\|_2 \\
&= \|(A'A + \delta^2 I)(x-z\|_2 \\
&\le \|A'A + \delta^2 I\|_2 \|x-z\|_2
\end{aligned}$$

and notice that $\|A'A + \delta^2 I\|_2 \le \|A'A\|_2 + \|\delta^2 I\|_2 \le \|A'A\|_1 + \|\delta^2 I\|_2$

$\therefore$ the upper bound is $\|A'A\|_1 + \delta^2$ (not singular values in expression)

(c) when $A = \begin{bmatrix} I_5 \\ 1_5 1_5' \end{bmatrix}$ and $\delta = 2$

$$A'A = [I_5 \quad 1_5 1_5'] \begin{bmatrix} I_5 \\ 1_5 1_5' \end{bmatrix} = I_5 + 5\, 1_5 1_5' = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$\therefore \|A'A\|_1 =$ max column sum $= 1 + 25 = 26$

$\therefore$ the upper bound is $26 + 2^2 = 30$

(d) $L = \|A'A + \delta^2 I\|_2 = 30$ which is exactly the same as upper bound

P7:

$X_1 = -2+i\sqrt{3}$

$P_1(x) = (x-(-2+i\sqrt{3}))(x-(-2-i\sqrt{3}))$

$= x^2+4x+7$

$X_2 = 5-\sqrt{7}$

$P_2(x) = (x-(5-\sqrt{7}))(x-(5+\sqrt{7}))$

$= x^2-(5+\sqrt{7})x-(5-\sqrt{7})x+(25-7)$

$= x^2-10x+18$

So the companion matrix is (corresponding x_1)

$$A = \begin{bmatrix} -4 & -7 \\ 1 & 0 \end{bmatrix}$$

Companion matrix of $P_2(x)$

$$B = \begin{bmatrix} 10 & -18 \\ 1 & 0 \end{bmatrix}$$

eigvalues of $A \oplus B$ are $\lambda_i(A)+\lambda_j(B)$

$A \oplus B = A \otimes I_N + I_M \otimes B$

$$= \begin{bmatrix} 6 & 0 & -25 & 0 \\ 0 & 6 & 0 & -25 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

the eigenvalues of $A \otimes B$ are $\lambda_i(A)\cdot\lambda_j(B)$

$A \otimes B$

$$= \begin{bmatrix} -40 & 72 & -70 & 126 \\ -4 & 0 & -7 & 0 \\ 10 & -18 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now use symbolic computation:

$P_3 = x^4 - 12x^3 + 136x^2 - 600x + 2500$

$P_4 = x^4 + 40x^3 + 736x^2 + 5040x + 15876$

P8:

$x_1 = a + \sqrt{b}$

$$P_1(x) = (x - (a+\sqrt{b}))(x - (a - \sqrt{b}))$$
$$= x^2 - (a-\sqrt{b})x - (a+\sqrt{b})x + (a^2 - b)$$
$$= x^2 - 2ax + a^2 - b$$

$C_1 = [a^2 - b \quad -2a]$

$x_2 = c - \sqrt{d}$

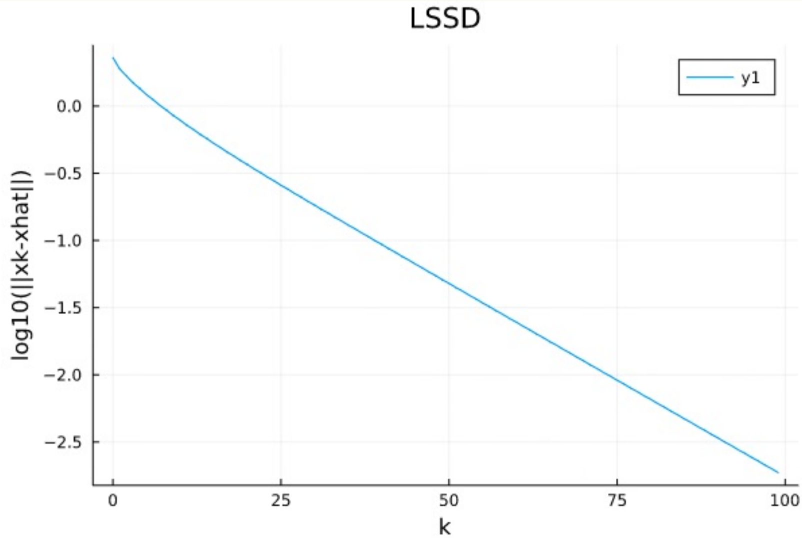
$$P_2(x) = (x - (c - \sqrt{d}))(x - (c+\sqrt{d}))$$
$$= x^2 - 2cx + c^2 - d$$

$C_2 = [c^2 - d \quad -2c]$

P9:

(b)

Out[170]:



In [170]:

```
using Random: seed!
using Plots
seed!(0) # seed random number generator
m = 100; n = 50; sigma = 0.1
A = randn(m, n); xtrue = rand(n)
b = A * xtrue + sigma * randn(m)

k=LinRange(0,99,100)
xhat = A \b
y1=zeros(100)
for i=1:100
    y1[i]=log10(norm(lssd(A,b,nIters=i)-xhat ))
end

plot(xlabel = "k")
plot!(ylabel = "log10(||xk-xhat||)")
plot!(k,y1)
plot!(title="LSSD")
```

(C)

```
function lssd(A, b ; x0=zeros(size(A,2)), nIters::Int=10)

    x_current = x0
    Ax = A * x_current
    for _ in 1:nIters
        gradient = A' * (Ax - b)
        direction = - gradient
        Ad = A * direction
        Ad_norm = norm(Ad)

        if(Ad_norm == 0)
            return x_crrent
        else
            step = - direction' * gradient / (Ad_norm.^2)
        end

        x_current = x_current + step * direction
        Ax = Ax + step * Ad
    end

    return x_current

end
```