EECS501: Solutions to Homework 11

1. Poisson Process: Direct Calculation

(a)

$$P(X(y_1)<1, X(y_2)<2) = P(X(y_1)=0, X(y_2)<2) = P(X(y_1)=0, X(y_2)-X(y_1)<2)$$

$$= P(X(y_1)=0)P(X(y_2)-X(y_1)<2) = e^{-\lambda y_1}\times e^{-\lambda(y_2-y_1)}(1+\lambda(y_2-y_1)) = e^{-\lambda y_2}(1+\lambda(y_2-y_1))$$

(b)

$$P(Y_1>y_1, Y_2>y_2) = P(X(y_1)<1, X(y_2)<2) = e^{-\lambda y_2}(1+\lambda(y_2-y_1))$$

$$P(Y_1\le y_1, Y_2\le y_2) = 1-P(Y_1>y_1 \cup Y_2>y_2) = 1-P(Y_1>y_1)-P(Y_2>y_2)+P(Y_1>y_1, Y_2>y_2)$$

$$F_{Y_1,Y_2}(y_1,y_2) = 1-P(X(y_1)<1)-P(X(y_2)<2)+e^{-\lambda y_2}(1+\lambda(y_2-y_1))$$

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = \frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2}F_{Y_1,Y_2}(y_1,y_2) = \lambda^2 e^{-\lambda y_2},\ \text{ for } y_2>y_1>0$$

(c) Use induction. As shown above, result is true for $k=2$. Now assume it is true for $k=K-1$.

$$(-1)^K\frac{\partial^K}{\partial Z}P(Z>z) = (-1)\frac{\partial}{\partial Z_K}(-1)^K\frac{\partial^{(K-1)}}{\partial Z^{K-1}}P(Z_K>z_K)P(\bigcap_{i=1}^{K-1} Z_i>z_i|Z_K>z_K)$$

$$= -\frac{\partial}{\partial Z_K}P(Z_K>z_K)f_{Z^{K-1}}(z^{K-1}|Z_K>z_K) = -\frac{\partial}{\partial Z_K}f_{Z^{K-1}}(z^{K-1}\cap Z_K>z_K)$$

$$= -\frac{\partial}{\partial Z_K}P(Z_K>z_K|Z^{K-1}=z^{K-1})f_{Z^{K-1}}(z^{K-1}) = f_{Z^K}(z^K|Z^{K-1}=z^{K-1})f_{Z^{K-1}}(z^{K-1})$$

$$= f_{Z^K}(z^K)$$

So it is also true for $k=K$. By induction, the result is proved.

(d) For $y_1\le y_2\le\cdots\le y_K$,

$$P(\underline{Y}>\underline{y}) = P(X(y_1)<1, X(y_2)<2,\ldots,X(y_K)<K)$$

$$= e^{-\lambda y_1}e^{-\lambda(y_2-y_1)}\ldots e^{-\lambda(y_K-y_{K-1})}(1+\lambda(y_2-y_1)+\lambda^2(y_2-y_1)(y_3-y_2)+\lambda^3(y_2-y_1)(y_3-y_2)(y_4-y_3)+\ldots$$
$$+\lambda^{K-1}(y_2-y_1)(y_3-y_2)(y_4-y_3)\ldots(y_K-y_{K-1}))$$

The only term that contains all variables is $\lambda^{K-1}(y_2-y_1)(y_3-y_2)(y_4-y_3)\ldots(y_K-y_{K-1}))$. Hence we get

$$f_{\underline{Y}}(\underline{y}) = (-1)^K\frac{\partial^K}{\partial\underline{Y}}P(\underline{Y}>\underline{y}) = \lambda^k e^{-\lambda y_K},\ \text{ for } y_K\ge y_{K-1}\ge\cdots\ge y_1\ge 0\,.$$

2. Wide-Sense Stationary

We need to calculate $E[X_t]$ and $R_X(t_1, t_2)$. We have

$$E[X_t] = E[A\sin(t+\Theta)] = E[A]E[\sin(t+\Theta)] = \frac{1}{4}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\sin(t+\theta)d\theta = \frac{1}{4}\frac{1}{2\pi}(-\cos(t+\theta))|_0^{2\pi} = 0,$$

where the second equality is due to A and Θ being independent. We can also write

$$\begin{aligned} R_X(t_1, t_2) &= E[X_{t_1}X_{t_2}] = E[A\sin(t_1+\Theta)A\sin(t_2+\Theta)] = E[A^2]E[\sin(t_1+\Theta)\sin(t_2+\Theta)] \\ &= \frac{1}{4}E[\frac{1}{2}(\cos(t_1-t_2) - cos(t_1+t_2+2\Theta))] = \frac{1}{8}\cos(t_1-t_2) - \frac{1}{8}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\cos(t_1+t_2+2\theta)d\theta \\ &= \frac{1}{8}\cos(t_1-t_2) - \frac{1}{8}\frac{1}{4\pi}\sin(t_1+t_2+2\theta)|_0^{2\pi} = \frac{1}{8}\cos(t_1-t_2) \end{aligned}$$

Therefore, we have $E[X_t] = 0$ and $R_X(t_1, t_2) = R_X(t_1 - t_2) = \frac{1}{8}\cos(t_1 - t_2)$. Since the mean function is a constant and autocorrelation function depends on t_1 and t_2 only through their difference, the process X_t is WSS.

3. Wide-Sense Stationary 2

We need to calculate $E[X_t]$ and $R_X(t_1, t_2)$. We have

$$E[X_t] = E[A\sin(t+\Theta) + B] = E[A]E[\sin(t+\Theta)] + E[B] = \frac{1}{2}$$

where we have used the results of the previous problem to derive the final equality.

$$\begin{aligned} R_X(t_1, t_2) &= E[X_{t_1}X_{t_2}] = E[(A\sin(t_1+\Theta) + B)(A\sin(t_2+\Theta) + B)] \\ &= E[A^2]E[\sin(t_1+\Theta)\sin(t_2+\Theta)] + E[A]E[B]E[\sin(t_1+\Theta)] + E[A]E[B]E[\sin(t_2+\Theta)] + E[B^2] \\ &= \frac{1}{8}\cos(t_1-t_2) + \frac{1}{2} \end{aligned}$$

Therefore, we have $E[X_t] = \frac{1}{2}$ and $R_X(t_1, t_2) = R_X(t_1 - t_2) = \frac{1}{8}\cos(t_1 - t_2) + \frac{1}{2}$. Since the mean function is a constant and autocorrelation function depends on t_1 and t_2 only through their difference, the process X_t is WSS.