University of Michigan Fall 2021

EECS501: Solution to Homework 10

1. Markov or Not Let $(Z_n)_n$ be the iid sequence of rv's denoting the face of the die at the n-th role (where $Z_n \in \{1,2,3,4,5,6\}$).

1. X_n can be represented through a recursive equation of the form

$$X_{n+1} = \max\{X_n, Z_{n+1}\} \qquad n = 1, 2, \ldots$$

and $X_1 = Z_1$. Since (i) X_{n+1} is a function of X_n and Z_{n+1}, and (ii) Z_{n+1} is independent of $X_n, X_{n-1}, \ldots, X_0$, we infer that X_{n+1} is conditionally independent of $X_{n-1}, \ldots, X_0$ given X_n. Hence X_n is a Markov chain.

2. Similarly, N_n can be expressed as

$$N_{n+1} = N_n + 1_{\{6\}}(Z_{n+1}) \qquad n = 1, 2, \ldots$$

and $N_1 = 1_{\{6\}}(Z_1)$, where $1_{\{6\}}(Z_{n+1})$ is an indicator random variable that indicates $Z_{n+1} = 6$. Hence N_n is a Markov chain.

3. C_n can be expressed as

$$C_{n+1} = (C_n + 1)1_{\{1,2,3,4,5\}}(Z_{n+1})$$

and $C_1 = 0$ (by convention) so for the same reason as above it is a Markov chain.

4. B_n can be expressed as

$$B_n = \min\{k > n : Z_k = 6\} - n.$$

To see the evolution of B_n consider a realization of $\{Z_n\}$ as follows: $Z_1 = 1$, $Z_2 = 3$, $Z_3 = 5$, $Z_4 = 6$, $Z_5 = 2$, $Z_6 = 1$, $Z_7 = 3$, $Z_8 = 5$, $Z_9 = 6$, $Z_{10} = 1, \ldots$ Then the corresponding realization for $\{B_n\}$ is $B_1 = 3$, $B_2 = 2$, $B_3 = 1$, $B_4 = 5$, $B_5 = 4$, $B_6 = 3$, $B_7 = 2$, $B_8 = 1, \ldots$ As can be seen $\{B_n\}$ is a decreasing sequence up until it reaches the value $B_n = 1$ which is equivalent to $Z_{n+1} = 6$. Thus we can write

$$B_{n+1} = \begin{cases} B_n - 1, & B_n \geq 2 \\ \min\{k > n+1 : Z_k = 6\} - (n+1), & B_n = 1 \Leftrightarrow Z_{n+1} = 6 \end{cases}$$

We now have

$$P(B_{n+1} = m | B_n = b_n, \ldots, B_1 = b_1) = \begin{cases} 1, & b_n \geq 2 \text{ and } m = b_n - 1 \\ 0, & b_n \geq 2 \text{ and } m \neq b_n - 1 \end{cases} \tag{1}$$

$$= P(B_{n+1} = m | B_n = b_n), \tag{2}$$

while for the case where $b_n = 1$ we have

$$P(B_{n+1} = m | B_n = 1, \ldots, B_1 = b_1)$$
$$= P(\min\{k > n+1 : Z_k = 6\} - (n+1) = m | B_n = 1, \ldots, B_1 = b_1). \tag{3}$$

Clearly the event $\min\{k > n+1 : Z_k = 6\} - (n+1) = m$ depends only on random variables $Z_{n+2}, Z_{n+3}, \ldots$ while the conditioning implies only information about $Z_{n+1} = 6$ and the value of previous Z_i's. Since the sequence of $\{Z_n\}$ consists of IID random variables, we have

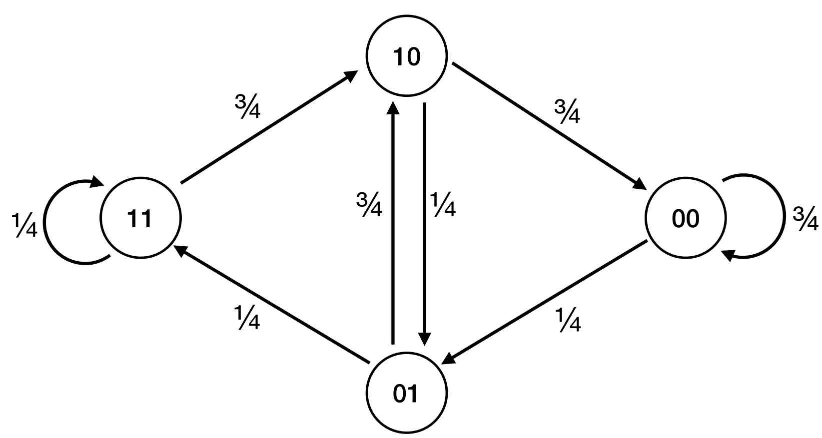
$$P(B_{n+1} = m | B_n = 1, \ldots, B_1 = b_1)$$
$$= P(\min\{k > n+1 : Z_k = 6\} - (n+1) = m) \tag{4}$$
$$= P(B_{n+1} = m). \tag{5}$$

Thus in all cases corresponding to the value of b_n the Markov property is satisfied.

2. Markov Chain

(a) We have 4 appearance of 11 and so we get 8 dollars.

(b) We have the following Markov chain:



If we call the state $(0,0)$ to be state 1, $(0,1)$ to be state 2, $(1,0)$ to be state 3, and $(1,1)$ to be state 4, we have the following transition probability matrix.

$$P = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 3/4 & 1/4 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \end{bmatrix}$$

(c) The stationary distribution of the Markov chain is calculated by $\pi = \pi P$ and $\sum_i \pi_i = 1$. We write

$$\pi_1 = 3/4\pi_1 + 3/4\pi_3$$

$$\pi_2 = 1/4\pi_1 + 1/4\pi_3$$

$$\pi_3 = 3/4\pi_2 + 3/4\pi_4$$

$$\pi_4 = 1/4\pi_2 + 1/4\pi_4$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

Therefore, we get $\pi_2 = \pi_3$ and $\pi_1 = 3\pi_2 = 3\pi_3$, and $\pi_2 = 3\pi_4$. Thus, we have $\pi = (9/16,\ 3/16,\ 3/16,\ 1/16)$.

The expected amount of dollars we win per flip at the steady-state is $2 \times P(11) = 2/16 = 1/8$.

3. PageRank We have the following transition probability matrix:

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore, we have $r = rP$ and $\sum_i r_i = 1$. We can write

$$r_1 = 1/3r_1 + 1/2r_2 + r_3$$

$$r_2 = 1/3r_1$$

$$r_3 = 1/3r_1 + 1/2r_2$$

$$r_1 + r_2 + r_3 = 1$$

Therefore, we have $r_1 = 3r_2$ and $r_3 = 3/2r_2$ and therefore, we have $r = (6/11,\ 2/11,\ 3/11)$.

4. Markov Molecules

X_n can take integer values between 0 and m. So there exist $m+1$ states with the following transition probabilities:

$$P(X_{n+1} = k+1 | X_n = k) = \frac{m-k}{m}\,,\ P(X_{n+1} = k-1 | X_n = k) = \frac{k}{m}\,,\ \text{for } k = 1, 2, \cdots m-1\,.$$

$$P(X_{n+1} = 1 | X_n = 0) = 1\,,\ P(X_{n+1} = m-1 | X_n = m) = 1\,.$$

One can easily draw the state diagram based on these transition probabilities. Assume the stationary distribution is π_i, for $i \in [0, m]$. They have to satisfy the following conditions:

$$\pi_0 = \frac{1}{m}\pi_1\,,\ \pi_m = \frac{1}{m}\pi_{m-1}\,,\ \text{and } \pi_k = \frac{m-k+1}{m}\pi_{k-1} + \frac{k+1}{m}\pi_{k+1}\,,\ \text{for } k = 1, 2, \cdots m-1\,.$$

$$\sum_{k=0}^{m} \pi_k = 1\,.$$

This implies that $\pi_1 = m\pi_0$, $\pi_2 = \frac{m(m-1)}{2}\pi_0$, and so on.

$$\Rightarrow \pi_k = \binom{m}{k} 2^{-m}\,,\ \text{for } k = 0, 1, 2, \cdots m\,.$$

5. **Generalization of Kelly's formula** If we bet α fraction of our money and we win, we get $1-\alpha+A\alpha$ and if we lose, we get $1-\alpha+B\alpha$. Therefore, we choose α according to the following optimization.

$$\max_{\alpha} p\log(1-\alpha+A\alpha) + (1-p)\log(1-\alpha+B\alpha).$$

By taking the derivative of the above equation with respect to α and setting it to 0, we can write

$$\frac{p(A-1)}{1-\alpha+A\alpha} - \frac{(1-p)(1-B)}{1-\alpha+B\alpha} = 0$$

Therefore, we have $\alpha = \frac{p(A-1)-(1-p)(1-B)}{(A-1)(1-B)}$.