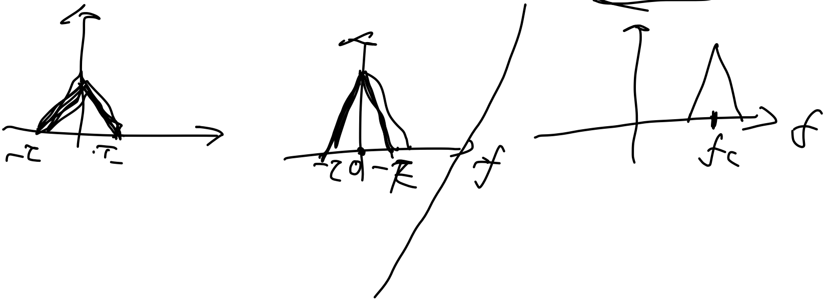
Lecture 5

Goals:

基带 传频通带

- Be able to convert a signal from baseband to passband and back.



Up and Down Conversion

generate signals

- In communication systems typically the signals are generated at baseband and then up converted to the desired carrier frequency.
- The baseband signals have some bandwidth W (e.g. concentrated from $-W$ to $+W$).
- The signals are mixed to a frequency f_c much greater than W.
- At the receiver this process is reversed.

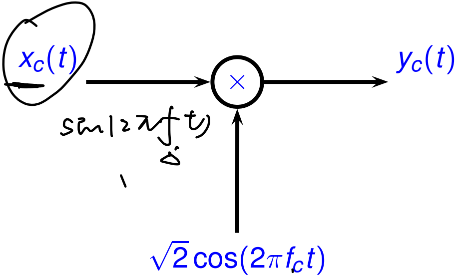
f_c 载波频率

$f_c >> W(\tau)$

Up and Down Conversion

Baseband signals

Passband signal

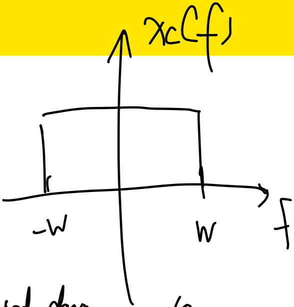


$\Rightarrow$

$f << f_c$.

Up and Down Conversion

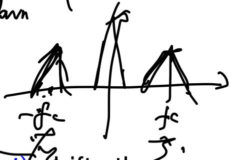
$X_c(f) = FT(x(t))$



$$y_c(t) = x_c(t)\sqrt{2}\cos(2\pi f_c t)$$

shifted up shifted down

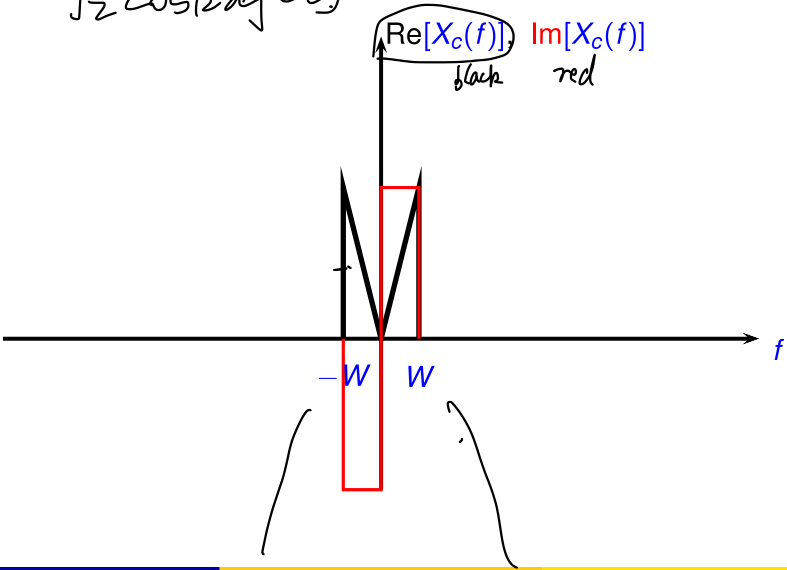
$$Y_c(f) = \frac{\sqrt{2}}{2}[X_c(f - f_c) + X_c(f + f_c)]$$



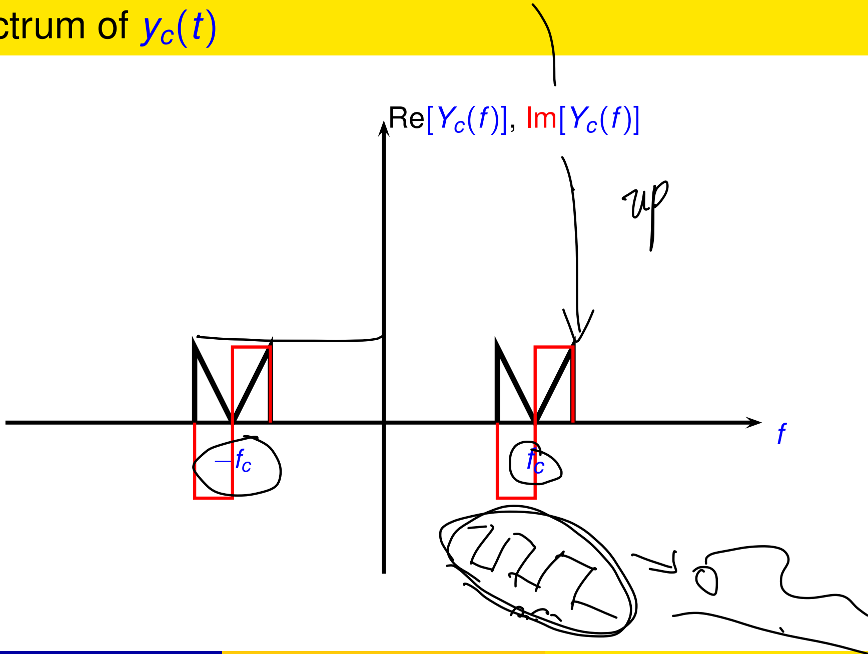
Thus multiplication in the time domain by $\sqrt{2}\cos(2\pi f_c t)$ shifts the spectrum up and down by f_c and reduces each part by 1/2.

Up and Down Conversion

$\sqrt{2}\cos(2\pi f_c t)$

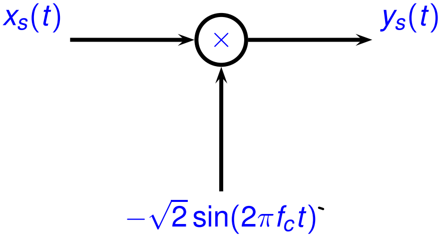


Spectrum of $y_c(t)$



Spectrum of $y_s(t)$

Now consider multiplication by $-\sqrt{2}\sin(2\pi f_c t)$.



$(x+jy)j$

$x - jy$

$(-x+jy)$ real → im

im → neg real

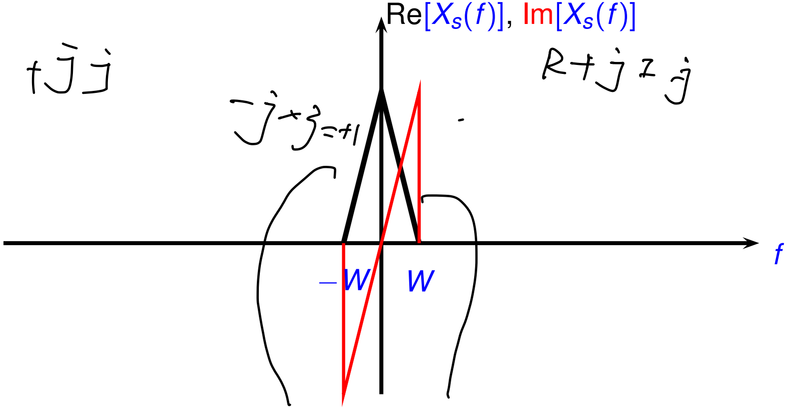
$= -x - y$

$$y_s(t) = -x_s(t)\sqrt{2}\sin(2\pi f_c t)$$

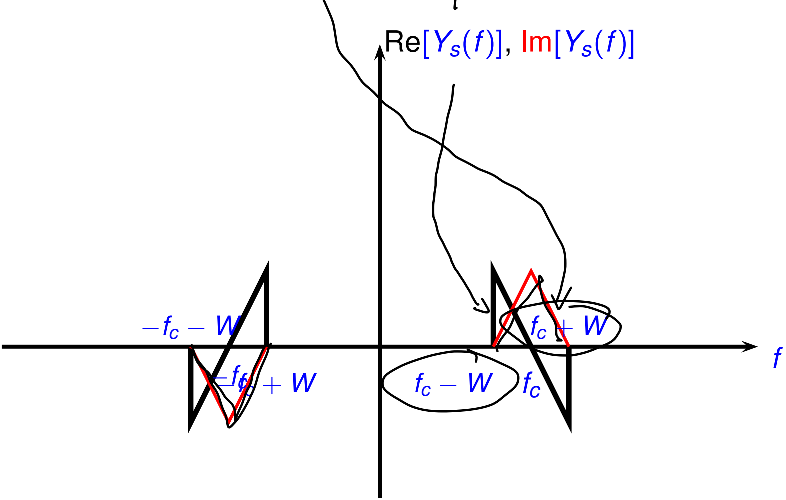
$$Y_s(f) = \frac{j\sqrt{2}}{2}[X_s(f-f_c) - X_s(f+f_c)]$$

Spectrum of $y_s(t)$

Thus multiplication by $-\sqrt{2}\sin(2\pi f_c t)$ shifts the spectrum up and down also except that the real part becomes the imaginary part and the imaginary part is inverted and becomes the real part in addition to a reduction by 1/2.

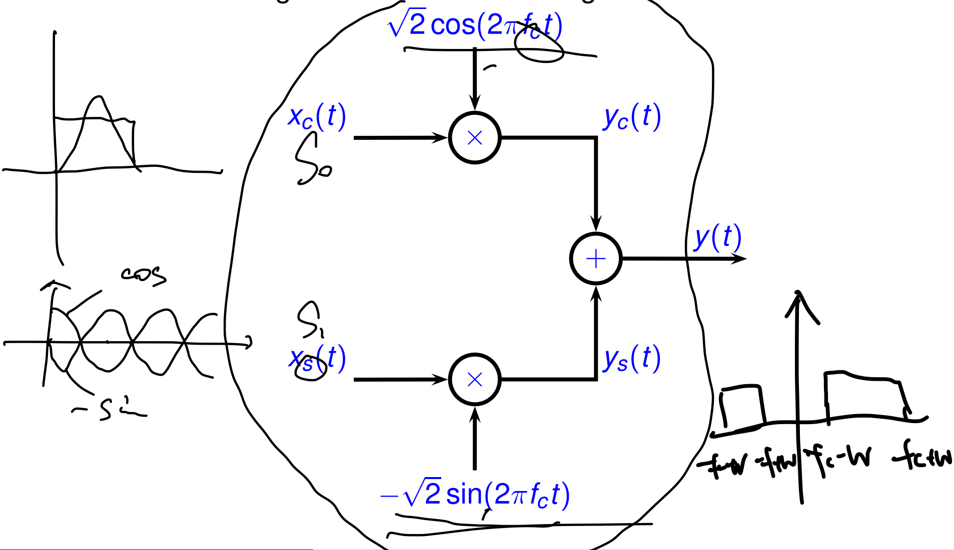


Spectrum of $y_s(t)$



IQ Modulation

Now consider adding these two functions together.



IQ Modulation

In-phase-quart:... (正交)

$$
\begin{aligned}
y(t) &= y_c(t) + y_s(t) \\
&= x_c(t)\sqrt{2}\cos(2\pi f_c t) - x_s(t)\sqrt{2}\sin(2\pi f_c t) \\
&= x_e(t)\cos(2\pi f_c t + \theta(t))
\end{aligned}
$$

The signal $x_e(t)$ is called the envelope and $\theta(t)$ is called the phase.

$e^{ix} = \cos x + i \sin x$

PSK

$$
\begin{aligned}
Y(f) &= Y_c(f) + Y_s(f) \\
\theta(t) &= \tan^{-1}\left[\frac{x_s(t)}{x_c(t)}\right] \\
x_e(t) &= \sqrt{2}(x_c^2(t) + x_s^2(t))^{1/2} \\
y(t) &= \mathrm{Re}[(x_c(t) + jx_s(t))\sqrt{2}e^{j2\pi f_c t}]
\end{aligned}
$$

The signal $x_c(t) + jx_s(t)$ is called the lowpass complex equivalent of the signal $x(t)$.

IEEE 802.11

This is from the IEEE 802.11 (WiFi) standard (page 555)

f → 调制

17.3.2.4 Mathematical conventions in the signal descriptions

The transmitted signals will be described in a complex baseband signal notation. The actual transmitted signal is related to the complex baseband signal by the following relation:

$$r_{(RF)}\langle t\rangle = Re\{r\langle t\rangle \exp\langle j2\pi f_c t\rangle\} \quad (17\text{-}1)$$

f_c

where

$Re(.)$	represents the real part of a complex variable
f_c	denotes the carrier center frequency

carrier 载波

GSM from 3GPP

This is from the 3GPP Standard for GSM [3GPP TS 05.04 Release 1999 V8.4.0 (2001-11)]

3.6 Modulation

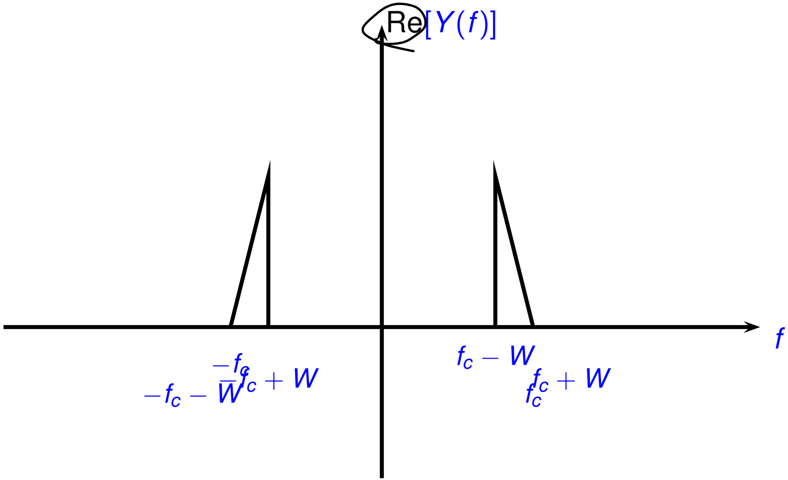
The modulated RF carrier during the useful part of the burst is therefore:

射频

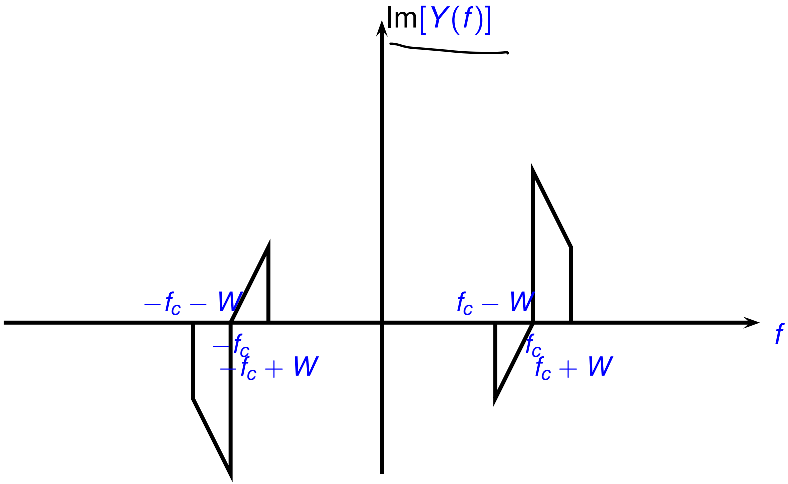
$$x(t') = \sqrt{\frac{2E_s}{T}}\,\mathrm{Re}\left[y(t')\cdot e^{j(2\pi f_0 t' + \varphi_0)}\right]$$

where E_s is the energy per modulating symbol, f_0 is the centre frequency and φ_0 is a random phase and is constant during one burst.

IQ Modulation



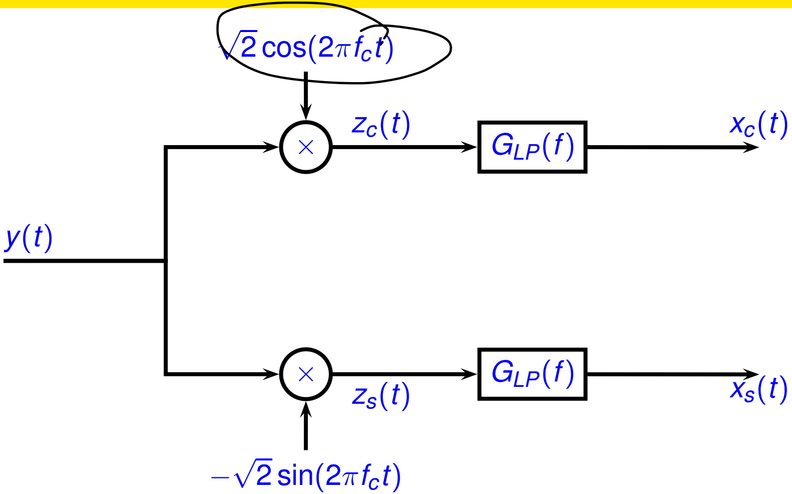
IQ Modulation



Signal Decomposition

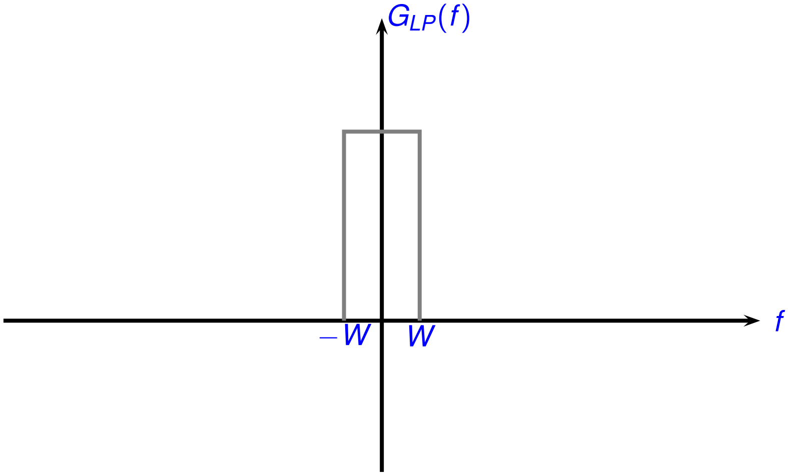
- The signals $x_c(t)$ and $x_s(t)$ can be recovered from $y(t)$ by mixing down to baseband and filtering out the double frequency terms.
- Note that we need the exact phase of the local oscillators to do this perfectly.

Signal Decomposition



Signal Decomposition

$G_{LP}(f)$ is an ideal low pass filter with transfer function $G_{LP}(f) = 1$ $|f| \leq W$ and $G_{LP}(f) = 0$ otherwise.



Signal Decomposition

Consider the spectrum of $z_c(t)$. This is given by

$$Z_c(f) = \frac{\sqrt{2}}{2}[Y(f - f_c) + Y(f + f_c)].$$

Similarly the spectrum of $z_s(t)$ is

$$Z_s(f) = \frac{j\sqrt{2}}{2}[Y(f - f_c) - Y(f + f_c)].$$

Signal Decomposition

$$
\begin{aligned}
Y_c(f) &= \frac{\sqrt{2}}{2}[X_c(f-f_c)+X_c(f+f_c)] \\
Y_s(f) &= \frac{j\sqrt{2}}{2}[X_s(f-f_c)-X_s(f+f_c)] \\
Y(f) &= [Y_c(f)+Y_s(f)] \\
&= \frac{\sqrt{2}}{2}[X_c(f-f_c)+X_c(f+f_c) \\
&\quad +j(X_s(f-f_c)-X_s(f+f_c))]
\end{aligned}
$$

quadrature

Signal Decomposition (Frequency Domain)

double frequency

$$
\begin{aligned}
Z_c(f) &= \frac{\sqrt{2}}{2}[Y(f-f_c)+Y(f+f_c)] \\
&= \frac{1}{2}[X_c(f-2f_c)+X_c(f)+j(X_s(f-2f_c)-X_s(f)) \\
&\quad +X_c(f)+X_c(f+2f_c)+j(X_s(f)-X_s(f+2f_c))] \\
&= \frac{1}{2}[X_c(f-2f_c)+2X_c(f)+X_c(f+2f_c) \\
&\quad +j(X_s(f-2f_c)-X_s(f+2f_c))] \\
\mathrm{LPF}[Z_c(f)] &= \frac{1}{2}\mathrm{LPF}[X_c(f-2f_c)+2X_c(f)+X_c(f+2f_c) \\
&\quad +j(X_s(f-2f_c)-X_s(f+2f_c))] \\
&= X_c(f)
\end{aligned}
$$

Lowpass Filter

共 f. H₂ ↑

Signal Decomposition (Time Domain)

带通

Suppose $s(t)$ is a bandpass signal with representation

$$s(t) = x_c(t)\sqrt{2}\cos(2\pi f_c t) - x_s(t)\sqrt{2}\sin(2\pi f_c t)$$

$$\begin{aligned}
z_c(t) &= s(t)\sqrt{2}\cos(2\pi f_c t) \\
&= 2\left[x_c(t)\cos(2\pi f_c t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t)\cos(2\pi f_c t)\right] \\
&= x_c(t)[1 + \cos(2\pi 2 f_c t)] - x_s(t)[1\sin(2\pi 2 f_c t) - \sin(0)] \\
&= x_c(t) + [x_c(t)\cancel{\cos(2\pi 2 f_c t)} - x_s(t)\cancel{\sin(2\pi 2 f_c t)}]
\end{aligned}$$

The first term above is the desired signal $x_s(t)$ while the second term is called the double frequency term. If $z_c(t)$ is then filtered with a filter that removes the double frequency term what remains is the message.

$$\text{LPF}[z_c(t)] = x_c(t)$$

Signal Decomposition

Similarly the spectrum of $z_s(t)$ is

$$
\begin{aligned}
Z_s(f) &= \frac{j\sqrt{2}}{2}[Y(f-f_c) - Y(f+f_c)] \\
&= \frac{j}{2}[X_c(f-2f_c) + X_c(f) + j(X_s(f-2f_c) - X_s(f)) \\
&\quad -X_c(f) - X_c(f+2f_c) - j(X_s(f) - X_s(f+2f_c))] \\
&= \frac{j}{2}[X_c(f-2f_c) + X_c(f+2f_c) \\
&\quad +j(X_s(f-2f_c) - 2X_s(f) + X_s(f+2f_c))] \\
\mathrm{LPF}[Z_s(f)] &= \frac{j}{2}\mathrm{LPF}[X_c(f-2f_c) + X_c(f+2f_c) \\
&\quad +j(X_s(f-2f_c) - 2X_s(f) + X_s(f+2f_c))] \\
&= X_s(f)
\end{aligned}
$$

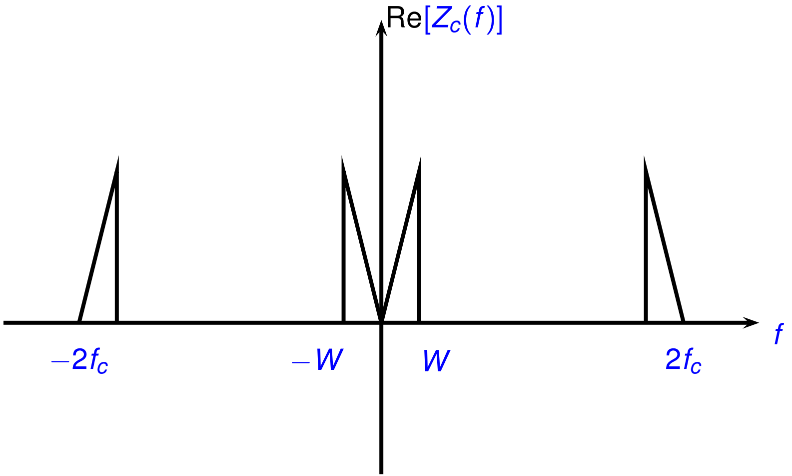
Signal Decomposition

Similarly

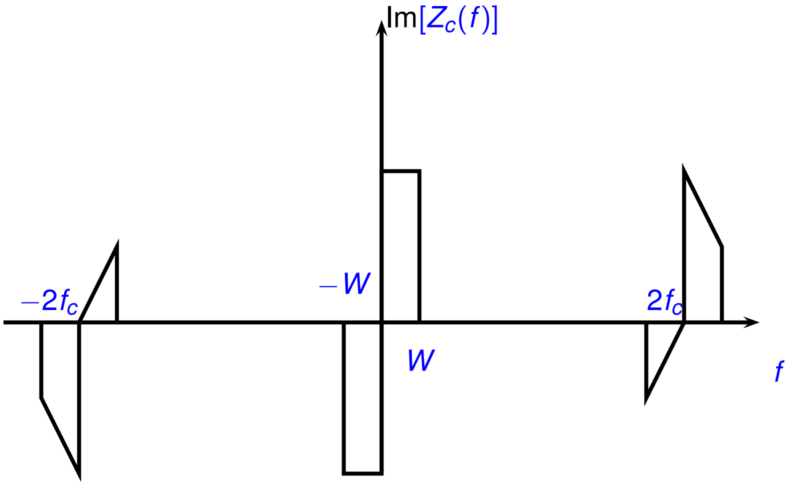
$$\begin{aligned} z_s(t) &= s(t)[-\sqrt{2}\sin(2\pi f_c t)] \\ &= 2[-x_s(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + x_s(t)\sin(2\pi f_c t)\sin(2\pi f_c t)] \\ &= x_c(t)[\sin(2\pi 2 f_c t) - \sin(0)] + x_s(t)[1 - \cos(2\pi 2 f_c t)] \\ &= x_s(t) + [x_c(t)\sin(2\pi 2 f_c t) - x_s(t)\cos(2\pi 2 f_c t)] \end{aligned}$$

$$\mathrm{LPF}[z_s(t)] = x_s(t)$$

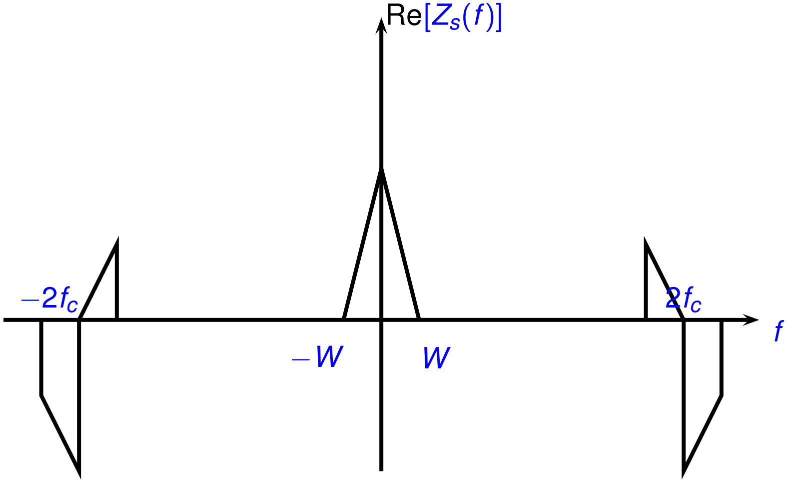
Signal Decomposition



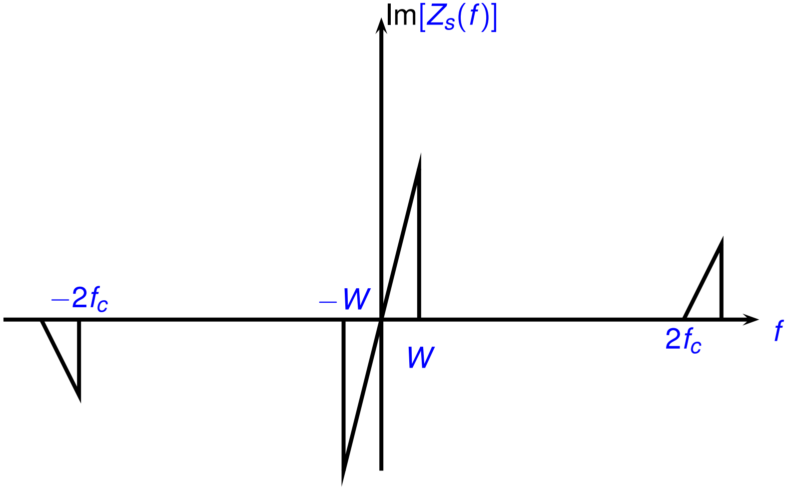
Signal Decomposition



Signal Decomposition



Signal Decomposition



Signal Decomposition: Example

$$
\begin{aligned}
x_c(t) &= a_{c,1}\cos(2\pi f_1 t) + a_{c,2}\sin(2\pi f_1 t) + a_{c,3}\cos(2\pi f_2 t) \\
x_s(t) &= a_{s,1}\cos(2\pi f_1 t) + a_{s,2}\sin(2\pi f_1 t) + a_{s,3}\sin(2\pi f_2 t)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{c,1} &= 0.25, a_{c,2} = 0.5, a_{c,3} = 1, \\
a_{s,1} &= -1.0, a_{c,2} = 0.25, a_{c,3} = 1, \\
f_1 &= 1, f_2 = 2.
\end{aligned}
$$

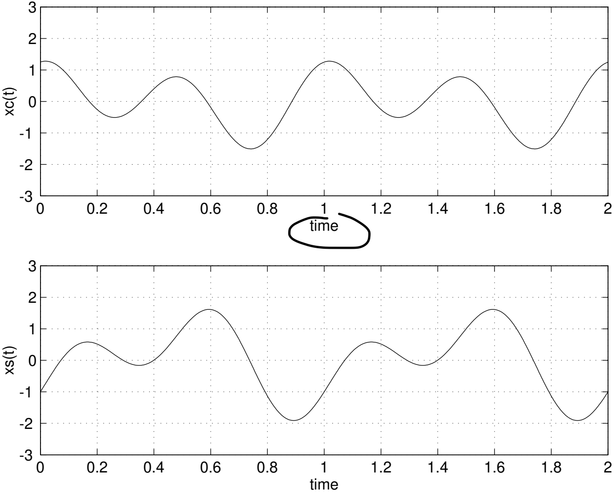
IQ mixer

The signals are upconverted with a quadrature modulator to produce

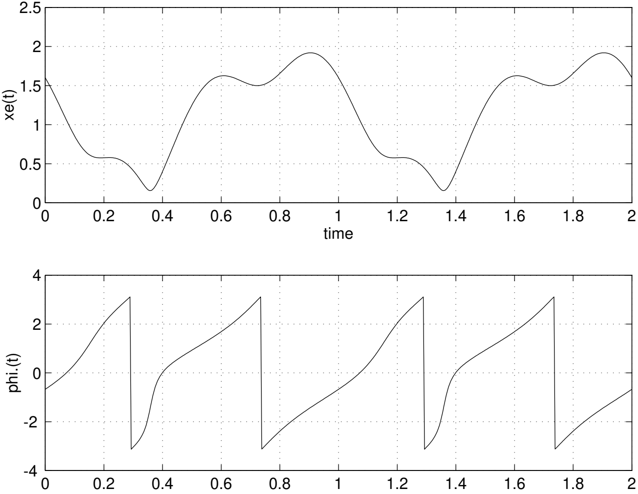
$$
\begin{aligned}
&= x_c(t)\sqrt{2}\cos(2\pi f_c t) - x_s(t)\sqrt{2}\sin(2\pi f_c t) \\
&= x_e(t)\cos(2\pi f_c t + \theta(t))
\end{aligned}
$$

where $f_c = 16$.

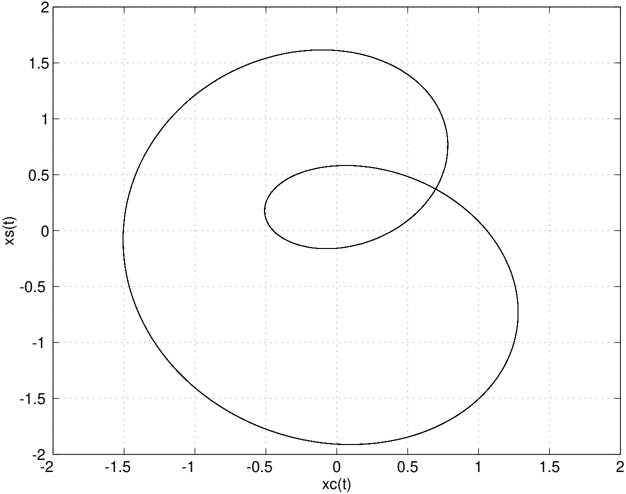
Example of Up and Down Conversion of Signals



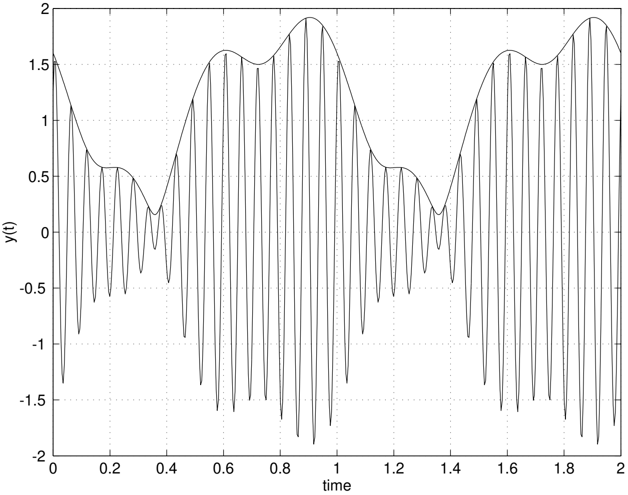
Example of Up and Down Conversion of Signals



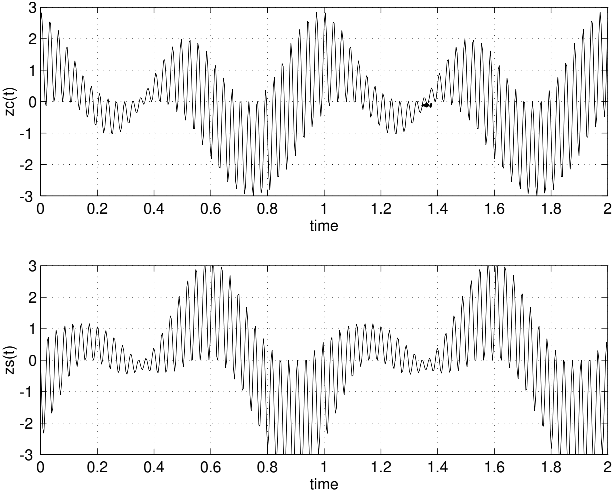
Example of Up and Down Conversion of Signals



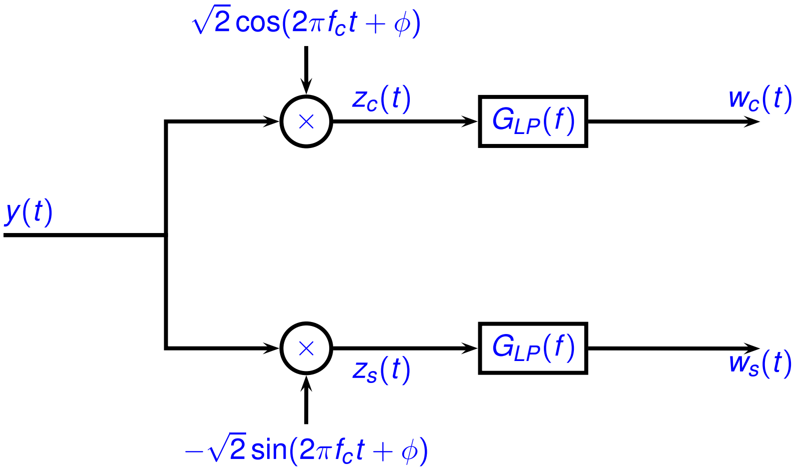
Example of Up and Down Conversion of Signals



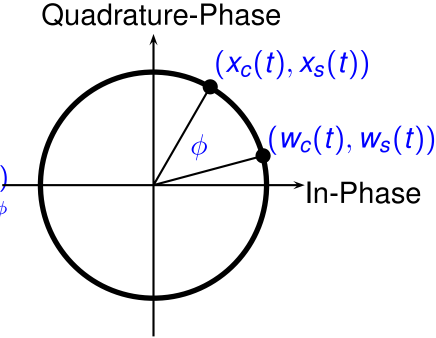
Example of Up and Down Conversion of Signals



Signal Decomposition: Imperfect Phase



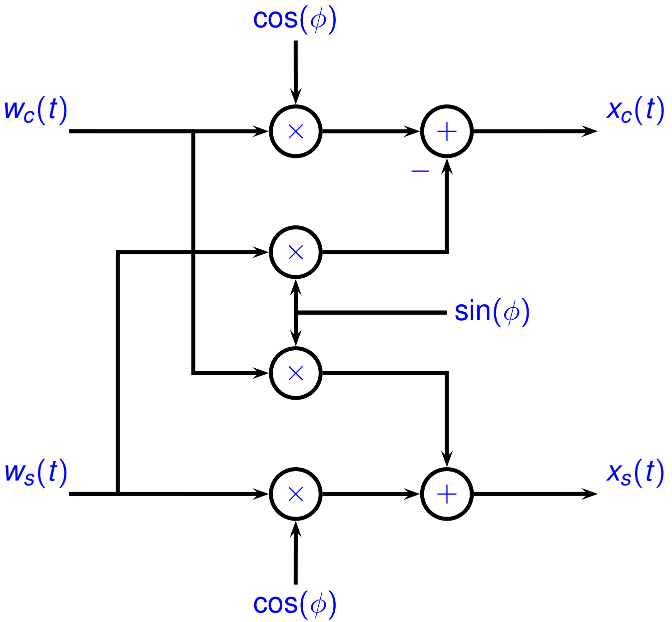
$$w_c(t) = x_c(t)\cos(\phi) + x_s(t)\sin(\phi)$$
$$w_s(t) = -x_c(t)\sin(\phi) + x_s(t)\cos(\phi)$$
$$w_c(t) + jw_s(t) = (x_c(t) + jx_s(t))e^{-j\phi}$$



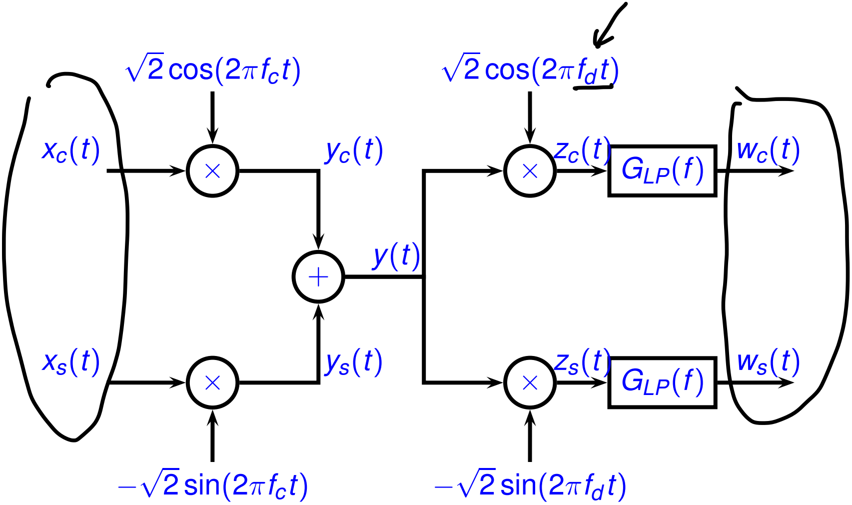
Note that this is equivalent to a phase rotation by angle $-\phi$.

We can recover the orignial signal by rotating the signal.

$$x_c(t) + jx_s(t) = (w_c(t) + jw_s(t))e^{+j\phi}$$


$\cos(\phi)$
$w_c(t)$
$x_c(t)$
$-$
$\sin(\phi)$
$w_s(t)$
$x_s(t)$
$\cos(\phi)$

Frequency Offset



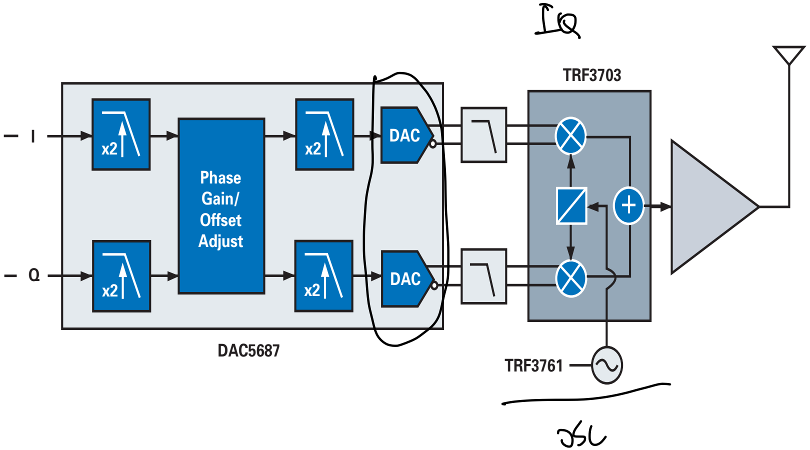
Frequency Offset

For frequency offset (that is sufficiently small relative to the bandwidth of the low pass filters) the output of the down conversion part is related to the input by

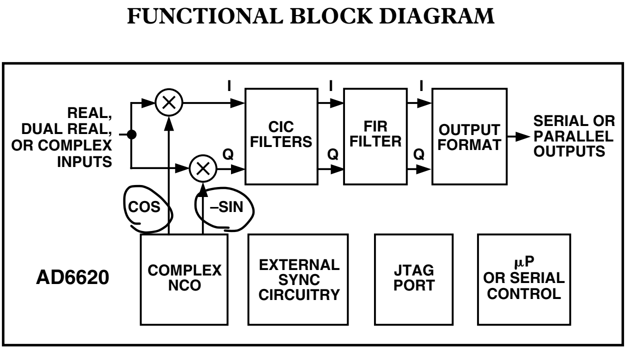
$$(w_c(t) + jw_s(t)) = (x_c(t) + jx_s(t))e^{j2\pi(f_c - f_d)t}.$$

Conceptually we can think about shifting up in frequency by f_c and down in frequency by f_d. Using a technique similar to the phase offset correction we can correct for frequency offset.

Typical IQ Modulator: Texas Instruments TRF3703



Typical IQ Demodulator: Analog Devices AD6620



Bandpass/Baseband Relations

Assume that $s_0(t), s_1(t), ...,$ are bandpass signals. That is, the bandwidth occupied is $[-f_c - W, -f_c + W]$ and $[f_c - W, f_c + W]$. Let $\hat{s}_0(t), \hat{s}_1(t), ...$ be the lowpass complex representation of the bandpass waveforms.

$$\begin{aligned}
\hat{s}_i(t) &= \int s_i(\tau)\sqrt{2}\cos(2\pi f_c\tau)g_{LP}(t-\tau)d\tau \\
&\quad -j\sqrt{2}\int s_i(\tau)\sin(2\pi f_c\tau)g_{LP}(t-\tau)d\tau \\
&= \int s_i(\tau)\sqrt{2}\exp\{-j2\pi f_c\tau\}g_{LP}(t-\tau)d\tau \\
s_i(t) &= \mathrm{Re}[\sqrt{2}\hat{s}_i(\tau)\exp\{j2\pi f_c t\}]
\end{aligned}$$

Signals

Claim:

$$\int s_0(t)s_1(t)dt = \mathrm{Re}[\int \hat{s}_0(t)\hat{s}_1^*(t)dt], \qquad \int \hat{s}_0(t)\hat{s}_1(t)dt \approx 0$$

Proof:

First note that if a and b are any complex numbers then $\mathrm{Re}[a]\mathrm{Re}[b] = \frac{1}{2}\mathrm{Re}[ab + ab^*]$. To show this just examine the following:

$$\begin{aligned}
ab &= (a_r + ja_i)(b_r + jb_i) \\
&= a_r b_r - a_i b_i + j(a_r b_i + a_i b_r) \\
ab^* &= (a_r + ja_i)(b_r - jb_i) \\
&= a_r b_r + a_i b_i + j(-a_r b_i + a_i b_r) \\
ab + ab^* &= 2a_r b_r + j2(a_i b_r) \\
\mathrm{Re}[ab + ab^*] &= 2a_r b_r \\
\frac{1}{2}\mathrm{Re}[ab + ab^*] &= a_r b_r = \mathrm{Re}[a]\mathrm{Re}[b]
\end{aligned}$$

Signals

Proof (cont.):

Now consider

$$\begin{aligned}
\int s_0(t)s_1(t)dt &= \int \mathrm{Re}[\sqrt{2}\hat{s}_0(t)\exp\{j2\pi f_c t\}]\mathrm{Re}[\sqrt{2}\hat{s}_1(t)\exp\{j2\pi f_c t\}]dt \\
&= \int \mathrm{Re}[\hat{s}_0(t)\exp\{j2\pi f_c t\}\hat{s}_1(t)\exp\{j2\pi f_c t\} \\
&\quad +\hat{s}_0(t)\exp\{j2\pi f_c t\}\hat{s}_1^*(t)\exp\{-j2\pi f_c t\}]dt \\
&= \int \mathrm{Re}[\hat{s}_0(t)\hat{s}_1(t)\exp\{j2\pi(2f_c)t\} + \hat{s}_0(t)\hat{s}_1^*(t)]dt
\end{aligned}$$

Now the first integral is zero since $\hat{s}_0(t)$ and $\hat{s}_1(t)$ are low pass functions while $\exp\{j2\pi(2f_c)t\}$ is a double frequency term. Thus

$$\begin{aligned}
\int s_0(t)s_1(t)dt &= \mathrm{Re}[\int \hat{s}_0(t)\hat{s}_1^*(t)dt] \\
&= \mathrm{Re}[\int \hat{s}_0^*(t)\hat{s}_1(t)dt]
\end{aligned}$$

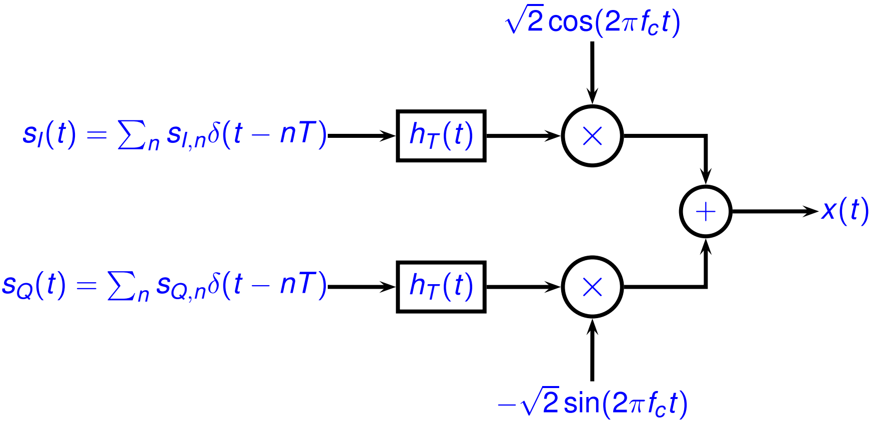
Signals

As a special case of the above result consider $s_0(t) = s_1(t)$.

Equal Energy

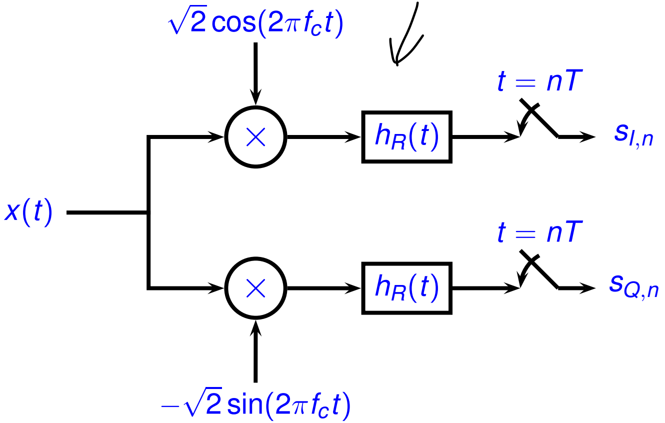
$$\int s_0^2(t)dt = \int |\hat{s}_0(t)|^2 dt$$

Mixing two baseband signals onto a carrier



Recovering data from bandpass signal

remove double frequency term



Matched Filter: $h_R(t) = h_T(-t)$

Up-Down Conversion Summary

- Two independent low-pass signals of bandwidth W can be up-converted to a signal of frequency f_c with bandwidth $2W$ and then individually recovered by mixing down to baseband.
- We can represent the two low-pass signals as a single complex-lowpass signal.
- Phase offset and frequency offset between the two oscillators can be corrected for by additional circuitry or signal processing.