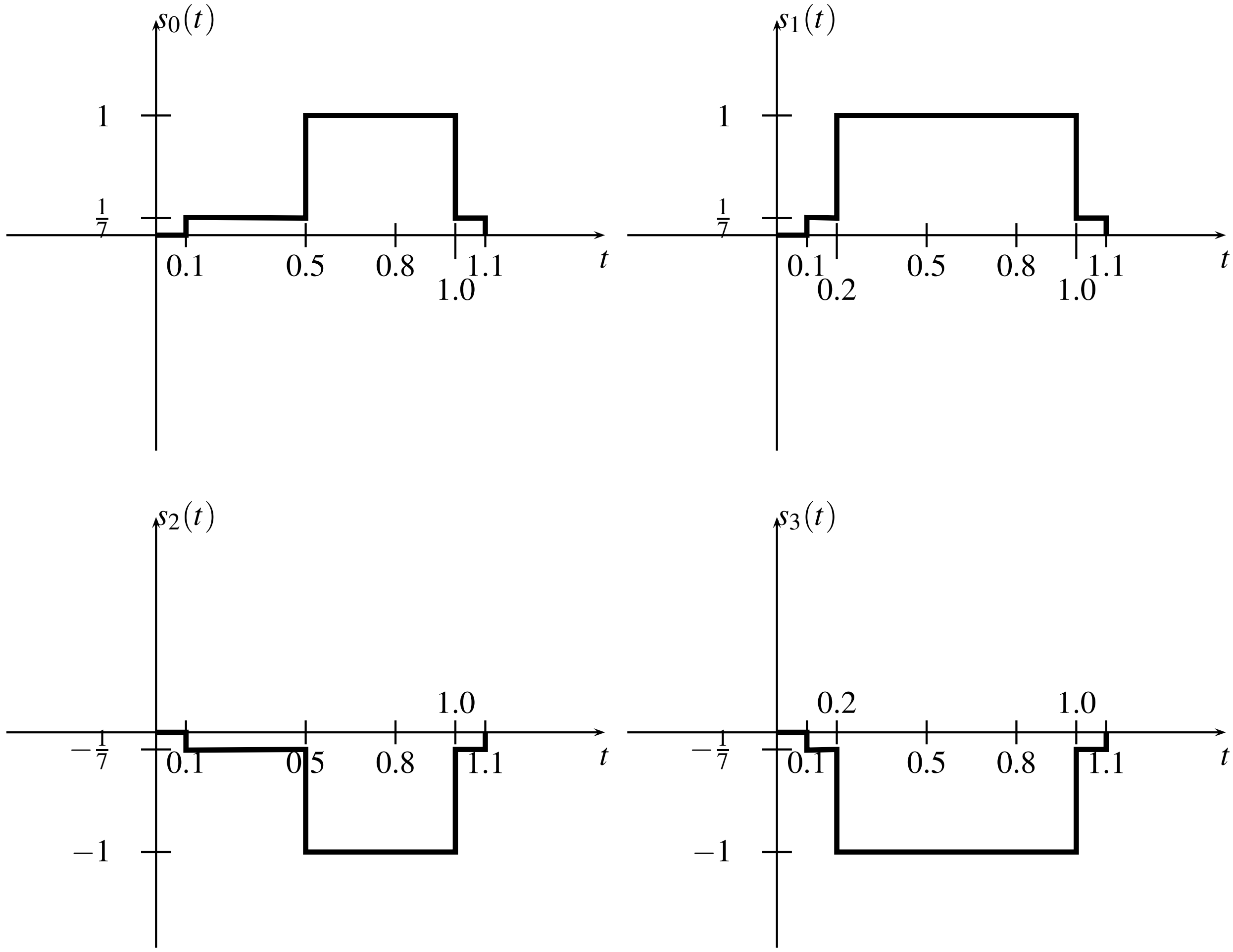
EECS 455: Solutions to Problem Set 7

1. (a) The following four signals are used in WWVB to transmit information. (See CTools for more information about signals in WWVB). Find two orthonormal signals for which these four signals can be represented as four vectors of dimension 2. Plot the four signals in a plane with the horizontal axis representing the first component and the vertical axis representing the second component.



Solution: First note that two of the signals are just the opposite of the other two so they can be expressed using the same orthogonal signals. Apply Gram-Schmidt procedure. Orthogonal signal one is a normalized version of $s_0(t)$. The energy of $s_0(t)$ is

$$E_0 = (\frac{1}{7})^2(0.4) + (1)^2(0.5) + (1/7)^2(0.1) = \frac{25}{49}$$

So the orthonormal signal is

$$\varphi_0(t) \quad = \quad \frac{1}{\sqrt{E_0}} s_0(t) = \frac{7}{5} s_0(t)$$

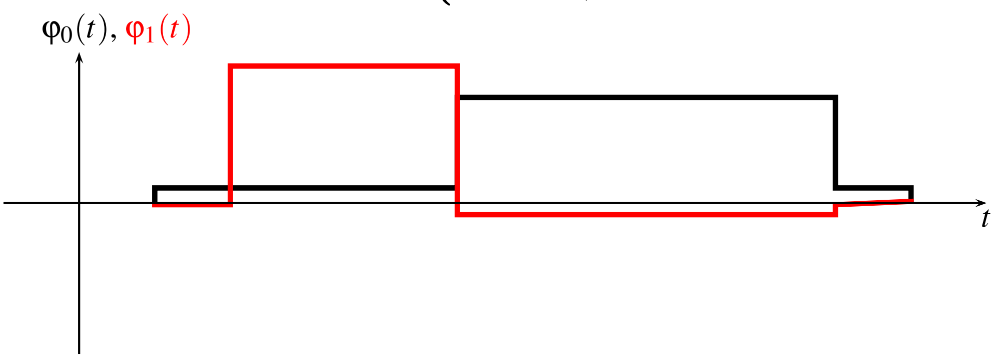
$$= \begin{cases} 0, & 0 \le t \le 0.1 \\ 1/5 & 0.1 \le t \le 0.5 \\ 7/5 & 0.5 \le t \le 1.0 \\ 1/5 & 1.0 \le t \le 1.1 \end{cases}$$

The correlation between $s_1(t)$ and $\varphi_0(t)$ is

$$\begin{aligned} (s_1(t), \varphi_0(t)) &= \int s_1(t)\varphi_0(t)dt \\ &= (\frac{1}{5})(\frac{1}{7})(0.1) + (\frac{1}{5})(1)(0.3) + (\frac{7}{5})(1)(0.5) + (\frac{1}{5})(\frac{1}{7})(0.1) \\ &= \frac{268}{350} \\ &= .7657 \end{aligned}$$

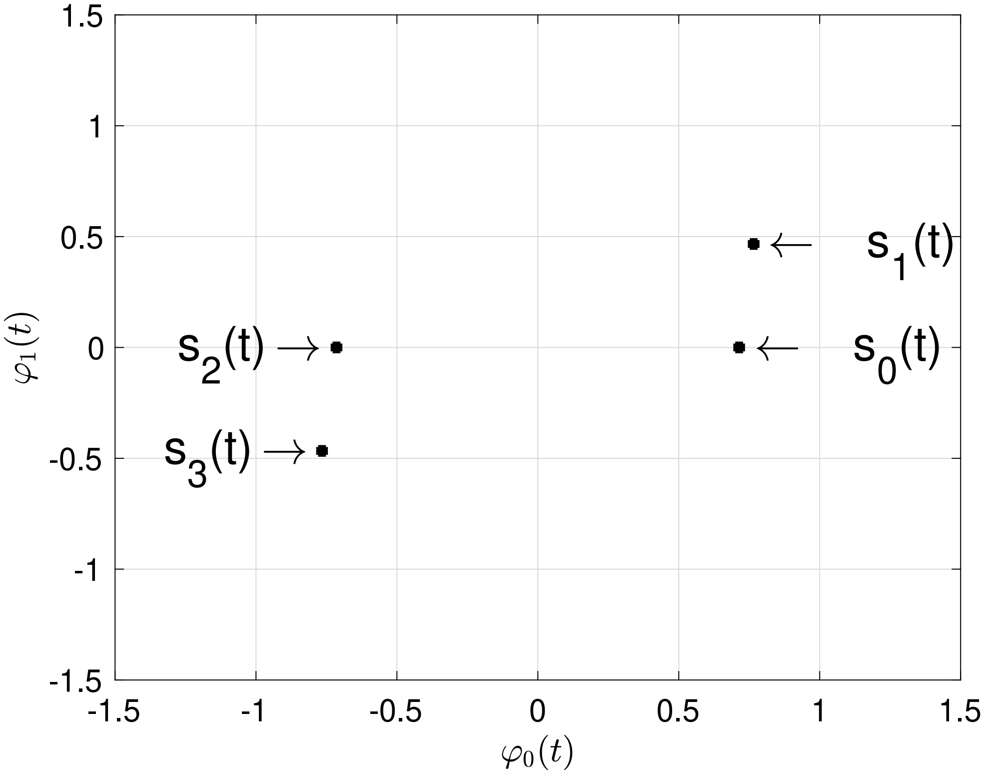
$$\begin{aligned} u_1(t) &= s_1(t) - s_{1,0}\varphi_0(t) \\ &= \begin{cases} 0, & 0 \le t \le 0.1 \\ +0.0103, & 0.1 \le t \le 0.2 \\ +0.8469, & 0.2 \le t \le 0.5 \\ -0.0720, & 0.5 \le t \le 1.0 \\ -0.0103, & 1.0 \le t \le 1.1 \end{cases} \end{aligned}$$

$$\varphi_1(t) = \begin{cases} 0, & 0 \le t \le 0.1 \\ -0.0220, & 0.1 \le t \le 0.2 \\ +1.8148, & 0.2 \le t \le 0.5 \\ -0.1543, & 0.5 \le t \le 1.0 \\ -0.0220, & 1.0 \le t \le 1.1 \end{cases}$$



$$\begin{aligned} s_0(t) &= \sqrt{E_0}\varphi_0(t) \\ s_1(t) &= s_{1,0}\varphi_0(t) + s_{1,1}\varphi_1(t) \\ s_2(t) &= -\sqrt{E_0}\varphi_0(t) \\ s_3(t) &= s_{3,0}\varphi_0(t) + s_{3,1}\varphi_1(t) \end{aligned}$$

$$
\begin{aligned}
s_0 &= \sqrt{E_0}(1,0) \\
s_1 &= (.7657, .4667) \\
s_2 &= \sqrt{E_0}(-1,0) \\
s_3 &= (-.7657, -.4667)
\end{aligned}
$$



(b) Assume there is an amplifier that multiplies the transmitted signal by A. Find the energy of each of the signals and the average energy. The answer should be in terms of A.

The energy of the signals are

$$
\begin{aligned}
E_0 &= A^2((1/7)^2(0.5)+(1)^2(0.5)) = .5102A^2 \\
E_1 &= A^2((1/7)^2(0.2)+(1)^2(0.8)) = 0.8041A^2 \\
E_2 &= A^2((1/7)^2(0.5)+(1)^2(0.5)) = .5102A^2 \\
E_3 &= A^2((1/7)^2(0.2)+(1)^2(0.8)) = 0.8041A^2 \\
\bar{E} &= (E_0+E_1+E_2+E_3)/4 = .6571A^2
\end{aligned}
$$

(c) Find the squared Euclidean distance between all the vectors. The answer should be in terms of A.

	s_0	s_1	s_2	s_3
s_0	0	$0.2727A^2$	$4A^2$	$3.3355A^2$
s_1	$0.2727A^2$	0	$3.3355A^2$	$4A^2$
s_2	$4A^2$	$3.3355A^2$	0	$0.2727A^2$
s_3	$3.3355A^2$	$4A^2$	$0.2727A^2$	0

(d) Describe the optimal receiver for deciding between these four signals.

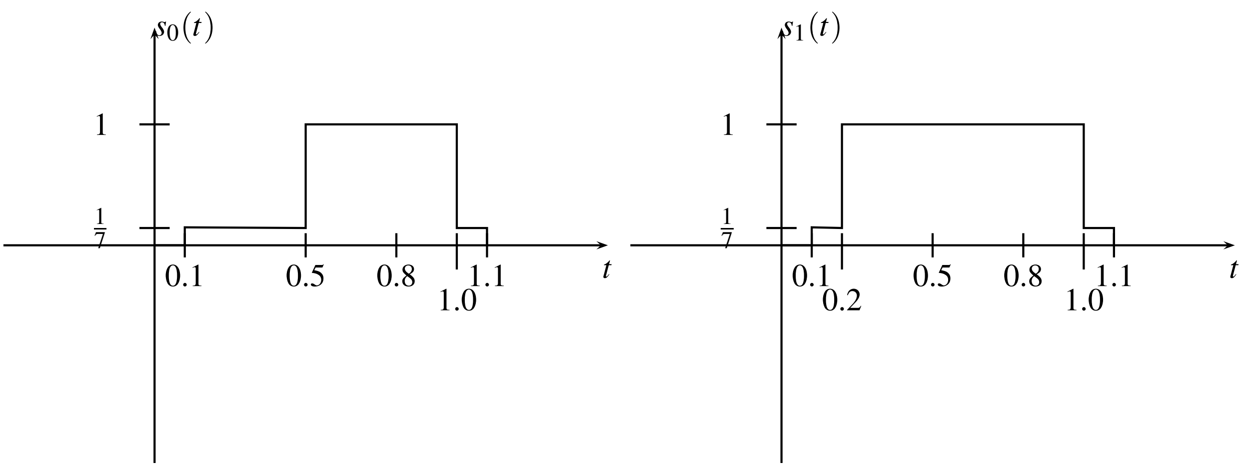
Solution: The optimal receiver correlates the received signal with $\varphi_0(t)$ and $\varphi_1(t)$ and then finds which of the four vectors s_0, s_1, s_2, s_3 is closest to the

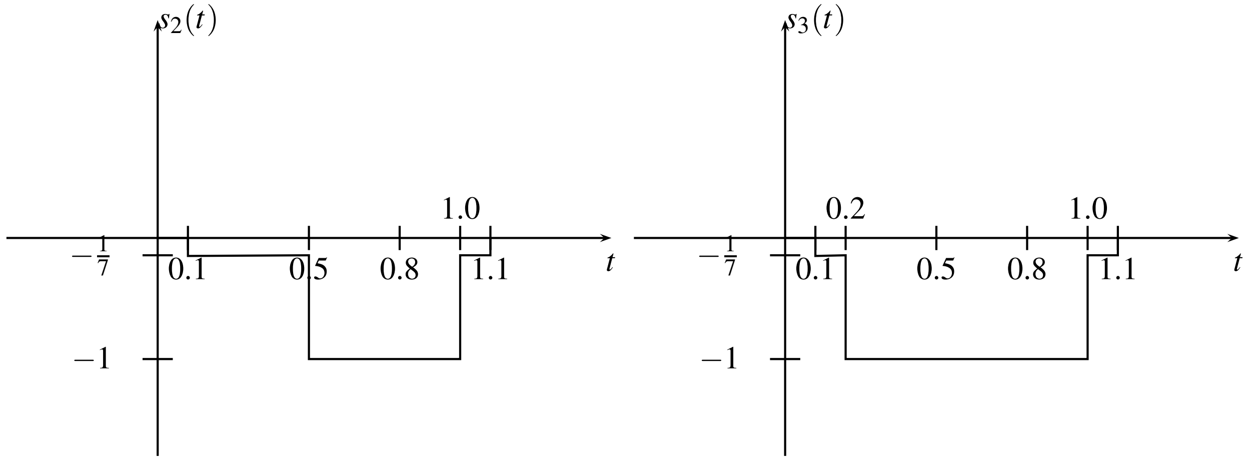
(e) Determine a bound on the symbol (2 bits) error probability of these signals in additive white Gaussian noise.

Solution: The union bound can be evaluated.

$$\begin{aligned}
P_{e,2} = P_{e,0} &= Q(\frac{d_{0,1}}{2\sqrt{N_0/2}}) + Q(\frac{d_{0,2}}{2\sqrt{N_0/2}}) + Q(\frac{d_{0,3}}{2\sqrt{N_0/2}}) \\
&= Q(\sqrt{\frac{.1363A^2}{N_0}}) + Q(\sqrt{\frac{2A^2}{N_0}}) + Q(\sqrt{\frac{1.6678A^2}{N_0}}) \\
&= Q(\sqrt{\frac{.2075\bar{E}}{N_0}}) + Q(\sqrt{\frac{3.0437\bar{E}}{N_0}}) + Q(\sqrt{\frac{2.5381\bar{E}}{N_0}})
\end{aligned}$$

$$\begin{aligned}
P_{e,3} = P_{e,1} &= Q(\frac{d_{1,0}}{2\sqrt{N_0/2}}) + Q(\frac{d_{1,2}}{2\sqrt{N_0/2}}) + Q(\frac{d_{1,3}}{2\sqrt{N_0/2}}) \\
&= Q(\sqrt{\frac{.1363A^2}{N_0}}) + Q(\sqrt{\frac{1.6678A^2}{N_0}}) + Q(\sqrt{\frac{2A^2}{N_0}}) \\
&= Q(\sqrt{\frac{.2075\bar{E}}{N_0}}) + Q(\sqrt{\frac{2.5381\bar{E}}{N_0}}) + Q(\sqrt{\frac{3.0437\bar{E}}{N_0}})
\end{aligned}$$





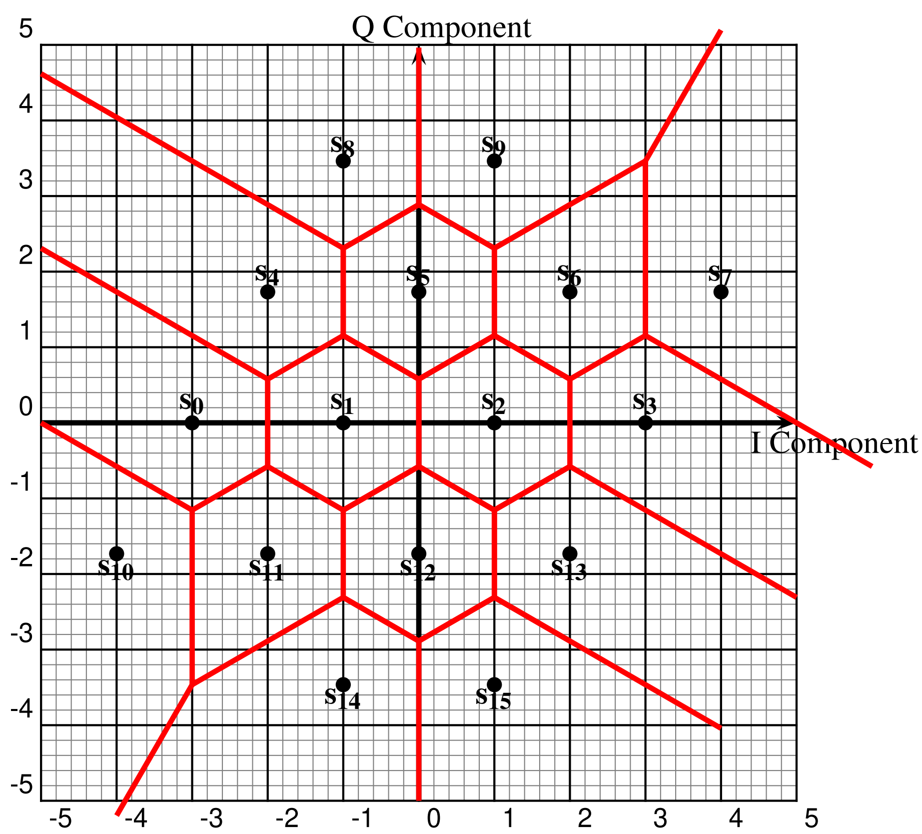
Matlab Code

```
s0=[1/7 1/7 1/7 1/7 1 1 1 1 1 1/7];
E0=sum(s0.^2)*0.1;
s1=[1/7 1 1 1 1 1 1 1 1 1/7];
E1=sum(s1.^2)*0.1;
u0=s0;
phi0=u0/(sqrt(sum(u0.^2)*0.1));
s10=s1*phi0'*0.1;
u1=s1-s10*phi0;
phi1=u1/(sqrt(sum(u1.^2)*0.1));
phi1*phi0'*0.1                      % This is a check for orthogonality
s11=s1*phi1'*0.1;

sp0=[1 0]
sp1=[s10 s11]
sp2=-sp0;
sp3=-sp1;

d01=sqrt(sum((sp0-sp1).^2))
d02=sqrt(sum((sp0-sp2).^2))
d03=sqrt(sum((sp0-sp3).^2))

figure(1)
hold off
plot(sp0(1),sp0(2),'+','LineWidth',2)
grid on
hold on
plot(sp1(1),sp1(2),'+','LineWidth',2)
plot(sp2(1),sp2(2),'+','LineWidth',2)
plot(sp3(1),sp3(2),'+','LineWidth',2)
```

```
axis([-1.5 1.5 -1.5 1.5])
set(gca,'FontSize',16)
xlabel('$\phi_0(t)$','Interpreter','Latex')
ylabel('$\phi_1(t)$','Interpreter','Latex')
str0='\leftarrow    s_0(t)';
str1='\leftarrow    s_1(t)';
str2='\leftarrow    s_2(t)';
str3='\leftarrow    s_3(t)';
text(sp0(1),sp0(2),str0,'FontSize',24)
text(sp1(1),sp1(2),str1,'FontSize',24)
text(sp2(1),sp2(2),str2,'FontSize',24)
text(sp3(1),sp3(2),str3,'FontSize',24)
```

2. Consider the following 16 signal vectors.

$$
\begin{array}{ll}
\mathbf{s_0} = (-3,0) & \mathbf{s_8} = (-1,2\sqrt{3}) \\
\mathbf{s_1} = (-1,0) & \mathbf{s_9} = (+1,2\sqrt{3}) \\
\mathbf{s_2} = (+1,0) & \mathbf{s_{10}} = (-4,-\sqrt{3}) \\
\mathbf{s_3} = (+3,0) & \mathbf{s_{11}} = (-2,-\sqrt{3}) \\
\mathbf{s_4} = (-2,\sqrt{3}) & \mathbf{s_{12}} = (0,-\sqrt{3}) \\
\mathbf{s_5} = (0,\sqrt{3}) & \mathbf{s_{13}} = (+2,-\sqrt{3}) \\
\mathbf{s_6} = (+2,\sqrt{3}) & \mathbf{s_{14}} = (-1,-2\sqrt{3}) \\
\mathbf{s_7} = (+4,\sqrt{3}) & \mathbf{s_{15}} = (+1,-2\sqrt{3})
\end{array}
$$



(a) Draw the optimum decision regions for this set of 16 signals.

Solution: See plot above.

(b) Determine and plot the union bound on the error probability for this signal set. You should plot the error probability as a function of E_b/N_0 in dB on a log scale for the vertical axis. (See Problem Set 2).

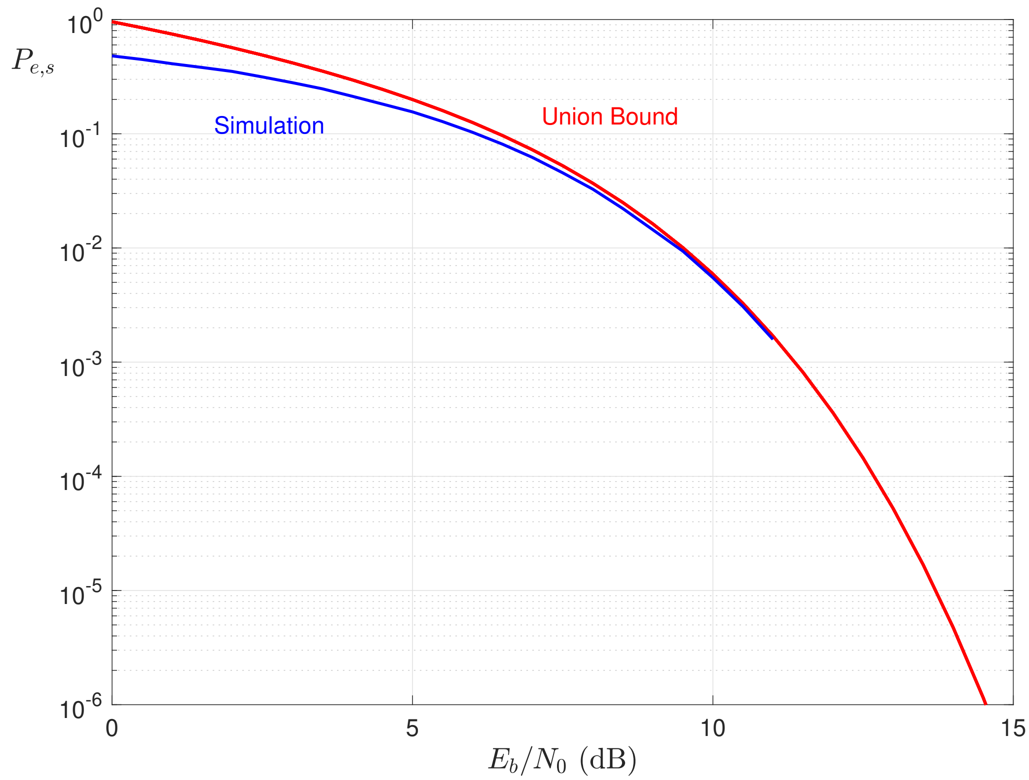
Solution: To determine the union bound we need all pairs of distances between signals. The pairs of distances are shown in the table below. The minimum Euclidean distance is 2.

0	2.00	4.00	6.00	2.00	3.4641	5.2915	7.2111	4	5.2915	2	2	3.4641	5.2915	4	5.2915
2	0	2	4	2	2	3.4641	5.2915	3.4641	4	3.4641	2	2	3.4641	3.4641	4
4	2	0	2	3.4641	2	2	3.4641	4	3.4641	5.2915	3.4641	2	2	4	3.4641
6	4	2	0	5.2915	3.4641	2	2	5.2915	4	7.2111	5.2915	3.4641	2	5.2915	4
2	2	3.4641	5.2915	0	2	4	6	2	3.4641	4	3.4641	4	5.2915	5.2915	6
3.4641	2	2	3.4641	2	0	2	4	2	2	5.2915	4	3.4641	4	5.2915	5.2915
5.2915	3.4641	2	2	4	2	0	2	3.4641	2	6.9282	5.2915	4	3.4641	6	5.2915
7.2111	5.2915	3.4641	2	6	4	2	0	5.2915	3.4641	8.7178	6.9282	5.2915	4	7.2111	6
4	3.4641	4	5.2915	2	2	3.4641	5.2915	0	2	6	5.2915	5.2915	6	6.9282	7.2111
5.2915	4	3.4641	4	3.4641	2	2	3.4641	2	0	7.2111	6	5.2915	5.2915	7.2111	6.9282
2	3.4641	5.2915	7.2111	4	5.2915	6.9282	8.7178	6	7.2111	0	2	4	6	3.4641	5.2915
2	2	3.4641	5.2915	3.4641	4	5.2915	6.9282	5.2915	6	2	0	2	4	2	3.4641
3.4641	2	2	3.4641	4	3.4641	4	5.2915	5.2915	5.2915	4	2	0	2	2	2
5.2915	3.4641	2	2	5.2915	4	3.4641	4	6	5.2915	6	4	2	0	3.4641	2
4	3.4641	4	5.2915	5.2915	5.2915	6	7.2111	6.9282	7.2111	3.4641	2	2	3.4641	0	2
5.2915	4	3.4641	4	6	5.2915	5.2915	6	7.2111	6.9282	5.2915	3.4641	2	2	2	0

The union bound is then

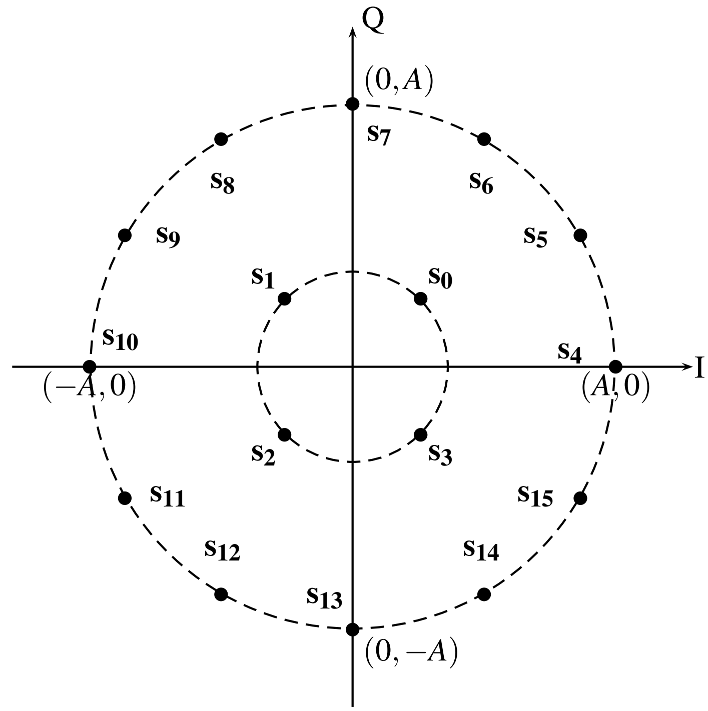
$$
\begin{aligned}
P_e &\leq \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i} P_2(s_i \rightarrow s_j) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i} Q(\frac{d_E(s_i,s_j)}{2\sigma})
\end{aligned}
$$

The union bound on the error probability and a simulation is shown below.



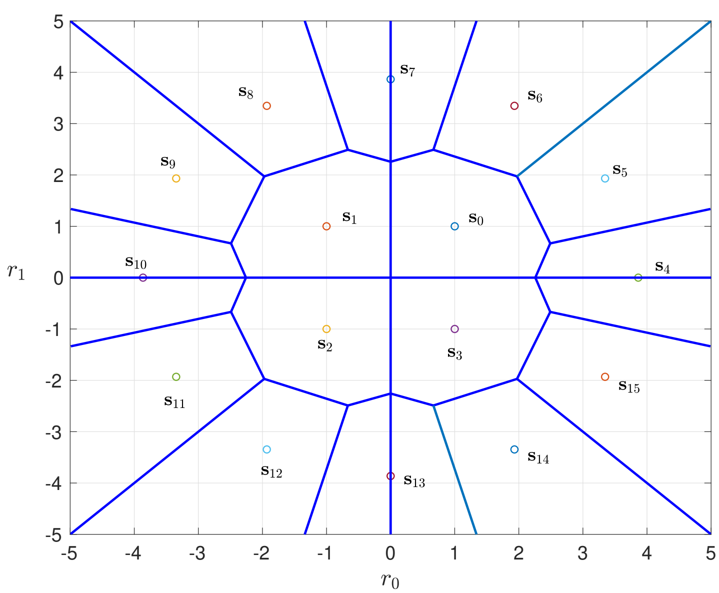
3. A communication system transmits one of 16 equally likely signals. The signals (waveforms) are represented by the vectors shown below. where $A = \sqrt{4(2+\sqrt{3})}$ so that the minimum Euclidean distance between any two signals is at least 2.

$$
\begin{aligned}
\mathbf{s_0} &= (+1,+1) \\
\mathbf{s_1} &= (-1,+1) \\
\mathbf{s_2} &= (-1,-1) \\
\mathbf{s_3} &= (+1,-1) \\
\mathbf{s_4} &= (A,0) \\
\mathbf{s_5} &= (A\cos(2\pi/12), A\sin(2\pi/12)) \\
\mathbf{s_6} &= (A\cos(2\pi 2/12), A\sin(2\pi 2/12)) \\
\mathbf{s_7} &= (0,A) \\
\mathbf{s_8} &= (A\cos(2\pi 4/12), A\sin(2\pi 4/12)) \\
\mathbf{s_9} &= (A\cos(2\pi 5/12), A\sin(2\pi 5/12)) \\
\mathbf{s_{10}} &= (-A,0) \\
\mathbf{s_{11}} &= (A\cos(2\pi 7/12), A\sin(2\pi 7/12)) \\
\mathbf{s_{12}} &= (A\cos(2\pi 8/12), A\sin(2\pi 8/12)) \\
\mathbf{s_{13}} &= (0,-A) \\
\mathbf{s_{14}} &= (A\cos(2\pi 10/12), A\sin(2\pi 10/12)) \\
\mathbf{s_{15}} &= (A\cos(2\pi 10/12), A\sin(2\pi 10/12))
\end{aligned}
$$



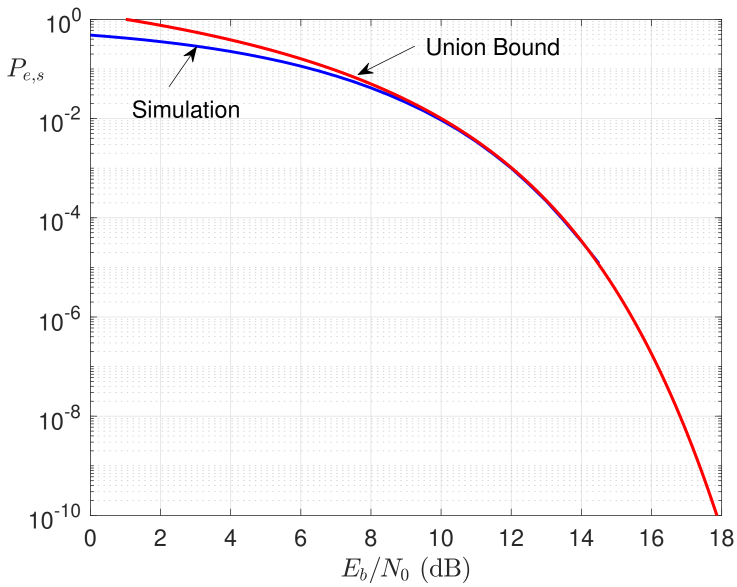
(a) Determine (draw) the optimum decision regions.

Solution:



(b) Determine the union bound on the symbol error probability. Plot the union bound versus E_b/N_0 in dB for values of E_b/N_0 between 0 and 18dB.

Solution:



4. Consider a communication system that transmits an infinite sequence of data bits $\{b_l\}_{l=-\infty}^{\infty}$ using two signals of duration T: $s_0(t) = -s_1(t) = Ap_T(t)$. Thus

$$s(t) = \sum_{l=-\infty}^{\infty} b_l A p_T(t - lT)$$

The signal $s(t)$ is transmitted over an additive white Gaussian noise channel with spectral density $N_0/2$. The receiver consists of a filter $h(t)$ the output of which is sampled at time iT and compared with a threshold of 0. If the output at time $iT > 0$ the receiver decides $b_{i-1} = +1$ otherwise the receiver decides $b_{i-1} = -1$. It is know that the filter is such that $\int_{-\infty}^{\infty} h^2(t)dt = 16$. It is also known that if the input to the filter is $p_T(t)$ ($X(t) = 0$) then the output at time iT is

$$Z(iT) = \begin{cases} 8 & i = 1 \\ 2 & i = 2 \\ 1 & i = 3 \\ 0 & i < 1,\ i > 3 \end{cases}$$

(a) Find the possible values for the output due to the desired signal (no noise) for different data bits.

Solution:

The possible values for the output in the absence of noise are 11,9,7,and 5.

(b) Find the upper and lower bounds for the $E[Z(iT)|b_{i-1} = +1, b_{i-2}, b_{i-3}, \ldots]$. That is, find the largest possible value for the output due to signal alone (no noise) at time iT for all possible previous data bits. Find upper and lower bounds on the probability of error for data bit b_{i-1} given that $b_{i-1} = +1$.

Solution:

If $b_{i-1} = +1$ then $E[Z(iT)|b_{i-1} = +, b_{i-2}, \ldots]$ can be as large as 8+2+1=11 while $E[Z(iT)]$ can be as small as 8-2-1=5.

$$5A < E[Z(iT)|b_{i-1} = +1, b_{i-2}, \ldots] < 11A$$
$$Q(\frac{11A}{\sigma}) < P_e < Q(\frac{5A}{\sigma})$$

where $\sigma^2 = \frac{N_0}{2}16 = 8N_0$. (c) Give an expression for the average probability of error for the data bit b_{i-1} if each data bit is equally likely to be +1 or -1 independently of all other data bits.

Solution: There are four possible outputs given $b_{i-1}+1$ (in the absence of noise). These are 11, 9, 7, and 5. These are equally likely. The probability of error is then

$$\begin{aligned}
\bar{P}_{e,+1} = \bar{P}_{e,-1} &= \frac{1}{4}\left[Q(\frac{11A}{\sigma}) + Q(\frac{9A}{\sigma}) + Q(\frac{7A}{\sigma}) + Q(\frac{5A}{\sigma})\right] \\
\bar{P}_e &= \frac{1}{2}\bar{P}_{e,+1} + \frac{1}{2}\bar{P}_{e,-1} \\
&= \frac{1}{4}\left[Q(\frac{11A}{\sigma}) + Q(\frac{9A}{\sigma}) + Q(\frac{7A}{\sigma}) + Q(\frac{5A}{\sigma})\right]
\end{aligned}$$

where $\sigma^2 = \frac{N_0}{2}16 = 8N_0$.

5. A communication uses the following eight signals where $f_cT = n/2$ for some large n.

$$\begin{aligned}
s_0(t) &= A\cos(2\pi f_c t),\ 0 \le t \le T \\
s_1(t) &= \frac{1}{2}A\cos(2\pi f_c t) + \frac{1}{2}A\sin(2\pi f_c t),\ 0 \le t \le T \\
s_2(t) &= \frac{1}{2}A\sin(2\pi f_c t),\ 0 \le t \le T \\
s_3(t) &= -\frac{1}{2}A\cos(2\pi f_c t) + \frac{1}{2}A\sin(2\pi f_c t),\ 0 \le t \le T \\
s_4(t) &= -A\cos(2\pi f_c t),\ 0 \le t \le T \\
s_5(t) &= -\frac{1}{2}A\cos(2\pi f_c t) - \frac{1}{2}A\sin(2\pi f_c t),\ 0 \le t \le T \\
s_6(t) &= -\frac{1}{2}A\sin(2\pi f_c t),\ 0 \le t \le T \\
s_7(t) &= \frac{1}{2}A\cos(2\pi f_c t) - \frac{1}{2}A\sin(2\pi f_c t),\ 0 \le t \le T
\end{aligned}$$

(a) Determine a suitable set of orthonormal functions to represent these 8 signals.

The orthogonal signals are $\varphi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t)$ and $\varphi_1(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t)$.

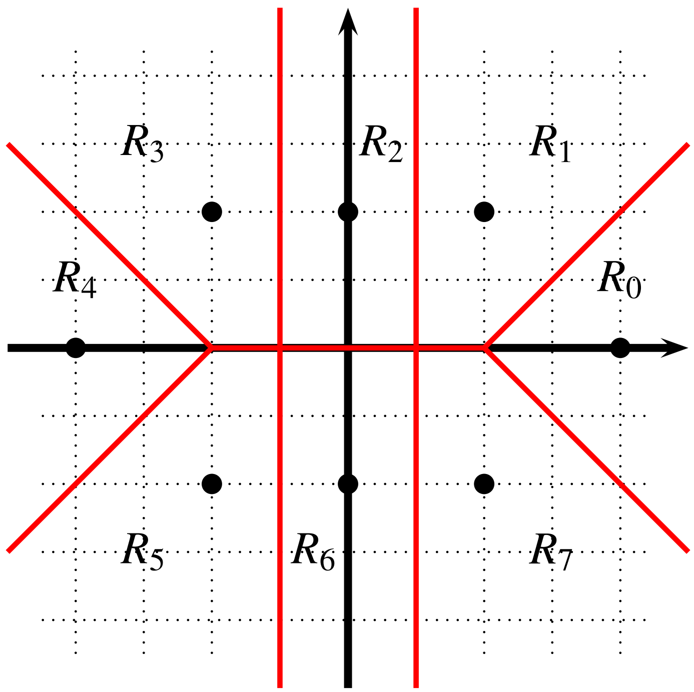
(b) Express the signals as a linear combination of the orthonormal signals.

Solution:

$$
\begin{aligned}
s_0(t) &= A\sqrt{\frac{T}{2}}\varphi_0(t) + 0\varphi_1(t) \\
s_1(t) &= \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_0(t) + \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_1 varphi(t) \\
s_2(t) &= 0\varphi_0(t) + \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_1(t) \\
s_3(t) &= -\frac{A}{2}\sqrt{\frac{T}{2}}\varphi_0(t) + \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_1(t) \\
s_4(t) &= -A\sqrt{\frac{T}{2}}\varphi_0(t) + 0\varphi_1(t) \\
s_5(t) &= -\frac{A}{2}\sqrt{\frac{T}{2}}\varphi_0(t) - \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_1(t) \\
s_6(t) &= 0\varphi_0(t) - \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_1(t) \\
s_7(t) &= \frac{A}{2}\sqrt{\frac{T}{2}}\varphi_0(t) - \frac{A}{2}\sqrt{\frac{T}{2}}\sqrt{\frac{T}{2}}\varphi_1(t)
\end{aligned}
$$

(c) Plot the signals (in signal space) and show the optimum decision regions when these signals are used with equal probability in an additive white Gaussian noise channel with two-side power spectral density $N_0/2$.

Solution:



(d) Find the probability of correct given signal 0 is transmitted in terms of the Φ function or Q function, A, T and N_0.

Solution:

The conditional error probability given signal $s_0(t)$ sent (or the conditional probability of correct) is determined as follows. In order to be correct the noise in the direction of s_1, call it n_A, and the noise in the direction of s_7, call it n_B must both be less than half the distance between $s_0(t)$ and $s_1(t)$. This Euclidean distance is $d_E(s_0, s_1)/2 = A\sqrt{T}/4$. Because n_A and n_B are orthogonal, they are independent. They are zero mean, Gaussian with variance $N_0/2$. Thus

$$
\begin{aligned}
P_{c,0} &= \Phi^2\left(\frac{A\sqrt{T}/4}{\sqrt{\frac{N_0}{2}}}\right) = \Phi^2\left(\sqrt{\frac{A^2T}{8N_0}}\right) \\
&= \left(1 - Q\left(\sqrt{\frac{A^2T}{8N_0}}\right)\right)^2 \\
&= 1 - 2Q\left(\sqrt{\frac{A^2T}{8N_0}}\right) + Q^2\left(\sqrt{\frac{A^2T}{8N_0}}\right) \\
P_{e,0} &= 2Q\left(\sqrt{\frac{A^2T}{8N_0}}\right) - Q^2\left(\sqrt{\frac{A^2T}{8N_0}}\right)
\end{aligned}
$$