EECS501: Solution to Homework 2

1. **Union Bound Refined**

   Note that we have two upper bounds on $P\left(\bigcup_{i=1}^n A_i\right)$: the first is the union bound of part (b), the second is 1. There are two possibilities: (i) $\sum_{i=1}^n P(A_i) \geq 1$, and (ii) $\sum_{i=1}^n P(A_i) \leq 1$. In the first case, the minimum is achieved when $\rho = 0$, and the minimum is 1. In second case, the minimum is achieved when $\rho = 1$, and the minimum is $\sum_{i=1}^n P(A_i)$. Hence we get the desired result.

2. **Shine or Rain**

   Define event R = rain and S = shine. So R^c = dry and S^c = cloudy.

   $P(R) = 0.2$, $P(S) = 0.8$ and $P(R^c \cap S^c) = 0.1$.

   $P(R^c|S^c) = \frac{P(R^c \cap S^c)}{P(S^c)} = \frac{P(R^c \cap S^c)}{1-P(S)} = \frac{0.1}{0.2} = 0.5$.

3. **Tennis Anyone?**

   Let A be the event that Alice wins the game, given that she is serving at deuce. Define the partition

$$
\begin{aligned}
B_1 &= \{\text{Alice wins the next two points}\} \\
B_2 &= \{\text{Alice wins the first point, Bob wins the second}\} \\
B_3 &= \{\text{Bob wins the first point, Alice wins the second}\} \\
B_4 &= \{\text{Bob wins the next two points}\}
\end{aligned}
$$

   By the law of total probability,

$$
\begin{aligned}
P(A) &= \sum_{i=1}^{4} P(A|B_i)P(B_i) \\
&= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + P(A|B_4)P(B_4) \\
&= 1 \cdot p^2 + P(A) \cdot p(1-p) + P(A) \cdot (1-p)p + 0 \cdot (1-p)^2 \\
&= p^2 + P(A) \cdot 2p(1-p).
\end{aligned}
$$

   Solving for $P(A)$ yields

$$
P(A) = \frac{p^2}{1-2p(1-p)} = \frac{p^2}{p^2+(1-p)^2}.
$$

Either of these last two forms is acceptable, but the second one is more intuitive. Think about it.

4. **Coin Tossing Game**

Let A_1, A_2, A_3 and A_4 denote the events that you get HH, HT, TH and TT on the first two rolls. Let W denote the event that you win. Note that the set of events A_1, A_2, A_3 and A_4 is a partition of the sample space. Note that $P(W|A_1) = \frac{1}{2}P(W|A_2)$, and $P(W|A_2) = \frac{1}{2}P(W|A_3) + \frac{1}{2}P(W|A_4)$. Further, $P(W|A_3) = \frac{1}{2} + \frac{1}{2}P(W|A_1)$, and $P(W|A_4) = \frac{1}{2}P(W|A_4) + \frac{1}{2}P(W|A_3)$. Solving these four linear equations, we get $P(W|A_1) = \frac{1}{3}$, $P(W|A_2) = P(W|A_3) = P(W|A_4) = \frac{2}{3}$. So we get using the law of total probability that $P(W) = \frac{7}{12}$.

5. **World Series**

There are 4 possibilities for the number of games played in the series. They are $4, 5, 6$ and 7. The sample space consists of blocks of game outcomes. For example $RCRCRR$ is an outcome in the sample space, denoting that Rangers win games $1, 3, 5$ and 6. So is $CCCC$, denoting that Cardinals win games $1, 2, 3$ and 4. When the number of games played is 4, Rangers win the series if they win all games. This will happen with probability p^4. When the number of games played is 5, the possibilities for the outcome of the sample space are $CRRRR$, $RCRRR$, $RRCRR$ and $RRRCR$. Each of these outcomes happens with probability $p^4(1-p)$. Similarly, when the number of games played is 6, there are $\binom{5}{2}$ possibilities, with the probability of each possibility is $p^4(1-p)^2$. Using this argument, we find that the probability that Rangers win the series is given by

$$p^4 + 4p^4(1-p) + 10p^4(1-p)^2 + 20p^4(1-p)^3.$$

6. **Dice Game** [10 points]

Define the following events:

$$\begin{aligned}
A_1 &= \{1 \text{ on first roll}\} \\
A_2 &= \{2, 3, 4, \text{ or } 6 \text{ on first roll}\} \\
A_3 &= \{5 \text{ on first roll}\} \\
A_4 &= \{6 \text{ on second roll}\} \\
A_5 &= \{5 \text{ on second roll}\} \\
A_6 &= \{1 \text{ on second roll}\} \\
A_7 &= \{2, 3, \text{ or } 4 \text{ on second roll}\} \\
B &= \{56 \text{ before } 1\}
\end{aligned}$$

First note that $P(A_1) = P(A_3) = P(A_4) = P(A_5) = P(A_6) = 1/6$, $P(A_2) = 2/3$, and $P(A_7) = 1/2$. $P(B|A_1) = 0$, and $P(B|A_2) = P(B)$ because rolling $2, 3, 4$, or 6 on the first roll resets the game to its initial state. A_1, A_2, A_3 form a partition so $P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$ by the law of total probability. Applying the law of total probability once again,

$$P(B|A_3) = P(B|A_4A_3)P(A_4|A_3) + P(B|A_5A_3)P(A_5|A_3) + P(B|A_6A_3)P(A_6|A_3) + P(B|A_7A_3)P(A_7|A_3).$$

$P(B|A_4A_3) = 1$, and $P(B|A_6A_3) = 0$. $P(B|A_5A_3) = P(B|A_3)$ because the game remains in the same state, requiring a 6 on the next roll to to get to event B, and $P(B|A_7A_3) = P(B)$ because rolling $2, 3$, or 4 resets the game to the initial state.

So

$$P(B) = \frac{1}{6}(0) + \frac{2}{3}P(B) + \frac{1}{6}P(B|A_3).$$

$$\begin{aligned}
P(B|A_3) &= \frac{1}{6}(1) + \frac{1}{6}P(B|A_3) + \frac{1}{6}(0) + \frac{1}{2}P(B) \\
\Rightarrow \frac{5}{6}P(B|A_3) &= \frac{1}{6} + \frac{1}{2}P(B) \\
\Rightarrow P(B|A_3) &= \frac{1}{5} + \frac{3}{5}P(B).
\end{aligned}$$

Thus

$$\begin{aligned}
P(B) &= \frac{2}{3}P(B) + \frac{1}{6}\left[\frac{1}{5} + \frac{3}{5}P(B)\right] \\
P(B)\left(1 - \frac{2}{3} - \frac{1}{10}\right) &= \frac{1}{30} \\
P(B) &= \frac{1}{7}.
\end{aligned}$$

Hence $P(B^c) = 6/7$.

7. **Independence**

We will use the following fact repeatedly: If G and H are independent, then (i) G and H^c are independent, and (ii) G^c and H^c are independent. Thus it remains to show $P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c)P(C^c)$. The idea is to start with independence of A, B, C, and apply rule (i) above three times, each time inserting an additional complement.

First step. Show that A, B, C^c are independent.

Notice that

$$\begin{aligned} P((A \cap B) \cap C) &= P(A \cap B \cap C) \\ &= P(A)P(B)P(C) \\ &= P(A \cap B)P(C). \end{aligned}$$

This shows that $A \cap B$ and C and independent. By (i), $A \cap B$ and C^c are independent and therefore

$$\begin{aligned} P(A \cap B \cap C^c) &= P((A \cap B) \cap C^c) \\ &= P(A \cap B)P(C^c) \\ &= P(A)P(B)P(C^c). \end{aligned}$$

This shows that A, B, C^c are independent.

Second step. Show that A, B^c, C^c are independent.

This proceeds in the same way as the previous step, except now we are grouping A and C^c. We could write this out in detail, just like in the first step, but I think it is easier to think about it conceptually. What we showed in step one is that if we have any three events that are independent, we can replace one of them by their complement, and the three sets are still independent. In the second step we just apply this property to A, C^c, and B. These are independent (by step one), and therefore so are A, C^c, and B^c (by the logic of step one).

Third step. Show that A^c, B^c, C^c are independent.

This now follows from step two, where we again apply the logic of step one. Since A, B^c, C^c are independent, we can replace A by A^c and the events are still independent.

8. **Independence**

$A = \{(HHHT), (HHTH), (HTHH), (THHH), (TTHH), (TTHT), (TTTH), (TTTT)\}$.

Note: $A =$ either T on the first 2 tosses or exactly 3 H's.

$B = \{H$ is observed on the first toss $\}$.

$C = \{H$ is observed on the second toss $\}$.

$P(A) = P(B) = P(C) = \frac{1}{2}$.

$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$.

$P(A \cap B) = \frac{3}{16} \neq P(A)P(B)$.

$P(A \cap C) = \frac{3}{16} \neq P(A)P(C)$.

9. **Badminton**

Let B denote the event that Ann wins the next point. Let A_1 denote the event that Ann wins the current serve. Let A_2 denote the event that Ann loses the current serve and Bob wins the next serve. Let A_3 denote the event that Ann loses the current serve and Ann wins the next serve. Note that $P(B|A_1) = 1$, $P(B|A_2) = 0$. Further, $P(A_1) = p$ and $P(A_2) = (1-p)^2$. Moreover, $P(A_3) = (1-p)p$, and $P(B|A_3) = P(B)$. Hence we can solve for $P(B)$ as follows:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) = p + 0 + p(1-p)P(B).$$

We get

$$P(B) = \frac{p}{1 - p(1-p)}.$$