EECS 501 Final Exam YUZHAN JIANG

P0:

I have neither given nor received aid on this exam, nor have I concealed any honor code violations.

P1:

First, we compute " 4 or 6 before observing 1"

$A_1 = \{1 \text{ on the first roll}\}$
$A_2 = \{2, 3, 5 \text{ on the first roll}\}$
$A_3 = \{4, 6 \text{ on the first roll}\}$

$A_4 = \{4, 6 \text{ on the second roll}\}$
$A_5 = \{1 \text{ on the second roll}\}$
$A_6 = \{2, 3, 5 \text{ on the second roll}\}$
$B = \{4 \text{ or } 6 \text{ before } 1\}$

$\frac{3}{12} + \frac{3}{12} + \frac{1}{12}$

By Law of total probability:

$$P(B) = \underbrace{P(B|A_1)}_{0} \cdot P(A_1) + P(B|A_2) \cdot P(A_1) + P(B|A_3) \cdot P(A_3)$$

$$= + \left(\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{12}\right) \cdot P(B|A_2) + \tfrac{1}{6} \cdot \underbrace{P(B|A_3)}_{1}$$

$$= \tfrac{7}{12} \cdot P(B|A_2) + \tfrac{1}{6}$$

$$P(B|A_2) = P(B|A_2 A_4) \cdot P(A_4|A_2) + \underbrace{P(B|A_2 A_5)}_{0} \cdot P(A_4|A_2)$$
$$+ P(B|A_2 A_6) \cdot P(A_6|A_2)$$
$$= \tfrac{1}{6} \cdot 1 + P(B|A_2) \cdot \tfrac{7}{12}$$

$$\tfrac{5}{12} P(B|A_2) = \tfrac{1}{6}$$
$$P(B|A_2) = \tfrac{1}{6} \times \tfrac{12}{5}$$
$$= \tfrac{2}{5}$$

$$\therefore P(B) = \tfrac{7}{12} \cdot \tfrac{2}{5} + \tfrac{1}{6}$$
$$= \tfrac{7}{30} + \tfrac{1}{6}$$
$$= \tfrac{7}{30} + \tfrac{5}{30}$$

$$\therefore \quad P(B) = \frac{12}{30}$$

$$= \frac{4}{10}$$

$$= \frac{2}{5}$$

$$\therefore \quad P(B^c) = 1 - \frac{2}{5}$$

$$= \frac{3}{5}$$

$\therefore$ P(observing "1" before "4" or "6") $= \frac{3}{5}$

P2: $Y = X + N$, where $X \sim \text{unif}[0,1]$, $N \sim \text{Gaussian}(0,1)$
X and N are independent $\Rightarrow E[XN] = E[X] \cdot E[N]$

The LMMSE estimator:

$$\hat{E}[X|Y] = E[X] + \frac{\text{cov}(X,Y)}{\text{Var}(Y)} \cdot (Y - E[Y])$$

$E[X] = \frac{1+0}{2} = \frac{1}{2}$ Since $X \sim \text{unif}(0,1)$

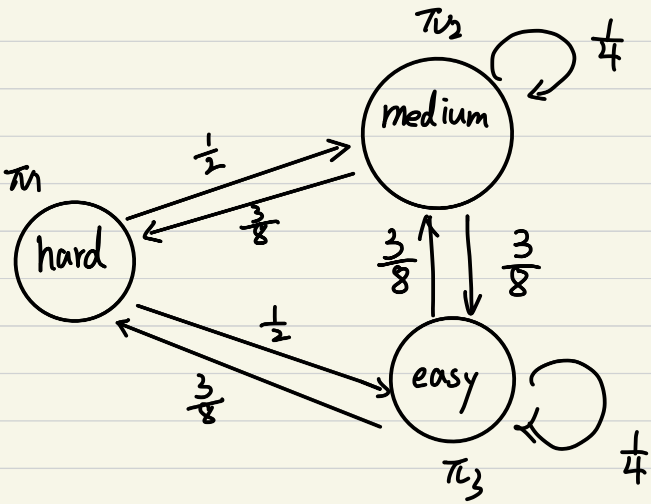
$E[Y] = E[X+N] = E[X] + E[N] = \frac{1}{2} + 0 = \frac{1}{2}$

$$
\begin{aligned}
\text{cov}(X,Y) &= \text{cov}(X, X+N) \\
&= \text{cov}(X,X) + \text{cov}(X,N) \\
&= E[X^2] - E^2[X] + 0 \quad \text{(since } X \text{ and } N \text{ are independent)} \\
&= \text{Var}(X) \\
&= \frac{1}{12}
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(Y) &= \text{Var}(X+N) \quad \text{(Since } X \text{ and } N \text{ are independent)} \\
&= \text{Var}(X) + \text{Var}(N) \\
&= \frac{1}{12} + 1 \\
&= \frac{13}{12}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \hat{E}[X|Y] &= \frac{1}{2} + \frac{1}{12} \cdot \left(\frac{13}{12}\right)^{-1} \cdot \left(Y - \frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{13} \cdot \left(Y - \frac{1}{2}\right) \\
&= \frac{1}{13} Y + \frac{1}{2} - \frac{1}{26} \\
&= \frac{1}{13} Y + \frac{12}{26} \\
&= \frac{1}{13} Y + \frac{6}{13}
\end{aligned}
$$

P3:



$$\pi = \pi P$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{3}{8}\pi_2 + \frac{3}{8}\pi_3 \\ \pi_2 = \frac{1}{4}\pi_2 + \frac{1}{2}\pi_1 + \frac{3}{8}\pi_3 \\ \pi_3 = \frac{1}{4}\pi_3 + \frac{1}{2}\pi_1 + \frac{3}{8}\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Solve these equations we get:

$$\Rightarrow \pi_1 = \frac{3}{11} \quad \pi_2 = \frac{4}{11}, \quad \pi_3 = \frac{4}{11}$$

$$E = \frac{3}{11} \times 70 + \frac{4}{11} \times 85 + \frac{4}{11} \times 100$$

$$= \frac{950}{11}$$

$$= 86.36$$

Therefore, the steady-state probabilies are $\pi_1 = \frac{3}{11}$, $\pi_2 = \frac{4}{11}$, $\pi_3 = \frac{4}{11}$, the expected score of the students is 86.36.

P4:

$$X_t = A\cdot \sin(t+\theta)$$

$A \sim unif[0,1] \Rightarrow E[A] = \frac{1}{2}$

$Var(A) = E[A^2] - E^2[A]$

$E[A^2] = Var(A) + E^2[A]$

$= \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

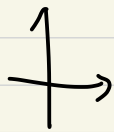
First, we need to calculate $E[X_t]$

$$E[X_t] = E[A\cdot \sin(t+\theta)]$$

$= E[A]\cdot E[\sin(t+\theta)]$ (Since A and θ are independent.)

$= \frac{1}{2}\cdot \int_0^{\pi} \frac{1}{\pi}\cdot \sin(t+\theta)\cdot d\theta$

$= \frac{1}{2\pi}\cdot \quad \cdot -\cos(t+\theta)\Big|_0^{\pi}$



$= \frac{1}{2\pi}\cdot \quad -\cos(t+\pi) + \cos(t)$

$= \frac{1}{2\pi}\cdot 2\cdot \cos t$

$= \frac{1}{\pi}\cdot \cos t$

$R_X(t_1,t_2) = E[X_{t_1} X_{t_2}]$

$= E[A\cdot \sin(t_1+\theta)]\cdot E[A\cdot \sin(t_2+\theta)]$

$= E[A^2]\cdot E[\sin(t_1+\theta)\sin(t_2+\theta)]$

$= \frac{1}{3}\cdot E[\frac{1}{2}\cdot(\cos(t_1-t_2) - \cos(t_1+t_2+2\theta)]$

$= \frac{1}{6}\cdot \cos(t_1-t_2) - \frac{1}{6}\cdot \int_0^{\pi} \frac{1}{\pi}\cdot \cos(t_1+t_2+2\theta)\cdot d\theta$

$> \frac{1}{6}\cdot \cos(t_1-t_2) - \frac{1}{12\pi}\cdot \sin(t_1+t_2+2\theta)\Big|_0^{\pi}$

$= \frac{1}{6}\cdot \cos(t_1-t_2)$

Although $R_X(t_1,t_2)$ depends on (t_1-t_2) their time difference, $E[X_t]$ is not a constant, thus it is not WSS