University of Michigan Fall 2021

EECS501: Solution to Homework 1

1. **Consistency of subjective probabilities**

   Let A be the event "rain on Saturday," and let B be the event "rain on Sunday." Then $A \cap B$ is the event "rain on both days," $A \cup B$ is the event "rain on at least one of the days," and $(A \cup B)^c$ is the event "rain on neither day."

   (a) By the inclusion-exclusion principle, we should have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .9 + .8 - .7 = 1$. However, we also should have $P(A \cup B) = 1 - P((A \cup B)^c) = 0.7$. This contradiction shows that the axioms of probability are not satisfied. There are possibly other valid arguments.

   (b) Since $A \cap B \subset A$, we expect to see $P(A \cap B) \leq P(A)$, but this is not the case here, so these probabilities are invalid (there are other similar inconsistencies as well).

   (c) The probabilities are valid. This follows from the inclusion exclusion principle: $P(A) + P(B) = P(A \cup B) + P(A \cap B)$, and the Law of the complements $P(A^c \cap B^c) = 1 - P(A \cup B)$. These are both valid for this case.

   (d) We should have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8$. Moreover, we must have $P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c) = 1 - 0.9 = 0.1$. Hence a contradiction. Not a valid probability assignment.

2. **High Probability Events**

   The proof uses inclusion-exclusion principle.

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 > 1 - \delta + 1 - \delta - 1 = 1 - 2\delta$$

   Note: $P(A \cup B) \leq 1$, $P(A) > 1 - \delta$ and $P(B) > 1 - \delta$.

3. **Inclusion Exclusion Principle**

   (a) We have derived the following result in the class.

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{1}$$

Now let $M = B \cup C$. Then using the above equation we get

$$\begin{aligned}
P(A \cup B \cup C) &= P(A \cup M) && (2)\\
&= P(A) + P(M) - P(A \cap M) && (3)\\
&= P(A) + P(M) - P(A \cap (B \cup C)) && (4)\\
&= P(A) + P(M) - P\left((A \cap B) \cup (A \cap C)\right) && (5)\\
&= P(A) + P(M) - P(A \cap B) - P(A \cap C) + P((A \cap B) \cap (A \cap C)) && (6)\\
&= P(A) + P(M) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) && (7)\\
&= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)
\end{aligned}$$

where the second equation follows from using $M = (B \cup C)$, the third follows from using equation (5) with M replacing B, the fourth follows from substituting M with $(B \cup C)$, the fifth from distributive law, the sixth from using equation (5) again, the seventh from noting that $(A \cap B) \cap (A \cap C) = (A \cap B \cap C)$, and the eighth from using equation (5) on M.

(b) $A = \{$ You get 4 or 5 on first roll.$\}$

$B = \{$ You get 4 or 5 on second roll.$\}$

$C = \{$ You get 4 or 5 on third roll.$\}$

Note: All numbers for each roll are equally likely to happen with probability $\frac{1}{6}$.

$P(A) = P(B) = P(C) = \frac{2}{6} = \frac{1}{3}$.

$P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{2^2}{6^2} = \frac{1}{9}$.

Note: $P(A \cap B) = \{$You get 4 or 5 on both first and second rolls.$\}$. Since all numbers are equally likely to happen, the probability is $\frac{2^2}{6^2} = \frac{1}{9}$.

$P(A \cap B \cap C) = \frac{2^3}{6^3} = \frac{1}{27}$.

Note: $P(A \cap B \cap C) = \{$You get 4 or 5 on first, second and third rolls.$\}$. Since all numbers are equally likely to happen, the probability is $\frac{2^3}{6^3} = \frac{1}{27}$.

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) = \frac{19}{27}$.

4. **Coin Tossing**

There are 16 outcomes in the sample space. Hence $|\Omega| = 16$. Let E denote the event of interest.

(i) $E = \{TTTT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$. Hence $P(E) = 11/16$.

(ii) $E = \{THTT, HHTT, HTTH, TTHH, TTHT, HTTT, TTTH, TTTT\}$. Hence $P(E) = 8/16$.

5. **Daughters**

   – Consider the first problem. The sample space is

   $$\Omega = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}.$$

   We view the outcomes as ordered because we are sampling with replacement (otherwise they are not equally likely). Since three of these outcomes involve a single girl, the probability is $3/8$.

   – Now consider the second problem. There are a couple of ways to proceed. The first is to view an outcome as an ordered pair xy where x is the sex of the older of the girl's two siblings, and y is the sex of the younger of the two. Then the sample space is

   $$\Omega = \{bb, bg, gb, gg\},$$

   where b denotes a boy and g a girl. These outcomes are equally likely, and only one outcome corresponds to the event in question, so the probability is $1/4$.

   Another approach is to view outcomes as ordered triples xyz, where x,y,z are the sexes of the three siblings listed in birth order. Now, let g^* denote the chosen girl, who must appear somewhere in each outcome. Then the sample space is

   $$\Omega = \{g^*bb, g^*bg, g^*gb, g^*gg, bg^*b, bg^*g, gg^*b, gg^*g, bbg^*, bgg^*, gbg^*, ggg^*\}.$$

   Now, three of the twelve outcomes correspond to the event in question, so again we find the probability is $1/4$.

6. **Shine or Rain**

   Define event R = rain and S = shine. So R^c = dry and S^c = cloudy.

   $P(R) = 0.2$, $P(S) = 0.8$ and $P(R^c \cap S^c) = 0.1$.

   $P(R^c|S^c) = \frac{P(R^c \cap S^c)}{P(S^c)} = \frac{P(R^c \cap S^c)}{1-P(S)} = \frac{0.1}{0.2} = 0.5$.

7. **Pairwise Independence**

   The sample space is $S = \{(i,j); i,j \leq 6\}$. The sets A, B and C are as follows:

   $$A = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$$

$$B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$$

$$C = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$$

Therefore, $P(A) = P(B) = P(C) = \frac{1}{6}$.

Their intersections are as follows: $A \cap B = \{(3,4)\}$, $A \cap C = \{(3,4)\}$, $B \cap C = \{(3,4)\}$, $A \cap B \cap C = \{(3,4)\}$.

So we have

$$\begin{aligned}
P(A \cap B) = \frac{1}{36} &= \tfrac{1}{6} \times \tfrac{1}{6} = P(A) \times P(B) \\
P(A \cap C) = \frac{1}{36} &= \tfrac{1}{6} \times \tfrac{1}{6} = P(A) \times P(C) \\
P(B \cap C) = \frac{1}{36} &= \tfrac{1}{6} \times \tfrac{1}{6} = P(B) \times P(C)
\end{aligned}$$

and it means A,B,C are pairwise independent. On the other hand,

$$P(A \cap B \cap C) = \frac{1}{36} = \frac{1}{6} \times \frac{1}{6} \neq P(A) \times P(B) \times P(C)$$

which means that A,B,C are not independent as a triplet.