EECS 455: Solutions to Problem Set 2

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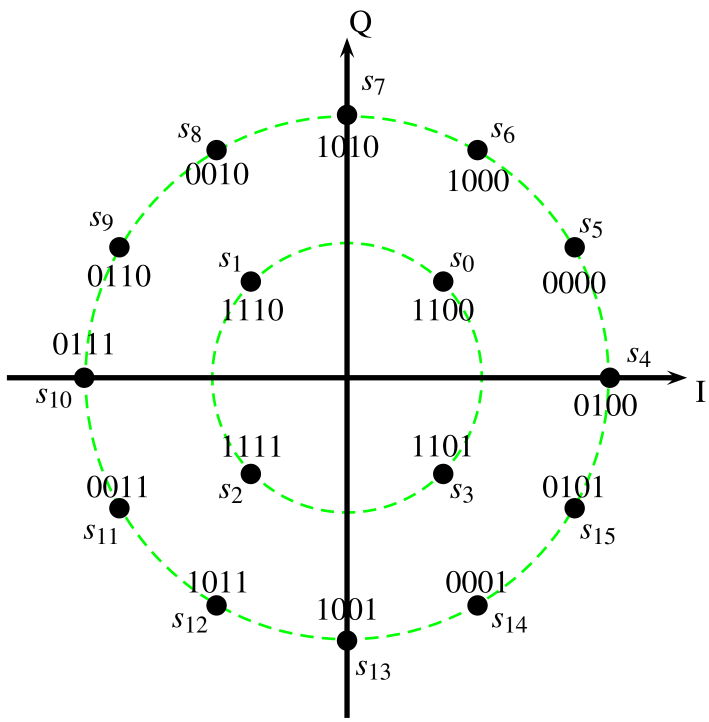
1. A communication system transmits one of 16 equally likely signals using 16APSK modulation. The signal vectors of length 2 (or 1 complex dimension) lie on two circles with 4 points on the inner circle spaced evenly in phase and 12 points on the outer circle also evenly spaced as shown in the diagram below.



(a) Suppose the desired minimum squared Euclidean distance is 4 . Determine the radius of the outer circle so that the minimum squared Euclidean distance is 4 between signals on the outer circle.

Solution: Let r_2 be the radius of the outer circle. Consider the two signals $s_4 = (r_2, 0)$ and $s_5 = (r_2 \cos(\pi/6), r_2 \sin(\pi/6))$. Then the squared Euclidean distance between these signals is

$$\begin{aligned} d_E^2(s_1, s_2) &= (r_2 - r_2 \cos(\pi/6))^2 + (0 - r_2 \sin(\pi/6))^2 \\ &= r_2^2(1 - 2\cos(\pi/6) + \cos^2(\pi/6) + \sin^2(\pi/6)) \\ &= r_2^2(2 - 2\cos(\pi/6)) \end{aligned}$$

Since the squared distance should be 4, the radius of the outer circle should be

$$\begin{aligned} r_2 &= \frac{2}{\sqrt{(2 - 2\cos(\pi/6))}} \\ &= \frac{2}{\sqrt{2 - \sqrt{3}}} \\ &= 3.8637 \end{aligned}$$

(b) Determine the radius of the inner circle so that the minimum squared Euclidean distance between any point on the inner circle and points on the outer circle is also 4.

Solution: Let r_1 be the radius of the inner circle. Consider the two signals $s_0 = (r_1 \cos(\pi/4), r_1 \sin(\pi/4))$ and $s_5 = (r_2 \cos(\pi/6), r_2 \sin(\pi/6))$. Then the squared Euclidean distance between these points is

$$\begin{aligned} d_E^2(s_0, s_5) &= (r_1 \cos(\pi/4) - r_2 \cos(\pi/6))^2 + (r_1 \sin(\pi/4) - r_2 \sin(\pi/6))^2 \\ &= (r_1(\sqrt{2}/2) - r_2(\sqrt{3}/2))^2 + (r_1(\sqrt{2}/2) - r_2(1/2))^2 \\ &= \frac{1}{4}(r_1\sqrt{2} - r_2\sqrt{3})^2 + (r_1\sqrt{2} - r_2)^2 \end{aligned}$$

Since the squared Euclidean distance should be 4, the radius of the outer circle should be

$$\begin{aligned} 4 &= \frac{1}{4}(r_1\sqrt{2} - r_2\sqrt{3})^2 + (r_1\sqrt{2} - r_2)^2 \\ 16 &= 2r_1^2 - 2\sqrt{6}r_1r_2 + 3r_2^2 + 2r_1^2 - 2\sqrt{2}r_1r_2 + r_2^2 \\ 0 &= ar_1^2 + br_1 + c \\ a &= 4 \\ b &= -2(\sqrt{6} + \sqrt{2})r_2 \\ c &= 4r_2^2 - 16 \\ r_1 &= \frac{2(\sqrt{6} + \sqrt{2})r_2 \pm \sqrt{4(\sqrt{6} + \sqrt{2})^2 r_2^2 - 4(2)(4r_2^2 - 16)}}{8} \\ &= 2 \end{aligned}$$

(c) With these radii, is the minimum squared Euclidean distance of two points on the inner circle at least 4?

Solution: Yes, the squared Euclidean distance between s_0 and s_1 is 8

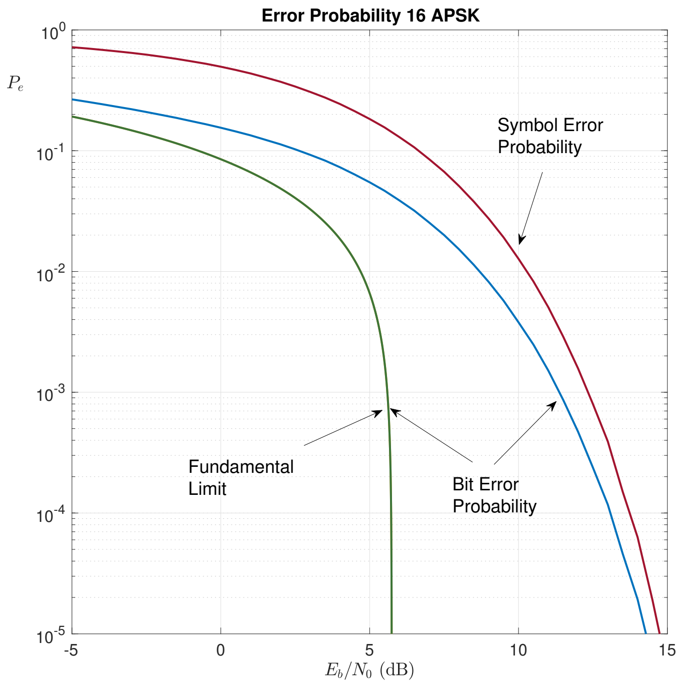
(d) Determine the average energy, E and the energy per bit E_b.

Solution:

$$\begin{aligned} \bar{E} &= (4 + 12\frac{4}{2 - \sqrt{3}})/16 \\ &= 12.196 \\ \bar{E}_b &= \bar{E}/4 \\ &= 3.049 \end{aligned}$$

(e) Modify the simulation for 32APSK to simulate the error probability for 16APSK. Generate the symbol (a symbol is 4 bits) error probability and the bit error probability.

Solution:




```
clear all
N=500000;
A1=sqrt(2);
A2=2/(sqrt(2-2*cos(pi/6)));
s(1:4)=A1*[1+j -1+j -1-j 1-j];
phi1=pi/6*(0:11);
s(5:16)=A2*(cos(phi1)+j*sin(phi1));
for m=1:16;
    for l=1:16
        d(m,l)=abs(s(m)-s(l));
    end
end
figure(1)
plot(s,'*','LineWidth',2)
grid on
Es=sum(abs(s).^2)/16;
Eb=Es/4;
tic
for j1=1:42
    EbN0dB(j1)=(j1-11)/2
    EbN0=10.^(EbN0dB(j1)/10);
    N0=Eb/EbN0;
    sigma=sqrt(N0/2);
    error=0;
    biterror=0;
    for n=1:N
        data=round(15*rand(1,1)+1);
        r=s(data)+sigma*(randn(1,1)+j*randn(1,1));
```

```
        dmin=10000;
        for k=1:16
            d=abs(r-s(k));
            if (d<dmin) info=k; dmin=d; end
        end
        if (info ~= data) error =error+1;
        end
        bhat=de2bi(bitmap(info),5);
        b=de2bi(bitmap(data),5);
        biterror =biterror+sum(abs(b-bhat));
    end
    Pe(j1)=error/N;
    Peb(j1)=biterror/N/4;
    [EbN0dB(j1),Pe(j1), Peb(j1)]
end
figure(2)
semilogy(EbN0dB,Pe)
hold on
semilogy(EbN0dB,Peb)
grid on
xlabel('$E_b/N_0$ (dB)','Interpreter','Latex')
ylabel('$P_e$','Rotation',0,'Interpreter','Latex')
set(gca,'FontSize',16)
axis([-5 15 .9991e-5 1])
toc

logpe=-5:0.01:-.301;
pe=10.^(logpe);
R1=1+pe.*log2(pe)+(1-pe).*log2(1-pe);
RsoW=4;
ebn0=(2.^(RsoW*R1)-1)/RsoW;
ebn0dB=10*log10(ebn0);
figure(2)
semilogy(ebn0dB,pe)
grid on


function f=bitmap(n)
switch n
    case 1
        f=12; %1100
    case 2
        f=14; %1110
    case 3
        f=15;  %1111
    case 4
        f=13;  %1101
```

```
    case 5
        f=4;  %0100
    case 6
        f=0;  %0000
    case 7
        f=8;  %1000
    case 8
        f=10; %1010
    case 9
        f=2; %0010
    case 10
        f=6;  %0110
    case 11
        f=7;   %0111
    case 12
        f=3;  %0011
    case 13
        f=11;   %1011
    case 14
        f=9;  %1001
    case 15
        f=1;  %0001
    case 16
        f=5;  %0101

end
end
```

2. Consider the following 16 signals in 2 dimensions.

$$
\begin{aligned}
\mathbf{s_0} &= (1.2028, -3.4096) \\
\mathbf{s_1} &= (2.8312, -2.2484) \\
\mathbf{s_2} &= (3.5934, -0.3994) \\
\mathbf{s_3} &= (3.2558, +1.5720) \\
\mathbf{s_4} &= (1.9220, +3.0622) \\
\mathbf{s_5} &= (0.0000, +3.6154) \\
\mathbf{s_6} &= (-1.9220, +3.0622) \\
\mathbf{s_7} &= (-3.2558, +1.5720) \\
\mathbf{s_8} &= (-3.5934, -0.3994) \\
\mathbf{s_9} &= (-2.8312, -2.2484) \\
\mathbf{s_{10}} &= (-1.2028, -3.4096) \\
\mathbf{s_{11}} &= (0.0000, -1.8116)
\end{aligned}
$$

$i \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	2	3.84	5.39	6.51	7.13	7.19	6.69	5.66	4.2	2.41	2	4.7	2.82	5.19	3.96
1	2	0	2	3.84	5.39	6.51	7.13	7.19	6.69	5.66	4.2	2.86	3.98	2.04	5.21	4.73
2	3.84	2	0	2	3.84	5.39	6.51	7.13	7.19	6.69	5.66	3.86	3.09	2	4.89	5.2
3	5.39	3.84	2	0	2	3.84	5.39	6.51	7.13	7.19	6.69	4.7	2.27	2.74	4.27	5.33
4	6.51	5.39	3.84	2	0	2	3.84	5.39	6.51	7.13	7.19	5.24	2	3.69	3.42	5.1
5	7.13	6.51	5.39	3.84	2	0	2	3.84	5.39	6.51	7.13	5.43	2.53	4.53	2.53	4.53
6	7.19	7.13	6.51	5.39	3.84	2	0	2	3.84	5.39	6.51	5.24	3.42	5.1	2	3.69
7	6.69	7.19	7.13	6.51	5.39	3.84	2	0	2	3.84	5.39	4.7	4.27	5.33	2.27	2.74
8	5.66	6.69	7.19	7.13	6.51	5.39	3.84	2	0	2	3.84	3.86	4.89	5.2	3.09	2
9	4.2	5.66	6.69	7.19	7.13	6.51	5.39	3.84	2	0	2	2.86	5.21	4.73	3.98	2.04
10	2.41	4.2	5.66	6.69	7.19	7.13	6.51	5.39	3.84	2	0	2	5.19	3.96	4.7	2.82
11	2	2.86	3.86	4.7	5.24	5.43	5.24	4.7	3.86	2.86	2	0	3.26	2	3.26	2
12	4.7	3.98	3.09	2.27	2	2.53	3.42	4.27	4.89	5.21	5.19	3.26	0	2	2	3.23
13	2.82	2.04	2	2.74	3.69	4.53	5.1	5.33	5.2	4.73	3.96	2	2	0	3.23	3.21
14	5.19	5.21	4.89	4.27	3.42	2.53	2	2.27	3.09	3.98	4.7	3.26	2	3.23	0	2
15	3.96	4.73	5.2	5.33	5.1	4.53	3.69	2.74	2	2.04	2.82	2	3.23	3.21	2	0

Table 1: $d_E(i,k)$

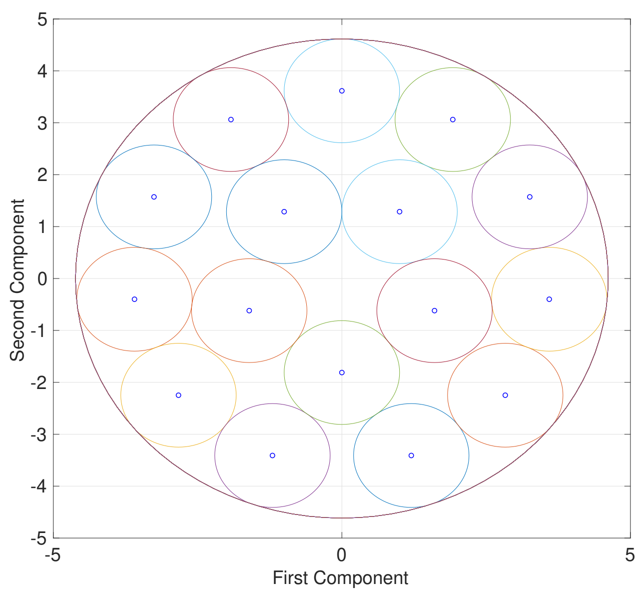
$$\begin{aligned}
\mathbf{s_{12}} &= (1.0000, +1.2874) \\
\mathbf{s_{13}} &= (1.6054, -0.6188) \\
\mathbf{s_{14}} &= (-1.0000, +1.2874) \\
\mathbf{s_{15}} &= (-1.6054, -0.6188)
\end{aligned}$$

(a) Calculate the Euclidean distance $d_E(i,k)$ between every pair (s_i, s_k) of distinct signals and the minimum Euclidean distance $d_{E,min}$ between distinct signals.

Solution: The pairs of distances are shown in the table below. The minimum Euclidean distance is 2.

(b) Plot the signals in the plane. On the sample plot draw circles around each signal point with radius being half the minimum distance $d_{E,min}$ calculated in part (a).

Solution:



(c) Calculate the average energy per information bit, E_b.

Solution:

The average energy per bit is 2.47.

(d) Calculate the normalized squared Euclidean distance $(d^2_{E,min}/E_b)$.

Solution:

The normalized squared Euclidean distance is

$$d_{E,min}^2/E_b = \frac{4}{2.47} = 1.62$$

(e) Calculate the peak-to-average power ratio for this constellation.

Solution:

The peak-to-average power ratio is

$$\Gamma_v = \frac{13.07}{9.85} = 1.32 = 1.21\text{dB}$$

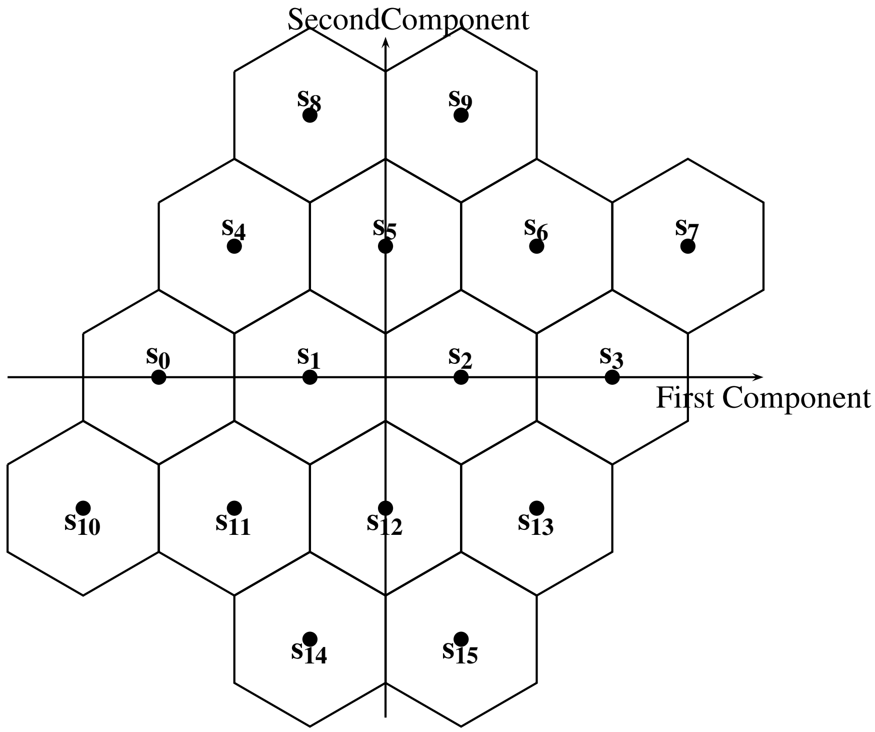
(f) If $\varphi_0 = \sqrt{2/T}\cos(2\pi f_c t)p_T(t)$ and $\varphi_1 = -\sqrt{2/T}\sin(2\pi f_c t)p_T(t)$ calculated the peak-to-average power ratio for the set of 16 signal waveforms.

Solution: The peak-to-average power ratio of the signal waveforms is

$$\Gamma_w = \Gamma_v + 3.01 = 4.22\text{dB}$$

3. Consider the following 16 signal vectors.

$$\begin{array}{ll}
\mathbf{s_0} = (-3,0) & \mathbf{s_8} = (-1,2\sqrt{3}) \\
\mathbf{s_1} = (-1,0) & \mathbf{s_9} = (+1,2\sqrt{3}) \\
\mathbf{s_2} = (+1,0) & \mathbf{s_{10}} = (-4,-\sqrt{3}) \\
\mathbf{s_3} = (+3,0) & \mathbf{s_{11}} = (-2,-\sqrt{3}) \\
\mathbf{s_4} = (-2,\sqrt{3}) & \mathbf{s_{12}} = (0,-\sqrt{3}) \\
\mathbf{s_5} = (0,\sqrt{3}) & \mathbf{s_{13}} = (+2,-\sqrt{3}) \\
\mathbf{s_6} = (+2,\sqrt{3}) & \mathbf{s_{14}} = (-1,-2\sqrt{3}) \\
\mathbf{s_7} = (+4,\sqrt{3}) & \mathbf{s_{15}} = (+1,-2\sqrt{3})
\end{array}$$



(a) Calculate the Euclidean distance $d_E(i,k)$ between every pair (s_i, s_k) of distinct signals and the minimum Euclidean distance $d_{E,min}$ between distinct signals.

Solution: The pairs of distances are shown in the table below. The minimum Euclidean distance is 2.

0	2.00	4.00	6.00	2.00	3.4641	5.2915	7.2111	4	5.2915	2	2	3.4641	5.2915	4	5.2915
2	0	2	4	2	2	3.4641	5.2915	3.4641	4	3.4641	2	2	3.4641	3.4641	4
4	2	0	2	3.4641	2	2	3.4641	4	3.4641	5.2915	3.4641	2	2	4	3.4641
6	4	2	0	5.2915	3.4641	2	2	5.2915	4	7.2111	5.2915	3.4641	2	5.2915	4
2	2	3.4641	5.2915	0	2	4	6	2	3.4641	4	3.4641	4	5.2915	5.2915	6
3.4641	2	2	3.4641	2	0	2	4	2	2	5.2915	4	3.4641	4	5.2915	5.2915
5.2915	3.4641	2	2	4	2	0	2	3.4641	2	6.9282	5.2915	4	3.4641	6	5.2915
7.2111	5.2915	3.4641	2	6	4	2	0	5.2915	3.4641	8.7178	6.9282	5.2915	4	7.2111	6
4	3.4641	4	5.2915	2	2	3.4641	5.2915	0	2	6	5.2915	5.2915	6	6.9282	7.2111
5.2915	4	3.4641	4	3.4641	2	2	3.4641	2	0	7.2111	6	5.2915	5.2915	7.2111	6.9282
2	3.4641	5.2915	7.2111	4	5.2915	6.9282	8.7178	6	7.2111	0	2	4	6	3.4641	5.2915
2	2	3.4641	5.2915	3.4641	4	5.2915	6.9282	5.2915	6	2	0	2	4	2	3.4641
3.4641	2	2	3.4641	4	3.4641	4	5.2915	5.2915	5.2915	4	2	0	2	2	2
5.2915	3.4641	2	2	5.2915	4	3.4641	4	6	5.2915	6	4	2	0	3.4641	2
4	3.4641	4	5.2915	5.2915	5.2915	6	7.2111	6.9282	7.2111	3.4641	2	2	3.4641	0	2
5.2915	4	3.4641	4	6	5.2915	5.2915	6	7.2111	6.9282	5.2915	3.4641	2	2	2	0

(b) Calculate the average energy per information bit, E_b.

Solution: The average energy per information bit is $E_b = 2.25$.

(c) Calculate the normalized squared Euclidean distance $(d_{E,min}^2/E_b)$.

Solution: The normalized squared Euclidean distance is $d_{E,min}^2/E_b = 4/2.25 = 1.78 =$ 2.5dB.

(d) Calculate the peak-to-average power ratio for this constellation.

Solution: The peak and average power for the constellation is

$$\max_i ||s_i||^2 = 19$$

$$\frac{1}{16}\sum_{i=0}^{1} 5||s_i||^2 = 9$$

The peak-to-average power ratio for the constellation is

$$\Gamma_v = 2.11 = 3.25\text{dB}$$

(e) If $\varphi_0(t) = \sqrt{2/T}\cos(2\pi f_c t)p_T(t)$ and $\varphi_1(t) = -\sqrt{2/T}\sin(2\pi f_c t)p_T(t)$ calculated the peak-to-average power ratio for the set of 16 signal waveforms.

Solution:

Consider a signal waveform $s_7(t)$. The waveform is

$$\begin{aligned}
s_7(t) &= 4\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
&= 4\sqrt{\frac{2}{T}}\cos(2\pi f_c t) - \sqrt{3}\sqrt{\frac{2}{T}}\sin(2\pi f_c t) \\
&= \sqrt{19}\sqrt{\frac{2}{T}}\cos(2\pi f_c t + .4086)
\end{aligned}$$

The peak power of this waveform is

$$\max_t |s_7(t)|^2 = (19)\frac{2}{T}.$$

This is also the largest peak power of every waveform.

$$\max_{t,i} |s_i(t)|^2 = (19)\frac{2}{T}.$$

The average power of this waveform is

$$\begin{aligned}
\frac{1}{T}\int_0^T s_7^2(t)dt &= 19(\frac{2}{T})\frac{1}{T}\int_0^T \cos^2(2\pi f_c t + .4086)dt \\
&= 19(\frac{2}{T})\frac{1}{2}
\end{aligned}$$

So for each waveform there is just a factor of 1/2 the peak to the average. So for these orthonormal waveforms the waveform peak-to-average power ratio is 3.01 dB more than the corresponding constellation peak-to-average ratio. The peak-to-average power ratio for the signal set is $\Gamma_w = 2\Gamma_c = 4.22 = 6.25\text{dB}$