University of Michigan Fall 2021

EECS501: Homework 6

Assigned: October 11, 2021
Due: October 26, 2021 at 11:59PM on gradescope
Text: "Probability and random processes" by J. A. Gubner

Reading assignment: Please read Chapter 3, 4 and 5.

1. Transformation of Two Exponentials [5 points]

Let X and Y be two independent exponential random variables with parameter $\lambda = 1$. Consider $Z_1 = \sqrt{X+Y}$ and $Z_2 = \frac{X}{X+Y}$. Find the joint PDF of Z_1 and Z_2.

2. Three Uniforms [5 points, 5 points, 10 points]

Let X, Y and Z be independent random variables that are uniformly distributed over $[0,1]$.

(a) Compute the probability that Y falls in between X and Z.

(b) Compute the probability that the largest of the three is greater than the sum of the other two.

(c) Let M be the sum of the largest and the smallest of the three. Find the PDF of M.

3. Points in a Disc [5 points, 5 points]

(a) A point is chosen at random in a disc of radius 1. Let R denote the distance of the point from the center of the disc. Find the PDF of R.

(b) Two points are chosen independently at random in a disc of radius 1. Let Z denote the distance between the points. Show that $E(Z^2) = 1$.
Hint: Use the Law of cosines.

4. Search Engine [10 points]

A search engine is looking for a particular document. Any given website carries a copy of the document with probability p independently of others. The engine spends a random amount of time, that is exponentially distributed with parameter λ, searching for a copy of the document in a given website until either the document is found and then the search is stopped or the document is not found and the next website is visited. Assume that the websites are listed as $1, 2, 3, \ldots,$ all the way to infinity (a reasonable model for the internet which has around 1 billion websites), and let U_i denote the time spent on website i. Let V denote the time spent by the engine to find a copy of the document. Observe that $V = U_1 + U_2 + \ldots + U_N$, $U_i \sim \exp(\lambda)$, and N is a geometric random variable with parameter p, and $U_1, U_2, \ldots,$ are mutually independent. Using MGF, find the PDF of V.

5. Gamma random variable (7.5 points each)

Let X be a gamma random variable with parameter α and β. The PDF of X is given by

$$f_X(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0 \\ 0 & x \leq 0 \end{cases},$$

where $\alpha > 0$ and $\beta > 0$, and

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

is a normalization constant. Show that MGF of X is given by

$$M_X(s) = \left(\frac{\beta}{\beta - s}\right)^\alpha.$$

Find $E(X)$ and $\mathrm{Var}(X)$ in terms of α and β using the MGF.

6. Moment generating function (5 points)

Let X_1 and X_2 are independent Gaussian random variables with means μ_1 and μ_2, respectively, and variances σ_1^2 and σ_2^2, respectively. Let $Y = aX_1 + bX_2 + c$ for some constants a, b, and c. Find the PDF of Y using the moment generating functions (MGF).

7. MMSE (5 points, 5 points)

(a) Let $Y_1, Y_2, \cdots, Y_n$ be independent identically distributed random variables and let $Y = Y_1 + Y_2 + \cdots + Y_n$. Show that

$$E[Y_1|Y] = \frac{Y}{n}.$$

(b) Let Θ and W be independent zero-mean Gaussian random variables, with positive integer variances k and m, respectively. Use the result of part (a) to find

$$E[\Theta|\Theta + W].$$