EECS501 Homework 5 YUZHAN JIANG

P1:

(a) "vec trick" $vec(AXB^T) = (B \otimes A)\, vec(X)$

Show that

$$S_x = F_M X F_N^T \quad \text{is 2D DFT of } X$$

proof:

$F_M \in F^{M\times M}$ and $F_N \in F^{N\times N}$ are DFT matrix.

$F_M' F_M = M I_M \qquad F_N' F_N = N I_N$

$\Longrightarrow$ implies that $\quad F_{ii}' F_{jj} = \begin{cases} M, & i=j \\ 0, & i\neq j \end{cases}$

$Vec(S_x) = Vec(F_M X F_N^T)$ where $X \in F^{M\times N}$, $X = [x_1 \ x_2 \cdots\cdots x_n]$, $x_i \in C^{m\times 1}$

$$= (F_N \otimes F_M)\, Vec(X)$$

$$= \underbrace{\begin{bmatrix} F_{11} F_M & \cdots & F_{1N} F_M \\ \vdots & & \vdots \\ F_{N1} F_M & \cdots & F_{NN} F_M \end{bmatrix}}_{MN\times MN} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{MN\times 1}$$

$F_M \otimes F_N^T \in F^{MN\times MN} \qquad Vec(X) \in F^{MN\times 1}$

$F_{MN} \mathbin{/\!/}$

$\therefore\ Vec(S_x) = F_{MN} Vec(x)$ where F_{MN} is the $MN\times MN$ DFT Matrix

$$\therefore\ F_{MN}' F_{MN} = \begin{bmatrix} F_{11} F_M & \cdots & F_{N1} F_M \\ \vdots & & \vdots \\ F_{1N} F_M & \cdots & F_{NN} F_M \end{bmatrix} \begin{bmatrix} F_{11} F_M & \cdots & F_{1N} F_M \\ \vdots & & \vdots \\ F_{N1} F_M & \cdots & F_{nn} F_M \end{bmatrix}$$

$$= \begin{bmatrix} F_M' F_{11}' F_{11} F_M & \cdots & \\ \vdots & \ddots & \\ F_M' F_{1N}' F_{11} F_M & \cdots & F_M' F_{NN}' F_{NN} F_M \end{bmatrix}$$

$$= \begin{bmatrix} MN\, I_M & & \\ & \ddots & \\ & & MN\, I_M \end{bmatrix}$$

$$= MN\, I_{MN}$$

$$\therefore\ X = \frac{1}{MN} F_M' S_x \bar{F}_N$$

1(b) Show that 2D inverse DTF of S_x:

$$X = \frac{1}{MN} F_M' S_x \bar{F}_N$$

$$Vec(X) = Vec\left(\frac{1}{MN} F_M' S_x \bar{F}_N\right)$$

$$= \frac{1}{MN} Vec(F_M' S_x \bar{F}_N)$$

$$= \frac{1}{MN} (F_N^T \otimes F_M) \cdot vec(S_x)$$

$$= \frac{1}{MN} \underbrace{\begin{bmatrix} F_{1,1}^T F_M & \cdots & F_{N,1}^T F_M \\ \vdots & & \vdots \\ F_{1,N}^T F_M & \cdots & F_{N,N}^T F_M \end{bmatrix}}_{\Downarrow} vec(S_x)$$

$$= \frac{1}{MN} F_{MN} Vec(S_x)$$

$\therefore$ $Vec(x) = \frac{1}{MN} F_{MN} Vec(S_x)$ where $F_{MN} \in F^{MN \times MN}$

$$F_{MN}' F_{MN} = \begin{bmatrix} F_{1,1}^T F_M & \cdots & F_{1,N}^T F_M \\ \vdots & & \vdots \\ F_{N,1}^T F_M & \cdots & F_{N,N}^T F_M \end{bmatrix} \cdot \begin{bmatrix} F_{1,1}^T F_M & \cdots & F_{N,1}^T F_M \\ \vdots & & \vdots \\ F_{1,N}^T F_M & \cdots & F_{N,N}^T F_M \end{bmatrix}$$

$$= \begin{bmatrix} F_M' F_{1,1} F_{1,1}^T F_M + 0 + \cdots & & \\ \vdots & \ddots & \\ & & \cdots 0 + F_M' F_{N,N} F_{N,N} F_M \end{bmatrix}$$

$$= \begin{bmatrix} MI_M & & \\ & \ddots & \\ & & MI_M \end{bmatrix}$$

$$= MN\, I_{MN}$$

$\therefore$ It can computes the 2D inverse DFT of S_x

P2.

(a) Find solutions that $\arg\min_{x \in \mathbb{R}^2} \|Ax - b\|_2$ when $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

we find $\text{rank}(A) = 1 < N = 2$, then there are multiple minimizers. All minimizers are given by

$$\hat{x} = V_r \Sigma_r^{-1} U_r' b + V_0 z_0 \quad \forall z \in F^{2-1}$$

Let's find SVD of A first.

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\det(A^T A - \lambda I_2) = 0 \implies (2-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 0$$

$$\therefore \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \Sigma_r = [2] \implies \Sigma_r^{-1} = \left[\tfrac{1}{2}\right]$$

Then we need to find eigenvector of $A^T A$

$$\begin{bmatrix} 2-4 & 2 \\ 2 & 2-4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies -2v_1 + 2v_2 = 0 \quad v_1 = v_2.$$

$$\therefore V_1 = V_r = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad V_2 = V_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$U_r = \frac{1}{\sigma_1} A V_1$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \text{Overall, } \hat{x} = V_r \Sigma_r^{-1} U_r' b + V_0 z_0$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left[\tfrac{1}{2}\right]\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} z_0$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} z_0$$

$$= \frac{1}{4}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} z_0 \quad \forall z_0 \in F^{2-1}$$

(b) $\arg\min_{x \in \mathbb{C}^2} \|Ax - b\|_2^2$

$$\hat{x} = \frac{1}{4}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} z_0, \quad \forall z_0 \in \mathbb{C}^1$$

P3:

(a) If we minimize $\|Wr\|_2$ where $r = b - Ax$ and $W = \begin{bmatrix} w_1 & & \\ & w_2 & \\ & & \ddots \\ & & & w_m \end{bmatrix}$

$\underset{x}{\text{Argmin}} : \|W(b - Ax)\|_2^2$

$$Wr = Wb - WAx$$

Let $\tilde{b} = Wb$ $\quad \tilde{A} = WA$

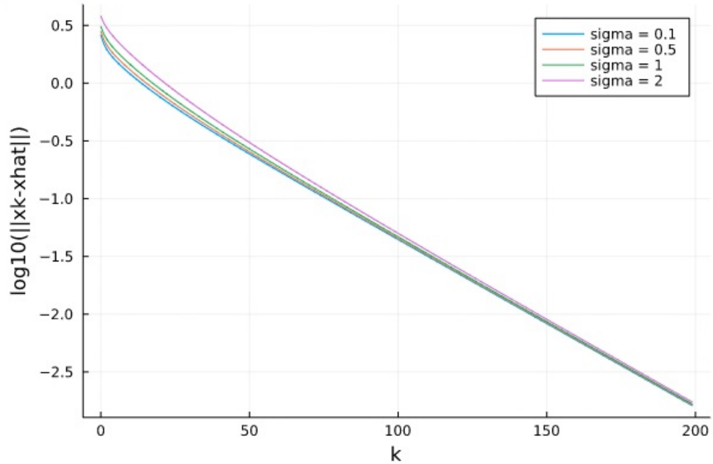
$$\therefore \quad \hat{x} = \underset{}{\text{Argmin}} \; \|W(b - Ax)\|_2^2$$
$$= \text{Argmin} \; \|Wb - WAx\|_2^2$$
$$= \text{argmin} \; \|\tilde{b} - \tilde{A}x\|_2^2$$

$$\therefore \quad \hat{x} = \tilde{A}^+ \tilde{b}$$
$$= \tilde{V}\tilde{\Sigma}^+\tilde{U}' Wb \qquad \text{where } \tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}'$$
$$\tilde{A}^+ = \tilde{V}\tilde{\Sigma}^+\tilde{U}'$$

P4.

(b)

Out[38]:



$\|x_k - \hat{x}\|$ decreases monotonically with k as the result shown.

P5:

(a) $\hat{x}(\delta) = \arg\min_x \frac{1}{2}\|Ax-b\|_2^2 + \delta^2 \frac{1}{2}\|x\|_2^2$

$$\frac{1}{2}\|Ax-b\|_2^2 + \delta^2\frac{1}{2}\|x\|_2^2 = \frac{1}{2}(Ax-b)'(Ax-b) + \delta^2\frac{1}{2}\|x\|_2^2$$
$$= \frac{1}{2}\left(x'A'Ax - 2b'Ax + b'b + \delta^2 x'x\right)$$
$$= \frac{1}{2}\left(x'\begin{bmatrix} A'A \\ \delta^2 \end{bmatrix}x - 2\begin{bmatrix} b' & 0 \end{bmatrix}\begin{bmatrix} A \\ \delta I \end{bmatrix}x + \begin{bmatrix} b' & 0 \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix}\right)$$
$$= \frac{1}{2}\left\|\begin{bmatrix} A \\ \delta I \end{bmatrix}x - \begin{bmatrix} b \\ 0 \end{bmatrix}\right\|_2^2 \quad \text{(new cost function)}$$
$$\hat{x} = \begin{bmatrix} A \\ \delta I \end{bmatrix}^+ \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} A \\ \delta I \end{bmatrix}'\begin{bmatrix} A \\ \delta I \end{bmatrix} = \begin{bmatrix} A' & \delta I \end{bmatrix}\begin{bmatrix} A \\ \delta I \end{bmatrix} = A'A + \delta^2 I$ is always invertible when $\delta > 0$

$$\therefore \begin{bmatrix} A \\ \delta I \end{bmatrix}^+ = \begin{bmatrix} A \\ \delta I \end{bmatrix}'\begin{bmatrix} A \\ \delta I \end{bmatrix}^{-1} \cdot \begin{bmatrix} A \\ \delta I \end{bmatrix}'$$

$$\therefore \hat{x} = (A'A + \delta^2 I) \cdot \begin{bmatrix} A' & \delta I \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix}$$
$$= (A'A + \delta^2 I)\, A'b$$

(b) $\hat{x}(\delta) \longrightarrow 0$ as $\delta \to \infty$.

This answer makes sense since $\delta \to \infty$, the $\delta^2\|x\|_2^2$ term dominates the cost function

(c) $$x_{k+1} = x_k - \mu\begin{bmatrix} A \\ \delta I \end{bmatrix}'\left(\begin{bmatrix} A \\ \delta I \end{bmatrix}x_k - \begin{bmatrix} b \\ 0 \end{bmatrix}\right)$$
$$= x_k - \mu\begin{bmatrix} A' & \delta I \end{bmatrix}\left(\begin{bmatrix} Ax_k - b \\ \delta I x_k \end{bmatrix}\right)$$
$$= x_k - \mu(A'Ax_k - A'b + \delta^2 x_k)$$

This iteration will converge to $\hat{x}$ whenever $\mu < \frac{2}{\sigma_1(\tilde{A})^2}$ where $\hat{A} = \begin{bmatrix} A \\ \delta I \end{bmatrix}$

(d) $$\sigma_1\left(\begin{bmatrix} A \\ \delta I \end{bmatrix}\right) = \sqrt{\sigma_1^2(A) + \delta^2}$$

$\therefore$ the range of step size:

$$0 < u < \frac{2}{\sigma_1(A) + \delta^2}$$

P6:

(a) We are given that $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix}$

$$A_1 = A + E_1 = \begin{bmatrix} 1 & \delta \\ 0 & 0 \end{bmatrix} = \underset{a}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{b'}{\begin{bmatrix} 1 & \delta \end{bmatrix}}$$

$$A_1 = \frac{a}{\|a\|} \|a\| \|b'\| \cdot \frac{b'}{\|b'\|}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1+\delta^2) \begin{bmatrix} \frac{1}{\sqrt{1+\delta^2}} & \frac{\delta}{\sqrt{1+\delta^2}} \end{bmatrix}$$

$$A_1 = \sigma_1 u_1 v_1^T \implies u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \sigma_1 = 1+\delta^2 \quad v_1 = \begin{bmatrix} 1/\sqrt{1+\delta^2} \\ \delta/\sqrt{1+\delta^2} \end{bmatrix}$$

Therefore, $z^* = \underset{z}{\operatorname{argmin}} \|A_1 z - b\| = A_1^+ b$

$$= \frac{1}{\sigma_1} v_1 u_1^T b \qquad (\text{since } \operatorname{rank}(A_1) = 1)$$

$$= \frac{1}{1+\delta^2} \begin{bmatrix} 1/\sqrt{1+\delta^2} \\ \delta/\sqrt{1+\delta^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{1+\delta^2} \begin{bmatrix} 1/\sqrt{1+\delta^2} & 0 \\ \delta/\sqrt{1+\delta^2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{1+\delta^2} \begin{bmatrix} 1/\sqrt{1+\delta^2} \\ \delta/\sqrt{1+\delta^2} \end{bmatrix}$$

As $\delta \to 0$, $z^* \to \frac{1}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore \|\hat{x} - z^*\| \to 0$ when $\delta \to 0$

6(b) Now $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$ and $A_2 = A + E_2$

$\therefore z^* = \underset{z}{\operatorname{argmin}} \|A_2 z - b\| = A_2^+ b$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\delta} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{\delta} \end{bmatrix}$$

(check: $A_2 A_2^+ A_2 = A_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\delta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \text{)}$$

$$\therefore \|\hat{x} - z^*\|_2 = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{\delta} \end{bmatrix} \right\|_2$$

$$= \sqrt{(1-1)^2 + (0 - \tfrac{1}{\delta})^2}$$

$$= \frac{1}{\delta}$$

$\therefore$ As $\delta \to 0$, $\|\hat{x} - z^*\|_2 = \frac{1}{\delta} \to \infty$

P7

(a) the plane $\{(x,y,z) \in \mathbb{R}^3 : ax+by+cz=0\}$

$$\Rightarrow \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax+bx+cz = 0$$

$\therefore$ we can see that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in N(\begin{bmatrix} a & b & c \end{bmatrix})$

Compact SVD of $\begin{bmatrix} a & b & c \end{bmatrix} = [1]\sqrt{a^2+b^2+c^2}\,\dfrac{\begin{bmatrix} a & b & c \end{bmatrix}}{\sqrt{a^2+b^2+c^2}} = \sigma_1 u_1 v_1^T$

where $u_1 = [1]$ $\sigma_1 = \sqrt{a^2+b^2+c^2}$,

$v_1^T = \dfrac{\begin{bmatrix} a & b & c \end{bmatrix}}{\sqrt{a^2+b^2+c^2}}$

Full SVD of $\begin{bmatrix} a & b & c \end{bmatrix}$ will have 3×3 matrix V $\{v_1\ v_2\ v_3\}$ which $N(\begin{bmatrix} a & b & c \end{bmatrix}) = \text{span}\{v_2, v_3\}$

Since v_2, v_3 are orthonormal, therefore $\{v_2, v_3\}$ are the orthonormal basis for the plane.

(b) Given point $(\alpha, \beta, \gamma) \in \mathbb{R}^3$

The point that is closest to this point

$$P_{R\{v_2,v_3\}}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = (v_2v_2^T + v_3v_3^T)\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$= (I - v_1v_1^T)\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

(c) Plane $x+2y+3z=0$ that is closet to the point $(4,5,6)$

$$x+2y+3z = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = [1]\sqrt{1^2+2^2+3^2}\,\frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}{\sqrt{1^2+2^2+3^2}}$$

$$= [1]\sqrt{13}\,\frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}{\sqrt{13}} \Rightarrow u_1 = [1] \quad \sigma_1 = \sqrt{13} \quad v_1^T = \frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}{\sqrt{13}}$$

$$v_1v_1^T = \frac{1}{13}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$\therefore$ This point is $(I - v_1v_1^T)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

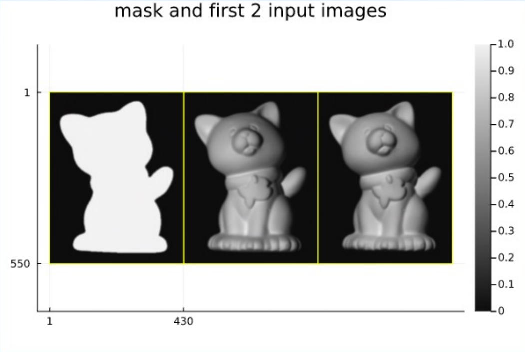
$$= \frac{1}{13}\begin{bmatrix} 12 & -2 & -3 \\ -2 & 9 & -6 \\ -3 & -6 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 20 \\ 1 \\ -18 \end{bmatrix}$$

P9

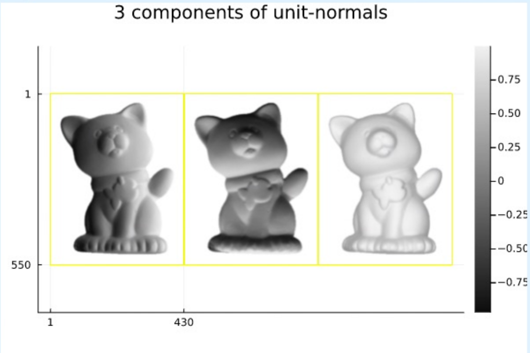
In [5]:
```
# show mask and p ≤ d of the input images
p = 2
# rowify = (I3) -> reshape(permutedims(I3, [1 3 2]), :, size(I3,2))
jim(cat(M, data[:,:,1:p], dims=3), "mask and first $p input images")
```

Out[5]:



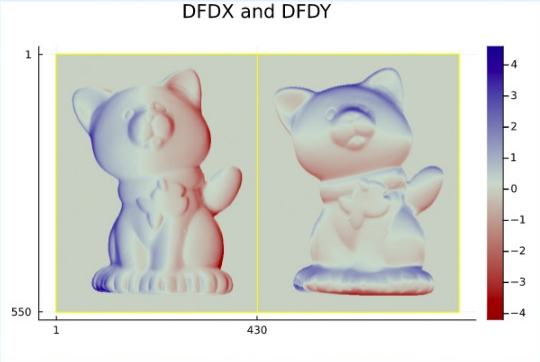
In [8]:
```
# display the 3 components of the normal vectors as images
jim(N, title = "3 components of unit-normals")
```

Out[8]:

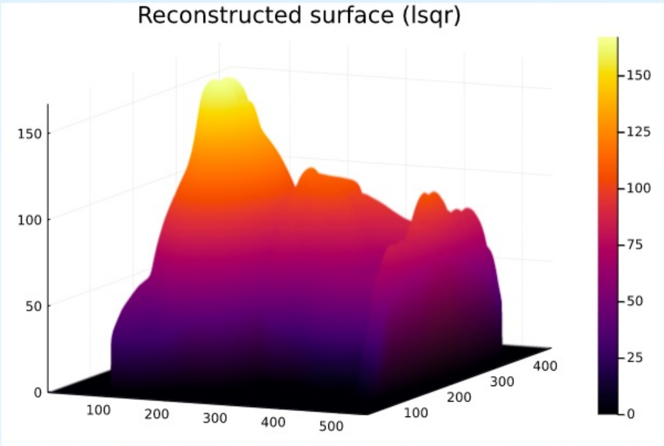


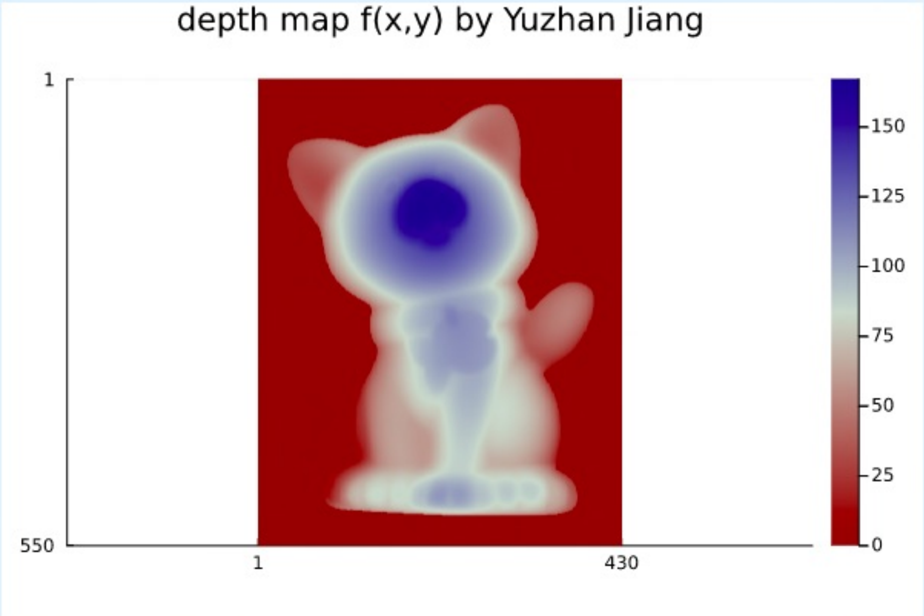
In [10]:
```
# display the gradients
jim(cat(DFDX, DFDY, dims=3), # feline pun intended?
    color=:redsblues,
    title = "DFDX and DFDY")
```

Out[10]:



Out[16]:


Reconstructed surface (lsqr)
150
100
50
0
100
200
300
400
500
100
200
300
400
150
125
100
75
50
25
0


depth map f(x,y) by Yuzhan Jiang
1
550
1
430
150
125
100
75
50
25
0