EECS 455: Solutions to Problem Set 3

1. Consider a baseband signal

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

where $x_I(t)$ and $x_Q(t)$ are baseband signals with frequency content limited to $[-W, +W]$. Let $X_I(f)$ and $X_Q(f)$ be the frequency content of the signals. So

$$X_I(f) = X_Q(f) = 0 \text{ for } f \notin [-W, W]$$

The energy of the lowpass complex signal is

$$E_l = \int |\tilde{x}(t)|^2 dt$$

The passband signal is

$$x(t) = x_I(t)\sqrt{2}\cos(2\pi f_c t) - x_Q(t)\sqrt{2}\sin(2\pi f_c t)$$

where $f_c > W$. The energy of the passband signals is

$$E_p = \int |x(t)|^2 dt$$

Show that $E_l = E_p$.

Hint: Derive expressions for the energy in the frequency domain for both $\tilde{x}(t)$ and $x(t)$. Use Parseval's Theorem

$$\int u(t)v^*(t)dt = \int U(f)V^*(f)df.$$

where $u(t) = x_I(t)$ and $v(t) = x_I(t)\cos(2\pi(2f_c)t)$.

Solution:

$$\begin{aligned} E_l &= \int |x_I(t) + jx_Q(t)|^2 dt \\ &= \int x_I^2(t) + x_Q^2(t) dt \\ &= \int X_I^2(f) + X_Q^2(f) dt \end{aligned}$$

$$\begin{aligned} E_p &= \int x^2(t) dt \\ &= 2\int x_I^2(t)\cos^2(2\pi f_c t) - x_I(t)x_Q(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + x_Q^2(t)\sin^2(2\pi f_c t) dt \\ &= \int x_I^2(t)[1 + \frac{1}{2}\cos(2\pi(2f_c)t)] - x_I(t)x_Q(t)\frac{1}{2}\sin(2\pi(2f_c)t) + x_Q^2(t)[1 - \frac{1}{2}\cos(2\pi(2f_c)t)] dt \\ &= \int (x_I^2(t) + x_Q^2(t) dt \\ &\quad + \frac{1}{2}\int [x_I^2(t)\cos^2(2\pi f_c t) - x_I(t)x_Q(t)\sin(2\pi(2f_c)t) - x_Q^2(t)\cos(2\pi(2f_c)t)] dt \\ &= E_l + \frac{1}{2}\int [x_I^2(t)\cos(2\pi(2f_c)t) - x_I(t)x_Q(t)\sin(2\pi(2f_c)t) - x_Q^2(t)\cos(2\pi(2f_c)t)] dt \end{aligned}$$

Consider just the term

$$\int x_I^2(t)\cos(2\pi(2f_c)t)dt = \int x_I(t)x_I(t)\cos(2\pi(2f_c)t)dt$$

Let $u(t) = x_I(t)$ and $v(t) = x_I(t)\cos(2\pi(2f_c)t)$. The frequency content of $u(t) = x_I(t)$ is $U(f) = X_I(f)$. The frequency content of $v(t) = x_I(t)\cos(2\pi(2f_c)t)$ is $V(f) = \frac{1}{2}[X_I(f - 2f_c) + X_I(f+2f_c)]$. The frequency content of $U(f)$ is limited to the interval $f \in [-W,W]$. The frequency content of $V(f)$ is limited to the intervals $f \in [-2f_c - W, -2f_c + W]$ and $f \in [2f_c - W, 2f_c + W]$. Then by Parseval

$$\int u(t)v(t)dt = \int U(f)V^*(f)df.$$

However because $f_c > W$ the product $U(f)V^*(f) = 0$. Thus

$$\int x_I^2(t)\cos(2\pi(2f_c)t)dt = \int x_I(t)x_I(t)\cos(2\pi(2f_c)t)dt = 0.$$

Similarly

$$\int x_Q^2(t)\cos(2\pi(2f_c)t)dt = 0.$$

$$\int x_I(t)x_Q(t)\sin(2\pi(2f_c)t)dt = 0.$$

Thus $E_l = E_p$.

2. Suppose you want to implement a square-root raised cosine pulse shape. The pulse shape theoretically last forever. You truncate the pulse to some number of samples. Suppose you generated 8 samples per T seconds where T is the time between pulses. You can assume $T = 1$. You can use Matlab. Do the following for $\alpha = 0.05, 0.15, 0.25$.

   (a) Determine how many samples you need for the maximum sample you ignore is 40dB down relative to the peak sample. That is the amplitude of the any sample you ignore is 0.01 times as small as the peak sample.

   Solution:

   For $\alpha = 0.05$ the number of samples is 245.
   For $\alpha = 0.15$ the number of samples is 113.
   For $\alpha = 0.25$ the number of samples is 79.

   Figure 1 shows the samples of the pulse shape for $\alpha = 0.25$.

   (b) Determine the frequency content of the signal when the pulse shape is truncated to the number of samples indicated in (a). Plot the frequency content in dB versus f.

   Solution:

   Figure 1 shows the samples when truncated according to part a (red circles) and the samples when truncated according to part b (green x). Notice that two extra samples are added to each side of the impulse response.

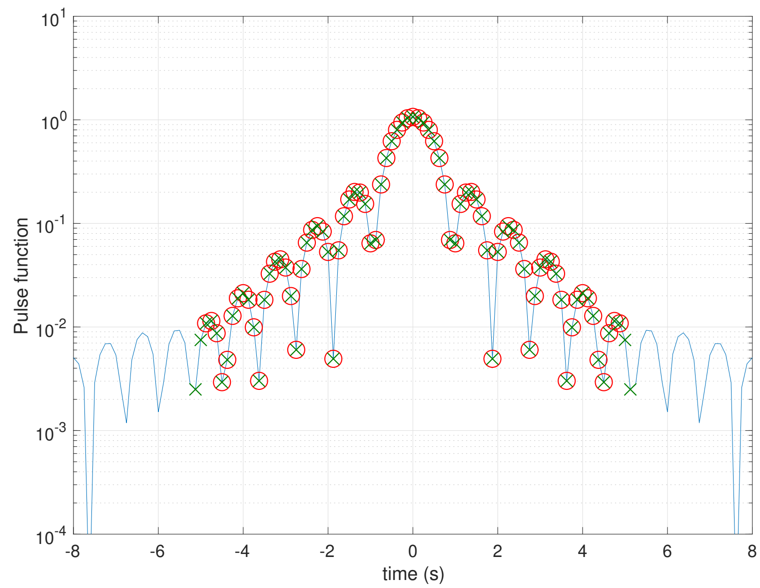


Figure 1: Samples of square-root raised cosine filter ($\alpha = 0.25$)

(c) Suppose you add samples to the truncated pulse. You want to add samples so that the last sample is as close as possible to 0. What is your new pulse length in samples. Compare the frequency response to that in part (b).

Solution:

For $\alpha = 0.05$ the number of samples is 253.
For $\alpha = 0.15$ the number of samples is 121.
For $\alpha = 0.25$ the number of samples is 83.

Figure 2 shows the frequency response for $\alpha = 0.05$ when the impulse response is truncated according to part a (blue) and the samples when truncated according to part b (red). Notice that the transfer function decays faster when truncated close to the point that the impulse response is 0.

Figure 3 shows the frequency response for $\alpha = 0.15$ when the impulse response is truncated according to part a (blue) and the samples when truncated according to part b (red).

Figure 4 shows the frequency response for $\alpha = 0.25$ when the impulse response is truncated according to part a (blue) and the samples when truncated according to part b (red).

3. (a) Consider a complex sequence of length $N = 26$;

$$\begin{aligned} x = (x_0, \ldots, x_{25}) = \quad & (-1+1j, -1+1j, -1+1j, 1-1j, 1-1j, -1+1j, \\ & -1+1j, -1+1j, 1-1j, -1+1j, 1-1j, 1-1j, \\ & 1+1j, 1-1j, 1+1j, -1-1j, 1+1j, -1-1j, \\ & -1-1j, 1+1j, 1+1j, -1-1j, 1+1j, 1+1j, 1+1j, 1+1j) \end{aligned}$$

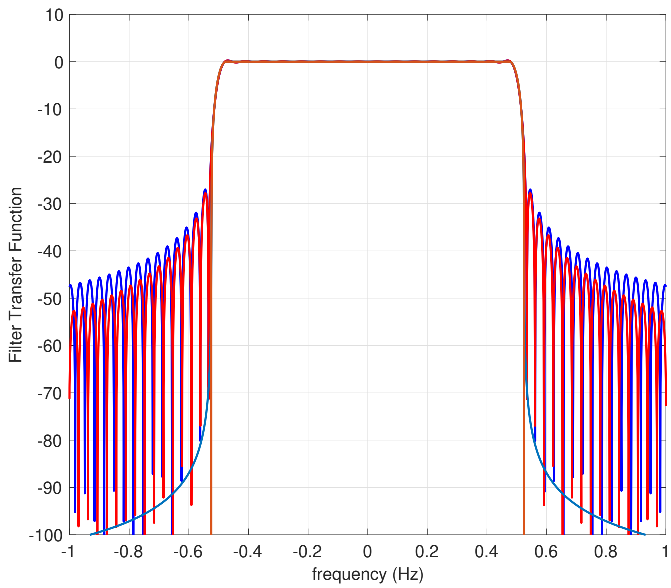


Figure 2: Samples of square-root raised cosine filter ($\alpha = 0.05$).

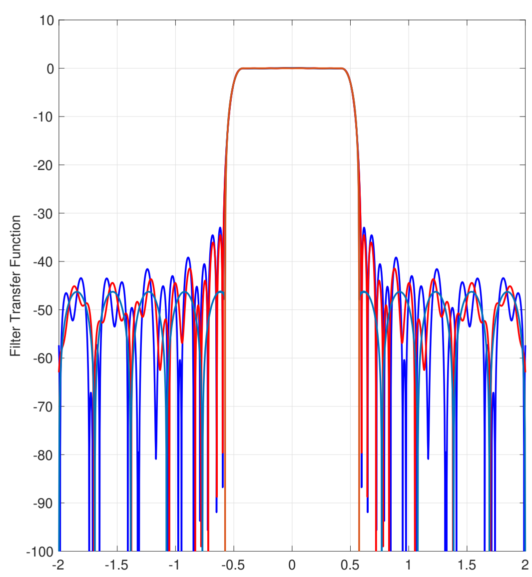


Figure 3: Samples of square-root raised cosine filter ($\alpha = 0.15$)

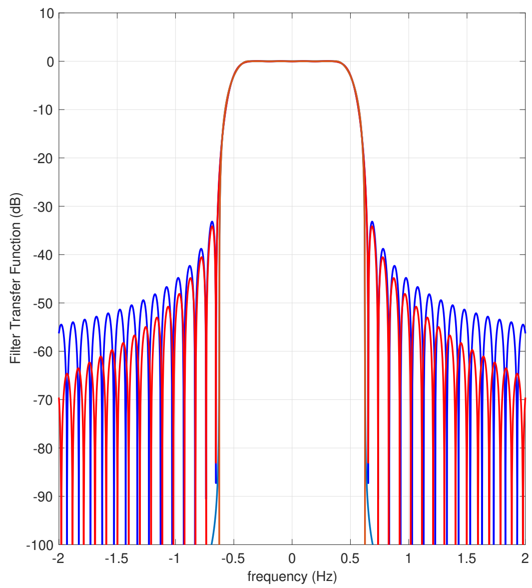


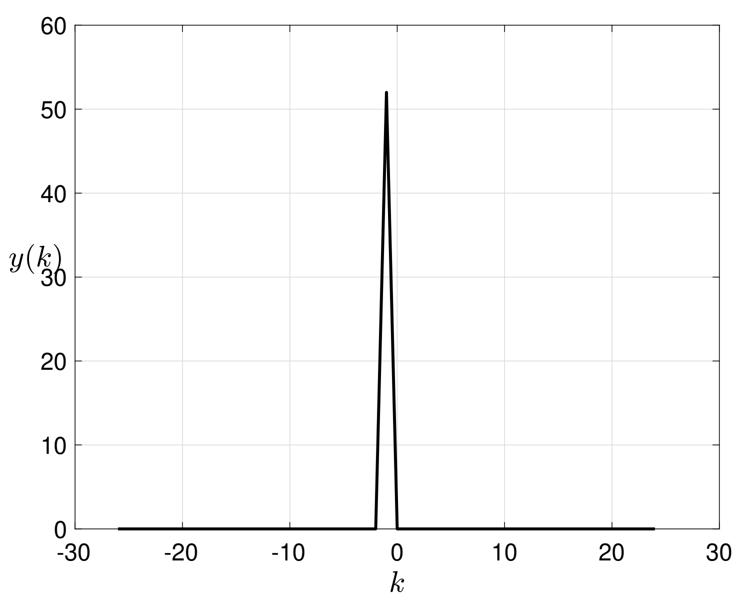
Figure 4: Samples of square-root raised cosine filter ($\alpha = 0.25$)

You can load this vector into Matlab from a file on Canvas HW3P1.mat. Now consider a matched fiter $h(n) = x^*((N-1)-n) = x^*(25-n)$. Use Matlab functions `fliplr` and `conj`. Determine the output of the matched filter when the input is x. That is determine y where

$$y(n) = \Re\left[\sum_{l=0}^{N-1} h(n-l)x(l)\right]$$

and $\Re(x)$ is the real part of x. Use Matlab's `conv` functjion. Plot the result.

Soution:

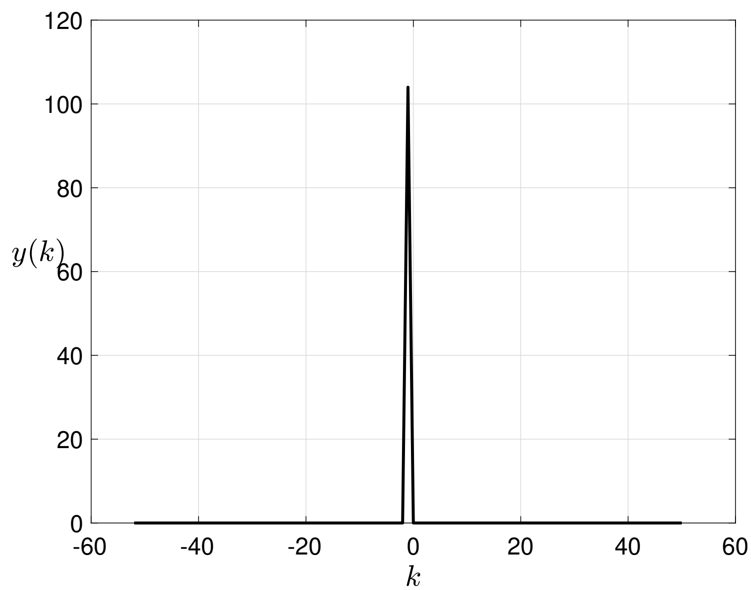


(b) Consider the sequence of length 52

$$\begin{aligned} \Re(u) &= [\Re(x), \quad +\Im(x)] \\ \Im(u) &= [\Re(x), \quad -\Im(x)] \\ u &= \Re(u) + j\Im(u) \end{aligned}$$

That is, the first half of the real part of u is the real part of x from part (a). The second half of the real part of u is the imaginary part of x. Similarly for the imaginary part of u. Determine the real part of the output of a filter matched to the sequence u. Plot the result.

Soution:



4. A communication system uses BPSK modulation to transmit data bits $b_l, l = 0, 1, 2, \ldots$. In the transmitter a sequence of rectangular pulses is mixed to a carrier frequency by multiplying the rectangular pulses by $\sqrt{2P}\cos(2\pi f_1 t)$.

$$s(t) = \sqrt{2P} \sum_{l=0,1,\ldots} b_l p_T(t - lT)\cos(2\pi f_1 t).$$

At the receiver the received signal is first mixed down to baseband by multiplying by $\sqrt{2/T}\cos(2\pi f_2 t)$ where $f_2 - f_1 = \Delta f$ is the offset of the two oscillators. After the signal is mixed down it is filtered with a matched filter (that is $h(t) = p_T(t)$). The filter is sampled at time $t = iT$ for $i = 1, 2, \ldots$. In addition the signal is mixed down by multiplying by $-\sqrt{2/T}\sin(2\pi f_2 t)$. Let $y_c(iT)$ denote the first output and $y_s(iT)$ denote the second output. Then

$$\begin{aligned} y_c(iT) &= \int_{-\infty}^{\infty} s(\tau)\sqrt{\frac{2}{T}}\cos(2\pi f_2 \tau)h(iT - \tau)d\tau \\ y_s(iT) &= -\int_{-\infty}^{\infty} s(\tau)\sqrt{\frac{2}{T}}\sin(2\pi f_2 \tau)h(iT - \tau)d\tau. \end{aligned}$$

(a) Evaluate the outputs $y_c(iT)$ and $y_s(iT)$ in terms of b_{i-1}, $E = PT$, ΔfT and i. Ignore double frequency terms in evaluating the output. That is, derive an expression for $y_c(iT)$ and $y_s(iT)$.

Solution

$$
\begin{aligned}
y_c(iT) &= \int_{-\infty}^{\infty} s(\tau)\sqrt{2/T}\cos(2\pi f_2\tau)h(iT-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \sqrt{2P}\sum_{l=0,1,\ldots} b_l p_T(\tau - lT)\cos(2\pi f_1\tau)\sqrt{2/T}\cos(2\pi f_2\tau)h(iT-\tau)d\tau \\
&= \sqrt{PT}(\frac{2}{T})\sum_{l=0,1,\ldots} b_l \int_{-\infty}^{\infty} p_T(\tau - lT)\cos(2\pi f_1\tau)p_T(iT-\tau)\cos(2\pi f_2\tau)d\tau \\
&= \sqrt{E}(\frac{2}{T})\sum_{l=0,1,\ldots} b_l \int_{(i-1)T}^{iT} p_T(\tau - lT)\cos(2\pi f_1\tau)\cos(2\pi f_2\tau)d\tau \\
&= \sqrt{E}(\frac{2}{T})b_{i-1}\int_{(i-1)T}^{iT} \cos(2\pi f_1\tau)\cos(2\pi f_2\tau)d\tau \\
&= \sqrt{E}(\frac{1}{T})b_{i-1}\int_{(i-1)T}^{iT} \cos(2\pi(f_2-f_1)\tau)+\cos(2\pi(f_1+f_2)\tau)d\tau \\
&= \sqrt{E}(\frac{1}{T})b_{i-1}\left[\frac{\sin(2\pi(f_2-f_1)\tau)}{2\pi(f_2-f_1)}+\frac{\sin(2\pi(f_1+f_2)\tau)}{2\pi(f_1+f_2)}\right]\Big|_{(i-1)T}^{iT} \\
&= \sqrt{E}(\frac{1}{T})b_{i-1}\left[\frac{\sin(2\pi(f_2-f_1)iT)}{2\pi(f_2-f_1)}-\frac{\sin(2\pi(f_2-f_1)(i-1)T)}{2\pi(f_2-f_1)}\right. \\
&\qquad \left.+\frac{\sin(2\pi(f_1+f_2)iT)}{2\pi(f_1+f_2)}-\frac{\sin(2\pi(f_1+f_2)(i-1)T)}{2\pi(f_1+f_2)}\right] \\
&= \sqrt{E}b_{i-1}\left[\cos(\pi(f_2-f_1)(2i-1)T)\frac{\sin(\pi(f_2-f_1)T)}{\pi(f_2-f_1)T}\right. \\
&\qquad \left.+\cos(\pi(f_1+f_2)(2i-1)T)\frac{\sin(\pi(f_1+f_2)T)}{\pi(f_1+f_2)T}\right] \\
&= \sqrt{E}b_{i-1}\left[\cos(\pi(2i-1)\Delta fT)\frac{\sin(\pi\Delta fT)}{\pi\Delta fT}+\cos(\pi(f_1+f_2)(2i-1)T)\frac{\sin(\pi(f_1+f_2)T)}{\pi(f_1+f_2)T}\right].
\end{aligned}
$$

Ignoring the double frequency term we get

$$
\begin{aligned}
y_c(iT) &= \sqrt{E}b_{i-1}\cos(\pi(2i-1)\Delta fT)\frac{\sin(\pi\Delta fT)}{\pi\Delta fT} \\
&= \sqrt{E}b_{i-1}\cos(\pi(2i-1)\Delta fT)\mathrm{sinc}(\Delta fT).
\end{aligned}
$$

Note that the last factor does not depend on i. The last factor is indicative of the fact that part of the signal does not pass through the low pass filter because of the frequency offset. If $\Delta fT = 1$ then the output becomes zero (the two sinusoids are orthogonal). Furthermore the output, if $\Delta f T$ is small, slowly changes. In a similar manner

$$\begin{aligned}
y_s(iT) &= -\int_{-\infty}^{\infty} s(\tau)\sqrt{2/T}\sin(2\pi f_2\tau)h(iT-\tau)d\tau \\
&= -\int_{-\infty}^{\infty}\sqrt{2P}\sum_{l=0,1,\ldots} b_l p_T(\tau-lT)\cos(2\pi f_1\tau)\sqrt{2/T}\sin(2\pi f_2\tau)h(iT-\tau)d\tau \\
&= -\sqrt{PT}(\frac{2}{T})\sum_{l=0,1,\ldots} b_l\int_{-\infty}^{\infty} p_T(\tau-lT)\cos(2\pi f_1\tau)p_T(iT-\tau)\sin(2\pi f_2\tau)d\tau \\
&= -\sqrt{E}(\frac{2}{T})\sum_{l=0,1,\ldots} b_l\int_{(i-1)T}^{iT} p_T(\tau-lT)\cos(2\pi f_1\tau)\sin(2\pi f_2\tau)d\tau \\
&= -\sqrt{E}(\frac{2}{T})b_{i-1}\int_{(i-1)T}^{iT}\cos(2\pi f_1\tau)\sin(2\pi f_2\tau)d\tau \\
&= -\sqrt{E}(\frac{1}{T})b_{i-1}\int_{(i-1)T}^{iT}\sin(2\pi(f_1+f_2)\tau)-\sin(2\pi(f_1-f_2)\tau)d\tau \\
&= -\sqrt{E}b_{i-1}\left[-\frac{\cos(2\pi(f_1+f_2)\tau)}{2\pi(f_1+f_2)T}+\frac{\cos(2\pi(f_1-f_2)\tau)}{2\pi(f_1-f_2)T}\right]\Big|_{(i-1)T}^{iT} \\
&= -\sqrt{E}b_{i-1}\left[-\frac{\cos(2\pi(f_1+f_2)iT)}{2\pi(f_1+f_2)T}+\frac{\cos(2\pi(f_1+f_2)(i-1)T)}{2\pi(f_1+f_2)T}\right. \\
&\qquad \left.+\frac{\cos(2\pi(f_1-f_2)iT)}{2\pi(f_1-f_2)T}-\frac{\cos(2\pi(f_1-f_2)(i-1)T)}{2\pi(f_1-f_2)T}\right] \\
&= -\sqrt{E}b_{i-1}\left[\frac{\sin(\pi(f_1+f_2)(2i-1)T)\sin(\pi(f_1+f_2)T)}{\pi(f_1+f_2)T}\right. \\
&\qquad \left.-\frac{\sin(\pi(f_1-f_2)(2i-1)T)\sin(\pi(f_1-f_2)T}{\pi(f_1-f_2)T}\right] \\
&= -\sqrt{E}b_{i-1}\left[\sin(\pi(f_2-f_1)(2i-1)T)\frac{\sin(\pi(f_2-f_1)T)}{\pi(f_2-f_1)T}\right. \\
&\qquad \left.-\sin(\pi(f_1+f_2)(2i-1)T)\frac{\sin(\pi(f_1+f_2)T)}{\pi(f_1+f_2)T}\right] \\
&= -\sqrt{E}b_{i-1}\left[\sin(\pi(2i-1)\Delta f T)\frac{\sin(\pi\Delta f T)}{\pi\Delta f T}-\sin(\pi(f_1+f_2)(2i-1)T)\frac{\sin(\pi(f_1+f_2)T)}{\pi(f_1+f_2)T}\right].
\end{aligned}$$

Ignoring the double frequency terms we get

$$y_s(iT) = -\sqrt{E}b_{i-1}\sin(\pi(2i-1)\Delta f T)\frac{\sin(\pi\Delta f T)}{\pi\Delta f T}.$$

(b) Assume you buy two crystal oscillators at a 10MHz nominal frequency that have $\pm$ 10 PPM accuracy. That is, $f_{\text{actual}} = f_{\text{nominal}}(1\pm 10/10^6)$. Assume that the data rate is 100kbps ($T = 10^{-5}$).

i. Are the double frequency terms negligible?

ii. Plot the output of the filters $y_c(iT)$ and $y_s(iT)$ as a function of i for $1 \leq i \leq 500$.

Solution: Since $f_1 \approx 10^7$Hz and $T = 10^{-5}$, $f_1 T \approx 100 \gg 1$. So the double frequency component is negligible. For the parameters given $\Delta f = 200$ Hz. So $\Delta f T = 2 \times 10^{-3}$.

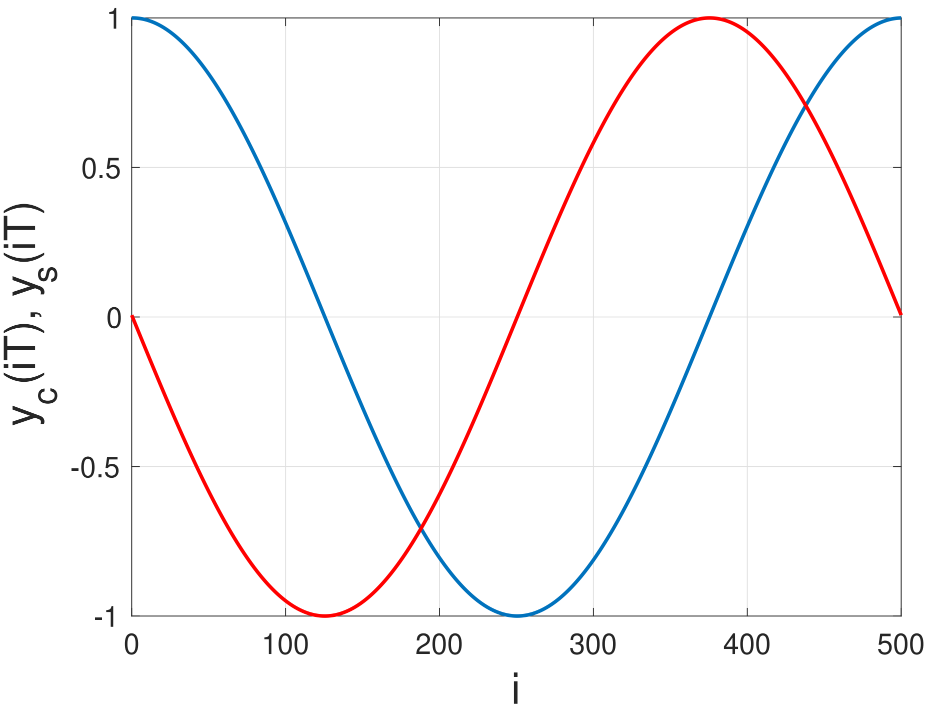


Figure 5: Filter Outputs

iii. Plot $y_s(iT)$ versus $y_c(iT)$ for the case where $\Delta f T = 0.1$, the data bits are all 1 for $i = 0, \ldots, 39$. Plot each point with a "marker". Use the Matlab command `plot(yc,ys,'+')` where yc is a vector of samples representing $y_c(iT)$ and `ys` is a vector of samples representing $y_s(iT)$ for $i = 0, 1, \ldots, 39$.

When the data sequence is $b_I(l) = [1, -1, 1, -1, 1, 1, -1, -1]$ the output of the I-branch and the output of the Q branch are shown below (Figure 7). This is shown for a frequency offset of $\Delta f = 0.04$, $T = 1$, and $f_c = 16$. These outputs at the sample time are plotted in the plane below (Figure 9).

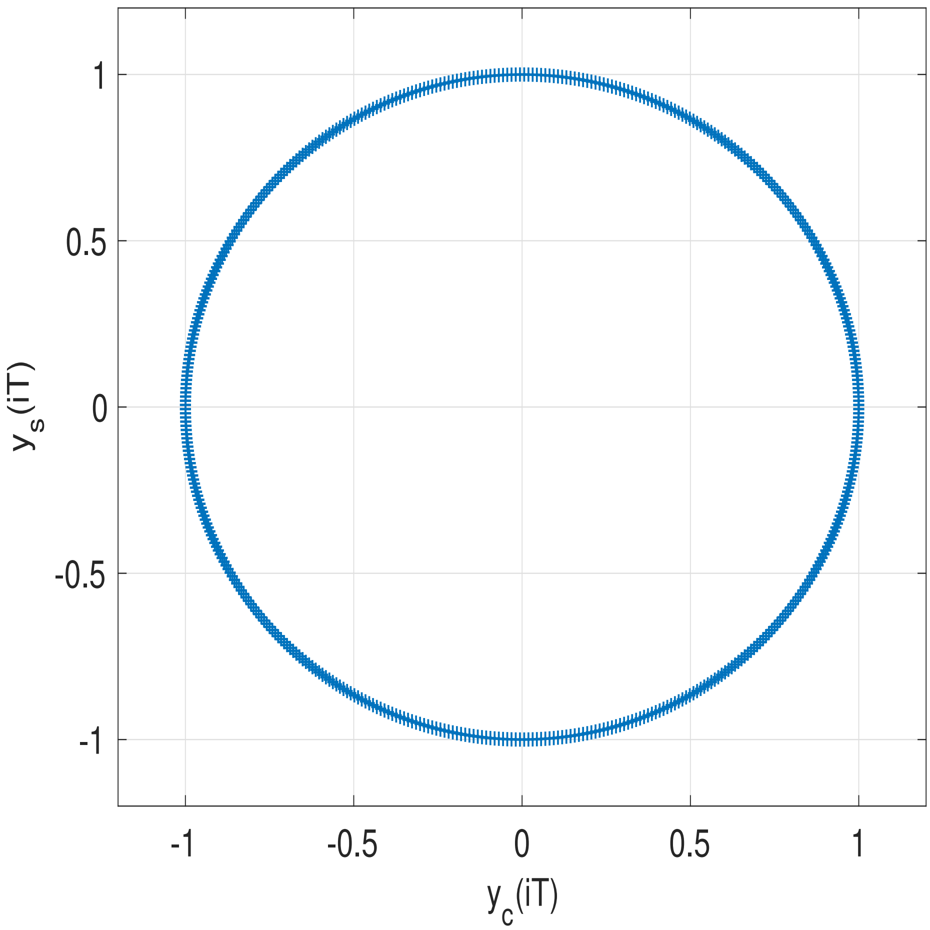


Figure 6: Filter Outputs

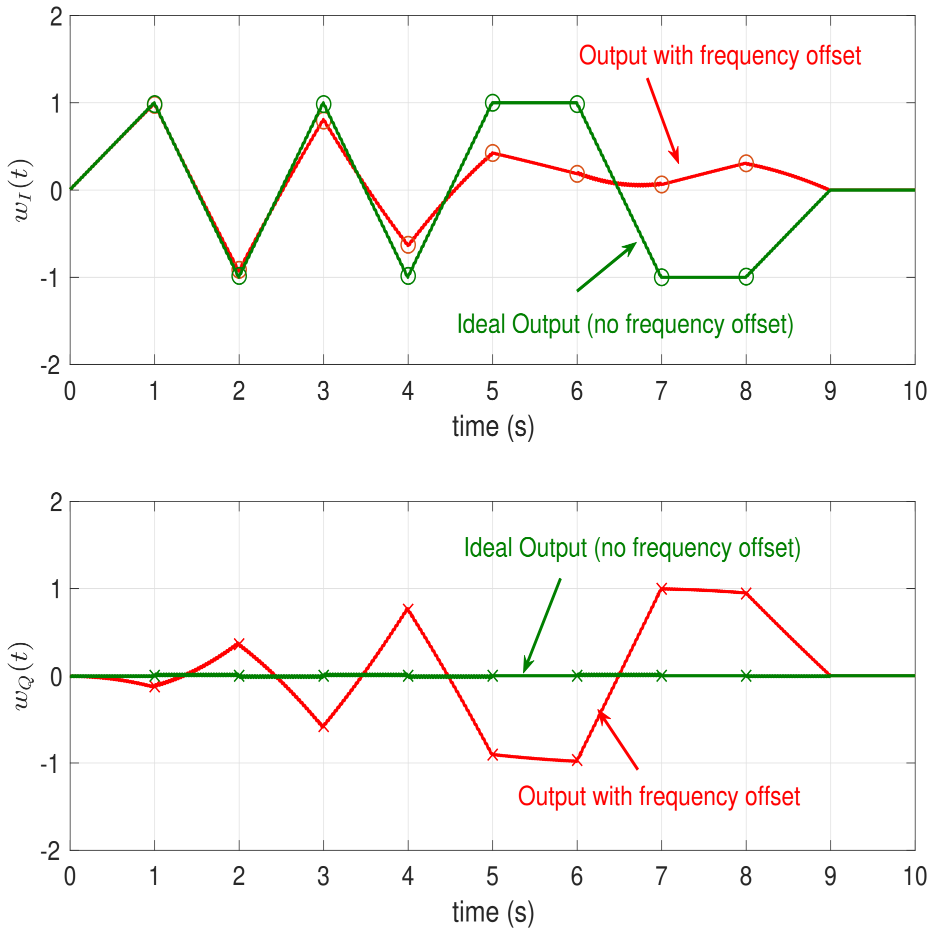


Figure 7: Filter Outputs

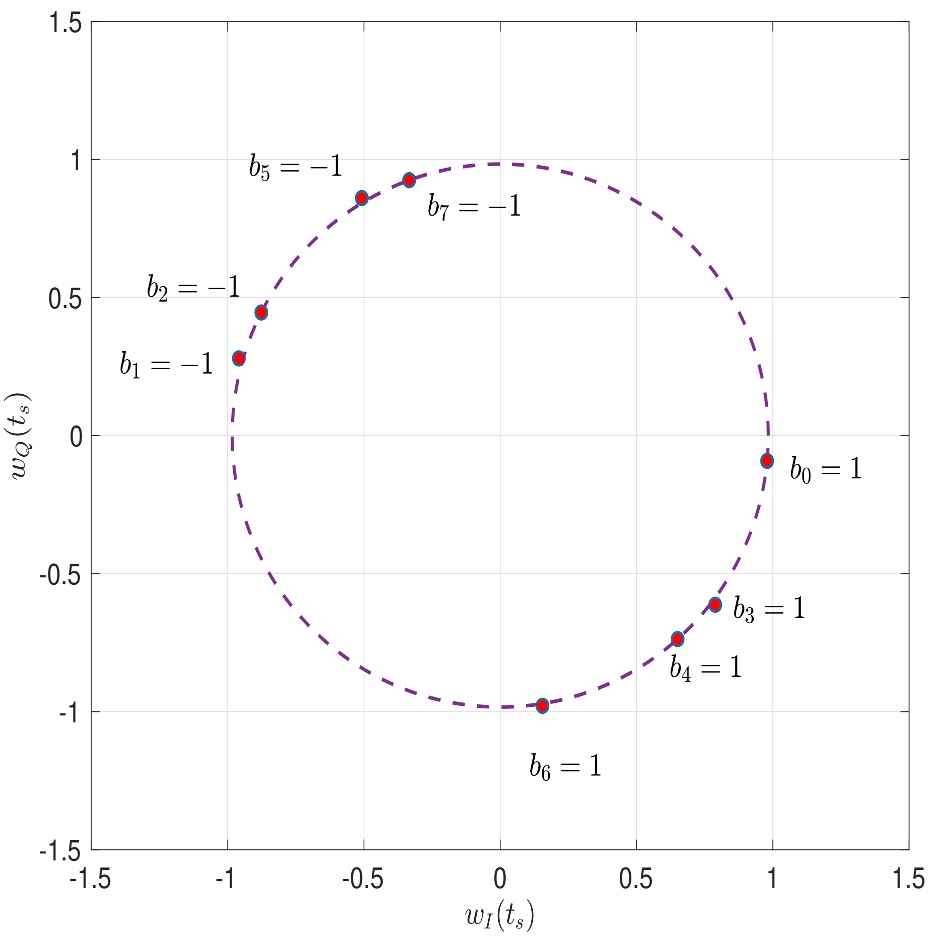


Figure 8: Filter Outputs (frequency offset=0.03

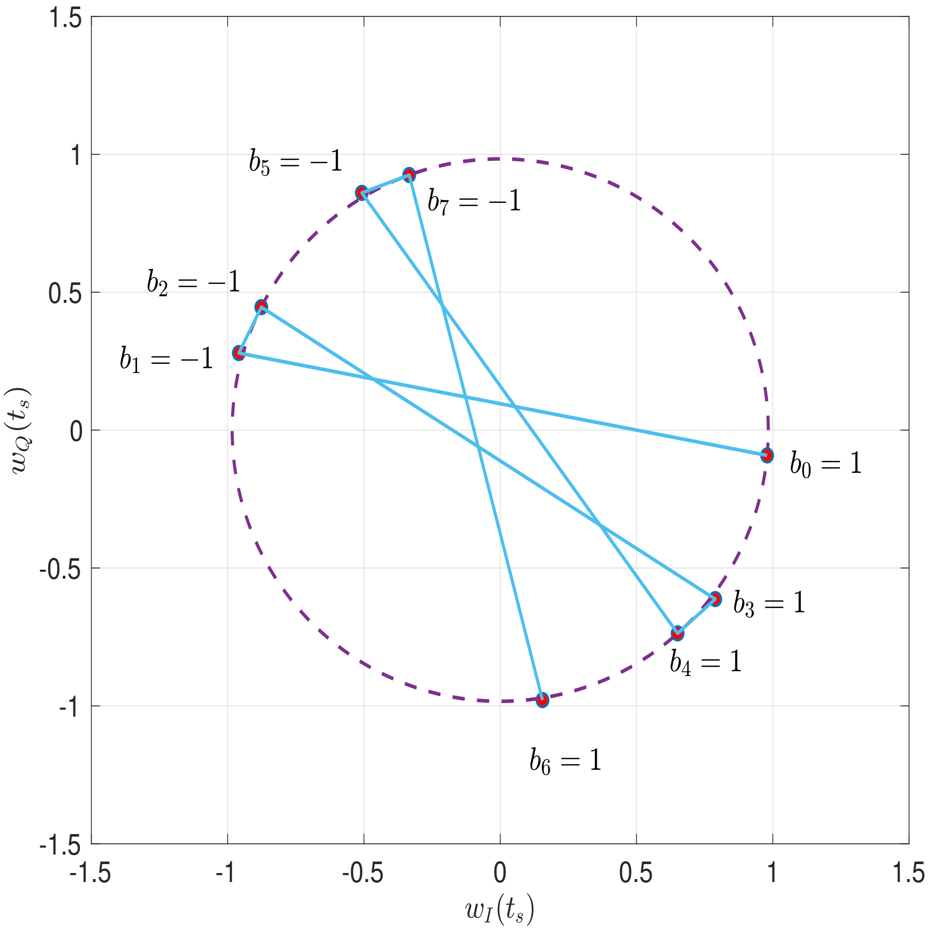


Figure 9: Filter Outputs (frequency offset=0.03