EECS 501 Homework 11 YUZHAN JIANG

Problem 1

(a) Observe that $(X(y_2) - X(y_1))$ is the number of arrivals in the intervals $(y_1, y_2]$, and hence $X(y_2) - X(y_1)$ is independent of $X(y_1)$

X_t is passion Process and $X(y_2) - X(y_1)$ is independent of $X(y_1)$

$P(X(y_1) < 1, X(y_2) < 2) = P(X(y_1) = 0, X(y_2) - X(y_1) < 2)$

$= P(X(y_1) = 0, X(y_2) - X(y_1) = 0) + P(X(y_1) = 0, X(y_2) - X(y_1) = 1)$

where

$$P(X(y_1) = 0) = \frac{(\lambda y_1)^0 \cdot e^{-\lambda y_1}}{0!} = e^{-\lambda y_1}$$

$$P(X(y_2) - X(y_1)) = \frac{[\lambda(y_2 - y_1)]^k e^{-\lambda(y_2 - y_1)}}{k!}$$

$$\therefore P(X(y_1) < 1, X(y_2) < 2) = e^{-\lambda y_1} \left(\frac{\lambda^0 (y_2 - y_1)^0 \cdot e^{-\lambda(y_2 - y_1)}}{0!} \right) + e^{-\lambda y_1} \frac{\lambda(y_2 - y_1) e^{-\lambda(y_2 - y_1)}}{1!}$$

$$= e^{-\lambda y_1} e^{-\lambda(y_2 - y_1)} + e^{-\lambda y_1} \cdot \lambda(y_2 - y_1) \cdot e^{-\lambda(y_2 - y_1)}$$

$$= e^{-\lambda y_2} + \lambda(y_2 - y_1) \cdot e^{-\lambda y_2}$$

$$= e^{-\lambda y_2} [1 + \lambda(y_2 - y_1)]$$

This is exactly the equation we need to show.

(b) $P(Y_1 > y_1, Y_2 > y_2) = P(X(y_1) < 1, X(y_2) < 2)$

$= e^{-\lambda y_2}[1 + \lambda(y_2 - y_1)]$ Based on part (a)

$\therefore$ and $P(Y_1 \le y_1, Y_2 \le y_2) = 1 - P(Y_1 > y_1 \cup Y_2 > y_2)$

where $P(Y_1 > y_1 \cup Y_2 > y_2) = P(Y_1 > y_1) + P(Y_2 > y_2) - P(Y_1 > y_1 \cap Y_2 > y_2)$

(By inclusion - exclusion principle)

$= P(X(y_1) < 1) + P(X(y_2) < 2) - e^{-\lambda y_2}[1 + \lambda(y_2 - y_1)]$

$= P(X(y_1) = 0) + (P(X(y_2) = 0) + P(X(y_2) = 1)) - \sim$

$= e^{-\lambda y_1} + e^{-\lambda y_2} + \lambda y_2 \cdot e^{-\lambda y_2} - e^{-\lambda y_2}[1 + \lambda(y_2 - y_1)]$

$= e^{-\lambda y_1} + \lambda y_1 e^{-\lambda y_2}$

$\therefore P(Y_1 \le y_1, Y_2 \le y_2) = 1 - e^{-\lambda y_1} - \lambda y_1 e^{-\lambda y_2}$ Which is the $F_{Y_1 Y_2}(Y_1 Y_2)$

$\therefore$ the joint PDF of Y_1 and Y_2 is

$$f_{Y_1 Y_2}(y_1\, y_2) = \frac{\partial F_{Y_1 Y_2}(Y_1 Y_2)}{\partial Y_1 \, \partial Y_2}$$

$$= \lambda^2 e^{-\lambda y_2}$$

$$\therefore \quad f_{Y_1 Y_2}(y_1 y_2) = \begin{cases} \lambda^2 e^{-\lambda y_2}, & y_2 > y_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) Based on part (a) and part (b), the result is true for $k = 2$.

For $k > 2$

$$P(Z_1 \le z_1, \dots Z_k \le z_k) = 1 - P(Z_1 > z_1 \cup Z_2 > z_2 \cdots \cup Z_k > z_k)$$

$$= 1 - \left(\sum_i^k P(Z_i > z_i) - \sum P(Z_i > z_i \cap Z_j > z_j) + \cdots + (-1)^{k-1} P(Z_1 > z_1 \cap Z_2 > z_2 \cdots \cap Z_k > z_k) \right)$$

Notice that only the last term contains all variables by taking derivatives

$$\frac{\partial P(Z \le z)}{\partial_1 \partial_2 \cdots \partial_k} = \frac{\partial}{\partial z}(-1)(-1)^{k-1} P(Z_1 > z_1, \dots Z_k > z_k)$$

$$f_{Z_1 Z_2 \cdots Z_k} = (-1)^k \frac{\partial}{\partial z} P(Z > z)$$

$$\therefore \quad f_Z(z) = (-1)^k \frac{\partial}{\partial z} P(Z > z)$$

1(d) $P(Y>y) = P(X(y_1)<1,\ X(y_2)<2,\ \cdots\cdots X(y_k)<k)$

$= e^{-\lambda y_1} e^{-\lambda(y_2-y_1)} \cdots\cdots e^{-\lambda(y_k-y_{k-1})} [1+\lambda(y_2-y_1)+\lambda^2(y_2-y_1)(y_3-y_2)+ \cdots\cdots \lambda^{k-1}(y_2-y_1)\cdots(y_k-y_{k-1})]$

(Based on part(a))

The only term that contains all variables is $\lambda^{k-1}(y_2-y_1)\cdots\cdots(y_k-y_{k-1})$

$\therefore$ when take paritial derivatives, the rest of terms will equal to 0

$\therefore$ $f_Y(y) = (-1)^k \frac{\partial k}{\partial Y} P(Y>y)$

$= \lambda^k e^{-\lambda y_k}$

$$\therefore\quad f_Y(y) = \begin{cases} \lambda^k e^{-\lambda y_k}, & \text{for } y_k \geq y_{k-1} \geq \cdots y_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

P2:

Random process $\{X_t\}$ such that

$$X_t = A\sin(t+\theta)$$

Where A is a Bernoulli random variable with mean $1/4$, θ is uniformly distributed over $[0, 2\pi]$ and A and θ are independent

$$\sin(t+\theta) = \sin(t)\cdot\cos\theta + \cos t \sin\theta$$

$\therefore$ Then, $\mu_{xt} = E[A](E[\cos\theta]\cdot\sin t + E[\sin\theta]\cos t)$

Since θ is uniformly distributed over $[0, 2\pi]$

$\therefore \quad E[\cos\theta] = E[\sin\theta] = 0$

$\therefore \quad \mu_x t = 0$

$$\begin{aligned}
R_{s,s+t} &= E[A^2]\ E[\sin(s+\theta)\sin(s+t+\theta)] \\
&= E[A^2]\left(\tfrac{1}{2}E[\cos(s+\theta-s-t-\theta) - \cos(s+\theta+s+t+\theta)]\right) \\
&= \tfrac{1}{2}E[A^2]\cdot E[\cos(-t) - \cos(2s+t+2\theta)] \\
&= \tfrac{1}{2}E[A^2]\cos(t) - \frac{E[A^2]}{2}E[\cos(2s+t+2\theta)] \\
&= \tfrac{1}{2}E[A^2]\cdot\cos(t) - \frac{E[A^2]}{2}\int_0^{2\pi}\cos(2s+t+2\theta)\cdot d\theta \cdot \frac{1}{2\pi} \\
&= \tfrac{1}{2}E[A^2]\cdot\cos(t) - \frac{E[A^2]}{4\pi}\left[\frac{\sin(2s+t+2\theta)}{2}\right]\Bigg|_0^{2\pi} \\
&= \tfrac{1}{2}E[A^2]\cos(t) \\
&= \tfrac{1}{8}\cos(t) \qquad \left(\text{Since } E[A^2] = Var[A] + E^2[A] = \tfrac{3}{4}\times\tfrac{1}{4} + \left(\tfrac{1}{4}\right)^2 = \tfrac{1}{4}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore\ R_X(t,0) &= E[X_t X_0] \\
&= E[A^2]\ E[\sin(t+\theta)\cdot\sin\theta] \\
&= \tfrac{1}{2}E[A^2]\cdot\cos(t)
\end{aligned}$$

$$R_X(s, s+t) = R_X(t,0)$$

$\therefore$ Therefore, $\{X_t\}$ is WSS

P3:

$X_t = A \sin(t+\theta) + B$, $A \sim$ Bernoulli r.v. $(\frac{1}{4})$

$B \sim$ Bernoulli $(\frac{1}{2})$

$\theta \sim$ Unif $[0, 2\pi]$

$$X_t = A \cdot \sin(t+\theta) + B$$
$$= A \cdot (\sin t \cdot \cos\theta + \cos t \cdot \sin\theta) + B$$
$$\mu_{X_t} = E[A] \cdot E[\sin t \cdot \cos\theta + \cos t \cdot \sin\theta] + E[B]$$
(since A, B and θ are independent from each other)
$$= 0 + E[B]$$
$$= \frac{1}{2}$$

$$R_{s,s+t} = E[X_s X_{s+t}]$$
$$= E\left[(A \cdot \sin(s+\theta) + B)(A \cdot \sin(s+t+\theta) + B)\right]$$
$$= E[A^2 \cdot \sin(s+\theta) \sin(s+t+\theta) + AB \sin(s+\theta) + AB \cdot \sin(s+t+\theta) + B^2]$$
$$= E[A^2 \tfrac{1}{2}(\cos t - \cos(s+s+t+2\theta)) + AB \sin(s+\theta) + AB \sin(s+t+\theta) + B^2]$$
$$= \frac{1}{2} E[A^2] \cos t + E[B^2]$$
$$= \frac{1}{8} \cos(t) + \frac{1}{2}$$
(Since $E[A^2] = Var(A) + E[A]^2 = \frac{1}{4}$ and $E[B^2] = Var(B) + E[B]^2 = \frac{1}{2}$)

and $R_X(t, 0) = E[X_t X_0]$
$$= E[(A \cdot \sin(t+\theta) + B)(A \cdot \sin\theta + B)]$$
$$= E[A^2]\, E[\sin(t+\theta) \sin\theta] + E[B^2]$$
$$= \frac{1}{2} E[A^2] \cdot \cos(t) + E[B^2]$$
$$= \frac{1}{8} \cos(t) + \frac{1}{2}$$

$\therefore$ $R_X(s, s+t) = R_X(s, 0)$ and $\mu_{X_t} = \frac{1}{2}$

Therefore, $\{X_t\}$ is WSS