EECS 455 Homework4 YUZHAN JIANG

1. $r(t) = S_i(t) + n(t)$

$$\hat{S}_i(t) = \int_{-\infty}^{\infty} \left(\sqrt{2P}\, (-1)^i \cos(2\pi f_c t)\, p_T(t) \right) \cdot \left(\sqrt{\tfrac{2}{T}} \cos(2\pi f_c t + \theta) \right) h(T-t) \cdot dt$$

$$\therefore \hat{S}_i(t) = (-1)^i\, 2\sqrt{\frac{P}{T}} \int_{-\infty}^{\infty} \cos(2\pi f_c t)\, \underline{p_T(t)} \cdot \cos(2\pi f_c t + \theta)\, \underbrace{p_T(T-t)}_{p_T(T-t)} \cdot dt$$

$$= (-1)^i\, 2\sqrt{\frac{P}{T}} \int_{-\infty}^{\infty} \frac{1}{2} \left[\cos(2\pi f_c t - 2\pi f_c t - \theta) + \underbrace{\cancel{\cos(2\pi 2 f_c t + \theta)}}_{0} \right] p_T(t)\, dt$$

$$= (-1)^i\, 2\sqrt{\frac{P}{T}} \int_{-\infty}^{\infty} \frac{1}{2} \left[\cos(-\theta) \right] \cdot dT$$

$$= (-1)^i \sqrt{\frac{P}{T}} \int_0^T \cos\theta \, dt$$

$$= (-1)^i \sqrt{\frac{P}{T}} \cos\theta \; T$$

$$= (-1)^i \sqrt{PT} \cdot \cos\theta$$

$$\therefore \hat{S}_0(t) = \sqrt{PT} \cdot \cos\theta \qquad \hat{S}_1(t) = -\sqrt{PT} \cdot \cos\theta$$

$$\therefore \bar{P}_e = \pi_0 \cdot Q\left(\frac{\hat{S}_0(t) - \gamma}{\sigma_N} \right) + \pi_1 Q\left(\frac{\gamma - \hat{S}_1(t)}{\sigma_N} \right) \qquad \text{where } \gamma = 0$$

$$= \pi_0\, Q\left(\frac{\sqrt{PT} \cdot \cos\theta}{\sigma_N} \right) + \pi_1 Q\left(\frac{\sqrt{PT} \cos\theta}{\sigma_N} \right)$$

where $\sigma_N^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t)\, dt = \frac{N_0}{2} \int_0^T p_T^2(t) \cdot dt$

$$= \frac{N_0}{2} T$$

P2.

(a) By def of probability of error:

$$\bar{P_e} = \pi_0 P_{e,0} + \pi_1 P_{e,1} + \pi_2 P_{e,2} + \pi_3 P_{e,3}$$

The probability of error given signal 0 transmitted is then

$$\begin{aligned} P_{e,0} &= P\{\hat{S}_0(T) + n < 2\} \\ &= P\{n < 2-3\} \\ &= P\{n < -1\} \\ &= \Phi\left(\frac{-1}{2}\right) \\ &= Q\left(\frac{1}{2}\right) \end{aligned}$$

$\sigma_n^2 = 4$

$\sigma_n = 2$

Similarly:

$$\begin{aligned} P_{e,1} &= 1 - P\{0 < \hat{S}_1(T) + n < 2\} \\ &= 1 - P\{0 < 1+n < 2\} \\ &= 1 - P\{-1 < n < 1\} \\ &= 1 - \left(\Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)\right) \\ &= 2Q\left(\frac{1}{2}\right) \end{aligned}$$

$\hat{S}_1(T) = +1$

$\hat{S}_2(T) = -1$

$$\begin{aligned} P_{e,2} &= 1 - P\{-2 < \hat{S}_2(T) + n < 0\} \\ &= 1 - P\{-2 < -1+n < 0\} \\ &= 1 - P\{-1 < n < 1\} \\ &= 1 - \left(\Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)\right) \\ &= 2Q\left(\frac{1}{2}\right) \end{aligned}$$

$\hat{S}_3(T) = -3$

$$\begin{aligned} P_{e,3} &= P(\hat{S}_3(T) + n > -2) \\ &= P(-3+n > -2) \\ &= P(n > 1) \\ &= Q\left(\frac{1}{2}\right) \end{aligned}$$

P2 (b) Let b_0 represent the first bit. Let $\hat{b_0}$ represent the decision on the first bit.
Then we need to compute:

$$P\{\text{descion on } S_2, S_3 \mid S_0, S_1 \text{ transmitted}\}$$

$$= P\{\hat{b_0} = 1 \mid b_0 = 0\}$$

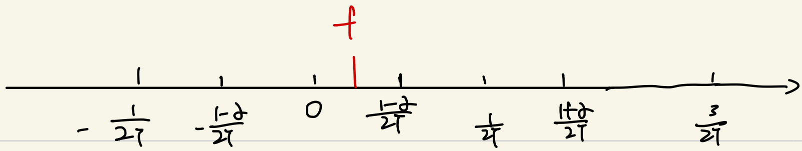
$$= P\{z(T) < 0 \mid b_0 = 0\}$$

$$= P\{\hat{S_0}(T) + n < 0\} + P\{\hat{S_1}(T) + n < 0\}$$

$$= P\{n < -3\} + P\{n < -1\}$$

$$= \Phi(-\tfrac{3}{2}) + \Phi(-\tfrac{1}{2})$$

$$= Q(\tfrac{3}{2}) + Q(\tfrac{1}{2})$$



P3:

① For $-\frac{1}{2T} < f < \frac{1}{2T}$, $\alpha \in [0,1]$

$$0 < |f| < \frac{1}{2T}$$

$$H(f) = \begin{cases} 0 & , 0 < |f| < \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left[f - \frac{1-\alpha}{2T}\right]\right)\right\} & , \frac{1-\alpha}{2T} < |f| < \frac{1}{2T} \end{cases}$$

② Let $f_1 = f - \frac{1}{T}$; if $-\frac{1-\alpha}{2T} < f_1 < \frac{1-\alpha}{2T}$

For f_1,

$$-\frac{1}{2T} - \frac{1}{T} < f_1 < \frac{1}{2T} - \frac{1}{T} \qquad -\frac{3-\alpha}{2T} < f_1 < -\left(\frac{1+\alpha}{2T}\right)$$

$$-\frac{3}{2T} < f_1 < -\frac{1}{2T} \qquad \frac{1+\alpha}{2T} < |f_1|$$

$$\frac{1}{2T} < |f_1| < \frac{3}{2T}$$

$$H(f_1) = H\left(f - \frac{1}{T}\right) = \begin{cases} \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left[f - \frac{1}{T} - \frac{1-\alpha}{2T}\right]\right)\right. & , \frac{1}{2T} < |f_1| < \frac{1+\alpha}{2T} \\ 0 & , \frac{1+\alpha}{2T} < |f_1| < \frac{3}{2T} \end{cases}$$

③ Let $f_2 = f + \frac{1}{T}$.

$$-\frac{1}{2T} + \frac{1}{T} < f_2 < \frac{1}{2T} + \frac{1}{T} \qquad -\frac{2}{2T} \quad -\frac{1}{2T} + \frac{\alpha}{2T}$$

$$\frac{1}{2T} < f_2 < \frac{3}{2T} \qquad -\frac{3}{2T} \quad -\frac{3-\alpha}{2T}$$

$$\therefore \frac{1}{2T} < |f_2| < \frac{3}{2T}$$

$$\therefore H(f_2) = H\left(f + \frac{1}{T}\right) = \begin{cases} \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left[f + \frac{1}{T} - \frac{1-\alpha}{2T}\right]\right)\right\} & , \frac{1}{2T} < |f_2| < \frac{1+\alpha}{2T} \\ 0 & , \frac{1+\alpha}{2T} < |f_2| < \frac{3}{2T} \end{cases}$$

$$\cos(x)+\cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Case ①: If $0<|f| < \frac{1-\alpha}{2T}$, $\Rightarrow$ $<|f-\frac{1}{T}| < \frac{1+\alpha}{2T}$

$$H(f) = 0$$

$$H(f_1) = \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left[f - \frac{3+\alpha}{2T}\right]\right)\right\}$$

$$H(f_2) = \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left[f + \frac{1+\alpha}{2T}\right]\right)\right\}$$

$$\therefore\ H(f)+H(f_1)+H(f_2) = T + \frac{T}{2}\left\{\cos\left(\frac{\pi T}{\alpha}\left[f-\frac{3+\alpha}{2T}\right]\right) + \cos\left(\frac{\pi T}{\alpha}\left[f+\frac{1+\alpha}{2T}\right]\right)\right\}$$

$$= T + \frac{T}{2}\left\{\cos\left(\frac{\pi T}{2\alpha}\left[2f-\frac{2}{2T}\right]\right)\cos\left(\frac{\pi T}{2\alpha}\left(-\frac{4}{2T}\right)\right)\right.$$

$$= T$$

Case ②: If $-\frac{1-\alpha}{2T} < f < \frac{1-\alpha}{2T}$

then $\frac{1+\alpha}{2T} < |f_1| < \frac{3}{2T}$ and $\frac{1+\alpha}{2T} < |f_2| < \frac{3-\alpha}{2T}$

$\therefore\ H(f_2) = 0$

$$\therefore\ H(f)+H(f_1)+H(f_2) = T + \frac{T}{2}\left\{\cos\left(\frac{\pi T}{\alpha}\left[f-\frac{3-\alpha}{2T}\right]\right) + \cos\left(\frac{\pi T}{\alpha}\left[f-\frac{1+\alpha}{2T}\right]\right)\right\}$$

$$= T + \frac{T}{2}\left\{\cos\left(\frac{\pi T}{2\alpha}\left(2f-\frac{4}{2T}\right)\right)\cdot\cos\left(\frac{\pi T}{2\alpha}\left(\frac{-2+2\alpha}{2T}\right)\right)\right\}$$

$$= T + \frac{T}{2}\left\{\cos\left(\frac{\pi T f}{\alpha} - \frac{\pi}{\alpha}\right)\cos\left(-\frac{\pi}{2\alpha}+\frac{\pi}{2}\right)\right\}$$

$$= T$$

P4:

$S_0 = A p_{T/2}(t) - A p_{T/2}(t - T/2)$, $S_1(t) = 0$

(a) $\bar{P_e} = \pi_0 Q\left(\frac{\hat{S_0}(T) - \gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma - \hat{S_1}(T)}{\sigma_N}\right)$ where $\pi_0 = \pi_1 = \frac{1}{2}$

$$\hat{S_0}(T) = \int_{-\infty}^{\infty} h(T-\tau) S_0(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) S_0(T-\tau) d\tau$$

$$\doteq \int_{-\infty}^{\infty} S_0^2(T-\tau) d\tau$$

$$\hat{S_1}(T) = \int_{-\infty}^{\infty} h(T-\tau) S_1(\tau) d\tau$$

$$= 0$$

$$\sigma_N^2 = \frac{N_0}{2} \int h^2(t) dt = \frac{N_0}{2} \int |H(f)|^2 df$$

When $\pi_0 = \pi_1 = \frac{1}{2}$, optimise with respect to $h(t)$

$\bar{P_e} = Q(\lambda)$ where $\lambda = \sqrt{\frac{E(1-r)}{N_0}}$

$E_1 = 0$ since $\hat{S_1}(t) = 0$

$$E_0 = \int_{-\infty}^{\infty} S_0^2(t) \cdot dt$$

$$= \int_{-\infty}^{\infty} \left(A P_{\frac{T}{2}}(t) - A P_{\frac{T}{2}}(t - \tfrac{T}{2})\right)^2 dt$$

$$= A^2 \int_{-\infty}^{\infty} \left(P_{\frac{T}{2}}^2(t) - 2 P_{\frac{T}{2}}(t) P_{\frac{T}{2}}(t - \tfrac{T}{2}) + P_{\frac{T}{2}}^2(t - \tfrac{T}{2})\right) dt$$

$$= A^2 \left[\int_0^{\frac{T}{2}} P_{\frac{T}{2}}^2(t) - 0 + \int_{\frac{T}{2}}^{T} P_{\frac{T}{2}}^2(t - \tfrac{T}{2}) \cdot dt \right.$$

$$= A^2 \left(\tfrac{T}{2} + \tfrac{T}{2}\right)$$

$$= A^2 T$$

$\therefore \bar{E} = \frac{E_0 + E_1}{2}$

$= \frac{A^2 T}{2}$

$\therefore \gamma = \frac{(S_0, S_1)}{\bar{E}}$

$= 0 \qquad \text{since } (S_0, S_1) = \int_{-\infty}^{\infty} S_0(t) S_1(t)\, dt = 0$

$\therefore Q = (\alpha) \quad \text{where} \quad \alpha = \sqrt{\frac{A^2 T}{2N_0}}$

(4b)

Optimum filter:

$h_{opt} = S_0(T-t) - S_1(T-t)$

$= A p_{T/2}(T-t) - A p_{T/2}(T/2 - t)$

Optimum Threshold.

$\gamma_{opt} = \frac{1}{2}(E_0 - E_1)$

$= \frac{A^2}{2} T$

4(c) $S_0(t) = CA P_{\frac{T}{2}}(t) - CA P_{\frac{T}{2}}(t-\frac{T}{2})$

$S_1(t) = 0$

$\bar{P}_e(\gamma, h(t), S_0(t), S_1(t)) = \pi_0 Q(\frac{\hat{S}_0(t) - \gamma}{\sigma_N}) + \pi_1 Q(\frac{\gamma - \hat{S}_1(t)}{\sigma_N})$

$$\hat{S}_0(t) = \int_{-\infty}^{\infty} h(\tau) S_0(T-\tau) \cdot d\tau$$

$$= \int_{-\infty}^{\infty} \left[A P_{T/2}(T-\tau) - A P_{\frac{T}{2}}(\tfrac{T}{2}-\tau)\right]\left[CA P_{\frac{T}{2}}(T-\tau) - CA P_{\frac{T}{2}}(\tfrac{T}{2}-\tau)\right] d\tau$$

$$= CA^2 \int_{-\infty}^{\infty} \left[P_{T/2}^2(T-\tau) + P_{\frac{T}{2}}^2(\tfrac{T}{2}-\tau) - 2P_{\frac{T}{2}}(\tfrac{T}{2}-\tau) P_{\frac{T}{2}}(T-\tau)\right] d\tau$$

$$= CA^2 \left(\frac{T}{2} + \frac{T}{2} + 0\right)$$

$$= CA^2 T$$

$\hat{S}_1(t) = 0$, $\gamma_{opt} = \frac{A^2}{2} T$ (from part (b))

$$\sigma_N^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t)\, dt$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} \left[A P_{\frac{T}{2}}(T-t) - A P_{\frac{T}{2}}(\tfrac{T}{2}-t)\right] dt$$

$$= \frac{N_0}{2} A^2 \int_{-\infty}^{\infty} \left[P_{\frac{T}{2}}^2(T-t) - 2P_{\frac{T}{2}}(T-t) P_{\frac{T}{2}}(\tfrac{T}{2}-t) + P_{\frac{T}{2}}^2(\tfrac{T}{2}-t)\right] dt$$

$$= \frac{N_0}{2} A^2 \left(\frac{T}{2} + \frac{T}{2}\right)$$

$$= \frac{A^2}{2} N_0 T$$

$$\therefore \bar{P}_e = \frac{1}{2} Q\left(\frac{CA^2T - \frac{A^2T}{2}}{\sigma_n}\right) + \frac{1}{2} Q\left(\frac{\frac{A^2T}{2}}{\sigma_n}\right)$$

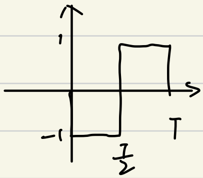
where $\sigma^2 = \frac{A^2}{2} N_0 T$

P5

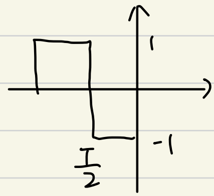
$S_0(t) = A_0 P_{T/2}(t) \qquad S_1(t) = A_1 P_{T/2}(t - T/2)$

(a) $h_R(t) = -P_{T/2}(t) + P_{T/2}(t - T/2)$

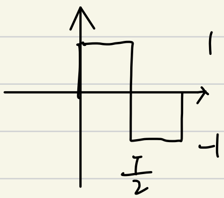
$h(t) =$



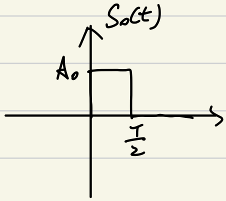
$\Rightarrow h(-t)$



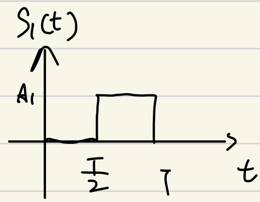
$h(T-t) =$



$S_0(t) =$



$S_1(t) =$



$$\hat{S}_0(T) = \int_{-\infty}^{\infty} h(T-\tau)\, S_0(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, S_0(T-\tau)\, d\tau$$

$$= \int_0^{\frac{T}{2}} A_0\, d\tau + \int_{\frac{T}{2}}^{T} 0(-1)\, d\tau$$

$$= \frac{A_0 T}{2}$$

$$\hat{S}_1(t) = \int_{-\infty}^{\infty} h(T-\tau)\, S_1(\tau)\, d\tau$$

$$= \int_0^{\frac{T}{2}} (-1)\cdot 0\, d\tau + \int_{\frac{T}{2}}^{T} (-1)\cdot A_1\, d\tau$$

$$= -A_1 T - \left(-\frac{A_1 T}{2}\right)$$

$$= -\frac{A_1 T}{2}$$

$\therefore \hat{S}_0(T) = \frac{A_0}{2} T$ and $\hat{S}_1(T) = -\frac{A_1}{2} T$

5(b) Assum $\pi_0 = \pi_1$

$$\gamma_{opt} = \frac{\sigma_N^2 \ln \frac{\pi_1}{\pi_0}}{\hat{S}_0(T) - \hat{S}_1(T)} + \frac{\hat{S}_0(T) + \hat{S}_1(T)}{2}$$

$$= \frac{\hat{S}_0(T) + \hat{S}_1(T)}{2}$$

$$= \frac{\frac{A_0 T}{2} - \frac{A_1}{2} T}{2}$$

$$= \frac{(A_0 - A_1)}{4} T$$

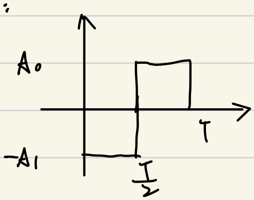
5(c) Assum that $\pi_0 = \pi_1$

$$\bar{P_e} = \pi_0 P_{e,0} + \pi_1 P_{e,1}$$

$$= Q(\alpha\lambda) \quad \text{where} \quad \lambda = \frac{(h, S_T)}{\|h\| \, \|S_T\|}, \quad \alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \bar{E} = \frac{E_0 + E_1}{2}, \quad r = \frac{(S_0, S_1)}{E}$$

$$S_T = S_0(T-\tau) - S_1(T-\tau)$$

$S_0(T-\tau) - S_1(T-\tau)$:



$$(h, S_T) = \int_{-\infty}^{\infty} h(\tau)[S_0(T-\tau) - S_1(T-\tau)] \, d\tau$$

$$= \int_0^{\frac{T}{2}} (-1)\cdot(-A_1) \, d\tau + \int_{\frac{T}{2}}^{T} A_0 \, d\tau$$

$$= A_1 \frac{T}{2} + \frac{1}{2} A_0 T$$

$$= \left(\frac{A_0 + A_1}{2}\right) T$$

(continued)

5(c)

$$\|h\| = \sqrt{\int_{-\infty}^{\infty} h^2(t)\, dt} \qquad \|S_T\| = \sqrt{\int_{-\infty}^{\infty} S_T^2(t)\, dt}$$

$$= \sqrt{\int_0^T 1 \, dt} \qquad = \sqrt{\int_0^{\frac{T}{2}} A_1^2\, dt + \int_{\frac{T}{2}}^{T} A_0^2\, dt}$$

$$= \sqrt{T} \qquad = \sqrt{\frac{(A_0^2 + A_1^2)}{2} T}$$

Therefore, we get $\lambda = \dfrac{\frac{(A_0 + A_1)}{2} T}{\sqrt{T} \cdot \sqrt{\frac{(A_0^2 + A_1^2)}{2} T}}$

$$d = \sqrt{\frac{\bar{E}(1-r)}{N_0}}$$

$$E_0 = \int_{-\infty}^{\infty} S_0(t)^2 \cdot dt = \int_0^{\frac{T}{2}} A_0^2 \cdot dt = \frac{A_0^2}{2} T$$

$$E_1 = \int_{-\infty}^{\infty} S_1^2(t)\, dt = \int_{\frac{T}{2}}^{T} A_1^2 \cdot dt = \frac{A_1^2}{2} T$$

$$\bar{E} = \frac{E_0 + E_1}{2}$$

$$= \left(\frac{A_0^2 + A_1^2}{4}\right) T$$

$$(S_0, S_1) = \int_{-\infty}^{\infty} S_0(t)\, S_1(t) . dt$$

$$= 0$$

$$\therefore\ d = \sqrt{\frac{(A_0^2 + A_1^2) T}{4N_0}}$$

Overall, $\bar{P_e} = Q(d\lambda)$, $\quad d = \sqrt{\dfrac{(A_0^2 + A_1^2) T}{4N_0}}$, $\quad \lambda = \dfrac{\frac{(A_0 + A_1)}{2} T}{\sqrt{T} \cdot \sqrt{\frac{(A_0^2 + A_1^2)}{2} \cdot T}}$

5(d)

$h_{opt} = S_0(T-t) - S_1(T-t)$ is the matched filter

$= A_0 P_{T/2}(T-t) - A_1 P_{T/2}(t)$

$y_{opt} = (h_{opt}, s_0, S_1)$

$= \frac{1}{2}(E_0 - E_1) + \underbrace{\frac{1}{2} N_0 \ln \frac{\pi_1}{\pi_0}}_{0}$

$= \frac{1}{2}(E_0 - E_1)$

$= \left(\frac{A_0^2 - A_1^2}{4}\right) T$

5(E)

Assume that $\pi_0 = \pi_1$,

$\bar{P}_e = Q(\alpha)$ where $\alpha = \sqrt{\frac{(A_0^2 + A_1^2)}{4N_0} T}$ (using α in part (c))

for the matched filter