EECS 501
Discussion 12
Solution

1 Review

- *Poisson Process*: (a) $N(0) = 0$ (b) $N(t_2) - N(t_1)$ is a Poisson random variable with parameter $\lambda(t_2 - t_1)$, i.e., $\mathbb{P}(N(t_2) - N(t_1) = n) = e^{-\lambda(t_2-t_1)} \frac{(\lambda(t_2-t_1))^n}{n!}$ and (c) for non-overlapping intervals $(t_1, t_2]$ and $(t_3, t_4]$ the increments $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are independent.
- *Bernoulli Process*: A discrete time i.i.d. process $\left(X_n\right)_{n\in\mathbb{N}}$ where each $X_i \sim \text{Bernoulli}(p)$.
- Given a random process X_t, we define
  1. Mean function: $\mu_t = E[X_t]$.
  2. Autocorrelation function $R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$
  3. Autocovariance function $C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_{t_1}\mu_{t_2}$.
- X is said to be wide-sense stationary (WSS) if

  $$\mu_X(t) = \mu_X, \forall t \text{ and } R_X(s+\tau, s) = R_X(\tau, 0), \forall s, t.$$

  If a process is WSS, we have

  $$R_X(\tau) = E[X_\tau X_0]$$

- A time continuous random process X_t is Gaussian if and only if for every finite set of indices $t_1, \ldots, t_k$, $(X_{t_1}, \ldots, X_{t_k})$ are jointly Gaussian random variables. That is the same as saying every linear combination of $(X_{t_1}, \ldots, X_{t_k})$ has a normal (or Gaussian) distribution.

  The Gaussian process X_t is white Gaussian noise if it has the following autocorrelation function,

  $$R_X(\tau) = r\delta(\tau)$$

- $X(t)$ is passed through an LTI system with impulse response h and the output is $Y(t)$. $X(t)$ and $Y(t)$ are J-WSS and we have

  $$E[Y(t)] = \mu_Y = c\int_{-\infty}^{\infty} h(\tau)d\tau$$

  $$R_{XY}(t_1 - t_2) = (\bar{h} * R_X)(t_1 - t_2)$$

  where where $\bar{h}(x) = h(-x)$.

  $$R_Y(t_1 - t_2) = (h * R_{XY})(t_1 - t_2)$$

2 Practice Problems

Problem 1 *Superposition*: We know that the sum of two Poisson random variables is another Poisson random variable, with parameter equal to the sum of the parameters. If $\big(N_1(t)\big)_{t\geq 0}, \big(N_2(t)\big)_{t\geq 0}$ are two independent Poisson processes with rates λ_1, λ_2 then is $N(t) = N_1(t) + N_2(t)$ also a Poisson process? If yes, what is the rate?

Solution First we check whether $N(t)$ has independent increments. Take any times $t_3 \geq t_2 \geq t_1 \geq t_0$, and define

$$X = N_1(t_1) - N_1(t_0) \qquad Y = N_1(t_3) - N_1(t_2) \tag{1}$$
$$W = N_2(t_1) - N_2(t_0) \qquad Z = N_2(t_3) - N_2(t_2) \tag{2}$$

As N_1, N_2 are Poisson processes, we know that X, Y are independent and W, Z are independent. Also since the two Poisson processes are independent we can now say that all RVs W, X, Y, Z are independent. Thus $W + X$ is independent of $Y + Z$. But note that

$$W + X = N_1(t_1) - N_1(t_0) + N_2(t_1) - N_2(t_0) = N(t_1) - N(t_0) \tag{3}$$
$$Y + Z = N_1(t_3) - N_1(t_2) + N_2(t_3) - N_2(t_2) = N(t_3) - N(t_2) \tag{4}$$

Thus the increments for the process $N(t)$ are independent.

Now we check the marginal distribution property

$$\mathbb{P}(N(t) = n) = \mathbb{P}(N_1(t) + N_2(t) = n) \tag{5}$$

here $N_1(t), N_2(t)$ are independent and $N_1(t) \sim \text{Poisson}(\lambda_1 t)$ and $N_2(t) \sim \text{Poisson}(\lambda_2 t)$. We know that $N_1(t) + N_2(t) \sim \text{Poisson}((\lambda_1 + \lambda_2)t)$. Hence from above we can write

$$\mathbb{P}(N(t) = n) = e^{-(\lambda_1+\lambda_2)t} \frac{((\lambda_1 + \lambda_2)t)^n}{n!} \tag{6}$$

Thus we have proved that $N(t)$ is a Poisson process and its rate is $\lambda_1 + \lambda_2$.

Problem 2 In the above problem verify that the distribution of inter-arrival times for the superposition $N(t) = N_1(t) + N_2(t)$ is exponential$(\lambda_1 + \lambda_2)$.

Solution If the most recent arrival for $N(t)$ comes from the process $N_1(t)$ then the time till the next arrival for N is the minimum of the next inter-arrival time of N_1, say I, and the residual/remaining of the current inter-arrival time of N_2, say L. Inter-arrival times for Poisson process are independent and exponentially distributed. By the memoryless property, statistically, we can treat L as an exponential(λ_2) random variable. Thus the next inter-arrival time for N is the minimum of an exponential(λ_1) RV with an exponential(λ_2) RV. We know that minimum of independent exponentials is an exponential with parameters added, thus inter-arrival time of N is distributed as exponential$(\lambda_1 + \lambda_2)$.

This is exactly what we expect since in the previous problem we proved that $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

Problem 3 If we have two independent Poisson processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ with parameters λ_1 and λ_2, respectively. What is the probability that the first n arrivals are all from $\{X_t\}$?

Solution: *First method:*

$$\begin{aligned}
&P(\text{at least first } n \text{ arrivals are all from } \{X_t\})\\
=&P(\text{at least } n \text{ arrivals from } \{X_t\} \text{ comes before the first arrival from } \{Y_t\})\\
=&\int_0^\infty P(\text{at least } n \text{ arrivals from } \{X_t\} \text{ before } s|T_1 = s) f_{T_1}(s)ds
\end{aligned}$$

where T_1 is the time of the first arrival from $\{Y_t\}$.
We know that $T_1 \sim \exp(\lambda_2)$ so $f_{T_1}(s) = \lambda_2 e^{-\lambda_2 s}$.
Since $\{X_t\}$ and $\{Y_t\}$ are independent, we have

$$\begin{aligned}
P(\text{at least } n \text{ arrivals from } \{X_t\} \text{ before } s|T_1 = s) =&P(\text{at least } n \text{ arrivals from } \{X_t\} \text{ before } s)\\
=&P(X_s \geq n)\\
=&\sum_{k=n}^\infty e^{-s\lambda_1} \frac{(s\lambda_1)^k}{k!}
\end{aligned}$$

Therefore, using integral by parts we get

$$\begin{aligned}
&P(\text{first } n \text{ arrivals are all from } \{X_t\})\\
=&\sum_{k=n}^\infty \int_0^\infty e^{-s\lambda_1} \frac{(s\lambda_1)^k}{k!} \lambda_2 e^{-\lambda_2 s} ds\\
=&\sum_{k=n}^\infty \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \frac{\lambda_2}{\lambda_1+\lambda_2}\\
=&\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n
\end{aligned}$$

Second method:
Try to compute first the probability that the first arrival is from $\{X_t\}$.
Let S_1 and T_1 be the first arrival time from $\{X_t\}$ and $\{Y_t\}$, respectively. Then, we know that $S_1 \sim \exp(\lambda_1)$ and $T_1 \sim \exp(\lambda_2)$. Consequently,

$$\begin{aligned}
&P(\text{the first arrival is from } \{X_t\})\\
=&P(S_1 < T_1)\\
=&\frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

Where the last equality follows by applying law of total probability on the values of $T_1 = t$. Since the inter-arrival times of a Poisson process are i.i.d. exponential variables, from the memoryless property of exponential variables we have

$$\begin{aligned}
&P(\text{first } n \text{ arrivals are all from } \{X_t\})\\
=&P(\text{the 2nd to } n\text{th arrivals are all from } \{X_t\}|\text{the first arrival is from } \{X_t\})\frac{\lambda_1}{\lambda_1+\lambda_2}\\
=&P(\text{first } n-1 \text{ arrivals are all from } \{X_t\})\frac{\lambda_1}{\lambda_1+\lambda_2}\\
&\vdots\\
=&\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n
\end{aligned}$$

Problem 4 Hits to the websites of EECS department and Economics department form two independent Poisson processes S_t and E_t respectively. Let λ and α be their respective rates. Find the probability that between two consecutive hits to EECS website there are m hits to the Economics website.

Solution: From the memoryless property of Poisson processes, this probability is equal to the probability that the first m hits are to the Economics website and the $m+1$th hit is to EECS website. From the first problem we get

$$\begin{aligned}
&P(\text{the first } m \text{ hits are to the Economics website and the } m+1\text{th hit is to EECS website})\\
=&P(\text{the } m+1\text{th hit is to EECS website}|\text{the first } m \text{ hits are to the Economics website})\\
&P(\text{the first } m \text{ hits are to the Economics website})\\
=&P(\text{the first hit is to EECS website})P(\text{the first } m \text{ hits are to the Economics website})\\
=&\left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{\alpha}{\lambda+\alpha}\right)^m
\end{aligned}$$

Problem 5

1. Prove that number of arrivals up to time N for a Bernoulli process is Binomial(N, p).
2. Prove that the inter-arrival times for a Bernoulli process $\sim$ geometric(p).

Solution

1. The number of arrivals up to time N is the same as the number of successes in N independent Bernoulli(p) trials. Recall that counting the number of successes in independent trials is indeed a Binomial RV. Moreover the success probability is p.
2. Inter-arrival times refer to the number of trials after a success until the next success. Let T_k denote the k^{th} arrival time i.e. the time at which a 1 is seen in the sequence $X_1, X_2, \ldots$ for the k^{th} time. Then the k^{th} inter-arrival time is $I_k = T_k - T_{k-1}$. By definition I_k depends only $X_{T_{k-1}+1}, X_{T_{k-1}+2}, \ldots, X_{T_k}$ and since the Bernoulli process comprises of independent RVs we have that inter-arrival times $\{I_k\}_{k\in\mathbb{N}}$ are independent of each other.

   We can calculate as follows

$$\begin{aligned}
\mathbb{P}(I_k = m) &= \mathbb{P}(X_{T_{k-1}+1} = 0, X_{T_{k-1}+2} = 0, \ldots, X_{T_{k-1}+m-1} = 0, X_{T_{k-1}+m} = 1) && (7)\\
&= \mathbb{P}(X_1 = 0, X_2 = 0, \ldots, X_{m-1} = 0, X_m = 1) && (8)\\
&= \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0)\cdots\mathbb{P}(X_{m-1} = 0)\mathbb{P}(X_m = 1) && (9)\\
&= (1-p)^{m-1}p && (10)
\end{aligned}$$

   where the second and third equality follow from the fact that X_j's are independent and identically distributed. Hence we have shown that $I_k \sim$ geometric(p).

Problem 6 The behavior of a Linear Time Invariant (LTI) system is characterized by its impulse response. If system is fed with a signal $x(t)$, the output would be $y(t) = x(t) * h(t)$. Suppose an LTI system has the following impulse response:

$$h(t) = 3e^{-2t}, t > 0.$$

Assume we are feeding the system with a zero mean white Gaussian noise with autocorrelation equal to $R_X(\tau) = 4\delta(\tau)$.

a) Is Y_t also a Gaussian process?

b) Find the auto correlation of Y_t and variance of Y_t.

Solution:

a) Yes, since the system is linear, the output is the limit of a summation of jointly Gaussian inputs (X_t) so the output is jointly Gaussian.

b) We first find the cross correlation of X_t and Y_t, which is given as below for $\tau > 0$:

$$R_{XY}(\tau) = (\bar{h} * R_X)(\tau) = \int_{-\infty}^{\infty} 4h(t)\delta(\tau + t)dt = 4h(-\tau)$$

and it is 0 otherwise. Now we can calculate $R_Y(\tau)$ as follows.

$$R_Y(\tau) = (h * R_{XY})(\tau) = \int_{\max(\tau,0)}^{\infty} 4h(t)h(-(\tau - t))dt = \int_{\max(\tau,0)}^{\infty} 4h(t)h(t - \tau))dt$$

For $\tau \geq 0$, we have

$$R_Y(\tau) = (h * R_{XY})(\tau) = 4\int_{\tau}^{\infty} 9e^{-2(2t-\tau)}dt = -9e^{-4t+2\tau}|_{\tau}^{\infty} = 9e^{-2\tau}$$

For $\tau < 0$, we have

$$R_Y(\tau) = 4\int_{0}^{\infty} 9e^{-2(2t-\tau)}dt = 9e^{2\tau}$$

Therefore, we have $R_Y(\tau) = 9e^{-2|\tau|}$. We have $Var(Y_t) = E(Y_t^2) = R_Y(0) = 9$.

Problem 7 Consider a WSS Gaussian Process $\{X_t\}_{t\geq 0}$ with mean 0 and autocorrelation function $R_X(\tau) = e^{-|\tau|}$. Find $\mathbb{P}(|X_2 - X_5| < 2)$.

Solution: A process $\{X_t\}_{t\geq 0}$ is Gaussian if for any $N \in \mathbb{N}$ and any $t_1, t_2, \dots, t_N > 0$ the vector $(X_{t_1}, \dots, X_{t_N})$ is a multivariate Gaussian random vector.

This definition implies that if we choose $N = 2$ and $t_1 = 2, t_2 = 5$ then (X_2, X_5) are jointly Gaussian. Let $Y = X_2 - X_5$. Then Y is a Gaussian random variable. We need to find the mean and variance of Y. We know $\mathbb{E}[Y] = 0$. Because the mean for (X_2, X_5) is $(0, 0)$. For the variance we have

$$\begin{aligned}
Var(Y) = \mathbb{E}[Y^2] &= \mathbb{E}[(X_2 - X_5)^2] = \mathbb{E}[X_2^2] + \mathbb{E}[X_5^2] - 2\mathbb{E}[X_2X_5] \\
&= 2R_X(0) - 2R_X(3) \\
&= 2 - 2e^{-3}
\end{aligned}$$

Therefore, the answer is

$$\mathbb{P}(|X_2 - X_5| < 2) = \mathbb{P}(|Y| < 2) = 1 - 2Q(\frac{2}{\sqrt{2 - 2e^{-3}}})$$

Problem 8 $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are uncorrelated, continuous-time, wide-sense stationary, random processes with means m_X, m_Y and autocorrelation functions $R_X(\tau)$, $R_Y(\tau)$, respectively. Let $Z_t = X_t + Y_t$ for any $t \geq 0$.

- Find the mean of Z_t at any time $t \geq 0$.
- Find the autocorrelation function $R_Z(t_1, t_2)$.
- Is $\{Z_t\}$ a wide-sense stationary random process?

Solution:

$$E(Z_t) = E(X_t + Y_t) = E(X_t) + E(Y_t) = m_X + m_Y$$

$$\begin{aligned}
R_Z(t_1, t_2) = E(Z_{t_1} Z_{t_2}) =& E[(X_{t_1} + Y_{t_1})(X_{t_2} + Y_{t_2})] \\
=& E[X_{t_1} X_{t_2} + X_{t_1} Y_{t_2} + Y_{t_1} X_{t_2} + Y_{t_1} Y_{t_2}] \\
=& E(X_{t_1} X_{t_2}) + E(X_{t_1} Y_{t_2}) + E(Y_{t_1} X_{t_2}) + E(Y_{t_1} Y_{t_2}) \\
=& R_X(t_1 - t_2) + E(X_{t_1}) E(Y_{t_2}) + E(Y_{t_1}) E(X_{t_2}) + R_Y(t_1 - t_2) \text{ (since } \{X_t\}, \{Y_t\} \text{ are uncorrelated)} \\
=& R_X(t_1 - t_2) + R_Y(t_1 - t_2) + 2 m_X m_Y
\end{aligned}$$

Since $E(Z_t)$ does not depend on t and $R_Z(t_1, t_2)$ depends only on $t_1 - t_2$, $\{Z_t\}$ is a wide-sense stationary random process.