Lecture 7

Goals:

- Error probability for arbitrary signals, filter, threshold
- Determine the optimal threshold, filter, signals for a binary communications problem
- Suboptimum receivers: intersymbol interference (ISI)

Schwartz's inequality:

Let $f(t)$ and $g(t)$ be any (finite energy) and real functions. Let

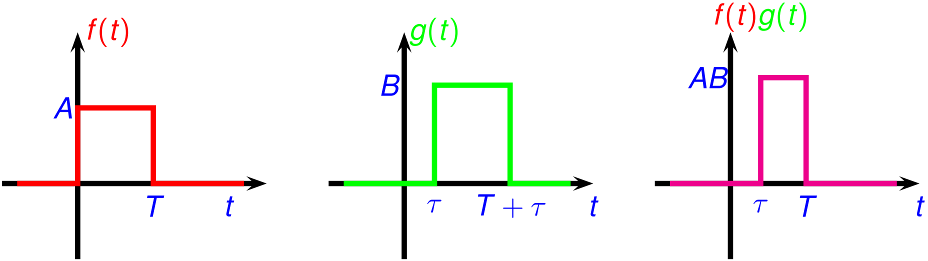
$$||f||^2 = \int f^2(t)dt, \quad (f,g) = \int f(t)g(t)dt$$

Claim:

$$-||f|| \; ||g|| \quad \leq \quad (f,g) \leq ||f|| \; ||g||$$

Schwartz's inequality: Example

Consider $f(t) = p_T(t)$ and $g(t) = p_T(t-\tau)$, $0 \le \tau \le T$.



$$||f||^2 = \int f^2(t)dt = A^2 T, \quad ||g||^2 = \int g^2(t)dt = B^2 T$$

$$(f,g) = \int f(t)g(t)dt = AB(T-\tau), 0 \le \tau \le T$$

$$-||f|| \cdot ||g|| \le (f,g) \le ||f|| \cdot ||g|| \Longrightarrow -ABT \le AB(T-\tau) \le ABT$$

Schwartz's inequality:

Proof:

For any real α

$$\begin{aligned} ||\alpha g - f||^2 \geq 0 \\ \alpha^2 ||g||^2 - 2\alpha(f,g) + ||f||^2 \geq 0. \end{aligned}$$

Since the polynomial in α is never negative there must be either no zeros or a double zero. Thus the discriminant ($b^2 - 4ac$) must be not be positive.

$$\begin{aligned} 4(f,g)^2 - 4||f||^2||g||^2 &\leq 0 \\ -||f|| \, ||g|| &\leq (f,g) \leq ||f|| \, ||g|| \end{aligned}$$

Equality occurs when $f(x) = Kg(x)$. If K is positive the inequality on the right side becomes equality and if K is negative the inequality on the right side becomes equality. This is Schwartz's inequality.

Arithmetic mean $\geq$ Geometric mean:

Claim:

Let a_0 and a_1 be real *nonnegative* numbers. Then

$$\frac{a_0 + a_1}{2} \quad \geq \quad \sqrt{a_0 a_1} \text{ with equaltiy if } a_0 = a_1.$$

Proof:

$$\begin{aligned}
(a_0 - a_1)^2 &\geq 0 \text{ with equaltiy if } a_0 = a_1 \\
a_0^2 - 2a_0 a_1 + a_1^2 &\geq 0 \\
a_0^2 + 2a_0 a_1 + a_1^2 &\geq 4a_0 a_1 \\
(a_0 + a_1)^2 &\geq 4a_0 a_1 \\
a_0 + a_1 &\geq 2\sqrt{a_0 a_1} \\
\frac{a_0 + a_1}{2} &\geq \sqrt{a_0 a_1} \text{ with equaltiy if } a_0 = a_1.
\end{aligned}$$

Gaussian Distribution

- If X is a Gaussian random variable with mean 0 and variance 1 then

$$\begin{aligned} P\{X \leq x\} &= \Phi(x) \\ P\{X > x\} &= 1 - \Phi(x) = Q(x). \end{aligned}$$

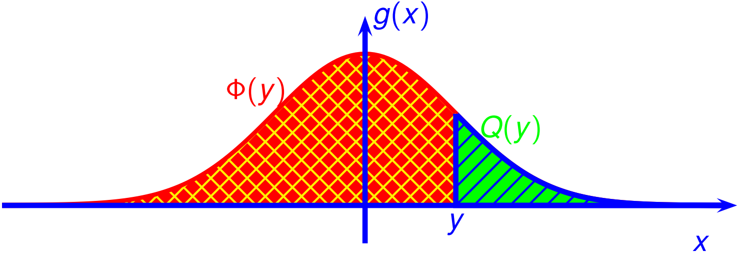
- If Y is a Gaussian random variable with mean μ and variance 1 then

$$\begin{aligned} P\{Y \leq y\} &= \Phi(y - \mu) \\ P\{Y > y\} &= 1 - \Phi(y - \mu) = Q(y - \mu). \end{aligned}$$

- If Z is a Gaussian random variable with mean μ and variance σ^2 then

$$\begin{aligned} P\{Z \leq z\} &= \Phi(\frac{z - \mu}{\sigma}) \\ P\{Z > z\} &= 1 - \Phi(\frac{z - \mu}{\sigma}) = Q(\frac{z - \mu}{\sigma}). \end{aligned}$$

Gaussian Distributions



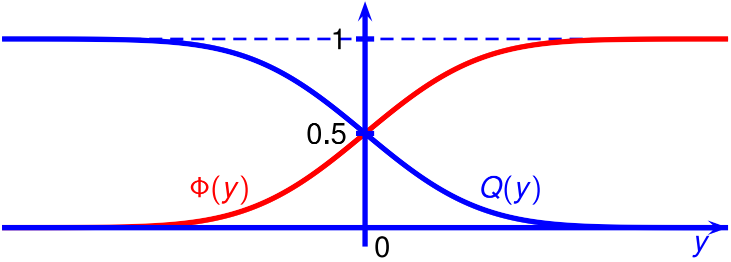
$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$Q(y) = \int_{y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$Q(y) = \Phi(-y)$$

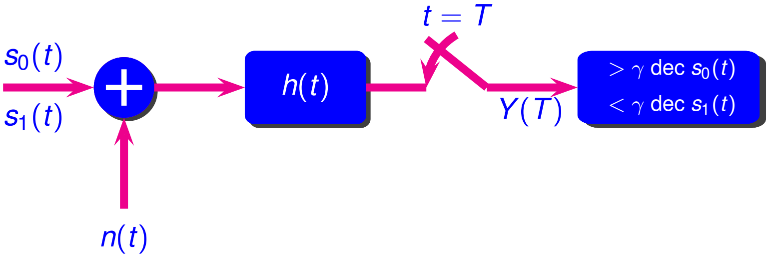
Gaussian Distributions



$$
\begin{aligned}
Q(y) &= 1 - \Phi(y) \\
Q(y) &= \Phi(-y) \\
\frac{\partial Q(y)}{\partial u} &= -\frac{1}{\sqrt{2\pi}} e^{-y^2/2} \frac{\partial y}{\partial u}
\end{aligned}
$$

Minimum Average Error Probability

Given: Transmitting one of two signals with probability π_0, π_1 over an additive white Gaussian noise channel with PSD $N_0/2$.
Problem: Find the optimum filter, threshold and signals to minimize the average error probability.



Minimum Average Error Probability

$$\begin{aligned} P_{e,0} &= P\{\text{error}|s_0 \text{ transmitted}\}. \\ P_{e,1} &= P\{\text{error}|s_1 \text{ transmitted}\}. \\ \pi_0 &= \text{Probability } s_0 \text{ transmitted.} \\ \pi_1 &= \text{Probability } s_1 \text{ transmitted.} \end{aligned}$$

$(\pi_0 + \pi_1 = 1)$.

The average probability of error is

$$\bar{P}_e = P_{e,0}\pi_0 + P_{e,1}\pi_1. \tag{1}$$

Analysis of error probability

- Receiver is a linear system.
- Determine the output of the receiver due to signal alone.
- Determine the output due to noise alone.
- Determine the probability of error.

Signal Alone Output

Let

$$\begin{aligned}\hat{s}_0(T) &= \int_{-\infty}^{\infty} h(T-\tau)s_0(\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)s_0(T-\tau)d\tau \leftarrow \text{output due to } s_0 \text{ alone,} \\ \hat{s}_1(T) &= \int_{-\infty}^{\infty} h(T-\tau)s_1(\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)s_1(T-\tau)d\tau \leftarrow \text{output due to } s_1 \text{ alone,}\end{aligned}$$

- Since we assume that the receiver will decide s_0 if the output of the filter is larger than a threshold and s_1 if it is smaller, we need to assume that $\hat{s}_0(T) > \hat{s}_1(T)$.
- Note, we are considering real signals (not complex) and real filters (not complex) so the output of the filter is real.

Noise Alone Output

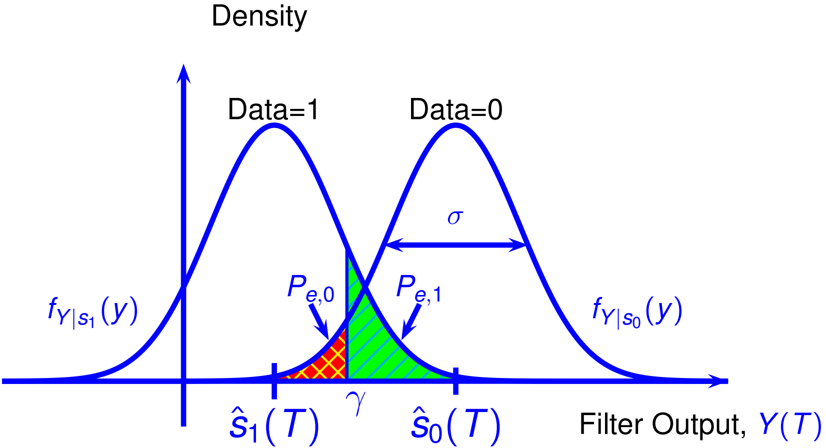
- The noise at the input is assumed to be WGN with PSD $N_0/2$.
- The output due to noise is $\eta = \int h(T-\tau)n(\tau)d\tau$.
- The mean of the output is $E[\eta] = 0$.
- The variance of the output is $\sigma_n^2 = \mathrm{Var}[\eta] = \frac{N_0}{2}\int h^2(\tau)d\tau$.
- The distribution of the noise is Gaussian.

Signal and Noise Output

$$Y(T) = \int h(\tau)\left(s_0(T-\tau) + n(T-\tau)\right)d\tau$$

- If $s_0(t)$ is transmitted the output is $Y(T) = \hat{s}_0(T) + \eta$.
- If $s_1(t)$ is transmitted the output is $Y(T) = \hat{s}_1(T) + \eta$.

Filter Output with Noise



- The outputs $\hat{s}_0(T)$ and $\hat{s}_1(T)$ depend on $h(t)$, $s_0(t)$ and $s_1(t)$.
- The variance of the noise depends on $h(t)$.

Error Probability Analysis

$$P_{e,0} = P\left\{Y(T) < \gamma | s_0 \text{ transmitted}\right\}.$$

If s_0 is transmitted then $Y(T)$ takes the form

$$Y(T) = \hat{s}_0(T) + \eta$$

where η is a Gaussian random variable with mean 0 and variance σ_N^2;

$$\sigma_N^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df.$$

time

Error Probability Analysis

Thus

$$
\begin{aligned}
P_{e,0} &= P\left\{\hat{s}_0(T)+\eta<\gamma\right\} \\
&= P\left\{\eta<\gamma-\hat{s}_0(T)\right\} \\
&= \Phi\left(\frac{\gamma-\hat{s}_0(T)}{\sigma_N}\right)=Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right). \qquad (2) \\
P_{e,1} &= P\left\{Y(T)>\gamma|s_1 \text{ transmitted}\right\} \\
&= P\left\{\hat{s}_1(T)+\eta>\gamma|s_1 \text{ transmitted}\right\} \\
&= P\left\{\eta>\gamma-\hat{s}_1(T)\right\} \\
&= Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right). \qquad (3)
\end{aligned}
$$

Substituting (2) and (3) into (1) yields

$$
\bar{P}_e(\gamma,h(t),s_0(t),s_1(t))=\pi_0 Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)+\pi_1 Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right). \qquad (4)
$$

Overall Optimization

The problem is to minimize the error probability over all choices of $\gamma, h(t)$ and $s_0(t), s_1(t)$.

Problem:

$$\min_{\gamma, h(t), s_0(t), s_1(t)} \pi_0 Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)+\pi_1 Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right).$$

$$\begin{aligned}
\hat{s}_0(T) &= \int h(\tau)s_0(T-\tau)d\tau = \int H(f)S_0(f)e^{j2\pi fT}df \\
\hat{s}_1(T) &= \int h(\tau)s_1(T-\tau)d\tau = \int H(f)S_1(f)e^{j2\pi fT}df \\
\sigma_N^2 &= \frac{N_0}{2}\int h^2(t)dt = \frac{N_0}{2}\int |H(f)|^2 df
\end{aligned}$$

Step 1: Minimize $\bar{P}_e$ over γ

Problem

$$P_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) = \min_{\gamma} \pi_0 Q\left(\frac{\hat{s}_0(T) - \gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma_N}\right).$$

Step 1: Minimize $\bar{P}_e$ over γ

Facts used:

$$
\begin{aligned}
Q(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, \\
Q'(x) &= \frac{-e^{-x^2/2}}{\sqrt{2\pi}}.
\end{aligned}
$$

Method: Set the derivative of $\bar{P}_e$ with respect to γ equal to 0.

$$
\begin{aligned}
\frac{d\bar{P}_e}{d\gamma} &= \pi_0 \left(\frac{-\exp\{-\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)^2/2\}}{\sqrt{2\pi}} \left(-\frac{1}{\sigma_N}\right) \right) \\
&\quad + \pi_1 \left(\frac{-\exp\{-\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)^2/2\}}{\sqrt{2\pi}} \left(\frac{1}{\sigma_N}\right) \right) = 0
\end{aligned}
$$

Step 1: Minimize $\bar{P}_e$ over γ

$$\pi_0 \exp\{-\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)^2/2\} = \pi_1 \exp\{-\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)^2/2\}$$

$$\exp\left\{\left[(\gamma-\hat{s}_1(T))^2-(\hat{s}_0(T)-\gamma)^2\right]/2\sigma_N^2\right\} = \frac{\pi_1}{\pi_0}$$

$$\gamma^2 - 2\gamma\hat{s}_1(T) + \hat{s}_1^2(T) - \hat{s}_0^2(T) + 2\gamma\hat{s}_0(T) - \gamma^2 = 2\sigma_N^2 \ln\frac{\pi_1}{\pi_0}$$

$$2\gamma\left[\hat{s}_0(T)-\hat{s}_1(T)\right] = 2\sigma_N^2\ln\frac{\pi_1}{\pi_0} + \hat{s}_0^2(T) - \hat{s}_1^2(T)$$

$$\gamma = \frac{\sigma_N^2\ln\frac{\pi_1}{\pi_0} + \frac{\hat{s}_0^2(T)-\hat{s}_1^2(T)}{2}}{\hat{s}_0(T)-\hat{s}_1(T)}$$

$$\boxed{\gamma_{opt} = \frac{\sigma_N^2\ln\frac{\pi_1}{\pi_0}}{\hat{s}_0(T)-\hat{s}_1(T)} + \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2}.} \qquad (5)$$

Step 1: Minimize $\bar{P}_e$ over γ

Special Case: If $\pi_1 = \pi_0 = 1/2$ then

$$\gamma_{opt} = \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}.$$

What is $\bar{P}_e$ for the optimal threshold?

$$\begin{aligned}
\hat{s}_0(T) - \gamma_{opt} &= \hat{s}_0(T) - \left[\frac{\sigma_N^2 \ln \frac{\pi_1}{\pi_0}}{\hat{s}_0(T) - \hat{s}_1(T)} + \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2} \right] \\
&= \frac{\hat{s}_0(T) - \hat{s}_1(T)}{2} - \frac{\sigma_N^2}{\hat{s}_0(T) - \hat{s}_1(T)} \ln \frac{\pi_1}{\pi_0} \\
\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} &= \frac{\hat{s}_0(T) - \hat{s}_1(T)}{2\sigma_N} - \frac{\sigma_N}{\hat{s}_0(T) - \hat{s}_1(T)} \ln \frac{\pi_1}{\pi_0}. \qquad (6)
\end{aligned}$$

Step 1: Minimize $\bar{P}_e$ over γ

Definition:

$$\begin{aligned}
(f(t), g(t)) &\stackrel{\Delta}{=} \int_{-\infty}^{\infty} f(t)g(t)dt \\
s_T(t) &\stackrel{\Delta}{=} s_0(T-t) - s_1(T-t) \\
\hat{s}_0(T) - \hat{s}_1(T) &= \int_{-\infty}^{\infty} h(\tau)\left[s_0(T-\tau) - s_1(T-\tau)\right] d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)s_T(\tau)d\tau = (h, s_T).
\end{aligned}$$

Thus from (5)

$$\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} = \frac{(h, s_T)}{2\sigma_N} - \frac{\sigma_N}{(h, s_T)} \ln \frac{\pi_1}{\pi_0}. \tag{7}$$

Similarly

$$\frac{\gamma_{opt} - \hat{s}_1(T)}{\sigma_N} = \frac{(h, s_T)}{2\sigma_N} + \frac{\sigma_N}{(h, s_T)} \ln \frac{\pi_1}{\pi_0}. \tag{8}$$

Step 1: Minimize $\bar{P}_e$ over γ

Remember that

$$\sigma_N^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt \tag{9}$$

$$= \frac{N_0}{2}\|h\|^2 \qquad (\|h\|^2 = \int_{-\infty}^{\infty} h^2(t)dt).$$

$$\text{Let } \lambda \stackrel{\Delta}{=} \frac{(h, s_T)}{\|h\|\ \|s_T\|} \tag{10}$$

Step 1: Minimize $\bar{P}_e$ over γ

$$
\begin{aligned}
\|s_T\|^2 &= \int_{-\infty}^{\infty} [s_0(T-t) - s_1(T-t)]^2\, dt \\
&= \int_{-\infty}^{\infty} s_0^2(T-t) - 2s_0(T-t)s_1(T-t) + s_1^2(T-t)dt \\
&= \int_{-\infty}^{\infty} s_0^2(t)dt - 2(s_0, s_1) + \int_{-\infty}^{\infty} s_1^2(t)dt \\
&= E_0 + E_1 - 2r\bar{E}. \\
r &= (s_0, s_1)/\bar{E}, \quad \bar{E} = \frac{E_0 + E_1}{2}.
\end{aligned}
$$

$$
\|s_T\|^2 = 2\bar{E}(1-r) \Rightarrow \|s_T\| = \sqrt{2\bar{E}(1-r)}. \qquad (11)
$$

Step 1: Minimize $\bar{P}_e$ over γ

Combining (6), (7), (9), (10), and (11)

$$\begin{aligned}\frac{\hat{s}_0(T)-\gamma_{opt}}{\sigma_N} &= \frac{(h,s_T)}{2\sqrt{\frac{N_0}{2}}\|h\|} - \frac{\sqrt{\frac{N_0}{2}}\|h\|}{(h,s_T)}\ln\frac{\pi_1}{\pi_0} \\ &= \frac{(h,s_T)}{\|h\|\,\|s_T\|}\frac{\sqrt{2\bar{E}(1-r)}}{\sqrt{2N_0}} - \sqrt{\frac{N_0}{4\bar{E}(1-r)}}\frac{\|h\|\,\|s_T\|}{(h,s_T)}\ln\frac{\pi_1}{\pi_0}.\end{aligned}$$

Step 1: Minimize $\bar{P}_e$ over γ

Let $\lambda = \frac{(h, s_T)}{\|h\| \, \|s_T\|}$. Then

$$\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} = \lambda\alpha - \beta\frac{1}{\lambda},$$

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln\frac{\pi_1}{\pi_0}.$$

Similarly

$$\frac{\gamma_{opt} - \hat{s}_1(T)}{\sigma_N} = \lambda\alpha + \beta\frac{1}{\lambda}.$$

Summary of Step 1:

$$\gamma_{opt} = \frac{\sigma_N^2 \ln \frac{\pi_1}{\pi_0}}{\hat{s}_0(T) - \hat{s}_1(T)} + \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}.$$

$$\bar{P}_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) = \pi_0 Q\left(\lambda\alpha - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\lambda\alpha + \frac{\beta}{\lambda}\right).$$

$$\begin{aligned}
\lambda &= \frac{(h, s_T)}{\|h\| \, \|s_T\|}. \\
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln \frac{\pi_1}{\pi_0}. \\
\bar{E} &= \frac{E_0 + E_1}{2}, \quad r = (s_0, s_1)/\bar{E}.
\end{aligned}$$

Notes on Step 1:

- The optimum threshold depends on the apriori probabilities π_0 and π_1.
- If $\pi_0 = \pi_1$ then $\gamma_{opt} = (\hat{s}_0(T) + \hat{s}_1(T))/2$. That is, the threshold is the average output in the absence of noise.
- If $\pi_1 \gg \pi_0$ then $\ln \frac{\pi_1}{\pi_0} > 0$ and the threshold will increase making it more likely for the receiver to decide $s_1(t)$ as the transmitted signal.
- Similarly if $\pi_1 \ll \pi_0$ then $\ln \frac{\pi_1}{\pi_0} < 0$ and the threshold will decrease making it more likely for the receiver to decide $s_0(t)$ as the transmitted signal.

Step 2:

Find the optimal filter $h(t)$ to minimize the average probability of error

Problem

$$\begin{aligned} P_e(\gamma_{opt}, h_{opt}(t), s_0(t), s_1(t)) &= \min_h P_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) \\ &= \min_h \pi_0 Q(\left(\alpha\lambda - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\alpha\lambda + \frac{\beta}{\lambda}\right). \end{aligned}$$

Method: First show that $\bar{P}_e$ is an decreasing function of λ by showing the derivative is negative. Then find the h that maximizes λ (thus minimizing $\bar{P}_e$).

$$\begin{aligned} \bar{P}_e(h, s_0, s_1) &= \bar{P}_e(\gamma_{opt}, h, s_0, s_1) \\ &= \pi_0 Q(\left(\alpha\lambda - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\alpha\lambda + \frac{\beta}{\lambda}\right). \end{aligned}$$

Step 2:

$$
\begin{aligned}
\frac{\partial \bar{P}_e}{\partial \lambda} &= \pi_0 \left[-e^{-(\alpha\lambda - \frac{\beta}{\lambda})^2/2} \frac{1}{\sqrt{2\pi}} \left(\alpha + \frac{\beta}{\lambda^2} \right) \right] \\
&\quad + \pi_1 \left[-e^{-(\alpha\lambda + \frac{\beta}{\lambda})^2/2} \frac{1}{\sqrt{2\pi}} \left(\alpha - \frac{\beta}{\lambda^2} \right) \right] \\
&= -\pi_0 \left[\left(\alpha + \frac{\beta}{\lambda^2} \right) \exp\left\{ -\frac{1}{2} \left(\alpha^2\lambda^2 - 2\alpha\beta + \beta^2/\lambda^2 \right) \right\} \right] \frac{1}{\sqrt{2\pi}} \\
&\quad -\pi_1 \left[\left(\alpha - \frac{\beta}{\lambda^2} \right) \exp\left\{ -\frac{1}{2} \left(\alpha^2\lambda^2 + 2\alpha\beta + \beta^2/\lambda^2 \right) \right\} \right] \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

Step 2:

$$
\begin{aligned}
&= -\frac{\exp\left\{-\frac{1}{2}\left(\alpha^2\lambda^2+\beta^2/\lambda^2\right)\right\}}{\sqrt{2\pi}}\left[\pi_0\left(\alpha+\frac{\beta}{\lambda^2}\right)e^{\alpha\beta}+\pi_1\left(\alpha-\frac{\beta}{\lambda^2}\right)e^{-\alpha\beta}\right. \\
\alpha\beta &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}\sqrt{\frac{N_0}{4\bar{E}(1-r)}}\ln\frac{\pi_1}{\pi_0} \\
&= \frac{1}{2}\ln\frac{\pi_1}{\pi_0} = \ln\sqrt{\frac{\pi_1}{\pi_0}}. \\
e^{\alpha\beta} &= \sqrt{\frac{\pi_1}{\pi_0}}, \quad e^{-\alpha\beta} = \sqrt{\frac{\pi_0}{\pi_1}}. \\
\pi_0 e^{\alpha\beta} &= \sqrt{\pi_0\pi_1}, \quad \pi_1 e^{-\alpha\beta} = \sqrt{\pi_0\pi_1}. \\
\frac{d\bar{P}_e}{d\lambda} &= -e^{-1/2(\alpha^2\lambda^2+\beta^2/\lambda^2)}\left[\sqrt{\pi_0\pi_1}\left(\alpha+\frac{\beta}{\lambda^2}+\alpha-\frac{\beta}{\lambda^2}\right)\right] \\
&= -\frac{1}{\sqrt{2\pi}}e^{-1/2(\alpha^2\lambda^2+\beta^2/\lambda^2)}\sqrt{\pi_0\pi_1}(2\alpha).
\end{aligned}
$$

Step 2:

Since $\alpha > 0$, $\frac{d\bar{P}_e}{d\lambda} < 0$ so that $\bar{P}_e$ is minimized by maximizing λ.

$$\lambda = \frac{(h, s_T)}{\|h\| \; \|s_T\|}.$$

From Schwartz's inequality

$$-\|h\| \; \|s_T\| \leq (h, s_T) \leq \|h\| \; \|s_T\|.$$

Thus $-1 \leq \lambda \leq 1$ with equality if $h = s_T$. Choose $\lambda = 1$ $(h = s_T = s_0(T-t) - s_1(T-t))$. For optimal threshold and optimal filter

$$\bar{P}_e(\gamma_{opt}, h_{opt}, s_0, s_1) = \pi_0 Q(\alpha - \beta) + \pi_1 Q(\alpha + \beta).$$

$h(t) = s_0(T-t) - s_1(T-t)$ is called the matched filter because it is matched to the signals.

$$\gamma_{opt}(h_{opt}, s_0, s_1) = \frac{1}{2}(E_0 - E_1) + \frac{1}{2} N_0 \ln \frac{\pi_1}{\pi_0}.$$

Step 2:

For the optimal filter the outputs due to signal alone are

$$\hat{s}_0(T) = E_0 - r\bar{E}$$

$$\hat{s}_1(T) = r\bar{E} - E_1.$$

If $\pi_0 = \pi_1$ then $\bar{P}_e = Q(\alpha) = Q\left(\sqrt{\frac{\bar{E}(1-r)}{N_0}}\right)$.

Step 3:

Find the optimal signals $s_0(t)$ and $s_1(t)$ to minimize the average probability of error.

Method: $\bar{P}_e$ depends on the signal only through $\bar{E}$ and r.

$$\left(\bar{E} = \frac{E_0 + E_1}{2}, \quad r = (s_0, s_1)/\bar{E}\right).$$

It is obvious that we could just increase the energy to infinity and get error probability 0. Instead we will fix $\bar{E}$ and vary the signals to vary r. Again we show that $\bar{P}_e$ is an increasing function of r and then choose the signals to minimize r.

Step 3:

$$
\begin{aligned}
\bar{P}_e &= \pi_0 Q(\alpha - \beta) + \pi_1 Q(\alpha + \beta). \\
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln \frac{\pi_1}{\pi_0}. \\
\frac{d\bar{P}_e}{dr} &= \pi_0 \left[\frac{-e^{-(\alpha-\beta)^2/2}}{\sqrt{2\pi}} (\frac{\partial \alpha}{\partial r} - \frac{\partial \beta}{\partial r}) \right] + \pi_1 \left[\frac{-e^{-(\alpha+\beta)^2/2}}{\sqrt{2\pi}} (\frac{\partial \alpha}{\partial r} + \frac{\partial \beta}{\partial r}) \right] \\
&= -e^{-(\alpha^2+\beta^2)/2} \left[\pi_0 e^{\alpha\beta} \left(\frac{\partial \alpha}{\partial r} - \frac{\partial \beta}{\partial r} \right) + \pi_1 e^{-\alpha\beta} \left(\frac{\partial \alpha}{\partial r} + \frac{\partial \beta}{\partial r} \right) \right] \\
&= -e^{-(\alpha^2+\beta^2)/2} \left[\sqrt{\pi_0 \pi_1} 2 \frac{\partial \alpha}{\partial r} \right].
\end{aligned}
$$

Step 3:

$$
\begin{aligned}
\frac{\partial \alpha}{\partial r} &= \sqrt{\frac{\bar{E}}{N_0}} \frac{1}{2} \left(\frac{-1}{\sqrt{1-r}} \right) < 0 \\
&\Rightarrow \frac{d\bar{P}_e}{dr} > 0
\end{aligned}
$$

Step 3:

From Schwartz's inequality

$$r = \frac{(s_0, s_1)}{\bar{E}} \geq \frac{-\|s_0\| \, \|s_1\|}{\bar{E}}$$

with equality if $s_0 = -Ks_1, \quad K > 0$. For $s_0 = -Ks_1$

$$\begin{aligned} r &= -\frac{\sqrt{E_0 E_1}}{\left(\frac{E_0+E_1}{2}\right)} \\ &\geq -1 \end{aligned}$$

with equality if $E_0 = E_1$. (Arithmetic mean $\geq$ Geometric mean).

Step 3:

Two signals $s_0(t)$ and $s_1(t)$ are said to be antipodal if

$$s_0(t) = -s_1(t).$$

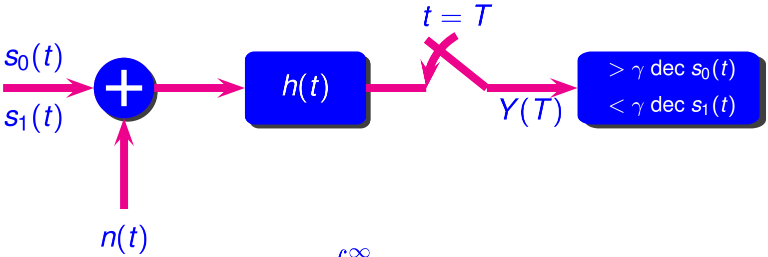
Optimal signals are antipodal.

$$r = -1 \quad \Rightarrow \quad \alpha = \sqrt{\frac{2E}{N_0}},$$
$$\beta = \sqrt{\frac{N_0}{8E}} \ln \frac{\pi_1}{\pi_0}.$$

If $\pi_1 = \pi_0 = 1/2$ then

$$\bar{P}_e = Q(\alpha) = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

Summary



$$\begin{aligned}
\hat{s}_0(T) &= \int_{-\infty}^{\infty} h(\tau)s_0(T-\tau)d\tau. \\
\hat{s}_1(T) &= \int_{-\infty}^{\infty} h(\tau)s_1(T-\tau)d\tau. \\
\sigma_N^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} h(\tau)d\tau = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df. \\
\bar{P}_e(\gamma, h(t), s_0(t), s_1(t)) &= \pi_0 Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right).
\end{aligned}$$

Summary

Step 1: Optimize with respect to γ.

$$
\begin{aligned}
\bar{P}_e(\gamma_{opt}, h, s_0, s_1) &= \pi_0 Q\left(\alpha\lambda - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\alpha\lambda + \frac{\beta}{\lambda}\right). \\
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \\
\beta &= \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln\frac{\pi_1}{\pi_0}, \\
\lambda &= \frac{(h, s_T)}{\|h\|\,\|s_T\|}, \\
s_T(t) &= s_0(T-t) - s_1(T-t), \\
\gamma_{opt} &= \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2} + \frac{\sigma_N^2}{\hat{s}_0(T) - \hat{s}_1(T)} \ln\frac{\pi_1}{\pi_0}.
\end{aligned}
$$

Summary

Step 2: Optimize with respect to $h(t)$.

$$
\begin{aligned}
\bar{P}_e(\gamma_{opt}, h_{opt}, s_0, s_1) &= \pi_0 Q(\alpha - \beta) + \pi_1 Q(\alpha + \beta). \\
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln \frac{\pi_1}{\pi_0}. \\
h_{opt}(t) &= s_0(T-t) - s_1(T-t), \ \textbf{the matched filter.} \\
\gamma_{opt}\big|_{h=h_{opt}} &= \frac{1}{2}(E_0 - E_1) + \frac{1}{2} N_0 \ln \frac{\pi_1}{\pi_0}.
\end{aligned}
$$

Summary

Step 3: Optimize with respect to s_0 and s_1.

$$
\begin{aligned}
\bar{P}_e(\gamma_{opt}, h_{opt}, s_{0,opt}, s_{1,opt}) &= \pi_0 Q\left(\hat{\alpha} - \hat{\beta}\right) + \pi_1 Q\left(\hat{\alpha} + \hat{\beta}\right). \\
\hat{\alpha} &= \sqrt{\frac{2\bar{E}}{N_0}}, \\
\hat{\beta} &= \sqrt{\frac{N_0}{8\bar{E}}} \ln \frac{\pi_1}{\pi_0}. \\
s_0(t) &= -s_1(t). \\
h_{opt}(t)\big|_{\substack{s_0 = s_{0,opt} \\ s_1 = s_{1,opt}}} &= 2s_0(T-t). \\
\gamma_{opt}\big|_{h=h_{opt}, s_{0,opt}, s_{1,opt}} &= \frac{1}{2}(N_0) \ln \frac{\pi_1}{\pi_0}.
\end{aligned}
$$

Special Case

$$
\begin{aligned}
\pi_0 &= \pi_1 = 1/2 \\
\bar{P}_e &= 1/2 Q\left(\frac{\hat{s}_0(T) - \gamma}{\sigma_N}\right) + 1/2 Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma_N}\right)
\end{aligned}
$$

Special Case $\pi_0 = \pi_1$

Step 1: Optimize with respect to γ

$$
\begin{aligned}
\bar{P}_e &= Q(\alpha\lambda), \\
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}} \quad \lambda = \frac{(h, s_T)}{\|h\| \, \|S_T\|} \\
\bar{E} &= \frac{E_0 + E_1}{2}, \quad r = (s_0, s_1)/\bar{E} \\
E_0 &= \int_{-\infty}^{\infty} s_0^2(t)dt \quad E_1 = \int_{-\infty}^{\infty} s_1^2(t)dt \\
\gamma_{opt} &= \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}
\end{aligned}
$$

Special Case $\pi_0 = \pi_1$

Step 2: Optimize with respect to $h(t)$

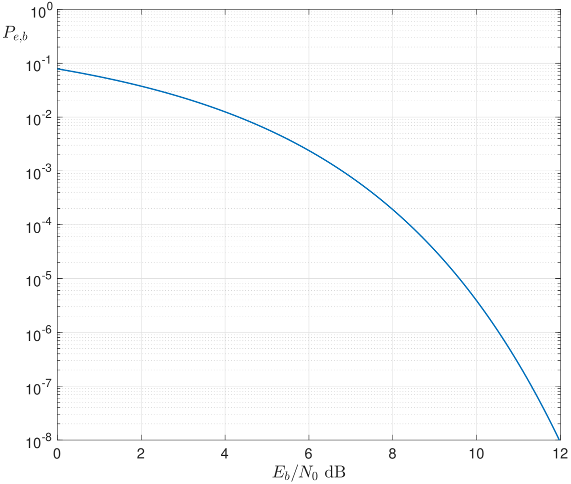
$$
\begin{aligned}
\bar{P}_e &= Q(\alpha), \quad \alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}} \\
h_{opt} &= s_0(T-t) - s_1(T-t) \text{ matched filter} \\
\gamma_{opt} &= 1/2(E_0 - E_1)
\end{aligned}
$$

Special Case $\pi_0 = \pi_1$

Step 3: Optimize with respect to $s_0(t)$ and $s_1(t)$.

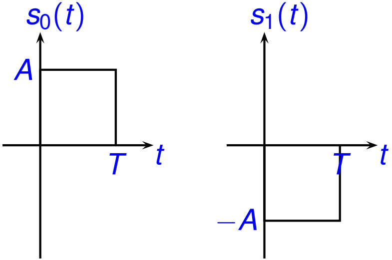
$$
\begin{aligned}
\bar{P}_e &= Q(\hat{\alpha}) \\
\hat{\alpha} &= \sqrt{\frac{2\bar{E}}{N_0}} \\
s_0(t) &= -s_1(t) \\
h_{opt} &= 2s_0(T-t), \\
\gamma_{opt} &= 0.
\end{aligned}
$$

Error Probability for Antipodal Signals (e.g. BPSK)



Example: Antipodal Rectangular Pulses

Assume equally likely signals ($\pi_0 = \pi_1$).

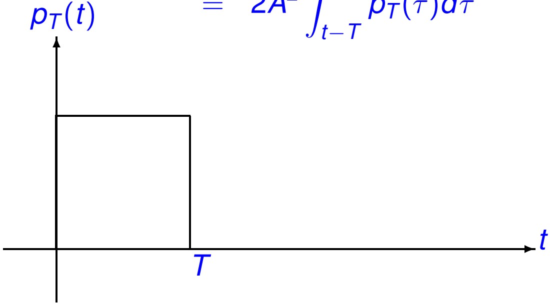


$$\begin{aligned} s_0(t) &= Ap_T(t) \\ s_1(t) &= -Ap_T(t) \\ \gamma_{opt} &= 0 \\ h_{opt}(t) &= 2Ap_T(t) \end{aligned}$$

Example: Antipodal Rectangular Pulses

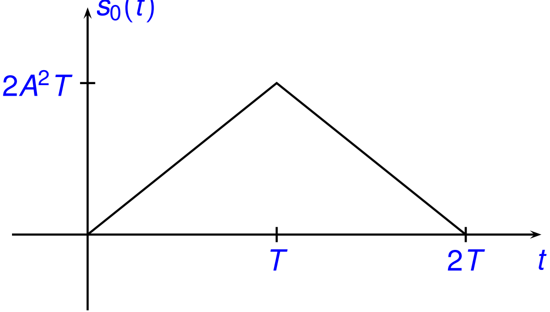
Assume $s_0(t)$ transmitted

$$\begin{aligned}
\int_{-\infty}^{\infty} h(t-\tau)s_0(\tau) &= \int_{-\infty}^{\infty} 2Ap_T(t-\tau)Ap_T(\tau)d\tau \\
&= 2A^2\int_{-\infty}^{\infty} p_T(t-\tau)p_T(\tau)d\tau \\
&= 2A^2\int_{t-T}^{t} p_T(\tau)d\tau
\end{aligned}$$



Example: Antipodal Rectangular Pulses

The output due to signal alone:



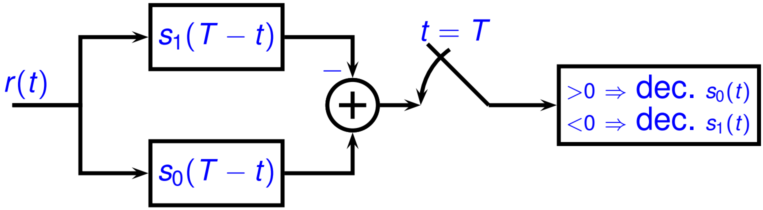
The output due to noise is a Gaussian random variable with mean zero and variance

$$\sigma_N^2 = \frac{N_0}{2}||h||^2 = \frac{N_0}{2}(4A^2T) = 2A^2N_0T$$

Example: Antipodal Rectangular Pulses

Let T_0 be the sampling time. Since the signal out is a maximum when $T_0 = T$ and the noise variance does not depend on the sample time the optimum sampling time is $T_0 = T$.

Equivalent form of optimal receiver



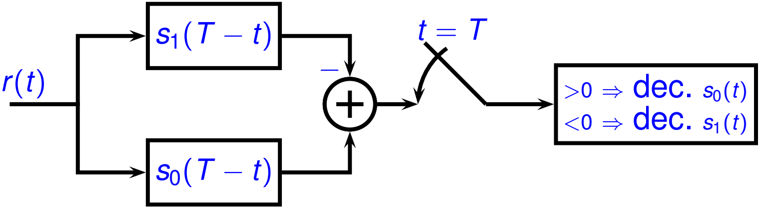
Example: Antipodal Rectangular Pulses

$$
\begin{aligned}
Z(t) &= \int_{-\infty}^{\infty} h(t-\tau) r(\tau) d\tau, \\
& \qquad h(t) = s_0(T-t) - s_1(T-t) \\
&= \int_{-\infty}^{\infty} r(\tau) \left[s_0(T-(t-\tau)) - s_1(T-(t-\tau)) \right] d\tau \\
&= \int_{-\infty}^{\infty} r(\tau) \left[s_0(\tau+T-t) - s_1(\tau+T-t) \right] d\tau \\
Z(T) &= \int_{-\infty}^{\infty} r(\tau) \left[s_0(\tau) - s_1(\tau) \right] d\tau
\end{aligned}
$$

If $s_0(t)$ and $s_1(t)$ are time limited to $[0, T]$ then

$$
Z(T) = \int_0^T r(\tau) \left[s_0(\tau) - s_1(\tau) \right] d\tau
$$

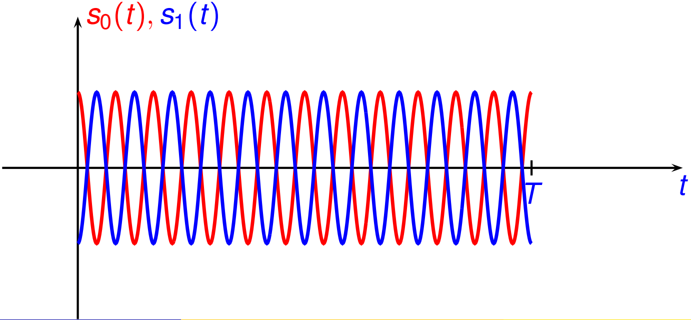
Example: Antipodal Rectangular Pulses



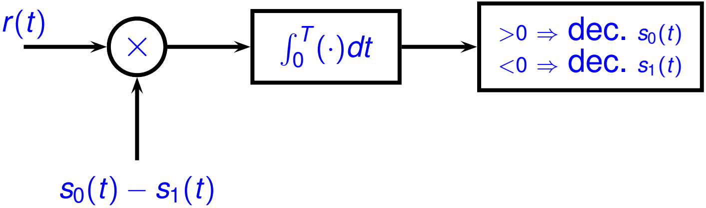
This is called the “correlation receiver.”

Example: Binary Phase Shift Keying (RF signals)

$$
\begin{aligned}
s_0(t) &= A\cos(2\pi f_c t)p_T(t) \\
s_1(t) &= -s_0(t) \\
s_i(t) &= (-1)^i A\cos(2\pi f_c t)p_T(t) \\
&= A\cos(2\pi f_c t + i\pi)p_T(t)
\end{aligned}
$$



Example: Binary Phase Shift Keying (RF signals)



Example: Binary Phase Shift Keying (RF signals)

Assume $f_c T >> 1$ or $2\pi f_c T = 2n\pi$

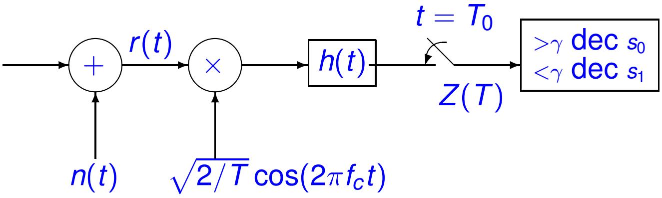
$$
\begin{aligned}
E_i = \int_{-\infty}^{\infty} s_i^2(t)dt &= \int_0^T A^2 \cos^2(2\pi f_c t)dt \\
&= A^2 \int_0^T 1/2 + 1/2\cos(2\pi 2 f_c t)dt \\
&= \frac{A^2 T}{2}[1 + \frac{\sin(2\pi 2 f_c T)}{2\pi 2 f_c T}] \\
&= \frac{A^2 T}{2}.
\end{aligned}
$$

$$
P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{A^2 T}{N_0}}\right) \quad (\pi_1 = \pi_0, \gamma = 0).
$$

Suboptimal Receivers: BPSK

Suppose that $h(t)$ in the figure below is not the matched filter (namely a filter with impulse response that is a rectangular pulse) but a more easily implementable filter.

$$s_i(t) = (-1)^i \sqrt{2P} \cos(2\pi f_c t) p_T(t), \quad f_c T >> 1$$



Example: Binary Phase Shift Keying (RF signals)

$$P_e = 1/2Q\left(\frac{\hat{s}_0(T_0)}{\sigma_N}\right) + 1/2Q\left(\frac{-\hat{s}_1(T_0)}{\sigma_N}\right)$$

Claim: $\sigma_n^2 = N_0\|h\|^2/(2T)$.

Proof:

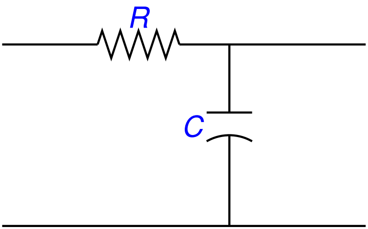
$$\begin{aligned}
\sigma_n^2 &= (2/T)E[\int\int h(\tau-t)n(t)\cos(2\pi f_c t)h(\tau-s)n(s)\cos(2\pi f_c s)dtds] \\
&= (2/T)\int\int E[n(t)n(s)]\cos(2\pi f_c t)h(\tau-t)\cos(2\pi f_c s)h(\tau-s)dtds \\
&= (2/T)\int\int \frac{N_0}{2}\delta(t-s)\cos(2\pi f_c t)h(\tau-t)\cos(2\pi f_c s)h(\tau-s)dtds \\
&= (2/T)\int \frac{N_0}{2}\cos^2(2\pi f_c t)h^2(\tau-t)dt \\
&= \frac{N_0}{2T}\int h^2(\tau-t)(1+\cos(2\pi 2 f_c t))dt \\
&= \frac{N_0}{2T}\int_{-\infty}^{\infty} h^2(\tau-t)dt = \frac{N_0}{2T}\int_{-\infty}^{\infty} h^2(s)ds = \frac{N_0}{2}(||h||^2/T)
\end{aligned}$$

provided that $f_c T >> 1$ or $2\pi 2 f_c T = n\pi$ ($f_c T = n/4$ for some integer).

Example: Binary Phase Shift Keying (RF signals)

$$
\begin{aligned}
\hat{s}_i(T_0) &= \int_{-\infty}^{\infty} h(T_0-\tau)(-1)^i\sqrt{2P}\sqrt{2/T}\cos^2(2\pi f_c\tau)p_T(\tau)d\tau \\
&= (-1)^i\sqrt{P/T}\int_0^T \underbrace{h(T_0-\tau)}_{\text{low pass}}[1+\underbrace{\cos(2\pi 2f_c\tau)}_{\text{double freq.}}]d\tau \\
&= (-1)^i\sqrt{PT}\left[\frac{1}{T}\int_0^T h(T_0-\tau)d\tau\right] \\
P_e &= Q\left(\frac{|\hat{s}_i(T_0)|}{\sigma_N}\right)
\end{aligned}
$$

Example: Single pole RC filter



$$
\begin{aligned}
h(t) &= \frac{1}{RC} e^{-t/RC} u(t) \\
&= \alpha e^{-\alpha t} u(t), \ \alpha = 1/RC \\
\|h\|^2 &= \int_{-\infty}^{\infty} h^2(t) dt = \int_0^{\infty} \alpha^2 e^{-2\alpha t} dt \\
&= \frac{\alpha^2}{-2\alpha} e^{-2\alpha t} |_0^{\infty} = \frac{\alpha}{2} \\
\sigma_N^2 &= \frac{N_0}{2} (\frac{\alpha}{2T})
\end{aligned}
$$

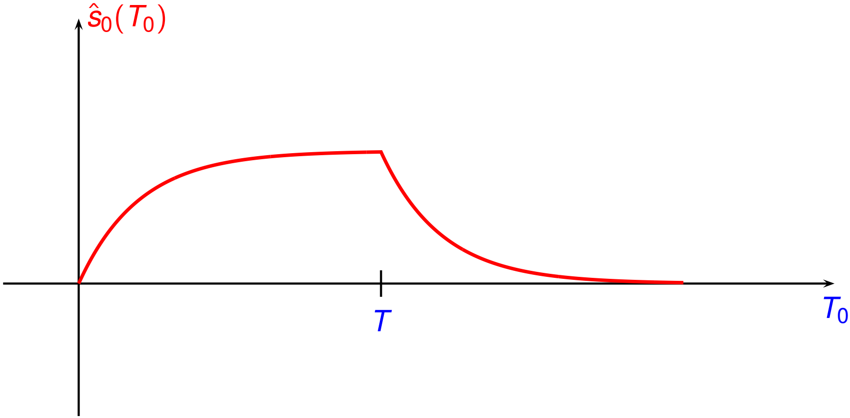
The larger RC the smaller the bandwidth of the filter (takes more time to charge capacitor). Small RC means large α. So large α means large bandwidth. Small α small bandwidth.

Example: Single pole RC filter

$$\begin{aligned} \int_0^T h(T_0-\tau)d\tau &= \int_0^T \alpha e^{-\alpha(T_0-\tau)} u(T_0-\tau)d\tau \\ u(T_0-\tau) &= \begin{cases} 0, & T_0-\tau<0 \quad (\tau>T_0) \\ 1, & T_0-\tau>0 \quad (\tau<T_0) \end{cases} \end{aligned}$$

$$\begin{aligned} \int_0^T h(T_0-\tau)d\tau &= \begin{cases} 0, & T_0<0 \\ \int_0^{T_0} \alpha e^{-\alpha(T_0-\tau)}d\tau, & 0\le T_0\le T \\ \int_0^{T} \alpha e^{-\alpha(T_0-\tau)}d\tau, & T_0>T \end{cases} \\ &= \begin{cases} 0 & T_0<0 \\ +e^{-\alpha(T_0-\tau)}\big|_0^{T_0}, & 0\le T_0\le T \\ +e^{-\alpha(T_0-\tau)}\big|_0^{T}, & T_0\ge T \end{cases} \\ &= \begin{cases} 0, & T_0<0 \\ 1-e^{-\alpha T_0}, & 0\le T_0\le T \\ (1-e^{-\alpha T})e^{-\alpha T_0-T}, & T_0>T \end{cases} \end{aligned}$$

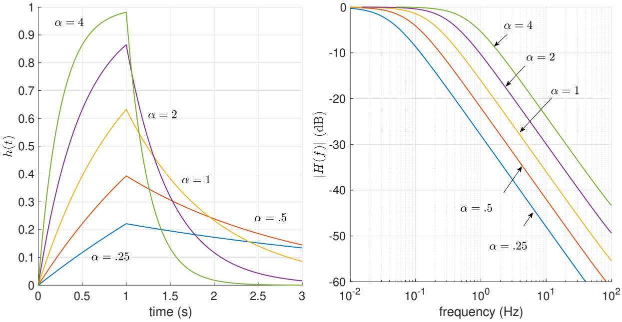
Example: Single pole RC filter



$$\begin{aligned}\hat{s}_0(T) &= \sqrt{PT}(1-e^{-\alpha T})/T \\ &= \sqrt{E}(1-e^{-\alpha T})/T\end{aligned}$$

Signal at Output of Filter

Consider a data waveform containing a positive pulse followed by a negative pulse. The left curve below shows the response of the filter to such a waveform. The right plot shows the frequency response of the filter.

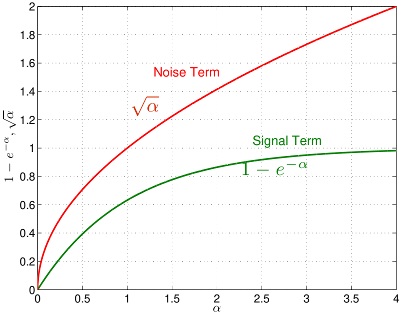


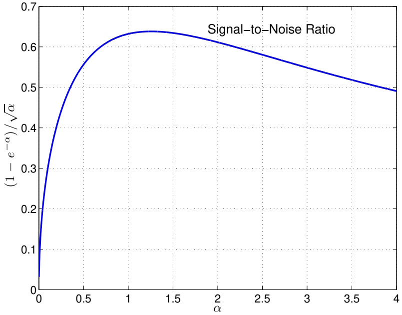
Signal and Noise

- Output due to signal at sampling time varies with α (bandwidth of filter).
- Output due to noise varies with α (bandwidth of filter).
- We would like to maximize $|\hat{s}_0(T_0)|/\sigma_n^2$.
- Since $|\hat{s}_0(T_0)|$ is maximized at $T_0 = T$ and $\sigma_N^2 = \frac{N_0}{4}\frac{\alpha}{2}$ does not depend on the sampling time, the optimal sampling time is $T_0 = T$.
- This results in a signal-to-noise ratio of

$$\begin{aligned} SNR &= \frac{(\hat{s}_0(T))}{\sigma_N} = \frac{\sqrt{E}(1-e^{-\alpha T})/T}{\sqrt{\alpha/(2T)}\sqrt{\frac{N_0}{2}}} \\ &= \sqrt{\frac{2E}{N_0}}\frac{(1-e^{-\alpha T})}{T\sqrt{\alpha/(2T)}} = \sqrt{\frac{4E}{N_0}}\frac{(1-e^{-\alpha T})}{\sqrt{\alpha T}} \end{aligned}$$

Optimization of Filter Parameter





Optimization of Filter Parameter

Goal: Maximize SNR with respect to α. Let $x = \alpha T$.

$$\begin{aligned} f(x) &= (1 - e^{-x})(x)^{-1/2} \\ f'(x) &= e^{-x}x^{-1/2} - 1/2(1 - e^{-x})x^{-3/2} = 0 \\ & xe^{-x} - 1/2(1 - e^{-x}) = 0 \\ & e^{-x}(x + 1/2) = 1/2 \\ & x + 1/2 = e^{x}/2 \\ & 2x = e^{x} - 1 \end{aligned}$$

Can only solve this numerically at $x = 1.256$.

Optimization of Filter Parameter

- We can numerically solve this to get $x = 1.256$.
- So $\alpha = \frac{1.256}{T}, \Rightarrow RC = .7962T$.
- This yields a signal-to-noise ratio of

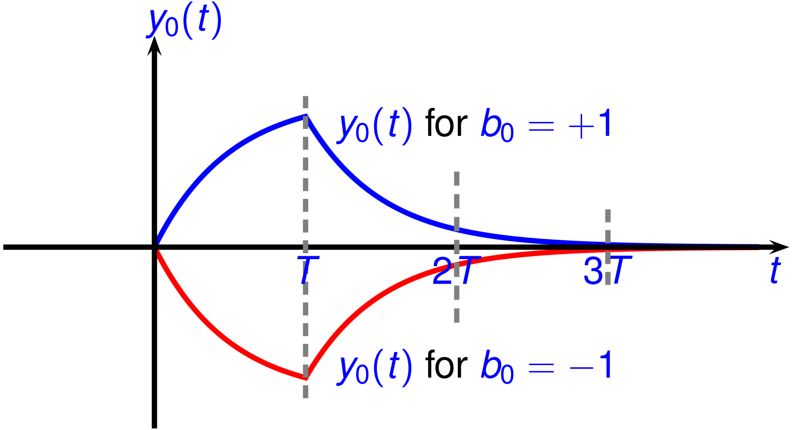
$$SNR = \sqrt{\frac{2E}{N_0}}\sqrt{2}(0.6382)$$

- Signal-to-noise ratio with the (optimum) matched filter is

$$SNR = \sqrt{\frac{2E}{N_0}}$$

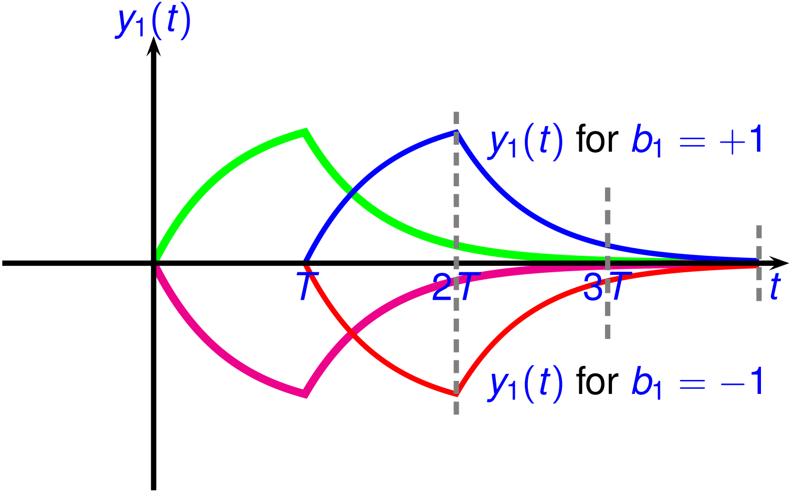
- Loss due to suboptimal receiver = $-10\log_{10}((0.6832\sqrt{2})^2)$=<u>0.89 dB</u>.
- This analysis ignores intersymbol interference.

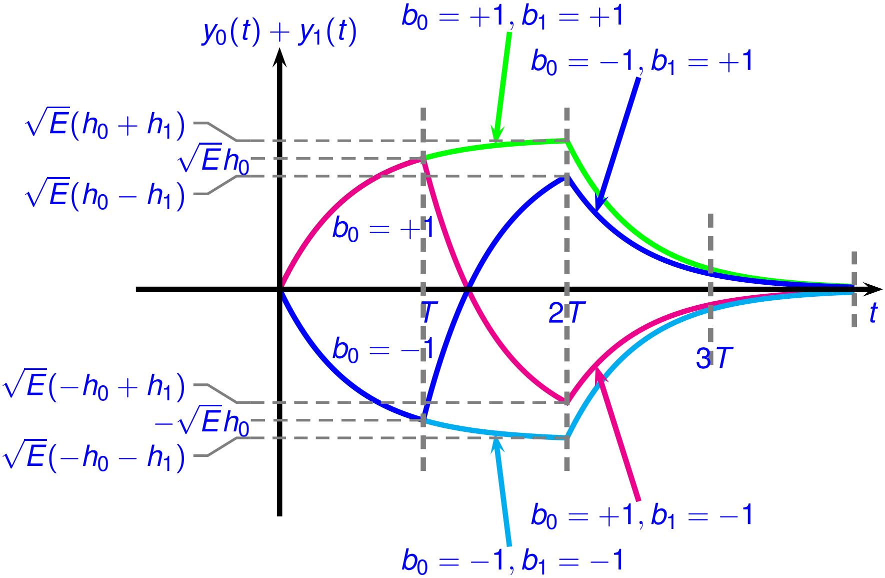
Output due to a single pulse



$$
\begin{aligned}
y(lT) &= \begin{cases} b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T}), & l=1 \\ b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{-\alpha(l-1)T}, & l\geq 2 \end{cases} \\
&= \begin{cases} b_0\sqrt{E}h_0, & l=1 \\ b_0\sqrt{E}h_{l-1}, & l\geq 2. \end{cases}
\end{aligned}
$$

Output due to a second single pulse



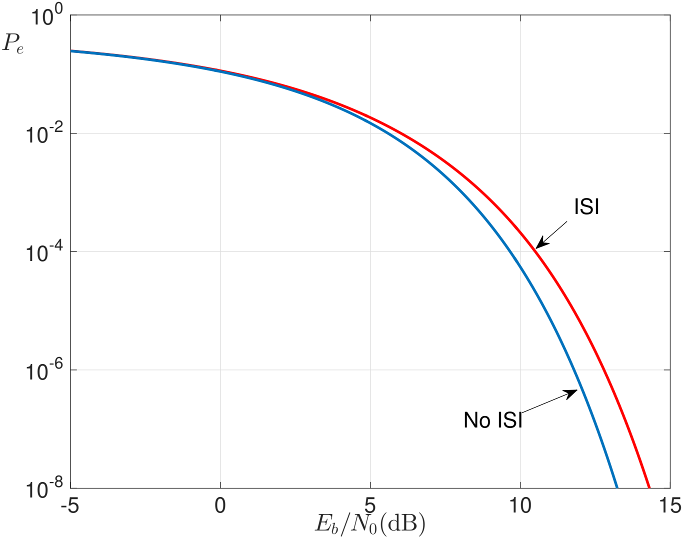

$y_0(t) + y_1(t)$
$b_0 = +1, b_1 = +1$
$b_0 = -1, b_1 = +1$
$\sqrt{E}(h_0 + h_1)$
$\sqrt{E}h_0$
$\sqrt{E}(h_0 - h_1)$
$b_0 = +1$
$b_0 = -1$
T
$2T$
$3T$
t
$\sqrt{E}(-h_0 + h_1)$
$-\sqrt{E}h_0$
$\sqrt{E}(-h_0 - h_1)$
$b_0 = +1, b_1 = -1$
$b_0 = -1, b_1 = -1$

$$y(2T) = b_0\sqrt{E}h_1 + b_1\sqrt{E}h_0 + \eta_1$$

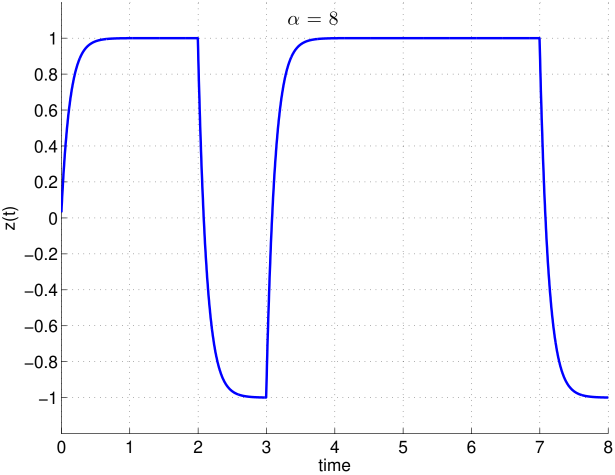
$$\begin{aligned} h_0 &= \sqrt{2/(\alpha T)}(1 - e^{-\alpha T}) \\ h_1 &= \sqrt{2/(/\alpha T)}(1 - e^{-\alpha T})e^{-\alpha T}. \end{aligned}$$

$$
\begin{aligned}
P_{e,+1}(1) &= P\{\text{error for bit } b_1 | b_1 = +1\} \\
&= P\{y(2T) < 0 | b_1 = +1\} \\
&= P\{b_0\sqrt{E}h_1 + b_1\sqrt{E}h_0 + \eta_2 < 0 | b_1 = +1\} \\
&= P\{b_0\sqrt{E}h_1 + b_1\sqrt{E}h_0 + \eta_2 < 0 | b_0 = +1, b_1 = +1\}\frac{1}{2} \\
&\quad + P\{b_0\sqrt{E}h_1 + b_1\sqrt{E}h_0 + \eta_2 < 0 | b_0 = -1, b_1 = +1\}\frac{1}{2} \\
&= P\{\sqrt{E}(h_1 + h_0) + \eta_1 < 0 | b_0 = +1, b_1 = +1\}\frac{1}{2} \\
&\quad + P\{\sqrt{E}(-h_1 + h_0) + \eta_1 < 0 | b_0 = -1, b_1 = +1\}\frac{1}{2} \\
&= P\{\eta_1 < \sqrt{E}(-h_0 - h_1)\}\frac{1}{2} + P\{\eta_1 < \sqrt{E}(-h_0 + h_1)\}\frac{1}{2} \\
&= \frac{1}{2}Q(\frac{\sqrt{E}}{\sigma}(h_0 + h_1)) + \frac{1}{2}Q(\frac{\sqrt{E}}{\sigma}(h_0 - h_1)) \\
&= \frac{1}{2}Q(\sqrt{\frac{2E}{N_0}}(h_0 + h_1)) + \frac{1}{2}Q(\sqrt{\frac{2E}{N_0}}(h_0 - h_1)).
\end{aligned}
$$

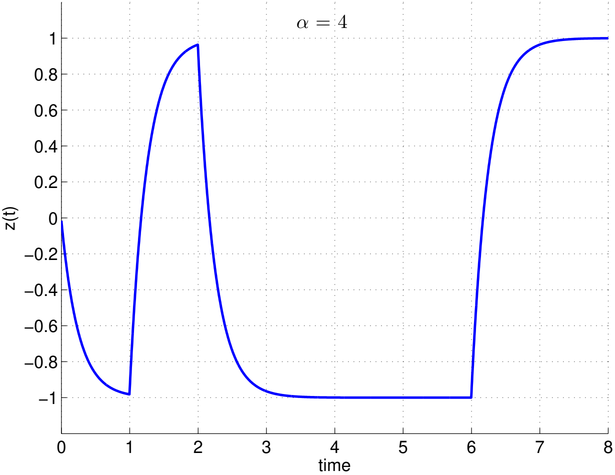
- Sometimes because of intersymbol interference the output is larger than the output due to a single pulse ($h_0 + h_1$).
- Sometimes because of intersymbol interference the output is larger than the output due to a single pulse ($h_0 - h_1$).
- Because the Q function is convex the average error probability will be larger than the error probability without intersymbol interference.


P_e
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
-5
0
5
10
15
$E_b/N_0(\mathrm{dB})$
ISI
No ISI

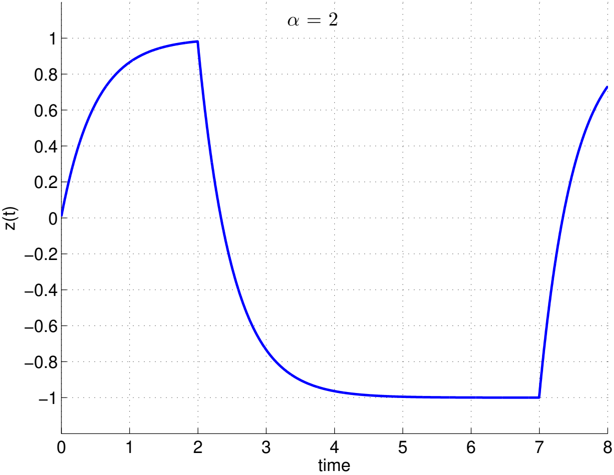
Intersymbol Interference ($T = 1$)



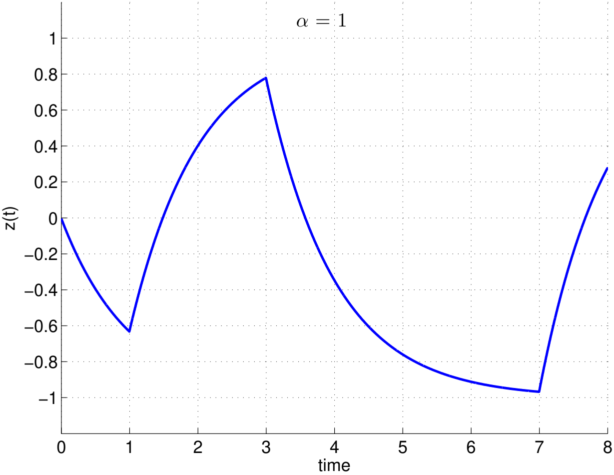
Intersymbol Interference ($T = 1$)



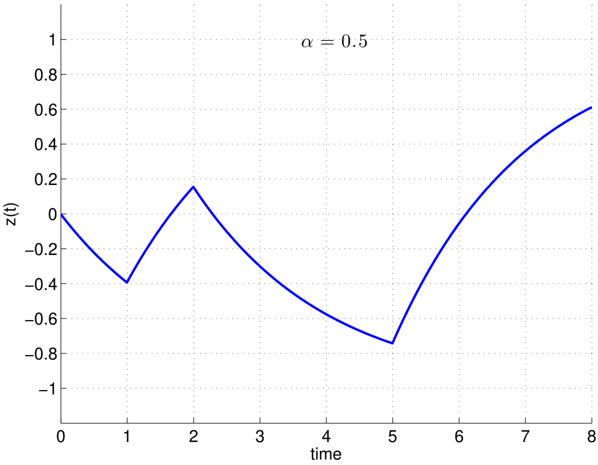
Intersymbol Interference ($T = 1$)



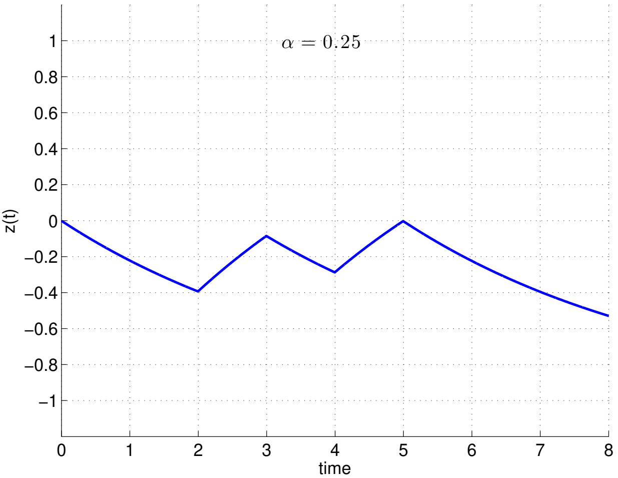
Intersymbol Interference ($T = 1$)



Intersymbol Interference ($T = 1$)



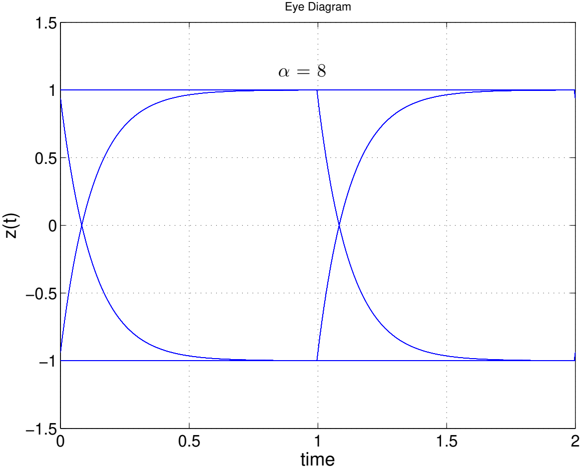
Intersymbol Interference ($T = 1$)



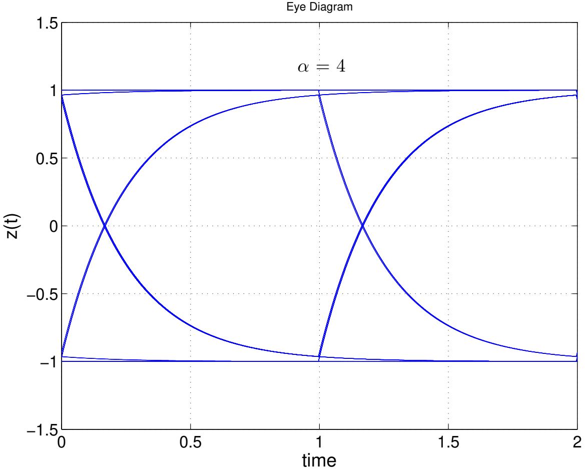
Intersymbol Interference

- The smaller α the more intersymbol interference.

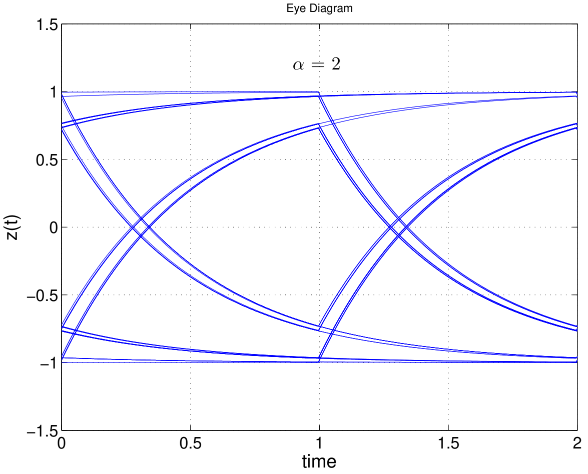
Intersymbol Interference ($T = 1$)



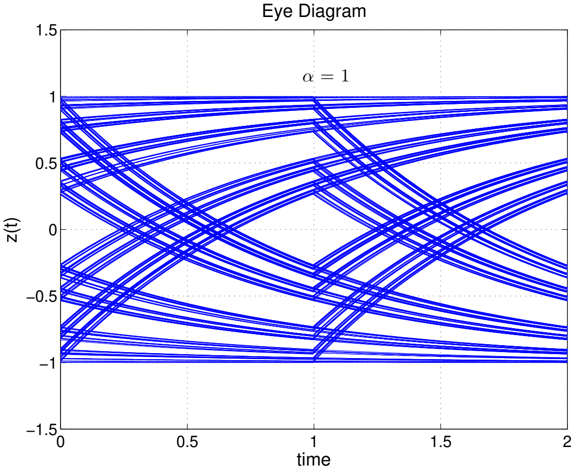
Intersymbol Interference



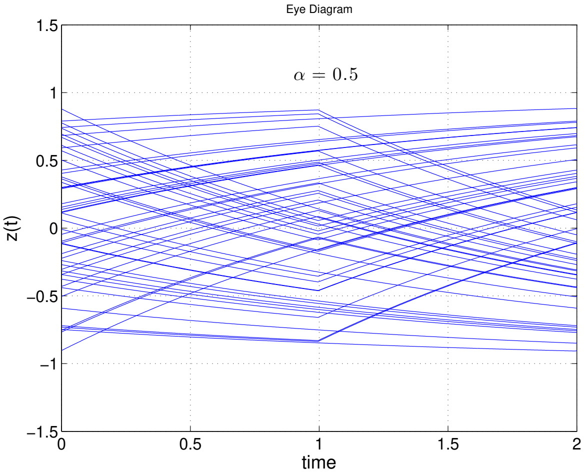
Intersymbol Interference



Intersymbol Interference



Intersymbol Interference



Intersymbol Interference

$$
\begin{aligned}
Z(T) &= \underbrace{b_0\sqrt{E}(1-e^{-\alpha T})/T}_{\textit{Desired Signal}} \\
&+ \underbrace{b_{-1}\sqrt{E}(1-e^{-\alpha T})e^{-\alpha T}/T + b_{-2}\sqrt{E}(1-e^{-\alpha T})e^{-\alpha 2T}/T + \cdots}_{\textit{Intersymbol Interference}} \\
&+ \underbrace{\eta}_{\textit{Noise}}
\end{aligned}
$$

Intersymbol Interference

$$\alpha = \frac{\Delta f}{f_H}$$

- The larger α the smaller the amount of inter symbol interference.
- The larger α the larger amount of noise that gets through the filter
- There is an optimal α (different from what we previously found that ignored inter symbol interference) that makes the error probability small.