EECS501: Solution to Homework 5

1. **(a)** For $y \geq 1$,

$$\begin{aligned} P(Y \leq y) &= P(1/X \leq y) && (1) \\ &= P(X \geq 1/y) && (2) \\ &= 1 - 1/y && (3) \end{aligned}$$

so $f_Y(y) = F_Y'(y) = 1/y^2$, $y \geq 1$, and $f_Y(y) = 0$ otherwise.

(b) For $y \geq 1$,

$$\begin{aligned} P(Y \leq y) &= P(1/\sqrt{X} \leq y) && (4) \\ &= P(X \geq 1/y^2) && (5) \\ &= 1 - 1/y^2 && (6) \end{aligned}$$

so $f_Y(y) = F_Y'(y) = 2/y^3$, $y \geq 1$, and $f_Y(y) = 0$ otherwise.

(c) For $y \leq 0$,

$$\begin{aligned} P(Y \leq y) &= P(\ln X \leq y) && (7) \\ &= P(X \leq e^y) && (8) \\ &= e^y && (9) \end{aligned}$$

so $f_Y(y) = F_Y'(y) = e^y$ for $y \leq 0$, and $f_Y(y) = 0$ otherwise.

(d)

$$P(Y \leq y) = P(2^{-y} \leq F_X(X)) = P(F_X^{-1}(2^{-y}) \leq X) = 1 - F_X(F_X^{-1}(2^{-y})) \quad (10)$$

$$= 1 - 2^{-y} = 1 - e^{-y \log_e(2)}. \quad (11)$$

so Y is exponential.

2. **Waiting times**

Let X be your total waiting time, and Y the total waiting time after navigating the menus. Then $X = 1 + Y$, so $E[X] = 1 + E[Y]$, by linearity of expectation.

To find $E[Y]$, let's partition the sample space into events

$$A := \{\text{your first operator is competent}\} \tag{12}$$

$$B := \{\text{your first operator is incompetent}\}. \tag{13}$$

By the law of total expectation,

$$E[Y] = 0.6E[Y|A] + 0.4E[Y|B].$$

Let Z be the time until you finish with your first operator. We can write $Z = U + V$, where U is the time spent waiting for the operator, and V the time spent talking to the operator. When A occurs, we finish after one time through, and so

$$E[Y|A] = E[Z|A] = E[U|A] + E[V|A] = 2 + 5 = 7,$$

where we used linearity of (conditional) expection, the fact that $E[U|A] = E[U]$, and the formula for the mean of an exponential. When B occurs, we can apply recursion:

$$E[Y|B] = E[Y] + E[Z|B] = E[Y] + E[U|B] + E[V|B] = E[Y] + 2 + 10 = E[Y] + 12.$$

Therefore

$$E[Y] = 0.6 \times 7 + 0.4 \times (E[Y] + 12)$$

which implies $E[Y] = 15$ minutes, and thus $E[X] = 16$ minutes.

3. **Transformation of Uniform**

CDF of Y is $F_Y(y)$:

$$F_Y(y) = \begin{cases} \frac{y^2}{2} & 0 \le y \le 1 \\ -\frac{y^2}{2} + 3y - \frac{7}{2} & 2 \le y \le 3 \\ \frac{1}{2} & 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow F_Y^{-1}(x) = g(x) = \begin{cases} \sqrt{2x} & 0 \le x \le \frac{1}{2} \\ 3 - \sqrt{2(1-x)} & \frac{1}{2} < x \le 1 \end{cases}$$

4. **Joint Probability Density**

(a)

$$P(X > Y) = \int_{y=0}^{1} \int_{x=y}^{1} \frac{6}{7}(x^2 + \frac{xy}{2})dxdy = \frac{15}{56}$$

(b)

$$f_Y(y) = \int_0^1 \frac{6}{7}(x^2 + \frac{xy}{2})dx = \frac{6}{7}(\frac{1}{3} + \frac{y}{4})$$

$$E(X|Y) = \frac{1}{f_Y(y)} \int_0^1 x \frac{6}{7}(x^2 + \frac{xy}{2}) dx = \frac{\frac{6}{7}(\frac{1}{3} + \frac{y}{6})}{\frac{6}{7}(\frac{1}{3} + \frac{y}{4})} = \frac{3+2y}{4+3y}$$