Pr. 1. (sol/hsj33)
We are given that $\|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ so $\boldsymbol{x}'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{C}^K$.
We *cannot* directly conclude from this equality that $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}$, however!
If we had $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y} = \boldsymbol{x}'\boldsymbol{B}\boldsymbol{y}$, $\forall \boldsymbol{x} \in \mathbb{F}^N, \boldsymbol{y} \in \mathbb{F}^M$, then it would be trivial to show that $\boldsymbol{A} = \boldsymbol{B}$, but that is not our situation here.
To see why some proof is needed, consider $\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{B} = (\boldsymbol{A} + \boldsymbol{A}')/2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For these two matrices, we have $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} = 2x_1x_2$, $\forall \boldsymbol{x} \in \mathbb{R}^2$, yet clearly $\boldsymbol{A} \neq \boldsymbol{B}$. Now you might argue that this situation here is different, because $\boldsymbol{Q}'\boldsymbol{Q}$ and $\boldsymbol{I}$ are Hermitian and we have $\forall \boldsymbol{x} \in \mathbb{C}^K$. Indeed it is different, but one must use these additional properties to *prove* the equality. Three proofs follow.

Approach 1
For $\boldsymbol{x} = \boldsymbol{e}_j$ we have $\boldsymbol{e}_j'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{e}_j = \boldsymbol{e}_j'\boldsymbol{e}_j = 1$, so the diagonal elements of $\boldsymbol{Q}'\boldsymbol{Q}$ are unity.
For $\boldsymbol{x} = \boldsymbol{e}_j + \boldsymbol{e}_k$ with $j \neq k$ we have $(\boldsymbol{e}_j + \boldsymbol{e}_k)'\boldsymbol{Q}'\boldsymbol{Q}(\boldsymbol{e}_j + \boldsymbol{e}_k) = (\boldsymbol{e}_j + \boldsymbol{e}_k)'(\boldsymbol{e}_j + \boldsymbol{e}_k) = 2$ so after expanding and simplifying: $\boldsymbol{e}_j'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{e}_k + \boldsymbol{e}_k'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{e}_j = 0$ *i.e.*, $\Re(\boldsymbol{e}_j'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{e}_k) = 0$, where $\Re(\cdot)$ denotes the real part of its argument.
Similarly, for $\boldsymbol{x} = \boldsymbol{e}_j + \imath\boldsymbol{e}_k$, where $\imath = \sqrt{-1}$, we find that $\Im(\boldsymbol{e}_j'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{e}_k) = 0$, where $\Im(\cdot)$ denotes the imaginary part of its argument. So the off-diagonal elements are $\boldsymbol{e}_j'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{e}_k = 0$ for $j \neq k$. So $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_K$.

Approach 2
We are given that $\forall \boldsymbol{x} \in \mathbb{C}^K$, $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = 0$ where $\boldsymbol{A} = \boldsymbol{Q}'\boldsymbol{Q} - \boldsymbol{I}$. Because this $\boldsymbol{A}$ is Hermitian, we can write $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}'$ where $\boldsymbol{V}$ is unitary and $\boldsymbol{\Lambda} = \operatorname{Diag}(\lambda_1, \ldots, \lambda_K)$ corresponds to real eigenvalues. Pick $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{e}_k$, then $0 = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \boldsymbol{e}_k'\boldsymbol{V}'\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}'\boldsymbol{V}\boldsymbol{e}_k = \lambda_k$ implying that $\lambda_k = 0$. This holds for every $k = 1, \ldots, K$, so $\boldsymbol{\Lambda} = \boldsymbol{0}$ and hence $\boldsymbol{A} = \boldsymbol{0}$ so $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_K$.

Approach 3
Taking $\boldsymbol{x} = \boldsymbol{e}_k$, we have $1 = \|\boldsymbol{x}\|_2 = \|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{Q}_{:,k}\|_2$ so $\boldsymbol{Q}$ has unit-norm columns.
Taking $\boldsymbol{x} = \boldsymbol{e}_k + \boldsymbol{e}_j$, for $k \neq j$, and using the fact that $\boldsymbol{Q}$ has unit-norm columns, we have $\|\boldsymbol{Q}_{:,k}\|_2^2 + \|\boldsymbol{Q}_{:,j}\|_2^2 = 1 + 1 = 2 = \|\boldsymbol{x}\|_2^2 = \|\boldsymbol{Q}\boldsymbol{x}\|_2^2 = \|\boldsymbol{Q}_{:,k} + \boldsymbol{Q}_{:,j}\|_2^2$, so the Pythagorean theorem says that $\boldsymbol{Q}_{:,k} \perp \boldsymbol{Q}_{:,j}$. Thus $\boldsymbol{Q}$ has orthonormal columns, so $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_K$.

Pr. 2. (sol/hs035)

(a) $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{v} : \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}\} \Rightarrow \mathcal{N}(\boldsymbol{A})$ is the space spanned by the basis vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\mathcal{R}(\boldsymbol{A}) = \{\boldsymbol{A}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{F}^2\} \Rightarrow \mathcal{R}(\boldsymbol{A})$ is the space spanned by the basis vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Here $\mathcal{N}(\boldsymbol{A}) \neq \mathcal{R}(\boldsymbol{A})$ and in general they are not equal. In fact they do not have even have the same dimension when $\boldsymbol{A}$ is rectangular!

However, there are some matrices for which $\mathcal{N}(\boldsymbol{A})$ and $\mathcal{R}(\boldsymbol{A})$ are equal, in the sense that they are the span of the same vector(s), such as $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, where $\mathcal{N}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}) = \operatorname{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$.

Pr. 3. (sol/hs033)

(a) Let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}'$. Then $\boldsymbol{C} = \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q} = \underbrace{\boldsymbol{W}\boldsymbol{U}}_{\triangleq \widetilde{\boldsymbol{U}}} \boldsymbol{\Sigma} \underbrace{\boldsymbol{V}'\boldsymbol{Q}}_{\triangleq \widetilde{\boldsymbol{V}}'} = \widetilde{\boldsymbol{U}}\boldsymbol{\Sigma}\widetilde{\boldsymbol{V}}'$, where $\widetilde{\boldsymbol{U}} \triangleq \boldsymbol{W}\boldsymbol{U}$ and $\widetilde{\boldsymbol{V}} \triangleq \boldsymbol{Q}'\boldsymbol{V}$ are unitary because they are the products of unitary matrices. In other words, $\widetilde{\boldsymbol{U}}\boldsymbol{\Sigma}\widetilde{\boldsymbol{V}}'$ is an SVD of $\boldsymbol{C}$. The diagonal matrix $\boldsymbol{\Sigma}$ containing the singular values is the same for both $\boldsymbol{A}$ and $\boldsymbol{C}$, implying they have the same singular values and thus the same rank.

(b) *Solution 1*

Use the inequality $\operatorname{rank}(\boldsymbol{A}\boldsymbol{B}) \leq \min(\operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{B}))$ that generalizes readily (by induction) to the inequality $\operatorname{rank}(\boldsymbol{A}_1\boldsymbol{A}_2 \cdots \boldsymbol{A}_K) \leq \min(\operatorname{rank}(\boldsymbol{A}_1), \ldots, \operatorname{rank}(\boldsymbol{A}_K))$

Because $\boldsymbol{W}$ and $\boldsymbol{Q}$ are non-singular here, their rank equals their size, so:
$\operatorname{rank}(\boldsymbol{D}) = \operatorname{rank}(\boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}) \leq \min(\operatorname{rank}(\boldsymbol{W}), \operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{Q})) = \min(M, \operatorname{rank}(\boldsymbol{A}), N) = \operatorname{rank}(\boldsymbol{A})$
because $\operatorname{rank}(\boldsymbol{A}) \leq \min(M, N)$.

Similarly, by the invertibilty of $\boldsymbol{W}$ and $\boldsymbol{Q}$:
$\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{W}^{-1}\boldsymbol{D}\boldsymbol{Q}^{-1}) \le \min(\operatorname{rank}(\boldsymbol{W}^{-1}), \operatorname{rank}(\boldsymbol{D}), \operatorname{rank}(\boldsymbol{Q}^{-1})) = \min(M, \operatorname{rank}(\boldsymbol{D}), N) = \operatorname{rank}(\boldsymbol{D})$

Thus $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{D})$

Solution 2

We first show that $\operatorname{rank}(\boldsymbol{W}\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A})$ when $\boldsymbol{W}$ is invertible.

Let $r = \operatorname{rank}(\boldsymbol{A}) = \dim(\operatorname{span}(\boldsymbol{A}))$. Then by definition of $\dim(\cdot)$ there exists a $M \times r$ basis matrix $\boldsymbol{B}$ such that $\boldsymbol{y} \in \operatorname{span}(\boldsymbol{A}) \Rightarrow \boldsymbol{y} = \boldsymbol{B}\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{F}^r$.

Now we show that $\boldsymbol{W}\boldsymbol{B}$ is a basis matrix for $\operatorname{span}(\boldsymbol{W}\boldsymbol{A})$. First, if $\boldsymbol{z} \in \operatorname{span}(\boldsymbol{W}\boldsymbol{A})$, then $\boldsymbol{z} = \boldsymbol{W}\boldsymbol{A}\boldsymbol{x}$ for some $\boldsymbol{x} \in \mathbb{F}^N$ so $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{F}^r$ because $\boldsymbol{B}$ is a basis matrix for the column space of $\boldsymbol{A}$. Thus $\boldsymbol{z} = \boldsymbol{W}\boldsymbol{B}\boldsymbol{\alpha} \in \operatorname{span}(\boldsymbol{W}\boldsymbol{B})$.

We also need to show that $\boldsymbol{W}\boldsymbol{B}$ has linearly independent columns: $\boldsymbol{W}\boldsymbol{B}\boldsymbol{\alpha} = \boldsymbol{0} \Rightarrow \boldsymbol{B}\boldsymbol{\alpha} = \boldsymbol{W}^{-1}\boldsymbol{0} = \boldsymbol{0} \Rightarrow \boldsymbol{\alpha} = \boldsymbol{0}$ because $\boldsymbol{B}$ is a basis matrix and thus had linearly independent columns. So $\boldsymbol{W}\boldsymbol{B}$ spans $\boldsymbol{W}\boldsymbol{A}$ and has r linearly independent columns so $\operatorname{rank}(\boldsymbol{W}\boldsymbol{A}) = \dim(\operatorname{span}(\boldsymbol{W}\boldsymbol{A})) = r = \operatorname{rank}(\boldsymbol{A})$.

A similar argument shows that if $\boldsymbol{Q}$ is invertible, then $\operatorname{rank}(\boldsymbol{A}\boldsymbol{Q}) = \operatorname{rank}(\boldsymbol{A})$.

Combining yields $\operatorname{rank}(\boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}) = \operatorname{rank}(\boldsymbol{A})$ when both $\boldsymbol{W}$ and $\boldsymbol{Q}$ are invertible.

(c) Here is a simple example that shows that singular values of $\boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ and $\boldsymbol{A}$ need not be the same.
Take $\boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{W} = 2\boldsymbol{I}$ and $\boldsymbol{Q} = \boldsymbol{I}$. Singular values of $\boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ equal 2 while those of $\boldsymbol{A}$ equal 1.

Pr. 4. (sol/hs055)
Approach 1:
If $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{A}')$, then $\boldsymbol{A}'\boldsymbol{y} = \boldsymbol{0}$, so $\boldsymbol{x}'\boldsymbol{A}'\boldsymbol{y} = 0$ for any vector $\boldsymbol{x}$, so $(\boldsymbol{A}\boldsymbol{x})'\boldsymbol{y} = 0$ and $\boldsymbol{A}\boldsymbol{x} \perp \boldsymbol{y}$. Thus $\boldsymbol{y} \in \mathcal{R}^\perp(\boldsymbol{A})$ because $\mathcal{R}(\boldsymbol{A}) = \{\boldsymbol{A}\boldsymbol{x}\}$. This shows that $\mathcal{N}(\boldsymbol{A}') \subseteq \mathcal{R}(\boldsymbol{A})^\perp$.
Conversely, if $\boldsymbol{y} \in \mathcal{R}^\perp(\boldsymbol{A})$ then $\boldsymbol{y} \perp \boldsymbol{A}\boldsymbol{x}$ for any $\boldsymbol{x}$ so $0 = (\boldsymbol{A}\boldsymbol{x})'\boldsymbol{y} = \boldsymbol{x}'(\boldsymbol{A}'\boldsymbol{y})$ so choosing $\boldsymbol{x} = \boldsymbol{A}'\boldsymbol{y}$ we have $\|\boldsymbol{A}'\boldsymbol{y}\| = 0$ so $\boldsymbol{y} \in \mathcal{N}(\boldsymbol{A}')$. Thus $\mathcal{N}(\boldsymbol{A}') = \mathcal{R}^\perp(\boldsymbol{A})$.

Approach 2:
Let $\boldsymbol{A} \in \mathbb{R}_r^{m \times n}$. Then $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}'$ and $\boldsymbol{A}' = \boldsymbol{V}\boldsymbol{\Sigma}'\boldsymbol{U}'$. Using our discussion on the anatomy of the SVD:

$$\boldsymbol{A} = \Big[\underbrace{\boldsymbol{u}_1 \ \ldots \ \boldsymbol{u}_r}_{\text{basis for } \mathcal{R}(\boldsymbol{A})} \quad \underbrace{\boldsymbol{u}_{r+1} \ \ldots \ \boldsymbol{u}_m}_{\text{basis for } \mathcal{R}(\boldsymbol{A})^\perp}\Big] \boldsymbol{\Sigma} \Big[\underbrace{\boldsymbol{v}_1 \ \ldots \ \boldsymbol{v}_r}_{\text{basis for } \mathcal{N}(\boldsymbol{A})^\perp} \quad \underbrace{\boldsymbol{v}_{r+1} \ \ldots \ \boldsymbol{v}_n}_{\text{basis for } \mathcal{N}(\boldsymbol{A})}\Big]'$$

$$\boldsymbol{A}' = \Big[\underbrace{\boldsymbol{v}_1 \ \ldots \ \boldsymbol{v}_r}_{\text{basis for } \mathcal{R}(\boldsymbol{A}')} \quad \underbrace{\boldsymbol{v}_{r+1} \ \ldots \ \boldsymbol{v}_n}_{\text{basis for } \mathcal{R}(\boldsymbol{A}')^\perp}\Big] \boldsymbol{\Sigma}' \Big[\underbrace{\boldsymbol{u}_1 \ \ldots \ \boldsymbol{u}_r}_{\text{basis for } \mathcal{N}(\boldsymbol{A}')^\perp} \quad \underbrace{\boldsymbol{u}_{r+1} \ \ldots \ \boldsymbol{u}_m}_{\text{basis for } \mathcal{N}(\boldsymbol{A}')}\Big]'$$

We see that $\mathcal{R}^\perp(\boldsymbol{A}) = \operatorname{span}(\{\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m\}) = \mathcal{N}(\boldsymbol{A}')$.

Approach 3:
If $r = 0$ then $\boldsymbol{A} = \boldsymbol{0}$ and the result is obvious, so focus on the case where $r \ge 1$.
Using the compact SVD, we have $\boldsymbol{A} = \boldsymbol{U}_r\boldsymbol{\Sigma}_r\boldsymbol{V}_r'$ and $\boldsymbol{A}' = \boldsymbol{V}_r\boldsymbol{\Sigma}_r\boldsymbol{U}_r'$.
Because $\boldsymbol{\Sigma}_r$ is invertible and $\boldsymbol{V}_r$ has linearly independent columns, $\boldsymbol{A}'\boldsymbol{y} = \boldsymbol{0}$ iff $\boldsymbol{U}_r'\boldsymbol{y} = \boldsymbol{0}$. Thus, because $\boldsymbol{U} = \begin{bmatrix}\boldsymbol{U}_r & \boldsymbol{U}_0\end{bmatrix}$ is unitary, $\mathcal{N}(\boldsymbol{A}') = \boldsymbol{U}_0$.
Because $\boldsymbol{\Sigma}_r$ is invertible and $\mathcal{R}(\boldsymbol{V}') = \mathbb{F}^r$, we have $\mathcal{R}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{U}_r)$. Because $\boldsymbol{U} = \begin{bmatrix}\boldsymbol{U}_r & \boldsymbol{U}_0\end{bmatrix}$ is unitary, $\mathcal{R}^\perp(\boldsymbol{A}') = \mathcal{R}^\perp(\boldsymbol{U}_r) = \boldsymbol{U}_0$.

Pr. 5. (sol/hs065)
Approach 1
Define the $N \times N$ diagonal matrix $\boldsymbol{S} = \operatorname{Diag}(\operatorname{sign}(\lambda_i))$ and define the permutation matrix $\boldsymbol{P}$ such that it sorts the eigenvalues in descending order according to the magnitudes. Then write the unitary eigendecomposition of $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{S}'\boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{Q}' = \underbrace{\boldsymbol{Q}\boldsymbol{S}'\boldsymbol{P}'}_{\boldsymbol{U}}\underbrace{\boldsymbol{P}\boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{P}}_{\boldsymbol{\Sigma}}\underbrace{\boldsymbol{P}'\boldsymbol{Q}'}_{\boldsymbol{V}'}.$$

The matrix $\boldsymbol{U}$ is the product of three unitary matrices so it is unitary, and $\boldsymbol{\Sigma}$ is a diagonal matrix with nonnegative elements sorted in descending order, and $\boldsymbol{V}$ is also a product of two unitary matrices hence unitary. So those components satisfy the requirements of a SVD.

Approach 2

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' \qquad (\text{where } \boldsymbol{Q} = \begin{bmatrix}\boldsymbol{q}_1 & \dots & \boldsymbol{q}_N\end{bmatrix})$$

$$= \underbrace{\begin{bmatrix}\boldsymbol{q}_1 & \dots, \boldsymbol{q}_k & -\boldsymbol{q}_{k+1} & \dots & -\boldsymbol{q}_N\end{bmatrix}}_{\tilde{\boldsymbol{U}}} \underbrace{\begin{bmatrix}\lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & -\lambda_{k+1} & & \\ & & & & \ddots & \\ & & & & & -\lambda_n\end{bmatrix}}_{\tilde{\boldsymbol{\Sigma}}} \underbrace{\boldsymbol{Q}'}_{\tilde{\boldsymbol{V}}'} = \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{V}}'.$$

It is easily checked that $\tilde{\boldsymbol{U}}$ and $\tilde{\boldsymbol{V}}$ are unitary matrices, and that $\tilde{\boldsymbol{\Sigma}}$ is a diagonal matrix with real, positive entries along its diagonal. However, these diagonal elements are not necessarily in descending order. Let $\sigma_1, \sigma_2, \dots, \sigma_N$ be the sequence $\{\lambda_1, \dots, \lambda_k, -\lambda_{k+1}, \dots, -\lambda_N\}$ sorted in descending order. That is, $\sigma_1 = \max\{\lambda_1, \dots, \lambda_k, -\lambda_{k+1}, \dots, -\lambda_N\}$, $\sigma_2 = \max\{\lambda_1, \dots, \lambda_k, -\lambda_{k+1}, \dots, -\lambda_N\} \backslash \{\sigma_1\}$, and so on. Let $\boldsymbol{u}_i, \boldsymbol{v}_i$ denote the eigenvectors of $\boldsymbol{A}$ ordered such that

$$\boldsymbol{u}_i = \begin{cases} \boldsymbol{q}_j & \text{if } \sigma_i = \lambda_j, \quad j \in \{1, \dots, k\}, \\ -\boldsymbol{q}_j & \text{if } \sigma_i = -\lambda_j, \quad j \in \{k+1, \dots, N\}, \end{cases}$$

and

$$\boldsymbol{v}_i = \begin{cases} \boldsymbol{q}_j & \text{if } \sigma_i = \lambda_j, \quad j \in \{1, \dots, k\}, \\ \boldsymbol{q}_j & \text{if } \sigma_i = -\lambda_j, \quad j \in \{k+1, \dots, N\}. \end{cases}$$

Then an SVD of $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_N\end{bmatrix} \begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_N\end{bmatrix} \begin{bmatrix}\boldsymbol{v}_1 & \dots & \boldsymbol{v}_N\end{bmatrix}' = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}',$$

because $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary (they are just column rearranged versions of $\tilde{\boldsymbol{U}}$ and $\tilde{\boldsymbol{V}}$), and $\boldsymbol{\Sigma}$ is a diagonal matrix, with real, positive entries along the diagonal, sorted in descending order.

Here we have flipped the signs on the negative eigenvalues, and the eigenvectors corresponding to the negative eigenvalues in the left-most matrix of the expression $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$. Alternatively, one could change the signs of the appropriate vectors in the right-most matrix of the same expression.

Pr. 6. (sol/hsj21)

Repeating for reference the Venn diagram for eigendecompositions:

$\boldsymbol{A}_1$ Rectangular SVD: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}'$	$\boldsymbol{A}_2$ Square $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$	$\boldsymbol{A}_3$ Diagonalizable $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ iff $\boldsymbol{V}$ linearly ind.	$\boldsymbol{A}_4$ Normal $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}'$ $\boldsymbol{V}$ unitary $\sigma_n = \lvert\lambda_{(n)}\rvert$	$\boldsymbol{A}_5$ Hermitian $\boldsymbol{\Lambda}$ real	PSD: $\boldsymbol{A} \succeq \boldsymbol{0}$ $\boldsymbol{\Sigma} = \boldsymbol{\Lambda} \succeq \boldsymbol{0}$ $\boldsymbol{U} = \boldsymbol{V}$

$\boldsymbol{A}_2 = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ is square but not diagonalizable.

$\boldsymbol{A}_3 = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ is diagonalizable but not normal.

$\boldsymbol{A}_4 = \begin{bmatrix}\imath\end{bmatrix}$ is normal but not Hermitian

$\boldsymbol{A}_5 = \begin{bmatrix}-1\end{bmatrix}$ is Hermitian but not PSD

Pr. 7. (sol/hs015)

(a) Here $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}'$ is written in outer product form. We can express $\boldsymbol{A}$ as

$$\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}' = \boldsymbol{x}\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} = \begin{bmatrix} y_1^*\boldsymbol{x} & \dots & y_n^*\boldsymbol{x} \end{bmatrix},$$

which has (at most) one linearly independent column when $n > 1$, because, for every $i \neq j$, if $\boldsymbol{A}_{:,i}$ and $\boldsymbol{A}_{:,j}$ denote the ith and jth column of $\boldsymbol{A}$, then

$$y_i^*\boldsymbol{A}_{:,j} - y_j^*\boldsymbol{A}_{:,i} = y_i^* y_j^* \boldsymbol{x} - y_j^* y_i^* \boldsymbol{x} = 0.$$

Because $\boldsymbol{y} \neq \boldsymbol{0}$ is given, we can choose i and j such that at least one of y_i or y_j is nonzero; thus the columns of $\boldsymbol{A}$ are linearly dependent when $n > 1$.

For $n = 1$, $\boldsymbol{A}$ has a single nonzero column (because $\boldsymbol{x}$ and $\boldsymbol{y}$ are nonzero), and a set consisting of a single nonzero vector is linearly independent by definition.

(b) Because $\boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{0}$ is given, $\boldsymbol{A}$ has rank one.
(If either of $\boldsymbol{x}$ or $\boldsymbol{y}$ were zero, then $\boldsymbol{A}$ would have rank zero.)

We can see this another way via the SVD. Write $\boldsymbol{A} = \sigma\boldsymbol{u}\boldsymbol{v}' = \boldsymbol{u}\sigma\boldsymbol{v}'$ where $\sigma \triangleq \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y}\rangle}$, $\boldsymbol{u} \triangleq \boldsymbol{x}/\sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$ and $\boldsymbol{v} \triangleq \boldsymbol{y}/\sqrt{\langle \boldsymbol{y}, \boldsymbol{y}\rangle}$. This is a compact SVD of $\boldsymbol{A}$ with rank 1, implying $\boldsymbol{A}$ has one linearly independent column.

(c) The given `Julia` code yields Figure 1, where there $n = 100$ non-zero singular values. The discrepany arises because of numerical round-off error due to representing the matrix $\boldsymbol{A}$ in finite-precision arithmetic. Typing `rank(A)` gives the theoretically correct answer despite the $n - 1$ singular values not being identically equal to zero. This is because `Julia`'s `rank` function first computes the singular values and counts how many are greater than $\min(m, n)\epsilon$ where ϵ is machine precision; here `eps(Float64)` is 2.2204×10^{-16}.

(d) Clicking on the link to the source code reveals the rank function:

```
function rank(A::AbstractMatrix; atol::Real = 0.0,
    rtol::Real = (min(size(A)...)*eps(real(float(one(eltype(A))))))*iszero(atol))
    s = svdvals(A)
    tol = max(atol, rtol*s[1])
    count(x -> x > tol, s)
end
```

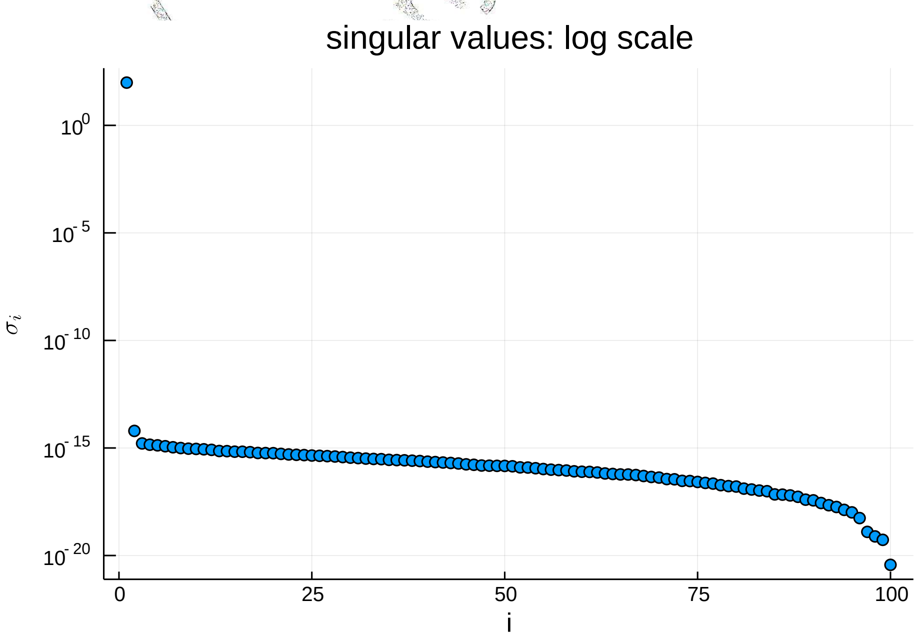

Figure 1: The singular values of $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}'$.

This reveals that (by default) the "numerical rank" of a matrix $\boldsymbol{A}$ is determined by the formula

$$r = \{\text{Number of } \sigma_i > \text{threshold}\}, \quad \text{threshold} = \epsilon \, \min(m,n)\sigma_1,$$

where ϵ is machine precision.

(e) For $\boldsymbol{A} = [9 \; 9]$, we have $\sigma_1 = 9\sqrt{2} \approx 12.73$ and $\min(m,n) = 1$, so the threshold is $\sigma_1 \varepsilon \approx 2.83 \cdot 10^{-15}$, on a typical 64-bit computer.

The "continuous-looking" spectrum in Figure 1 is a signature of noise-only singular values.

Pr. 8. (sol/hs036)

(a) We have that

$$\begin{aligned}
\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}' &= \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2} \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \frac{\boldsymbol{y}'}{\|\boldsymbol{y}\|_2} \\
&= \begin{bmatrix} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2} & \boldsymbol{U}_\perp \end{bmatrix} \begin{bmatrix} \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_2} & \boldsymbol{V}_\perp \end{bmatrix}'.
\end{aligned} \tag{1}$$

We know that $\boldsymbol{x}\boldsymbol{y}'$ is a rank–1 matrix, and thus has only one nonzero singular value. Furthermore, $\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$ and $\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_2}$ are unit norm vectors, and $\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2$ is a positive real number. Therefore, $\sigma_1 = \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2$ is the only nonzero singular value of $\boldsymbol{A}$, and $\boldsymbol{u}_1 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$, $\boldsymbol{v}_1 = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_2}$ are the corresponding left and right singular vectors. The full SVD of $\boldsymbol{A}$ is given by (1), where $\boldsymbol{U}_\perp \in \mathbb{F}^{M \times M-1}$ is any set of $M-1$ orthonormal vectors orthogonal to $\boldsymbol{u}_1$, and $\boldsymbol{V}_\perp \in \mathbb{F}^{N \times N-1}$ is any set of $N-1$ orthonormal vectors orthogonal to $\boldsymbol{v}_1$.

Using the definition of pseudoinverse:

$$\boldsymbol{A}^+ = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_2} \frac{1}{\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2} \frac{\boldsymbol{x}'}{\|\boldsymbol{x}\|_2} = \frac{\boldsymbol{y}\,\boldsymbol{x}'}{\|\boldsymbol{x}\|_2^2 \, \|\boldsymbol{y}\|_2^2}.$$

(b) This is a special case of (a) in which $\boldsymbol{y} = \boldsymbol{x}$. Thus:

$$\boldsymbol{A}^+ = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2} \frac{1}{\|\boldsymbol{x}\|_2\|\boldsymbol{x}\|_2} \frac{\boldsymbol{x}'}{\|\boldsymbol{x}\|_2} = \frac{\boldsymbol{x}\,\boldsymbol{x}'}{\|\boldsymbol{x}\|_2^4}.$$

Pr. 9. (sol/hsj32)

(a) $\{\boldsymbol{e}_1, \boldsymbol{e}_3\}$ is an orthonormal basis for the orthogonal complement of $\boldsymbol{v} = \operatorname{span}(2\boldsymbol{e}_2)$.

However this is not the only solution! Another orthonormal basis is $\{\boldsymbol{e}_1, -\boldsymbol{e}_3\}$.

But wait there are more! Another option is $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$.

For $\mathbb{R}^3$, every possible solution to this problem has the form $\left\{ \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}, \begin{bmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{bmatrix} \right\} = \{\boldsymbol{e}_1 \cos\theta + \boldsymbol{e}_3 \sin\theta, \boldsymbol{e}_1 \sin\theta - \boldsymbol{e}_3 \cos\theta\}$.

The three answers above simply correspond to different values of θ.

Graders: accept any solution that is a special case. Students do not need to give the general form.

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$, is an orthonormal basis for the orthogonal complement of $\operatorname{span}(\{\boldsymbol{z}\}) = \operatorname{span}(\left\{ \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\})$.

Again there is a family of possible solutions of the form $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos\phi + \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \sin\phi, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sin\phi - \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \cos\phi \right\}$.

(c) Using the general solution in the next part, the projection of $\boldsymbol{y}$ onto the orthogonal complement of $\mathcal{S} = \operatorname{span}(\{\boldsymbol{z}\})$ is

$$\boldsymbol{y} - \frac{\boldsymbol{z}'\boldsymbol{y}}{\boldsymbol{z}'\boldsymbol{z}} \boldsymbol{z} = \boldsymbol{y} - \frac{12}{8} \boldsymbol{z} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(d) For a nonzero vector $\boldsymbol{x}$, an orthonormal basis for $\mathcal{S} = \operatorname{span}(\{\boldsymbol{x}\})$ is $\boldsymbol{u} = \boldsymbol{x} / \|\boldsymbol{x}\|_2$, so $\boldsymbol{P}_{\mathcal{S}} = \boldsymbol{u}\boldsymbol{u}'$. Thus, the projection of $\boldsymbol{y}$ onto the orthogonal complement of $\mathcal{S} = \operatorname{span}(\{\boldsymbol{x}\})$ is

$$\boldsymbol{P}_{\mathcal{S}^\perp} \boldsymbol{y} = \boldsymbol{P}_{\mathcal{S}}^\perp \boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{P}_{\mathcal{S}}) \boldsymbol{y} = \boldsymbol{y} - \boldsymbol{P}_{\mathcal{S}} \boldsymbol{y} = \boldsymbol{y} - \boldsymbol{u}\boldsymbol{u}'\boldsymbol{y} = \boldsymbol{y} - \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2} \frac{\boldsymbol{x}'}{\|\boldsymbol{x}\|_2} \boldsymbol{y} = \boldsymbol{y} - \frac{\boldsymbol{x}'\boldsymbol{y}}{\boldsymbol{x}'\boldsymbol{x}} \boldsymbol{x}.$$

(e) A possible `Julia` implementation is

```julia
"""
    z = orthcomp1(y, x)

Project `y` onto the orthogonal complement of `Span({x})`

In:
* `y` vector
* `x` nonzero vector of same length, both possibly very long

Out:
* `z` vector of same length

For full credit, your solution should be computationally efficient.
"""
function orthcomp1(y, x)
    return y - ((x'y) / (x'x)) * x
end
```

This solution is likely to be the simplest possible solution and probably also the most computationally efficient solution because it uses only elementary vector arithmetic. Another solution would be to normalize $\boldsymbol{x}$ first:

`x = x / norm(x)`

`return y - (x’y) * x`

Non-graded problem(s) below

Pr. 10. (sol/hsj34)
Proof sketch for the case of distinct magnitudes, i.e., $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}'$ with $|\lambda_1| > \ldots > |\lambda_N|$.
If $\boldsymbol{x}$ is a RSV of $\boldsymbol{A}$, then $\boldsymbol{A}'\boldsymbol{A}\boldsymbol{x} = \alpha\boldsymbol{x}$, so $|\boldsymbol{\Lambda}|^2\boldsymbol{y} = \alpha\boldsymbol{y}$ where $\boldsymbol{y} = \boldsymbol{V}'\boldsymbol{x}$. Because $\boldsymbol{\Lambda}$ is diagonal with distinct magnitudes along its diagonal, then we must have $\boldsymbol{y} = \gamma\boldsymbol{e}_n$, where $\gamma = y_n$, and where n is the unique index where $|\lambda_n|^2 = \alpha$. So $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{y} = \gamma\boldsymbol{v}_n$ is an eigenvector of $\boldsymbol{A}$.
Now suppose that $\lambda_1 = \lambda_2$ and all other eigenvalues have distinct magnitudes. Then because $\boldsymbol{\Lambda}$ is diagonal, $|\boldsymbol{\Lambda}|^2\boldsymbol{y} = \alpha\boldsymbol{y}$ implies that $\boldsymbol{y} = \gamma_1\boldsymbol{e}_1 + \gamma_2\boldsymbol{e}_2$, where $\gamma_1 = y_1$ and $\gamma_2 = y_2$, so $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{y} = \gamma_1\boldsymbol{v}_1 + \gamma_2\boldsymbol{v}_2$. Thus $\boldsymbol{A}\boldsymbol{x} = \gamma_1\boldsymbol{A}\boldsymbol{v}_1 + \gamma_2\boldsymbol{A}\boldsymbol{v}_2 = \gamma_1\lambda_1\boldsymbol{v}_1 + \gamma_2\lambda_2\boldsymbol{v}_2 = \lambda_1(\gamma_1\boldsymbol{v}_1 + \gamma_2\boldsymbol{v}_2) = \lambda_1\boldsymbol{x}$. The idea generalizes to more repeated eigenvalues having the same value.

Pr. 11. (sol/hsj23)
For $x \in \mathbb{R}$: $x\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \preceq \begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$ iff $\begin{bmatrix}a & -x\\ -x & 0\end{bmatrix} \succeq \boldsymbol{0}$, *i.e.*, iff $\begin{bmatrix}a & -x\\ -x & 0\end{bmatrix}$ has nonnegative eigenvalues.
Now $\det\left(\begin{bmatrix}(z-a) & x\\ x & z\end{bmatrix}\right) = z^2 - az - x^2$ so the eigenvalues are $z = (a \pm \sqrt{a^2 + 4x^2})/2$, and both eigenvalues are nonnegative iff $x = 0$. So the set in the problem statement is simply $\{0\}$.

Pr. 12. (sol/hs122)

(a)

$$\begin{aligned}
&(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y)(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)'\\
=&(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y)(\boldsymbol{V}_x' \otimes \boldsymbol{V}_y')\\
=&(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)((\boldsymbol{\Sigma}_x\boldsymbol{V}_x') \otimes (\boldsymbol{\Sigma}_y\boldsymbol{V}_y'))\\
=&(\boldsymbol{U}_x\boldsymbol{\Sigma}_x\boldsymbol{V}_x') \otimes (\boldsymbol{U}_y\boldsymbol{\Sigma}_y\boldsymbol{V}_y') = \boldsymbol{X} \otimes \boldsymbol{Y}.
\end{aligned}$$

To show that $\boldsymbol{U}_x \otimes \boldsymbol{U}_y$ is unitary, observe that

$$\begin{aligned}
&(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)'(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)\\
=&(\boldsymbol{U}_x' \otimes \boldsymbol{U}_y')(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)\\
=&(\boldsymbol{U}_x'\boldsymbol{U}_x) \otimes (\boldsymbol{U}_y'\boldsymbol{U}_y) = \boldsymbol{I}_m \otimes \boldsymbol{I}_p = \boldsymbol{I}_{mp}.
\end{aligned}$$

The above argument can be repeated to show that

- $(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)' = \boldsymbol{I}_{mp}$
- $(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)'(\boldsymbol{V}_x \otimes \boldsymbol{V}_y) = \boldsymbol{I}_{nq}$
- $(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)' = \boldsymbol{I}_{nq}$.

A technicality here is that $\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y$ is a block rectangular diagonal matrix with rectangular diagonal blocks, which is not our usual arrangement of the "$\boldsymbol{\Sigma}$" for an SVD. To make a "proper" SVD we will need to introduce permutation matrices:

$$\underbrace{(\boldsymbol{U}_x \otimes \boldsymbol{U}_y)\boldsymbol{P}_1}_{\boldsymbol{U}}\,\underbrace{\boldsymbol{P}_1'(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y)\boldsymbol{P}_2}_{\boldsymbol{\Sigma}}\,\underbrace{\boldsymbol{P}_2'(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)'}_{\boldsymbol{V}'}$$

where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ permute things to put the singular values in descending order in the proper places.

(b)

$$\begin{aligned}
&(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \otimes \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'\\
=&(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \otimes \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a' \otimes \boldsymbol{Q}_b')\\
=&(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a\boldsymbol{Q}_a' \otimes \boldsymbol{\Lambda}_b\boldsymbol{Q}_b')\\
=&\boldsymbol{Q}_a\boldsymbol{\Lambda}_a\boldsymbol{Q}_a' \otimes \boldsymbol{Q}_b\boldsymbol{\Lambda}_b\boldsymbol{Q}_b' = \boldsymbol{A} \otimes \boldsymbol{B}.
\end{aligned}$$

To show that $\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b$ is unitary, observe that

$$\begin{aligned}
&(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)\\
=&(\boldsymbol{Q}_a' \otimes \boldsymbol{Q}_b')(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)\\
=&(\boldsymbol{Q}_a'\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b'\boldsymbol{Q}_b) = \boldsymbol{I}_N \otimes \boldsymbol{I}_M = \boldsymbol{I}_{MN}.
\end{aligned}$$

The above argument can be repeated to show that $(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)' = \boldsymbol{I}_{MN}$.

(c)

$$
\begin{aligned}
\boldsymbol{A} \oplus \boldsymbol{B} &= \boldsymbol{A} \otimes \boldsymbol{I}_M + \boldsymbol{I}_N \otimes \boldsymbol{B} \\
&= \boldsymbol{A} \otimes (\boldsymbol{Q}_b \boldsymbol{I}_M \boldsymbol{Q}_b') + (\boldsymbol{Q}_a \boldsymbol{I}_N \boldsymbol{Q}_a') \otimes \boldsymbol{B} \\
\text{by (b)} \to &= (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \otimes \boldsymbol{I}_M)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)' + (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{I}_N \otimes \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)' \\
&= (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \otimes \boldsymbol{I}_M + \boldsymbol{I}_N \otimes \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)' \\
&= (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \oplus \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'.
\end{aligned}
$$

As in (b), $\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b$ is unitary.

Pr. 13. (sol/hsj10)
I welcome feedback on how helpful this tutorial was for you.
