HW01

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1. Let A be $M \times N$ matrix having elements: $A_{i,j} = i 2^j$

(for $i \in 1 \dots M$, $j \in 1 \dots N$)

(a) Let $a = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ m \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 2^2 \\ \vdots \\ 2^n \end{bmatrix}$

So the outer product of a and b is:

$$A = ab' = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ m \end{bmatrix} \begin{bmatrix} 2 & 2^2 & \cdots & 2^n \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2 & 1\times 2^2 & \cdots & 1\times 2^n \\ 2\times 2 & 2\times 2^2 & \cdots & 2\times 2^n \\ \vdots & \vdots & & \vdots \\ m\times 2 & m\times 2^2 & \cdots & m\times 2^n \end{bmatrix}$$

Thus, Matrix A is an outer product ab'

(b) One-line Julia expression for creating A:

```
A = reshape([i * 2.^j for j in 1:N for i in 1:M], (M,N))
```

2. Determinant properties:

① $\det(A) = \det(A^T)$ for $A \in R^{n\times n}$

② $\det(AB) = \det(A)\det(B)$ for $A, B \in R^{n\times n}$

Therefore, we have: $\det(AA^T) = \det(A)\det(A^T)$ by ②

$= \det(A)\det(A)$ by ①

$= \det(A)^2$

If A is orthogonal, the $AA^T = I$ by the definition of orthogonal matrix

Thus, $\det(AA^T) = \det(I)$

$= 1$

$\therefore$ $\det(A)^2 = 1$

$\therefore$ $\det(A) = 1$ or -1

The possible values of $\det(A)$ is 1 or -1

3.(a) Prove that $\det(I - xy') = 1 - y'x$ for $x, y \in F^N$

Proof:

First, decomposing as lower · upper and upper · lower gives

$$\begin{pmatrix} 1 & y' \\ x & I \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix} \cdot \begin{pmatrix} 1 & y' \\ 0 & I - xy' \end{pmatrix} = \begin{pmatrix} 1 - y'x & y' \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix}$$

Thus, $\det(LHS) = \det(RHS)$

$$\det(LHS) = \det\left(\begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix}\begin{pmatrix} 1 & y' \\ 0 & I - xy' \end{pmatrix}\right)$$

$$= \underbrace{\det\begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix}}_{1} \cdot \det\begin{pmatrix} 1 & y' \\ 0 & I - xy' \end{pmatrix}$$ by the determinant properties

$$= \det\begin{pmatrix} 1 & y' \\ 0 & I - xy' \end{pmatrix}$$

$$= \det(1) \det(I - xy')$$ since $1 \in F^{1 \times 1}$ is invertible,

(by the properties:

$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A) \det(D - CA^{-1}B)$

if $A \in F^{N \times N}$ is invertible)

$$\det(RHS) = \det\left(\begin{pmatrix} 1 - y'x & y' \\ 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 1 - y'x & y' \\ 0 & I \end{pmatrix} \underbrace{\det\begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix}}_{1}$$ by the determinant property

$$= I \cdot (1 - y'x) \cdot 1$$

$$= 1 - y'x$$

Thus, $\det(I - xy') = 1 - y'x$

3.

(b)

① when $\lambda \neq 0$

$$\det(\lambda I_N - xy') = \det\left(\lambda\left(I_N - \frac{xy'}{\lambda}\right)\right)$$
$$= \det\left((\lambda I_N)\left(I_N - \frac{xy'}{\lambda}\right)\right) \quad \text{when } \lambda \neq 0$$
$$= \det(\lambda I_N)\det\left(I_N - \frac{xy'}{\lambda}\right) \quad \text{by the determinant properties}$$
$$= \lambda^N \det(I_N) \cdot \det\left(I_N - \frac{xy'}{\lambda}\right)$$
$$= \lambda^N \cdot \left(1 - \frac{y'x}{\lambda}\right) \quad \text{by problem (a)}$$
$$= \lambda^N - \lambda^{N-1}(y'x)$$

② when $\lambda = 0$

$$\det(\lambda I_N - xy') = \det(-xy')$$
$$= 0$$

since xy' has rank 1 and it is singular for $n > 1$

(C) By question (b)

when $\lambda \neq 0$

$$\det(\lambda I - xy') = \lambda^N - \lambda^{N-1}(y'x) = 0$$

$$\therefore \lambda^N - \lambda^{N-1} y'x = 0$$

$$\lambda^N = \lambda^{N-1} y'x$$

$$\lambda = y'x$$

and $\lambda = 0$ is the other eigenvalues

(d) when matrix $xy' \in F^{N \times N}$ is not equal to zero, find the eigenvalues of matrix xy'

$$|xy' - \lambda I_N| = 0$$

$$\det(xy' - \lambda I_N) = 0$$

by the question (c):

$\lambda = y'x$ is the eigenvalues of the matrix xy'

4.

(a) $A, B \in F^{N\times N}$

$$Tr(\alpha A + \beta B) = \sum_{i=1}^{N}(\alpha a_{ii} + \beta b_{ii})$$ by the definition of trace

$$= \sum_{i=1}^{N}(\alpha a_{ii}) + \sum_{i=1}^{N}(\beta b_{ii})$$

$$= \alpha\left(\sum_{i=1}^{N} a_{ii}\right) + \beta\left(\sum_{i=1}^{N} b_{ii}\right)$$

$$= \alpha Tr(A) + \beta Tr(B)$$

(b) Let $A \in F^{M\times N}$ and $B \in F^{N\times M}$ so that $AB \in F^{M\times M}$ and $BA \in F^{N\times N}$

By definition,

$$Tr(AB) = \sum_{i=1}^{m} AB_{ii}$$

$$= (AB)_{11} + (AB)_{22} + \cdots + (AB)_{mm}$$

$$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1} +$$
$$a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + \cdots + a_{2n}b_{n2} +$$
$$\cdots$$
$$a_{m1}b_{1m} + a_{m2}b_{2m} + a_{m3}b_{3m} + \cdots + a_{mn}b_{nm}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij}$$

$$= b_{11}a_{11} + \cdots + b_{1m}a_{m1} +$$
$$\cdots$$
$$b_{n1}a_{1n} + \cdots + b_{nm}a_{mn}$$

$$= (BA)_{11} + (BA)_{22} + \cdots + (BA)_{nn}$$

$$= \sum_{i=1}^{n} BA_{ii}$$

$$= Tr(BA)$$

4.

(c) $S \in R^{N \times N}$ is skew-symmetric that is $S^T = -S$

By the definition of skew-symmetric, the digonals are zeros

Thus $Tr(S) = 0$

And we also can prove in this way,

$Tr(S) = Tr(S^T)$ by the def of transpose

$= Tr(-S)$ by the def of skew-symmetric

$= -Tr(S)$ by part (a)

$Tr(S) = -Tr(S)$

$2Tr(S) = 0$

$\therefore Tr(S) = 0$

(d) A counterexample to show that if $Tr(S) = 0$, then $S \in R^{N \times N}$ skew-symmetric

let S be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$Tr(S) = 0$; however, $S^T \neq -S$, S is not skew-symmetric

(e) $A = \frac{1}{2}\begin{bmatrix} 2\cos^2(\theta) & \sin(2\theta) \\ \sin(2\theta) & 2\sin^2(\theta) \end{bmatrix}$

Let $v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\therefore vv' = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2(\theta) & \cos\theta\sin\theta \\ \cos\theta \cdot \sin\theta & \sin^2(\theta) \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 2\cos^2(\theta) & \sin(2\theta) \\ \sin(2\theta) & 2\sin^2(\theta) \end{bmatrix}$

$= A$

4(e) contioue....

Then, we compute A^2:

$$
\begin{aligned}
A^2 &= (vv')(vv') \\
&= v\cdot(v'v)\cdot v' \\
&= v\cdot\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}\cdot v' \\
&= v\cdot(\cos^2\theta+\sin^2\theta)\cdot v' \\
&= v\cdot 1\cdot v' \\
&= v\cdot v' \\
&= A
\end{aligned}
$$

$\therefore$ A is idempotent for all θ.

6.

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ Find the eigenvalues of A..

start with $|A - \lambda I| = 0$ which is $\left| \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$

$$\left| \begin{bmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{bmatrix} \right| = 0$$

$$(6-\lambda)(-4-\lambda) + 16 = 0$$

$$-24 - 6\lambda + 4\lambda + \lambda^2 + 16 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = 4 \text{ or } -2$$

$\therefore$ there are two eigenvalues 4 and -2

6.

(b) $\det(A) = \det\left(\begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\right)$

$= -24 + 16$

$= -8$

$Tr(A) = 6 - 4$

$= 2$

The product of two eigenvalues of A:

$4 \times (-2) = -8$ which is the same as $\det(A)$

The sum of two eigenvalues of A:

$4 + (-2) = 2$ which is the same as $Tr(A)$

6.

(c)

In [3]:
```
using LinearAlgebra
```

In [13]:
```
A = [6 16;-1 -4]
```

Out[13]:
```
2×2 Matrix{Int64}:
  6  16
 -1  -4
```

In [14]:
```
lambda, V = eigen(A)
display(lambda)
display(V)
```

```
2-element Vector{Float64}:
 -2.0
  4.0

2×2 Matrix{Float64}:
 -0.894427   0.992278
  0.447214  -0.124035
```

In [16]:
```
display(V' * V)

```

```
2×2 Matrix{Float64}:
  1.0       -0.94299
 -0.94299    1.0
```

In [ ]:
```
#The eigenvectors (columns of V ) are not orthogonal
```

7.

Rewrite $y = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^* A_{ij} x_j$ where $A \in F^{N \times N}$ and $x \in C^N$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & & A_{2n} \\ \vdots & & \vdots \\ A_{n1} & & A_{nn} \end{bmatrix}$

$$y = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^* A_{ij} x_j$$

$$= \sum_{i=1}^{n} x_i^* (A_{i1} x_1 + A_{i2} x_2 + A_{i3} x_3 + \cdots + A_{in} x_n)$$

$$= x_1^* A_{11} x_1 + x_1^* A_{12} x_2 + \cdots + x_1^* A_{1n} x_n +$$

$$x_2^* A_{21} x_1 + x_2^* A_{22} x_2 + \cdots + x_2^* A_{2n} x_n +$$

$$\cdots,$$

$$x_n^* A_{n1} x_1 + x_n^* A_{n2} x_2 + \cdots + x_n^* A_{nn} x_n$$

$$= [x_1^* \ x_2^* \ \cdots x_n^*] \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x' A X$$

8.

Let the matrix U be $[u_1, u_2 \cdots u_m]$

Let the matrix V be $[V_1 \ V_2 \cdots V_n]$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} =$$

$$3\times 3 \qquad 3\times 4$$

(a) when $M<N$:

$$A = U \Sigma V'$$

$$= \underset{M\times M}{\begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1m} \\ U_{21} & U_{22} & \cdots & U_{2m} \\ \vdots & \vdots & & \vdots \\ U_{m1} & U_{m2} & \cdots & U_{mm} \end{bmatrix}} \underset{M\times N}{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \cdots \end{bmatrix}} \underset{N\times N}{\begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ \vdots & & & \\ V_{n1} & V_{n2} & \cdots & V_{nn} \end{bmatrix}'}$$

$$= \underset{M\times N}{\left[\begin{array}{cccc|c} U_{11}\sigma_1 & & & & \\ & U_{22}\sigma_2 & & & 0 \\ & & \ddots & & \\ & & & U_{mm}\sigma_m & \cdots \end{array}\right]} \underset{N\times N}{\begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ \vdots & \vdots & & \vdots \\ V_{n1} & V_{n2} & & V_{nn} \end{bmatrix}'}$$

(Since $M<N$, there are $N-M$ diagonals are zeros)

$$= \left[\begin{array}{cccc|c} U_{11}\sigma_1 V_{11} & & & & \\ & U_{22}\sigma_2 V_{22} & & & 0 \\ & & \ddots & & \\ & & & U_{mm}\sigma_m V_{mm} & \cdots \end{array}\right]$$

$\therefore \ A = \sum_{i=1}^{M} \sigma_i U_i V_i^T$, the rests are zeros

$\therefore$ M outer-product matrice are added to form A

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$2\times 2 \qquad 2\times 3$

(b) Suppose $M > N$

$$A = U \Sigma V'$$

$$A = \underbrace{\begin{bmatrix} U_{11} & \cdots & U_{1m} \\ \vdots & & \\ U_{m1} & \cdots & U_{mm} \end{bmatrix}}_{M\times M} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \\ \hline & 0 & \end{bmatrix}}_{\substack{M\times N \\ (\text{Since } M>N)}} \underbrace{\begin{bmatrix} V_{11} & \cdots & V_{1n} \\ & & \\ V_{n1} & \cdots & V_{nn} \end{bmatrix}'}_{N\times N}$$

$$= \underbrace{\begin{bmatrix} U_{11}\sigma_1 & & \\ & U_{22}\sigma_2 & \\ & & \ddots \\ & & & U_{nn}\sigma_n \\ \hline & 0 & \end{bmatrix}}_{M\times N} \underbrace{\begin{bmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & & \\ V_{n1} & \cdots & V_{nn} \end{bmatrix}'}_{N\times N}$$

$$\therefore \quad A = \sum_{i=1}^{N} \sigma_i U_i V_i'$$

$\therefore$ There are n outer-product are added to form A

(c) Based on (a) and (b)

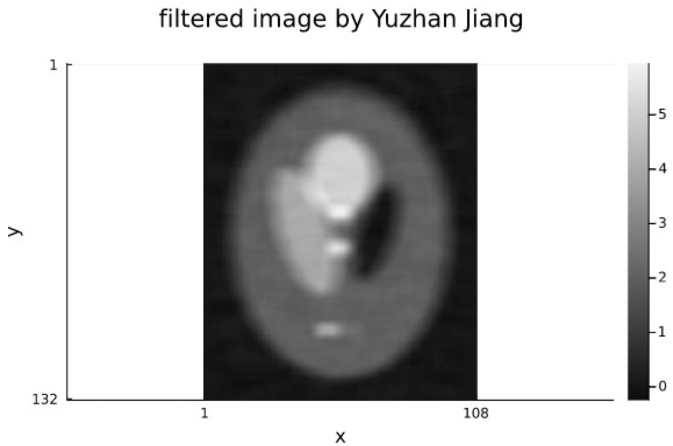
$$A = \sum_{i=1}^{\min(M,N)} \sigma_i U_i V_i'$$

$\therefore$ The maximum number of outer-product is $\min(M, N)$

10.

In [105]:
```
# display filtered image - the noise is greatly reduced!
# it is "smeared out" (filtered or smoothed) more along x than y
# do you know why?
#whoami = ENV["USER"] # if this fails, put your name in it manually
jim(Y, xlabel="x", ylabel="y", "filtered image by Yuzhan Jiang")
```

Out[105]:



11.

Vectors u, v, x, y, z are all in R^N

Compute $z * u' * v * x' * y$

(a) $z * ((u' * v) * (x' * y))$

is the most efficient way to compute.

(b)

① $u' * v$: N multiplication operations

② $x' * y$: N multiplication operations

③ $(u' * v) * (x' * y)$: 1 $*$ operations

(since $u' * v$ and $x' * y$ both are a number)

④ $z * (\cdots)$: N multiplication operations

$\therefore$ Total multiplication operations : $N + N + 1 + N = \underline{\underline{3N+1}}$

12.

Since $U_1, U_2 \cdots U_k \in F^{N\times N}$ are unitary matrices,

we have $U_i' U_i = U_i U_i' = I$ for $i \in 1 \cdots k$

$$(U_1 U_2 \cdots U_k)(U_1 U_2 \cdots U_k)' = (U_1 U_2 \cdots U_k)(U_k' \cdots U_2' U_1')$$
$$= U_1 U_2 \cdots (U_k U_k') \cdots U_2' U_1'$$
$$= U_1 U_2 \cdots I \cdots U_2' U_1'$$
$$= U_1 U_2 \cdots (U_{k-1} U_{k-1}') \cdots U_2' U_1'$$
$$= U_1 U_2 \cdots I \cdots U_2' U_1'$$
$$\cdots$$
$$= U_1 U_1'$$
$$= I$$

$\therefore$ The product $U_1 U_2 \cdots U_k$ is a unitary Matrix