Lecture 13

Goals

1. Random Coding Bound.

akin to Shannon's

Error Probability for M Signals

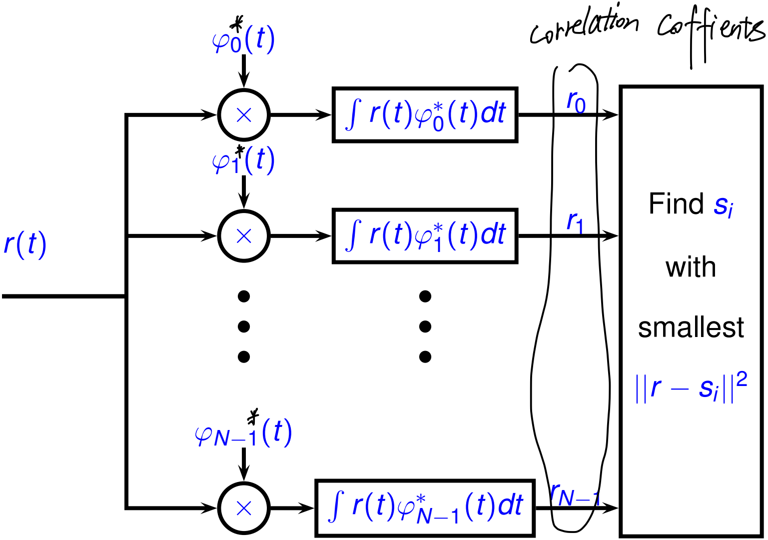
vectors → waveforms

The general signal set we consider has the form

$$s_i(t) = \sum_{j=0}^{N-1} s_{i,j}\varphi_j(t), \quad i = 0, 1, ..., M-1$$

The number of orthonormal basis functions is less than the number of signals ($N \leqslant M$). The optimum receiver does a correlation with the N orthonormal waveforms to form the decision variables.

Optimum Receiver in Additive White Gaussian Noise



Error Probability for Arbitrary Signals

$p_i(r)$ = density r given S_i trans

= H_i = event s_i trans

$$r_j = \int_0^T r(t)\varphi_j^*(t)dt, j = 0, 1, ..., N-1.$$

The decision regions are for equally likely signals given by

$$R_i = \{\mathbf{r} : p_i(\mathbf{r}) > p_j(\mathbf{r}), \forall j \neq i\}$$

H_i = event S_i transmits

The error probability is then determined by

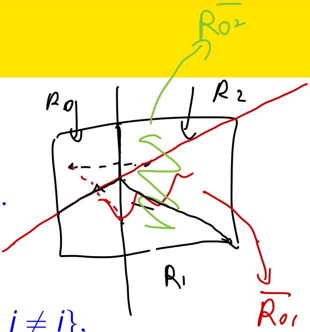
R_i = region decide H_i is true

$$P_{e,i} = P(\bigcup_{j=0,j\neq i}^{M-1} R_j | H_i).$$

For all but a few small dimensional signals or signals with special structures (such as orthogonal signal sets) the exact error probability is very difficult to calculate.

Union Bound

$$\boxed{\overline{R_0} = \overline{R_{01}} \cup \overline{R_{02}}}$$



Assume

$$\pi_i = \frac{1}{M}, \quad 0 \le i \le M-1.$$

Let

$$\begin{aligned}
R_i &= \{\mathbf{r} : p_i(\mathbf{r}) > p_j(\mathbf{r}) \quad \text{for all } \; j \neq i\}, \\
\overline{R}_i &= \{\mathbf{r} : p_i(\mathbf{r}) \le p_j(\mathbf{r}) \quad \text{for some } \; \mathbf{j} \neq \mathbf{i}\}, \\
&= \bigcup_{j=0, j\neq i}^{M-1} \{\mathbf{r} : p_i(\mathbf{r}) \le p_j(\mathbf{r})\}, \\
\overline{R}_{ij} &= \{\mathbf{r} : p_i(\mathbf{r}) \le p_j(\mathbf{r})\}.
\end{aligned}$$

region decide j rather than i for only i,j

$P(A \cup B) \leq P(A) + P(B)$

Then

$$
\begin{aligned}
P_{e,i} &= P(\mathbf{r} \in \overline{R}_i \mid H_i) \\
&= P(\mathbf{r} \in \bigcup_{j \neq i} \overline{R}_{ij} \mid H_i) \quad \text{Union bound} \\
&\leq \sum_{j \neq i} P(\overline{R}_{ij} \mid H_i)
\end{aligned}
$$

where

$$
P(\overline{R}_{ij} \mid H_i) = P\left\{ \frac{p_i(\mathbf{r})}{p_j(\mathbf{r})} \leq 1 \mid H_i \right\}
$$

This is the union bound.

We now consider the bound for an arbitrary signal set in additive white Gaussian noise.
Let

$$s_i(t) = \sum_{l=0}^{N-1} s_{il}\varphi_l(t), \quad 0 \le i \le M-1.$$

For additive white Gaussian noise

$$p_i(\mathbf{r}) = \prod_{l=0}^{N-1} \left(\frac{\exp\{-\frac{1}{N_0}(r_l - s_{il})^2\}}{\sqrt{\pi N_0}}\right)$$

$$\begin{aligned} \frac{p_i(\mathbf{r})}{p_j(\mathbf{r})} &= \prod_{l=0}^{N-1} \exp\{-\frac{1}{N_0}[(r_l - s_{il})^2 - (r_l - s_{jl})^2]\} \\ &= \exp\left\{\frac{2}{N_0}(\underline{r}, \underline{s}_i - \underline{s}_j) + \frac{E_j - E_i}{N_0}\right\} \end{aligned}$$

where $(\underline{r}, \underline{s}_i - \underline{s}_j) = \sum_{l=0}^{N-1} r_l(s_{il} - s_{jl})$ and $E_k = \sum_{l=0}^{N-1} s_{kl}^2$ for $0 \le k \le M-1$.

Thus

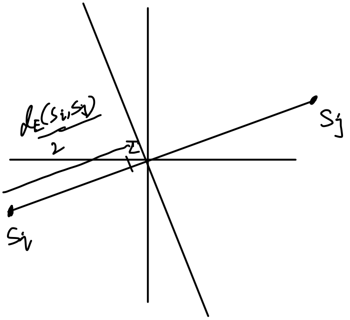
$$P(\overline{R}_{ij} \mid H_i) \quad = \quad P\left\{\frac{2}{N_0}(\mathbf{r}, \underline{s}_i - \underline{s}_j) \leq \frac{E_i - E_j}{N_0} \mid H_i\right\}.$$

To do this calculation we need to calculate the statistics of the random variable $(r, s_i - s_j)$. The mean and variance are calculated as follows.

$$\begin{aligned}
E[(\mathbf{r}, \underline{s}_i - \underline{s}_j) \mid H_i] &= E[(\underline{n} + \underline{s}_i, \underline{s}_i - \underline{s}_j] \\
&= E_i - (\underline{s}_i, \underline{s}_j). \\
\mathrm{Var}[(\mathbf{r}, \underline{s}_i - \underline{s}_j) \mid H_i] &= \mathrm{Var}\ [(\underline{n} + \underline{s}_i, s_i - s_j)] \\
&= \frac{N_0}{2} \parallel \underline{s}_i - \underline{s}_j \parallel^2 .
\end{aligned}$$

Also $(\mathbf{r}, \underline{s}_i - \underline{s}_j)$ is a Gaussian random variable. Thus

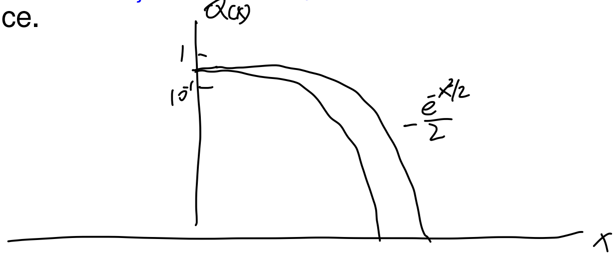
$$
\begin{aligned}
P\left\{(\mathbf{r}, \underline{s}_i - \underline{s}_j) \le \frac{E_i - E_j}{2}\right\} &= \Phi\left(\frac{\frac{E_i - E_j}{2} - (E_i - (s_i, s_j))}{\sqrt{\frac{N_0}{2}}\,\|\underline{s}_i - \underline{s}_j\|}\right) \\
&= Q\left(\frac{E_i - 2(s_i, s_j) + E_j}{\sqrt{2N_0}\,\|\underline{s}_i - \underline{s}_j\|}\right) \\
&= Q\left(\frac{\|\underline{s}_i - \underline{s}_j\|}{\sqrt{2N_0}}\right). \\
&= Q\left(\frac{d_E(s_i, s_j)}{2\sigma}\right)
\end{aligned}
$$



Thus the union bound on the error probability is given as

$$P_{e,i} \le \sum_{j \neq i} Q\left(\frac{\|\underline{s}_j - \underline{s}_i\|}{\sqrt{2N_0}}\right). = \sum_{j \neq i} Q\left(\frac{d_E(s_i, s_j)}{2\sigma}\right)$$

Note that $\|\underline{s}_i - \underline{s}_j\|^2 = d_E^2(s_i, s_j)$, i.e. the square of the Euclidean distance.



Union-Bhattacharyya Bound

We now use the following to derive the Union-Bhattacharyya bound.

Fact: $Q(x) \leq \frac{1}{2}e^{-x^2/2} \leq e^{-x^2/2}, \quad x \geq 0$.

To prove this let X_1 and X_2 be independent Gaussian random variables mean 0 variance 1. Then show
$Q^2(x) = P(X_1 \geq x, X_2 \geq x) \leq \frac{1}{4}P(X_1^2 + X_2^2 \geq \sqrt{2}\, x)$. Use the fact the $X_1^2 + X_2^2$ has Rayleigh density.
Using this fact leads to the bound

$$P_{e,i} \leq \sum_{j \neq i} \exp\left\{-\frac{\|\underline{s}_i - \underline{s}_j\|^2}{4N_0}\right\}. = \sum_{j \neq i} e^{-\frac{d_E^2(s_i,s_j)}{4N_0}}$$

This is the Union Bhattacharyya bound for an additive white Gaussian noise channel.

Random Coding

M signals, N dimensions MNcoef

Now consider 2^{NM} communication systems corresponding to all possible signals where

E = energy per dimension

$$s_{ij} = \pm\sqrt{E}$$

this is energy

$$\|s_i\|^2 = NE \qquad 0 \le i \le M-1$$

Consider the average error probability, averaged over all possible selections of signal sets

For example: Let $N = 3,\ M = 2 \Rightarrow$ there are $2^{3\times 2} = 2^6 = 64$ possible sets of 2 signals with each signal a linear combination of three orthogonal signals with the coefficients required to be one of two values.

Bad choice $\Longrightarrow$ Because no matter what data bit, we send the same Signal. $P = \frac{1}{2}$ we do only guessing

Set number 0	$s_0(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t)$
$S_0(t) = S_1(t)$	$s_1(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t)$
Set number 1	$s_0(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t)$
	$s_1(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t)$
Set number 2	$s_0(t)$	$= -\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_2(t)$
	$s_1(t)$	$= -\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_2(t)$
Set number 3	$s_0(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t)$
Bad choice	$s_1(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t)$
Set number 4	$s_0(t)$	$= -\sqrt{E}\,\varphi_0(t) - \sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t)$
So-So choice	$s_1(t)$	$= -\sqrt{E}\,\varphi_0(t) + \sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t)$
$\vdots$	$\vdots$	Good is $S_0(t) = -S_1(t)$
Set number 63	$s_0(t)$	$= +\sqrt{E}\,\varphi_0(t) + \sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t)$
Bad choice	$s_1(t)$	$= +\sqrt{E}\,\varphi_0(t) + \sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t)$

Let $P_{e,i}(k)$ be the error probability of signal set k given H_i. Then

$$\overline{P}_{e,i} = \frac{1}{2^{NM}} \sum_{k=0}^{2^{NM-1}} P_{e,i}(k)$$

and

$$\overline{P}_e = \frac{1}{M} \sum_{i=0}^{M-1} \overline{P}_{e,i}$$

and

$$P_e(k) = \frac{1}{M} \sum_{i=0}^{M-1} P_{e,i}(k).$$

If $\overline{P}_e \leq \alpha$ then at least one of the of the 2^{NM} signals sets must have $P_e(k) \leq \alpha$ (otherwise $P_e(k) > \alpha$ for all $k \Rightarrow \overline{P}_e > \alpha$; contradiction). In other words, there exists a signal set with $P_e \leq \alpha$. This is known as the random coding argument. Let $s_{i,j},\ 0 \leq i \leq M-1,\ 0 \leq j \leq N-1$ be independent identically distributed random variables with

Average

$$P(s_{i,j} = +\sqrt{E}) = P(s_{i,j} = -\sqrt{E}) = \frac{1}{2}$$

and $\overline{P}_e = E[P_e(s)]$ where the expectation is with respect to the random variables s.

with respect to S.

$$\overline{P}_{e,i} = E[P_{e,i}(s)] \le \sum_{j \ne i} E\left[\exp \left\{-\frac{\|s_i - s_j\|^2}{4N_0}\right\}\right].$$

Let $X_{ij} = \|s_i - s_j\|^2 = \sum_{l=1}^{N}(s_{il} - s_{jl})^2$. Then

$$\begin{aligned} P(X_{ij} = 4Em) &= P(s_i \text{ and } s_j \text{ differ in } m \text{ places out of } N) \\ &= \binom{N}{m} 2^{-N} \end{aligned}$$

since $P(s_{il} = s_{jl}) = P(s_{il} \ne s_{jl}) = \frac{1}{2}$.

$S_i = (+\sqrt{E}, -\sqrt{E}, -\sqrt{E}, +\sqrt{E})$

$S_j = (-\sqrt{E}, -\sqrt{E}, +\sqrt{E}, -\sqrt{E})$

$d_E^2(s_i, s_j) = 3((2\sqrt{E})^2)$

$= 3 \cdot 4E$

# of place where s_i, s_j diff

$d_E(s_i, s_j)$

So

$$E[\exp\left\{-\frac{\|s_i - s_j\|^2}{4N_0}\right\}] = E[e^{-X_{ij}/4N_0}]$$

$$\begin{aligned}
E[e^{-X_{ij}/4N_0}] &= \sum_{m=0}^{N} \binom{N}{m} 2^{-N} e^{-m4E/4N_0} \\
&= 2^{-N} \sum_{m=0}^{N} \binom{N}{m} (e^{-E/N_0})^m \\
&= 2^{-N} (1 + e^{-E/N_0})^N \\
&= \exp_2[-N\left(1 - \log_2(1 + e^{-E/N_0})\right)] = e^{-NR_0}
\end{aligned}$$

$a = e^{-E/N_0}$

$b = 1$

$\exp_2(x) = 2^x$

$(a+b)^N = \sum_{m=0}^{N} \binom{N}{m} a^m b^{N-m}$ — Binomial expansion cut off rate

$M = 2^{Nr}$

Let $R_0 = 1 - \log_2\left(1 + e^{-E/N_0}\right)$ then

$$\overline{P}_{e,i} \le \sum_{j \ne i} 2^{-NR_0} = (M-1)2^{-NR_0} \le M2^{-NR_0} = 2^{-N(R_0 - \overset{r}{\cancel{R}})}$$

where $r = \frac{\log_2 M}{N}$ is the number of bits transmitted per dimension (not to be confused with the received signal) and E is the signal energy per dimension. Averaging over the different signals sent we obtain

$r = \frac{\log_2 M}{N}$ $\frac{\text{bits}}{\text{dimension}}$

$$\boxed{\begin{aligned} \bar{P}_e &= \frac{1}{M}\sum_{i=0}^{M-1} \bar{P}_{e,i} \\ &\le 2^{-N(R_0 - r)} \end{aligned}}$$

- We have shown that there exist a signal set for which the average value of the error probability for the $i-th$ signal is small.
- Thus we have shown that as N goes to ∞ the error probability given s_i was transmitted goes to zero if the rate (bits/dimension) is less than the cutoff rate R_0.
- This however does not imply that there exist a single code (signal set) $s_0, ..., s_{M-1}$ such that $P_{e,0}, ..., P_{e,M-1}$ are simultaneously small. It is possible that $P_{e,i}$ is small for some code for which $P_{e,j}$ is large.
- We now show that we can simultaneously make each of the error probabilities small simultaneously.

- First choose a code with $M = (2)2^{rN}$ codewords for which the average error probability is less than say $\epsilon_N/2$ for large N.
- If more than 2^{NR} of these codewords has $P_{e,i} \geq \epsilon_N$ then the average error probability would be greater than $\epsilon_N/2$, a contradiction.
- Thus at least $M/2 = 2^{Nr}$ of the codewords must have $P_{e,i} \leq \epsilon_N$.
- So delete the codewords that have $P_{e,i} \geq \epsilon_N$ (less than half).
- We obtain a code with (at least) 2^{Nr} codewords with $P_{e,i} \to 0$ as $N \to \infty$ for $r < R_0$ for $i = 0, 1, ..., M - 1$.

$$\frac{\log_2 M}{N} = r \qquad \log_2 M = Nr$$
$$M = 2^{Nr}$$

Thus we have proved the following.

Theorem

There exist a signal set with M signals in N dimensions with $P_e \leq 2^{-N(R_0 - r)}$ ($\Rightarrow P_e \to 0$ *as* $N \to \infty$ *provided* $r < R_0$).

Note: E is the energy per dimension. Each signal then has energy NE and is transmitting $\log_2 M$ bits of information so that $E_b = \frac{NE}{\log_2 M} = E/r$ is the energy per bit of information.

$$P_e \leq 2^{-N(R_0 - r)} \text{ if } R_0 - r > 0 \Rightarrow P_e \to 0 \text{ as } N \to \infty$$

$$\boxed{P_e \to 0 \quad r < C \quad \text{Shannon}} \qquad R_0 < C$$

Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 1/8 codes

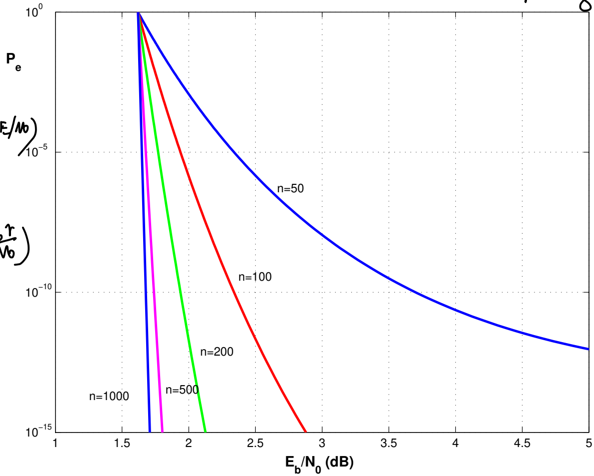
$r = \frac{1}{8}$

$P_e \le 2^{-N(R_0 - r)}$

$r_0 = 1 - \log_2(1 + e^{-E/N_0})$

$E = E_b \cdot r$

$R_0 = 1 - \log_2\left(1 + e^{-\frac{E_b r}{N_0}}\right)$

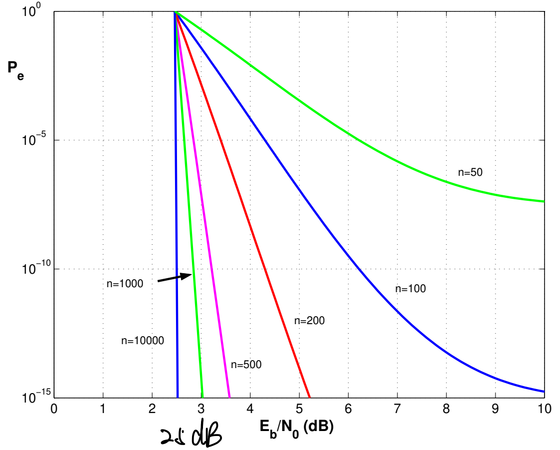


$E_b = \frac{E N}{\log_2 M}$

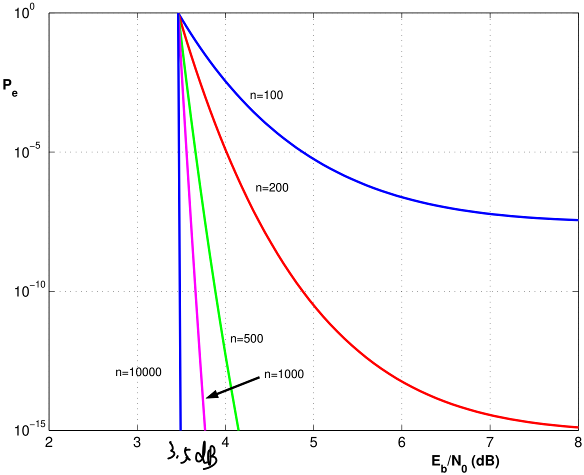
$= \frac{E}{r}$

$M = 2^{rN}$

Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 1/2 codes



Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 3/4 codes



From the theorem, reliable communication ($P_e \to 0$) is possible provided $r < R_0 \leq 1$, i.e.

$$\underbrace{1 - \log_2 \left(1 + \exp\ \{-E_b r/N_0\}\right)}_{R_0} > r$$

$$1 - r > \log_2(1 + e^{-E_b r/N_0})$$

$$2^{1-r} > 1 + e^{-E_b r/N_0} \Rightarrow e^{-E_b r/N_0} < 2^{1-r} - 1$$

$$-E_b r/N_0 < \ln(2^{1-r} - 1) \Rightarrow \boxed{\frac{E_b}{N_0} \geq -\frac{\ln(2^{1-r} - 1)}{r}}$$

$r = \frac{3}{4}$ $\quad \frac{E_b}{N_0} \geq 3.5\ dB$

$-1.59 + 3$ dB

For

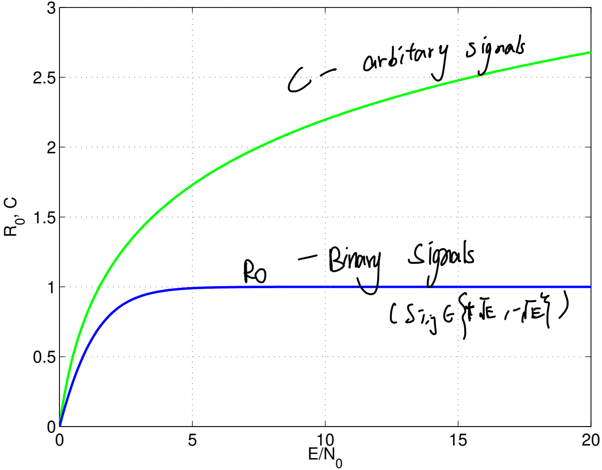
$$r\ \cancel{R} \to 0 \qquad -\frac{\ln(2^{1-\cancel{R}^{\,r}}-1)}{r} \to 2\ln 2 \Rightarrow P_e \to 0 \text{ if } E_b/N_0 > 2\ln 2$$

Note: M orthogonal signals have $P_e \to 0$ if $E_b/N_0 > \ln 2$. The rate of orthogonal signals is

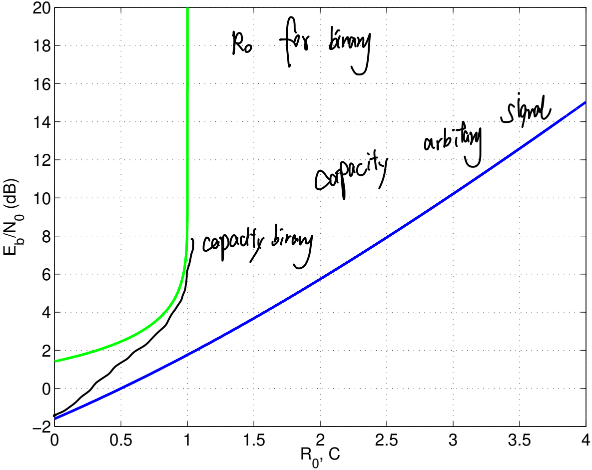
$$r = \frac{\log_2 M}{N} = \frac{\log_2 M}{M} \to 0 \text{ as } M \to \infty$$

- The theorem guarantees existence of signals with $\frac{\log_2 M}{N} = r > 0$ and $P_e \rightarrow 0$ as $M \rightarrow \infty$.
- $r < R_0$ means there exists (binary coefficient) signals with arbitrarily small error probability.

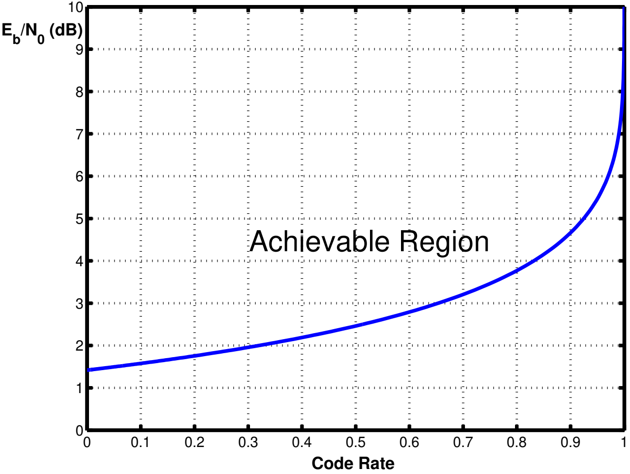
Cutoff Rate for Binary Input-Continuous Output Channel



Cutoff Rate for Binary Input-Continuous Output Channel



Cutoff Rate for Binary Input-Continuous Output Channel



- The parameter R_0 derived determined an region of rates ($r < R_0$) such that there exists signals with error probability that can be made arbitrarily small as the number of dimensions becomes large.
- It was derived for the case that the signals are restricted to have binary coefficients.
- It is not the largest region of rates where the error probability can be made arbitrarily small (the capacity is the largest region of rates where the error probability can be made arbitrarily small).