EECS 455 Exam I: Fall 2020

Instructions:
Print your name and sign the honor code.

Name ____________________

Honor code: _____________________

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

Trig. Identities

$$\begin{aligned}
\sin(u)\cos(v) &= \frac{1}{2}[\sin(u+v)+\sin(u-v)] \\
\cos(u)\cos(v) &= \frac{1}{2}[\cos(u-v)+\cos(u+v)] \\
\cos^2(u) &= \frac{1}{2}[1+\cos(2u)] \\
\sin^2(u) &= \frac{1}{2}[1-\cos(2u)] \\
\int_b^c \cos(ax)dx &= \frac{1}{a}\sin(ax)|_b^c \\
\int_b^c \sin(ax)dx &= -\frac{1}{a}\cos(ax)|_b^c
\end{aligned}$$

1. A communication system transmits one of 8 equally likely signals. The signal (waveforms) are represented by the vectors shown below by some suitable set of orthonormal signals.

   (a) Determine how many information bits can be sent using these signals.

$$
\begin{aligned}
s_0 &= (-1,-1,-1,-1,-1)\\
s_1 &= (-1,-1,+3,-3,+3)\\
s_2 &= (-1,+3,-3,+3,-1)\\
s_3 &= (-1,+3,+1,+1,+3)\\
s_4 &= (+3,-3,+3,-1,-1)\\
s_5 &= (+3,-3,-1,-3,+3)\\
s_6 &= (+3,+1,+1,+3,-1)\\
s_7 &= (+3,+1,-3,+1,+3)
\end{aligned}
$$

   Solution: Since there are eight signals $\log_2(8) = 3$ bits can be sent using these signals.

   (b) Determine the energy of each of the signals and the average energy per information bit.

   Solution:

$$
\begin{aligned}
E_0 &= 5\\
E_1 &= 29\\
E_2 &= 29\\
E_3 &= 21\\
E_4 &= 29\\
E_5 &= 37\\
E_6 &= 21\\
E_7 &= 29\\
\bar{E} &= 25\\
\bar{E}_b &= 25/3 = 8.33
\end{aligned}
$$

   (c) Determine the Euclidean distance (or Euclidean distance squared) between signals s_0 and and all the other signals.

   Solution:

$$
\begin{aligned}
d_E^2(s_0,s_1) = 36, &\quad d_E(s_0,s_1) = \sqrt{36} = 6\\
d_E^2(s_0,s_2) = 36, &\quad d_E(s_0,s_2) = \sqrt{36} = 6\\
d_E^2(s_0,s_3) = 40, &\quad d_E(s_0,s_3) = \sqrt{40} = 6.32\\
d_E^2(s_0,s_4) = 36, &\quad d_E(s_0,s_4) = \sqrt{36} = 6
\end{aligned}
$$

$$
\begin{aligned}
d_E^2(s_0,s_5) = 40, \quad d_E(s_0,s_5) = \sqrt{40} = 6.32 \\
d_E^2(s_0,s_6) = 40, \quad d_E(s_0,s_6) = \sqrt{40} = 6.32 \\
d_E^2(s_0,s_7) = 44, \quad d_E(s_0,s_7) = \sqrt{44} = 6.63
\end{aligned}
$$

(d) Determine the rate of communication in bits/dimension for these signals.

$$
\begin{aligned}
s_0 &= (-1,-1,-1,-1,-1) \\
s_1 &= (-1,-1,+3,-3,+3) \\
s_2 &= (-1,+3,-3,+3,-1) \\
s_3 &= (-1,+3,+1,+1,+3) \\
s_4 &= (+3,-3,+3,-1,-1) \\
s_5 &= (+3,-3,-1,-3,+3) \\
s_6 &= (+3,+1,+1,+3,-1) \\
s_7 &= (+3,+1,-3,+1,+3)
\end{aligned}
$$

The rate of communication is 3 bits/ 5 dimensions or 0.6 bits/dimension.

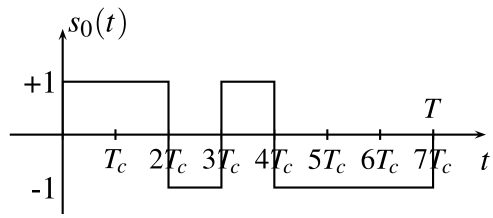
2. (a) Find the filter output when the input is a sequence of four pulses each of duration $T_c = T/7$ as shown below

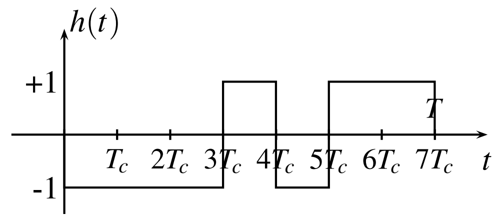
$$s(t) = p_{T_c}(t) + p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c) - p_{T_c}(t - 4T_c) - p_{T_c}(t - 5T_c) - p_{T_c}(t - 6T_c)$$

and the filter is given by.

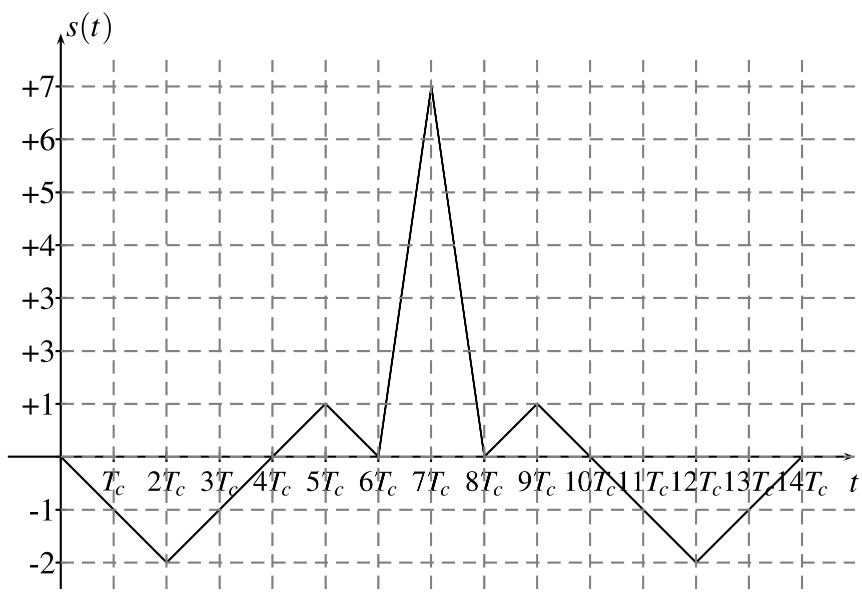
$$h(t) = -p_{T_c}(t) - p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c) - p_{T_c}(t - 4T_c) + p_{T_c}(t - 5T_c) + p_{T_c}(t - 6T_c)$$

as shown below. The output should be a function of time beginning at time 0 and ending at time $2T = 14T_c$.

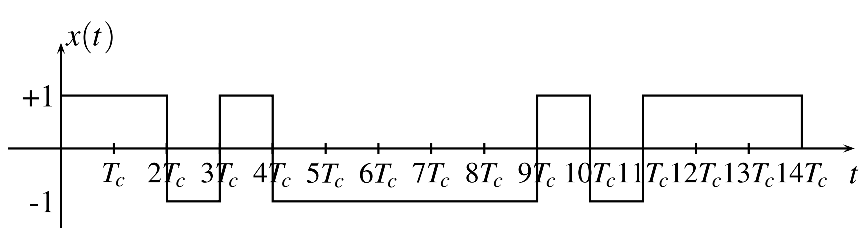




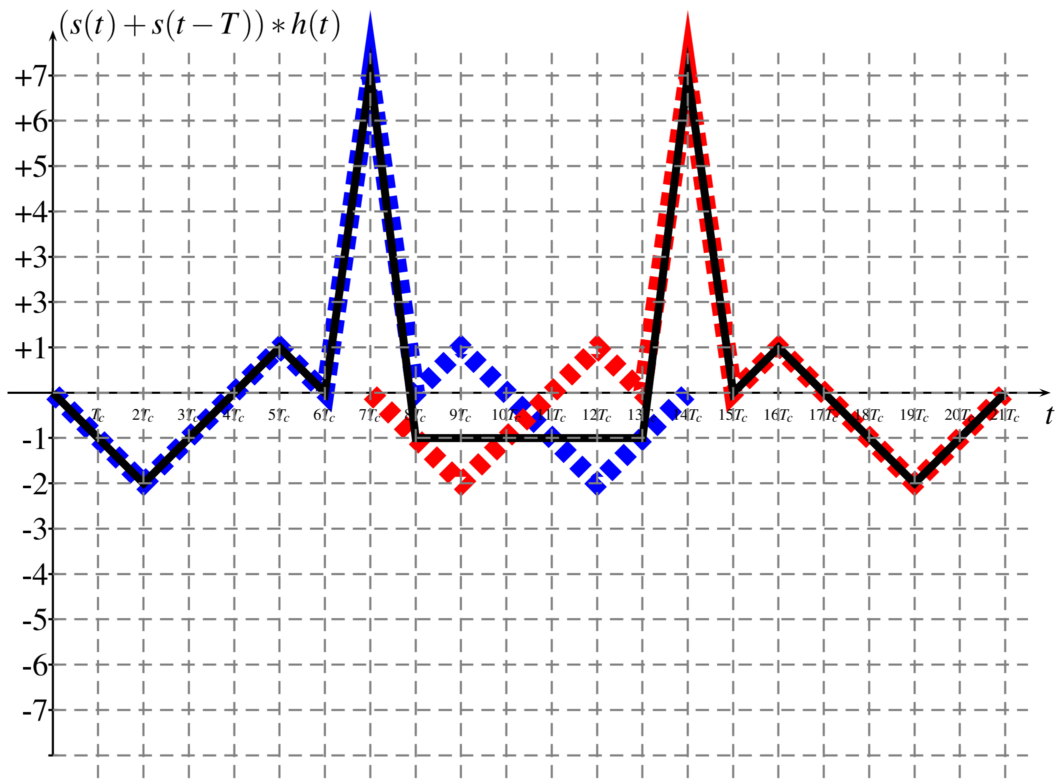
Solution:



(b) Find the filter output (for the same filter) when the input is $x(t) = s(t) - s(t - T)$. The output is a function beginning at time 0 and ending at time $21T_c = 3T$.



Solution: The blue (dashed) curve represents the output due to $s(t)$ and the red (dotted) curve represents the output due to $-s(t-T)$. The total output is the sum of the two and is show by the solid black line.



3. A communication system transmits one of three signals:

$$s_0(t) = A\cos(2\pi f_c t)p_T(t)$$

$$s_1(t) = 0$$

$$s_2(t) = -A\cos(2\pi f_c t)p_T(t)$$

over an additive white Gaussian noise channel with spectral density $N_0/2$. Let $r(t)$ denote the received signal ($r(t) = s_i(t) + n(t)$). The receiver computes the quantity

$$Z = \int_0^T r(t)\cos(2\pi f_c t)dt.$$

Assume $2\pi f_c T = 2\pi n$ for some integer n. Z is compared with a threshold γ and a threshold $-\gamma$. If $Z > \gamma$, the decision is made that $s_0(t)$ was sent. If $Z < -\gamma$, the decision is made that $s_2(t)$ was sent. If $-\gamma < Z < \gamma$ the the decision is made in favor of $s_1(t)$

(a) Determine the three conditional probabilities of error: $P_{e,0} =$ probability of error given s_0 sent, $P_{e,1}$ =probability of error given s_1 sent, and $P_{e,2}$

Solution: Assume signal 0 is transmitted. The decision variable is

$$\begin{aligned}
Z &= \int_0^T r(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt. \\
Z &= \int_0^T (s_0(t) + n(t))\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt. \\
&= \int_0^T A\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\cos(2\pi f_c t)dt + \eta. \\
&= A\sqrt{\frac{2}{T}}\int_0^T [1/2 + 1/2\cos(2\pi f_c t)]dt + \eta. \\
&= A\sqrt{\frac{2}{T}}\int_0^T [1/2 + 1/2\cos(2\pi f_c t)]dt + \eta. \\
&= AT/2\sqrt{\frac{2}{T}} + \eta. \\
&= \sqrt{\frac{A^2 T}{2}} + \eta. \\
&= \sqrt{E} + \eta.
\end{aligned}$$

where η is a Gaussian random variable. The mean of η is zero and the variance of η is calculated as

$$\begin{aligned}
\sigma^2 = \text{Var}\{\eta\} &= E[\int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \int_0^T n(s)\sqrt{\frac{2}{T}}\cos(2\pi f_c s)ds] \\
&= \int_0^T \int_0^T E[n(t)n(s)]\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\sqrt{\frac{2}{T}}\cos(2\pi f_c s)dtds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{T}\int_0^T\int_0^T \frac{N_0}{2}\delta(t-s)\cos(2\pi f_c t)\cos(2\pi f_c s)\,dt\,ds \\
&= \frac{2}{T}\int_0^T \frac{N_0}{2}\cos^2(2\pi f_c t)\,dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T [1/2+1/2\cos(2\pi f_c t)]\,dt \\
&= \frac{2}{T}\frac{N_0}{2}T/2 \\
&= \frac{N_0}{2}
\end{aligned}
$$

The probability of error given signal 0 transmitted is then

$$
\begin{aligned}
P_{e,0} &= P\{\sqrt{E}+\eta<\gamma\} \\
&= P\{\eta<\gamma-\sqrt{E}\} \\
&= \int_{-\infty}^{\gamma-\sqrt{E}} \frac{1}{\sqrt{2\pi}\sigma}e^{-u^2/(2\sigma^2)}\,du \\
&= Q(\frac{\sqrt{E}-\gamma}{\sigma})
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P_{e,2} &= P\{-\sqrt{E}+\eta>-\gamma\} \\
&= Q(\frac{\sqrt{E}-\gamma}{\sigma}).
\end{aligned}
$$

Finally

$$
\begin{aligned}
P_{e,1} &= 1-P\{-\gamma<\eta<\gamma\} \\
&= 1-[\Phi(\gamma/\sigma)-\Phi(-\gamma/\sigma)] \\
&= 1-Q(-\gamma/\sigma)+Q(\gamma/\sigma) \\
&= 2Q(\gamma/\sigma)
\end{aligned}
$$

(b) Determine the average error probability assuming that all three signals are equally probable of being transmitted.

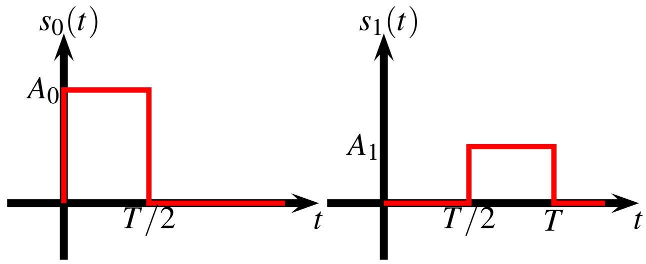
Solution: The average error probability is

$$
\bar{P}_e = \frac{1}{3}P_{e,0}+\frac{1}{3}P_{e,1}+\frac{1}{3}P_{e,2}
$$

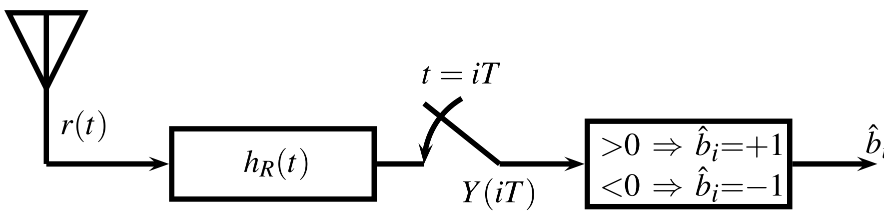
4. Consider a binary communication system that transmits one of two signal $s_0(t)$ and $s_1(t)$ over an additive white Gaussian noise channel (spectral density $N_0/2$) where

$$s_0(t) = A_0 p_{T/2}(t), \quad s_1(t) = A_1 p_{T/2}(t - T/2);$$

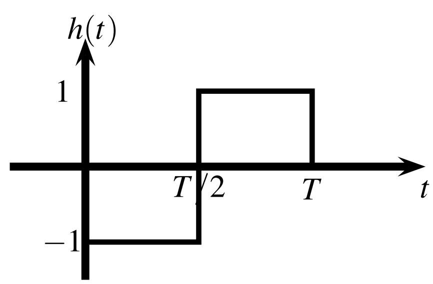
that is $s_0(t)$ is a pulse of amplitude A_0 from 0 to $T/2$ and $s_1(t)$ is a pulse of amplitude A_1 from $T/2$ to T.



The receiver shown below consist of a filter $h(t)$ which is sampled at time T and a threshold device.



(a) If $h(t) = -p_{T/2}(t) + p_{T/2}(t - T/2)$ find the output of the filter $\hat{s}_0(T)$ due to signal $s_0(t)$ at time T and the output of the filter $\hat{s}_1(T)$ due to signal $s_1(t)$ at time T



Solution: The output due to signal 0 is

$$\begin{aligned}
\hat{s}_0(T) &= \int h(T-\tau)s_0(\tau)d\tau \\
&= \int_0^{T/2} s_0(\tau)d\tau - \int_{T/2}^{T} s_0(\tau)d\tau \\
&= \int_0^{T/2} A_0 d\tau \\
&= A_0 T/2.
\end{aligned}$$

The output due to signal 1 is

$$
\begin{aligned}
\hat{s}_1(T) &= \int h(T-\tau)s_1(\tau)d\tau \\
&= \int_0^{T/2} s_1(\tau)d\tau - \int_{T/2}^{T} s_1(\tau)d\tau \\
&= -\int_{T/2}^{T} A_1 d\tau \\
&= -A_1T/2.
\end{aligned}
$$

(b) Find the threshold γ that will minimize the average of the error probabilities $P_{e,0}$ and $P_{e,1}$ for the given signals and filter. Assume $\pi_0 = \pi_1$.

Solution: The threshold γ that will minimize the average error probability is

$$
\begin{aligned}
\gamma_{opt} &= \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2} \\
&= (A_0 - A_1)T/4.
\end{aligned}
$$

(c) Find the error average error probability $\bar{P}_e = \pi_0 P_{e,0} + \pi_1 P_{e,1}$ for the threshold found in the previous part. Assume $\pi_0 = \pi_1$.

The corresponding error probability is

$$\bar{P}_e = Q(\alpha\lambda)$$

$$
\begin{aligned}
\lambda &= \frac{(h, s_T)}{||h||||s_T||} \\
&= \frac{(A_0+A_1)T/2}{\sqrt{T}\sqrt{A_0^2T/2 + A_1^2T/2}} \\
&= \frac{(A_0+A_1)}{\sqrt{2}\sqrt{A_0^2+A_1^2}}
\end{aligned}
$$

$$
\begin{aligned}
E_0 &= A_0^2T/2 \\
E_1 &= A_1^2T/2 \\
\bar{E} &= (A_0^2+A_1^2)T/4 \\
r &= (s_0(t), s_1(t))/\bar{E} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}} \\
&= \sqrt{\frac{T(A_0^2+A_1^2)}{4N_0}}
\end{aligned}
$$

$$\begin{aligned}\alpha\lambda &= \sqrt{\frac{T(A_0^2+A_1^2)}{4N_0}}\frac{(A_0+A_1)}{\sqrt{2}\sqrt{A_0^2+A_1^2}} \\ &= \sqrt{\frac{(A_0+A_1)^2T}{8N_0}}\end{aligned}$$

So

$$\bar{P}_e = Q(\sqrt{\frac{(A_0+A_1)^2T}{8N_0}}).$$

(d) Find the matched filter for the same signals and find the corresponding threshold that minimizes $\bar{P}_e$. Assume $\pi_0 = \pi_1$.

Solution:

$$h_{opt} = s_0(T-t) - s_1(T-t) = -A_1 p_{T/2}(t) + A_0 p_{T/2}(t - t/2)$$

$$\gamma_{opt} = \frac{T}{4}(A_0^2 - A_1^2)$$

(e) Find $\bar{P}_e$ for the matched filter with the optimum threshold. Assume $\pi_0 = \pi_1$.

Solution: For the matched filter $\lambda = 1$ so the error probability is

$$\begin{aligned}\bar{P}_e &= Q(\alpha) \\ &= Q(\sqrt{\frac{T(A_0^2+A_1^2)}{4N_0}})\end{aligned}$$