Homework 6 EECS 501 YUZHAN JIANG

P1: Given that $Z_1 = \sqrt{X+Y}$ and $Z_2 = \frac{X}{X+Y}$.

Using change variable formula,

$(X, Y) = (z_1^2 z_2,\ z_1^2(1-z_2))$

then we can get Jacobi =

$$J(z_1, z_2) = \begin{bmatrix} \frac{\partial x}{\partial z_1} & \frac{\partial x}{\partial z_2} \\ \frac{\partial y}{\partial z_1} & \frac{\partial y}{\partial z_2} \end{bmatrix} = \begin{bmatrix} 2z_1 z_2 & z_1^2 \\ 2z_1(1-z_2) & -z_1^2 \end{bmatrix}$$

$$\therefore |\det(J(z_1, z_2))| = |-2z_1^3 z_2 - 2z_1^3(1-z_2)|$$
$$= |-2z_1^3|$$
$$= 2z_1^3$$

Since X and Y are ind. exponential random variable

$$f_{XY}(X, Y) = \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

By using changing variable formula,

$$f_{z_1 z_2}(z_1, z_2) = f_{XY}(z_1^2 z_2,\ z_1^2(1-z_2))\,|\det(J(z_1, z_2))|$$

$$= e^{-(z_1^2 z_2 + z_1^2(1-z_2))} \cdot 2z_1^3$$

$$= e^{-(z_1^2)}\, 2 \cdot z_1^3$$

$$\therefore f_{z_1 z_2}(z_1, z_2) = \begin{cases} e^{-z_1^2} \cdot 2z_1^3, & z_1 \geq 0,\ 0 \leq z_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

P2:

$X \sim U[0,1]$, $Y \sim U[0,1]$, $Z \sim U[0,1]$

(a) Let A = Y falls in between X and Z,

B = $0 \le X < Y < Z \le 1$ or $0 \le Z < Y < X \le 1$

$$P(A) = 2P(B) = 2\int_0^1 \int_x^1 \int_y^1 1 \cdot dz \cdot dy \cdot dx$$

$$= 2\int_0^1 \int_x^1 1-y \cdot dy \cdot dx$$

$$= 2\int_0^1 y - \frac{1}{2}y^2 \Big|_x^1 \cdot dx$$

$$= 2\int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2 \, dx$$

$$= 2\left(\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \Big|_0^1\right)$$

$$= \frac{1}{3}$$

(b) Let A = largest of three is greater than the sum of the other two

B = $X > Y+Z$ if X is largest or $Y > X+Z$ if Y is largest or $Z > X+Y$ if Z is largest

$$P(A) = 3P(B) = 3 \cdot P(X > Y+Z) \qquad 0 \le Y+Z < X \le 1$$

$$= 3 \cdot \int_0^1 \int_0^x \int_0^{x-Y} 1 \cdot dz \, dy \cdot dx$$

$$= 3 \int_0^1 \int_0^x x - Y \cdot dy \cdot dx$$

$$= 3 \int_0^1 xy - \frac{1}{2}y^2 \Big|_0^x \cdot dx$$

$$= 3 \int_0^1 \frac{1}{2}x^2 \cdot dx$$

$$= 3 \cdot \frac{1}{6}x^3 \Big|_0^1$$

$$= \frac{1}{2}$$

P2 (c)

Assume that X is the smallest number and Y is the largest number

$\therefore$ the CDF of M is

$$P(M \le m) = P(X+Y \le m)$$

$$= \int_0^m \int_0^{m-x} 1 \cdot dy \cdot dx$$

$$= \int_0^m m-x \cdot dx$$

$$= \frac{1}{2}m^2$$

For Any pair of max-min, we can the same CDF of M

Thus, the PDF of M.

$$f_M(m) = \begin{cases} m, & 0 \le m \le 2 \\ 0, & \text{otherwise} \end{cases}$$

P3:

(a) The CDF of R:

$$F_R = P(R \le r)$$
$$= \frac{\pi r^2}{\pi 1^2}$$
$$= r^2 \quad , \quad 0 \le r \le 1$$

then the PDF of F_R

$$f_R = f(R=r) = 2r$$

$$\therefore \quad f_R(r) = \begin{cases} 2r, & 0 \le r \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Assume the disc is centered at 0, and Point is located as (x, y)

Then X and $Y \sim U(-1, 1)$ since the radius of disc is 1

Assume the two points are chosen with (x_1, y_1) and (x_2, y_2)

Then we have $E[Z^2] = E[(X_1 - X_2)^2 + (y_1 - y_2)^2]$ $\quad (Z = \sqrt{(X_1-X_2)^2 - (y_1-y_2)^2}$ By the Euclidean distance$)$

$$= E[x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2]$$
$$= E[x_1^2 + y_1^2] + E[x_2^2 + y_2^2] - 2E[x_1]E[x_2] - 2E[y_1]E[y_2] \quad \text{(By linearity of Expectation)}$$
$$= E[R^2] + E[R^2] \quad (\text{Since } X^2 + Y^2 = R^2 \text{ and } E[X_1] = E[X_2] = E[Y_2] = E[Y_1] = \tfrac{1+(-1)}{2} = 0)$$
$$= 2E[R^2]$$
$$= 2$$

P4:

U_i denote the time spent on website i. Let $V = U_1 + U_2 + \cdots + U_N$ Where $U_i \sim \exp(\lambda)$, $N \sim Geo(p)$

Since
$$
\begin{aligned}
M_U(t) &= E[\exp(tU)] \\
&= \int_{-\infty}^{\infty} \exp(tu)\, f_U(u) \cdot du \\
&= \int_0^{\infty} \exp(t-\lambda)\, u \cdot du \\
&= \frac{\lambda}{\lambda - t}
\end{aligned}
$$

and
$$
\begin{aligned}
M_V(t) &= E[e^{tV}] \\
&= E_N[E_{V|N}(e^{tV}|N) \quad \text{(By law of iterated expentation)} \\
&= E_N(M_U^N(t)) \\
&= E_N((\tfrac{\lambda}{\lambda-t})^n) \\
&= \sum_{n=1} (1-p)^{n-1} p \cdot (\tfrac{\lambda}{\lambda-t})^n \\
&= \frac{\lambda p}{\lambda p - t} \quad \text{(By formula of sum of geometric)}
\end{aligned}
$$

Therefore, we can get $V \sim Exp(\lambda p)$

$$
\therefore \quad f_V(v) = \begin{cases} \lambda p\, e^{-\lambda p}\, v, & v \geq 0 \\ 0, & \text{otherwise} \end{cases}
$$

P5:

$$M_X(t) = E[e^{tx}]$$
$$= \int_0^\infty e^{tx} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \cdot dx$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{(t-\beta)x} \cdot dx$$
$$= \left(\frac{\beta}{\beta-t}\right)^\alpha \quad \text{which is given}$$

$$E[X] = \frac{d M_X(s)}{ds}\Big|_{s=0}$$
$$= \beta^\alpha \alpha \left(\frac{1}{\beta-s}\right)^{\alpha+1}\Big|_{s=0}$$
$$= \frac{\alpha\beta^\alpha}{\beta^{\alpha+1}}$$
$$= \frac{\alpha}{\beta}$$

$$E[X^2] = \frac{d'' M_X(s)}{ds}\Big|_{s=0}$$
$$= \beta^\alpha \alpha \cdot (\alpha+1) \left(\frac{1}{\beta-s}\right)^{(\alpha+2)}\Big|_{s=0}$$
$$= \frac{\alpha(\alpha+1)}{\beta^2}$$

Thus, $Var(X) = E[X^2] - (E[X])^2$
$$= \frac{(\alpha+1)\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2}$$
$$= \frac{\alpha}{\beta^2}$$

P6:

We are given that $Y = aX_1 + bX_2 + C$

$$M_Y(s) = E[e^{sY}]$$
$$= E[e^{sax_1 + sbx_2 + sc}]$$
$$= E[e^{sax_1} \cdot e^{sbx_2} \cdot e^{sc}]$$
$$= E[e^{sax_1}] E[e^{sbx_2}] \cdot E[e^{sc}] \quad \text{(Since } X_1 \text{ and } X_2 \text{ are independent Gaussian r.v.)}$$
$$= M_{x_1}(sa) \cdot M_{x_2}(sb) \cdot e^{sc}$$

Since the MGF of Gaussian r.v is

$$M_X(s) = e^{\mu s + \frac{1}{2}\sigma^2 s^2} \quad \text{where } X \sim (\mu, \sigma^2)$$

Therefore, $M_Y(s) = e^{(\mu_1 a + \mu_2 b + c)s + \frac{1}{2}(\sigma_1^2 a^2 + \sigma_2^2 b^2)s^2}$

Y is also Gaussian r.v $Y \sim N(\mu_1 a + \mu_2 b + c, \sigma_1^2 a^2 + \sigma_2^2 b^2)$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 a^2 + \sigma_2^2 b^2}} \cdot \exp\left(-\frac{1}{2}\left(\frac{y - (\mu_1 a + \mu_2 b + c)}{\sqrt{\sigma_1^2 a^2 + \sigma_2^2 b^2}}\right)^2\right)$$

P7:

(a) $Y = Y_1 + Y_2 + \cdots + Y_n$

$E[Y|Y] = Y$

$E[Y_1 + Y_2 + \cdots + Y_n | Y] = Y$

$E[Y_1] + E[Y_2] + \cdots E[Y_n] = Y$ (Since $Y_1, Y_2, \cdots Y_n$ are independent r.v)

$n E[Y_1] = Y$ (Since $Y_1, Y_2, \cdots Y_n$ are identically r.v)

$E[Y_1] = \frac{Y}{n}$

(b) Given $\theta \sim N(0, k)$ and $W \sim N(0, m)$

Based on part(a):

If we let $\theta = \theta_1 + \theta_2 + \cdots + \theta_k$ where $\theta_1, \cdots \theta_k$ be independent identically $\theta_i \sim N(0,1)$

$W = W_1 + W_2 + \cdots + W_m$ Where $W_1 \cdots W_m$ be independent identically $W_i \sim N(0,1)$

$E[\theta | \theta + W] = E[\theta_1 + \theta_2 \cdots \theta_k | \theta + W]$

$= k\, E[\theta_1 | \theta + W]$

$= \frac{k}{m+k}(\theta + W)$ (By Part (a))