EECS 501: Probability
Final Exam
12/16/2011

• Exam location: 1013 DOW (A-R) & 1010 DOW (S-Z).

• No calculators allowed.

• This is a 2 hour exam, which is closed book. There are 4 problems (section II), worth 40 points total, and one multiple choice section (section I) worth 10 points.

• You are permitted to use two 8 1/2 inch by 11 inch sheet of notes written on both sides. You may also use my chapter 9-12 lecture notes, plus Appendix D. A Fourier transform table is attached to the end of the exam.

• Show <u>all</u> of your work for problems 1-4. No work shown, then <u>no</u> credit given!

• Please write and sign the following honor pledge on your bluebook.

I have neither given nor received aid on this exam.

• Please print your name on this exam sheet and return it with your bluebook.

• I will be sitting outside the exam room during the exam to answer questions.

Do Not Turn Over This Page Until Told to Do So,

Good Luck!

Print Your Name Here:______________________________

Section I - Multiple Choice Section (10 points total)

There is only <u>one</u> correct answer for each problem.
<u>No</u> partial credit will be given.
You <u>do not</u> need to show any work.
You <u>must</u> place your answer for each multiple choice question onto the appropriate line of the table on page 4 to receive credit.

M1. The power spectral density of a wide-sense stationary random process is $S_X(\omega) = \dfrac{1}{1+\omega^2}$. The average power in this process is

(a) ∞
(b) 1.5
(c) 1
(d) 1/2
(e) None of the above.

M2. A zero mean, wide-sense-stationary Gaussian random process, $X(t)$, has autocorrelation function $R_X(\tau) = e^{-\tau^4}$. Then

(a) The average power of this process is $t^2 e^{-t}$.
(b) $X(3) - X(1)$ is a zero mean Gaussian random variable with variance $2 - 2e^{-16}$
(c) $X(2)$ and $X(1)$ are independent random variables
(d) $X(t_1)X(t_2)$ is a Gaussian random variable for all t_1, t_2.
(e) None of the above.

M3. A wide-sense stationary random process with power spectral density

$$S_X(\omega) = \begin{cases} 1+\omega^2, |\omega| \le 2 \\ 0, |\omega| > 2 \end{cases}$$

is the input to a linear time-invariant system described by the following differential equation $\dfrac{dy(t)}{dt} + y(t) = x(t)$, where $x(t)$ is the input and $y(t)$ is the output. The average power at the output of this system is

(a) $6/\pi$
(b) $1/\pi$
(c) $2/\pi$
(d) $3/\pi$
(e) None of the above.

M4. A continuous-time random process has a sample space with four sample points s_1, s_2, s_3, s_4 that occur with probabilities 1/4, 1/2, 1/8 and 1/8, respectively. The sample functions corresponding to these sample points are given by
$X(t,s_1)=\cos(2t+\pi/4),\ X(t,s_2)=\sin(2t+\pi/4),\ X(t,s_3)=-\cos(2t+\pi/4),\ X(t,s_4)=-\sin(2t+\pi/4)$
Then $P\{X(0)\geq 0 \text{ and } X(\pi/8)\geq 0\}$ is equal to
(a) 7/8
(b) 1/4
(c) 3/8
(d) 3/4
(e) None of the above.

M5. The renewal function for a stationary and independent increment, arrival process equals
(a) The expected inter-arrival time.
(b) The expected number of arrivals in a time interval of duration t.
(c) The autocorrelation function of $N(t)$, where $N(t)$ equals the number of arrivals in an interval of time t.
(d) The second moment of $N(t)$, where $N(t)$ equals the number of arrivals in an interval of time t.
(e) None of the above.

M6. A finite state Markov chain has the following state transition matrix

$$\underline{P}=\begin{bmatrix}1/2 & 1/4 & 1/4\\ 0 & 1/4 & 3/4\\ 1/2 & 1/2 & 0\end{bmatrix}$$

Then the Markov chain
(a) Has unique stationary probability distribution.
(b) Is not irreducible.
(c) Is not indecomposable.
(d) Contains transient states.
(e) None of the above.

M7. A finite state Markov chain has the following state transition matrix

$$\underline{P}=\begin{bmatrix}3/4 & 1/4 & 0 & 0\\ 1/3 & 1/3 & 1/3 & 0\\ 0 & 0 & 2/5 & 3/5\\ 0 & 0 & 1/2 & 1/2\end{bmatrix}$$

Then this Markov chain
(a) Does not have stationary distribution.
(b) Does not have a steady state distribution.
(c) Is irreducible.
(d) Has two transient states.
(e) None of the above.

M8. A finite state Markov chain has the following state transition matrix

$$\underline{P} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Then

(a) $\lim_{n\to\infty}(\underline{P}^n)_{23} = 1/2$

(b) The stationary distribution is not unique.

(c) There are two equivalence classes of communication states.

(d) The expected number of times that state 2 is revisited starting form state 2 is 4.

(e) None of the above.

M9. Consider a Poisson process with arrival times $Y_1, Y_2, \ldots$ Then

(a) The arrival times are independent random variables.

(b) $P\{Y_5 - Y_3 > Y_{11} - Y_9\} = 1/2$.

(c) $Y_{10} - Y_7$ is exponentially distributed.

(d) $P\{0 < Y_2 < 1/3 | 3 \text{ arrivals occured in the time interval } (0,1)\} = 2/3$.

(e) None of the above.

M10. Let $X_i, i = 1,2,\ldots$ be a Bernoulli random process. Define a new random process $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i,\ n = 1,2,\ldots$ Then

(a) S_n converges in probability but does not converge almost surely.

(b) S_n converges in mean-square.

(c) S_n converges almost surely but not in mean-square.

(d) S_n converges in mean-square but not almost surely.

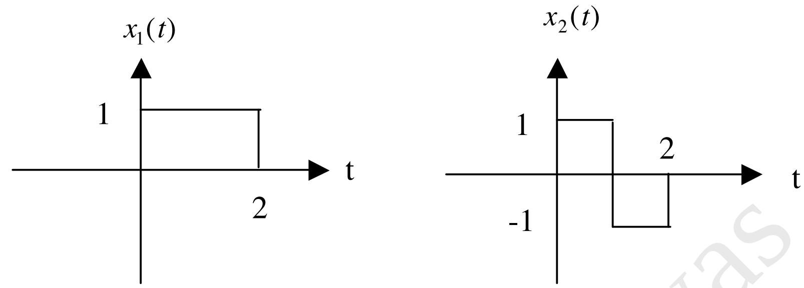
(e) None of the above.

You must place your answers in the table below.

Problem	**Fill in the one correct letter on each line below**
M1	
M2	
M3	
M4	
M5	
M6	
M7	
M8	
M9	
M10	

Section II - Problems (40 points total)
You <u>must</u> show your work to get credit.

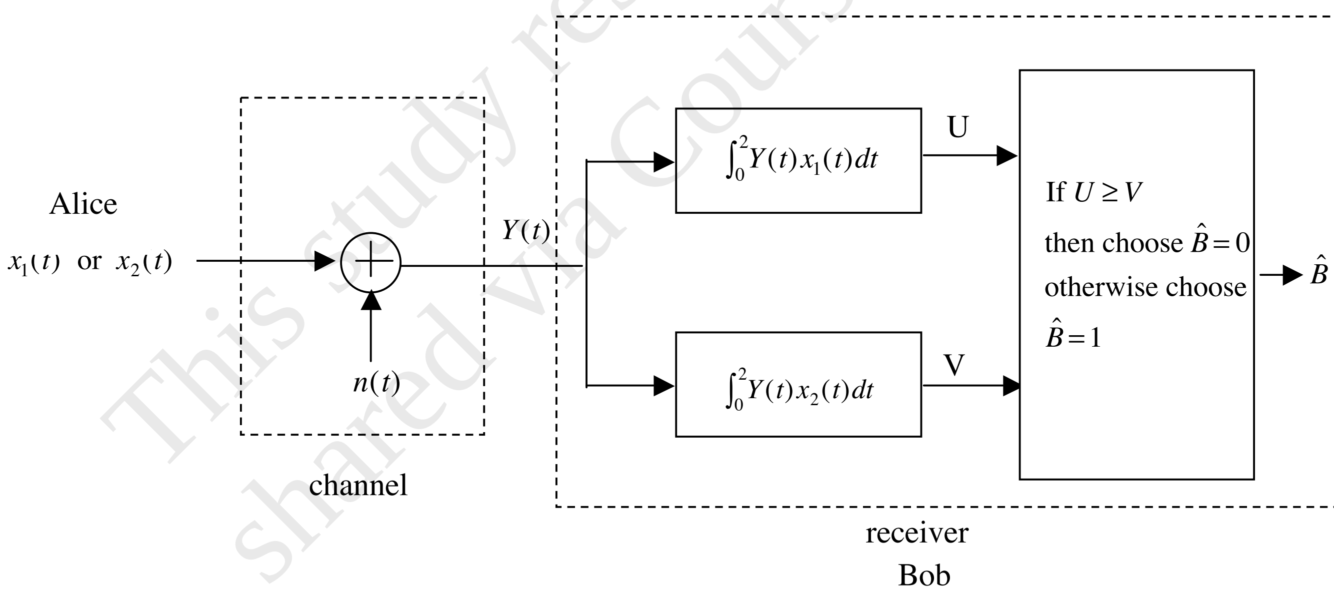
Problem 1 (10 points)
Consider a binary communication system that sends a bit (i.e., $B=0$ or $B=1$) across a channel from Alice to Bob using the two waveforms, $x_1(t)$ and $x_2(t)$, shown below



$B=0 \Rightarrow$ transmit $x_1(t)$

$B=1 \Rightarrow$ transmit $x_2(t)$

Zero mean, white Gaussian noise, $n(t)$, with power spectral density $S_n(\omega)=1, \quad |\omega|<\infty$ is added to the transmitted waveform resulting in the random process $Y(t)$ shown in the figure below. Bob receives $Y(t)$ and processes it as shown below with his receiver to produce an estimate, $\hat{B}$, of Alice's transmitted bit B.



(a) Find the conditional probability density, $f_{U,V|B}(u,v\,|\,B=0)$, of U,V given $B=0$.
(b) Find the probability of error given that Alice transmits the bit $B=0$, i.e., find $P(\hat{B}\neq B\,|\,B=0)=P(U<V\,|\,B=0)$.

Problem 2 (10 points)
A jar initially contains 2 balls, each of which may be either white or black. At a set of N discrete times $1,2,3,...,N$ a ball is randomly chosen from the jar and replaced by a white ball with probability p or with a black ball with probability $1-p$. Let λ_n denote the probability that at time n the jar has 2 white balls. Does $\lim_{n\to\infty} \lambda_n$ exist? You must justify your answer. If the limit does exist then find its value.

Problem 3 (12 points)
A data server serves two independent data streams. The arrivals of the data packets in the two streams are independent Poisson processes with rates λ_1 and λ_2, respectively. Each packet arriving in the first data stream has probability p of being corrupted, while packets arriving in the second stream have been corrupted with probability q. Different packets are corrupted independently of one another.

(a) The server closes down for 1 ms at 12 midnight and then resumes operation. After operation resumes, find the probability that server will encounter a corrupted packet in stream 1 before it encounters a corrupted packet in stream two.
(b) Find the expected time that elapses between the arrival of two consecutive corrupted packets at the server.
(c) Given that 2 corrupted data packets arrive in data stream one in a time interval of duration η, find the probability that in stream one 3 uncorrupted data packets also arrive in this same time interval.
(d) The server will temporarily shut down when it detects that two consecutive corrupted packets arrive within τ_0 units of time of one another. What is the expected time until the server shuts down?

Problem 4 (8 points)
Consider a continuous-time stochastic process, $X(t)$, that satisfies the following three conditions (Note such a process does exist):
(i) $X(0)=0$;
(ii) the process $\{X(t), t\geq 0\}$ has stationary and independent increments;
(iii) for every $t>0$, $X(t)$ is a Gaussian random variable with mean 0 and variance $\sigma^2 t$, where σ is a constant.
(a) Show that $E[X(t_1)X(t_2)] = \sigma^2 \min(t_1,t_2)$.
Hint: $X(t_1)X(t_2) = X^2(t_1) + X(t_1)[X(t_2) - X(t_1)]$
(b) Is $X(t)$ wide-sense-stationary? Justify your answer.
(c) Derive a expression for the joint probability density of $X(t_1), X(t_2), X(t_3)$ for the case when $\sigma = 1$.