EECS 501 Homework 7 YUZHAN JIANG

P1: We are given that Gaussian Vector (X_1, X_2, X_3) with zero mean and Cov matrix

$$\begin{bmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix} \qquad E[X_1] = E[X_2] = E[X_3] = 0$$

(a) $E[X_1 | X_2, X_3]$ directly from $f_{X_1|X_2,X_3}(x_1|x_2,x_3)$

$$f_{X_1|X_2,X_3}(x_1|x_2,x_3) = \frac{\frac{1}{(\sqrt{2\pi})^3\sqrt{\det(C)}} e^{-\frac{1}{2}([x_1\ x_2\ x_3]C^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix})}}{\frac{1}{(\sqrt{2\pi})^2\sqrt{\det C_{2\times2}}} e^{-\frac{1}{2}([x_2\ x_3]\cdot C_{2\times2}^{-1}\begin{bmatrix}x_2\\x_3\end{bmatrix})}}$$

$$= \frac{\frac{1}{\sqrt{8\pi^3}\cdot\sqrt{\det(C)}}\cdot e^{-\frac{1}{2}([x_1\ x_2\ x_3]\cdot\frac{1}{27}\begin{bmatrix}7 & -2 & -2\\ -2 & 7 & -2 \\ -2 & -2 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix})}}{\frac{1}{\sqrt{4\pi^2}\sqrt{\det C_{2\times2}}}\cdot e^{-\frac{1}{2}([x_2\ x_3]\cdot\frac{1}{21}\begin{bmatrix}5 & -2\\ -2 & 5\end{bmatrix}\begin{bmatrix}x_2\\x_3\end{bmatrix})}}$$

$$= \sqrt{\frac{7}{54\pi}}\, e^{-\frac{1}{2}([x_1\ x_2\ x_3]\cdot\frac{1}{27}\begin{bmatrix}7 & -2 & -2\\ -2 & 7 & -2 \\ -2 & -2 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \frac{1}{21}[x_2\ x_3]\begin{bmatrix}5 & -2\\ -2 & 5\end{bmatrix}\begin{bmatrix}x_2\\x_3\end{bmatrix})}$$

$$= \sqrt{\frac{7}{54\pi}}\, e^{-\frac{1}{2}(\frac{7}{27}(x_1 - \frac{2}{7}(x_2+x_3))^2}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{\frac{27}{7}}}\cdot e^{-\frac{1}{2}\frac{1}{(\sqrt{\frac{27}{7}})^2}(x_1 - \frac{2}{7}(x_2+x_3))^2}$$

Therefore, $f_{X_1|X_2,X_3}$ is also Gaussian distribution with mean $= \frac{2}{7}(x_2+x_3)$ and $\text{Var} = \sqrt{\frac{27}{7}}$

$\therefore\ E[X_1|X_2,X_3] = \frac{2}{7}(X_2+X_3)$

1(b) The LMMSE of x_1 from x_2 and x_3:

From the formula:

$$E[x|Y] = E[x] + Cov(x,Y)\,Var(Y)^{-1}(Y - E[Y])$$

$$\therefore E[x_1|x_2\,x_3] = A\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} E[x_2] \\ E[x_3] \end{bmatrix}\right) + m[x_1]$$

$$= A\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}\right)$$

$$= A\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

$$\Longrightarrow A \cdot C_{x_2,x_3} = C_{x_1,x_2x_3}$$

$$A\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 \end{bmatrix}\left(\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\right)^{-1}$$

$$A = \left[\frac{2}{7}, \frac{2}{7}\right]$$

Therefore, $E[x_1|x_2,x_3] = \left[\frac{2}{7}, \frac{2}{7}\right]\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$

$$= \underbrace{\frac{2}{7}x_2 + \frac{2}{7}x_3}$$

LMMSE of x_1 which is the same as $E[x_1|x_2,x_3]$

1(c) $Var(x_1|x_2,x_3) = E[(x_1 - g(x_2,x_3))^2]$

$$= E\left[\left(x_1 - \left(\frac{2}{7}x_2 + \frac{2}{7}x_3\right)\right)^2\right]$$

$$= E\left[x_1^2 - 2x_1\left(\frac{2}{7}x_2 + \frac{2}{7}x_3\right) + \left(\frac{2}{7}x_2 + \frac{2}{7}x_3\right)^2\right]$$

$$= E\left[x_1^2 - \frac{4}{7}x_1x_2 - \frac{4}{7}x_1x_3 + \frac{4}{49}x_2^2 + \frac{8}{49}x_2x_3 + \frac{4}{49}x_3^2\right]$$

$$= E[x_1^2] - \frac{4}{7}E[x_1x_2] - \frac{4}{7}E[x_1x_3] + \frac{4}{49}E[x_2^2] + \frac{8}{49}E[x_2x_3] + \frac{4}{49}E[x_3^2]$$

$$= Var(x_1^2) - \frac{4}{7}cov(x_1,x_2) - \frac{4}{7}cov(x_1,x_3) + \frac{4}{49}Var(x_2^2) + \frac{8}{49}cov(x_2,x_3) + \frac{4}{49}Var(x_3^2)$$

$$= 5 - \frac{4}{7}\cdot 2 - \frac{4}{7}\cdot 2 + \frac{4}{49}\cdot 5 + \frac{8}{49}\cdot 2 + \frac{4}{49}\cdot 5$$

$$= \frac{27}{7}$$

P2.

We are given that $f_{XZ}(x,z) = f_X(x) f_Y(z-x)$, the joint PDF of X and Z, where $Z = X+Y$. And $X_1 \sim exp(1)$ and $X_2 \sim exp(1)$, $Y_1 = X_1$, $Y_2 = X_1 + X_2$

$R(Y_1) = [0, \infty)$, $R(Y_2) = [0, \infty)$

$f_{X_1}(x) = e^{-x_1}$ $\quad f_{X_2}(x) = e^{-x_2}$

Therefore, by formula

$$E[Y_1|Y_2] = \int_0^\infty Y_1 \cdot f_{Y_1|Y_2}(y_1|y_2) \cdot dy_1$$
$$= \int_0^\infty Y_1 \cdot \frac{f_{Y_1Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} \cdot dy_1$$

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1}(x_1)\, f_{X_2}(y_2 - x_1)$$
$$= f_{X_1}(x_1)\, f_{X_2}(x_2)$$
$$= e^{-x_1} \cdot e^{-x_2}$$
$$= e^{-(x_1+x_2)}$$

and

$$f_{Y_2}(y_2) = \int_0^{y_2} f_{Y_1Y_2}(y_1, y_2) \cdot dy_1$$
$$= \int_0^{y_2} e^{-y_2}\, dy_1$$
$$= y_2 e^{-y_2}$$

Overall,

$$E[Y_1|Y_2] = \int_0^\infty Y_1 \cdot f_{Y_1|Y_2}(y_1|y_2) dy_1$$
$$= \int_0^\infty Y_1 \frac{f_{Y_1Y_2}(Y_1, Y_2)}{f_{Y_2}(y_2)} dy_1$$
$$= \int_0^\infty Y_1 \cdot \frac{e^{-(x_1+x_2)}}{y_2 e^{-y_2}} \cdot dy_1$$
$$= \int_0^\infty Y_1 \cdot \frac{e^{-y_2}}{y_2 e^{-y_2}} dy_1$$
$$= \int_0^{y_2} y_1 \frac{1}{y_2} dy_1$$
$$= \frac{1}{2} y_1^2 \frac{1}{y_2} \Big|_0^{y_2}$$
$$= \frac{y_2}{2}$$

$\therefore$ the MMSE estimate of Y_1 using Y_2 is $\frac{Y_2}{2}$

P3.

(a) $f(x) = E[|X-a|]$

$$f(x) = \int_{-\infty}^{a} -(x-a) f_X(x) \cdot dx + \int_{a}^{\infty} (x-a) f_X(x) \cdot dx$$

$$\Rightarrow f'(x) = \int_{-\infty}^{a} f_X(x) \cdot dx + a f_X(a) - a f_X(a) - a f_X(a) - \int_{a}^{\infty} f_X(x) \cdot dx + a f_X(a)$$

$$= \int_{-\infty}^{a} f_X(x) \cdot dx - \int_{a}^{\infty} f_X(x) \cdot dx$$

let $f'(x) = 0$

$$\Rightarrow \int_{-\infty}^{a} f_X(x) \cdot dx - \int_{a}^{\infty} f_X(x) \cdot dx = 0$$

$$\Rightarrow \int_{-\infty}^{a} f_X(x) \cdot dx = \int_{a}^{\infty} f_X(x)\, dx$$

$$\Rightarrow F_X(x) \Big|_{-\infty}^{a} = F_X(x) \Big|_{a}^{\infty}$$

$$\Rightarrow F_X(a) - F_X(-\infty) = F_X(\infty) - F_X(a)$$

$$\Rightarrow 2F_X(a) = \underbrace{F_X(-\infty) + F_X(\infty)}_{1}$$

Since we are given that $F_X(a^*) = \frac{1}{2}$

$\therefore$ $F_X(a) = \frac{1}{2}$

$a = F_X^{-1}(\frac{1}{2})$

3(b)

$$\min_{g(\cdot)} E[X - g(Y)]$$

$$E[|x - g(Y)|] = \int f_Y(y) \cdot dy \left(\int f_{X|Y}(x) \cdot dx \mid x - g(y) | \right)$$

Based on part (a)

$$\int_{-\infty}^{g(y)} f_{X|Y}(X|Y) \cdot dx = \int_{g(y)}^{\infty} f_{X|Y}(X|Y) \cdot dx$$

$$F_{X|Y}(g^*(Y)|Y) = \frac{1}{2}$$

P4

(a) X, Y and Z are three random variables with known Var and Cov

By the formula of LMSE,

$$g(X,Y) = A\left(\begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} E[X] \\ E[Y] \end{bmatrix}\right) + E[Z]$$

$$A\, C_{XY} = C_{Z,XY}$$

$$A = C_{X,XY} \cdot C_{XY}^{-1}$$

We are given that X and Y are uncorrelated

$$\therefore \ \mathrm{Cov}(X,Y) = 0$$

$$\therefore \ C_{XY} = \begin{bmatrix} \mathrm{Var}(X) & 0 \\ 0 & \mathrm{Var}(Y) \end{bmatrix}$$

And since $\mathrm{Var}(X) > 0$ and $\mathrm{Var}(Y) > 0$

$\therefore$ C_{XY} is invertible and

$$C_{XY}^{-1} = \begin{bmatrix} \frac{1}{\mathrm{Var}(X)} & 0 \\ 0 & \frac{1}{\mathrm{Var}(Y)} \end{bmatrix}$$

$$\therefore \ A = \begin{bmatrix} \mathrm{Cov}(Z,X) & \mathrm{Cov}(Z,Y) \end{bmatrix} \begin{bmatrix} \frac{1}{\mathrm{Var}(X)} & 0 \\ 0 & \frac{1}{\mathrm{Var}(Y)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mathrm{Cov}(Z,X)}{\mathrm{Var}(X)} , & \frac{\mathrm{Cov}(Z,Y)}{\mathrm{Var}(Y)} \end{bmatrix}$$

$$\therefore \ g(X,Y) = \frac{\mathrm{Cov}(Z,X)}{\mathrm{Var}(X)} \cdot (Z - E(X)) + \frac{\mathrm{Cov}(Z,Y)}{\mathrm{Var}(Y)} \cdot (Y - E[Y]) + E[Z]$$

b) In general case, X, Y are not uncorrelated

Thus,

$$C_{XY} = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$$

and $\det(C)$ is not 0 in general case, thus C_{XY}^{-1} exists.

$$\therefore \quad g(X,Y) = \begin{bmatrix} Cov(Z,X) & Cov(Z,Y) \end{bmatrix} \left(\begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix} \right)^{-1} \begin{bmatrix} X - E[X] \\ Y - E[Y] \end{bmatrix} + E[Z]$$

P5.

We are given that

$$\begin{pmatrix} 1 & 0.5 & 0.5 & 0 \\ 0.5 & 1 & 0.5 & 0.25 \\ 0.5 & 0.5 & 1 & 0.25 \\ 0 & 0.25 & 0.25 & 1 \end{pmatrix}$$

and $E[Y_1] = E[Y_2] = E[Y_3] = E[X] = 0$

(a)

$$\tilde{Y}_1 = Y_1 - E[Y_1] = Y_1$$

$$\tilde{Y}_2 = Y_2 - E[Y_2] - \operatorname{cov}(Y_2, \tilde{Y}_1) \operatorname{Var}(\tilde{Y}_1)^{-1} \tilde{Y}_1$$

$$= Y_2 - \operatorname{cov}(Y_2, Y_1) \operatorname{Var}(Y_1)^{-1} Y_1$$

$$= Y_2 - \frac{1}{2} 1 \cdot Y_1$$

$$= Y_2 - \frac{1}{2} Y_1$$

$$\tilde{Y}_3 = Y_3 - E[Y_3] - \sum_{k=1}^{2} \operatorname{cov}(Y_3, \tilde{Y}_k) \cdot \operatorname{Var}(\tilde{Y}_k)^{-1} \tilde{Y}_k$$

$$= Y_3 - \operatorname{cov}(Y_3, \tilde{Y}_2) \cdot \operatorname{Var}(\tilde{Y}_2)^{-1} \tilde{Y}_2 - \operatorname{cov}(Y_3, \tilde{Y}_1) \operatorname{Var}(\tilde{Y}_1)^{-1} \tilde{Y}_1$$

$$= Y_3 - \underline{\operatorname{cov}(Y_3, (Y_2 - \tfrac{1}{2} Y_1))} \cdot \underline{\operatorname{Var}(Y_2 - \tfrac{1}{2} Y_1)^{-1}} (Y_2 - \tfrac{1}{2} Y_1) - \underline{\operatorname{cov}(Y_3, Y_1)} \operatorname{Var}(Y_1)^{-1} Y_1$$

$$\operatorname{Cov}(Y_3, (Y_2 - \tfrac{1}{2} Y_1)) = E[Y_3 (Y_2 - \tfrac{1}{2} Y_1)] - E[Y_3] E[Y_2 - \tfrac{1}{2} Y_1]$$

$$= E[Y_3 Y_2 - \tfrac{1}{2} Y_3 Y_1]$$

$$= E[Y_3 Y_2] - \tfrac{1}{2} E[Y_3 Y_1]$$

$$= \operatorname{Cov}(Y_3, Y_2) - \tfrac{1}{2} \operatorname{Cov}(Y_3, Y_1)$$

$$= \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

$$\operatorname{Var}(Y_2 - \tfrac{1}{2} Y_1) = \operatorname{Var}(Y_2) + \tfrac{1}{4} \operatorname{Var}(Y_1) - E[(Y_2 - \underbrace{E[Y_2]}) (Y_1 - \underbrace{E[Y_1]})]$$

$$= 1 + \frac{1}{4} - \frac{1}{2}$$

$$= \frac{3}{4}$$

$$\therefore \tilde{Y}_3 = Y_3 - \frac{1}{4} \cdot \frac{4}{3} \cdot (Y_2 - \tfrac{1}{2} Y_1) - \frac{1}{2} \cdot 1 \cdot Y_1$$

$$= Y_3 - \frac{1}{3} Y_2 - \frac{1}{3} Y_1$$

(b) Compute the LMSE of X

$$\hat{E}[X|\hat{Y}_1, \hat{Y}_2, \tilde{Y}_3] = E[X] + \sum_{i=1}^{3} \hat{E}[X - E[X] \,|\, \hat{Y}_i]$$
$$= \hat{E}[X|\hat{Y}_1] + \hat{E}[X|\hat{Y}_2] + \hat{E}[X|\tilde{Y}_3]$$

Where
$$\hat{E}[X|\hat{Y}_1] = \hat{E}[X|Y_1]$$
$$= cov(X, Y_1)\, Var(Y_1)^{-1}\, Y_1$$
$$= 0.$$

$$\hat{E}[X|\hat{Y}_2] = cov(X, \hat{Y}_2)\, Var(\hat{Y}_2)^{-1}\, \tilde{Y}_2$$
$$= Cov(X, Y_2 - \tfrac{1}{2}Y_1) \cdot Var(Y_2 - \tfrac{1}{2}Y_1)^{-1}\, (Y_2 - \tfrac{1}{2}Y_1)$$
$$= \left(Cov(X, Y_2) - \tfrac{1}{2} Cov(X, Y_1)\right) \left(\tfrac{3}{4}\right)^{-1} \left(Y_2 - \tfrac{1}{2}Y_1\right) \quad \text{By part (a)}$$
$$= \tfrac{1}{4} \cdot \tfrac{4}{3} (Y_2 - \tfrac{1}{2}Y_1)$$
$$= \tfrac{1}{3}(Y_2 - \tfrac{1}{2}Y_1)$$

$$\hat{E}[X|\hat{Y}_3] = Cov(X, Y_3 - \tfrac{1}{3}Y_2 - \tfrac{1}{3}Y_1) \cdot Var(\hat{Y}_3)^{-1}\, \tilde{Y}_3$$

$$Cov(X, \hat{Y}_3) = Cov(X, Y_3) - \tfrac{1}{3} Cov(X, Y_2) - \tfrac{1}{3} Cov(X, Y_1)$$
$$= \tfrac{1}{4} - \tfrac{1}{3} \cdot \tfrac{1}{4} - \tfrac{1}{3} \cdot 0 = \tfrac{1}{6}$$

$$Var(\hat{Y}_3) = Var(Y_3 - \tfrac{1}{3}Y_2 - \tfrac{1}{3}Y_1)$$
$$= Var(Y_3) + \tfrac{1}{9}Var(Y_2) + \tfrac{1}{9}Var(Y_1) - \tfrac{2}{3}Cov(Y_3, Y_2) - \tfrac{2}{3}Cov(Y_1, Y_3) + \tfrac{2}{9}Cov(Y_1, Y_2)$$
$$= 1 + \tfrac{1}{9} \cdot 1 + \tfrac{1}{9} - \tfrac{2}{3} \cdot \tfrac{1}{2} - \tfrac{2}{3} \cdot \tfrac{1}{2} + \tfrac{2}{9} \cdot \tfrac{1}{2}$$
$$= \tfrac{11}{9} - \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{9}$$
$$= \tfrac{2}{3}$$

$$\therefore \quad \hat{E}[X|\hat{Y}_3] = \tfrac{1}{6} \cdot \tfrac{3}{2} \cdot (Y_3 - \tfrac{1}{3}Y_2 - \tfrac{1}{3}Y_1)$$
$$= \tfrac{1}{4}(\hat{Y}_3)$$

Overall,
$$\hat{E}[X|\hat{Y}_1, \hat{Y}_2, \hat{Y}_3] = \tfrac{2}{3}(Y_2 - \tfrac{1}{2}Y_1) + \tfrac{1}{4}(Y_3 - \tfrac{1}{3}Y_2 - \tfrac{1}{3}Y_1)$$
$$= \tfrac{2}{3}Y_2 - \tfrac{1}{3}Y_1 + \tfrac{1}{4}Y_3 - \tfrac{1}{12}Y_2 - \tfrac{1}{12}Y_1$$
$$= \tfrac{1}{4}Y_3 + \tfrac{1}{4}Y_2 - \tfrac{1}{4}Y_1$$