1. Poisson Process: Direct Calculation [5 points each]

Consider a Poisson Process with intensity λ. Let Y_K denote the time to see the Kth arrival. Let us use the following direct approach to calculate the joint PDF of $Y_1, Y_2, \ldots, Y_K$.

(a) Consider any two positive numbers $y_1 < y_2$. Observe that $(X(y_2) - X(y_1))$ is the number of arrivals in the interval $(y_1, y_2]$, and hence $X(y_2) - X(y_1)$ is independent of $X(y_1)$. Using this show

$$P(X(y_1) < 1, X(y_2) < 2) = e^{-\lambda y_2}(1 + \lambda(y_2 - y_1))$$

(a) $P(N=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ $\quad P(X(y_1)=0,\ X(y_2)-X(y_1)<2) = P(X(y_1)=0,\ X(y_2)-X(y_1)<2)$

For $P(X(y_1)=0)$: $k=0$ $\tilde{\lambda} = \lambda y_1$ $P(X(y_1)=0) = \dfrac{(\lambda y_1)^0 e^{-\lambda y_1}}{0!} = e^{-\lambda y_1}$

For $P(X(y_2)-X(y_1)=1)$: $k=1$ $\tilde{\lambda} = \lambda(y_2-y_1)$ $P(X(y_2)-X(y_1)=1) = \dfrac{e^{-\lambda(y_2-y_1)} \cdot \lambda(y_2-y_1)}{1!}$

$P(X(y_2)-X(y_1)=0)$: $k=0$ $\tilde{\lambda} = \lambda(y_2-y_1)$ $P(X(y_2)-X(y_1)=0) = e^{-\lambda(y_2-y_1)}$

$P(X(y_1)=0,\ X(y_2)-X(y_1)<2) = e^{-\lambda y_1} \cdot e^{-\lambda(y_2-y_1)}(\lambda(y_2-y_1)+1) = e^{-\lambda y_2}(\lambda(y_2-y_1)+1)$

(b) Use part (a) to find $P(Y_1 > y_1, Y_2 > y_2)$, and then find the joint PDF of Y_1 and Y_2 using the inclusion-exclusion principle.

(b) $P(Y_1 > y_1,\ Y_2 > y_2) = P(X(y_1) < 1,\ X(y_2) < 2) = e^{-\lambda y_2}(1+\lambda(y_2-y_1))$

$P(Y_1 \leq y_1,\ Y_2 \leq y_2) = 1 - P(Y_1 > y_1 \cup Y_2 > y_2) = 1 - P(Y_1 > y_1) - P(Y_2 > y_2) + P(Y_1 > y_1,\ Y_2 > y_2)$

$F_{Y_1 Y_2}(y_1 y_2) = 1 - P(X(y_1) < 1) - P(X(y_2) < 2) + P(X(y_1) < 1,\ X(y_2) < 2)$

$e^{-\lambda y_2}(1+\lambda(y_2-y_1))\ dy_1$

(c) To extend this idea to the general case, first prove the following result (again using the inclusion-exclusion principle) for any K-dimensional random vector $\mathbf{Z}$:

$$f_{\mathbf{Z}}(\mathbf{z}) = (-1)^K \frac{\partial^K}{\partial \mathbf{z}} P(\mathbf{Z} > \mathbf{z}).$$

(d) Using ideas from (a), (b) and (c), find the joint PDF of $Y_1, Y_2, \ldots, Y_K$.
Hint: Use the inclusion-exclusion principle one more time and evaluate only that term in $P(\mathbf{Y} \geq \mathbf{y})$ that contains all variables $y_1, y_2, \ldots, y_K$. The rest of the terms will be canceled by the partial derivatives.

2. Wide-Sense Stationary [15 points]

Consider a random process $\{X_t\}$ such that

$$X_t = A\sin(t + \Theta),$$

where A is a Bernoulli random variable with mean 1/4, Θ is uniformly distributed over $[0, 2\pi]$, and A and Θ are independent.

- Is $\{X_t\}$ WSS?

$A\sin(t+\theta) = A\sin t\cos\theta + A\cos t\sin\theta \qquad \mu[X_t] = E[A]\cdot(E(\cos\theta)\cdot\sin t + E(\sin\theta)\cdot\cos t)$

Since θ is uniformly distributed over $[0, 2\pi]$: $E(\cos\theta) = E(\sin\theta) = 0$

$\mu_{X_t} = 0$

$$\begin{aligned}
R_{s,s+t} &= E[A^2]\, E[\sin(s+\theta)\sin(s+t+\theta)] \\
&= E[A^2]\cdot\tfrac{1}{2}\cdot E[\cos(s+\theta-s-t-\theta) - \cos(s+\theta+s+t+\theta)] \\
&= E[A^2]\cdot\tfrac{1}{2}\cdot E[\cos(-t) - \cos(2s+t+2\theta)] \\
&= \frac{E[A^2]}{2}\cos(t) - \frac{E[A^2]}{2} E[\cos(s+s+t+2\theta)] \\
&= \frac{E[A^2]}{2}\cos(t) - \frac{E[A^2]}{2}\int_0^{2\pi} \cos(2s+t+2\theta)\cdot d\theta\cdot\frac{1}{2\pi} \\
&= \frac{E[A^2]}{2}\cos(t) - \frac{E[A^2]}{2\times 2\pi}\left[\frac{\sin(2s+t+2\theta)}{2}\right]\Bigg|_0^{2\pi} \\
&= \frac{E[A^2]}{2}\cos(t) = \frac{1}{8}\cos(t)
\end{aligned}$$

Therefore, $\{X_t\}$ is WSS.

3. Wide-Sense Stationary 2 [15 points]

Consider a random process $\{X_t\}$ such that

$$X_t = A\sin(t+\Theta) + B,$$

where A is a Bernoulli random variable with mean $1/4$, B is a Bernoulli random variable with mean $1/2$, Θ is uniformly distributed over $[0, 2\pi]$, and A, B, and Θ are independent from each other.

- Is $\{X_t\}$ WSS?

$X_t = A\sin(t+\theta) + B = A(\sin t \cos\theta + \cos t \sin\theta) + B$

$\mu[X_t] = E[A]\,E(\sin t\cos\theta + \cos t\sin\theta) + E[B]$ Based on the above question 2:

$\mu[X_t] = 0 + E[B] = \frac{1}{2}$ (a Constant)

$R_{s,s+t} = E[X_s X_{s+t}] = E\{(A\sin(s+\theta)+B)(A\sin(s+t+\theta)+B)\}$

$= E\{A^2\sin(s+\theta)\sin(s+t+\theta) + AB\sin(s+\theta) + AB\sin(s+t+\theta) + B^2\}$

Based on $\sin A\cdot\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, Besides, A, B are Bernoulli distri, $E(A^2) = \frac{1}{4}$ $E(B^2) = \frac{1}{2}$

$R_{s,s+t} = E\{A^2\cdot\frac{1}{2}[\cos t - \cos(s+s+t+2\theta)] + AB\sin(s+\theta) + AB\sin(s+t+\theta) + B^2\}$

$= \frac{E[A^2]}{2}\cdot E[\cos t - \cos(s+s+t+2\theta)] + E[AB\sin(s+\theta)] + E[AB\sin(s+t+\theta)] + E[B^2]$

$= \frac{E[A^2]}{2}\cdot\cos t + E[B^2] = \frac{1}{8}\cos t + \frac{1}{2}$

Therefore $\{X_t\}$ is WSS