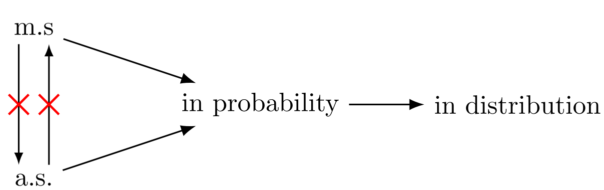
EECS 501
Discussion 9
Solution

Review

Convergence Notions of Random Sequences:

- Almost Sure Convergence:
  $X_n \to X$ almost surely (a.s.) or with probability 1 (w.p.1) if
  $$\mathbb{P}(\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1.$$
- Mean-Square Convergence:
  $X_n \to X$ in mean-square if $\mathbb{E}[X_n^2] < \infty \;\; \forall n$ and $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$.
- Convergence in Probability:
  $X_n \to X$ in probability if $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0$ for all $\epsilon > 0$.
- Convergence in Distribution:
  $X_n \to X$ in distribution if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ for any x such that $F_X(x)$ is continuous at x.
- Relation between the notions of convergence:



- If $X_n \to X$ in any one sense, then if it converges in any other sense, it must converge to the same limit.
- Suppose X_n is Gaussian random variable for each n and $X_n \to X$ in any of the four sense (a.s.,m.s.,d.,p.), then X is a Gaussian random variable.
- Define $A_n(\epsilon) = \{|X_n - X| \geq \epsilon\}$ (the event that $|X_n - X| \geq \epsilon$). If $\sum_{n=1}^{\infty} P(A_n(\epsilon)) < \infty$ holds for every $\epsilon > 0$, then $X_n \to X$ a.s.

Practice Problems

Problem 1 Let X_n converge in distribution to X, and let $c(x)$ be a continuous function. Show that $c(X_n)$ converges in distribution to $c(X)$.

Hint: Use the Skorohod representation which says that if X_n converges in distribution to X, then we can construct random variables Y_n and Y with $F_{Y_n} = F_{X_n}$, $F_Y = F_X$, and such that Y_n converges almost surely to Y.

Solution:

Let Y_n and Y be as given by the Skorohod representation. Since Y_n converges almost surely to Y, for the set $G = \{\omega : Y_n(\omega) \to Y(\omega)\}$, we have $P(G) = 1$. For any fixed $\omega \in G$, we have $Y_n(\omega) \to Y(\omega)$ and since c is a continuous function, we have $c(Y_n(\omega)) \to c(Y(\omega))$. Therefore, $c(Y_n)$ converges almost surely to $c(Y)$. Recall that almost-sure convergence implies convergence in distribution. Hence, $c(Y_n)$ converges in distribution to $c(Y)$. To conclude, observe that since Y_n and X_n have the same cumulative distribution function and so $c(Y_n)$ and $c(X_n)$ have the same cumulative distributions. Similarly, $c(Y)$ and $c(X)$ have the same cumulative distribution function. Thus, $c(X_n)$ converges in distribution to $c(X)$.

Problem 2 Show that if X_n converges in mean square to X, and if Y_n converges in mean square to Y, then $X_n + Y_n$ converges in mean square to $X + Y$.

Solution:

We have

$$(X_n + Y_n - (X + Y))^2 = ((X_n - X) + (Y_n - Y))^2 \leq 2((X_n - X)^2 + (Y_n - Y)^2)$$

Therefore, we have

$$\mathbb{E}[(X_n + Y_n - (X + Y))^2] \leq 2\mathbb{E}[((X_n - X)^2 + (Y_n - Y)^2)$$

and

$$0 \leq \lim_{n\to\infty} \mathbb{E}[(X_n + Y_n - (X + Y))^2] \leq 2 \lim_{n\to\infty} \mathbb{E}[((X_n - X)^2 + (Y_n - Y)^2) = 0$$

and so we have $\lim_{n\to\infty} \mathbb{E}[(X_n + Y_n - (X + Y))^2] = 0$, thus $X_n + Y_n$ converge to $X + Y$ in m.s.

Problem 3 Suppose $X_n \to X$ a.s. and $Y_n \to Y$ a.s. Show that if $X_n \leq Y_n$ a.s. for all n, then $X \leq Y$ a.s. (The statement $X_n \leq Y_n$ a.s. means $P(X_n > Y_n) = 0$.)

Solution:

Let $G_X = \{X_n \to X\}$, $G_Y = \{Y_n \to Y\}$, and $G_I = \cap_{n=1}^{\infty}\{X_n \leq Y_n\}$. We have $P(G_X^c) = P(G_Y^c) = 0$. We also have

$$P(G_I^c) \leq \sum_{n=1}^{\infty} P(X_n > Y_n) = 0.$$

Let $G = \{X \leq Y\}$. We must show $P(G^c) = 0$. For any $w \in G_X \cap G_Y \cap G_I$ we have $X_n(\omega) \to X(\omega)$, $Y_n(\omega) \to Y(\omega)$, and for all n, $X_n(w) \leq Y_n(w)$. Therefore, by properties of sequences of real numbers, for such w, we must have $X(\omega) \leq Y(\omega)$. Therefore $G_X \cap G_Y \cap G_I \subset G$ which will imply $G^c \subset G_X^c \cup G_Y^c \cup G_I^c$. We have

$$P(G^c) \leq P(G_X^c \cup G_Y^c \cup G_I^c) \leq P(G_X^c) + P(G_Y^c) + P(G_I^c) = 0.$$

Problem 4 Show that if X_n converges to X in mean square, then X_n converges to X in mean, i..e., $\lim_{n\to\infty} E[X_n] = E[X]$.

Solution:

Using Jensen's inequality ($\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$ where $g(\cdot)$ is a convex function), we have

$$\mathbb{E}[(X_n - X)^2] \geq \mathbb{E}[X_n - X]^2$$

Since we have $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$, we have

$$\lim_{n\to\infty} \mathbb{E}[X_n - X]^2 \leq \lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$$

Therefore, we have $\lim_{n\to\infty} \mathbb{E}[X_n - X] = 0$, which means $\lim_{n\to\infty} E[X_n] = E[X]$.

Problem 5 Let $X_1,\ X_2,\ X_3, \cdots$ be independent random variables that are uniformly distributed over $[-1, 1]$. Show that the sequence $Y_1,\ Y_2,\ Y_3, \cdots$ converges in probability to some limit and identify the limit, for each of the following cases:

(a) $Y_n = X_n/n$

(b) $Y_n = (X_n)^n$

(c) $Y_n = X_1 X_2 \cdots X_n$.

(d) $Y_n = \max\{X_1, \cdots, X_n\}$

Solution:

(a) For any $\epsilon > 0$ we have

$$P(|Y_n| \geq \epsilon) = P(|X_n| \geq n\epsilon) = 0, \qquad \text{for } n\epsilon > 1 \text{ or } n > \frac{1}{\epsilon}.$$

So for any $\epsilon > 0$, $\lim_{n\to\infty} P(|Y_n - 0| \geq \epsilon) = 0$ and so Y_n converges to 0 in p.

(b) For all $\epsilon \in (0, 1)$, we have

$$P(|Y_n| \geq \epsilon) = P(|X_n|^n \geq \epsilon) = P(X_n \geq \epsilon^{1/n}) + P(X_n \leq -\epsilon^{1/n}) = 1 - \epsilon^{1/n} \to 0$$

where we have the result since $\epsilon^{1/n} \to 1$. For $\epsilon > 1$, we clearly have $P(|Y_n| \geq \epsilon) = 0$. Therefore, Y_n converges to 0 in p.

(c) Since $X_1, X_2, \cdots$ are independent random variables, we have

$$\mathbb{E}[Y_n] = \mathbb{E}[X_1]\mathbb{E}[X_2] \cdots \mathbb{E}[X_n] = 0$$

Also

$$Var(Y_n) = \mathbb{E}[Y_n^2] = \mathbb{E}[X_1^2]\mathbb{E}[X_2^2] \cdots \mathbb{E}[X_n^2] = Var(X_1)^n = (\frac{4}{12})^n \to 0$$

Therefore, from Chebyshev's inequality we have

$$P(|Y_n - 0| > \epsilon) \leq \frac{Var(Y_n)}{\epsilon^2}$$

and at limit we have

$$\lim_{n\to\infty} P(|Y_n - 0| > \epsilon) \leq \lim_{n\to\infty} \frac{Var(Y_n)}{\epsilon^2} = 0$$

(d) For all $\epsilon \in (0,1)$, using the independence of $X_1, X_2 \cdots$, we have

$$
\begin{aligned}
P(|Y_n - 1| \geq \epsilon) &= P(\max\{X_1, \cdots, X_n\} \geq 1 + \epsilon) + P(\max\{X_1, \cdots, X_n\} \leq 1 - \epsilon) \\
&= P(X_1 \leq 1 - \epsilon, \cdots, X_n \leq 1 - \epsilon) = P(X_1 \leq 1 - \epsilon)^n = (1 - \frac{\epsilon}{2})^n \to 0
\end{aligned}
$$

Therefore, Y_n converges to 1 in p.

Problem 6 Show that the convergence in mean square implies convergence in probability.

Solution:

If $X_n \to X$ in m.s., we have $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$. Using Jensen's inequality, we have

$$\mathbb{E}[(X_n - X)^2] = \mathbb{E}[|X_n - X|^2] \geq \mathbb{E}[|X_n - X|]^2$$

Hence, we have

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|]^2 \leq \lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$$

and so $\lim_{n\to\infty} \mathbb{E}[|X_n - X|] = 0$. Using Markov inequality, we have

$$P(|X_n - X| > \epsilon) \leq \frac{\mathbb{E}[|X_n - X|]}{\epsilon} \to 0$$

Therefore, $X_n \to X$ in p.

Problem 7 Consider a sequence $\{X_n, n = 1, 2, 3, \cdots\}$

$$X_n = \begin{cases} -\frac{1}{n} & \text{with probability } \frac{1}{2} \\ \frac{1}{n} & \text{with probability } \frac{1}{2} \end{cases}$$

Show that $X_n \to 0$ a.s.

Solution:

In order to show a.s. convergence, it suffices to show that

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) < \infty$$

Since $|X_n| = \frac{1}{n}$, we have $|X_n| > \epsilon$ if and only if $n < \frac{1}{\epsilon}$. Therefore,

$$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) = \sum_{n=1}^{\lfloor \frac{1}{\epsilon} \rfloor} P(|X_n| > \epsilon) = \lfloor \frac{1}{\epsilon} \rfloor < \infty$$

Hence, X_n converges to 0 a.s.

Problem 8 Let $X_1, X_2, \cdots$ be a sequence of random variables such that $X_n \sim \mathit{Geometric}(\frac{\lambda}{n})$, where $\lambda > 0$ is a constant. Define the sequence $Y_n = \frac{X_n}{n}$. Show that Y_n converges in distribution to $\mathit{Exponential}(\lambda)$.

Solution:

Note that if $W \sim Geometric(p)$, then for any positive integer l, we have

$$P(W \leq l) = \sum_{k=1}^{l} (1-p)^{k-1} p = p \frac{1-(1-p)^l}{1-(1-p)} = 1-(1-p)^l$$

Since $Y_n = \frac{X_n}{n}$, for any positive real number, we have

$$P(Y_n \leq y) = P(X_n \leq ny) = P(X_n \leq \lfloor ny \rfloor) = 1 - (1 - \frac{\lambda}{n})^{\lfloor ny \rfloor},$$

where $\lfloor ny \rfloor$ is the largest integer less than or equal to ny. We have

$$\lim_{n \to \infty} F_{Y_n}(y) = \lim_{n \to \infty} 1 - (1 - \frac{\lambda}{n})^{\lfloor ny \rfloor} = 1 - \lim_{n \to \infty} (1 - \frac{\lambda}{n})^{\lfloor ny \rfloor} = 1 - e^{-\lambda y}$$

where the last equality holds because $ny - 1 \leq \lfloor ny \rfloor \leq ny$ and

$$\lim_{n \to \infty} (1 - \frac{\lambda}{n})^{ny} = e^{-\lambda y}$$

We know $1 - e^{-\lambda y}$ is the CDF of an exponential random variable with parameter λ.