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EECS 501 F09

Midterm # 2 Solutions

1. $f_{X,Y}(x,y) = \exp[-(x+y)]$ $x, y \geq 0$ (0, otherwise)

$S \equiv X + Y$, $T \equiv \dfrac{X}{X+Y}$

(a) $\therefore$ $X = ST$ and $Y = S(1-T)$

$$f_{S,T}(s,t) = f_{X,Y}(ST, S(1-T))\left(|\det \underline{J}|\right)^{-1}$$

$$\underline{J} = \begin{bmatrix} \partial S/\partial x & \partial S/\partial y \\ \partial T/\partial x & \partial T/\partial y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \dfrac{y}{(x+y)^2} & \dfrac{-x}{(x+y)^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ \dfrac{1-T}{S} & -\dfrac{T}{S} \end{bmatrix} \Longrightarrow \det(\underline{J}) = -\frac{1}{S}$$

$$\therefore f_{S,T}(s,t) = \exp[-(ST + S(1-T))]\, S \qquad S \geq 0,\ 0 \leq T \leq 1 \text{ (0, o.w.)}$$

$$= \begin{cases} Se^{-S}, & S \geq 0,\ 0 \leq t \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_S(s) = \int_{-\infty}^{\infty} f_{S,T}(s,t)\,dt = Se^{-S}, \quad S \geq 0 \text{ (0, o.w.)}$$

$$f_T(t) = \int_{-\infty}^{\infty} f_{S,T}(s,t)\,ds = 1, \quad 0 \leq t \leq 1 \text{ (0, o.w.)}$$

$\therefore$ $f_{S,T}(s,t) = f_S(s) f_T(t) \Longrightarrow$ S and T are statistically independent

(1 continued)

(b) $P\{T \geq S\} = \int_{-\infty}^{\infty} \underbrace{P\{T \geq s \mid S=s\}}^{1-s,\ 0\leq s\leq 1\ (0,\ \text{o.w.})} f_S(s)\,ds$

$= \int_0^1 (1-s)se^{-s}ds$

$\therefore\ a=0,\ b=1,\ g(x)=(1-x)xe^{-x}$

or $\int_0^1\int_0^t f_{S,T}(s,t)\,ds\,dt = \int_0^1\int_0^t se^{-s}ds\,dt = \int_0^1 [1-(t+1)e^{-t}]dt$

$\therefore\ a=0$
$b=1$
$g(x) = [1-(x+1)e^{-x}]$

2. (a) $M_{U,V}(t_1, t_2) = E[e^{t_1 U + t_2 V}]$

$= E[e^{t_1 U}]E[e^{t_2 V}]$ since U & V are indep.

$= E[e^{t_1 X} e^{t_1 Y}]\, E[e^{t_2 X} e^{-t_2 Y}]$

since $U = X + Y$ & $V = X - Y$

$= E[e^{t_1 X}]E[e^{t_1 Y}]E[e^{t_2 X}]E[e^{-t_2 Y}]$

since X & Y are indep

$= g^2(t_1)\, g(t_2) g(-t_2)$

$\therefore\ M_{U,V}(t, t) = g^3(t) g(-t)$

but also observe $M_{U,V}(t, t) = E[e^{tU + tV}]$

$= E[e^{2tX}]$

$= g(2t)$

$\therefore\ g^3(t)\, g(-t) = g(2t)$

Q.E.D.

Gaussian

2.(b) If X is a zero mean unit variance r.v., then

$$g(t) = M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx} e^{-x^2/2} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2} dx \; e^{t^2/2}$$

$$= e^{t^2/2}$$

note now that

$$\underbrace{(e^{t^2/2})^3}_{g^3(t)} \underbrace{e^{(-t)^2/2}}_{g(-t)} = \underbrace{e^{(2t)^2/2}}_{g(2t)}$$

3. Let N_K = number of arriving cars when K consecutive cars have exact change for the first time

$$E[N_K] = E\left(E[N_K | N_{K-1}]\right) \quad \text{by Law of iterated expectation}$$

$$E[N_K | N_{K-1} = n] = (n+1)p + (n+1+E[N_K])(1-p), \; K \geq 1$$

$$\therefore E[N_K | N_{K-1}] = (N_{K-1}+1)p + (N_{K-1}+1+E[N_K])(1-p)$$

$$\therefore E[N_K] = E[N_{K-1}] + (1-p)E[N_K] + 1$$

(note $E[N_0] = 0$)

$$\therefore E[N_K] = p^{-1}(1 + E[N_{K-1}])$$

$$E[N_0] = 0$$

$$E[N_1] = p^{-1}$$

$$E[N_2] = p^{-1}(1+p^{-1}) = p^{-1} + p^{-2}$$

$$E[N_3] = p^{-1}(1+p^{-1}+p^{-2}) = p^{-1}+p^{-2}+p^{-3}$$

$$\vdots$$

$$E[N_K] = \sum_{m=1}^{K} p^{-m} = \frac{p^{-K}-1}{1-p}$$

Let T_K = time between breaks, i.e., time for K consecutive cars to arrive with exact change. Then $T_K = \sum_{i=1}^{N_K} X_i$ where X_i are i.i.d exponential random variables with parameter λ.

$$E[T_K] = E[\underbrace{E[T_K | N_K]}_{N_K/\lambda}] = E[N_K/\lambda] = \lambda^{-1}(p^{-K}-1)/(1-p)$$

4. Let

event $A \equiv \frac{1}{2} \le \min(X_1, X_2, X_3) \le 3/4$

event $B \equiv \frac{7}{8} \le \max(X_1, X_2, X_3) \le 15/16$

event $C \equiv X_2 > X_1$

Note:

All 3! orderings (smallest to largest) of X_1, X_2 & X_3 are equally likely. These orderings are $\rightarrow O_1, O_2, \ldots, O_6$

	$X_{(1)}$	$X_{(2)}$	$X_{(3)}$ = order statistics
O_1	X_1	X_2	X_3
O_2	X_1	X_3	X_2
O_3	X_2	X_1	X_3
O_4	X_2	X_3	X_1
O_5	X_3	X_1	X_2
O_6	X_3	X_2	X_1

We need to compute

$$P\{AB \mid C\} = \frac{P\{ABC\}}{P\{C\}}$$

$$= 2P\{ABC\} \quad (\text{clearly } P\, C = 1/2)$$

$$= 2P\{AB(O_1 \cup O_2 \cup O_5)\}$$

$$= 2\left[P\{ABO_1\} + P\{ABO_2\} + P\{ABO_5\}\right]$$

$$= 2P\left\{\tfrac{1}{2} \le X_1 \le \tfrac{3}{4},\ \tfrac{7}{8} \le X_3 \le \tfrac{15}{16},\ X_1 < X_2 < X_3\right\}$$

$$+ 2P\left\{\tfrac{1}{2} \le X_1 \le \tfrac{3}{4},\ \tfrac{7}{8} \le X_2 \le \tfrac{15}{16},\ X_1 < X_3 < X_2\right\}$$

$$+ 2P\left\{\tfrac{1}{2} \le X_3 \le \tfrac{3}{4},\ \tfrac{7}{8} \le X_2 \le \tfrac{15}{16},\ X_3 < X_1 < X_2\right\}$$

$$= 6P\left\{\tfrac{1}{2} \le X_1 \le \tfrac{3}{4},\ \tfrac{7}{8} \le X_3 \le \tfrac{15}{16},\ X_1 < X_2 < X_3\right\}$$

by symmetry

$$= 6\int_{1/2}^{3/4}\int_{7/8}^{15/16} P\{x_1 < X_2 < x_3 \mid X_1 = x_1, X_3 = x_3\} f_{X_1 X_3}(x_1, x_3)\,dx_3\,dx_1$$

$$= 6\int_{1/2}^{3/4}\int_{7/8}^{15/16} (x_3 - x_1)\,dx_3\,dx_1$$

$$= 6\int_{1/2}^{3/4}\int_{7/8}^{15/16} x_3\,dx_3\,dx_1 - 6\int_{1/2}^{3/4}\int_{7/8}^{15/16} x_1\,dx_3\,dx_1$$

$$= \underline{\underline{27/1024}}$$

5. Let

$$X_i = \begin{cases} 1, & \text{if a queen is chosen for card } i \\ 0, & \text{otherwise} \end{cases} \quad (i = 1, 2, \ldots, 5)$$

$$Y_i = \begin{cases} 1, & \text{if a spade is chosen for card } i \\ 0, & \text{otherwise} \end{cases} \quad (i = 1, 2, \ldots, 5)$$

$$\therefore \quad X = \sum_{i=1}^{5} X_i \;\; \& \;\; Y = \sum_{k=1}^{5} Y_k$$

$$E[X] = \sum_{i=1}^{5} E[X_i] = 5 \cdot \frac{4}{52} = 5/13$$

$$E[Y] = \sum_{k=1}^{5} E[Y_i] = 5 \cdot \frac{13}{52} = 5/4$$

$$E[XY] = \sum_{i=1}^{5} \sum_{k=1}^{5} E[X_i Y_k]$$

$$E[X_i Y_i] = 1/52$$

queen which is not also a spade; a spade; a queen which is also spade

$$E[X_i Y_k] = \left[\frac{3}{52}\frac{13}{51} + \frac{1}{52}\frac{12}{51}\right] = \frac{1}{52} \quad i \neq k$$

a spade which is not a queen

$$\therefore E[XY] = 5 \cdot (1/52) + 20 \cdot (1/52) = 25/52$$

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

$$= \frac{25}{52} - \frac{25}{52}$$

$$= \underline{\underline{0}}$$

6. There are $6!$ equally likely orderings of $X_1 X_2 \ldots X_6$. $3/4$ of these orderings, i.e. those with $X_3 > X_5$ or $X_1 > X_4$ are discarded leaving $\frac{6!}{4} = 180$ equally like orderings to be sorted.

The entropy of 180 equally likely outcomes is

$$\log_2 180 \simeq 7.5$$

Thus a minimum of ~ 7.5 yes/no questions need to be asked on average to determine the ordering.

$\therefore$ 7 is a lower bound on the average number of comparisons required.