Pr. 1. (sol/hs010)
If λ is an eigenvalue of $\boldsymbol{A}$, then there exists a vector $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.
Thus $\boldsymbol{B}\boldsymbol{x} = (\boldsymbol{A} - 10\boldsymbol{I})\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} - 10\boldsymbol{x} = \lambda\boldsymbol{x} - 10\boldsymbol{x} = (\lambda - 10)\boldsymbol{x}$.
Thus $\lambda - 10$ is an eigenvalue of $\boldsymbol{B}$ with the same eigenvector $\boldsymbol{x}$.
Conversely, if z is an eigenvalue of $\boldsymbol{B}$ with eigenvector $\boldsymbol{y}$, then using $\boldsymbol{A} = \boldsymbol{B} + 10\boldsymbol{I}$ the same logic shows that $\boldsymbol{y}$ is also an eigenvector of $\boldsymbol{A}$ with eigenvalue $z + 10$.
In summary, $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvectors and $\lambda_B = \lambda_A - 10$.
The problem statement did not specify that $\boldsymbol{A}$ is symmetric, therefore $\boldsymbol{A}$ may not have an eigendecomposition. What follows here is an answer that applies when $\boldsymbol{A}$ is diagonalizable. Student answers that match the following earn partial but not full credit because the problem did not state that $\boldsymbol{A}$ is diagonalizable.
Let $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$ be an eigendecomposition of $\boldsymbol{A}$. Then we may write

$$\begin{aligned} \boldsymbol{B} = \boldsymbol{A} - 10\boldsymbol{I} &= \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' - 10\boldsymbol{Q}\boldsymbol{Q}' \qquad (\boldsymbol{I} = \boldsymbol{Q}\boldsymbol{Q}' \text{ since } \boldsymbol{Q} \text{ is unitary}) \\ &= \boldsymbol{Q}(\boldsymbol{\Lambda} - 10\boldsymbol{I})\boldsymbol{Q}'. \end{aligned} \tag{1}$$

Notice that $\boldsymbol{\Lambda} - 10\boldsymbol{I}$ is a diagonal matrix, and since $\boldsymbol{Q}$ is unitary, the right hand side of (1) gives us the eigendecomposition of $\boldsymbol{B}$. So we conclude that the eigenvectors of $\boldsymbol{B}$ are identical to the eigenvectors of $\boldsymbol{A}$. Further, if λ_B is an eigenvalue of $\boldsymbol{B}$, and λ_A is an eigenvalue of $\boldsymbol{A}$, then $\lambda_B = \lambda_A - 10$.

Pr. 2. (sol/hs013)
Only if:
We are given that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are orthonormal and that $\boldsymbol{A}\boldsymbol{v}_i, i = 1, \ldots, n$ are orthonormal. We want to show $\boldsymbol{A}$ is an orthogonal matrix. By assumption, the matrix $\boldsymbol{A}\Big[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\Big] := \boldsymbol{A}\boldsymbol{V}$ is orthogonal. Thus

$$\begin{aligned} \boldsymbol{I} &= (\boldsymbol{A}\boldsymbol{V})^T(\boldsymbol{A}\boldsymbol{V}) = \boldsymbol{V}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{V} \\ \implies \boldsymbol{V}\boldsymbol{I}\boldsymbol{V}' &= \boldsymbol{V}\boldsymbol{V}'\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{V}\boldsymbol{V}' \implies \boldsymbol{I} = \boldsymbol{A}^T\boldsymbol{A}, \end{aligned}$$

using the orthogonality of $\boldsymbol{V}$. Using similar steps, we can show that $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$, and conclude that $\boldsymbol{A}$ is orthogonal.
If:
We are given that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are orthonormal and that $\boldsymbol{A}$ is orthogonal. Then

$$(\boldsymbol{A}\boldsymbol{v}_i)^T(\boldsymbol{A}\boldsymbol{v}_j) = \boldsymbol{v}_i^T \underbrace{\boldsymbol{A}^T\boldsymbol{A}}_{=\boldsymbol{I}} v_j = \boldsymbol{v}_i^T\boldsymbol{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \text{(since } \boldsymbol{v}_i \text{ are orthonormal).}$$

Thus $\{\boldsymbol{A}\boldsymbol{v}_i\}$ are also orthonormal.

Pr. 3. (sol/hs117)
If $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ and $\boldsymbol{A}$ has full rank then $\boldsymbol{A}^{-1} = \boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^T$ since $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^T = \boldsymbol{I}$.

Pr. 4. (sol/hs016)
Here $\boldsymbol{T}$ is an invertible (square) matrix and $\boldsymbol{B} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$. Let λ be an eigenvalue of $\boldsymbol{B}$ associated with the eigenvector $\boldsymbol{v}$. By definition:

$$\boldsymbol{B}\boldsymbol{v} = \lambda\boldsymbol{v} \implies \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{v} = \lambda\boldsymbol{v} \implies \underbrace{\boldsymbol{T}\boldsymbol{T}^{-1}}_{=\boldsymbol{I}}\boldsymbol{A}\boldsymbol{T}\boldsymbol{v} = \lambda\boldsymbol{T}\boldsymbol{v} \implies \boldsymbol{A}(\boldsymbol{T}\boldsymbol{v}) = \lambda(\boldsymbol{T}\boldsymbol{v}).$$

Letting $\boldsymbol{u} \triangleq \boldsymbol{T}\boldsymbol{v}$, we have $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$, so $\boldsymbol{u}/\|\boldsymbol{u}\|$ is a unit-norm eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ. Because this is true for *every* eigenvector $\boldsymbol{v}$ of $\boldsymbol{B}$ associated with *every* eigenvalue λ, this implies that the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical.
Thus for $i = 1, \ldots, n$, the unit-norm eigenvector $\boldsymbol{v}_i$ of $\boldsymbol{B}$ associated with the eigenvalue λ_i is related to the unit-norm eigenvector $\boldsymbol{u}_i$ of $\boldsymbol{A}$ associated with the same λ_i by :

$$\boldsymbol{v}_i = \frac{\boldsymbol{T}^{-1}\boldsymbol{u}_i}{||\boldsymbol{T}^{-1}\boldsymbol{u}_i||_2}.$$

When speaking of the *set* of eigenvectors associated with a particular eigenvalue it is important to normalize the eigenvectors to have unit-norm. If we did not normalize $\boldsymbol{v}_i$ as above then we would have to multiply the ith eigenvalue of $\boldsymbol{B}$ by $||\boldsymbol{T}^{-1}\boldsymbol{u}_i||_2^2$ so that $\boldsymbol{B} = \sum_{i=1}^n \lambda_i\boldsymbol{v}_i\boldsymbol{v}_i^T$ as needed.

Pr. 5. (sol/hs018)
This is a straight-forward application of the properties of the SVD. Let $\boldsymbol{X} = \boldsymbol{U\Sigma V}^T$. Then $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{V\Sigma}^T\boldsymbol{\Sigma V}^T = 0 \Rightarrow \boldsymbol{\Sigma}^T\boldsymbol{\Sigma} = 0$. Since the singular values of $\boldsymbol{X}$ are non-negative numbers, the matrix $\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}$ is a diagonal matrix with non-negative numbers equal to the singular values squared along the diagonal. Thus $\boldsymbol{\Sigma}^T\boldsymbol{\Sigma} = 0 \Rightarrow \boldsymbol{\Sigma} = 0 \Rightarrow \boldsymbol{X} = \boldsymbol{U\Sigma V}^T = 0$.

Pr. 6. (sol/hs022)
Here $\boldsymbol{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$, so $\boldsymbol{A}'\boldsymbol{A} = \begin{bmatrix} \sum_{i=1}^m |a_{i1}|^2 & & \text{off diagonal elements} \\ & \ddots & \\ \text{off diagonal elements} & & \sum_{i=1}^m |a_{in}|^2 \end{bmatrix}$.

$$\Longrightarrow \operatorname{Tr}(\boldsymbol{A}'\boldsymbol{A}) = \sum_{j=1}^n \sum_{i=1}^m |a_{ij}|^2 = \sum_{(i,j)} |a_{ij}|^2 = \|\boldsymbol{A}\|_{\mathrm{F}}^2 .$$

Thus

$$\begin{aligned} \|\boldsymbol{A}\|_{\mathrm{F}} &= \sqrt{\operatorname{Tr}(\boldsymbol{A}'\boldsymbol{A})} = \sqrt{\operatorname{Tr}(\boldsymbol{U\Sigma\Sigma}'\boldsymbol{U}')} && \text{(using the SVD } \boldsymbol{A} = \boldsymbol{U\Sigma V}'\text{)} \\ &= \sqrt{\operatorname{Tr}(\boldsymbol{U}'\boldsymbol{U\Sigma\Sigma}')} && \text{(because } \operatorname{Tr}(\boldsymbol{AB}) = \operatorname{Tr}(\boldsymbol{BA}) \text{ as shown in HW 1)} \\ &= \sqrt{\operatorname{Tr}(\boldsymbol{\Sigma\Sigma}')} = \sqrt{\sum_{i=1}^r \sigma_i^2}, \end{aligned}$$

where r is the rank of the matrix $\boldsymbol{A}$.

Pr. 7. (sol/hs028)
In a previous problem we showed that

$$\|\boldsymbol{A}\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^r \sigma_i^2} \geq \sigma_1,$$

where $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0$ are the singular values of $\boldsymbol{A}$, and r is the rank of $\boldsymbol{A}$. We also know that

$$\begin{aligned} r = &\text{ row rank of } \boldsymbol{A} = \#\text{ linearly independent rows of } \boldsymbol{A} \leq m \\ = &\text{ column rank of } \boldsymbol{A} = \#\text{ independent rows of } \boldsymbol{A} \leq n \\ \leq &\min(m, n). \end{aligned}$$

Finally

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^r \sigma_i^2} \leq \sqrt{\sum_{i=1}^r \sigma_1^2} \leq \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_1^2} \leq \sqrt{\min(m,n)}\,\sigma_1.$$

Optional: The upper bound is tight; consider $\boldsymbol{A} = \boldsymbol{I}_n$ for which $\sigma_1 = 1$ and $\|\boldsymbol{A}\|_{\mathrm{F}} = \sqrt{n}$.

Pr. 8. (sol/hs033)

(a) Let $\boldsymbol{A} = \boldsymbol{U\Sigma V}'$. Then $\boldsymbol{WAQ} = \underbrace{\boldsymbol{WU}}_{=\widetilde{\boldsymbol{U}}}\boldsymbol{\Sigma}\underbrace{\boldsymbol{V}'\boldsymbol{Q}}_{=\widetilde{\boldsymbol{V}}'} = \widetilde{\boldsymbol{U}}\boldsymbol{\Sigma}\widetilde{\boldsymbol{V}}'$, where $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$ are unitary because they are the products of unitary matrices. In other words, $\widetilde{\boldsymbol{U}}\boldsymbol{\Sigma}\widetilde{\boldsymbol{V}}'$ is an SVD of $\boldsymbol{WAQ}$. The diagonal matrix containing the singular values, $\boldsymbol{\Sigma}$ is the same for both $\boldsymbol{A}$ and $\boldsymbol{WAQ}$, implying they have the same singular values and thus the same rank.

(b) We first show that $\operatorname{rank}(\boldsymbol{WA}) = \operatorname{rank}(\boldsymbol{A})$ when $\boldsymbol{W}$ is invertible.

Let $r = \operatorname{rank}(\boldsymbol{A}) = \dim(\operatorname{span}(\boldsymbol{A}))$. Then by definition of $\dim(\cdot)$ there exists a $M \times r$ basis matrix $\boldsymbol{B}$ such that $\boldsymbol{y} \in \operatorname{span}(\boldsymbol{A}) \implies \boldsymbol{y} = \boldsymbol{B\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{F}^r$.

Now we show that $\boldsymbol{WB}$ is a basis matrix for $\operatorname{span}(\boldsymbol{WA})$. First, if $\boldsymbol{z} \in \operatorname{span}(\boldsymbol{WA})$, then $\boldsymbol{z} = \boldsymbol{WAx}$ for some $\boldsymbol{x} \in \mathbb{F}^N$ so $\boldsymbol{Ax} = \boldsymbol{B\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{F}^r$ because $\boldsymbol{B}$ is a basis matrix for the column space of $\boldsymbol{A}$. Thus $\boldsymbol{z} = \boldsymbol{WB\alpha} \in \operatorname{span}(\boldsymbol{WB})$.

We also need to show that $\boldsymbol{WB}$ has linearly independent columns: $\boldsymbol{WB\alpha} = \boldsymbol{0} \implies \boldsymbol{B\alpha} = \boldsymbol{W}^{-1}\boldsymbol{0} = \boldsymbol{0} \implies \boldsymbol{\alpha} = \boldsymbol{0}$ because $\boldsymbol{B}$ is a basis matrix and thus had linearly independent columns. So $\boldsymbol{WB}$ spans $\boldsymbol{WA}$ and has r linearly independent columns so $\operatorname{rank}(\boldsymbol{WA}) = \dim(\operatorname{span}(\boldsymbol{WA})) = r = \operatorname{rank}(\boldsymbol{A})$.

A similar argument shows that if $\boldsymbol{Q}$ is invertible, then $\operatorname{rank}(\boldsymbol{AQ}) = \operatorname{rank}(\boldsymbol{A})$.

Combining yields $\operatorname{rank}(\boldsymbol{WAQ}) = \operatorname{rank}(\boldsymbol{A})$ when both $\boldsymbol{W}$ and $\boldsymbol{Q}$ are invertible.

(c) Here is a simple example that shows that singular values of $\boldsymbol{WAQ}$ and $\boldsymbol{A}$ need not be the same. Take $\boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{W} = 2\boldsymbol{I}$ and $\boldsymbol{Q} = \boldsymbol{I}$. Singular values of $\boldsymbol{WAQ}$ equal 2 while those of $\boldsymbol{A}$ equal 1.

Pr. 9. (sol/hs035)

(a) $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{v} : \boldsymbol{Av} = \boldsymbol{0}\} \implies \mathcal{N}(\boldsymbol{A})$ is the space spanned by the basis vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\mathcal{R}(\boldsymbol{A}) = \{\boldsymbol{Ax} : \boldsymbol{x} \in \mathbb{F}^2\} \implies \mathcal{R}(\boldsymbol{A})$ is the space spanned by the basis vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Here $\mathcal{N}(\boldsymbol{A}) \neq \mathcal{R}(\boldsymbol{A})$ and in general they are not equal. In fact they do not have even have the same dimension when $\boldsymbol{A}$ is rectangular!

However, there are some matrices for which $\mathcal{N}(\boldsymbol{A})$ and $\mathcal{R}(\boldsymbol{A})$ are equal, in the sense that they are the span of the same vector(s), such as $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, where $\mathcal{N}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}) = \operatorname{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$.

Pr. 10. (sol/hs074)
We are given that

$$\texttt{FXY} = \begin{bmatrix} f(x_1, y_1) & \dots & f(x_1, y_n) \\ \vdots & \dots & \vdots \\ f(x_m, y_1) & \dots & f(x_m, y_n) \end{bmatrix}$$

$$\texttt{DFDX} = \begin{bmatrix} f(x_2, y_1) - f(x_1, y_1) & \dots & f(x_2, y_n) - f(x_1, y_n) \\ f(x_3, y_1) - f(x_2, y_1) & \dots & f(x_3, y_n) - f(x_2, y_n) \\ \vdots & \dots & \vdots \\ f(x_m, y_1) - f(x_{m-1}, y_1) & \dots & f(x_m, y_n) - f(x_{m-1}, y_n) \\ f(x_1, y_1) - f(x_m, y_1) & \dots & f(x_1, y_n) - f(x_m, y_n) \end{bmatrix}$$

$$\texttt{DFDY} = \begin{bmatrix} f(x_1, y_2) - f(x_1, y_1) & \dots & f(x_1, y_n) - f(x_1, y_{n-1}) & f(x_1, y_1) - f(x_1, y_n) \\ \vdots & \dots & \vdots & \vdots \\ f(x_m, y_2) - f(x_m, y_1) & \dots & f(x_m, y_n) - f(x_m, y_{n-1}) & f(x_m, y_1) - f(x_m, y_n) \end{bmatrix}.$$

The vec($\cdot$) versions are:

$$
\texttt{fxy} = \begin{bmatrix} f(x_1,y_1) \\ \vdots \\ f(x_{m-1},y_1) \\ f(x_m,y_1) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ f(x_1,y_n) \\ \vdots \\ f(x_{m-1},y_n) \\ f(x_m,y_n) \end{bmatrix}, \quad
\texttt{dfdx} = \begin{bmatrix} f(x_2,y_1)-f(x_1,y_1) \\ \vdots \\ f(x_m,y_1)-f(x_{m-1},y_1) \\ f(x_1,y_1)-f(x_m,y_1) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ f(x_2,y_n)-f(x_1,y_n) \\ \vdots \\ f(x_m,y_n)-f(x_{m-1},y_n) \\ f(x_1,y_n)-f(x_m,y_n) \end{bmatrix}, \quad
\texttt{dfdy} = \begin{bmatrix} f(x_1,y_2)-f(x_1,y_1) \\ \vdots \\ f(x_m,y_2)-f(x_m,y_1) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ f(x_1,y_n)-f(x_1,y_{n-1}) \\ \vdots \\ f(x_m,y_n)-f(x_m,y_{n-1}) \\ \cdots\cdots\cdots \\ f(x_1,y_1)-f(x_1,y_n) \\ \vdots \\ f(x_m,y_1)-f(x_m,y_n) \end{bmatrix}.
$$

Thus:

$$
\begin{bmatrix} \texttt{dfdx} \\ \cdots \\ \texttt{dfdy} \end{bmatrix} = \begin{bmatrix} f(x_2,y_1)-f(x_1,y_1) \\ \vdots \\ f(x_1,y_1)-f(x_m,y_1) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ f(x_1,y_n)-f(x_m,y_n) \\ \vdots \\ f(x_1,y_n)-f(x_m,y_n) \\ \cdots\cdots\cdots \\ f(x_1,y_2)-f(x_1,y_1) \\ \vdots \\ f(x_m,y_2)-f(x_m,y_1) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ f(x_1,y_1)-f(x_1,y_n) \\ \vdots \\ f(x_m,y_1)-f(x_m,y_n) \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} f(x_1,y_1) \\ \vdots \\ f(x_m,y_1) \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ f(x_1,y_n) \\ \vdots \\ f(x_m,y_n) \end{bmatrix},
$$

where

$$
\boldsymbol{A} = \left[\begin{array}{cccccccccccccccc}
-1 & 1 & & & & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & -1 & 1 & & & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & \ddots & \ddots & & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
1 & & & & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & & & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & -1 & 1 & & & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & \ddots & \ddots & & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & & -1 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & & & & -1 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & & & \ddots & \ddots & & & & & \dots & -1 & 1 & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & \dots & & -1 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & \dots & & & \ddots & \ddots & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & \dots & & & & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & \dots & 1 & & & & -1 \\
\hline
-1 & & \dots & & 1 & & & & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & -1 & & \dots & & 1 & & & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & \ddots & & & & \ddots & & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & & -1 & & \dots & & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
 & & & & \ddots & & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & -1 & & & & & \dots & & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & \ddots & & & & \ddots & & & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & -1 & \ddots & & \dots & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & & & & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\
 & & \ddots & & \ddots & & & & & & \ddots & & & & & 0 \\
1 & & & & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & -1 & 0 & & \\
 & 1 & & & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & & -1 & & \\
 & & \ddots & & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & & & \ddots & \\
 & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & & & & -1
\end{array}\right].
$$

Using the definition of $\boldsymbol{D}_n$ given in the problem statement and Kronecker product, we write this as

$$
\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_n \otimes \boldsymbol{D}_m \\ \boldsymbol{D}_n \otimes \boldsymbol{I}_m \end{bmatrix}.
$$

A possible `Julia` implementation is

```
using LinearAlgebra: I
using SparseArrays: sparse, spdiagm

function first_diffs_2d_matrix(m, n)
#
# Syntax:       A = first_diffs_2d_matrix(m, n)
#
# Inputs:       m and n are positive integers
#
# Outputs:      A is a 2mn x mn sparse matrix such that A * X[:] computes the
#               first differences down the columns (along x direction)
#               and across the (along y direction) of the m x n matrix X
#

D = [kron(sparse(I, n, n), D(m));   # First differences down columns (x)
    kron(D(n), sparse(I, m, m))]    # First differences across rows (y)
return D
end


function D(n)
#
# Syntax:       Dn = D(n)
#
# Inputs:       n is a positive integer
#
# Outputs:      Dn is an n x n sparse circulant first differences matrix
#

Dn = spdiagm(0 => -ones(n), 1 => ones(n-1), 1-n => [1])
return Dn
end
```