Lecture 10

Goals

m

- Signals as Vectors, Noise as Vectors
- Optimum Detection in AWGN

opt rcv: find the signal closeset to rec signal

Composition of Signals

- A set of functions $\varphi_i(t)$ is said to be orthonormal if

$$\int \varphi_i(t)\varphi_j^*(t)dt = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

- Given a set of orthogonal signals $\{\varphi_i(t), i = 0, 1, ..., N-1\}$ and a set of M vectors $s_m = (s_{m,0}, ..., s_{m,N-1}), m = 0, 1, ..., M-1$ we can construct a set of M signals as

$$s_m(t) = \sum_{i-0}^{N-1} s_{m,n}\varphi_n(t)$$

Decomposition of Signals (Gram-Schmidt)

- Given a set of signals $s_0(t), ..., s_{M-1}(t)$ there exists a set of orthonormal signals $\varphi_0(t), \varphi_1(t), ..., \varphi_{N-1}(t)$ with $N \leq M$ such that

$$s_i(t) = \sum_{m=0}^{N-1} s_{i,m} \varphi_m(t).$$

- The coefficients are determined as

$$s_{i,l} = \int s_i(t) \varphi_l^*(t) dt.$$

Gram-Schmidt: Step 1, Step 2

$(A, B) = B^T A$

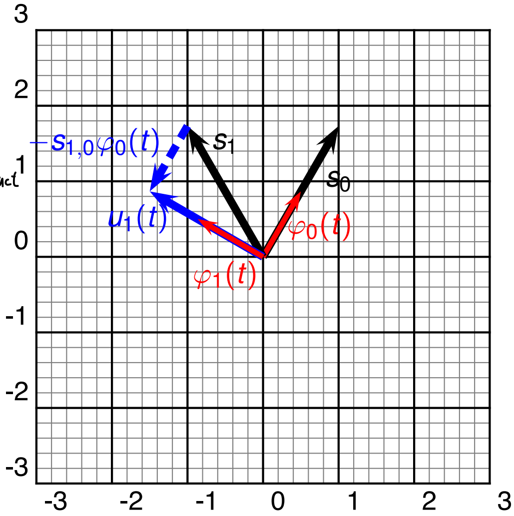
$A^T B$

$V'A$

$$
\begin{aligned}
u_0(t) &= s_0(t) \\
\varphi_0(t) &= u_0(t)/||u_0(t)|| \\
\hline
s_{1,0} &= (s_1(t), \varphi_0(t)) \\
u_1(t) &= s_1(t) - s_{1,0}\varphi_0(t) \\
\varphi_1(t) &= u_1(t)/||u_1(t)||
\end{aligned}
$$

inner product

The waveform $s_{1,0}\varphi_0(t)$ is the component of $s_1(t)$ in direction of $\varphi_0(t)$



Gram-Schmidt: Step 3

$$
\begin{aligned}
s_{2,1} &= (s_2(t), \varphi_1(t)) \\
s_{2,0} &= (s_2(t), \varphi_0(t)) \\
u_2(t) &= s_2(t) - s_{2,1}\varphi_1(t) - s_{2,0}\varphi_0(t) \\
\varphi_2(t) &= u_2(t)/||u_2(t)||
\end{aligned}
$$

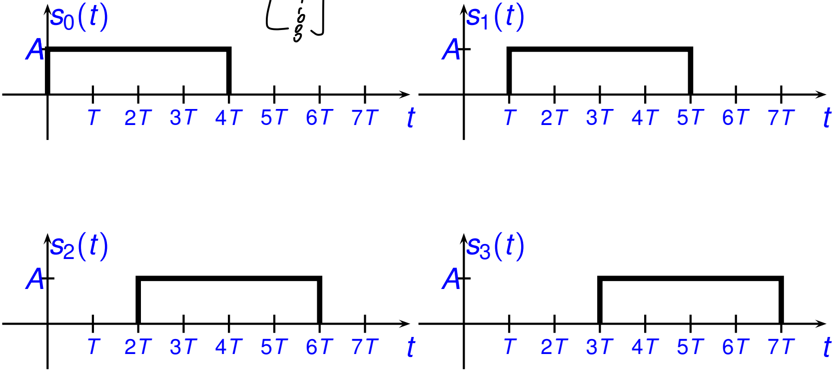
$$
\begin{aligned}
s_{3,2} &= (s_3(t), \varphi_2(t)) \\
s_{3,1} &= (s_3(t), \varphi_1(t)) \\
s_{3,0} &= (s_3(t), \varphi_0(t)) \\
u_3(t) &= s_3(t) - s_{3,2}\varphi_2(t) - s_{3,1}\varphi_1(t) - s_{3,0}\varphi_0(t) \\
\varphi_3(t) &= u_3(t)/||u_3(t)||
\end{aligned}
$$

Example 1

$$\varphi_0 = \frac{s_0}{\|s_0\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{bmatrix}$$

Consider the following set of four signals.

$$s_0 = \begin{bmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{bmatrix}$$



$$\varphi_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad S_1(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$S_{1,0} = (S_1(t), \varphi_0(t))$$

$$= \frac{1}{2}[1\ 1\ 1\ 1\ 0\ 0\ 0]\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\cdot 3$$

$$= \frac{3}{2}$$

$$u_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \frac{3}{2}\cdot\frac{1}{2}\begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

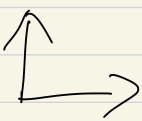
$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} 0 \\ \frac{1}{7} \\ \vdots \\ \frac{1}{7} \end{bmatrix}$$

$\varphi_0 \quad \varphi_1$

$$\begin{bmatrix} 0 \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \\ \vdots \\ \frac{1}{7} \end{bmatrix}$$

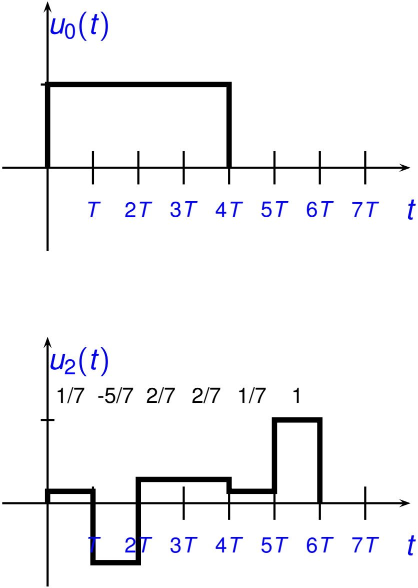


$x:\ (3, 4)$

$y:\ (3, 4)$

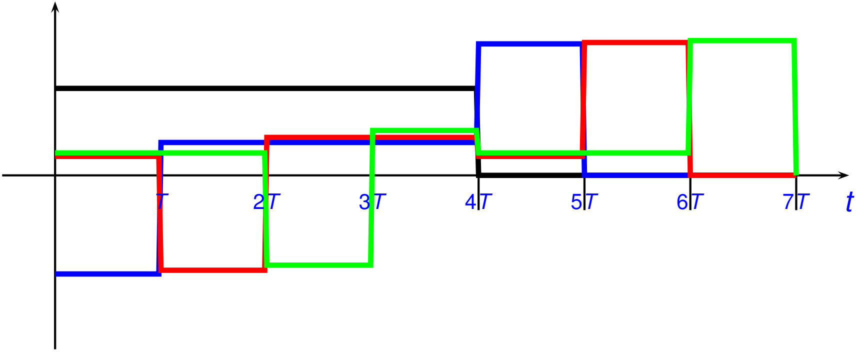
$(P((x, y)) \in R_{1.5} \mid H_0)$

Example 1: Orthogonal Basis (not orthonormal)



Example 1: Orthonormal Basis

$\varphi_0(t), \varphi_1(t), \varphi_2(t), \varphi_3(t)$



$$
\begin{aligned}
s_0(t) &= 2.00\varphi_0(t) + 0.00\varphi_1(t) + 0.00\varphi_2(t) + 0.00\varphi_3(t) \\
s_1(t) &= 1.50\varphi_0(t) + 1.32\varphi_1(t) + 0.00\varphi_2(t) + 0.00\varphi_3(t) \\
s_2(t) &= 1.00\varphi_0(t) + 1.13\varphi_1(t) + 1.31\varphi_2(t) + 0.00\varphi_3(t) \\
s_3(t) &= 0.50\varphi_0(t) + 0.94\varphi_1(t) + 1.09\varphi_2(t) + 1.29\varphi_3(t)
\end{aligned}
$$

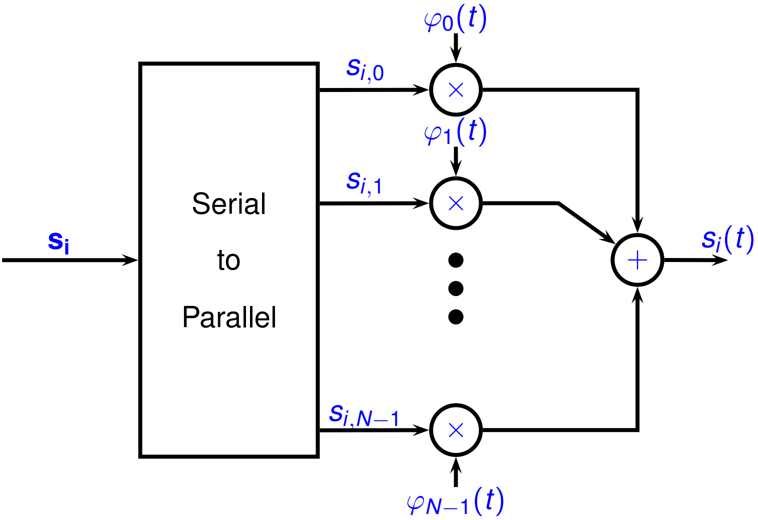
Properties

1. $(s_i, s_j) = \int s_i(t) s_j^*(t) dt = \sum_{l=0}^{N-1} s_{i,l} s_{j,l}^*$
2. $||s_i||^2 = \int |s_i(t)|^2 dt = \sum_{l=0}^{N-1} |s_{i,l}|^2$
3. $d_E^2(s_i, s_j) = ||s_i - s_j||^2 = \int |s_i(t) - s_j(t)|^2 dt = \sum_{l=0}^{N-1} |s_{i,l} - s_{j,l}|^2$

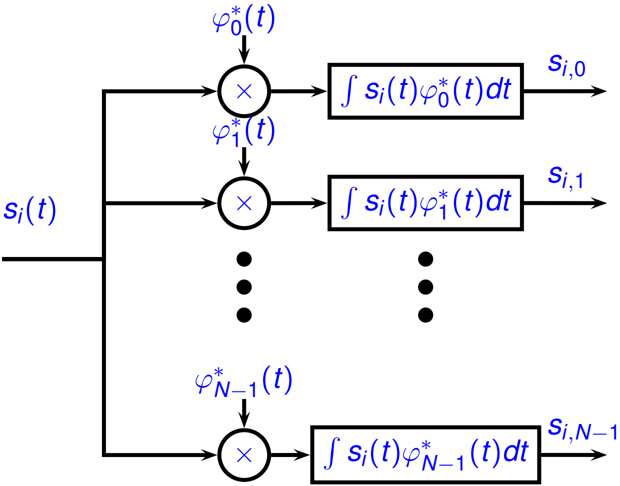
Proof of 1

$$
\begin{aligned}
\int s_i(t)s_j^*(t)dt &= \int \sum_{l=0}^{N-1} s_{i,l}\varphi_l(t) \sum_{m=0}^{N-1} s_{j,m}^*\varphi_m^*(t)dt \\
&= \sum_{l=0}^{N-1} s_{i,l} \sum_{m=0}^{N-1} s_{j,m}^* \int \varphi_l(t)\varphi_m^*(t)dt \\
&= \sum_{l=0}^{N-1} s_{i,l}s_{j,l}^*
\end{aligned}
$$

From a Vector to a Signal (Signal Composition)



From a Signal to a Vector (Signal Decomposition)



$r_0 = s_{i,0} + n_0$

$r_1 = s_{i,1} + n_1$

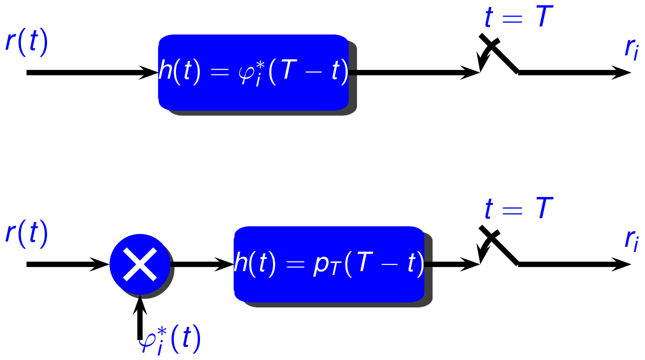
Correlation vs. Filtering

- Consider the computation of $\int r(t)\varphi_i^*(t)dt$.
- A filter with input $r(t)$ and impulse response $h(t) = \varphi_i^*(T-t)$ sampled at time T has output

$$\begin{aligned} r_i &= \int h(T-t)r(t)dt = \int \varphi_i^*(T-(T-t))r(t)dt \\ &= \int \varphi_i^*(t)r(t)dt \end{aligned}$$

- So either a correlator whereby the received signal is correlated with the orthonormal signal can be used to obtain r_i OR a matched filter with impulse reponse $h(t) = \varphi_i^*(T-t)$ which is sampled at time $t = T$ can be used to obtain r_i.

Correlation vs. Filtering

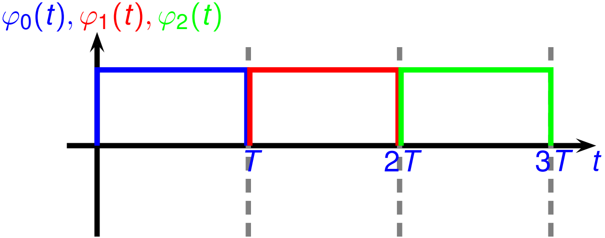


Example 1: Time orthogonal

$$\varphi_0(t) = \sqrt{\frac{1}{T}}p_T(t)$$

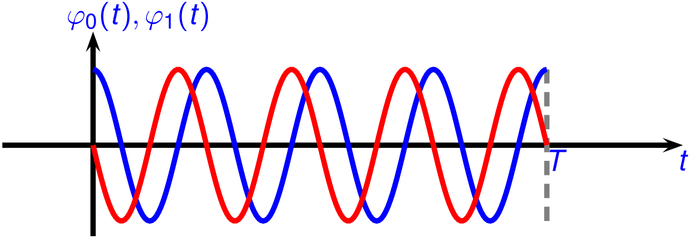
$$\varphi_1(t) = \sqrt{\frac{1}{T}}p_T(t-T)$$

$$\varphi_2(t) = \sqrt{\frac{1}{T}}p_T(t-2T)$$



Example 2: Phase Orthogonal

$$\begin{aligned} \varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t) \\ \varphi_1(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t) \end{aligned}$$



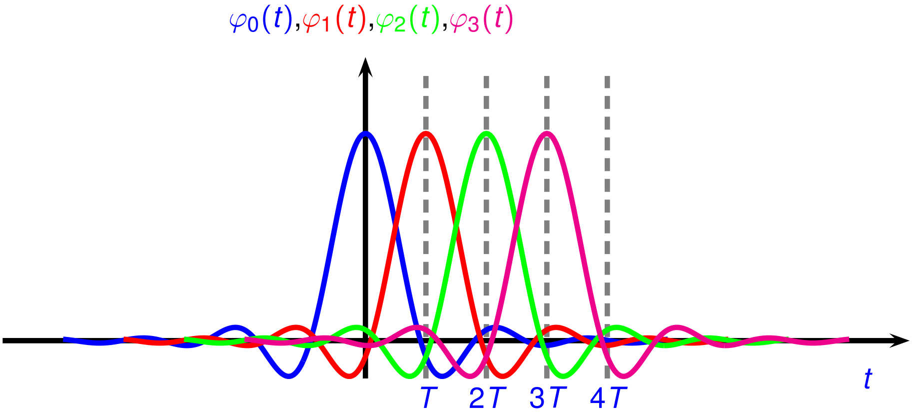
Example 3: Square-Root Raised Cosine Pulses

Let

$$x(t) = \frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T \cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T}.$$

$$\begin{aligned}
\varphi_0(t) &= x(t) \\
\varphi_1(t) &= x(t-T) \\
\varphi_2(t) &= x(t-2T) \\
\varphi_3(t) &= x(t-3T)
\end{aligned}$$

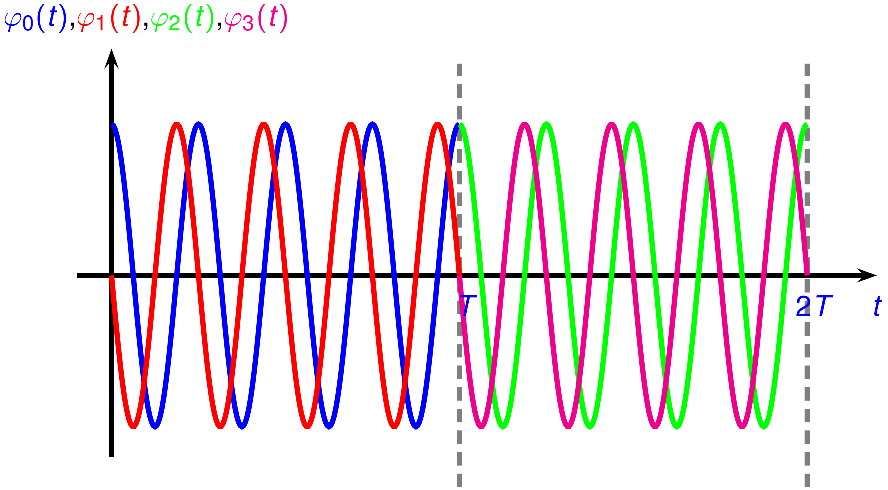
Example 3: Square-Root Raised Cosine Pulses



Example 4: Time, Phase Orthogonal

$$
\begin{aligned}
\varphi_0(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t) \\
\varphi_1(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t) \\
\varphi_2(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t-T) \\
\varphi_3(t) &= -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t-T)
\end{aligned}
$$

Example 4: Time, Phase Orthogonal



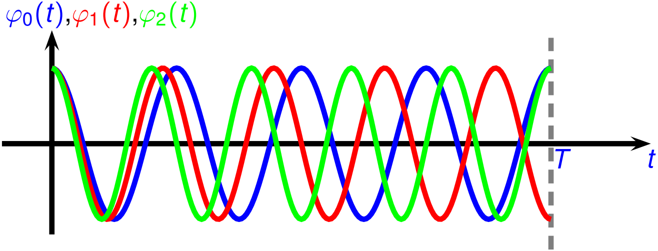
Example 5: Frequency Orthogonal

$$\varphi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)p_T(t)$$

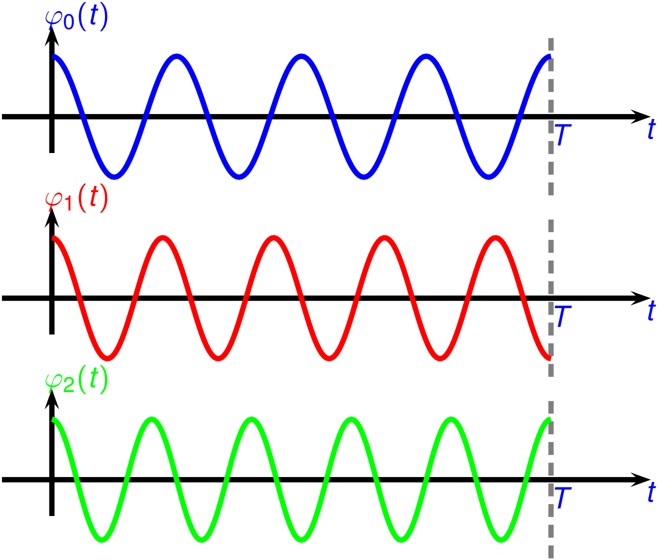
$$\varphi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)p_T(t)$$

$$\varphi_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)p_T(t)$$

$$(f_i - f_j) = \frac{n}{2T}$$



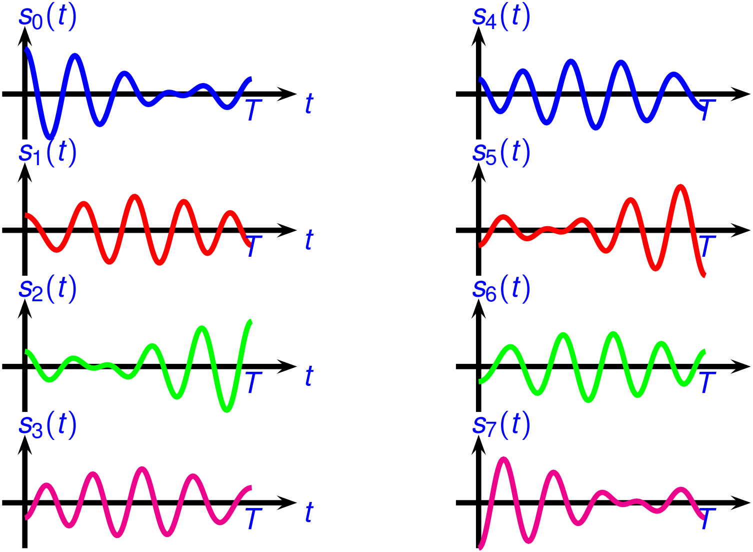
Example 5: Frequency Orthogonal



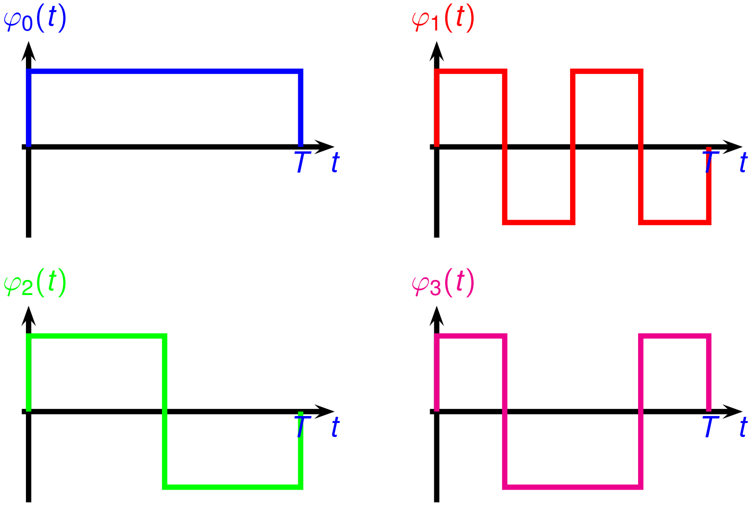
Example 5: Frequency Orthogonal

$$
\begin{aligned}
s_0 &= (+\sqrt{E}, +\sqrt{E}, +\sqrt{E}) \\
s_1 &= (+\sqrt{E}, +\sqrt{E}, -\sqrt{E}) \\
s_2 &= (+\sqrt{E}, -\sqrt{E}, +\sqrt{E}) \\
s_3 &= (+\sqrt{E}, -\sqrt{E}, -\sqrt{E}) \\
s_4 &= (-\sqrt{E}, +\sqrt{E}, +\sqrt{E}) \\
s_5 &= (-\sqrt{E}, +\sqrt{E}, -\sqrt{E}) \\
s_6 &= (-\sqrt{E}, -\sqrt{E}, +\sqrt{E}) \\
s_7 &= (-\sqrt{E}, -\sqrt{E}, -\sqrt{E})
\end{aligned}
$$

Example 5: Frequency Orthogonal



Example 6: Walsh/Hadamard Orthogonal



Recursive calculation for Walsh/Hadamard

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} A_2 & A_2 \\ A_2 & -A_2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$A_{2n} = \begin{bmatrix} A_n & A_n \\ A_n & -A_n \end{bmatrix}$$

A_{64} used in 2 Gen
cell phone

Straight forward to show any two rows in A_n are orthogonal.

Decomposition of Noise

Any finite energy signal can be written as linear combination of the orth signals

For any complete orthonormal set of signals $\varphi_0(t), \varphi_1(t), \ldots$ we can represent a noise process as random variables and deterministic orthonormal functions

r.v. random part

$$n(t) = \sum_{m=0}^{\infty} n_m \varphi_m(t), \quad n_m = \int n(t)\varphi_m^*(t)dt.$$

The noise variables n_m and n_l are independent if $n(t)$ is white Gaussian noise.

deterministic function of time

Decomposition of Noise

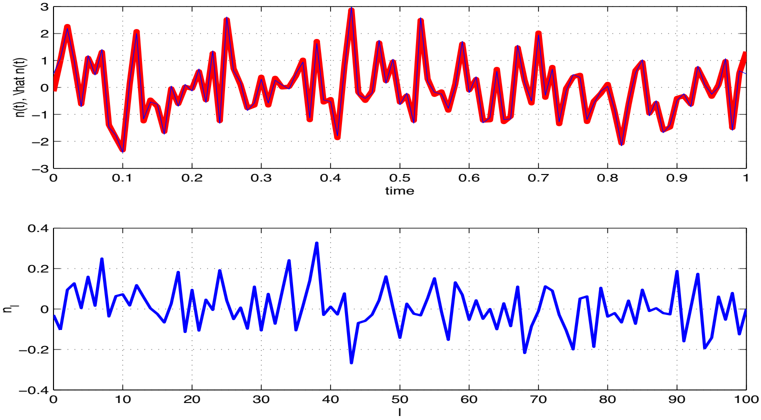
Consider the case of *real* white Gaussian noise with power spectral density $N_0/2$.

0 mean

$$
\begin{aligned}
E[n_m n_l] &= E[\int n(t)\varphi_m(t)dt \int n(s)\varphi_l(s)ds \\
&= \int\int E[n(t)n(s)]\varphi_m(t)\varphi_l(s)dtds \\
&= \int\int \frac{N_0}{2}\delta(t-s)\varphi_m(t)\varphi_l(s)dtds \\
&= \int \frac{N_0}{2}\varphi_m(t)\varphi_l(t)dt \\
&= \begin{cases} \frac{N_0}{2}, & m = l \\ 0, & m \neq l \end{cases}
\end{aligned}
$$

Thus n_m and n_l are uncorrelated for $m \neq l$. Because they are also Gaussian they are independent.

Decomposition of Noise



Decomposition of Signal and Noise

Consider a communication system that transmits one of M signals. $s_0(t), ..., s_{M-1}(t)$ in additive white Gaussian noise. s Then given $s_i(t)$ was transmitted the received signal is

$$\begin{aligned} r(t) &= s_i(t) + n(t) \\ &= \sum_{m=0}^{\infty} (s_{i,m} + n_m)\varphi_m(t). \end{aligned}$$

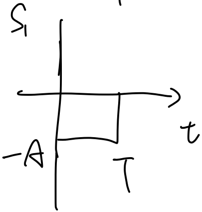
Define $r_m = s_{i,m} + n_m$. Then

$$r(t) = \sum_{m=0}^{\infty} r_m \varphi_m(t).$$

We can determine the (random) variable r_m by

$$r_m = \int r(t)\varphi_m^*(t)dt.$$

Example 1



$$
\begin{aligned}
s_0(t) &= Ap_T(t) \\
s_1(t) &= -Ap_T(t)
\end{aligned}
$$

a complete orthonormal set

$\underline{e^{j7\pi/3}} \neq 1$

$l=0,1,\cdots$

$e^{j2\pi \cdot l t} = 1$

- Let $\varphi_l(t) = \sqrt{\frac{1}{T}} \exp\{\underline{j2\pi l f_0 t}\} p_T(t)$ where $f_0 = 1/T$.
- Note that $\varphi_0(t) = \sqrt{\frac{1}{T}} p_T(t)$.

Example 1

With this set of orthonormal functions we can write

$$
\begin{aligned}
s_0(t) &= \sqrt{E}\varphi_0(t), \quad s_1(t) = -\sqrt{E}\varphi_0(t) \\
n(t) &= \sum_{m=0}^{\infty} n_m \varphi_m(t) \\
r(t) &= \sum_{m=0}^{\infty} r_m \varphi_m(t) \\
r_m &= \int r(t)\varphi_m^*(t)dt = \int (s_i(t) + n(t))\varphi_m^*(t)dt \\
&= s_{i,m} + n_m, \quad m = 0, 1, 2, \ldots
\end{aligned}
$$

$$s_{i,m} = \begin{cases} \pm\sqrt{E}, & m=0 \\ 0, & m\neq 0 \end{cases}$$

Example 1

$r_0 > 0 \quad s_0$
$r_0 < 0 \quad s_1$

$s_0(t)$	$s_1(t)$
$r_0 = \sqrt{E} + n_0$	$r_0 = -\sqrt{E} + n_0$
$r_1 = n_1$	$r_1 = n_1$
$r_2 = n_2$	$r_2 = n_2$

rest part just noises , make decision on r_0 alone

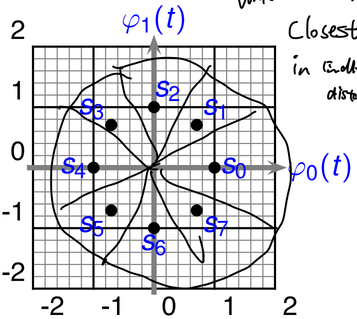
Example 2: 8PSK

$r_0 = s_{i,0} + n_0$
$r_1 = s_{i,1} + n_1$
$r_2 = 0 + n_2$
$r_3 = n_3$ . . .
} these are independent

$$\varphi_0(t) = \sqrt{2/T}\cos(2\pi f_c t)p_T(t)$$
$$\varphi_1(t) = -\sqrt{2/T}\sin(2\pi f_c t)p_T(t)$$

These are orthogonal if $f_c T \gg 1$. $s_m = e^{(j2\pi m/8)}, m = 0, 1, ..., 7$

Data bits	Signal	s_m
000	s_0	$(1, 0)$
001	s_1	$(\sqrt{2}/2, \sqrt{2}/2)$
011	s_2	$(0, 1)$
010	s_3	$(-\sqrt{2}/2, \sqrt{2}/2)$
110	s_4	$(-1, 0)$
111	s_5	$(-\sqrt{2}/2, -\sqrt{2}/2)$
101	s_6	$(0, -1)$
100	s_7	$(\sqrt{2}/2, -\sqrt{2}/2)$



Which one is Closest in Euclidean distance

Optimal Receiver

$r_m = s_{i,m} + n_m$

- Note that we can recover completely $r(t)$ if we know the coefficients $r_m,\ m = 0, 1, \ldots$.
- So the optimal decision based on observing $r_0, r_1, \ldots$ is also the optimal decision based on observing $r(t)$.
- Given signal $s_i(t)$ is transmitted we can determine the probability density of r_m as follows.
- First, r_m is Gaussian since it is the result of integrating Gaussian noise.
- Second the mean of r_m, conditioned on signal $s_i(t)$ transmitted is $s_{i,m}$ and the variance is $N_0/2$.

Optimal Receiver

- So the probability density of r_m conditioned on signal $s_i(t)$ transmitted (event H_i) is

density function

$\sigma^2 = \frac{N_0}{2}$

mean

$$\begin{aligned} p_i(r_m) &= f_{r_m|H_i}(r_m) \\ &= \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} \exp\{-\frac{(r_m - s_{i,m})^2}{2(N_0/2)}\} \end{aligned}$$

- Next note that r_m is independent of r_n for $m \neq n$.
- Thus

joint density function

product

$$\begin{aligned} f_{r_0,r_1,\ldots,r_k|H_i}(x_0, x_1, x_2, \ldots, x_k) &= \prod_{m=0}^{k} f_{r_m|H_i}(x_m) \\ &= \prod_{m=0}^{k} p_i(x_m) \end{aligned}$$

M-ary Detection Problem

$H_0, H_1, \cdots H_{M-1}$

- Consider the problem of deciding which of M hypotheses is true based on observing a random variable (vector) r. which H is true
- The performance criteria we consider is the average error probability. That is, the probability of deciding anything except hypothesis H_j when hypothesis H_j is true.
- The underlying model is that there is a conditional probability density (mass) function of the observation r given each hypothesis H_j.

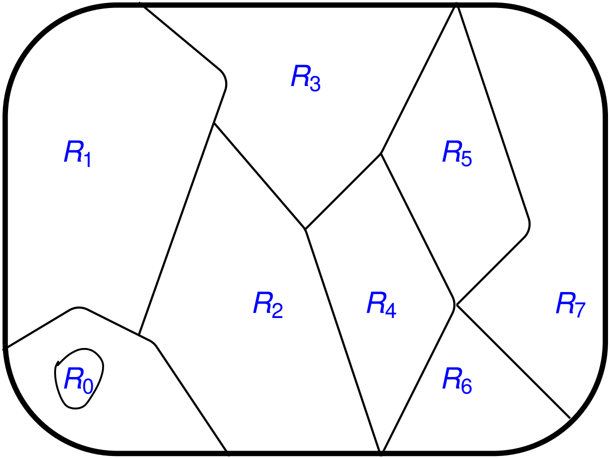
region where decision is H_m

$$P\{r \in R_m | H_i\} = \int_{R_m} p_i(r) dr$$

Based on r.

- There are disjoint decision regions $R_0, R_1, ..., R_{M-1}$. When $r \in R_m$ the receiver decides H_m.

Decision Regions



Objective

Our goal is to find the decision regions $R_0, R_1, ..., R_{M-1}$ that minimize the error probability.

p(error/H_i)

given H_i is true

$$
\begin{aligned}
E[P_e] = \sum_{i=0}^{M-1} P_{e,i}\pi_i &= \sum_{i=0}^{M-1} P\{\text{don't decide } H_i | H_i\}\pi_i \\
&= \sum_{i=0}^{M-1} [1 - P\{\text{decide } H_i | H_i \text{ true}\}]\,\pi_i \\
&= \sum_{i=0}^{M-1} \pi_i - \sum_{i=0}^{M-1} \int_{R_i} p_i(r)\pi_i dr \\
&= 1 - \sum_{i=0}^{M-1} \int_{R_i} p_i(r)\pi_i dr \quad .
\end{aligned}
$$

$p(H_i) = \pi_i$

apriori prob.

without r knowing

$\pi_i = \frac{1}{M}$ (mostly)

minimize this

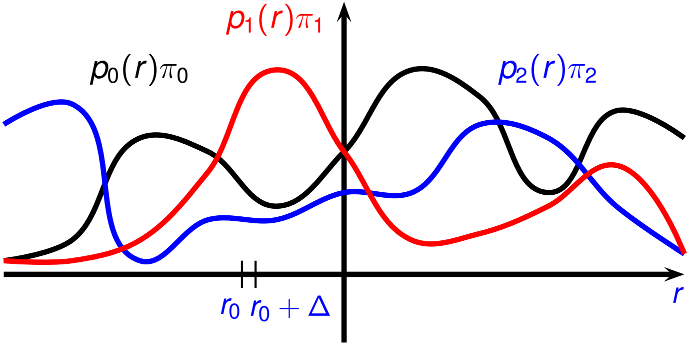
Objective

- The decision rule that minimizes the average error probability is the decision rule that maximizes

$$\Gamma = \sum_{i=0}^{M-1} \int_{R_i} p_i(r)\pi_i dr = \int_{-\infty}^{\infty} \sum_{i=0}^{M-1} p_i(r)\pi_i I(r \in R_i) dr.$$

- Consider a small region for $r \in A = (r_0, r_0 + \Delta)$ where $p_i(r)$ is nearly constant.
- If $r \in A$ then the contribution to Γ is either $p_0(r_0)\pi_0\Delta$ if we have a decision rule so that $r_0 \in R_0$, or the contribution to Γ is $p_1(r_0)\pi_1\Delta$ if we have a decision rule so that $r_0 \in R_1$, or the contribution to Γ is $p_2(r_0)\pi_2\Delta$ if we have a decision rule so that $r_0 \in R_2$.
- In order to make the largest contribution to Γ we should have a decision rule such that $r \in R_i$ if $p_i(r)\pi_i > p_j(r)\pi_j$ for all $j \neq i$.

Objective

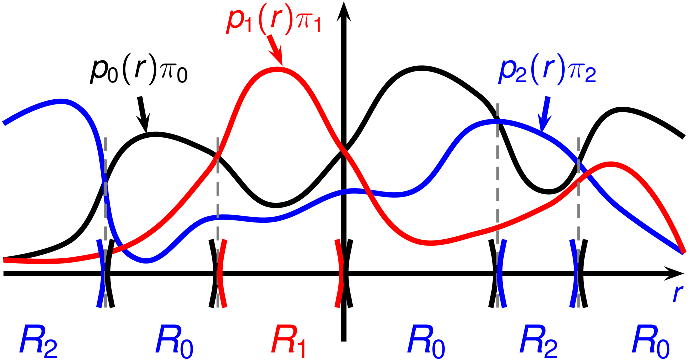


Objective

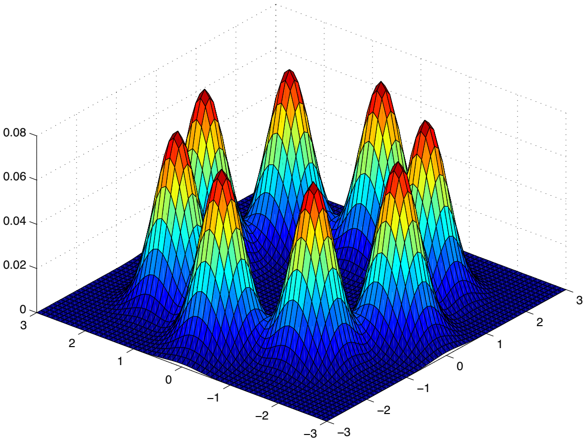
The decision rule that minimizes the average error probability is the decision rule that maximizes

maximize

$$\sum_{i=0}^{M-1} \int_{R_i} p_i(r)\pi_i dr \quad .$$



Example of Two-Dimensional Densities



Optimal Receiver

The decision rule that minimizes average probability of error assigns r to R_i if $p_i(r)\pi_i = \max_{0 \le j \le M-1} p_j(r)\pi_j$.

To understand this consider the case of $M = 2$ where $R_0 \cup R_1$ is the enitre observation region. Then

$$E[P_e] = 1 - \int_{R_0} p_0(r)\pi_0 dr - \int_{R_1} p_1(r)\pi_1 dr \quad .$$

If for a particular r, $p_0(r)\pi_0 > p_1(r)\pi_1$ then the error probability will be smaller if that value of r is included in the decision region for R_0 rather than the decision region for R_1.

Alternate Forms of the Optimal Receiver

Let $p(r)$ be an arbitrary density function that is nonzero everywhere $p_i(r)$ is nonzero then an equivalent decision rule is to assign r to R_i if

$$\frac{p_i(r)}{p(r)}\pi_i = \max_{0 \le j \le M-1} \frac{p_j(r)}{p(r)}\pi_j.$$

likelihood ratio

pairwise comp

Thus for M hypotheses the decision rule that minimizes average error probability is to choose i so that $p_i(r)\pi_i > p_j(r)\pi_j, \;\; \forall \;\; j \neq i$. Let

$$\Lambda_{i,j} = \frac{p_i(r)}{p_j(r)}$$

where $i = 0, 1, \ldots, M-1$, $j = 0, 1, \ldots, M-1$. Then the optimal decision rule is:

$$\text{Choose } i \text{ if } \Lambda_{i,j} > \frac{\pi_j}{\pi_i} \text{ for all } j \neq i.$$

Alternate Forms of the Optimal Receiver

- We will usually assume $\pi_i = \frac{1}{M} \; \forall i$. (If not we should do source encoding to reduce the entropy (rate)).
- For this case the optimal decision rule is

$$\text{Choose } i \text{ if } \Lambda_{i,j} > 1 \;\; \forall \;\; j \neq i.$$

- Note that the optimum receiver does a pair-wise comparison between two potential signals (for every pair).
- So if we know the optimum receiver for any two signals we can find the optimal receiver for M signals.

Example 2: Additive White Gaussian Noise

Consider three signals in additive white Gaussian noise. For additive white Gaussian noise $K(s,t) = \frac{N_0}{2}\delta(t-s)$. Let $\{\varphi_i(t)\}_{i=0}^{\infty}$ be any complete orthonormal set on $[0, T]$. Consider the case of 3 signals. Find the decision rule to minimize average error probability. First expand the noise using orthonormal set of functions and random variables.

$$n(t) = \sum_{i=0}^{\infty} n_i \varphi_i(t)$$

where $E[n_i] = 0$ and $\mathrm{Var}[n_i] = N_0/2$ and $\{n_i\}_{i=0}^{\infty}$ is an independent identically distributed (i.i.d.) sequence of random variables with Gaussian density functions.

Example 2: Additive White Gaussian Noise

Let $M=3$, $N=2$ dim

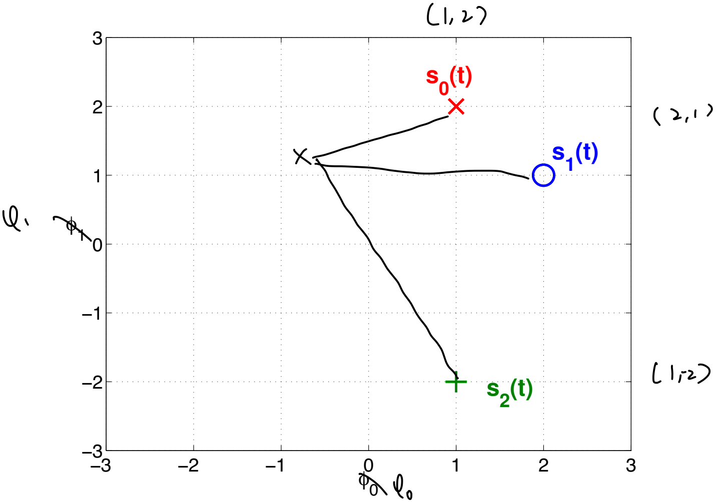
$$
\begin{aligned}
s_0(t) &= \varphi_0(t) + 2\varphi_1(t) \\
s_1(t) &= 2\varphi_0(t) + \varphi_1(t) \\
s_2(t) &= \varphi_0(t) - 2\varphi_1(t)
\end{aligned}
$$

Note that the energy of each of the three signals is the same, i.e. $\int_0^T s_i^2(t)dt = ||s_i||^2 = 5$. Then we have a three hypothesis testing problem.

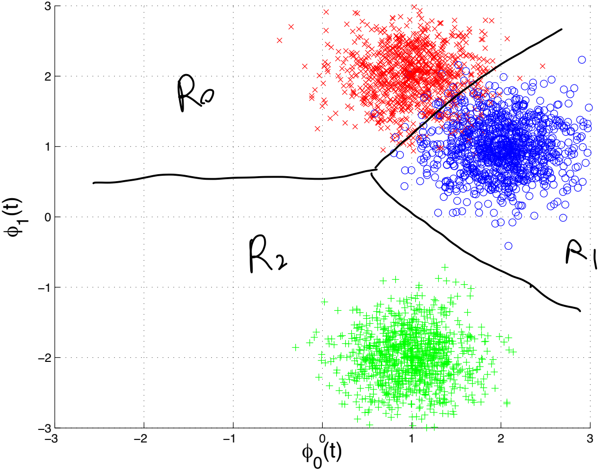
$s_{m,i}=0 \quad i \geq 2$

$$
\begin{aligned}
H_0 : r(t) &= s_0(t) + n(t) = \sum_{i=0}^{\infty}(s_{0,i} + n_i)\varphi_i(t) \\
H_1 : r(t) &= s_1(t) + n(t) = \sum_{i=0}^{\infty}(s_{1,i} + n_i)\varphi_i(t) \\
H_2 : r(t) &= s_2(t) + n(t) = \sum_{i=0}^{\infty}(s_{2,i} + n_i)\varphi_i(t)
\end{aligned}
$$

Example 1: Additive White Gaussian Noise



Example 1: Additive White Gaussian Noise



Example 1: Decision Rule

- The decision rule to minimize the average error probability is given as follows

$$\text{Decide } H_i \text{ if } \pi_i p_i(\mathbf{r}) = \max_j \pi_j p_j(\mathbf{r})$$

- Suppose the desired signal corresponds to an N dimensional signal. That is $s_{i,n} = 0$ for $n \geq N$.
- First let us consider the first L variables where $L > N$ and normalize each side by the a constant times the density function for the noise alone, but only in the received signal dimensions that has no signal.

- The noise density function for $L - N$ variables is

$$
\begin{aligned}
p^{(L)}(\mathbf{r}) &= \left(\frac{1}{\sqrt{2\pi N_0/2}} \right)^{L-N} \exp\{-\frac{1}{2\frac{N_0}{2}} \sum_{n=N}^{L-1} r_n^2\} \\
f^{(L)}(\mathbf{r}) &= \left(\frac{1}{\sqrt{2\pi N_0/2}} \right)^{N} p^{(L)}(r) \\
&= \left(\frac{1}{\sqrt{2\pi N_0/2}} \right)^{L} \exp\{-\frac{1}{2\frac{N_0}{2}} \sum_{n=N}^{L-1} r_n^2\}
\end{aligned}
$$

$$\underbrace{r_0, r_1, r_2 \cdots r_{N-1}}_{\text{have signal term}},\ \underbrace{r_N, \cdots r_{m-1}}_{\text{noise term}}$$

Optimal Decision Rule

The optimal decision rule is equivalent to

$$\text{Decide } H_i \text{ if } \pi_i \frac{p_i(\mathbf{r})}{f^{(L)}(\mathbf{r})} = \max_j \pi_j \frac{p_j(\mathbf{r})}{f^{(L)}(\mathbf{r})}.$$

As usual assume $\pi_i = 1/M$. Then $r_0, \ldots, r_{N-1}, r_N, \ldots, r_{n-1}$

$$\begin{aligned}
\frac{p_i^{(L)}(\mathbf{r})}{f^{(L)}(\mathbf{r})} &= \frac{\cancel{\left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^L} \exp\{-\frac{1}{2\frac{N_0}{2}}[\sum_{n=0}^{N-1}(r_n - s_{i,n})^2 \cancel{+ \sum_{n=N}^{L-1} r_n^2}]\}}{\cancel{\left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^L} \cancel{\exp\{-\frac{1}{2\frac{N_0}{2}} \sum_{n=N}^{L-1} r_n^2\}}} \\
&= \exp\{-\frac{1}{N_0}[\sum_{n=0}^{N-1}(r_n - s_{i,n})^2]\} = \exp\{-\frac{1}{N_0}[d_E^2(r, s_i)]\}
\end{aligned}$$

min this

max

Now since the above doesn't depend on L we can let $L \to \infty$ and the result is the same. The optimal rule is to find the signal s_i closest to the received signal.

Example 1

$s_0 = (+1, 2)$
$s_1 (+2, +1)$
$s_2 = (+1, -2)$

$$
\begin{aligned}
d_E^2(r, s_0) &= (r_0 - 1)^2 + (r_1 - 2)^2 \\
&= r_0^2 - 2r_0 + 1 + r_1^2 - 4r_1 + 4 \\
&= r_0^2 + r_1^2 - 2r_0 - 4r_1 + 5
\end{aligned}
$$

Similarly

$$
\begin{aligned}
d_E^2(r, s_1) &= (r_0 - 2)^2 + (r_1 - 1)^2] \\
&= r_0^2 + r_1^2 - 4r_0 - 2r_1 + 5 \\
d_E^2(r, s_2) &= r_0 + 1)^2 + (r_1 + 2)^2 \\
&= r_0^2 + r_1^2 + 2r_0 + 4r_1 + 5
\end{aligned}
$$

it's the energy on the circle

Example 1

- We need to find the smallest of these three quantities.

$$
\begin{aligned}
d_E^2(r, s_0) &= r_0^2 + r_1^2 - 2r_0 - 4r_1 + 5 \\
d_E^2(r, s_1) &= r_0^2 + r_1^2 - 4r_0 - 2r_1 + 5 \\
d_E^2(r, s_2) &= r_0^2 + r_1^2 + 2r_0 + 4r_1 + 5
\end{aligned}
$$

- Clearly we can ignore the $r_0^2 + \cancel{2}r_1^2 + 5$ terms.
- So we need to find the ~~largest~~ of smallest

$$
\begin{aligned}
C_0 = &= -2r_0 - 4r_1 + \cancel{5} \\
C_1 &= -4r_0 - 2r_1 \\
C_2 &= 2r_0 + 4r_1
\end{aligned}
$$

Example 1

c_0 $\qquad\qquad$ c_1

$$-2r_0 - 4r_1 \underset{s_1}{\overset{s_0}{\gtrless}} -4r_0 - 2r_1$$

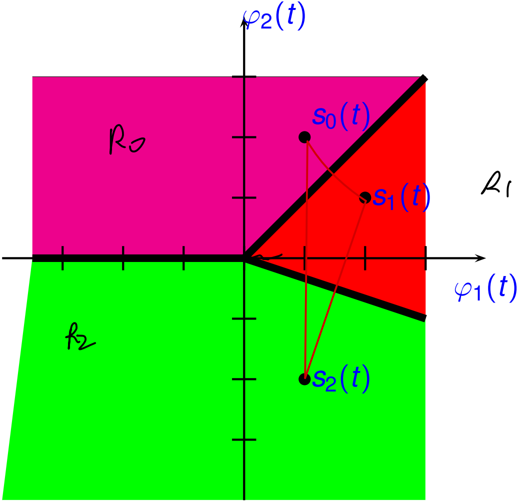
$$2r_0 \underset{s_1}{\overset{s_0}{\gtrless}} 2r_1$$

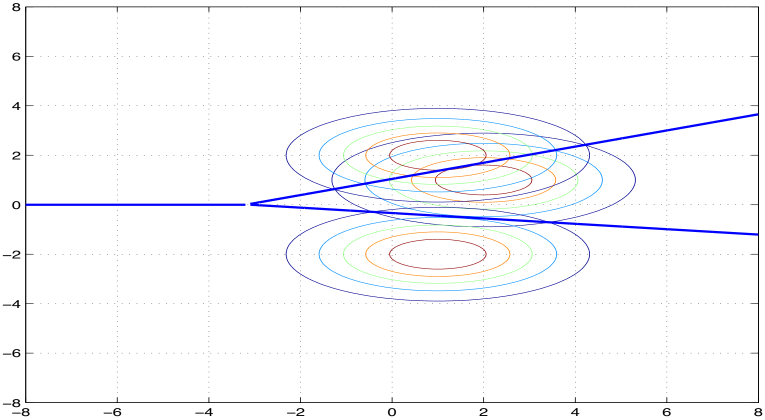
$$r_0 \underset{s_1}{\overset{s_0}{\gtrless}} r_1$$

If LHS > RHS decide s_0, otherwise, decide s_1

- To determine the region where the distance to s_0 is smallest we can draw the perpendicular bisector between the point s_0 and s_1 and between the points s_0 and s_2.
- Similarly for the region where the distance is closest to s_1 and the region where the received signal is closest to s_2.

Example 1




8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8

Likelihood Ratio for Real Signals in AWGN

Assume two signals in Gaussian noise.

$$\begin{aligned} H_0 &: \quad r(t) = s_0(t) + n(t) \\ H_1 &: \quad r(t) = s_1(t) + n(t) \end{aligned}$$

Goal: Find decision rule to minimize the average error probability.
Let $n(t)$ autocorrelation function $R((s,t) = \frac{N_0}{2}\delta(t-s)$. We assume that $n(t)$ is a zero mean white Gaussian noise random process.

Karhunen-Loeve Expansion

By Karhunen-Loeve expansion

$$n(t) = \sum_{m=0}^{\infty} n_m \varphi_i(t)$$

where n_i are Gaussian random variables with mean 0 variance $\frac{N_0}{2}$ and $E[n_m n_k] = 0,\ m \neq k$. Thus n_m and n_k are independent. Since $\{\varphi_m(t); m = 0, 1, ...\}$ is a complete orthonormal set and we assume $s_j(t)$ has finite energy we have

$$s_j(t) \ = \ \sum_{m=0}^{\infty} s_{j,m}\varphi_m(t) = \sum_{m=0}^{N-1} s_{j,m}\varphi_m(t).$$

This last equality is because we only need a finite ($N \leq M$) orthonormal waveforms to represent a set of M signals. Equivalently $s_{j,i} = 0$ for $i \geq N$.

Karhunen-Loeve Expansion

Thus

$$
\begin{aligned}
H_j : r(t) &= \sum_{m=0}^{\infty} (s_{j,m} + n_m)\varphi_m(t) \\
& r_m = s_{j,m} + n_m, \quad m = 0, 1, 2, ...
\end{aligned}
$$

Define

$$\Lambda_{j,i}(L) = \frac{p_j(r_0, r_1, \ldots, r_L)}{p_i(r_0, r_1, \ldots, r_L)} \quad .$$

$$\Lambda_{j,i}(r(t)) = \lim_{L\to\infty} \Lambda_{j,i}(L)$$

where r_m is Gaussian with mean $s_{j,m}$ variance $N_0/2$.

Karhunen-Loeve Expansion

$$
\begin{aligned}
p_j(r_m) &= \frac{1}{\sqrt{N_0\pi}} \exp\left\{-\frac{1}{N_0}(r_m - s_{j,m})^2\right\} \\
p_j(\underline{r}) = \prod_{m=0}^{L} p_j(r_m) &= \prod_{m=0}^{L} (\sqrt{N_0\pi})^{-1} \exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}(r_m - s_{j,m})^2\right\} \\
\Lambda_{j,l}(L) = \frac{p_j^L(\underline{r})}{p_l^L(\underline{r})} &= \frac{\cancel{\prod_{m=0}^{L}(\sqrt{N_0\pi})^{-1}}}{\cancel{\prod_{m=0}^{L}(\sqrt{N_0\pi})^{-1}}} \frac{\exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}(r_m - s_{j,m})^2\right\}}{\exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}(r_m - s_{l,m})^2\right\}} \\
&= \exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}[r_m^2 - 2r_m s_{j,m} + s_{j,m}^2 - r_m^2 + 2r_m s_{l,m} - s_{l,m}^2]\right\} \\
&= \exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}[\overset{\text{energy } j}{s_{j,m}^2} - \overset{\text{energy } l}{s_{l,m}^2} + 2r_m(s_{l,m} - s_{j,m})]\right\}.
\end{aligned}
$$

likelihood

Karhunen-Loeve Expansion

If we take the limit as $L \to \infty$ we get

$$\Lambda_{j,l}(r(t)) \quad = \quad \exp\left\{-\frac{1}{N_0}(E_j - E_l + 2(r, s_l - s_j)]\right\}.$$

$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{N_0}[(s_j, s_j) - (s_l, s_l) + 2(r, s_l) - 2(r, s_j)]\right\}.$$

or equivalently

$+||r||^2$

$$\begin{aligned}\Lambda_{j,l}(r(t)) &= \exp\left\{-\frac{1}{N_0}[||s_j||^2 - ||s_l||^2 + 2(r, s_l - s_j)]\right\} \\ &= \exp\left\{-\frac{1}{N_0}[||r - s_j||^2 - ||r - s_l||^2]\right\}\end{aligned}$$

$d(r, s_j)$ $d(r, s_l)$

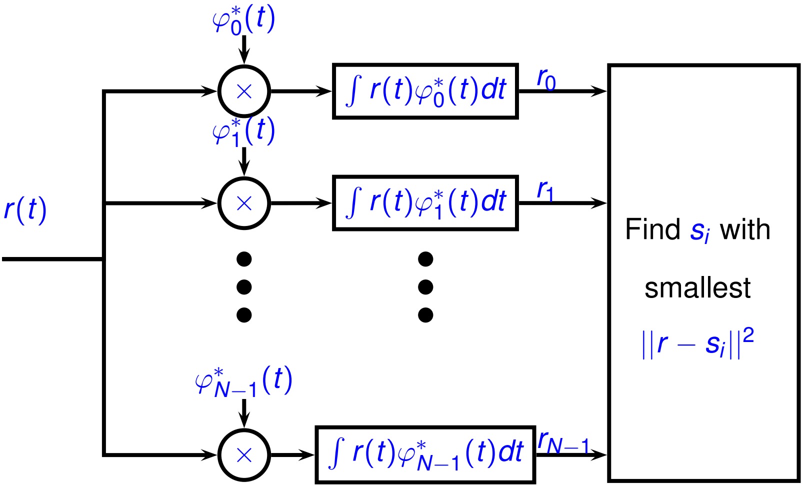
Optimum Receiver Principles

Claim:

The optimum decision rule to decide among M equally likely possible transmitted signals for additive white Gaussian noise is to choose i if

$$\|s_i - r\|^2 = \min_{0 \le j \le M-1} \|s_j - r\|^2.$$

Demodulator



Demodulator

Since the optimum receiver computes the squared Euclidean distance between the observation and the M signals this can be implemented by noting that

min $\max_i$ this

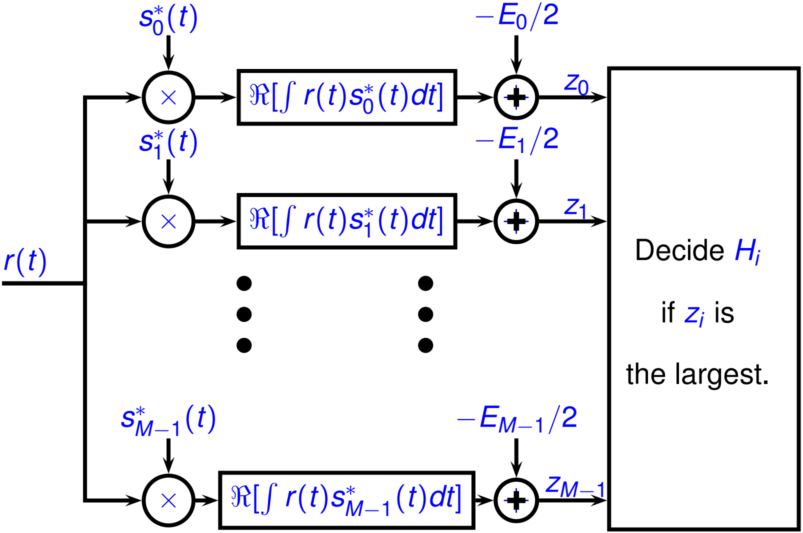
$$||r - s_i||^2 = ||r||^2 - 2\Re[(r, s_i)] + ||s_i||^2.$$

Thus minimizing the distance squared is equivalent to maximizing

$$z_i = \Re[(r, s_i)] - E_i/2 \;\; \text{where} \;\; E_i = ||s_i||^2.$$

This can be implemented as follows.

Demodulator



Example: M equal energy signals

Now consider the optimum receiver for M-ary equally likely signals and the associated error probability. Assume the M signals are equienergy signals and equiprobable. The decision rule derived previously for AWGN in this case simplifies to

$$\text{Decide } H_i \text{ if } ||s_i - r||^2 = \min_{0 \leq j \leq M-1} ||s_j - r||^2.$$

Now since the M signals are equienergy we can write this as

$$||s_i - r||^2 = ||s_i||^2 - 2\Re[(r, s_i)] + ||r||^2.$$

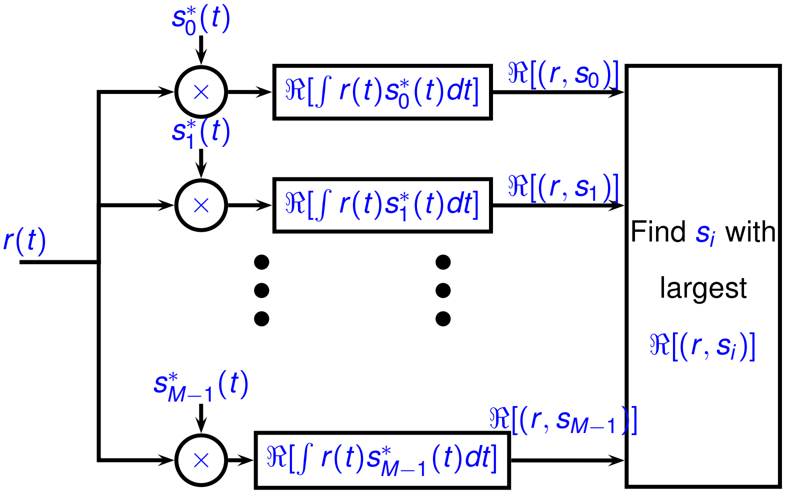
The first term above is constant for each i as is the last term. Thus finding the minimum is equivalent to finding the maximum of

$$\Re[(r, s_i).$$

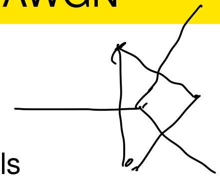
Example: M equal energy signals

- Thus the receiver should compute the inner product between the M different signals and find the largest such correlation.
- If the signals are all of duration T, i.e. zero outside the interval $[0, T]$ then this is also equivalent to filtering the received signal with a filter with impulse response $s_j(T - t)$, sampling the output of the filter at time T and choosing the largest.

Demodulator (Equal Energy Case)



Notes about Optimum Receiver in AWGN



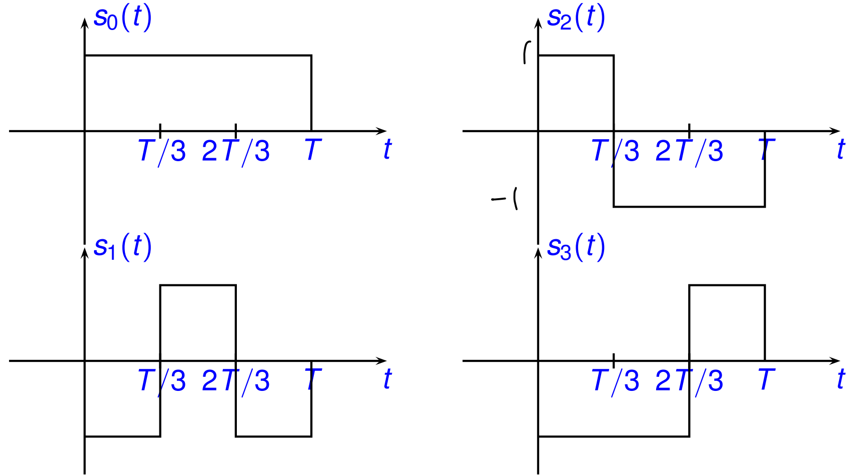
- Consider the case of equally likely signals ($\pi_0 = ... = \pi_{M-1} = 1/M$).
- The optimum receiver first maps the received signal into a N dimensional vector. ($r(t) \to r$).
- The decision region is determined by the perpendicular bisectors of the signal points.
- Then the receiver finds which signal is closest (in Euclidean distance) to the received vector. (Find i for which $r \in R_i$).

Example

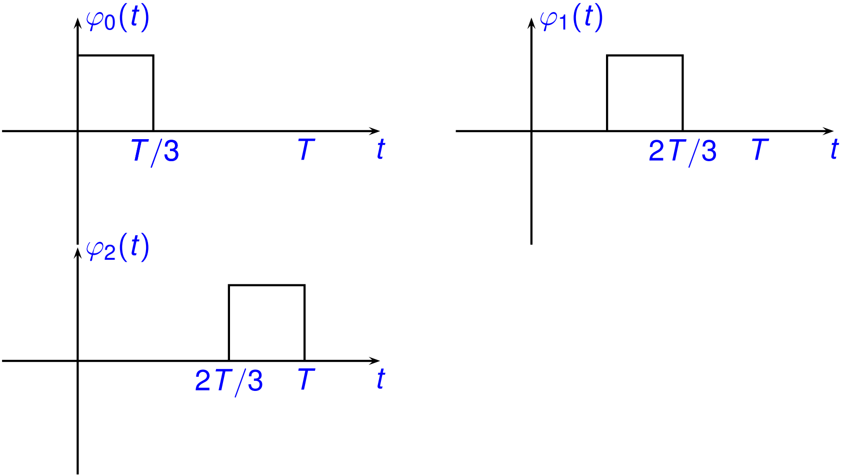
$\pi_i = \frac{1}{4}$

$M=3, N=3$

equal energy case



Orthonormal Basis Functions



Signal Vectors

$$A_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$A_4 = \left[\begin{array}{cc|cc} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ \hline +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{array}\right] = \left[\begin{array}{c|c} A_2 & A_2 \\ \hline A_2 & -A_2 \end{array}\right]$$

useless here

$$
\begin{aligned}
s_0 &= (+1, +1, +1) \\
s_1 &= (-1, +1, -1) \\
s_2 &= (+1, -1, -1) \\
s_3 &= (-1, -1, +1)
\end{aligned}
$$

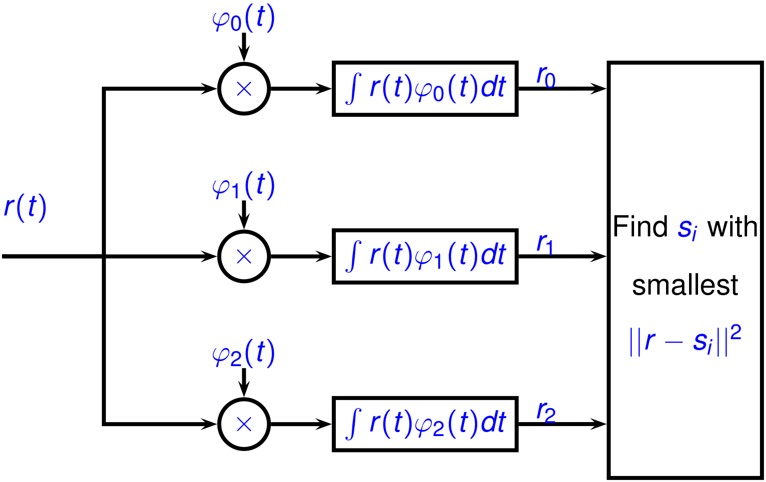
$(r, s_0) = (r_0 + r_1 + r_2)$

$(r, s_1) = (-r_0 + r_1 - r_2)$

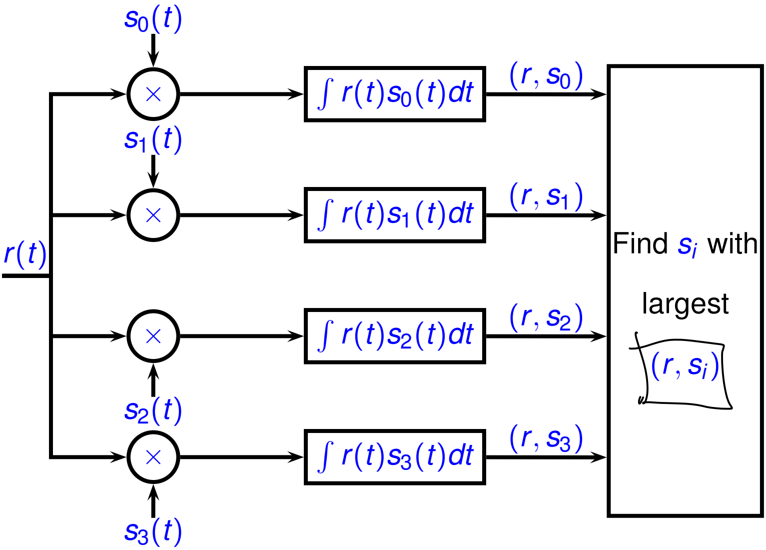
$(r, s_2) = (r_0 - r_1 - r_2)$

$(r, s_3) = (-r_0 - r_1 + r_2)$

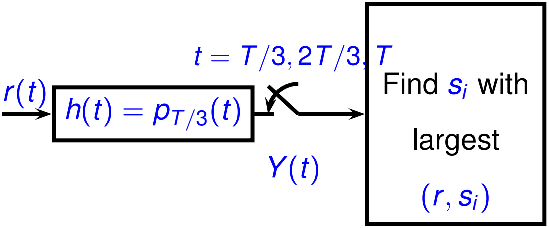
Optimum Receiver 1



Optimum Receiver 2



Optimum Receiver 3



$$
\begin{aligned}
r_0 = Y(T/3) &= \int r(t)\varphi_0(t)dt = \int_0^{T/3} r(t)dt \\
r_1 = Y(2T/3) &= \int r(t)\varphi_1(t)dt = \int_{T/3}^{2T/3} r(t)dt \\
r_2 = Y(T) &= \int r(t)\varphi_2(t)dt = \int_{2T/3}^{T} r(t)dt
\end{aligned}
$$

Simplified Calculation

$$\begin{aligned}(r, s_0) &= +r_0 + r_1 + r_2\\(r, s_1) &= -r_0 + r_1 - r_2\\(r, s_2) &= +r_0 - r_1 - r_2\\(r, s_3) &= -r_0 - r_1 + r_2\end{aligned}$$

8 additions

First calculate x_0, x_1, x_2, x_3 as follows

$$\begin{aligned}x_0 &= +r_0\\x_1 &= -r_0\\x_2 &= r_1 + r_2\\x_3 &= r_1 - r_2\end{aligned}$$

1
2

Simplified Calculation

Then

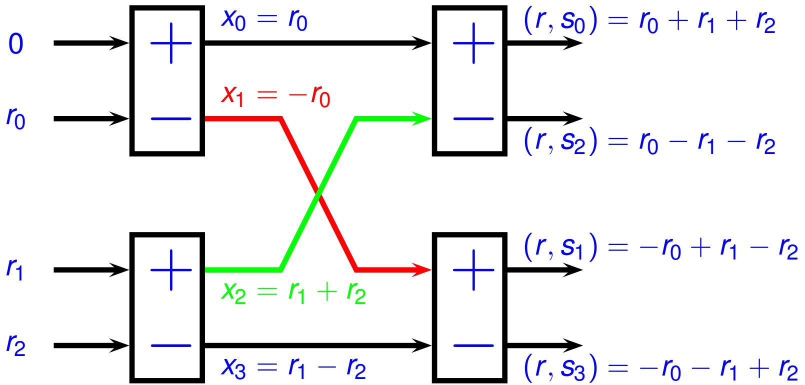
$$
\begin{aligned}
(r, s_0) &= x_0 + x_2 \quad 3 \\
(r, s_1) &= x_1 + x_3 \quad 4 \\
(r, s_2) &= x_0 - x_2 \quad 5 \\
(r, s_3) &= x_1 - x_3 \quad 6
\end{aligned}
$$

Thus the calculation requires only 6 additions/subtractions.

Implementation

saving in terms of add/sub

Fast Hadamard Transform



Performance

324

$M = 2$

$N = 648$

- The performance of the optimum demodulation is usually very difficult to evaluate exactly.
- Usually upper bounds on the error probability are employed.
- One bound is called the union bound.

$P(A \cup B) \leq P(A) + P(B)$

Example 3

$M = 8$

$N = 2$

$$
\begin{aligned}
s_0(t) &= 1\varphi_0(t) + 0\varphi_1(t) \\
s_1(t) &= -1\varphi_0(t) + 0\varphi_1(t) \\
s_2(t) &= 2\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
s_3(t) &= 0\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
s_4(t) &= -2\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
s_5(t) &= 2\varphi_0(t) - \sqrt{3}\varphi_1(t) \\
s_6(t) &= 0\varphi_0(t) - \sqrt{3}\varphi_1(t) \\
s_7(t) &= -2\varphi_0(t) - \sqrt{3}\varphi_1(t)
\end{aligned}
$$

Example 3

different energy

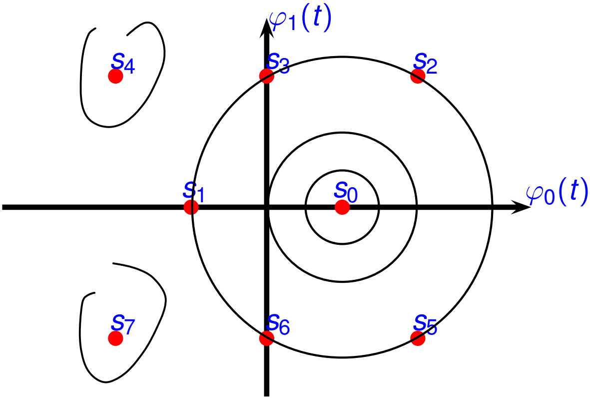
- The energy of the signals are

$$\begin{aligned} \underbrace{E_0 = 1,}\quad & E_1 = 1, \\ \underbrace{E_2 = 7,} \quad & \underbrace{E_3 = 3,} \\ E_4 = 7, \quad & E_5 = 7, \\ E_6 = 3, \quad & E_7 = 7. \end{aligned}$$

- The average energy is $E = 36/8 = 4.5$.
- The energy per bit is $E_b = E/3 = 1.5$.

Example 3: Contours of equal probability

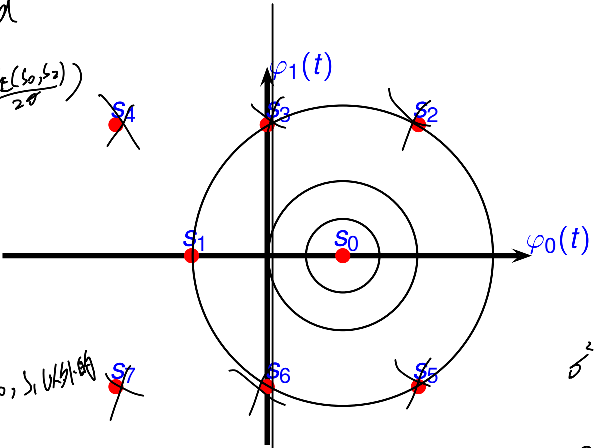
don't have same energy



Example 3: Contours of equal probability

Union bound

$P_2(s_0 \to s_2) = Q\left(\frac{d_E(s_0,s_2)}{2\sigma}\right)$



ignoring 除了 s_0, s_1 以外的

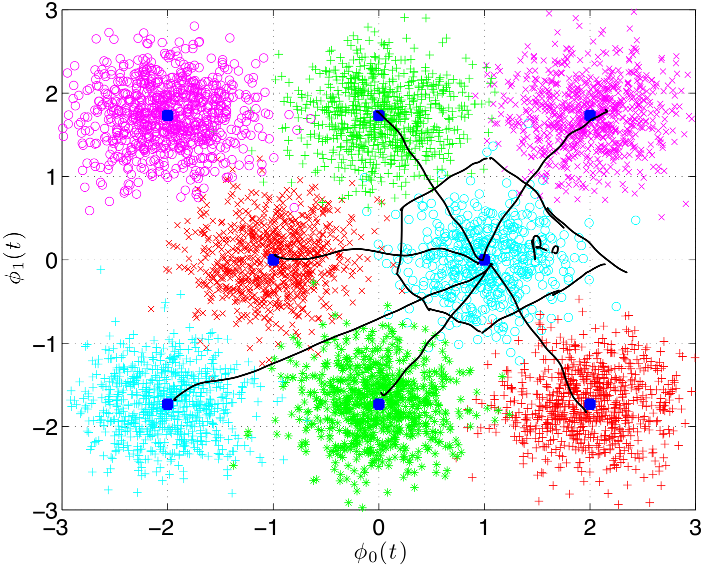
$\sigma^2 = \frac{N_0}{2}$

$P_2(s_0 \to s_1) = P(\text{decide } s_1 \text{ over } s_0 \mid s_0 \text{ transmitted}) = Q\left(\frac{d_E(s_0,s_1)}{2\sigma}\right)$

Pairwise error prob

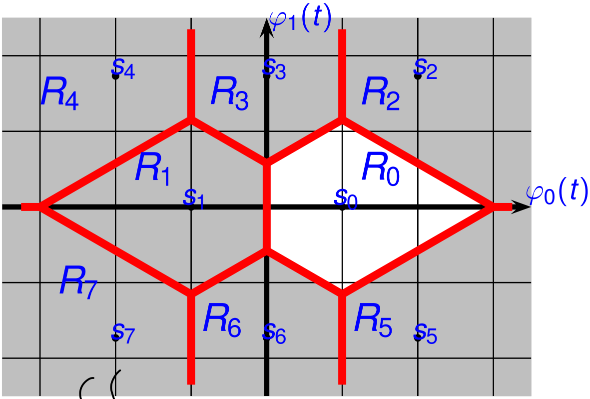
(s_0 trans but decide s_1)

Example 3



Example 3: Decision Region for s_0

$P_{e,0} = P(\text{error} \mid H_0)$



$$P_{e,0} = \iint_{r \notin R_0} p_i(r_0, r_1)\, dr_0\, dr_1$$

(matlab)

Union Bound

Another way

- To calculate the probability of error (exactly) we need to determine the probability that a two dimensional Gaussian random vector, centered at the point s_0 is not in the decision region for signal s_0.
- This is a two dimensional integration over a subset of the plane of the Gaussian density.
- Let R_i be the region of received signals where it is decided that signal i is transmitted.
- Let $R_{i,j}$ be the region where signal j is chosen when compared only to signal i.

Union Bound

Assume S_0 transmit,
but decide S_1

- Then

$$
\begin{aligned}
R_1 \quad &\cup \quad R_2 \cup \cdots \cup R_{M-1} = R_{\underline{1,0}} \cup R_{2,0} \cdots \cup R_{M-1,0} \\
P_{e,0} \quad &= \quad P\{\text{error}|s_0 \text{ transmitted}\} \\
&= \quad P\{r \in R_1 \cup R_2 \cup R_3 \cup \cdots \cup R_{M-1}|s_0 \text{ transmitted}\} \\
&= \quad P\{r \in R_{0,1} \cup R_{0,2} \cup R_{0,3} \cup \cdots \cup R_{0,M-1}|s_0 \text{ transmitted}\} \\
&\leq \quad \sum_{j=1}^{M-1} P\{r \in R_{0,j}|s_0 \text{ transmitted}\} \\
&= \quad \sum_{j=1}^{M-1} P_2(s_0 \to s_j) = \boxed{\sum_{j=1}^{M-1} Q\left(\frac{d_E(s_0, s_j)}{2\sqrt{N_0/2}}\right)}
\end{aligned}
$$

懂了!

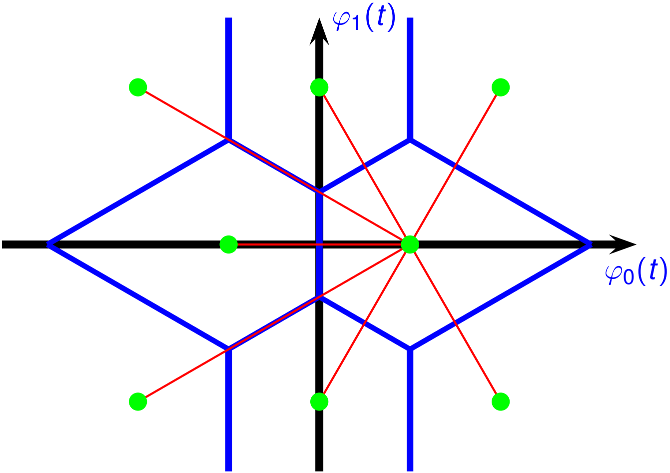
$$\sum_{j=1}^{M-1} Q\left(\frac{d_E(S_0,S_j)}{2\sigma}\right)$$

where $P_2(s_0 \to s_j)$ be the pair-wise error probability of deciding s_j given s_0 was transmitted when the *receiver assumes there is only two possible decisions, either* s_0 *or* s_j.

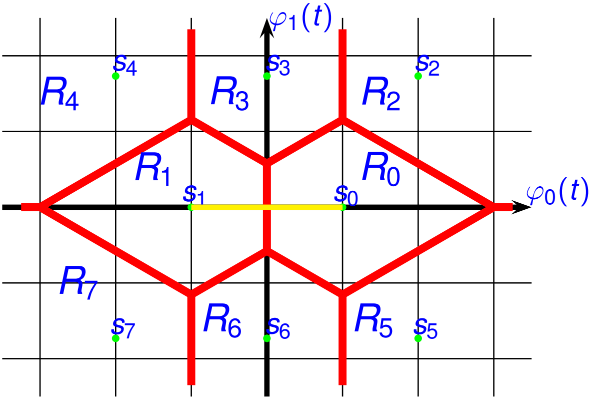
Example

$$
\begin{aligned}
s_0(t) &= 1\varphi_0(t) + 0\varphi_1(t) \\
s_1(t) &= -1\varphi_0(t) + 0\varphi_1(t) \\
s_2(t) &= 2\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
s_3(t) &= 0\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
s_4(t) &= -2\varphi_0(t) + \sqrt{3}\varphi_1(t) \\
s_5(t) &= 2\varphi_0(t) - \sqrt{3}\varphi_1(t) \\
s_6(t) &= 0\varphi_0(t) - \sqrt{3}\varphi_1(t) \\
s_7(t) &= -2\varphi_0(t) - \sqrt{3}\varphi_1(t)
\end{aligned}
$$

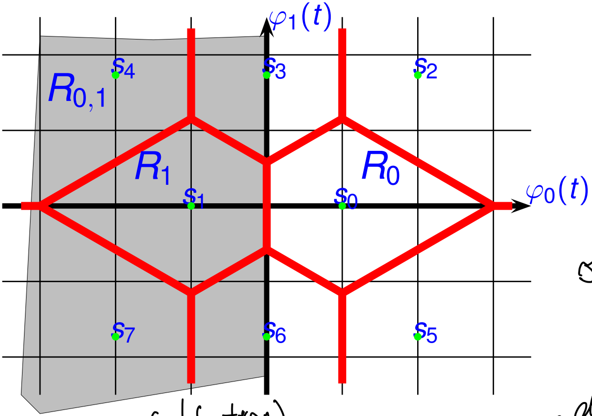
Example



Example



Pair-Wise Decision Regions



$\sigma^2 = \frac{N_0}{2}$

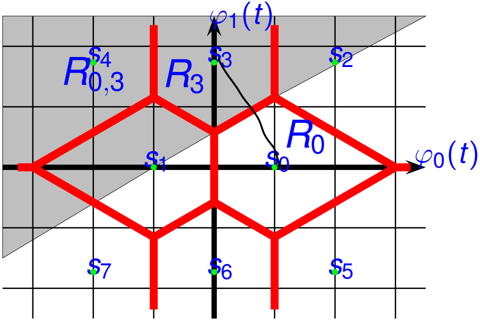
pair-wise error prob

P_2(decide s_2 over s_1 | s_1 tran)

$Q\left(\frac{d(s_1, s_0)}{2\sigma}\right)$

$$P_2(s_0 \to s_1) = Q\left(\frac{1}{\sigma}\right) = Q\left(\sqrt{\frac{2}{N_0}}\right)$$

Pair-Wise Decision Regions



Pairwise Distance

$$
\begin{aligned}
P_2(s_0 \to s_3) &= Q(\frac{d_{0,3}}{2\sigma}) = Q(\frac{2}{2\sigma}) \\
&= Q(\sqrt{\frac{4}{4\sigma^2}}) = Q(\sqrt{\frac{2}{N_0}}) \\
P_2(s_0 \to s_4) &= Q(\frac{d_{0,4}}{2\sigma}) = Q(\frac{\sqrt{12}}{2\sigma}) \\
&= Q(\sqrt{\frac{12}{4\sigma^2}}) = Q(\sqrt{\frac{6}{N_0}})
\end{aligned}
$$

Pairwise Distance

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_0	0	2	2	2	$2\sqrt{3}$	2	2	$2\sqrt{3}$
s_1	2	0	$2\sqrt{3}$	2	2	$2\sqrt{3}$	2	2
s_2	2	$2\sqrt{3}$	0	2	4	$2\sqrt{3}$	4	$2\sqrt{7}$
s_3	2	2	2	0	2	4	$2\sqrt{3}$	4
s_4	$2\sqrt{3}$	2	4	2	0	$2\sqrt{7}$	4	$2\sqrt{3}$
s_5	2	$2\sqrt{3}$	$2\sqrt{3}$	4	$2\sqrt{7}$	0	2	4
s_6	2	2	4	$2\sqrt{3}$	4	2	0	2
s_7	$2\sqrt{3}$	2	$2\sqrt{7}$	4	$2\sqrt{3}$	4	2	0

Union Bounds

$$
\begin{aligned}
P_{e,0} &= P_{e,1} \\
&\leq Q(\frac{2}{2\sigma}) + Q(\frac{2}{2\sigma}) + Q(\frac{2}{2\sigma}) + Q(\frac{2\sqrt{3}}{2\sigma}) + Q(\frac{2}{2\sigma}) + Q(\frac{2}{2\sigma}) + Q(\frac{2\sqrt{3}}{2\sigma}) \\
&= 5Q(\frac{2}{2\sigma}) + 2Q(\frac{2\sqrt{3}}{2\sigma}) \\
P_{e,2} &= P_{e,4} = P_{e,5} = P_{e,7} \\
&\leq 2Q(\frac{2}{2\sigma}) + 2Q(\frac{2\sqrt{3}}{2\sigma}) + 2Q(\frac{4}{2\sigma}) + Q(\frac{2\sqrt{7}}{2\sigma}) \\
P_{e,3} &= P_{e,6} \\
&\leq 4Q(\frac{2}{2\sigma}) + Q(\frac{2\sqrt{3}}{2\sigma}) + 2Q(\frac{4}{2\sigma})
\end{aligned}
$$

Union Bounds

$$
\begin{aligned}
P_e &= \frac{1}{M}\sum_{i=0}^{M-1} P_{e,i} \\
&\leq \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i} P_2(s_i \rightarrow s_j) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j\neq i} Q\left(\frac{d_E(s_{\overset{i}{\cancel{0}}}, s_j)}{2\sqrt{N_0/2}}\right)
\end{aligned}
$$

Union Bounds

$\sigma^2 = \frac{N_0}{2}$

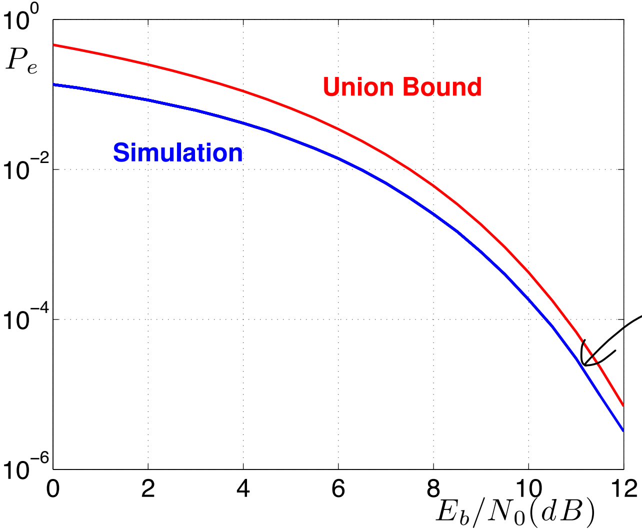
$$
\begin{aligned}
P_e &= \frac{1}{8}\sum_{i=0}^{7} P_{e,i} \\
&\leq \frac{1}{8}[26Q(\frac{2}{2\sigma}) + 14Q(\frac{2\sqrt{3}}{2\sigma}) + 12Q(\frac{4}{2\sigma}) + 4Q(\frac{2\sqrt{7}}{2\sigma})]
\end{aligned}
$$

Union Bounds

$$\frac{E_b}{N_0} = \frac{1.5}{N_0}$$

$$\begin{aligned} P_e &\leq \frac{1}{8}\left[26Q(\sqrt{\frac{4E_b}{3N_0}}) + 14Q(\sqrt{\frac{12E_b}{3N_0}}) + 12Q(\sqrt{\frac{16E_b}{3N_0}}) + 4Q(\sqrt{\frac{28E_b}{3N_0}})\right] \\ &\leq \left[3.25Q(\sqrt{\frac{4E_b}{3N_0}}) + 2.75Q(\sqrt{\frac{12E_b}{3N_0}}) + 1.5Q(\sqrt{\frac{16E_b}{3N_0}}) + .5Q(\sqrt{\frac{28E_b}{3N_0}})\right] \end{aligned}$$

Union Bounds



It hard to time consuming