Pr. 1.
If $\boldsymbol{Q}$ is a $M \times K$ matrix having **orthonormal** columns, then $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_K$ so $\|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{C}^K$. Show that the converse is true, *i.e.*, if $\boldsymbol{Q}$ is a $M \times K$ matrix for which $\|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{C}^K$, then $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_K$. Your proof should be general enough to cover the case where $\boldsymbol{Q}$ has complex elements.
Hint. Examine products with standard unit vectors $\boldsymbol{e}_j$ and combinations like $\boldsymbol{e}_j + \boldsymbol{e}_k$, or consider an eigendecomposition of $\boldsymbol{Q}'\boldsymbol{Q} - \boldsymbol{I}$.

Pr. 2.
Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Determine the **nullspace** of $\boldsymbol{A}$, denoted by $\mathcal{N}(\boldsymbol{A})$, and the **range space** or column space of $\boldsymbol{A}$ denoted by $\mathcal{R}(\boldsymbol{A})$.

(b) Are they equal? Does your answer hold in general? If not, provide a counterexample.

Pr. 3.
Let $\boldsymbol{A} \in \mathbb{F}^{M \times N}$.

(a) Suppose $\boldsymbol{W} \in \mathbb{F}^{M \times M}$ and $\boldsymbol{Q} \in \mathbb{F}^{N \times N}$ are each **unitary** matrices.
Show that $\boldsymbol{A}$ and $\boldsymbol{C} \triangleq \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ have the same singular values.
Consequently, $\boldsymbol{A}$ and $\boldsymbol{C}$ have the same **rank**, the same Frobenius norm and the same operator norm. This is why the Frobenius norm and the operator norm are called **unitarily invariant** norms; their value does not change when the matrix is multiplied from the left and/or the right by a unitary matrix. Any norm that depends only on the singular values of $\boldsymbol{A}$ will, by definition, be unitarily invariant.

(b) Suppose that $\boldsymbol{W}$ and $\boldsymbol{Q}$ are **nonsingular** but not necessarily unitary matrices. Do $\boldsymbol{A}$ and $\boldsymbol{D} \triangleq \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ have the same **rank**? Prove or give a counter-example.

(c) Continuing (b), do $\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ have the same singular values? Prove or give a counter-example.

Pr. 4.
Prove that the **orthogonal complement** of the **range** of a matrix $\boldsymbol{A}$ equals the **nullspace** of $\boldsymbol{A}'$:

$$\mathcal{R}^{\perp}(\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A}').$$

Pr. 5.
Let $\boldsymbol{A}$ be a $N \times N$ Hermitian matrix with unitary eigendecomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ and $0 \geq \lambda_{k+1} \geq \ldots \geq \lambda_N$, for some $1 < k < N$. Determine an **SVD** of $\boldsymbol{A}$ in terms of the given components.

Pr. 6.
The Ch. 2 notes show the following **Venn diagram** of matrices and **eigendecompositions**:

$\boldsymbol{A}_1$ Rectangular SVD: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}'$	$\boldsymbol{A}_2$ Square $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$	$\boldsymbol{A}_3$ Diagonalizable $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ iff $\boldsymbol{V}$ linearly ind.	$\boldsymbol{A}_4$ Normal $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}'$ $\boldsymbol{V}$ unitary $\sigma_n = \lvert\lambda_{(n)}\rvert$	$\boldsymbol{A}_5$ Hermitian $\boldsymbol{\Lambda}$ real	PSD: $\boldsymbol{A} \succeq \boldsymbol{0}$ $\boldsymbol{\Sigma} = \boldsymbol{\Lambda} \succeq \boldsymbol{0}$ $\boldsymbol{U} = \boldsymbol{V}$

Each of the categories shown above is a *strict superset* of the categories nested with in it.
Provide example matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_5$ that belong to the each of the categories above but *not* the next category nested within it. Try to provide the simplest possible example in each case.
For example, the matrix $\boldsymbol{A}_1 = [0\ 1]$ is rectangular, but not square. Now you do $\boldsymbol{A}_2, \ldots, \boldsymbol{A}_5$.
Hint. All the examples can be 1×1 or 2×2, often with simple "0" and "1" elements.

Pr. 7.
Let $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}'$, where neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ is $\mathbf{0}$.

(a) How many **linearly independent** columns does $\boldsymbol{A}$ have? I.e., of all the sets of columns from this matrix where the set is linearly independent, what is the maximum cardinality?

(b) What is the **rank** of $\boldsymbol{A}$?

(c) Enter (cut and paste) this code into `Julia`:

```
using LinearAlgebra: svdvals
using Plots
using LaTeXStrings
n = 100
x = randn(n); y = randn(n); A = x*y'
s = svdvals(A)
scatter(s, yscale = :log10, label= "", title = "singular values: log scale",
    xlabel="i", ylabel=L"\sigma_i") # to make σ_i
```

(Ideally the ylabel should look like σ_i, but if it does not, it is OK. Comment out the `ylabel` part if needed.)

- How many numerically computed singular values of $\boldsymbol{A}$ are non-zero?
- What answer do you get when you type `rank(A)` ?
- Turn in your plot of the singular values and the answers.

You will have just seen that a theoretically rank-one matrix will have multiple non-zero singular values when expressed in machine precision arithmetic; this property is due to roundoff errors (finite numerical precision). (Round-off errors should not be thought of as "wrong answers," but just a fact of real-life computing to be understood and not feared.)

(d) To help locate the code for the `rank` function, you can type this into `Julia`: `which(rank, (Matrix,))`

From the output you can (with a little web search) find the source code here (because all of `Julia` is open source): `https://github.com/JuliaLang/julia/blob/master/stdlib/LinearAlgebra/src/generic.jl`

Examine the code for the `rank` function and write down the formula used to determine the threshold (aka tolerance) below which, due to roundoff errors, the code considers the singular value to be "zero."

(e) Determine the numerical value of the threshold when $\boldsymbol{A} = [9\ 9]$.

(f) Optional. Check your answer to the previous part by using the `Julia` debugger to step into the rank function and examine the tolerance. In Atom/Juno, use: `Juno.@enter rank([9 9])`

Pr. 8.
When $\boldsymbol{D}$ is an $M \times N$ (rectangular) diagonal matrix, its **pseudo-inverse** $\boldsymbol{D}^+$ is an $N \times M$ (rectangular) diagonal matrix whose non-zero entries are the reciprocals $1/d_k$ of the non-zero diagonal entries of $\boldsymbol{D}$. A matrix $\boldsymbol{A}$ having SVD $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}'$ has $\boldsymbol{A}^+ = \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}'$. Working with just this equality, determine by hand (experimenting / checking with code is OK) the pseudo-inverse of

(a) $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}'$, where neither $\boldsymbol{x} \in \mathbb{F}^M$ nor $\boldsymbol{y} \in \mathbb{F}^N$ is $\mathbf{0}$, (your answer should depend only on $\boldsymbol{x}$ and $\boldsymbol{y}$),

(b) $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{x}'$, where $\boldsymbol{x} \neq \mathbf{0}$.

Pr. 9.
(Projection onto orthogonal complement of a 1D subspace)

(a) Determine an orthonormal basis for the orthogonal complement of the span of the vector $\boldsymbol{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

(b) Determine an orthonormal basis for the orthogonal complement of the span of the vector $\boldsymbol{z} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.

(c) Determine (by hand, no `Julia`) the projection of the vector $\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ onto the orthogonal complement of $\text{span}(\{\boldsymbol{z}\})$ *without* using the orthonormal basis you found in the previous part.

Hint. You may want to derive the general mathematical expression needed in the next part first, and then use that expression to solve this problem by hand.

(d) Write a function called `orthcomp1` that projects an input vector $\boldsymbol{y}$ onto the orthogonal complement of $\text{span}(\{\boldsymbol{x}\})$ for an (nonzero) input vector $\boldsymbol{x}$ of the same length as $\boldsymbol{y}$.

For full credit, your final version of the code must be computationally efficient, and it should be able to handle input vectors of length 10 million without running out of memory. Your code must *not* call `svd` or `eig` or `I` and the like. This problem can be solved in one line with elementary vector operations.

In `Julia`, your file should be named `orthcomp1.jl` and should contain the following function:

```
"""
    z = orthcomp1(y, x)

Project `y` onto the orthogonal complement of `Span({x})`

In:
* `y` vector
* `x` nonzero vector of same length, both possibly very long

Out:
* `z` vector of same length

For full credit, your solution should be computationally efficient.
"""
function orthcomp1(y, x)
```

Email your solution as an attachment to `eecs551@autograder.eecs.umich.edu`.

Test your code yourself using the example above (and others) *before* submitting to the autograder. Be sure to test it for very long input vectors.

(e) Submit your code (a screen capture is fine) to gradescope so that the grader can verify that your code is computationally *efficient*. (The autograder checks only correctness, not efficiency.)

Non-graded problem(s) below
(Solutions will be provided for self check; do not submit to gradescope.)

Pr. 10.
Let $\boldsymbol{A}$ be a **normal** matrix of the (unnamed) type where each eigenvalue either has a different magnitude than all either eigenvalues, or has the same value as all other eigenvalues with its magnitude. In other words, having both $|\lambda_j| = |\lambda_i|$ and $\lambda_j \neq \lambda_i$ is not allowed. For example, $\boldsymbol{A}$ might have eigenvalues $(3, 4\imath, 4\imath, -5)$ but cannot have eigenvalues $(3, 4, 4\imath, -5)$.
Prove that every right singular vector of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{A}$.
This problem finishes the story on relating **SVD** and **eigendecomposition**.

Pr. 11.
We write $\boldsymbol{A} \preceq \boldsymbol{B}$ iff $\boldsymbol{B} - \boldsymbol{A} \succeq \boldsymbol{0}$, *i.e.*, iff $\boldsymbol{B} - \boldsymbol{A}$ is **positive semi-definite**.
For $a > 0$, determine $\left\{ x \in \mathbb{R} : x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

Pr. 12.
Suppose that $\boldsymbol{X} = \boldsymbol{U}_x \boldsymbol{\Sigma}_x \boldsymbol{V}_x'$ and $\boldsymbol{Y} = \boldsymbol{U}_y \boldsymbol{\Sigma}_y \boldsymbol{V}_y'$ denote **SVD** s of the matrices $\boldsymbol{X} \in \mathbb{F}^{m \times n}$ and $\boldsymbol{Y} \in \mathbb{F}^{p \times q}$.

(a) Show that
$$\boldsymbol{X} \otimes \boldsymbol{Y} = (\boldsymbol{U}_x \otimes \boldsymbol{U}_y)(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y)(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)'$$
is an SVD of $\boldsymbol{X} \otimes \boldsymbol{Y}$ (up to a permutation of the singular values). Note that the matrix $\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y$ is something like a diagonal matrix containing the pairwise *products* of the singular values of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Hint: First show that the formula is correct, then argue that it is an SVD by showing that $\boldsymbol{U}_x \otimes \boldsymbol{U}_y$ and $\boldsymbol{V}_x \otimes \boldsymbol{V}_y$ are unitary matrices.

Hint: You may find the following properties of **Kronecker product** useful:

- $(\boldsymbol{X} \otimes \boldsymbol{Y})' = \boldsymbol{X}' \otimes \boldsymbol{Y}'$
- $(\boldsymbol{X} \otimes \boldsymbol{Y})(\boldsymbol{W} \otimes \boldsymbol{Z}) = (\boldsymbol{X}\boldsymbol{W}) \otimes (\boldsymbol{Y}\boldsymbol{Z})$

(b) Now suppose that $\boldsymbol{A} = \boldsymbol{Q}_a \boldsymbol{\Lambda}_a \boldsymbol{Q}_a'$ and $\boldsymbol{B} = \boldsymbol{Q}_b \boldsymbol{\Lambda}_b \boldsymbol{Q}_b'$ are **eigendecompositions** of the (square Hermitian) matrices $\boldsymbol{A} \in \mathbb{F}^{N \times N}$ and $\boldsymbol{B} \in \mathbb{F}^{M \times M}$.

Show that
$$\boldsymbol{A} \otimes \boldsymbol{B} = (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \otimes \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'$$
is an eigendecomposition of $\boldsymbol{A} \otimes \boldsymbol{B}$. Note that the matrix $\boldsymbol{\Lambda}_a \otimes \boldsymbol{\Lambda}_b$ is a diagonal matrix whose diagonal contains the pairwise *products* of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$!

Hint: First show that the formula is correct, then argue that it is an eigendecomposition by showing that $\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b$ is a unitary matrix.

(c) Continuing (b), now show that
$$\boldsymbol{A} \oplus \boldsymbol{B} = (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \oplus \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'$$
is an eigendecomposition of the **Kronecker sum** $\boldsymbol{A} \oplus \boldsymbol{B} \triangleq (\boldsymbol{A} \otimes \boldsymbol{I}) + (\boldsymbol{I} \otimes \boldsymbol{B})$, where you must determine the size(s) of the "$\boldsymbol{I}$" in that expression. Note that the matrix $\boldsymbol{\Lambda}_a \oplus \boldsymbol{\Lambda}_b$ is a diagonal matrix whose diagonal contains the pairwise *sums* of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Hint: Start by writing $\boldsymbol{I}_N = \boldsymbol{Q}_a \boldsymbol{I}_N \boldsymbol{Q}_a'$ and $\boldsymbol{I}_M = \boldsymbol{Q}_b \boldsymbol{I}_M \boldsymbol{Q}_b'$ in the definition of $\boldsymbol{A} \oplus \boldsymbol{B}$, and then apply (b).

Pr. 13.
Complete the "Introduction to Matrix Math" tutorial at `https://pathbird.com/courses/register`. (Use the registration code shown on the EECS 551 Canvas home page.) This tutorial should be helpful for anyone who wants deeper understanding of `Julia` for operations with matrices and vectors. Such operations are a major part of EECS 551.
