EECS 455: Solutions to Problem Set 1

Due: Wednesday, September 13, 2015.

1. A communication system transmits one of 8 equally likely signals. The signal (waveforms) are represented by the vectors shown below by some. suitable set of orthonormal signals.

   (a) Determine how many information bits can be sent using these signals.

$$
\begin{aligned}
s_0 &= (-1,-1,-1,-1,-1)\\
s_1 &= (-1,-1,+3,-3,+3)\\
s_2 &= (-1,+3,-3,+3,-1)\\
s_3 &= (-1,+3,+1,+1,+3)\\
s_4 &= (+3,-3,+3,-1,-1)\\
s_5 &= (+3,-3,-1,-3,+3)\\
s_6 &= (+3,+1,+1,+3,-1)\\
s_7 &= (+3,+1,-3,+1,+3)
\end{aligned}
$$

   Solution: Since there are eight signals $\log_2(8) = 3$ bits can be sent using these signals.

   (b) Determine the energy of each of the signals and the average energy per information bit.

   Solution:

$$
\begin{aligned}
E_0 &= 5\\
E_1 &= 29\\
E_2 &= 29\\
E_3 &= 21\\
E_4 &= 29\\
E_5 &= 37\\
E_6 &= 21\\
E_6 &= 29\\
\bar{E} &= 25\\
\bar{E}_b &= 25/3 = 8.33
\end{aligned}
$$

   (c) Determine the Euclidean distance between signals s_0 and and all the other signals.

   Solution:

$$
\begin{aligned}
d_E(s_0,s_1) &= \sqrt{36} = 6\\
d_E(s_0,s_2) &= \sqrt{36} = 6
\end{aligned}
$$

$$
\begin{aligned}
d_E(s_0, s_3) &= \sqrt{40} = 6.32 \\
d_E(s_0, s_4) &= \sqrt{36} = 6 \\
d_E(s_0, s_5) &= \sqrt{40} = 6.32 \\
d_E(s_0, s_6) &= \sqrt{40} = 6.32 \\
d_E(s_0, s_7) &= \sqrt{44} = 6.63
\end{aligned}
$$

(d) Determine the rate of communication in bits/dimension for these signals.

$$
\begin{aligned}
s_0 &= (-1, -1, -1, -1, -1) \\
s_1 &= (-1, -1, +3, -3, +3) \\
s_2 &= (-1, +3, -3, +3, -1) \\
s_3 &= (-1, +3, +1, +1, +3) \\
s_4 &= (+3, -3, +3, -1, -1) \\
s_5 &= (+3, -3, -1, -3, +3) \\
s_6 &= (+3, +1, +1, +3, -1) \\
s_7 &= (+3, +1, -3, +1, +3)
\end{aligned}
$$

The rate of communication is 3 bits/ 5 dimensions or 0.6 bits/dimension.

2. A first signal set with $M = 16$ signals in two dimensions that can transmit 4 bits of information has the following signals.

$$
\begin{aligned}
s_0 &= A(-3, -3) \\
s_1 &= A(-3, -1) \\
s_2 &= A(-3, +1) \\
s_3 &= A(-3, +3) \\
s_4 &= A(-1, -3) \\
s_5 &= A(-1, -1) \\
s_6 &= A(-1, +1) \\
s_7 &= A(-1, +3) \\
s_8 &= A(+1, -3) \\
s_9 &= A(+1, -1) \\
s_{10} &= A(+1, +1) \\
s_{11} &= A(+1, +3) \\
s_{12} &= A(+3, -3) \\
s_{13} &= A(+3, -1)
\end{aligned}
$$

$$
\begin{aligned}
s_{14} &= A(+3,+1)\\
s_{15} &= A(+3,+3)
\end{aligned}
$$

(a) Determine the average energy of this set of signals assuming each signal is used equally likely in terms of A.

Solution:

$$
\begin{aligned}
E_0 &= A^2(9+9) = 18A^2\\
E_1 &= A^2(9+1) = 10A^2\\
E_2 &= A^2(9+1) = 10A^2\\
E_3 &= A^2(9+9) = 18A^2\\
E_4 &= A^2(1+9) = 10A^2\\
E_5 &= A^2(1+1) = 2A^2\\
E_6 &= A^2(1+1) = 2A^2\\
E_7 &= A^2(1+9) = 10A^2\\
E_8 &= A^2(1+9) = 10A^2\\
E_9 &= A^2(1+1) = 2A^2\\
E_{10} &= A^2(1+1) = 2A^2\\
E_{11} &= A^2(1+9) = 10A^2\\
E_{12} &= A^2(9+9) = 18A^2\\
E_{13} &= A^2(9+1) = 10A^2\\
E_{14} &= A^2(9+1) = 10A^2\\
E_{15} &= A^2(9+9) = 18A^2
\end{aligned}
$$

The average energy is $\frac{1}{15}\sum_{i=0}^{15} E_i = 10A^2$

(b) Determine the average energy per information bits E_b of this set of signals in terms of A.
Solution:

$E_b = 10A^2/4 = 2.5A^2$.

(c) Determine the minimum squared Euclidean distance d_E^2 between any distinct pair of signals.

Solution: $d_E^2(s_0, s_1) = 4A^2$

(d) Determine the ratio of minimum squared Euclidean distance to energy per bit d_E^2/E_b.

Solution: $\frac{d_E^2}{E_b} = \frac{4A^2}{2.5A^2} = 8/5$

(e) Determine the rate in terms of bits per dimension.

Solution: $R = \log_2(16)/2 = 2$ bits/dimension

A second signal set with $M = 16$ signals in two dimensions that can transmit 4 bits of information has the following signals.

$$
\begin{aligned}
s_0 &= A(+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1) \\
s_1 &= A(+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1) \\
s_2 &= A(+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1) \\
s_3 &= A(+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,+1) \\
s_4 &= A(+1,+1,+1,+1,-1,-1,-1,-1,+1,+1,+1,+1,-1,-1,-1,-1) \\
s_5 &= A(+1,-1,+1,-1,-1,+1,-1,+1,+1,-1,+1,-1,-1,+1,-1,+1) \\
s_6 &= A(+1,+1,-1,-1,-1,-1,+1,+1,+1,+1,-1,-1,-1,-1,+1,+1) \\
s_7 &= A(+1,-1,-1,+1,-1,+1,+1,-1,+1,-1,-1,+1,-1,+1,+1,-1) \\
s_8 &= A(+1,+1,+1,+1,+1,+1,+1,+1,-1,-1,-1,-1,-1,-1,-1,-1) \\
s_9 &= A(+1,-1,+1,-1,+1,-1,+1,-1,-1,+1,-1,+1,-1,+1,-1,+1) \\
s_{10} &= A(+1,+1,-1,-1,+1,+1,-1,-1,-1,-1,+1,+1,-1,-1,+1,+1) \\
s_{11} &= A(+1,-1,-1,+1,+1,-1,-1,+1,-1,+1,+1,-1,-1,+1,+1,-1) \\
s_{12} &= A(+1,+1,+1,+1,-1,-1,-1,-1,-1,-1,-1,-1,+1,+1,+1,+1) \\
s_{13} &= A(+1,-1,+1,-1,-1,+1,-1,+1,-1,+1,-1,+1,+1,-1,+1,-1) \\
s_{14} &= A(+1,+1,-1,-1,-1,-1,+1,+1,-1,-1,+1,+1,+1,+1,-1,-1) \\
s_{15} &= A(+1,-1,-1,+1,-1,+1,+1,-1,-1,+1,+1,-1,+1,-1,-1,+1)
\end{aligned}
$$

(f) Determine the average energy of this set of signals assuming each signal is used equally likely in terms of A.

Solution:

$E_i = A^2 16$

(g) Determine the average energy per information bits E_b of this set of signals in terms of A.

Solution:

$E_b = 16A^2/4 = 4A^2$.

(h) Determine the minimum squared Euclidean distance d_E^2 between any distinct pair of signals.

Solution:

$d_E^2(s_0, s_1) = 32A^2$

(i) Determine the ratio of minimum squared Euclidean distance to energy per bit d_E^2/E_b. (This should be independent of A.

Solution:

$\frac{d_E^2(s_0,s_1)}{E_b} = \frac{32A^2}{4A^2} = 8$

(j) Determine the rate in terms of bits per dimension.

Solution:

$R = \log_2(16)/16 = 4/16 = 0.25$

(k) Which of the two signal sets is better from a bandwidth efficiency point of view and which is better from an energy efficiency point of view?

Solution: The first signal set is better from a bandwidth efficiency point of view (more bits/dimension) but the second signal set is better from an energy efficiency point of view (more squared Euclidean distance per unit energy).

3. Consider the UWB channel which goes from 3.1GHz to 10.6 GHz. Suppose the noise power spectral density is $N_0 = kT = (1.38 \times 10^{-23})(290) = 4 \times 10^{-21}$ Watts/Hz. Here k is Boltzman's constant and T is the temperature in Kelvin. A temperature of 290 K corresponds to 62 degree Fahrenheit. The allowed transmitted power *density* is -41.3dBm/MHz =-71.3dB/MHz. (Note 0dBm=1mW, 30dBm=1W, -30dBm=1 μW).

   (a) For the given frequency band determine the total power that can be transmitted.

   Solution:

   The total transmitted power is determined as follows. First determine the power in a 1 MHz bandwidth. Then multiply by 7500 to get the power in 7.5 GHz.

$$\begin{aligned} P_t|_{1MHz} &= 10^{(-71.3/10)} \\ &= 7.41 \times 10^{-8} W/MHz \end{aligned}$$

$$\begin{aligned} P_t &= 7.41 \times 10^{-8}(7500) \\ &= .556 \times 10^{-3} W \\ &= 556 \mu\text{Watts.} \end{aligned}$$

   Suppose the received power is related to the transmitted power by

$$P_r = P_t h_t^2 h_r^2 / d^4$$

   where the d is the distance in meters (independent of frequency), h_t is the height of the transmitting antenna (in meters) and h_r is the height of the receiving antenna (in meters).

   (b) Compute the largest possible data rate that can be communicated reliably with both antennas at a height of 1m at a distance of 100 m and 1000 m.

   Solution:

$$\begin{aligned} P_r &= .556 \times 10^{-3}/d^4 \\ &= \begin{cases} 5.556 \times 10^{-12} & d = 100 \\ 5.556 \times 10^{-16} & d = 1000. \end{cases} \end{aligned}$$

The capacity is then

$$\begin{aligned} C &= W\log_2(1+\frac{P_r}{N_0W}) \\ &= \begin{cases} 1.8\text{Gbps} & d=100 \\ 200\text{kbps} & d=1000. \end{cases} \end{aligned}$$

4. Consider a Wi-Fi (802.11) system. The bandwidth is 20MHz, and the data rates are 3,6,9,12, 18,24,36,48,54 Mbps. Determine the minimum required signal-to-noise ratio E_b/N_0 in dB for each of these data rates and the bandwidth specified.

Solution:

$$\begin{aligned} E_b/N_0 &> \frac{2^{R/W}-1}{R/W} \\ E_b/N_0(dB) &= 10\log_{10}(E_b/N_0) \end{aligned}$$

R/W	E_b/N_0 (dB)
3/20	-1.36
6/20	-1.13
9/20	-.90
12/20	-.66
18/20	-.17
24/20	.34
36/20	1.40
48/20	2.51
54/20	3.09