EECS501 YUZHAN JIANG H4

P1:

Since Y is a uniform r.v over the interval $[0,1]$ and X is a Binomial r.v with parameters N, Y

$Y \sim U(0,1)$

$\therefore E[Y] = \frac{1}{2}(0+1)$

$= \frac{1}{2}$

$X|Y \sim B(N, Y)$

$\therefore E[X|Y] = NY$

By the law of iterated Expecation:

$E[X] = E[E(X|Y)]$

$= E[NY]$

$= N \cdot E[Y]$

$= N \cdot \frac{1}{2}$

$= \frac{N}{2}$

P2.

We are given that $V = U_1 + U_2 + \cdots + U_N$ where V denote the time spent by the engine to find a copy

U_i denote the time spent on website i

$U_i \sim \exp(\lambda)$

$P(U_i) = \lambda \cdot e^{-\lambda u_i}, \quad u_i > 0$

$E[U_i] = \int_0^\infty U_i \cdot \lambda e^{-\lambda u_i} du_i$

$= \frac{1}{\lambda} \int_0^\infty y e^{-y} \cdot dy \qquad$ let $y = \lambda U_i$

$= \frac{1}{\lambda} [-e^{-y} - ye^{-y}]$

$= \frac{1}{\lambda}$

N is geometric r.v with p

$f(n) = p(1-p)^{n-1}, \quad n = 1, 2, 3 \cdots\cdots$

$E(N) = \frac{1}{p}$

$U_1, U_2, \cdots\cdots$ are independent to each other

$V = U_1 + U_2 + U_3 + \cdots + U_N$

$\therefore E[V] = E[U_1 + U_2 + \cdots\cdots + U_N]$

$= E_N(E_{V|N}(V|N)) \qquad$ (By Gem #1: Smoothing — $E_X[E[Y|X]] = E[Y]$

$= E_N E[U_1 + U_2 + \cdots + U_N | N]$

$= E_N\left(\frac{N}{\lambda}\right)$

$= \frac{1}{\lambda} \cdot E_N(N)$

$= \frac{1}{\lambda} \cdot \frac{1}{p}$

$= \frac{1}{\lambda p}$

P3:

We are given that
$$f(x) = \begin{cases} 0 & x<100 \\ \frac{100}{x^2} & x>100 \end{cases}$$
and assume that the event of failure are independent.

By the formula of CDF:
$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$
$$F(x) = \int_{0}^{100} 0\,dx + \int_{100}^{x} \frac{100}{x^2}\cdot dx \quad \text{for } x>100$$
$$= -\frac{100}{x}\Big|_{100}^{x}$$
$$= -\frac{100}{x}+1$$
$$= 1-\frac{100}{x}$$
$$\therefore\ P(X<150) = 1-\frac{100}{150}$$
$$= \frac{1}{3}$$

Let A denote that exactly 2 of the lamps fail within the first 150 hours of operation
$$P(A) = \binom{5}{2}\left(\frac{1}{3}\right)^2\cdot\left(1-\frac{1}{3}\right)^3$$
$$= 0.329$$

P4.

(a) $E(X) = \int_0^\infty P(X>t) \cdot dt$ where X is non-negative r.v.

proof:

Let $F(x) = P(X \leq x)$ is the CDF of X, and $f(x) = \frac{d\,F(x)}{dx}$ by definition of PDF and CDF

$\Downarrow$

$d\,F(x) = f(x) \cdot dx$

$$LHS = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_0^\infty x f(x) \cdot dx \quad (X \text{ is non-negative})$$

$$= \int_0^\infty x \, d\,F(x)$$

Let $1 - F(x) = R(x)$

$$\therefore LHS = E(X) = \int_0^\infty x \, d(1 - R(x))$$

$$= -\int_0^\infty x \cdot d\,R(x)$$

$$= -\int_0^\infty x\, R(x) \cdot dx + \int_0^\infty R(x) \cdot dx$$

By the def of CDF, we have

$F(0) = 0$ and $F(\infty) = 1$

$\Rightarrow R(0) = 1 \qquad R(\infty) = 0$

$\Rightarrow -\int_0^\infty x\, R(x) \cdot dx = 0$

$$\therefore LHS = E(X) = \int_0^\infty R(x) \cdot dx$$

$$= \int_0^\infty P(X > x) \cdot dx$$

$$= \int_0^\infty P(X > t) \cdot dt$$

4(b) we are given that CDF of x $P(X \le x) = 1 - e^{-x^2}$ for $x \ge 0$

Since X here is non-negative r.v., we can use part (a)

$$E(X) = \int_0^\infty P(X > t) \cdot dt$$

$$= \int_0^\infty (1 - P(X \le t)) dt$$

$$= \int_0^\infty 1 - (1 - e^{-t^2}) \cdot dt$$

$$= \int_0^\infty e^{-t^2} \cdot dt$$

Let $x = t^2$ $\quad dx = 2t \cdot dt \Rightarrow dt = \frac{1}{2t} \cdot dx = \frac{1}{2\sqrt{x}} \cdot dx$

$$\therefore E(X) = \int_0^\infty \frac{1}{2\sqrt{x}} \cdot e^{-x} \cdot dx$$

$$= \frac{1}{2} \int_0^\infty \underbrace{x^{\frac{1}{2}-1} e^{-x}} \cdot dx$$ (By integral of Gamma function)

$$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \sqrt{\pi}$$

P5.

(a) We are given that X and Y are independent r.v. with $Var(X)=1$ and $Var(Y)=2$

Let $Z = 3X+4Y$.

By the property of variance:

$$
\begin{aligned}
Var(Z) &= Var(3X+4Y) \\
&= 3^2 Var(X) + 4^2 Var(Y) \\
&= 9\cdot 1 + 16\cdot 2 \\
&= 41
\end{aligned}
$$

(b) Y is r.v. which is uniformly distributed over the set $\{1,2,3\}$

$E[X|Y=i] = i$ and $E(X^2|Y=i) = i^2+1$

By formula of variance:

$$
\begin{aligned}
Var(X|Y=i) &= E[X^2|Y=i] - E[X|Y=i]^2 \\
&= i^2+1 - i^2 \\
&= 1
\end{aligned}
$$

$E[Y] = \sum_y y\, Pr(y)$ where $Pr(Y=i) = \frac{1}{3}$ for $i \in \{1,2,3\}$ since Y is uniformly distributed

$$
\begin{aligned}
&= \tfrac{1}{3}\cdot 1 + \tfrac{1}{3}\times 2 + \tfrac{1}{3}\times 3 \\
&= 2
\end{aligned}
$$

Similarly, $E[Y^2] = \sum_y y^2 Pr(y)$

$$
\begin{aligned}
&= \tfrac{1}{3}\times 1^2 + \tfrac{1}{3}\times 2^2 + \tfrac{1}{3}\times 3^2 \\
&= \frac{14}{3}
\end{aligned}
$$

$\Longrightarrow$ $\therefore$ $Var(Y) = E[Y^2] - E[Y]^2$

$$
\begin{aligned}
&= \frac{14}{3} - 2^2 \\
&= \frac{2}{3}
\end{aligned}
$$

By Law of Total Variance:

$$
\begin{aligned}
Var(X) &= Var(E(X|Y=i)) + E(Var(X|Y=i)) \\
&= Var(i) + E[1] \\
&= Var(Y) + 1 \\
&= \frac{2}{3} + 1 \\
&= \frac{5}{3}
\end{aligned}
$$

5(c) Since $X_1, X_2, \cdots X_n$ are independent r.v. and $Var(X_1) = Var(X_2) = \cdots = Var(X_n) = \sigma^2$

Thus, $Var(Y) = Var(\frac{1}{n}(X_1 + X_2 + \cdots + X_n))$

$= Var(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots \frac{1}{n}X_n)$

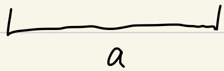
$= \frac{1}{n^2}Var(X_1) + \frac{1}{n^2}Var(X_2) + \cdots + \frac{1}{n^2}Var(X_n)$ ($cov(X_i, X_j) = 0 \quad \forall\ i \neq j$)

$= \frac{1}{n^2}(n \cdot \sigma^2)$

$= \frac{\sigma^2}{n}$

Pb:

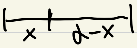
(a) Assume the length of yard stick is a



A point is chosen at random on the yard stick

$X \sim U(0, a)$, $f(x) = \frac{1}{a}$ for $0 < x < a$

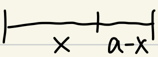
①



If The length of shorter piece is x,

$$P\left(\frac{\text{shorter}}{\text{longer}} < \frac{1}{4}\right) = P\left(\frac{x}{a-x} < \frac{1}{4}\right)$$
$$= P(4x < a - x)$$
$$= P(5x < a)$$
$$= P\left(x < \frac{a}{5}\right)$$
$$= \int_0^{\frac{a}{5}} \frac{1}{a} \cdot dx$$
$$= \frac{1}{a}x \Big|_0^{\frac{a}{5}}$$
$$= \frac{1}{5}$$

②



If the length of shorter piece is a-x

$$P\left(\frac{\text{shorter}}{\text{longer}} < \frac{1}{4}\right) = P\left(\frac{a-x}{x} < \frac{1}{4}\right)$$
$$= P(4a - 4x < x)$$
$$= P(4a < 5x)$$
$$= P\left(x > \frac{4}{5}a\right)$$
$$= 1 - P\left(x \leq \frac{4}{5}a\right)$$
$$= 1 - \int_0^{\frac{4}{5}a} \frac{1}{a} dx$$
$$= 1 - \frac{1}{a}x \Big|_0^{\frac{4}{5}a}$$
$$= \frac{1}{5}$$

Overall, by the law of probability addition,

$$P\left(\frac{\text{shorter}}{\text{longer}} < \frac{1}{4}\right) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

(b) Y is uniformly distributed r.v. over $(0,5)$

$$Y \sim U(0,5) \qquad f(y) = \frac{1}{5} \text{ for } 0<y<5$$

$$4X^2 + 4XY + Y + 2 = 0$$

By formula of solution to equation,

$$X = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-8Y \pm \sqrt{16Y^2 - 4\cdot 4(Y+2)}}{8}$$

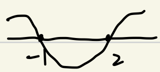
$$= \frac{-8Y \pm \sqrt{16Y^2 - 16Y - 32}}{8}$$

If the roots need to be read, then $16Y^2 - 16Y - 32 \geq 0$

$$16Y^2 - 16Y - 32 \geq 0$$

$$Y^2 - Y - 2 \geq 0$$

$$(Y-2)(Y+1) \geq 0$$



$$\therefore \quad Y \leq -1 \text{ or } Y \geq 2$$

But $0 < Y < 5$, $\therefore \quad Y \geq 2$

$$P(Y \geq 2) = 1 - P(Y < 2)$$

$$= 1 - \int_0^2 \frac{1}{5} \cdot dx$$

$$= 1 - \frac{1}{5}x \Big|_0^2$$

$$= 1 - \frac{2}{5}$$

$$= \frac{3}{5}$$

$\therefore$ the probility of having real roots is $\frac{3}{5}$