Chapter 4

Error Probability for Binary Signals

In this chapter we consider a communication system that transmits a single bit of information using one of two signals. The receiver filters the received signal, samples the filter output and then makes a decision about which of the two signals was transmitted. We first consider an example in which the two signals are just rectangular pulses with opposite sign. For those signals in AWGN we analyze the probability of error for a matched filter at the receiver. Second we consider optimizing the system over all possible filters, signals and decision rules. The optimal filter, and signals are derived for binary modulation (one of two signals transmitted). Finally the effect of imperfect receivers are considered and approaches to analyzing a system with intersymbol interference is discussed..

The performance of a communication system is typically the error probability which is a function of the received energy of the transmitted signals, the amount of noise in the system and the receiver processing. The goal is to find the relation between the error probability and the signal-to-noise ratio. Because the filter in the receiver is a linear time-invariant system, we can analyze the output of the receiver filter due to the signal alone and then analyze the output due to noise alone. The probability of error is then the probability that the noise at the output causes the filter output to be in the wrong decision region. If the filter is not perfectly matched then it is possible that one bit can have a nonzero output at the sampling time corresponding to make a decision about another bit. This is called intersymbol interference. At the end of the chapter we describe and analyze the performance with intersymbol interference for a specific example.

4.1 System Model: Transmitter, Channel and Receiver

The system considered in this chapter has a transmitter that communicates a bit of information by sending a single pulse with one of two amplitudes. We will assume the pulse is of duration T starting at $t = 0$ at the receiver. The received signal is the sum of the transmitted signal and white Gaussian noise (WGN). The receiver filters the received signal and samples the output. The purpose of filtering is to remove as much as the noise as possible without removing the desired signal. The sampled output is compared to a threshold to determine which of the two signals was transmitted. The analysis of error probability is done by first determining the receiver output due to the signal alone and then determining the receiver output due to noise alone. The output due to signal alone will depend on the power used and the time used to transmit the signal and the filter. The output due to noise alone will be a Gaussian random variable with zero mean. The variance of the output of the filter due to noise will depend on the noise level and the filter used in the receiver. One goal is to determine which filter minimizes the probability of making an error. A filter might remove so much noise that it removes also much of the signal will not be best. Neither will a filter that does not remove hardly any noise.

The structure of the transmitter is shown below in Figure 4.1. The data source produces a sequence of impulses of amplitude 1 or -1. The impulses are filtered by a transmit filter with impulse response $h_T(t) = \varphi(t)$ which has unit energy. The data signal is amplified by multiplying by $\sqrt{E}$ so that the power of the signal is $P = E/T$. The resulting signal is transmitted as indicated by the antenna. As an example the filter impulse response could be $h(t) = \sqrt{1/T}\,p_T(t)$ where

$$p_T(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

in which case the signal would consist of a sequence of rectangular pulses. We will be concerned with just one bit (e.g. b_0) and the probability of error for that bit. The pulse-shaped data signal is

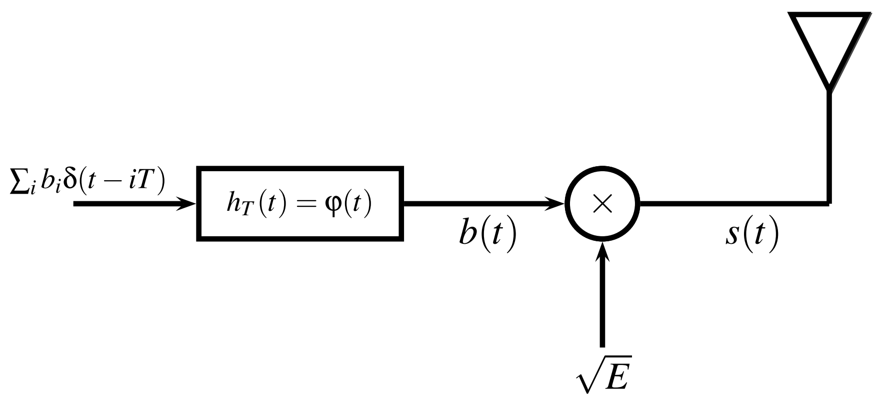

Figure 4.1: Model of Transmitter

$$b(t) = \sum_{l=-\infty}^{\infty} b_l\, \varphi(t - lT), \qquad b_l \in \{+1, -1\}.$$

Notice that if we let $\varphi_i(t) = \varphi(t - iT)$, that is, time-shifted orthonormal signals, then we see that $b(t) = \sum_{l=\infty}^{\infty} b_i \sqrt{E}\varphi_i(t)$. So the signal is really composition of (time-shifted) orthonormal signals with energy E per bit. The transmitted signal $s(t)$ is the data waveform amplified by $\sqrt{E}$.

$$\begin{aligned} s(t) &= \sqrt{E}\, b(t) \\ &= \sqrt{E} \sum_i b_i \varphi(t - iT). \end{aligned}$$

For the case of rectangular pulses

$$\begin{aligned} s(t) &= \sqrt{E} \sum_i b_i \sqrt{1/T}\, p_T(t - iT) \\ &= \sqrt{P} \sum_i b_i p_T(t - iT) \end{aligned}$$

where $E = PT$. The energy of a single transmitted bit is

$$E = \int_0^T s^2(t)dt = \int_0^T P dt = PT = E.$$

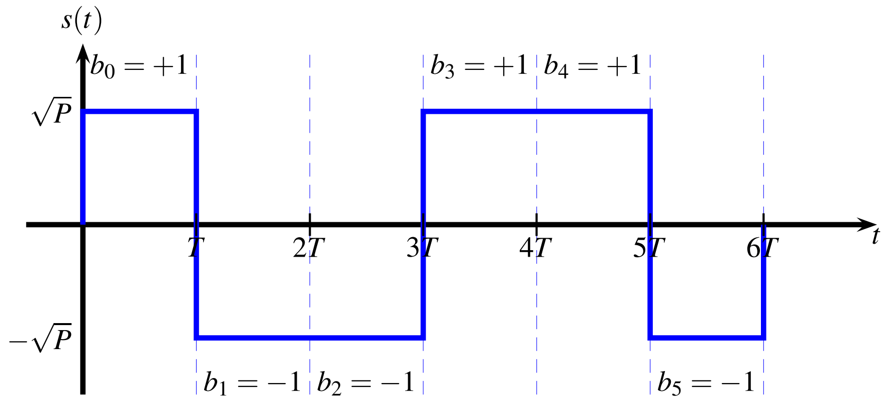

Figure 4.2: Realization of transmitted signal

Figure 4.2 shows a realization of the transmitted signal for the case of $\varphi(t)$ being a normalized rectangular pulse and for there being just 6 data bits. Thus the transmitted signal is a sum of pulses of various amplitudes (either $+\sqrt{P}$ or $-\sqrt{P}$). The channel simply adds noise to the transmitted signal. The received signal is

$$r(t) = s(t) + n(t).$$

The noise is white Gaussian noise with power spectral density $N_0/2$ W/Hz. One realization of the received signal, for the case $P = 1$ and $T = 1$ using the same data sequence as in Figure 4.2 is shown in Figure 4.3. Figure 4.4 shows the power spectral density of the desired signal, for the case of 2047

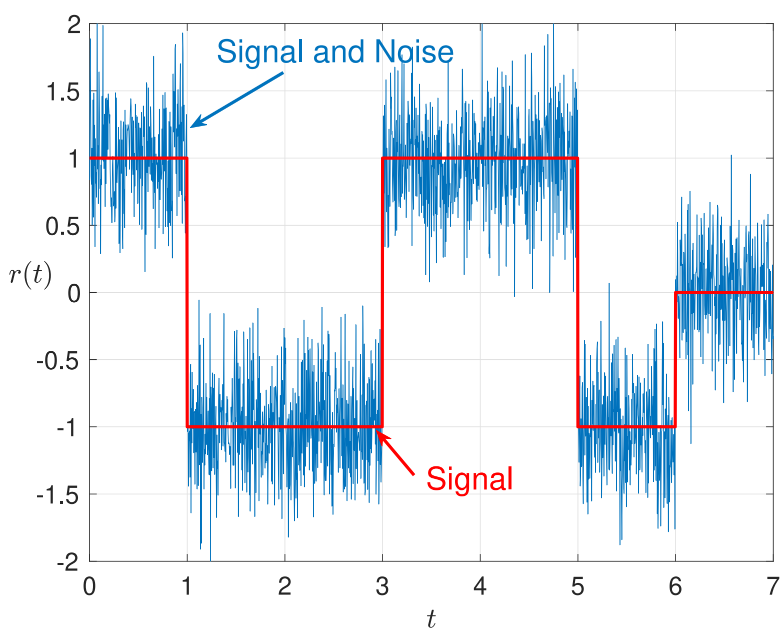

Figure 4.3: Signal and noise for rectangular pulses

transmitted randomly chosen bits with duration 1 second and amplitude 1. Also shown is the frequency response of the filter and the noise power spectral density. As can be seen from the figure the filter is matched to the spectral characteristics of the signal and attenuates the noise where the signal is small (for example high frequencies or nulls in the signal spectrum). Of course, the receiver can only processes the sum of the signal term and the noise term.

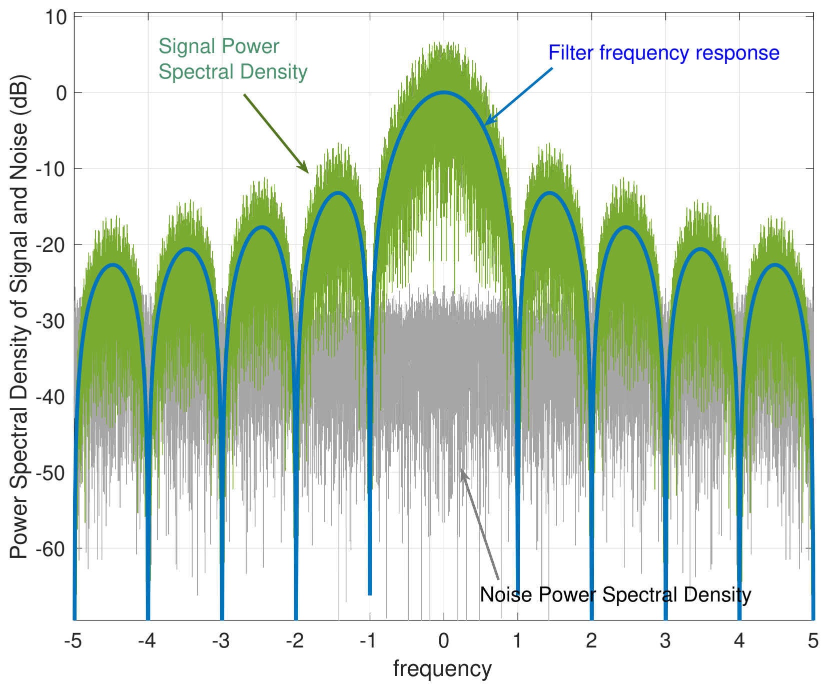

Figure 4.4: Power spectral density of transmitted signal and filter characteristics

In the time domain the filter is essentially finding the average value of the input over a window of T seconds. At time T the average is from 0 to T. At time $2T$ the average is from T to $2T$. The filter output will be sampled at time $T, 2T, ..., 6T$ to determine the data bits $b_0, b_1, ...b_5$. If the output is greater than 0 the decision will be that the data bit is +1. Otherwise the decision is that the data bit is -1. There is a certain probability that the noise will cause an error in the decision that will depend on the signal-to-noise ratio. The probability of error is the main performance criteria. Clearly the receiver must know when to sample the filter output. This is part of a synchronization subsystem that often relies of know data symbols, or pilots, to determine the correct timing.

The receiver consists of a filter with impulse response $h(t) = \varphi(T-t)$, and a sampling and decision device. This is shown in Figure 4.5. The output of the filter is used to decide the data bit transmitted. We let $\hat{b}_{i-1}$ denote the decision as to which bit was transmitted during the interval $[(i-1)T, iT)$. If $\hat{b}_{i-1} = b_{i-1}$ then the correct decision was made. If $\hat{b}_{i-1} \neq b_{i-1}$ then an error was made. The filter is "matched" to the baseband signal being transmitted. For simple rectangular type signals this is just a rectangular pulse of duration T. The impulse response is $h_R(t) = \varphi_0(T-t)$ which if sampled at times $T, 2T, ...$ will, in the absence of noise, reproduce the original bits scaled by $\sqrt{E}$.

The receiver filter $h_R(t) = \varphi(T-t)$ will be a noncausal filter if $\varphi(t) = 0$ for $t > T$. If the waveform $\varphi(t)$ lasts a finite time interval, T_I, such as a truncated square-root raised cosine pulse shape then we can use a filter $h_R(t) = \varphi(T_I - t)$ and then the output will be delayed by T_I seconds relative to the output for

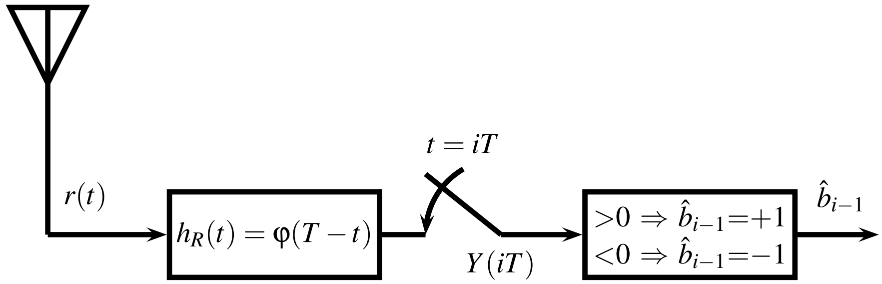

Figure 4.5: Model of receiver

a filter $h_R(t) = \varphi(-t)$. This just means we will sample the signal later and make a decision T_I seconds later. The filter is a low pass filter and removes much of the noise, especially high frequency noise.

To analyze the performance we will determine the output due to signal alone. Then we determine the output due to noise alone which will be a Gaussian random variable with mean zero and variance depending on the filter. The overall output is the sum of the two. Then the probability of the noise causing an incorrect decision will be analyzed.

The output due to signal will be determined in two steps. First the output due to a single pulse will be determined. Then, using linearity and time-invariance, the output of the filter due to a sequence of pulses will be analyzed. The output of the filter, $\hat{s}(t)$, due only to the transmitted signal is given by

$$\begin{aligned}
\hat{s}(t) &= \int_{-\infty}^{\infty} h_R(t-\tau)r(\tau)d\tau \\
&= \int_{-\infty}^{\infty} \varphi(T-(t-\tau))r(\tau)d\tau \\
&= \int_{-\infty}^{\infty} \varphi(\tau-(t-T))r(\tau)d\tau.
\end{aligned}$$

That is, the filter output at time t is a correlation of the received signal with a delayed (by $t-T$) copy of the transmitted pulse shape. In the case that $\varphi(t) = \sqrt{1/T}\,p_T(t)$ the filter output is

$$\hat{s}(t) = \int_{t-T}^{t} \sqrt{1/T}\,r(\tau)d\tau.$$

That is, the filter is essentially a sliding integrator, with the output at time t being an integration over the past T seconds (i.e. from time $t-T$ to time t). This is because we considered a non causal filter for $h(t)$.

Consider first the output due to transmitting just a single pulse (beginning at $t=0$ and ending at $t=T$) with amplitude $\sqrt{P}b_0$, where $b_0 \in \{+1,-1\}$. The output of the causal filter $h_T(t) = \varphi(T-t)$ due to signal alone, that is $r(t) = s(t)$ is

$$\hat{s}(t) = b_0\sqrt{P/T}\int_{t-T}^{t} p_T(\tau)d\tau.$$

That is, the filter output at time t is essentially integral over the past T seconds of the a square wave over T second intervals. The output would be maximum when the filter did an integration from 0 to T, i.e. at $t=T$ in which case the output would be $b_0(\sqrt{P/T})T = b_0\sqrt{E}$. In general the output would look like

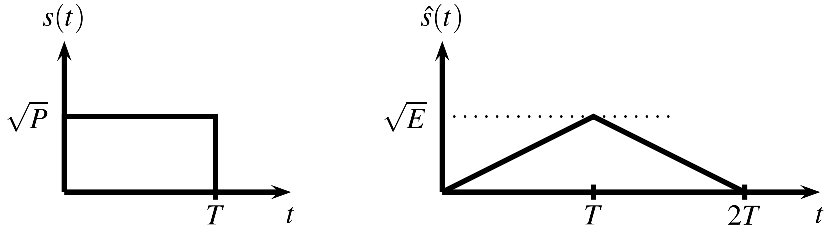

Figure 4.6: Filter input and output

a triangular function shown in Figure 4.6 for $b_0 = +1$. This is just the result that the convolution of a rectangular pulse with a rectangular pulse is a triangular pulse. The output of the filter at time $t = T$ is

$$\begin{aligned}
\hat{s}(T) &= b_0\sqrt{1/T}\int_0^T \sqrt{P}d\tau \\
&= b_0\sqrt{1/T}\sqrt{P}T \\
&= b_0\sqrt{PT} \\
&= b_0\sqrt{E}.
\end{aligned}$$

We can construct a sequence of pulses by adding shifted versions of a single pulse. Because filtering is a linear and time-invariant operation the output due to the sum of shifted versions of the input is the corresponding sum of the shifted version of the output due to a single pulse. The input and If now the transmitted signal was a sequence of pulses of various amplitudes the corresponding outputs would look like a sequence of triangles added together. Figure 4.7 shows a sequence of pulses and the output of the filter due to each pulse. The total output is the sum of the output due to each of the pulses. This is shown in Figure 4.8 for the case of rectangular pulses. The filter output will be sampled at times $T, 2T, 3T, \ldots$. Notice that when one of the signals reaches its peak (at a sampling time) all other signals are zero. That is, there is no interference from adjacent symbols at times that are multiples of T. We say there is no intersymbol interference.

Now consider the output due to noise alone at time iT, one of the sampling times. The output is

$$\eta_i = \int h_R(iT-\tau)n(\tau)d\tau.$$

The output due to noise alone at time iT is a random variable with mean 0 and variance

$$\begin{aligned}
\sigma^2 &= E[\eta_i^2] \\
&= E[\int_\tau h_R(iT-\tau)n(\tau)d\tau \int_\alpha h_R(iT-\alpha)n(\alpha)d\alpha] \\
&= \int_\tau\int_\alpha h_R(iT-\tau)h_R(iT-\alpha)E[n(\tau)n(\alpha)]d\tau d\alpha \\
&= \int_\tau\int_\alpha h_R(iT-\tau)h_R(iT-\alpha)\frac{N_0}{2}\delta(\tau-\alpha)d\tau d\alpha \\
&= \frac{N_0}{2}\int_\tau h_R^2(iT-\tau)d\tau \\
&= \frac{N_0}{2}\int_\tau h_R^2(\tau)d\tau.
\end{aligned}$$

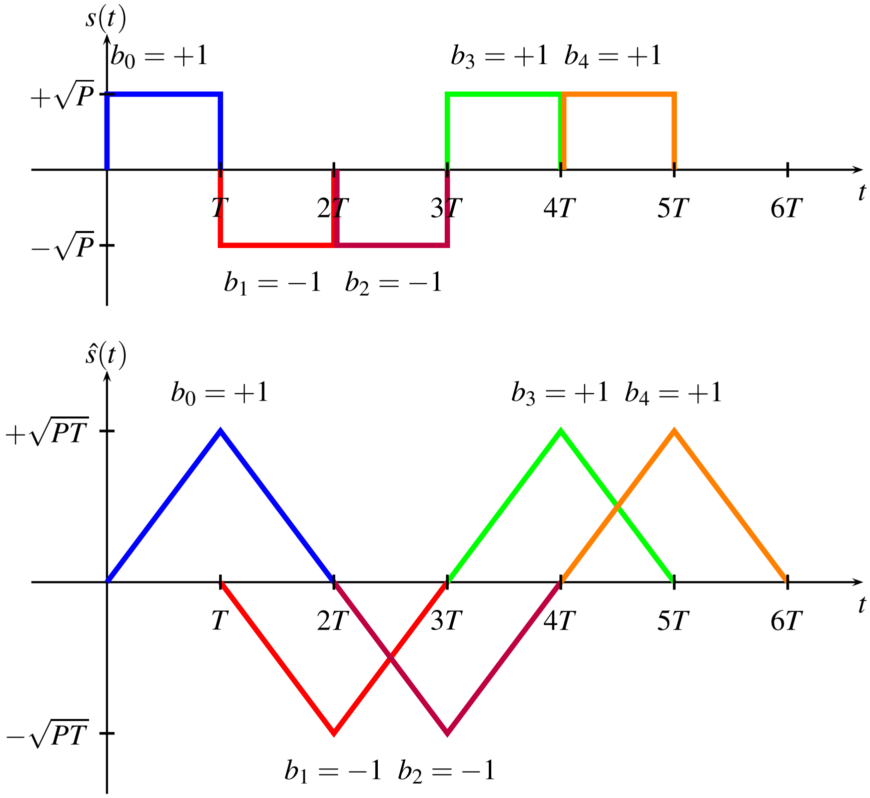

Figure 4.7: Transmitted signal and signal output of filter in receiver

For the case when the receiver filter is just matched to the transmit pulse, i.e. $\varphi(t) = \sqrt{\frac{1}{T}}p_T(t)$ then then

$$\begin{aligned}\sigma^2 &= \frac{N_0}{2}\int h_R^2(\tau)d\tau \\ &= \frac{N_0}{2}\frac{1}{T}\int_0^T 1 d\tau \\ &= \frac{N_0}{2}.\end{aligned}$$

The output of the filter in the time domain is shown in Figure 4.9.[1]

4.1.1 Performance Analysis

To analyze the performance we compute the output of the filter due to signal alone and the output of the filter due to noise alone. The overall output is the sum of these two outputs. For the communication system described earlier the output of the filter $h(t)$ at time T due to signal has value either $\pm\sqrt{E}$ depending on the information bit. If we just examine the output at multiples of the data symbol duration T then we can simply express the output of the filter as

$$Y(iT) = \sqrt{E}b_{i-1} + \eta_i$$

[1] Matlab and Python code for simulation of signals and noise is in Appendix D.

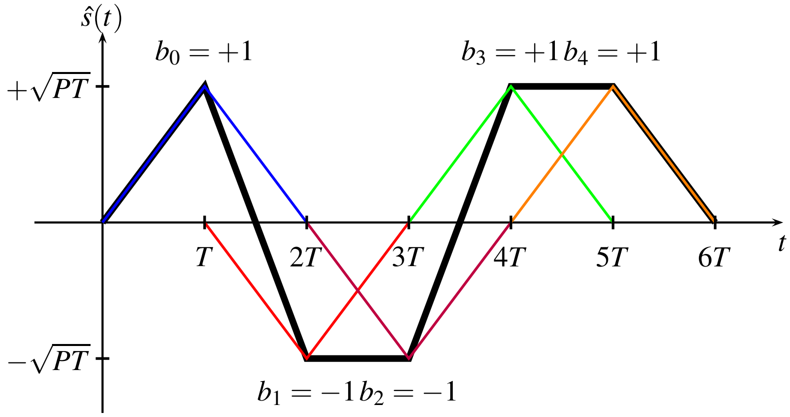

Figure 4.8: Output due to multiple rectangular pulses

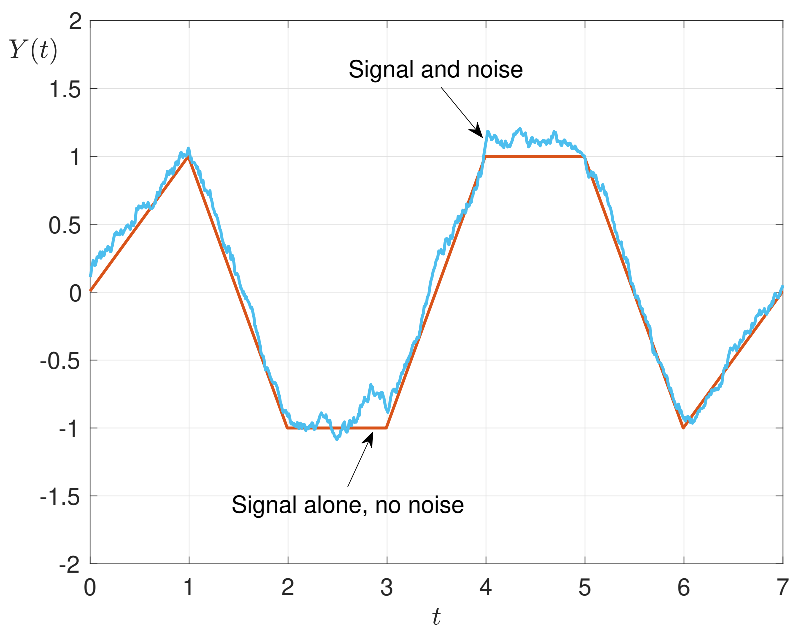

Figure 4.9: Signal and noise at filter output for rectangular pulses

where η_i is the output due to the noise and has a Gaussian density with mean zero and variance $N_0/2$. This is often written as $\eta_i \sim N(0, \frac{N_0}{2})$ meaning that the variable η_i has a normal or Gaussian distribution with zero mean and variance $N_0/2$. The filter $h(t)$ that is matched to the signal is optimum in that it lets in the most signal and the least amount of noise and is called the *matched filter*. In Section 4.3 the optimality of the matched filter is derived. We define two conditional probabilities of error: namely the conditional probability of error given the data bit is +1, $P_{e,+1}$, and the conditional probability of error given the data bit is -1, $P_{e,-1}$, The average probability of error can be calculated as follows

$$\begin{aligned}
\bar{P}_e &= P_{e,+1}P\{b_{i-1} = +1\} + P_{e,-1}P\{b_{i-1} = -1\} \\
P_{e,+1} &= P\{Y(iT) < 0 | b_{i-1} = +1\}, \\
P_{e,-1} &= P\{Y(iT) > 0 | b_{i-1} = -1\}
\end{aligned}$$

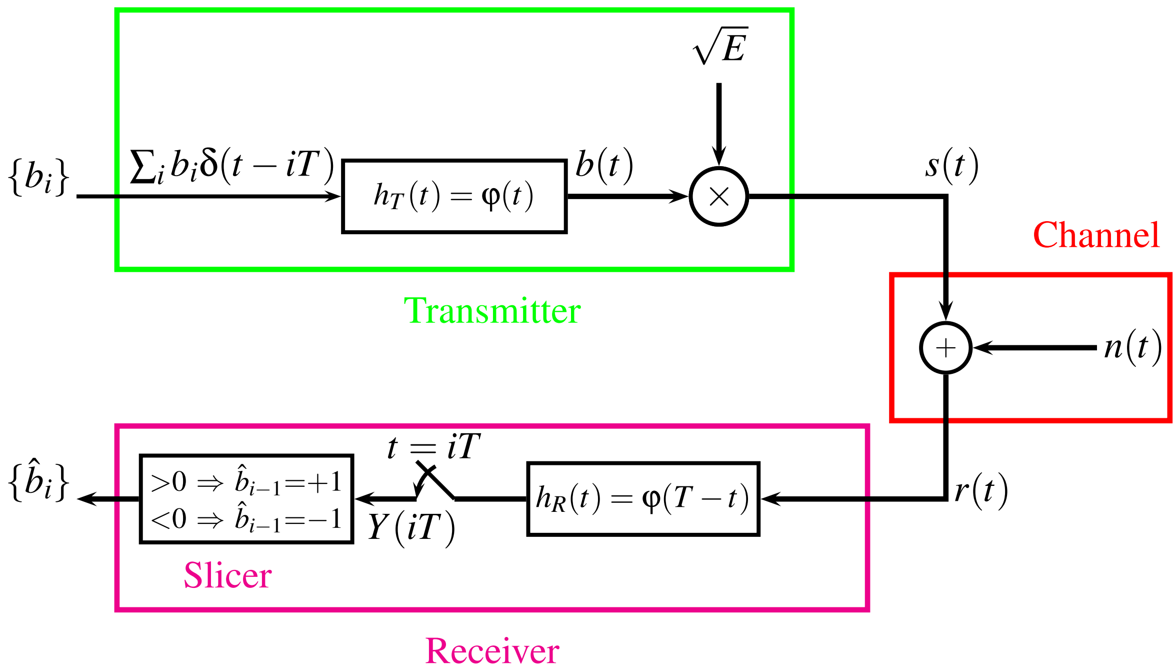

Figure 4.10: Block diagram of communication system

Consider first $P_{e,+1}$.

$$\begin{aligned} P_{e,+1} &= P\{Y(iT)<0|b_{i-1}=+1\}, \\ &= P\{\sqrt{E}b_{i-1}+\eta_i<0|b_{i-1}=+1\}, \\ &= P\{\sqrt{E}+\eta_i<0|b_{i-1}=+1\}, \\ &= P\{\sqrt{E}+\eta_i<0\}. \end{aligned}$$

This last line follows since the noise is independent of the data being transmitted.

$$\begin{aligned} P_{e,+1} &= P\{\eta_i<-\sqrt{E}\}, \\ &= \Phi(\frac{-\sqrt{E}}{\sqrt{N_0/2}}) = \Phi(-\sqrt{\frac{2E}{N_0}}) = Q(\sqrt{\frac{2E}{N_0}}). \end{aligned}$$

The calculation of $P_{e,-1}$ is similar. The result is

$$\begin{aligned} P_{e,-1} &= P\{Y(iT)>0|b_{i-1}=-1\}, \\ &= Q(\sqrt{\frac{2E}{N_0}}). \end{aligned}$$

The average error probability is then

$$\begin{aligned} \bar{P}_e &= P\{b_{i-1}=+1\}Q(\sqrt{2E/N_0})+P\{b_{i-1}=-1\}Q(\sqrt{2E/N_0}). \\ &= Q(\sqrt{2E/N_0}). \end{aligned}$$

Figure 4.11 shows the two conditional density functions of the filter output when conditioned on the data bit being +1 and -1 for $E_b/N_0=0$ dB. The conditional probability of error given the data bit is +1 is the

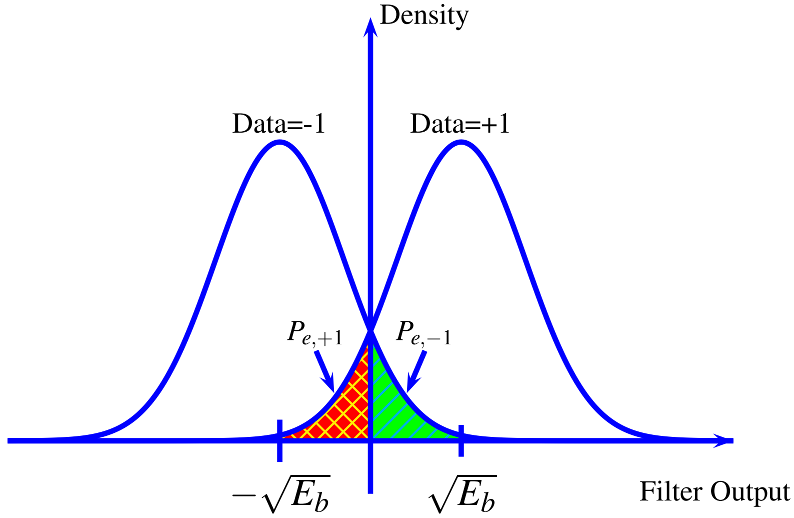

Figure 4.11: Filter output with noise ($E_b/N_0 = 0$ dB)

area under the curve of the conditional density function given the data is +1 to the left of 0. Similarly the conditional probability of error given the data is -1 is the area under the curve of the conditional density function given the data is -1 to the right of 0. Similarly Figure 4.12 shows the same functions for $E_b/N_0 = 3$ dB.

The output due to signal alone is $\pm\sqrt{E}$. The error probability can be expressed in terms of the Euclidean distance $d_E = 2\sqrt{E}$ between these two outputs and the noise variance σ^2. The output due to noise alone has variance $\sigma^2 = N_0/2$. If the noise is Gaussian then the probability of error is just the probability that the noise level causes the output to be on the opposite side of zero relative to the input. That is, the noise must be larger than half the Euclidean distance between the two signals. This is given by

$$P_{e,b} = Q\left(\frac{d_E}{2\sigma}\right) = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where $Q(x) = \int_x^\infty \frac{1}{2\pi} e^{-u^2/2} du$.

4.1.2 Bit Error Probability

For binary signals the energy transmitted per information bit E_b is equal to the energy per signal E. For $P_{e,b} = 10^{-5}$ we need a bit-energy, E_b to noise density N_0 ratio of $E_b/N_0 = 9.6$dB. Note that $Q(x)$ is a decreasing function which is 1/2 at $x = 0$. There are efficient algorithms (based on Taylor series expansions) to calculate $Q(x)$. There is a function in Matlab (`qfunc`) to calculate the Q function. In Python using `scipy` the Q function can be calculated from the error (erf) function.[2] Since $Q(x) \leq e^{-x^2/2}/2$ the error probability can be upper bounded by

$$P_{e,b} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \leq \frac{1}{2} e^{-E_b/N_0}$$

[2] The error function function is $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and $Q(x) = 1/2[1 - \mathrm{erf}(x/\sqrt{2})]$. In Python, after importing special from scipy (`from scipy import special`), the Q function is `0.5-0.5*special.erf(x/sqrt(2))`.

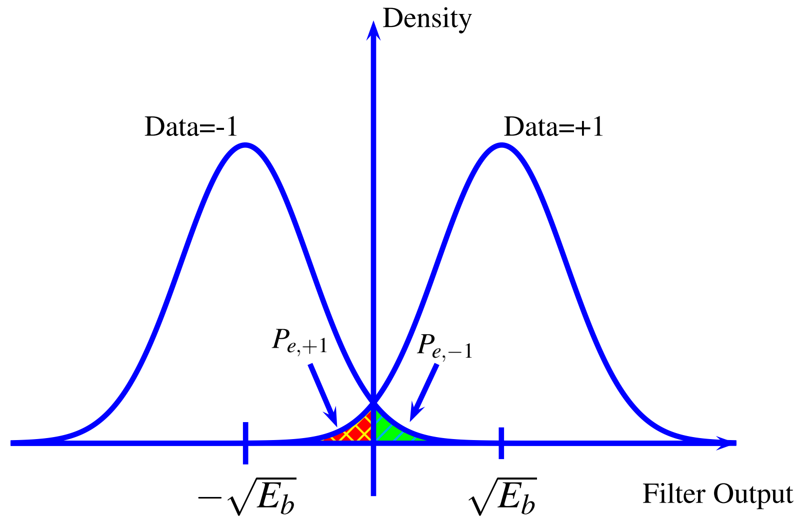

Figure 4.12: Filter output with noise ($E_b/N_0 = 3$ dB)

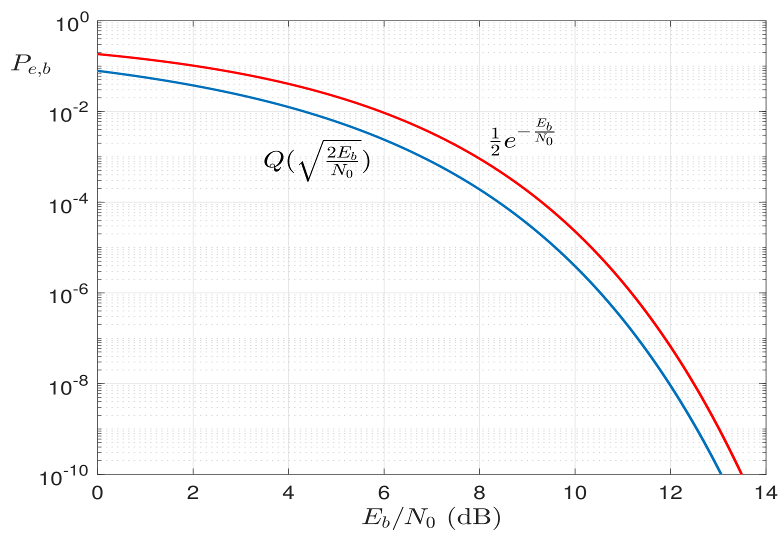

Figure 4.13: Error probability of BPSK

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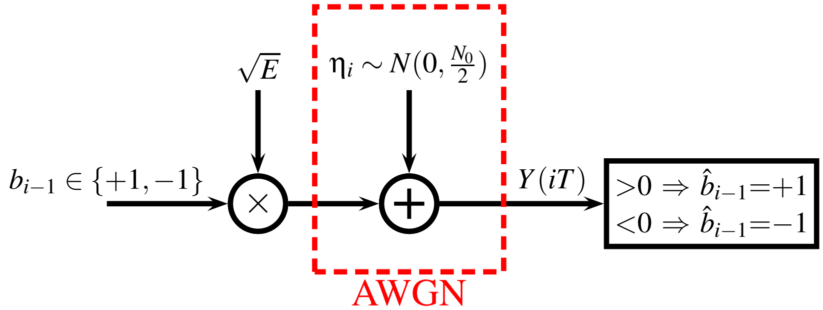


Figure 4.14: Additive white Gaussian noise channel

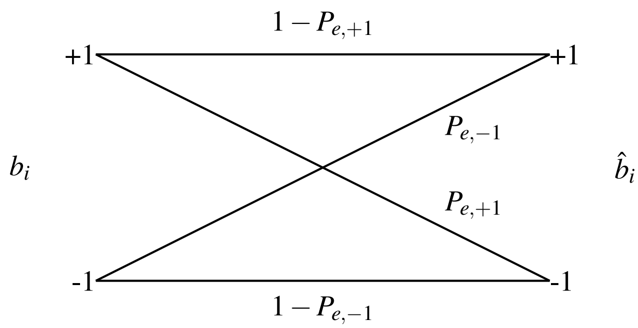


Figure 4.15: Binary symmetric channel

which decreases exponentially with signal-to-noise ratio. Both the exact error probability and the upper bound are plotted in Figure 4.13. The upper bound will be useful in analyzing a random collection of communications systems in Chapter 6. For binary signals this is the smallest bit error probability possible with any signals and any filter. This will be shown in the next section. The receiver (matched filter) shown above is optimum (in additive white Gaussian noise).

The models that we make for this modulation are the following. The first model is an additive white Gaussian noise channel (AWGN). This is illustrated in Figure 4.14. This channel is an additive Gaussian noise channel with a decision device. We will call this the binary input, binary output additive white Gaussian noise (BIBO-AWGN) channel. An equivalent model overall is that of a binary symmetric channel (BSC) in which the input values are +1 or -1 and the output values are +1 or -1. If the input to the channel is a +1 then there is one transmitted signal and a different signal if the input is -1. The received signal is demodulated and a decision is made as to what channel input most likely caused the output. There is a probability that an input that is +1 but the decision at the receiver is that the received signal was caused by a -1 input. Similarly there is a probability that a receiver decision of a +1 was caused by a -1 input. This is the probability of error. Often these two probabilities are identical. The result is a binary symmetric channel shown in Figure 4.15.

In this section we considered a transmitted signal consisting of a sequence of pulses (e.g. rectangular pulses or square-root raised cosine pulses). The channel adds white Gaussian noise. The receiver filters the received signal to remove as much noise as possible without removing the signal. We calculated the output due to signal alone and the output due to noise alone then added the results to get the total output.

The bit error probability depends on the ratio of the energy of the received desired signal and the noise power spectral density. The optimum filter (derived in the next section) is matched to the desired pulse shape used at the transmitter. The error probability for the optimum receiver for the rectangular pulse shapes is

$$P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

For this modulation (binary) the energy E for each signal is the same as the energy per bit E_b. The required signal-to-noise ratio E_b/N_0 for different error probabilities is shown in Table 4.1.

P_e	E_b/N_0 (dB)
10^{-2}	4.32
10^{-3}	6.79
10^{-4}	8.40
10^{-5}	9.59
10^{-6}	10.53
10^{-7}	11.31
10^{-8}	11.97
10^{-9}	12.55
10^{-10}	13.06

Table 4.1: E_b/N_0 (dB) for different error probabilities

4.2 Optimum Receiver, Optimum Signals

The goal of this section is to derive the optimal threshold, filter, and signals for a binary communications system that consists of transmitter that communicates one bit of information using one of two real signals and a receiver that uses a filter that is sampled and compared to a threshold to make a decision as to which bit was transmitted.

The communication system considered in this section shown in Figure 4.16. One of two signals $s_0(t)$ or $s_1(t)$ is transmitted. Noise is added to form the received signal $r(t)$. The received signal is filtered and the output of the filter is sampled. A comparison of the sampled output to a threshold is done to make a decision about which signal was transmitted. The goal is to find the optimum filter, threshold and signals to minimize the average error probability.

4.2.1 Error Probability Analysis

The error probability is the average of two conditional error probabilities; the conditional error probability given $s_0(t)$ is transmitted ($P_{e,0}$) and the conditional error probability given $s_1(t)$ is transmitted ($P_{e,1}$). Suppose that π_0 is the probability that s_0 is transmitted and π_1 is the probability that s_1 is transmitted. Then $\pi_0+\pi_1=1$. We will only consider the case where $\pi_0=\pi_1=1/2$ although the method for deriving the optimum receiver is the same for when $\pi_0 \neq \pi_1$ (see Problem XYZ).

$$\begin{aligned}
P_{e,0} &= P\{\text{error}|s_0 \text{ transmitted}\}.\\
P_{e,1} &= P\{\text{error}|s_1 \text{ transmitted}\}.
\end{aligned}$$

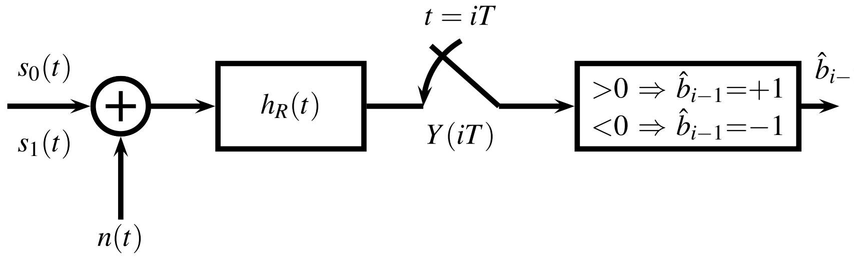


Figure 4.16: Model of binary communication system

The average probability of error is

$$\bar{P}_e = P_{e,0}\pi_0 + P_{e,1}\pi_1. \tag{4.1}$$

We will first derive an expression for P_e that depends on the signals, $s_0(t)$ and $s_1(t)$, the filter $h(t)$, and the threshold γ. Consider first the output just due to each of the transmitted signals. Let $\hat{s}_0(T)$ and $\hat{s}_1(T)$ be the output of the filter at the sample time T due $s_0(t)$ and $s_1(t)$.

$$\begin{aligned}
\hat{s}_0(T) &= \int_{-\infty}^{\infty} h(T-\tau)s_0(\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)s_0(T-\tau)d\tau \leftarrow \text{output due to } s_0 \text{ alone,} \\
\hat{s}_1(T) &= \int_{-\infty}^{\infty} h(T-\tau)s_1(\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)s_1(T-\tau)d\tau \leftarrow \text{output due to } s_1 \text{ alone,}
\end{aligned}$$

Since we assume that the receiver will decide s_0 if the output of the filter is larger than a threshold and s_1 if it is smaller, we need to assume that $\hat{s}_0(T) > \hat{s}_1(T)$.

The noise at the input to the receiver is assumed to be white Gaussian noise with two-sided power spectral density $N_0/2$. The output of the filter due to noise alone is $\eta = \int h(T-\tau)n(\tau)d\tau$. The mean of the output is $E[\eta] = 0$. The variance of the output is

$$\sigma_n^2 = \mathrm{Var}[\eta] = \frac{N_0}{2}\int h^2(\tau)d\tau.$$

The distribution of the noise is Gaussian. We now can combine the output due to noise and due to signal. If $s_0(t)$ is transmitted the output is $Y(T) = \hat{s}_0(T) + \eta$. If $s_1(t)$ is transmitted the output is $Y(T) = \hat{s}_1(T) + \eta$. The outputs $\hat{s}_0(T)$ and $\hat{s}_1(T)$ depend on $h(t)$, $s_0(t)$ and $s_1(t)$. The variance of the noise depends on $h(t)$.

The error probability analysis is as follows.

$$P_{e,0} = P\{Y(T) < \gamma | s_0 \text{ transmitted}\}.$$

If s_0 is transmitted then $Y(T)$ takes the form

$$Y(T) = \hat{s}_0(T) + \eta$$

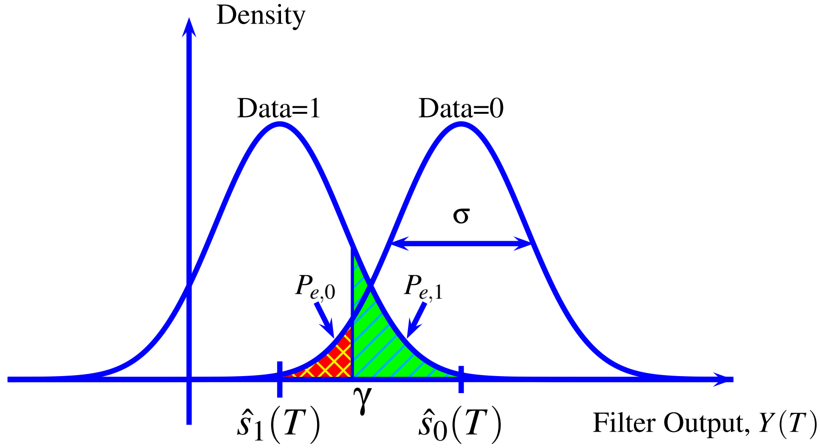


Figure 4.17: Filter Output with Noise

where η is a Gaussian random variable with mean 0 and variance σ_N^2;

$$\sigma_N^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df.$$

Thus

$$\begin{aligned} P_{e,0} &= P\{\hat{s}_0(T)+\eta<\gamma\} \\ &= P\{\eta<\gamma-\hat{s}_0(T)\} \\ &= \Phi\left(\frac{\gamma-\hat{s}_0(T)}{\sigma_N}\right) = Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right). \end{aligned} \tag{4.2}$$

$$\begin{aligned} P_{e,1} &= P\{Y(T)>\gamma|s_1 \text{ transmitted}\} \\ &= P\{\hat{s}_1(T)+\eta>\gamma|s_1 \text{ transmitted}\} \\ &= P\{\eta>\gamma-\hat{s}_1(T)\} \\ &= Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right). \end{aligned} \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1) yields

$$\bar{P}_e(\gamma, h(t), s_0(t), s_1(t)) = \pi_0 Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right). \tag{4.4}$$

The problem is to minimize the error probability over all choices of $\gamma, h(t)$ and $s_0(t), s_1(t)$.

4.2.2 Error Probability Optimization

To optimize the system we proceed in several steps. We first find the optimal threshold that minimizes the error probability for any signals and any filter. Second, we find the optimum filter for any signals. Finally we find the optimum signals.

$$\min_{\gamma, h(t), s_0(t), s_1(t)} \left[\pi_0 Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)\right].$$

$$\begin{aligned}\hat{s}_0(T) &= \int h(\tau)s_0(T-\tau)d\tau = \int H(f)S_0(f)e^{j2\pi fT}df \\ \hat{s}_1(T) &= \int h(\tau)s_1(T-\tau)d\tau = \int H(f)S_1(f)e^{j2\pi fT}df\end{aligned}$$

Step 1: Minimize $\bar{P}_e$ over γ. We will consider the special case, but important case, that $\pi_0 = \pi_1 = 1/2$. See the problems for the more general case.

$$P_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) = \min_{\gamma} \frac{1}{2}Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right) + \frac{1}{2}Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right).$$

We use the following facts to optimize over the threshold γ.

$$\begin{aligned}Q(x) &= \int_x^{\infty} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}du, \\ Q'(x) &= \frac{-e^{-x^2/2}}{\sqrt{2\pi}}.\end{aligned}$$

The method used to optimize over γ is taking the derivative of $\bar{P}_e$ with respect to γ and setting it equal to 0.

$$\frac{d\bar{P}_e}{d\gamma} = \frac{1}{2}\left(\frac{-\exp\{-\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)^2/2\}}{\sqrt{2\pi}}\left(-\frac{1}{\sigma_N}\right)\right) + \frac{1}{2}\left(\frac{-\exp\{-\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)^2/2\}}{\sqrt{2\pi}}\left(\frac{1}{\sigma_N}\right)\right) = 0$$

$$\begin{aligned}\frac{1}{2}\exp\{-\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)^2/2\} &= \frac{1}{2}\exp\{-\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)^2/2\} \\ \exp\left\{\left[(\gamma-\hat{s}_1(T))^2-(\hat{s}_0(T)-\gamma)^2\right]/2\sigma_N^2\right\} &= 1 \\ \gamma^2-2\gamma\hat{s}_1(T)+\hat{s}_1^2(T)-\hat{s}_0^2(T)+2\gamma\hat{s}_0(T)-\gamma^2 &= 0 \\ 2\gamma[\hat{s}_0(T)-\hat{s}_1(T)] &= \hat{s}_0^2(T)-\hat{s}_1^2(T) \\ \gamma &= \frac{1}{2}\left[\frac{\hat{s}_0^2(T)-\hat{s}_1^2(T)}{\hat{s}_0(T)-\hat{s}_1(T)}\right] \\ &= \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2}\end{aligned}$$

$$\boxed{\gamma_{opt} = \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2} \text{ for } \pi_0 = \pi_1} \tag{4.5}$$

This threshold is half way between the outputs for the different signals.

For the optimal value of γ the error probability $\bar{P}_e$ is a function of $h(t), s_0(t)$ and $s_1(t)$.

$$\begin{aligned}\hat{s}_0(T)-\gamma_{opt} &= \hat{s}_0(T)-\left[\frac{\hat{s}_0(T)+\hat{s}_1(T)}{2}\right] \\ &= \frac{\hat{s}_0(T)-\hat{s}_1(T)}{2} \\ \frac{\hat{s}_0(T)-\gamma_{opt}}{\sigma_N} &= \frac{\hat{s}_0(T)-\hat{s}_1(T)}{2\sigma_N}. \qquad (4.6)\end{aligned}$$

Also

$$\frac{\hat{s}_1(T)-\gamma_{opt}}{\sigma_N} = \frac{\hat{s}_0(T)-\hat{s}_1(T)}{2\sigma_N}. \tag{4.7}$$

In order to express the error probability in terms of the transmitted signals and the filter we make a few definitions. We well express the error probability in terms of inner products and norms. From Chapter 2 and for real signals

$$\begin{aligned}(f(t),g(t)) &\triangleq \int_{-\infty}^{\infty} f(t)g(t)dt \\ ||f(t)||^2 &\triangleq \int |f(t)|^2 dt\end{aligned}$$

Now let E_i be the energy of signal $s_i(t)$, $\bar{E}=(E_0+E_1)/2$ the average energy and $r=(s_0,s_1)/\bar{E}$ the normalized inner product of the signals. Let

$$s_T(t) \triangleq s_0(T-t)-s_1(T-t)$$

Then

$$\begin{aligned}\hat{s}_0(T)-\hat{s}_1(T) &= \int_{-\infty}^{\infty} h(\tau)\left[s_0(T-\tau)-s_1(T-\tau)\right]d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)s_T(\tau)d\tau = (h,s_T). \end{aligned} \tag{4.8}$$

Also,

$$\begin{aligned}\|s_T\|^2 &= \int_{-\infty}^{\infty}\left[s_0(T-t)-s_1(T-t)\right]^2 dt \\ &= \int_{-\infty}^{\infty} s_0^2(T-t)-2s_0(T-t)s_1(T-t)+s_1^2(T-t)dt \\ &= \int_{-\infty}^{\infty} s_0^2(t)dt-2(s_0,s_1)+\int_{-\infty}^{\infty} s_1^2(t)dt \\ &= E_0+E_1-2r\bar{E} \\ &= 2\bar{E}(1-r). \end{aligned} \tag{4.9}$$

The noise variance can be written in terms of the norm of $h(t)$.

$$\sigma_N^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0}{2}\|h\|^2. \tag{4.10}$$

Thus from (4.6), (4.7), (4.8) and (4.9)

$$\frac{\hat{s}_0(T)-\gamma_{opt}}{\sigma_N} = \frac{\gamma_{opt}-\hat{s}_1(T)}{\sigma_N} = \frac{(h,s_T)}{2\sigma_N} \tag{4.11}$$

$$= \frac{(h,s_T)}{2\sqrt{N_0/2}||h||} \tag{4.12}$$

$$= \frac{(h,s_T)}{\|h\|\ \|s_T\|}\sqrt{\frac{\bar{E}(1-r)}{N_0}}. \tag{4.13}$$

Let

$$\lambda \triangleq \frac{(h, s_T)}{\|h\|\|s_T\|}$$

and

$$\alpha \triangleq \sqrt{\frac{\bar{E}(1-r)}{N_0}},$$

then

$$\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} = \frac{\gamma_{opt} - \hat{s}_1(T)}{\sigma_N} = \lambda\alpha$$

so

$$\bar{P}_e = Q(\alpha\lambda)$$

Summary of Step 1:

$$\gamma_{opt} = \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}.$$

$$\bar{P}_e\left(\gamma_{opt}, h(t), s_0(t), s_1(t)\right) = Q\left(\alpha\lambda\right).$$

$$\begin{aligned}
\lambda &= \frac{(h, s_T)}{\|h\|\ \|s_T\|}. \\
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \\
\bar{E} &= \frac{E_0 + E_1}{2}, \quad r = (s_0, s_1)/\bar{E}.
\end{aligned}$$

Step 2: Minimize P_e over $h(t)$

Find the optimal filter $h(t)$ to minimize the average probability of error

$$\begin{aligned}
P_e(\gamma_{opt}, h_{opt}(t), s_0(t), s_1(t)) &= \min_h P_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) \\
&= \min_h Q\left(\alpha\lambda\right).
\end{aligned}$$

Method: The error probability is a function of $h(t)$ the impulse response of the filter. However, the error probability depends on $h(t)$ only through the variable λ. First show that $\bar{P}_e$ is an decreasing function of λ by showing the derivative is negative. Then find the h that maximizes λ using Schwartz's inequality and thus minimize $\bar{P}_e$ with respect to $h(t)$.

$$\begin{aligned}
\bar{P}_e(h, s_0, s_1) &= \bar{P}_e(\gamma_{opt}, h, s_0, s_1) \\
&= Q\left(\alpha\lambda\right) \\
\frac{\partial \bar{P}_e}{\partial \lambda} &= \left[-e^{-(\alpha\lambda)^2/2}\frac{1}{\sqrt{2\pi}}\left(\alpha\right)\right] < 0
\end{aligned}$$

since $\alpha > 0$. So $\frac{d\bar{P}_e}{d\lambda} < 0$ which means $\bar{P}_e$ is minimized by maximizing λ.

$$\lambda = \frac{(h, s_T)}{\|h\|\ \|s_T\|}.$$

From Schwartz's inequality (see Appendix 4A)

$$-\|h\|\,\|s_T\| \leq (h, s_T) \leq \|h\|\,\|s_T\|.$$

Thus $-1 \leq \lambda \leq 1$ with equality if $h = s_T$. Choose $\lambda = 1$ and $h_{opt}(t) = s_T(t) = s_0(T-t) - s_1(T-t))$. For the optimal threshold and optimal filter

$$\bar{P}_e(\gamma_{opt}, h_{opt}, s_0, s_1) = Q(\alpha) = Q\left(\sqrt{\frac{\bar{E}(1-r)}{N_0}}\right).$$

The optimum filter $h_{opt}(t) = s_0(T-t) - s_1(T-t)$ is called the matched filter because it is matched to the signals. For the optimum filter, the optimum threshold becomes

$$\gamma_{opt}(h_{opt}, s_0, s_1) = \frac{1}{2}(E_0 - E_1).$$

and the outputs due to signal alone are

$$\hat{s}_0(T) = E_0 - r\bar{E}$$
$$\hat{s}_1(T) = r\bar{E} - E_1.$$

Step 3: Minimize P_e over $s_0(t)$ and $s_1(t)$.

Find the optimal signals $s_{0,opt}(t)$ and $s_{1,opt}(t)$ to minimize the average probability of error.

$$\begin{aligned} P_e(\gamma_{opt}, h_{opt}(t), s_{0,opt}(t), s_{1,opt}(t)) &= \min_{s_0,s_1} P_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) \\ &= \min_{s_0,s_1} Q(\alpha). \end{aligned}$$

Method: $\bar{P}_e$ depends on the signal only through $\bar{E}$ and r.

$$\bar{E} = \frac{E_0 + E_1}{2}, \quad r = (s_0, s_1)/\bar{E}.$$

It is obvious that we could just increase the energy to infinity and get error probability 0. Instead we will fix $\bar{E}$ and vary the signals to vary r. We show that $\bar{P}_e$ is an increasing function of r and then choose the signals to minimize r.

$$\begin{aligned} \bar{P}_e &= Q(\alpha) \\ \alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}. \\ \frac{d\bar{P}_e}{dr} &= \frac{-e^{-(\alpha)^2/2}}{\sqrt{2\pi}}(\frac{\partial \alpha}{\partial r}) \\ &= -\frac{e^{-(\alpha^2)/2}}{\sqrt{2\pi}}\left(\frac{\partial \alpha}{\partial r}\right) \\ \frac{\partial \alpha}{\partial r} &= \sqrt{\frac{\bar{E}}{N_0}}\frac{1}{2}\left(\frac{-1}{\sqrt{1-r}}\right) < 0 \\ &\Rightarrow \frac{d\bar{P}_e}{dr} > 0 \end{aligned}$$

Thus the error probability is an increasing function of r. To find the smallest value of r we employ Schwartz's inequality again:

$$r = \frac{(s_0, s_1)}{\bar{E}} \geq \frac{-\|s_0\|\,\|s_1\|}{\bar{E}}$$

with equality if $s_0 = -Ks_1,\ \ K > 0$. For $s_0 = -Ks_1$

$$\begin{aligned} r &= -\frac{\sqrt{E_0 E_1}}{\left(\frac{E_0+E_1}{2}\right)} \\ &\geq -1 \end{aligned}$$

with equality if $E_0 = E_1$. (Arithmetic mean $\geq$ Geometric mean). Two signals $s_0(t)$ and $s_1(t)$ are said to be antipodal if

$$s_0(t) = -s_1(t).$$

So the optimal signals are antipodal and

$$r = -1 \quad \Rightarrow \quad \alpha = \sqrt{\frac{2E}{N_0}}$$

$$P_e(\gamma_{opt}, h_{opt}(t), s_{0,opt}(t), s_{1,opt}(t)) = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

Summary: For arbitrary signals $(s_0(t), s_1(t))$, arbitrary filter $(h(t))$, arbitrary threshold (γ) the error probability for equally likely $(\pi_0 = \pi_1 = 1/2)$ signals is

$$\bar{P}_e(\gamma, h_(t), s_0(t), s_1(t)) = 1/2Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right) + 1/2Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma_N}\right)$$

Step 1: Optimize with respect to γ:

$$\begin{aligned} \gamma_{opt} &= \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2} \\ \bar{P}_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) &= Q(\alpha\lambda), \\ \alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \lambda = \frac{(h, s_T)}{\|h\|\ \|S_T\|} \\ \bar{E} &= \frac{E_0+E_1}{2}, \quad r = (s_0, s_1)/\bar{E} \\ E_0 &= \int_{-\infty}^{\infty} s_0^2(t)dt, \quad E_1 = \int_{-\infty}^{\infty} s_1^2(t)dt. \end{aligned}$$

Step 2: Optimize with respect to $h(t)$:

$$\begin{aligned} h_{opt}(t) &= s_0(T-t) - s_1(T-t) \\ \bar{P}_e(\gamma_{opt}, h_{opt}(t), s_0(t), s_1(t)) &= Q(\alpha), \\ \alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \\ \gamma_{opt} &= 1/2(E_0 - E_1). \end{aligned}$$

Step 3: Optimize with respect to $s_0(t)$ and $s_1(t)$.

$$\begin{aligned} s_{0,opt}(t) &= -s_{1,opt}(t) \\ \bar{P}_e(\gamma_{opt}, h_{opt}(t), s_{0,opt}(t), s_{1,opt}(t)) &= Q\left(\sqrt{\frac{2\bar{E}}{N_0}}\right), \\ h_{opt}(t) &= 2s_0(T-t), \\ \gamma_{opt} = 0. \end{aligned}$$

4.2.3 Equivalent form of optimal receiver

There are several ways the optimal receiver can be implemented. Because the matched filter has impulse response $h(t) = s_0(T-t) - s_1(T-t)$ this can be implemented with two filters as shown in Figure 4.18. Another form of the

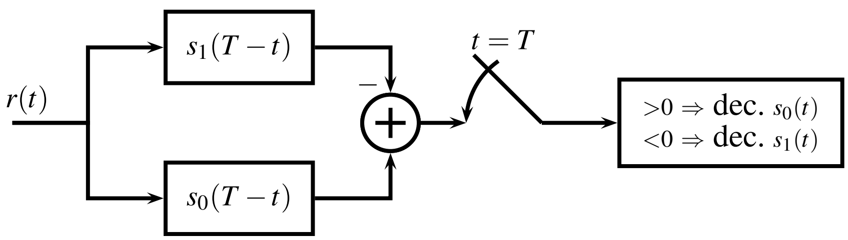


Figure 4.18: Alternate form of optimal receiver

optimal receiver can be found be examining the output of the filter at the appropriate sampling time.

$$\begin{aligned} Z(t) &= \int_{-\infty}^{\infty} h(t-\tau)r(\tau)d\tau, \\ &\quad h(t) = s_0(T-t) - s_1(T-t) \\ &= \int_{-\infty}^{\infty} r(\tau)\left[s_0(T-(t-\tau)) - s_1(T-(t-\tau))\right]d\tau \\ &= \int_{-\infty}^{\infty} r(\tau)\left[s_0(\tau+T-t) - s_1(\tau+T-t)\right]d\tau \\ Z(T) &= \int_{-\infty}^{\infty} r(\tau)\left[s_0(\tau) - s_1(\tau)\right]d\tau \end{aligned}$$

If $s_0(t)$ and $s_1(t)$ are time limited to $[0,T]$ then

$$Z(T) = \int_0^T r(\tau)\left[s_0(\tau) - s_1(\tau)\right]d\tau$$

The implementation of this, known as a correlation receiver, is shown in Figure 4.19. It is important to recognize

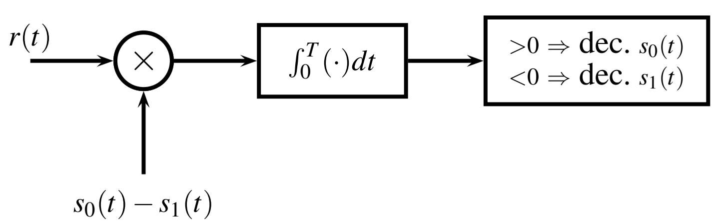


Figure 4.19: Correlation receiver

that the signal $s_0(t) - s_1(t)$ at the input to the mixer (multiplier) must be synchronized in time to the desired signal. This is known as synchronization.

4.2.4 Examples

In this section we go through several examples of signals and receivers. First we consider the simple rectangular pulses. Second we consider binary phase shift keying.

Antipodal Rectangular Pulses

Consider the two equally likely signals ($\pi_0 = \pi_1$)

$$\begin{aligned} s_0(t) &= +\sqrt{P}p_T(t) \\ s_1(t) &= -\sqrt{P}p_T(t). \end{aligned}$$

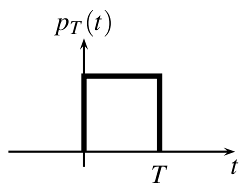
These signals are optimal for minimizing the probability of error. In this case the optimal threshold and filter are

$$\begin{aligned} \gamma_{opt} &= 0 \\ h_{opt}(t) &= 2Ap_T(t). \end{aligned}$$

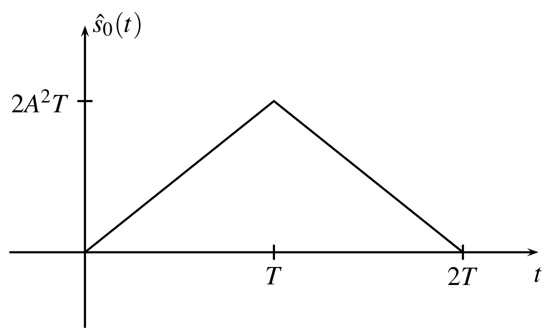
Note that any gain associated with the filter does not change the error probability. So the factor of $2A$ in $h_{opt}(t)$ could be dropped. This is because the gain affects the desired signal and the noise in the same way and does not change the signal-to-noise ratio.

Consider the output of the receiver due to $s_0(t)$ being transmitted and no noise.

$$\begin{aligned} \int_{-\infty}^{\infty} h(t-\tau)s_0(\tau) &= \int_{-\infty}^{\infty} 2Ap_T(t-\tau)Ap_T(\tau)d\tau \\ &= 2A^2\int_{-\infty}^{\infty} p_T(t-\tau)p_T(\tau)d\tau \\ &= 2A^2\int_{t-T}^{t} p_T(\tau)d\tau \end{aligned}$$



The output due to signal alone is



The output due to noise is a Gaussian random variable with mean zero and variance

$$\sigma_N^2 = \frac{1}{2}N_0T(4A^2) = 2A^2N_0T$$

Let T_0 be the sampling time. Since the signal out is a maximum when $T_0 = T$ and the noise variance does not depend on the sample time the optimum sampling time is $T_0 = T$.

Binary Phase Shift Keying (RF signals)

As a second example consider the two signals below and shown in Figure 4.20 for the case of $f_c = 12/T$.

$$\begin{aligned} s_0(t) &= A\cos(2\pi f_c t)p_T(t) \\ s_1(t) &= -s_0(t) \\ s_i(t) &= (-1)^i A\cos(2\pi f_c t)p_T(t) \\ &= A\cos(2\pi f_c t + i\pi)p_T(t) \end{aligned}$$

These signals are known as binary phase shift keying (BPSK) since the information is communicated by changing

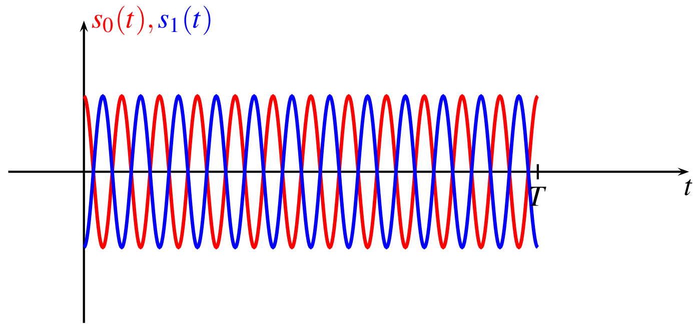


Figure 4.20: Binary phase shift keying (BPSK) signals

the phase of the carrier frequency. The transmitter can be implemented as shown in Figure 4.21. Here b_0 is either +1 or -1 and determines whether the transmitted signal is $s_0(t)$ or $s_1(t)$, respectively. The optimal receiver, based

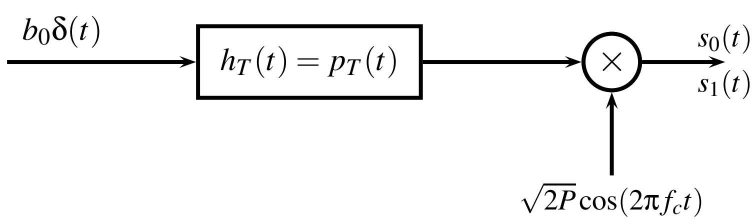


Figure 4.21: BPSK Transmitter

on the correlation receiver, is shown in Figure 4.22. For this receiver to be optimal the phase of the sinusoidal signal at the input to the mixer must match the phase of the received signal due to the transmitted signal $s_0(t)$, or equivalently be 180 degrees out of phase with the received signal due to the transmitted signal $s_1(t)$. This is known as a coherent receiver. Phase synchronization is needed for a coherent receiver. Also, the frequency f_c used in the receiver must match the frequency of the transmitted signal. Frequency synchronization is needed to achieve this goal. Finally, the timing of the integration must match the starting and ending of the received signal due to the transmitted signal. This is time synchronization. Notice that the receiver in Figure 4.22 first mixes to baseband by multiplying by $\cos(2\pi f_c t)$ followed by an integrator which is a low pass filter. This is the same structure we

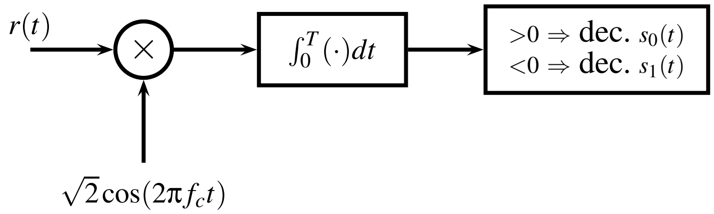


Figure 4.22: Correlation receiver for BPSK

discussed in Section 2.9 regarding baseband and passband signals. Assume $f_c T >> 1$ or $2\pi f_c T = 2n\pi$

$$
\begin{aligned}
E_i = \int_{-\infty}^{\infty} s_i^2(t)dt &= \int_0^T A^2\cos^2(2\pi f_c t)dt \\
&= A^2\int_0^T 1/2 + 1/2\cos(2\pi 2 f_c t)dt \\
&= \frac{A^2T}{2}[1+\frac{\sin(2\pi 2 f_c T)}{2\pi 2 f_c T}] \\
&= \frac{A^2T}{2}.
\end{aligned}
$$

$$
P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{N_0}}\right) \quad (\pi_1 = \pi_0, \gamma = 0).
$$

4.3 Suboptimal Receivers

In this section we analyze the performance of a communication system with a receiver that is not optimal because the filter is not matched to the signals. The signals are rectangular pulses (antipodal) but the receiver is a low pass RC filter. The time constant of the filter or the filter bandwidth can be changed to change how much signal and how much noise are passed. The filter can be optimized with respect to the bandwidth/time constant in order to maximize the ratio of the amount of signal passing to the noise power at the output of the filter. We first analyze this for the case of a single bit transmitted and optimize over the filter parameter. Second we consider multiple bits transmitted and consider the effect of intersymbol interference.

Suppose that $h(t)$ in Figure 4.16 is not the optimal matched filter but a suboptimal filter, perhaps a more easily implementable filter. Suppose the signals correspond to BPSK, namely

$$
s_i(t) = (-1)^i\sqrt{P}p_T(t), \quad i = 0, 1.
$$

The signal power is P and the energy is $E = PT$. Suppose the decision rule is still to compare the filter output with 0 which is the optimal threshold for equally likely antipodal signals with an arbitrary filter. The error probability in this case is

$$
P_e = 1/2Q\left(\frac{\hat{s}_0(T_0)}{\sigma_N}\right) + 1/2Q\left(\frac{-\hat{s}_1(T_0)}{\sigma_N}\right)
$$

where $\sigma_n^2 = \frac{N_0}{2}\|h\|^2$, $\hat{s}_i(T),\ i = 0,1$ is the output of the filter at time T. Suppose the filter is a simple resistor-capacitor filter as shown in Figure 4.23. The impulse response of the RC filter is

$$
\begin{aligned}
h(t) &= \frac{1}{RC}e^{-t/RC}u(t) \\
&= \alpha e^{-\alpha t}u(t),
\end{aligned}
$$

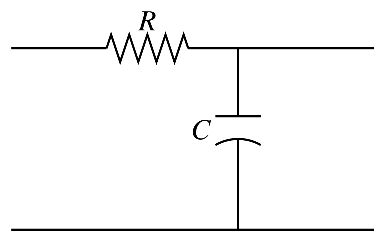


Figure 4.23: RC Filter

where $\alpha = 1/RC$. Consider first the output due to noise. The variance of the noise at the output of the filter can be calculated as follows.

$$
\begin{aligned}
\|h\|^2 &= \int_{-\infty}^{\infty} h^2(t)dt = \int_0^{\infty} \alpha^2 e^{-2\alpha t} dt \\
&= \frac{\alpha^2}{-2\alpha} e^{-2\alpha t}|_0^{\infty} = \frac{\alpha}{2} \\
\sigma_N^2 &= \frac{N_0}{2}||h||^2 \\
&= \frac{N_0}{2}(\frac{\alpha}{2}) = \frac{N_0\alpha}{4}.
\end{aligned}
$$

The larger RC the smaller the bandwidth of the filter. A smaller bandwidth requires more time to charge capacitor. Large RC means small α. So small α means small bandwidth and small noise power Larger α means larger noise variance because the bandwidth is larger.

Now consider the output due to a signal. The output of the filter at time T_0 due to a single transmitted pulse from 0 to T can be calculated as follows. The transmitted pulse has power P over a duration T seconds so the energy in the pulse is $E = PT$.

$$
\begin{aligned}
\hat{s}_0(t) &= \int_{-\infty}^{\infty} h(t-\tau)s_0(\tau)d\tau \\
&= \sqrt{P}\int_0^T h(t-\tau)d\tau \\
\hat{s}_0(T_0) &= \sqrt{P}\int_0^T h(T_0-\tau)d\tau \\
&= \sqrt{P}\int_0^T \alpha e^{-\alpha(T_0-\tau)}u(T_0-\tau)d\tau.
\end{aligned}
$$

Note that

$$
u(T_0-\tau) = \begin{cases} 0, & T_0-\tau<0 \quad (\tau>T_0) \\ 1, & T_0-\tau>0 \quad (\tau<T_0). \end{cases}
$$

Thus

$$\int_0^T h(T_0-\tau)d\tau = \begin{cases} 0, & T_0<0 \\ \int_0^{T_0} \alpha e^{-\alpha(T_0-\tau)}d\tau, & 0\le T_0\le T \\ \int_0^{T} \alpha e^{-\alpha(T_0-\tau)}d\tau, & T_0>T \end{cases}$$

$$= \begin{cases} 0 & T_0<0 \\ +e^{-\alpha(T_0-\tau)}|_0^{T_0}, & 0\le T_0\le T \\ +e^{-\alpha(T_0-\tau)}|_0^{T}, & T_0\ge T \end{cases}$$

$$= \begin{cases} 0, & T_0<0 \\ 1-e^{-\alpha T_0}, & 0\le T_0\le T \\ (1-e^{-\alpha T})e^{-\alpha(T_0-T)}, & T_0>T \end{cases}$$

The output of the filter is largest at time T.

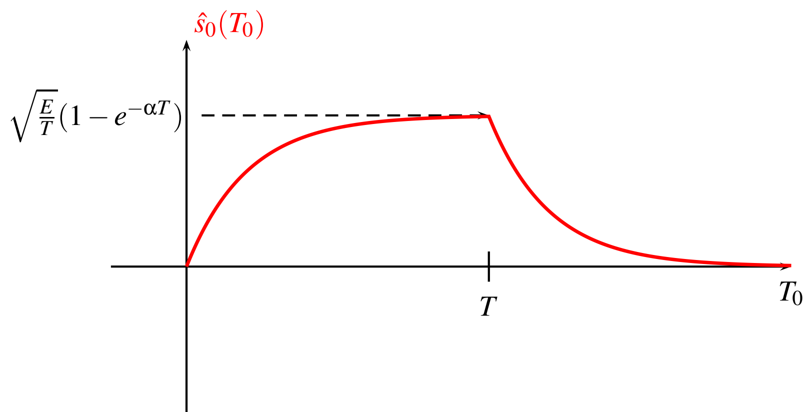


Figure 4.24: Response of an RC filter due to a pulse beginning at $t=0$ ending at time T.

$$\begin{aligned} \hat{s}_0(T) &= b_0\sqrt{PT}(1-e^{-\alpha T})/\sqrt{T} \\ &= b_0\sqrt{E}(1-e^{-\alpha T})/\sqrt{T}. \end{aligned}$$

Figure 4.25 shows the response of the filter due to a single pulse and the transfer function of the filter. The left curve below shows the time response of the filter due to a pulse. The right plot shows the frequency response of the filter. Note that the larger value of α gives a larger response to a pulse. The output due to noise also varies with α (bandwidth of filter). Larger α means larger bandwidth of the filter which means more noise passes through the filter. We would like to maximize the ratio $|\hat{s}_0(T_0)|/\sigma_n^2$. Since $|\hat{s}_0(T_0)|$ is maximized at $T_0=T$ and $\sigma_N^2=\frac{N_0}{4}\frac{\alpha}{2}$ does not depend on the sampling time, the optimal sampling time is $T_0=T$. This results in a signal-to-noise ratio of

$$\begin{aligned} SNR &= \frac{|\hat{s}_0(T)|}{\sigma_N} \\ &= \sqrt{\frac{E}{T}}(1-e^{-\alpha T})\sqrt{\frac{4}{N_0\alpha}} \\ &= \sqrt{\frac{4E}{N_0}}\frac{(1-e^{-\alpha T})}{\sqrt{\alpha T}}. \end{aligned}$$

Optimization of Filter Parameter

Figure 4.26 shows the dependence of the signal and the noise term on α in the left side figure and shows the dependence of the signal-to-noise ratio on the right hand side of the figure. The goal of the receiver design is to maximize the signal-to-noise ratio with respect to α. Let $x=\alpha T$. Then the signal-to-noise ratio depends on α

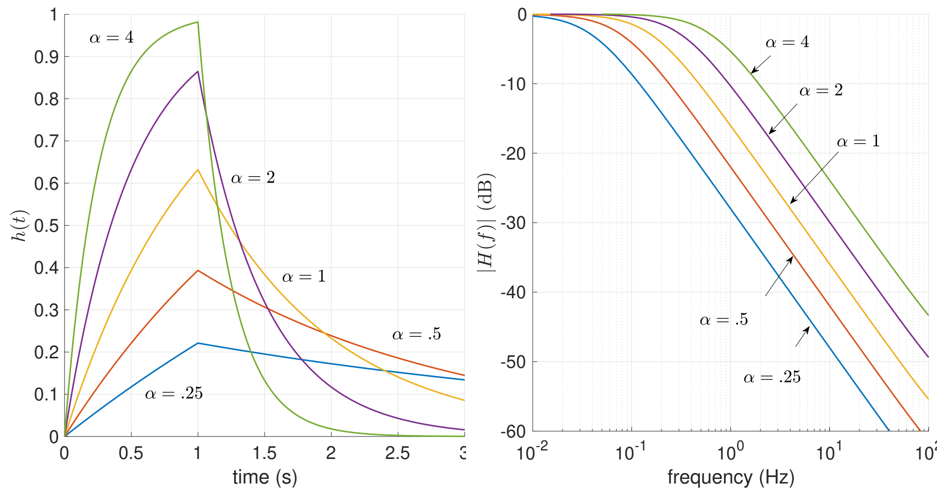


Figure 4.25: Filter output.

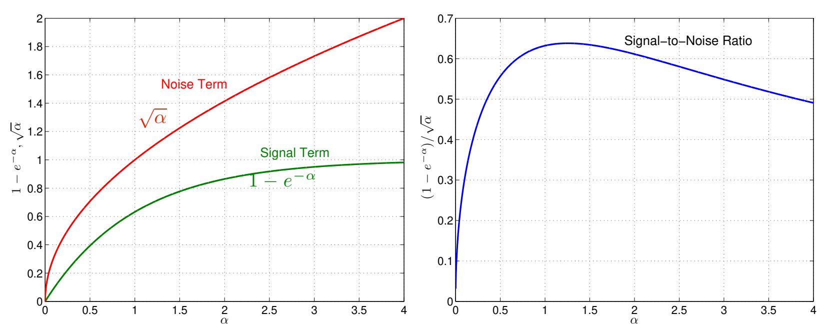


Figure 4.26: Optimization of Filter Parameter

through x. We can maximize the signal-to-noise ratio by maximizing $f(x)$ with respect to x.

$$
\begin{aligned}
f(x) &= (1-e^{-x})(x)^{-1/2} \\
f'(x) &= e^{-x}x^{-1/2} - 1/2(1-e^{-x})x^{-3/2} = 0 \\
& \quad xe^{-x} - 1/2(1-e^{-x}) = 0 \\
& \quad e^{-x}(x+1/2) = 1/2 \\
& \quad x + 1/2 = e^{x}/2 \\
& \quad 2x = e^{x} - 1.
\end{aligned}
$$

We can numerically solve this to get $x = 1.256$. So $\alpha = \frac{1.256}{T}, \Rightarrow RC = .7962T$. This yields a signal-to-noise ratio of

$$SNR = \sqrt{\frac{2E}{N_0}}\sqrt{2}(0.6382).$$

The signal-to-noise ratio with the (optimum) matched filter is

$$SNR_{\text{opt}} = \sqrt{\frac{2E}{N_0}}.$$

The loss due to the suboptimal receiver = $-10\log_{10}((0.6832\sqrt{2})^2)$=0.89 dB.

Now consider instead of just a single bit transmitted a sequence of bits being transmitted. To analyze this system we consider the output just due to noise and the output just due to signal. The overall output is the sum of the two individual outputs. First consider the output due to just a single transmitted pulse. That is

$$s(t) = \sqrt{P}b_0 p_T(t) = \sqrt{E}b_0\varphi_0(t)$$

where $\varphi_0(t) = \sqrt{1/T}\,p_T(t)$ and is a normalized (energy=1) signal. The output of the filter with normalized to unit energy impulse response $h(t) = \sqrt{2\alpha}e^{-\alpha t}u(t)$ due to this pulse is

$$\begin{aligned}
y_0(t) &= \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \\
&= \int_0^T h(t-\tau)\sqrt{P}b_0 d\tau \\
&= b_0\sqrt{P}\int_0^T \sqrt{2\alpha}e^{-\alpha(t-\tau)}u(t-\tau)d\tau \\
&= \begin{cases} b_0\sqrt{P}\sqrt{2\alpha}\int_0^t e^{-\alpha(t-\tau)}d\tau & t<T \\ b_0\sqrt{P}\sqrt{2\alpha}\int_0^T e^{-\alpha(t-\tau)}d\tau & t>T \end{cases} \\
&= \begin{cases} b_0\sqrt{P}\sqrt{2\alpha}e^{-\alpha t}\int_0^t e^{\alpha\tau}d\tau & t<T \\ b_0\sqrt{P}\sqrt{2\alpha}e^{-\alpha t}\int_0^T e^{\alpha\tau}d\tau & t>T \end{cases} \\
&= \begin{cases} b_0\sqrt{P}\sqrt{2\alpha}e^{-\alpha t}\frac{1}{\alpha}(e^{\alpha t}-1) & t<T \\ b_0\sqrt{P}\sqrt{2\alpha}e^{-\alpha t}\frac{1}{\alpha}(e^{\alpha T}-1) & t>T \end{cases} \\
&= \begin{cases} b_0\sqrt{PT}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha t}) & 0\le t\le T \\ b_0\sqrt{PT}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{-\alpha(t-T)} & t\ge T \end{cases} \\
&= \begin{cases} b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha t}) & 0\le t\le T \\ b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{-\alpha(t-T)} & t\ge T. \end{cases}
\end{aligned}$$

The output at time $t = T, 2T, \ldots$ due to a single pulse transmitted from time 0 to T is

$$\begin{aligned}
y_0(T) &= b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T}) = b_0\sqrt{E}h_0 \\
y_0(2T) &= b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{-\alpha T} = b_0\sqrt{E}h_1 \\
y_0(3T) &= b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{-\alpha 2T} = b_0\sqrt{E}h_2
\end{aligned}$$

where

$$\begin{aligned}
h_0 &= \sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T}) \\
h_1 &= \sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{\alpha T} \\
h_2 &= \sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{\alpha 2T}
\end{aligned}$$

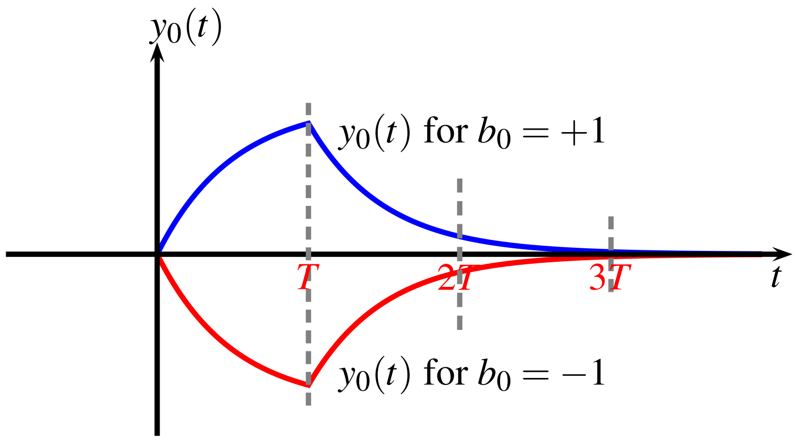


Figure 4.27: Output of RC filter due to single pulse input starting at time 0

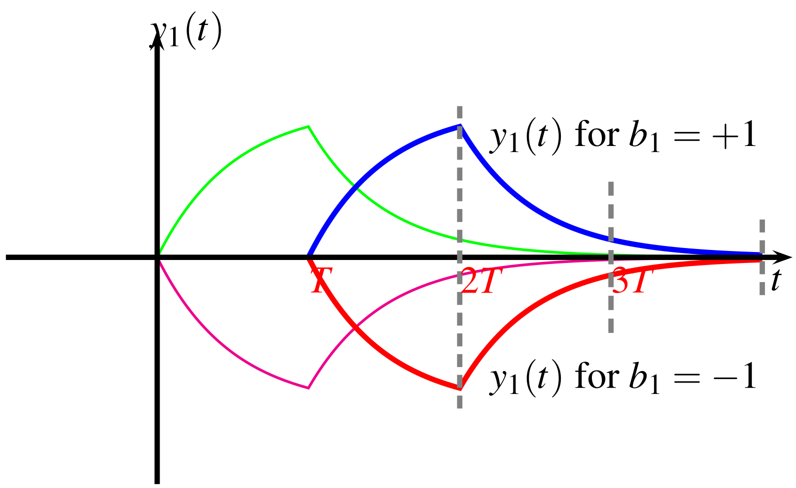


Figure 4.28: Output of RC filter due to single pulse input starting at time T

The output due to a single pulse is shown in Figure 4.27. Now consider the case of transmitting two bits of information. The first bit b_0 is transmitted using a pulse of duration T seconds starting at time 0 and ending at time T with amplitude $b_0\sqrt{P}$. The second pulse is transmitted using a pulse starting at time T and ending at time $2T$. of amplitude $b_1\sqrt{P}$. The receiver uses a filter with impulse response $h(t) = e^{-\alpha t}u(t)$. Because of linearity the output due to two consecutive pulses is the sum of the outputs due to each pulse. Because of time-invariance, the output due to the second pulse is a time delayed version of the output due to the first pulse. The outputs due $b_0 = +1$ and $b_0 = -1$ are shown in Figure 4.27. We are interested in the outputs at time lT. These outputs are

$$
\begin{aligned}
y(lT) &= \begin{cases} b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T}), & l = 1 \\ b_0\sqrt{E}\sqrt{\frac{2}{\alpha T}}(1-e^{-\alpha T})e^{-\alpha(l-1)T}, & l \geq 2 \end{cases} \\
&= \begin{cases} b_0\sqrt{E}h_0, & l = 1 \\ b_0\sqrt{E}h_{l-1}, & l \geq 2. \end{cases}
\end{aligned}
$$

Now consider the output due to bit b_1. This is shown in Figure 4.28. This is just a time delayed version of the output due to bit b_0.

Now consider the probability of error for bit b_1. We assume that the bits b_0 and b_1 are independent and equally

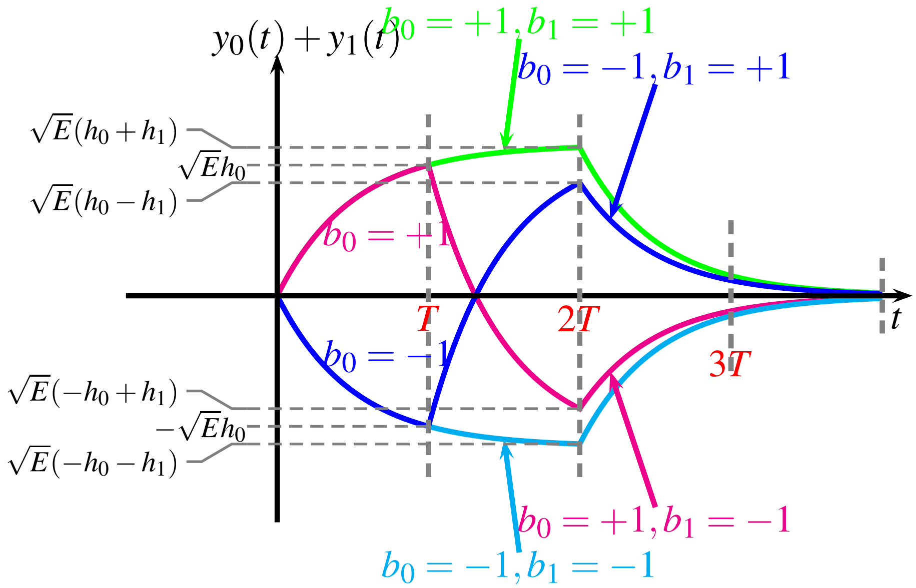


Figure 4.29: Output of RC filter due to two pulses

likely to be +1 and -1.

$$
\begin{aligned}
P_{e,+1}(1) &= P\{\text{error for bit } b_1|b_1=+1\} \\
&= P\{y(2T)<0|b_1=+1\} \\
&= P\{b_0\sqrt{E}h_1+b_1\sqrt{E}h_0+\eta_1<0|b_1=+1\} \\
&= P\{b_0\sqrt{E}h_1+b_1\sqrt{E}h_0+\eta_1<0|b_0=+1,b_1=+1\}\frac{1}{2} \\
&\quad +P\{b_0\sqrt{E}h_1+b_1\sqrt{E}h_0+\eta_1<0|b_0=-1,b_1=+1\}\frac{1}{2} \\
&= P\{\sqrt{E}(h_1+h_0)+\eta_1<0|b_0=+1,b_1=+1\}\frac{1}{2}+P\{\sqrt{E}(-h_1+h_0)+\eta_1<0|b_0=-1,b_1=+1\}\frac{1}{2} \\
&= P\{\eta_1<\sqrt{E}(-h_0-h_1)\}\frac{1}{2}+P\{\eta_1<\sqrt{E}(-h_0+h_1)\}\frac{1}{2} \\
&= \frac{1}{2}Q(\frac{\sqrt{E}}{\sigma}(h_0+h_1))+\frac{1}{2}Q(\frac{\sqrt{E}}{\sigma}(h_0-h_1)) \\
&= \frac{1}{2}Q(\sqrt{\frac{2E}{N_0}}(h_0+h_1))+\frac{1}{2}Q(\sqrt{\frac{2E}{N_0}}(h_0-h_1)).
\end{aligned}
$$

Here $h_0=\sqrt{2/(\alpha T)}(1-e^{-\alpha T})$ and $h_1=\sqrt{2/(\alpha T)}(1-e^{-\alpha T})e^{-\alpha T}$. This shows that the error probability is sometimes bigger (when the previous bit is the same as the current bit) and sometimes smaller (when the previous bit is different than the current bit). The error probability for the first bit b_0 when there are no transmitted bits before time 0 is

$$P_{e,+1}(0)=Q(\sqrt{\frac{2E}{N_0}}h_0).$$

In Figure 4.30 the error probability for the first bit (with no ISI) and the second bit (with one symbol of ISI) is shown. Adding more bits of intersymbol interference does not change significantly the error probability. Also shown in Figure 4.30 is the performance with the optimum filter $h_R(t)=p_T(t)$. Sometimes the ISI improves the error probability (when the previous symbol is the same sign as the current symbol). Sometimes the ISI degrades the error probability (when the previous symbol is a different sign than the current symbol). Overall the

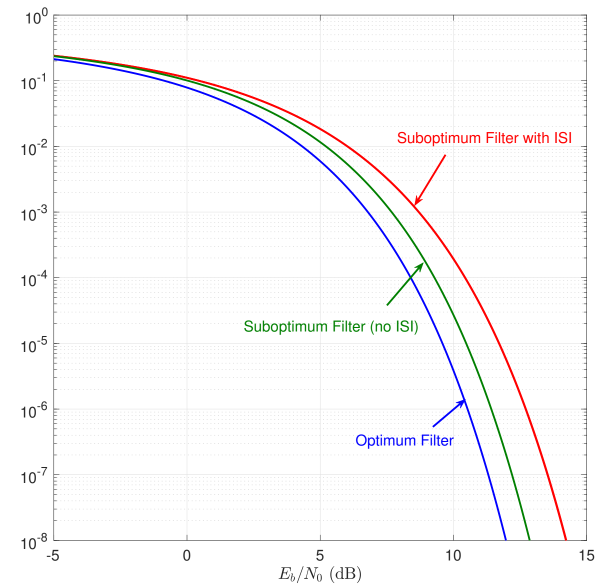


Figure 4.30: Error probability with and without intersymbol interference

average performance degrades. Note that for a single pulse the optimum filter parameter derived above results in $RC = 0.7962T$. However, with multiple pulses the optimum parameter $\alpha = 1/RC$ will be different with different values of E_b/N_0. A larger value of $\alpha = 1/RC$ has less intersymbol interference, larger desired signal but, as before, larger noise. However the fact that the intersymbol interference decreases with α results in a larger value of α being optimum.

4.3.1 Eye Diagram

Often the effect of suboptimal filters and imperfect sampling is best illustrated with what is called the eye diagram. The eye diagram is obtained by examining the filter output and displaying many traces of finite duration (say $2T$). Imagine examining the filter output on an oscilloscope. In Figure 4.31 we show the eye diagram for the case of a rectangular pulse shape and the optimal receiver. The horizontal axis is time and the vertical axis represents the filter output. In this diagram the output due to two consecutive pulse is shown. The pulse shape here is a rectangular pulse and the filter is the matched filter, that is the filter is a sliding integrator with integration window of T seconds corresponding the the length of the transmitted pulse. Since the pulse is rectangular and the filter is matched (also a rectangular pulse) the output of the matched filter due to a single input pulse is a triangular shape.

Now consider replacing the matched filter with a simple RC lowpass filter. The smaller the bandwidth of the filter the less noise that will appear at the filter output. With smaller filter bandwidth however, less signal will also pass through the filter and there will be more intersymbol interference. The normalized impulse response of the filter is

$$h(t) = \sqrt{2\alpha}e^{-\alpha t}u(t).$$

Here $u(t)$ is the unit step function that is 0 before time 0 and 1 after time 0. It is easy to verify that $h(t)$ has unit energy. Thus the output of the filter when the input is just noise is a Gaussian random variable with mean 0 and variance 1. The parameter α controls the bandwidth of the filter. However, it is the bandwidth relative to the data rate that is important. As we will see it is the product of α with T that controls the signal-to-noise ratio. The above analysis assumed a single pulse of duration T was transmitted in order to communicate a single bit of information.

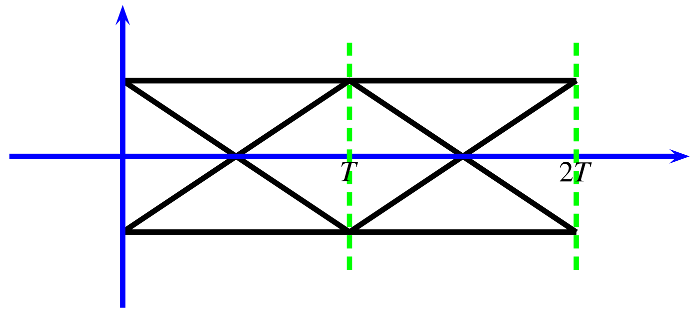


Figure 4.31: Eye diagram for rectangular pulses and the optimal matched filter

Now consider the case where an long stream of data bits is transmitted using a sequence of delayed pulses. The output of the filter for various values of α are shown in the next few figures.

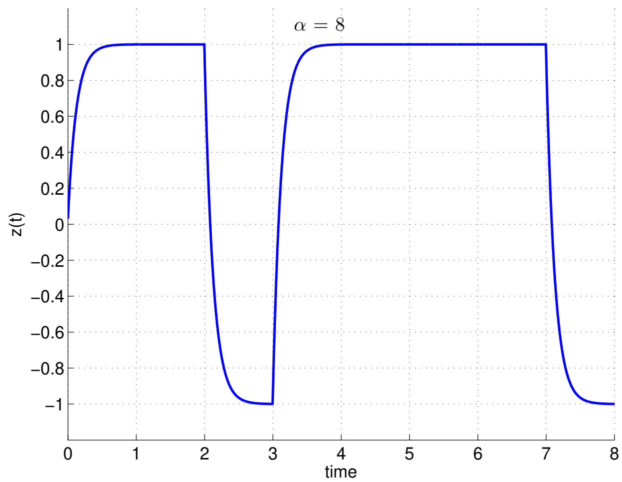


Figure 4.32: Intersymbol interference ($T = 1$)

The smaller α the more intersymbol interference.

$$Z(T) = \underbrace{b_0\sqrt{E}(1-e^{-\alpha T})/T}_{\text{Desired Signal}} + \underbrace{b_{-1}\sqrt{E}(1-e^{-\alpha T})e^{-\alpha T}/T + b_{-2}\sqrt{E}(1-e^{-\alpha T})e^{-\alpha 2T}/T + \cdots}_{\text{Intersymbol interference}} + \underbrace{\eta}_{\text{Noise}}$$

The larger α the smaller the amount of inter symbol interference. The larger α the larger amount of noise that gets through the filter. There is an optimal α (different from what we previously found that ignored inter symbol interference) that minimizes the error probability.

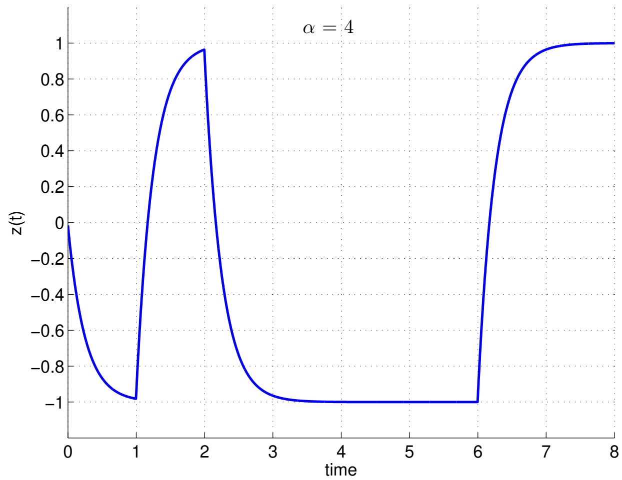


Figure 4.33: Intersymbol interference ($T = 1$)

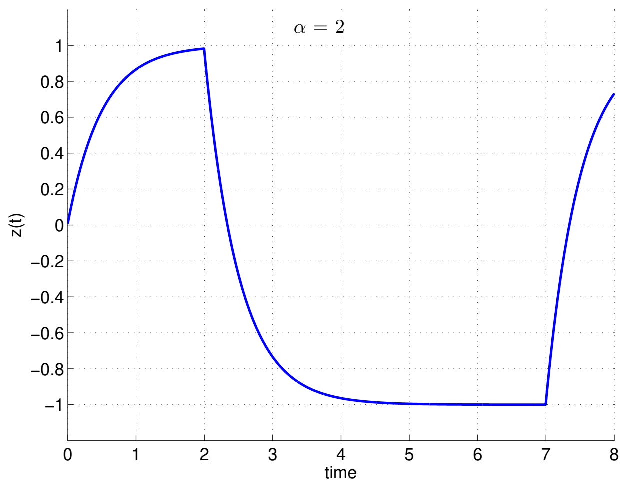


Figure 4.34: Intersymbol interference ($T = 1$)

4.4 Summary of Chapter 4 Concepts

- The transmitted signal consisted of a sequence of pulses (e.g. rectangular pulses or square-root raised cosine pulses).
- The channel adds white Gaussian noise.
- The receiver filters the received signal to remove as much noise as possible without removing the signal.
- We can calculate the output due to signal alone and the output due to noise alone then add the results to get the total output.
- The bit error probability depends on the ratio of the energy of the received desired signal and the noise power spectral density.
- The optimum filter is matched to the desired pulse shape used at the transmitter. That is $h(t) = s_0(T-t) - s_1(T-t)$ is the optimum filter.

$$P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

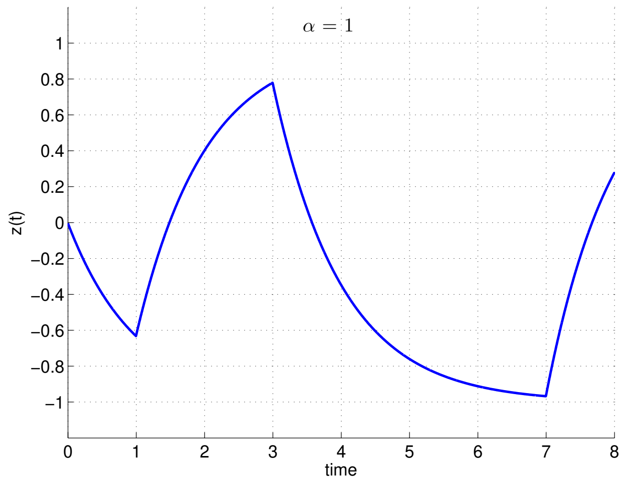


Figure 4.35: Intersymbol interference ($T = 1$)

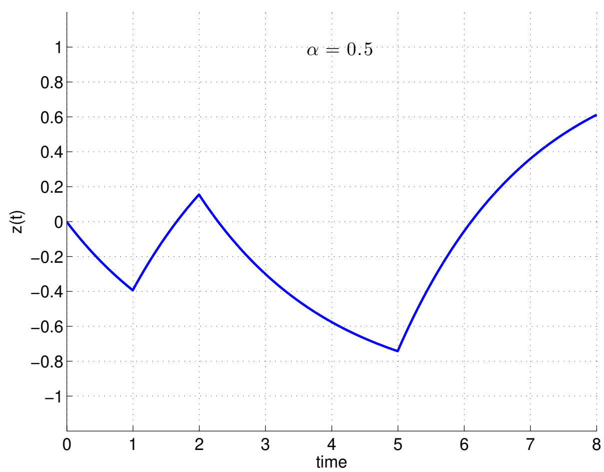


Figure 4.36: Intersymbol interference ($T = 1$)

- Nonideal filters at the receiver can cause intersymbol interference.
- Intersymbol interference on average degrades the system performance.

4.5 Appendix 4A: Schwartz's inequality:

Let $f(t)$ and $g(t)$ be any (finite energy) but real (not complex) functions. Let

$$||f||^2 = \int f^2(t)dt, \quad (f,g) = \int f(t)g(t)dt$$

Claim:

$$-||f||\,||g|| \quad \leq \quad (f,g) \leq ||f||\,||g||$$

with equality if $f(t) = kg(t)$ for some constant k.

As an example consider $f(t) = p_T(t)$ and $g(t) = p_T(t-\tau)$, $0 \leq \tau \leq T$.

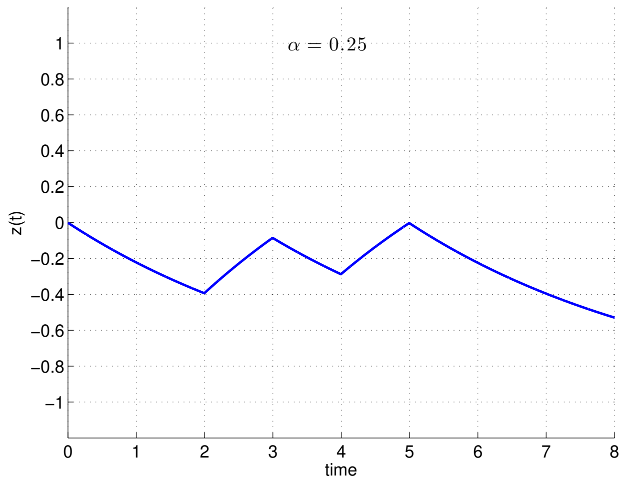


Figure 4.37: Intersymbol interference ($T = 1$)

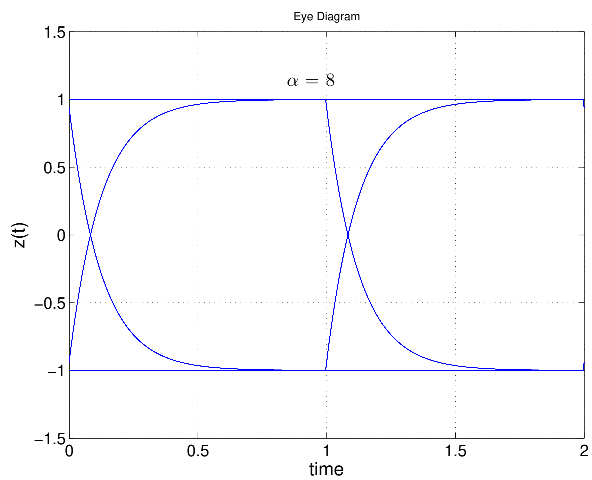


Figure 4.38: Intersymbol Interference ($T = 1$)

$$||f||^2 = \int f^2(t)dt = A^2T, \quad ||g||^2 = \int g^2(t)dt = B^2T$$
$$(f,g) = \int f(t)g^*(t)dt = AB(T-\tau), 0 \le \tau \le T$$
$$-||f|| \cdot ||g|| \le (f,g) \le ||f|| \cdot ||g|| \Longrightarrow -ABT \le AB(T-\tau) \le ABT, 0 \le \tau \le T$$

Proof: For any α

$$||f - \alpha g||^2 \ge 0$$
$$||f||^2 - 2\alpha(f,g) + \alpha^2||g||^2 \ge 0.$$

Since the polynomial in α is never negative there must be either no zeros or a double zero of the quadratic in α. Thus the discriminant must be not be positive.

$$\begin{aligned} 4(f,g)^2 - 4||f||^2||g||^2 &\le 0 \\ -||f||\ ||g|| &\le (f,g) \le ||f||\ ||g|| \end{aligned}$$

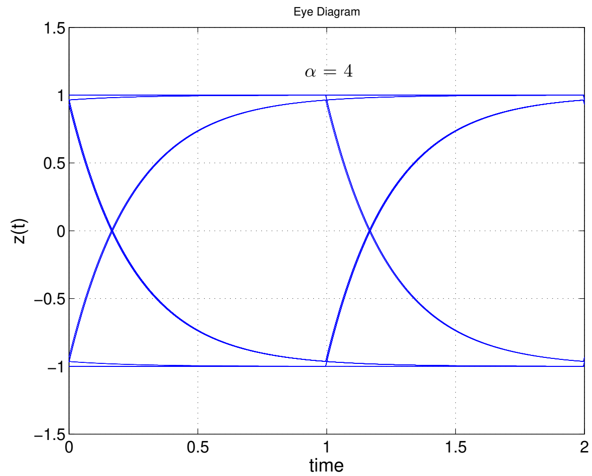


Figure 4.39: Intersymbol interference ($T = 1$)

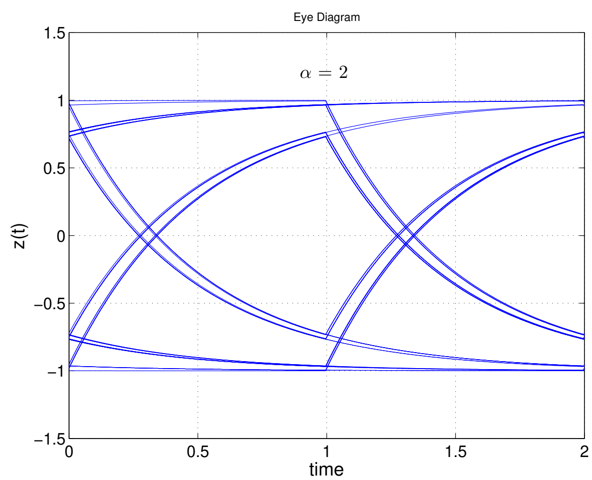


Figure 4.40: Intersymbol interference

Equality occurs when $f(x) = Kg(x)$. If K is positive the inequality on the right side becomes equality and if K is negative the inequality on the right side becomes equality. This is Schwartz's inequality.

4.6 Appendix 4B: Arithmetic mean $\geq$ Geometric mean:

Claim: Let a_0 and a_1 be real *nonnegative* numbers. Then

$$\frac{a_0 + a_1}{2} \geq \sqrt{a_0 a_1} \text{ with equality if } a_0 = a_1.$$

Proof:

$$\begin{aligned}
(a_0 - a_1)^2 &\geq 0 \text{ with equality if } a_0 = a_1 \\
a_0^2 - 2a_0a_1 + a_1^2 &\geq 0 \\
a_0^2 + 2a_0a_1 + a_1^2 &\geq 4a_0a_1 \\
(a_0 + a_1)^2 &\geq 4a_0a_1 \\
a_0 + a_1 &\geq 2\sqrt{a_0a_1} \\
\frac{a_0 + a_1}{2} &\geq \sqrt{a_0a_1} \text{ with equality if } a_0 = a_1.
\end{aligned}$$

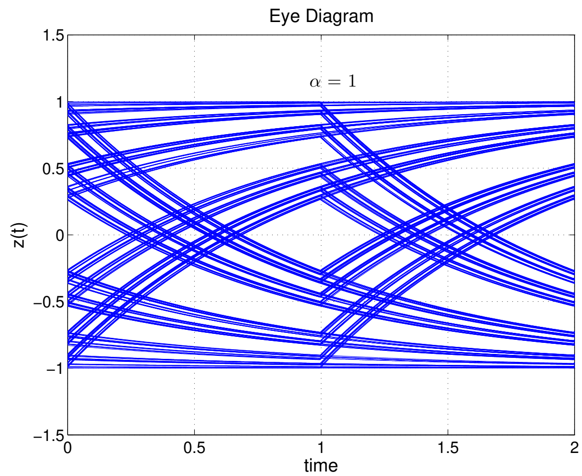


Figure 4.41: Intersymbol interference

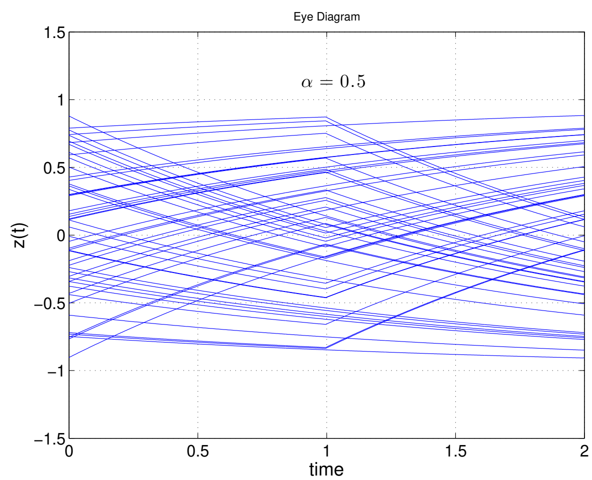


Figure 4.42: Intersymbol interference

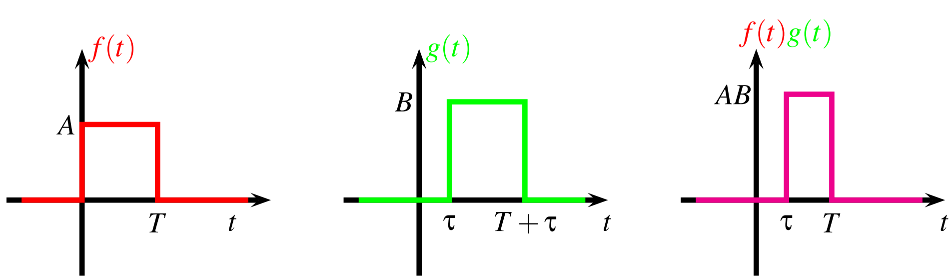


Figure 4.43: Example Signals.

4.7 Appendix 4C: Gaussian Distribution

- If X is a Gaussian random variable with mean 0 and variance 1 then

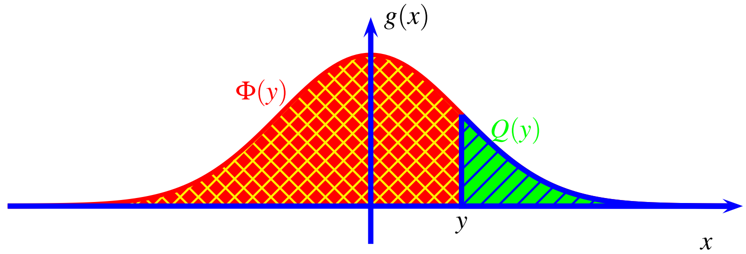
$$\begin{aligned} P\{X \leq x\} &= \Phi(x) \\ P\{X > x\} &= 1-\Phi(x) = Q(x). \end{aligned}$$

- If Y is a Gaussian random variable with mean μ and variance 1 then

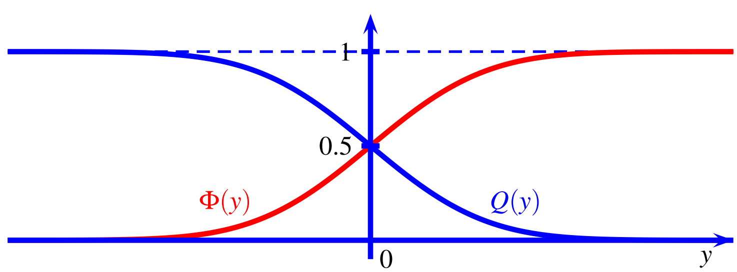
$$\begin{aligned} P\{Y \leq y\} &= \Phi(y-\mu) \\ P\{Y > y\} &= 1-\Phi(y-\mu) = Q(y-\mu). \end{aligned}$$

- If Z is a Gaussian random variable with mean μ and variance σ^2 then

$$\begin{aligned} P\{Z \leq z\} &= \Phi(\frac{z-\mu}{\sigma}) \\ P\{Z > z\} &= 1-\Phi(\frac{z-\mu}{\sigma}) = Q(\frac{z-\mu}{\sigma}). \end{aligned}$$



$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ \Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ Q(y) &= \int_{y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \end{aligned}$$

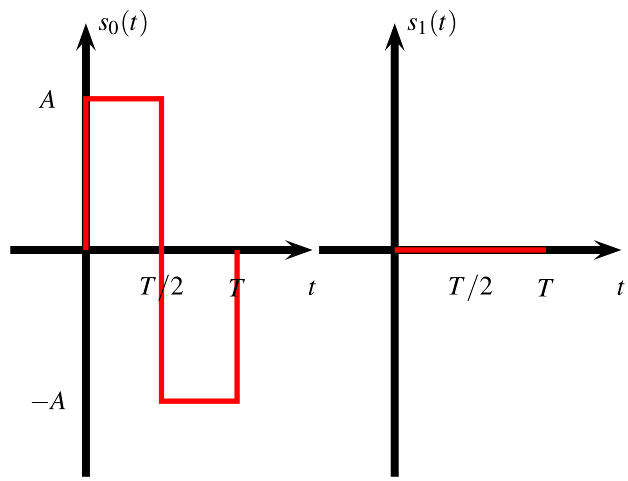


$$\begin{aligned} Q(y) &= 1-\Phi(y) \\ Q(y) &= \Phi(-y) \\ \frac{\partial Q(y)}{\partial u} &= -\frac{1}{\sqrt{2\pi}} e^{-y^2/2} \frac{\partial y}{\partial u} \end{aligned}$$

4.8 Problems

1. A binary communications system operates over an AWGN channel with power spectral density $N_0/2$. The transmitted signals are given by

$$s_0(t) = Ap_{T/2}(t) - Ap_{T/2}(t - T/2)$$
$$s_1(t) = 0$$



The receiver has the following structure.

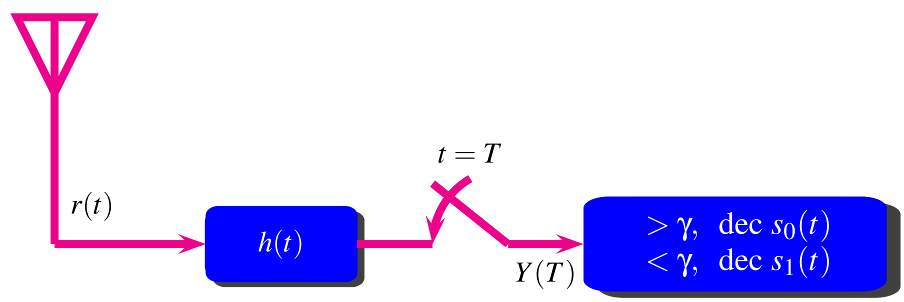


Figure 4.44: Receiver Structure

(a) Give an expression (in terms of A, T, N_0, and $Q(x)$) for the minimum average error probability when the two signals are transmitted with equal probabilities (i.e. $\pi_0 = \pi_1$).

(b) What is the optimum filter for minimizing the error probability and the optimum threshold?

(c) Assume that the optimum filter $h(t)$ and the optimum threshold is used for the signals above (i.e. the answer to part (b)) but the signal $s_0(t)$ is actually given by

$$s_0(t) = cAp_{T/2}(t) - cAp_{T/2}(t - T/2)$$

instead of the one given above while $s_1(t)$ is the same where $c > 0$. Give an expression (in terms of c, A, T, N_0 and Φ or Q) for the average error probability if $\pi_0 = \pi_1$.

2. A binary communications system operates over an AWGN channel with spectral density $N_0/2$. The transmitted signals are given by

$$s_0(t) = Ap_{T/2}(t) - Ap_{T/2}(t - T/2)$$
$$s_1(t) = 0$$

(a) Give an expression (in terms of A, T, N_0, and $Q(x)$) for the minimum average error probability when the two signals are transmitted with equal probabilities (i.e. $\pi_0 = \pi_1$).

(b) Assume that the optimum filter $h(t)$ is used for the signals above but the signal $s_0(t)$ is actually given by

$$s_0(t) = cAp_{T/2}(t) - cAp_{T/2}(t - T/2)$$

instead of the one given above while $s_1(t)$ is the same. Give an expression (in terms of c, A, T, N_0 and Φ) for the average error probability if $\pi_0 = \pi_1$.

3. Suppose $s_0(t) = A\sin(\omega_0 t + \theta_0 + \theta)p_T(t)$ and $s_1(t) = A\sin(\omega_0 t + \pi + \theta_1 + \theta)$ where θ is arbitrary but known to the receiver and the phase angles θ_i $(i = 0,1)$ satisfy $|\theta_i| < \pi/2$. Assume that $\omega_0 T = 2\pi n$ for some integer n.

   (a) Find the error probabilities $P_{e,0}$ and $P_{e,1}$ if the phase angles θ_0 and θ_1 are known and a matched filter (matched to $\mathbf{s}_0 - \mathbf{s}_1 1$) is used. Assume an optimum minmax threshold ($\gamma = \gamma_m$), and use sampling at $T_0 = T$.

   (b) Find $P_{e,0}$ and $P_{e,1}$ if θ_0 and θ_1 are not known and the filter is matched to the same signals with $\theta_i = 0$. Assume a minmax threshold is used for the signals with $\theta_i = 0$ and use sampling time $T_0 = T$.

4. Consider a binary communication system that uses two signals bfs_0 and $\mathbf{s}_1$ with

$$bfs_0 = Ap_T(t)$$

and $\mathbf{s}_1 = 0$. Find the filter $h(t)$ and the threshold γ that minimizes the maximum of the error probabilities $P_{e,0}$ and $P_{e,1}$ if the channel is an additive white Gaussian noise channel with power spectral density $N_0/2$. Also find the minmax error probability.

5. Let $s_i(t) = (-1)^i Ap_T(t)$, $i = 0,1$ be the signals for a binary communications system with an additive white Gaussian noise (AWGN) channel. Let $h(t)$ be the impulse response of the matched filter; that is

$$h(\lambda) = s_0(T - \lambda) - s_1(T - \lambda)\ .$$

The noise process $X(t)$ has spectral density $N_0/2$

   (a) For the system shown in Figure 1. Find $P_{e,0}$ and $P_{e,1}$ when the sampling time is $T_0 = \alpha T$ where α is an arbitrary number in the interval (0,2). Express your answer in terms of α, A, T, and $N_0/2$.

   (b) For the same signals and AWGN channel, find $P_{e,0}$ and $P_{e,1}$ for the system shown in Figure 2. Compare your answer to the answer for part (a).

6. For each signal set below, find the average error probability for binary communication via an AWGN channel (spectral density $N_0/2$). Assume for each signal that the receiver consists of an ideal matched filter, a sampler which samples at an optimal time, and a threshold device with the optimum threshold.

   (a)

$$\mathbf{s}_0 = \begin{cases} A & 0 \le t < T/3 \\ -A & 2T/3 \le t < T \\ 0 & \text{elsewhere} \end{cases}$$

$$s_1(t) = \begin{cases} A & 2T/3 \le t < T \\ -A & T/3 \le t < 2T/3 \\ 0 & \text{elsewhere} \end{cases}$$

   (b)

$$\begin{aligned} bfs_0 &= A|\cos(2\pi f_0 t)|p_T(t) \\ \mathbf{s}_1 &= A|\sin(2\pi f_0 t)|p_T(t) \end{aligned}$$

(c)

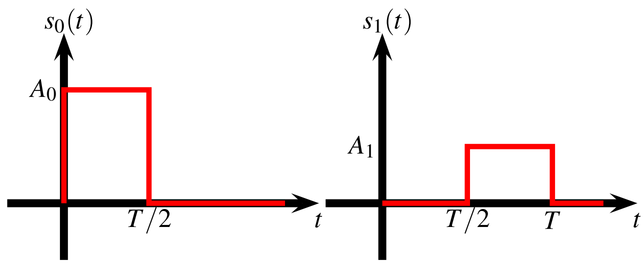
$$\begin{aligned}\mathbf{s}_0 &= A(1+\cos(2\pi f_0 t))p_T(t)\\ \mathbf{s}_1 &= A(1+\sin(2\pi f_0 t))p_T(t)\end{aligned}$$

In parts (b) and (c), assume $f_0 T = n$ for some integer n.

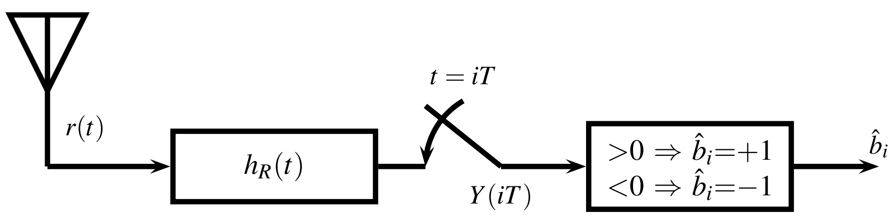
7. Consider a binary communication system that transmits one of two signal $s_0(t)$ and $s_1(t)$ over an additive white Gaussian noise channel (power spectral density $N_0/2$) where

$$s_0(t) = A_0 p_{T/2}(t), \quad s_1(t) = A_1 p_{T/2}(t - T/2);$$

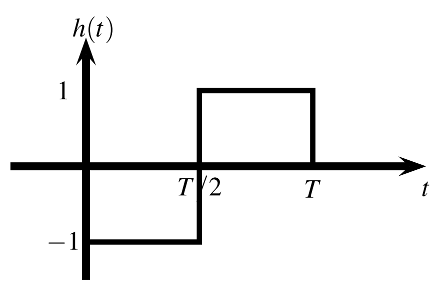
that is $s_0(t)$ is a pulse of amplitude A_0 from 0 to $T/2$ and $s_1(t)$ is a pulse of amplitude A_1 from $T/2$ to T as shown below.



The received signal, $r(t)$, is the transmitted signal with additive white Gaussian noise. The receiver shown below consist of a filter $h(t)$ which is sampled at time T and a threshold device.



(a) If $h_R(t) = -p_{T/2}(t) + p_{T/2}(t - T/2)$ shown below, find the output of the filter $\hat{s}_0(T)$ due to signal $s_0(t)$ at time T and the output of the filter $\hat{s}_1(T)$ due to signal $s_1(t)$ at time T



(b) Find the threshold γ that will minimize the average of the error probabilities $P_{e,0}$ and $P_{e,1}$ for the given signals and filter. Assume $\pi_0 = \pi_1$.

(c) Find the error average error probability $\bar{P}_e = \pi_0 P_{e,0} + \pi_1 P_{e,1}$ for the threshold found in the previous part. Assume $\pi_0 = \pi_1$.

(d) Find the matched filter for the same signals and find the corresponding threshold that minimizes $\bar{P}_e$. Assume $\pi_0 = \pi_1$.

(e) Find $\bar{P}_e$ for the matched filter with the optimum threshold. Assume $\pi_0 = \pi_1$.

8. Consider a communication system that transmits an infinite sequence of data bits $\{b_l\}_{l=-\infty}^{\infty}$ using two signals of duration T: $s_0(t) = -s_1(t) = Ap_T(t)$. Thus

$$s(t) = \sum_{l=-\infty}^{\infty} b_l A p_T(t - lT)$$

The signal $s(t)$ is transmitted over an additive white Gaussian noise channel with spectral density $N_0/2$. The receiver consists of a filter $h(t)$ the output of which is sampled at time iT and compared with a threshold of 0. If the output at time $iT > 0$ the receiver decides $b_{i-1} = +1$ otherwise the receiver decides $b_{i-1} = -1$. It is know that the filter is such that $\int_{-\infty}^{\infty} h^2(t)dt = 16$. It is also known that if the input to the filter is $p_T(t)$ ($n(t) = 0$, noise is 0) then the output at time iT is

$$Z(iT) = \begin{cases} 8 & i = 1 \\ 2 & i = 2 \\ 1 & i = 3 \\ 0 & i < 1,\ i > 3 \end{cases}$$

(a) Find the possible values for the output due to the desired signal (no noise) for different data bits.

(b) Find the upper and lower bounds for the $E[Z(iT)|b_{i-1} = +1, b_{i-2}, b_{i-3}, \ldots]$. That is, find the largest possible value for the output due to signal alone (no noise) at time iT for all possible previous data bits. Find upper and lower bounds on the probability of error for data bit b_{i-1} given that $b_{i-1} = +1$.

(c) Give an expression for the average probability of error for the data bit b_{i-1} if each data bit is equally likely to be +1 or -1 independently of all other data bits.

9. Consider a binary communication system that transmits one of two signals over an additive white Gaussian noise channel with power spectral density $N_0/2$. The two signals are $s_0(t) = A\cos(\pi t/2T)p_T(t)$ and $s_1(t) = -A\cos(\pi t/2T)p_T(t)$. The receiver shown below uses a filter $h(t) = p_T(t)$ which is sampled at time T and a threshold γ.

   (a) For these signals and this filter what is the threshold γ that minimizes the maximum of $P_{e,0}$, and $P_{e,1}$. What is $P_{e,m}$ in this case.

   (b) What is the filter and threshold that minimize the maximum of $P_{e,0}$, and $P_{e,1}$ and what is $P_{e,m}$.

   (c) If s_0 is transmitted with probability $\pi_0 = 1/4$ and s_1 is transmitted with probability $\pi_1 = 3/4$ what is the threshold that minimizes the average error probability for the filter of part (a). What is the average error probability for this case?

   (d) Repeat part (c) if the matched filter (optimum filter) is used. (i.e. find the optimum threshold and the resulting error probability)

10. Consider the communication system shown below. It is known that $\int_{-\infty}^{\infty} H_R^2(f)df = 8$. The noise is white Gaussian noise with power spectral density $N_0/2 = 1/2$. It is also know that if there is no noise ($n(t) = 0$) and a single pulse $\delta(t)$ is the input to $h_T(t)$ that the output $v(t)$ at time jT is

$$v(jT) = \begin{cases} 0 & j \le 0 \\ 5 & j = 1 \\ 2 & j = 2 \\ 1 & j = 3 \\ 1 & j = 4 \\ 0 & j > 4 \end{cases}.$$

   (a) If now an infinite data stream of the form

$$s(t) = \sum_{l=-\infty}^{\infty} b_l \delta(t - lT)$$

is transmitted write down an expression for the output of the filter at time T. ($h_R(t)$ is a linear time-invariant filter).

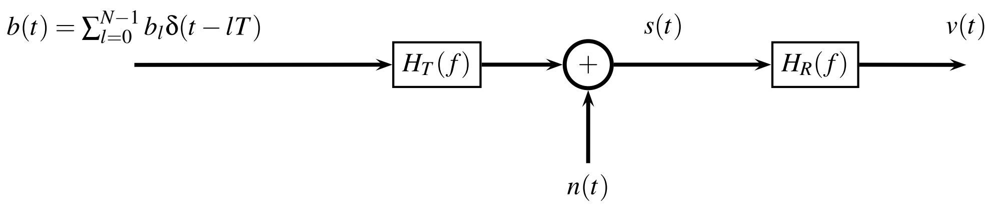
If a decision is to be made at time T about data bit b_0 identify which terms in the expression correspond to the signal, to the intersymbol interference, and to the noise.

(b) Write down an expression for the average probability of error given $b_0 = +1$.

(c) Find upper and lower bounds to the average probability of error.

(d) Find an approximation to the error probability by assuming the interference to be an appropriate Gaussian random variable.

Your answers to part (b), (c), and (d) should be written in terms of the Q function or Φ function.



11. Consider a binary communication system that transmits one of two signals $s_0(t)$ and $s_1(t)$ over an additive white Gaussian channel with power spectral density $N_0/2$ where

$$s_0(t) = A p_T(t)$$

and $s_1(t) = 0$.

(a) Find the matched filter for the signals and the threshold that minimizes the maximum of the error probabilities $P_{e,0}$ and $P_{e,1}$. What is the maximum error probability $P_{e,m} = \max\{P_{e,0}, P_{e,1}\}$.

(b) If $s_0(t)$ is sent with probability 1/4 and $s_1(t)$ is sent with probability 3/4 find the threshold that minimizes the average probability of error $\bar{P}_e = \pi_0 P_{e,0} + \pi_1 P_{e,1}$. What is the average error probability?

12. A communication system transmits one bit of information using one of two signals. The signals are

$$s_0(t) = \sqrt{2P}\cos(2\pi f_c t) p_T(t)$$

and

$$s_1(t) = \sqrt{2P}\sin(2\pi f_c t) p_T(t)$$

where $2\pi f_c T = 2\pi n$ for some (large) integer n. Assume the two signals are equally likely. The signals are received in the presence of additive white Gaussian noise with two-sided power spectral density $N_0/2$. The receiver consists of a filter which is sampled at time T and compared to a threshold γ.

(a) Determine the optimum threshold and filter to minimize the average probability of error.

(b) Determine the minimum probability of error for this set of signals. (Express your answer in terms of the Q function, the power P and the signal duration T.

Trig. Identities

$$\begin{aligned}
\sin(u)\cos(v) &= \frac{1}{2}[\sin(u+v)+\sin(u-v)] \\
\cos(u)\cos(v) &= \frac{1}{2}[\cos(u-v)+\cos(u+v)] \\
\cos^2(u) &= \frac{1}{2}[1+\cos(2u)] \\
\sin^2(u) &= \frac{1}{2}[1-\cos(2u)] \\
\int_b^c \cos(ax)dx &= \frac{1}{a}\sin(ax)|_b^c \\
\int_b^c \sin(ax)dx &= -\frac{1}{a}\cos(ax)|_b^c
\end{aligned}$$

13. A communication system transmits a bit of information using one of two signals $s_0(t)$ and $s_1(t)$ of duration T.

$$s_0(t) = +\sqrt{\frac{P(2\alpha+1)}{T^{2\alpha}}}t^{\alpha},\ 0 \le t \le T$$
$$s_1(t) = -\sqrt{\frac{P(2\alpha+1)}{T^{2\alpha}}}t^{\alpha},\ 0 \le t \le T.$$

At the receiver a filter with impulse response

$$h(t) = (T-t)^{\beta},\ 0 \le t \le T.$$

is used and the output is sampled at time $t = T$.

(a) Suppose $b_0 = +1$. Determine the output of the filter $\hat{s}_0(T)$ at the sampling time due to the signal alone.

(b) Determine the variance σ^2 of the noise at the output of the filter when the input to the filter is AWGN with PSD $N_0/2$.

(c) Determine the signal-to-noise ratio

$$\frac{\hat{s}_0^2(T)}{\sigma^2}$$

as a function of P, T, α, β.

(d) Find the β that makes the signal-to-noise ratio largest.

14. A communication system transmits one of three signals:

$$s_0(t) = A\cos(2\pi f_c t)p_T(t)$$
$$s_1(t) = 0$$
$$s_2(t) = -A\cos(2\pi f_c t)p_T(t)$$

over an additive white Gaussian noise channel with power spectral density $N_0/2$. Let $r(t)$ denote the received signal ($r(t) = s_i(t) + n(t)$). The receiver computes the quantity

$$Z = \int_0^T r(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt.$$

Assume $2\pi f_c T = 2\pi n$ for some large integer n (to ignore double frequency terms). The receiver output Z is compared with a threshold γ and a threshold $-\gamma$. If $Z > \gamma$, the decision is made that $s_0(t)$ was sent. If $Z < -\gamma$, the decision is made that $s_2(t)$ was sent. If $-\gamma < Z < \gamma$ the the decision is made in favor of $s_1(t)$

(a) Determine the three conditional probabilities of error: $P_{e,0}$ = probability of error given s_0 sent, $P_{e,1}$ =probability of error given s_1 sent, and $P_{e,2}$ = probability of error given s_2 sent.

(b) Determine the average error probability assuming that all three signals are equally probable of being transmitted.

15. A binary communication system transmits one of two equally likely signals $s_0(t)$ and $s_1(t)$ of duration T given by

$$s_i(t) = \sqrt{2P}(-1)^i \cos(2\pi f_c t) p_T(t), \quad i = 0, 1.$$

The noise in the system is white Gaussian noise with power spectral density $N_0/2$. The receiver shown in Figure 4.45 is used to demodulate the signal. The filter impulse response is a rectangular pulse, $h(t) = p_T(t)$. However, as shown, the phase of the received signal is not know completely accurately. In fact, there is a discrepancy of θ radians between the received signal (in the absence of noise) and the local reference signal. Determine the error probability at the output of the demodulator as a function of θ. (Assume $2\pi f_c T = 2\pi n$ for some integer n or $f_c T \gg 1$. That is, ignore double frequency terms).

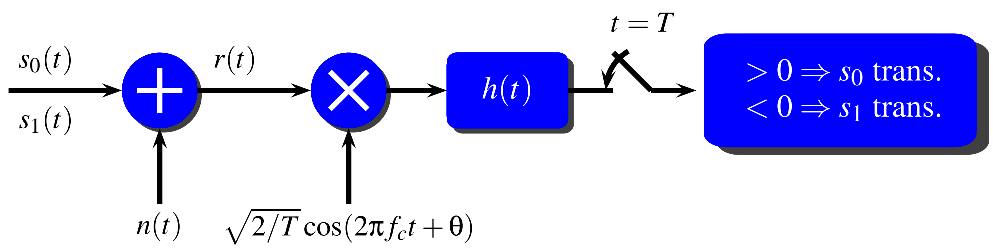


Figure 4.45: Receiver with Phase Offset.

16. A communication system uses $100kHz = 10^5 Hz$ of bandwidth (null-to-null). If $BPSK$ modulation is employed, determine the required power P such that the error probability is less than $Q\left(\sqrt{20}\right) = 3.876 \times 10^{-6}$. Assume an additive white Gaussian channel with noise power spectral density $N_0/2$ with $N_0/2 = 4 \times 10^{-21}$ watts/Hz. What is the data rate of this system (in bits/second)?

17. Consider a binary communications system with $s_i(t) = (-1)^i A p_T(t), \ i = 0, 1$ and an additive white Gaussian noise (AWGN) channel. Let $h(t)$ be the impulse response of the receiver filter with

$$h(t) = \frac{1}{\sqrt{2\pi}\alpha} \exp\{-t^2/2\alpha^2\}.$$

The noise process $n(t)$ has spectral density $N_0/2$.

(a) Find the optimal sampling time T_0.

(b) Find the optimal parameter α. Hint: The maximum with respect to x of $(2\Phi(x) - 1)/\sqrt{x}$ occurs at $x = 1.40$ and takes the value .70865.

(c) Find $P_{e,i}$ for E/N_0= 8,9,10,11,12 dB.

(d) Compare your answer of part (c) to the error probability for the optimal receiver (filter, sampler, threshold). You may assume that $\pi_0 = \pi_1$.

18. A communication system transmits information using one of two signals. The signals are

$$s_0(t) = \sqrt{2P}\cos(\omega_0 t) p_T(t),$$

and

$$s_1(t) = \sqrt{2P}\cos(\omega_1 t) p_t(t).$$

Assume $\omega_0 T = 2\pi n$ for some (large) integer n and that $\omega_1 = \omega_0 + u/T$ where u varies from 0 to 5. Also assume the two signals are equally likely. Derive and plot the error probability of the optimum receiver as a function of u for $E_b/N_0 = 5dB$. Find the value of u that minimizes the error probability. Be careful in Matlab because Matlab defines $\text{sinc}(u) = sin(\pi * u)/(\pi u)$.

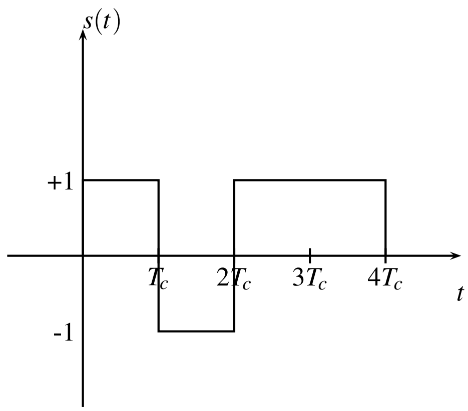
19. Consider a signal $s(t)$ as an input to a filter with impulse response $h(t)$. (a) Find the filter output when the input is a sequence of four pulses each of duration $T_c = T/4$ as shown below

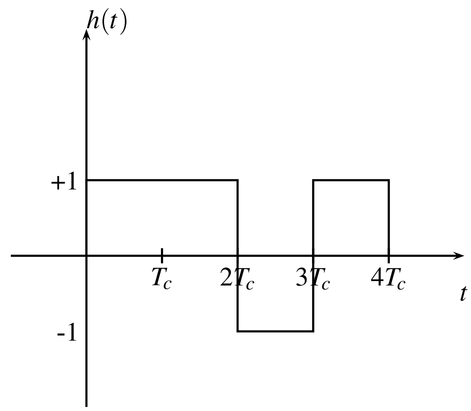
$$s(t) = p_{T_c}(t) - p_{T_c}(t - T_c) + p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c)$$

and the filter is given by.

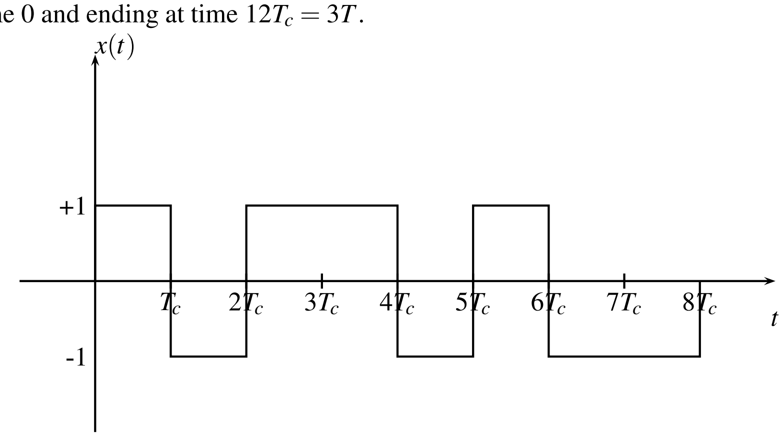
$$h(t) = p_{T_c}(t) + p_{T_c}(t - T_c) - p_{T_c}(t - 2T_c) + p_{T_c}(t - 3T_c)$$

as shown below. The output should be a function of time beginning at time 0 and ending at time $2T = 8T_c$.





(b) Find the filter output (for the same filter) when the input is $x(t) = s(t) - s(t - T)$. The output is a function beginning at time 0 and ending at time $12T_c = 3T$.



(c) If the input to the filter is white Gaussian noise with two-sided power spectral density $N_0/2$ find the variance of the output of the filter.

20. (a) Is it possible to have reliable communication (error probability approximately 0) with a data rate of 2.5Mbps if the received power is $P = 3 \times 10^{-12}$ watts using a (absolute) bandwidth of $W = 1$ MHz and a noise power spectral density of $\frac{N_0}{2} = \frac{10^{-18}}{2}$ watts/Hz?

(b) A communication system uses BPSK (rectangular pulses). The null-to-null bandwidth of BPSK with pulses of duration T is $2/T$. The power of the signal is $P = 5 \times 10^{-12}$ Watts. The available null-to-null bandwidth is $W = 1$ MHz. There is white Gaussian noise with two-side power spectral density $\frac{N_0}{2} = \frac{10^{-18}}{2}$. An error probability of $Q(\sqrt{20})$ is desired. What data rate is possible?

(c) For the same parameters as part (b) except the error probability requirement is $Q(\sqrt{2})$ what data rate would be possible?

(d) Consider two different signal sets with M signals for communication in additive white Gaussian noise. In signal set 1 the signals are

$$\begin{aligned} s_0^{(1)}(t) &= -3A\cos(2\pi f_c t)p_T(t), \\ s_1^{(1)}(t) &= -A\cos(2\pi f_c t)p_T(t), \\ s_2^{(1)}(t) &= +A\cos(2\pi f_c t)p_T(t), \\ s_3^{(1)}(t) &= +3A\cos(2\pi f_c t)p_T(t), \end{aligned}$$

where T is the duration of the signals. In signal set 2 the signals are

$$\begin{aligned} s_0^{(2)}(t) &= B\cos(2\pi f_c t)p_T(t), \\ s_1^{(2)}(t) &= -B\cos(2\pi f_c t)p_T(t), \end{aligned}$$

Which signal set is more bandwidth efficient? Which signal set is more energy efficient? Explain briefly.

21. Determine approximation and bounds to the error probability for a communication system operating in white Gaussian noise. The transmitter uses rectangular pulses of amplitude ± 1 to transmit data. The receiver uses an RC filter to detect the data. Plot the bounds and approximations and compare to the simulation in Homework 5.

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