Lecture 4: Goals

- Be able to calculate the output of a linear system (convolution) when the input is a specified function of time (or frequency).
- Be able to work in both time domain and frequency domain.
- ISI Free Pulses

Filtering, Convolution, Correlation and Noise

In most receivers in a digital communication system the received signal is filtered before a decision is made as to the data bit that is transmitted. The purpose of filtering is to remove as much of the noise as possible without removing any of the signal.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

impulse

Convolution

- Mathematically, *filtering* is the convolution of the input signal with the impulse response of the filter.
- That is, if the input to the filter is the signal $x(t)$ and the impulse response of the filter is $h(t)$ the output of the filter $y(t)$ is given by

Filter, Convolution

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\alpha)h(\alpha)d\alpha \\ &= \int_{-\infty}^{\infty} h(t-\alpha)x(\alpha)d\alpha \end{aligned}$$

- The above mathematical operation on $x(t)$ and $h(t)$ is called the convolution of h with x.

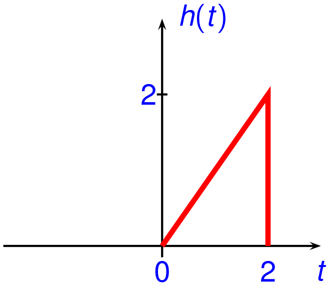
Graphical Convolution

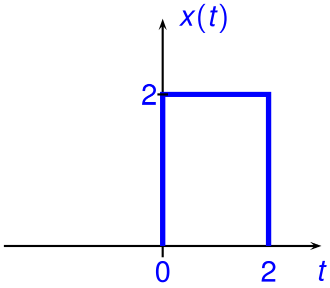
The convolution operation can be understood graphically. To do this we compute the output at different times. Combining the output at different times gives the total output. Consider the convolution of a rectangular pulse with a triangular impulse response filter.

$$y(t_1) = \int h(t_1 - \alpha)x(\alpha)d\alpha$$

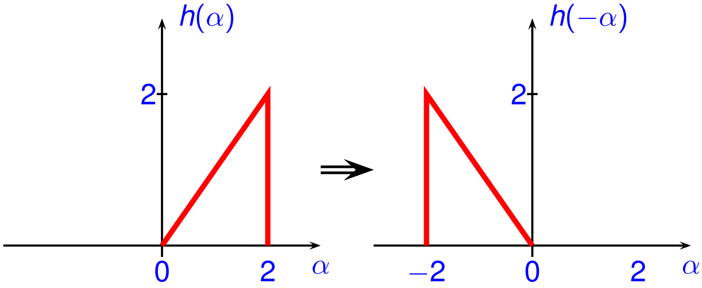
- First the function $h(\alpha)$ is flipped right to left to yield $h(-\alpha)$.
- Second, the function $h(-\alpha)$ is shifted to the right by t_1 seconds.
- Third, the flipped, shifted function h is correlated (multiplied and integrated) with the input x to give the output $y(t_1)$ at time t_1.

Graphical Convolution: Starting functions





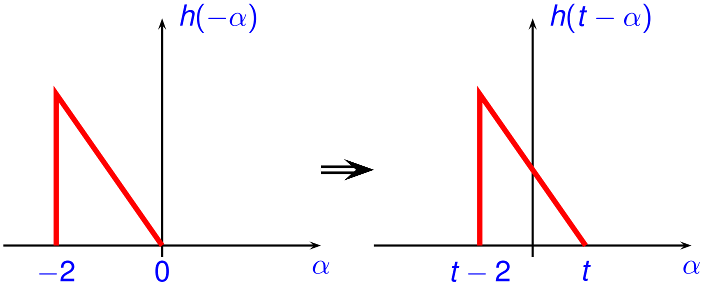
First Step: Flipping h



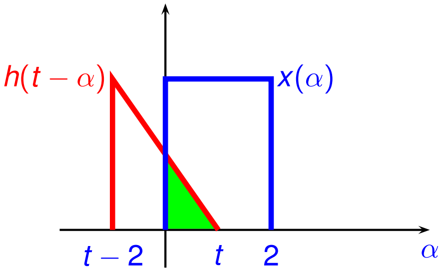
Second Step: Shifting $h(-\alpha) \to h(t-\alpha)$

$h(-(\alpha - t)) = h(t - \alpha)$

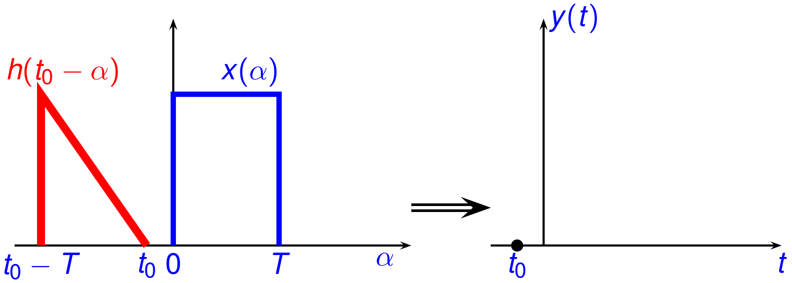
$-h$



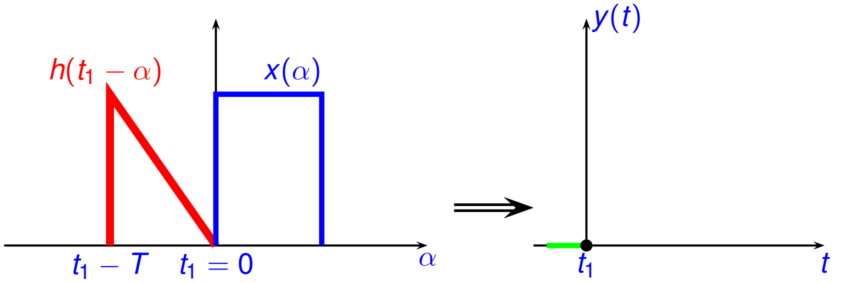
Third Step: Multiply and integrate $\int_{-\infty}^{\infty} h(t-\alpha)x(\alpha)d\alpha$



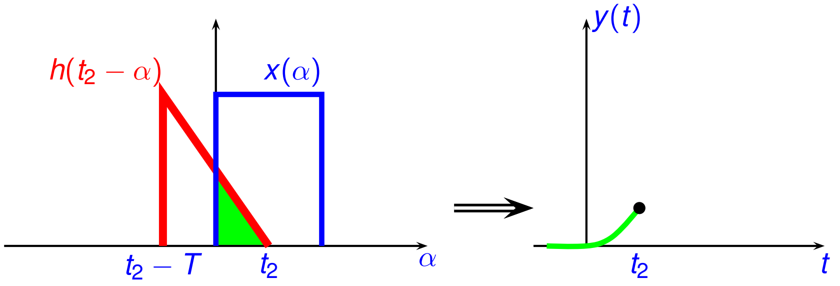
Output at $t = t_0 < 0$



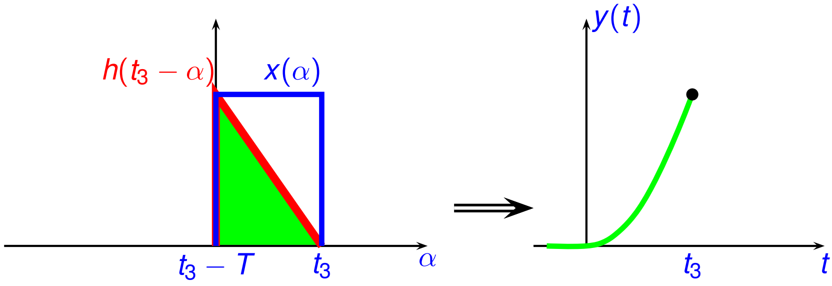
Output at $t_1 = 0$



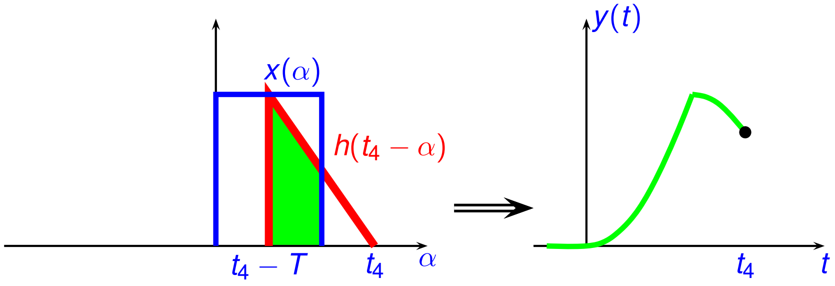
Output at $t = t_2$



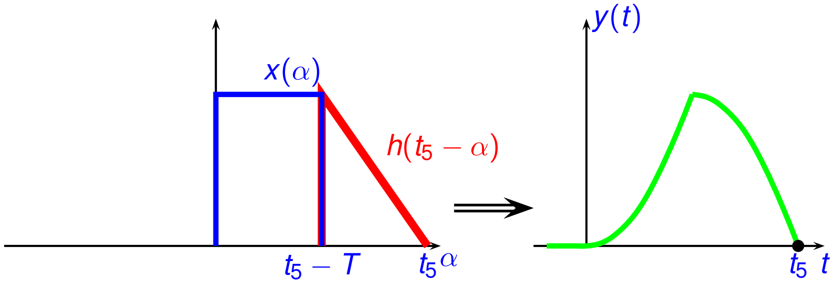
Output at $t = t_3$



Output at $t = t_4$



Output at $t = t_5$



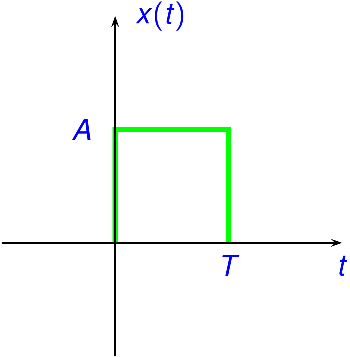
Convolution and Correlation

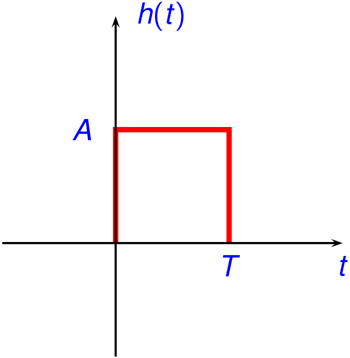
- The output of a filter is computed by convolving the input with the impulse response.
- Convolution is many correlations (multiply and integrate) done with different delays for the impulse response of the filter.

Rectangular-Rectangular Convolution

For example if $x(t)$ and $h(t)$ are rectangular pulses of amplitude A and duration T beginning at $t = 0$.

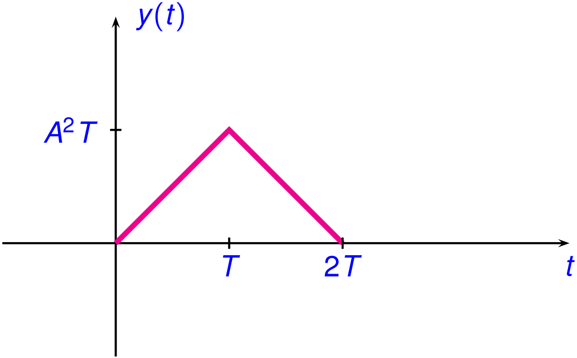
$$x(t) = h(t) = Ap_T(t) = \begin{cases} A, & 0 \leq t \leq T \\ 0, & \text{otherwise.} \end{cases}$$





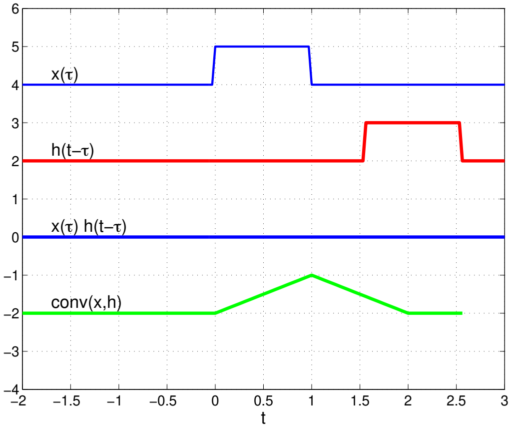
Rectangular-Rectangular Convolution

The output of the filter with rectangular impulse response of duration T to an input that is also rectangular of duration T is a triangular pulse of duration $2T$.

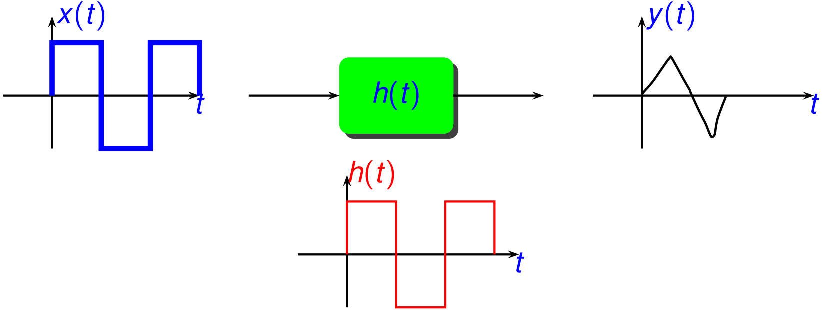


Convolution of a rectangular pulse with a rectangular pulse yields a triangular pulse

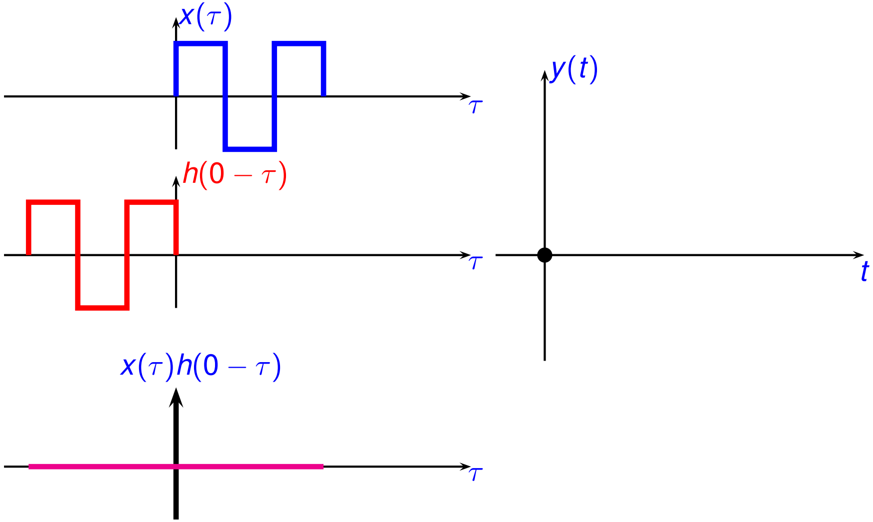
Rectangular Pulse



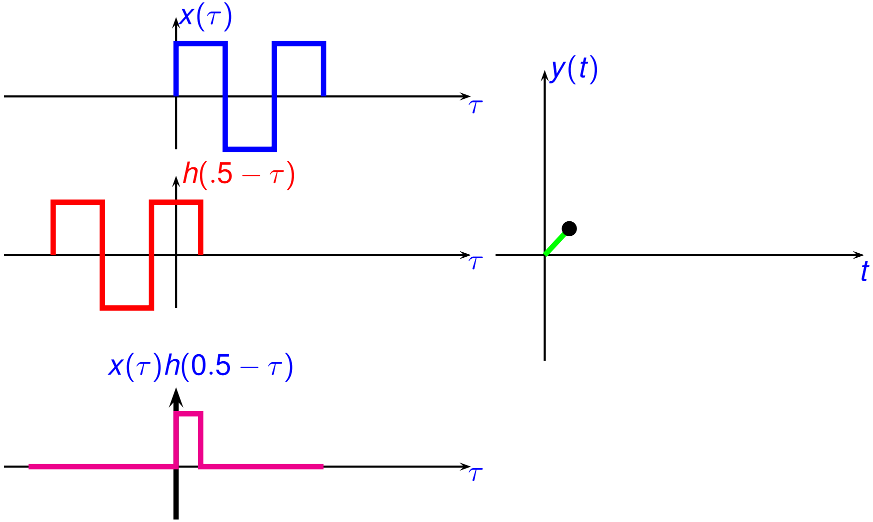
Example of Graphical Convolution



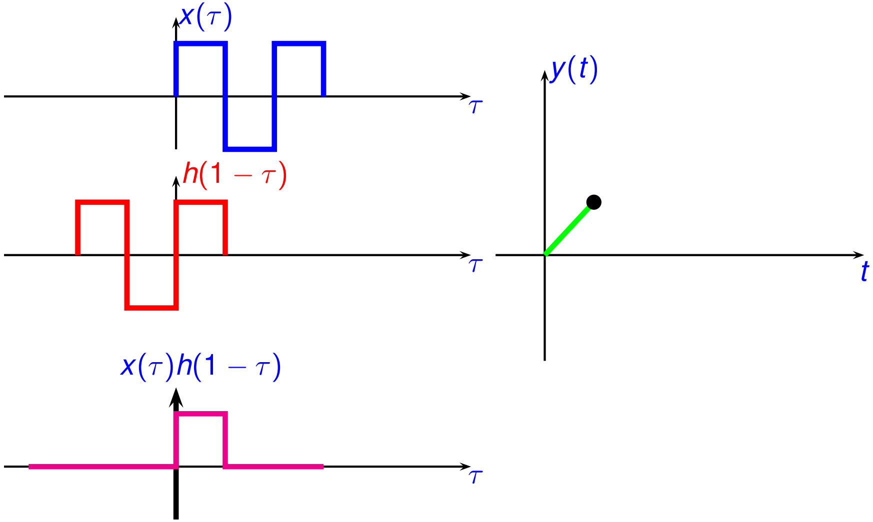
Example of Graphical Convolution (1)



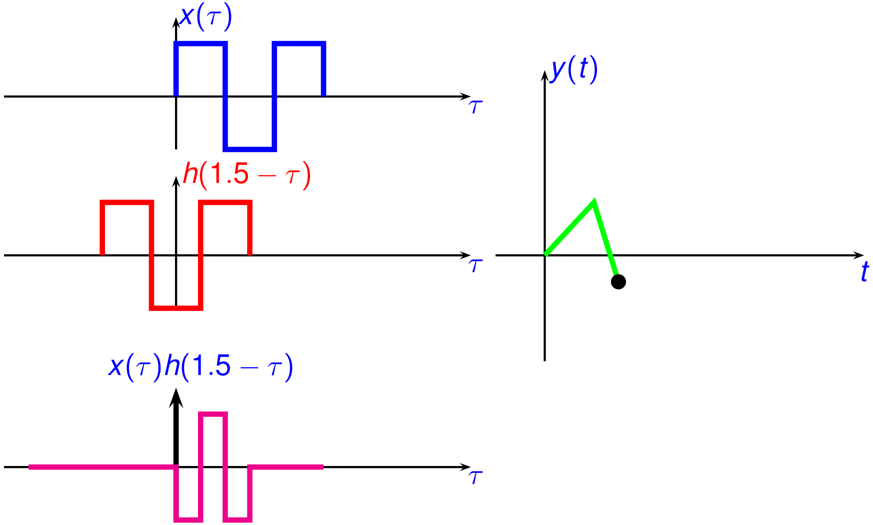
Example of Graphical Convolution (2)



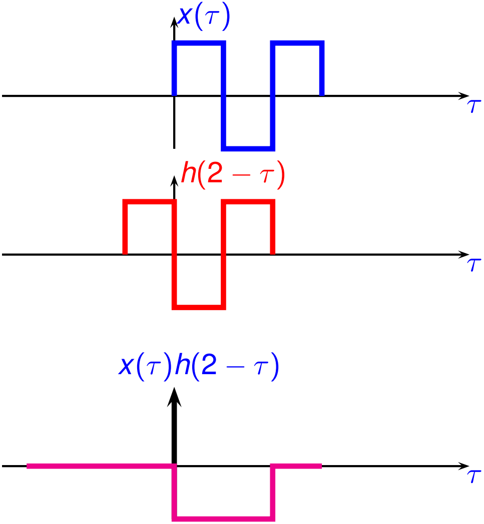
Example of Graphical Convolution (3)

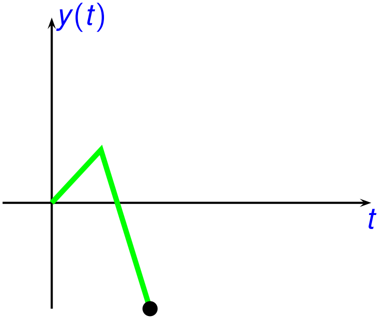


Example of Graphical Convolution (4)

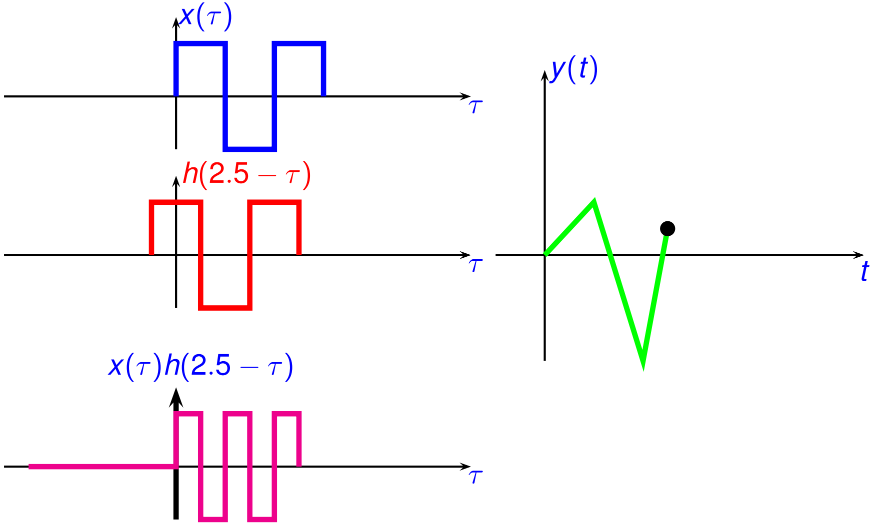


Example of Graphical Convolution (5)

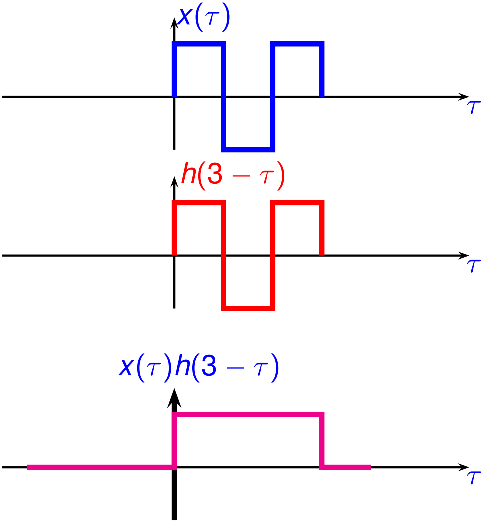


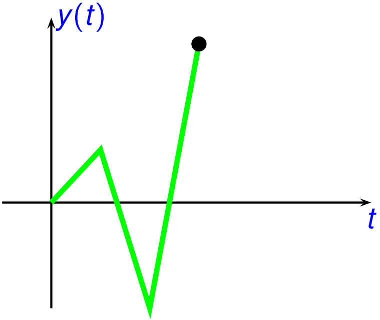


Example of Graphical Convolution (6)

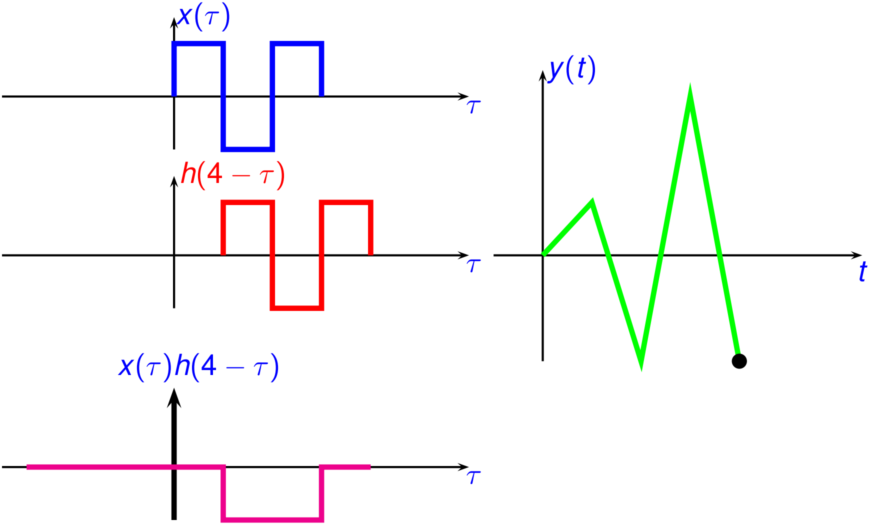


Example of Graphical Convolution (7)

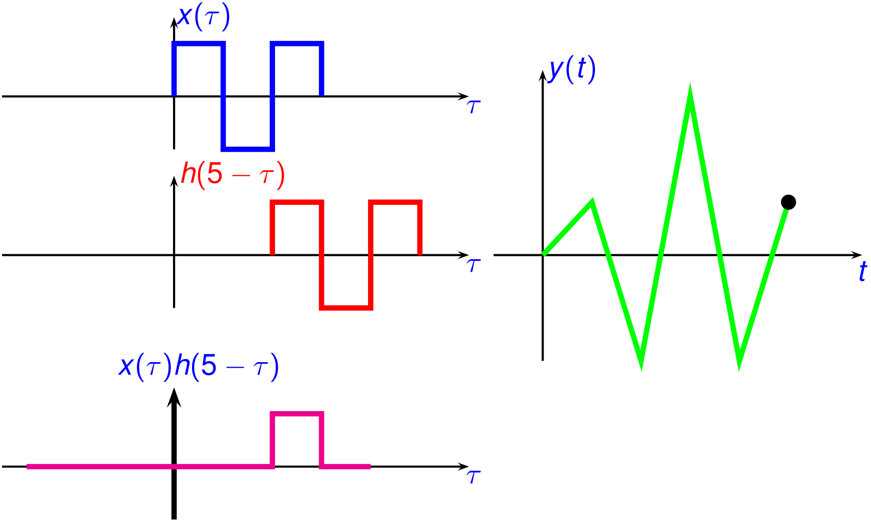




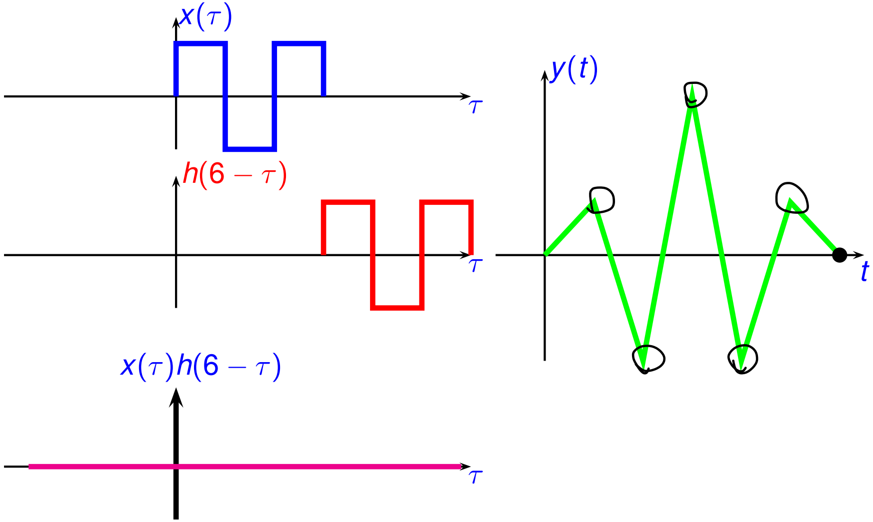
Example of Graphical Convolution (8)



Example of Graphical Convolution (9)



Example of Graphical Convolution (10)



- Convolution of rectangular pulses with rectangular pulse results in a piecewise linear function.
- Just need to evaluation the convolution at the break points and then connect the dots.

Properties of Linear Time-Invariant (LTI) Systems

- **Linearity:** If the output of a linear system is $y_1(t)$ when $x_1(t)$ is the input and the output is $y_2(t)$ when $x_2(t)$ is the input then the output due to $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is $\alpha_1 y_1(t) + \alpha_2 y_2(t)$.

$$\begin{array}{rcl} x_1(t) & \rightarrow & y_1(t) \\ x_2(t) & \rightarrow & y_2(t) \\ & \Downarrow & \\ \alpha_1 x_1(t) + \alpha_2 x_2(t) & \rightarrow & \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{array}$$

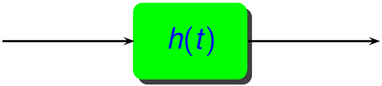
- **Time-Invariance:** If the output of a linear system is $y(t)$ when $x(t)$ is the input then the output due to $x(t-\tau)$ is $y(t-\tau)$.

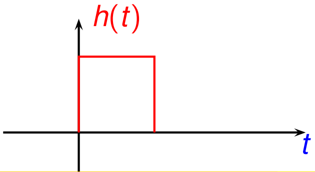
Example of Linearity, Time Invariance

Consider a filter with impulse response that is rectangular of duration T. Find the output due to a sequence of rectangular pulses.

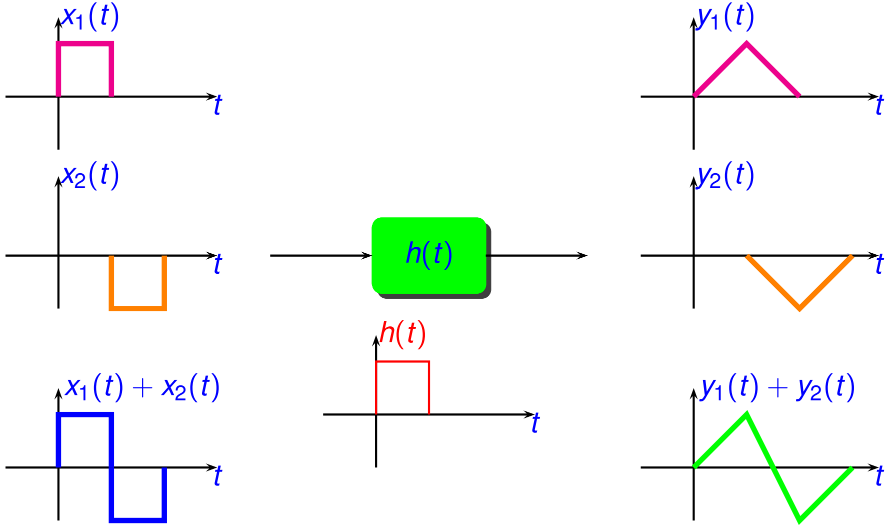
Linearity, Time Invariance

- If the input is shifted in time, the output is shifted in time
- The output due to the sum of two inputs is the sum of the two outputs

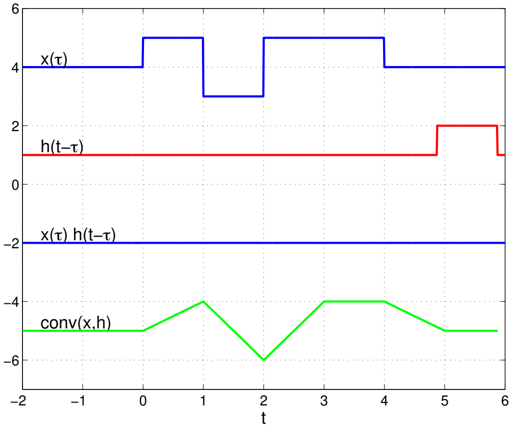




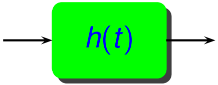
Example of Linearity, Time Invariance



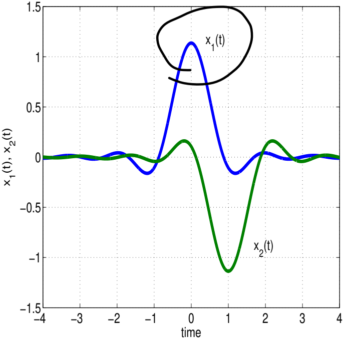
Four rectangular pulses

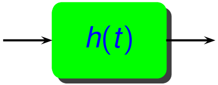


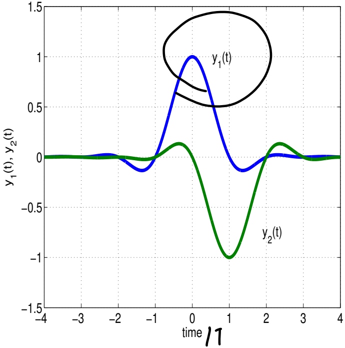
Example of Linearity, Time Invariance

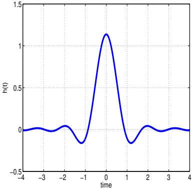


Example of Linearity, Time Invariance



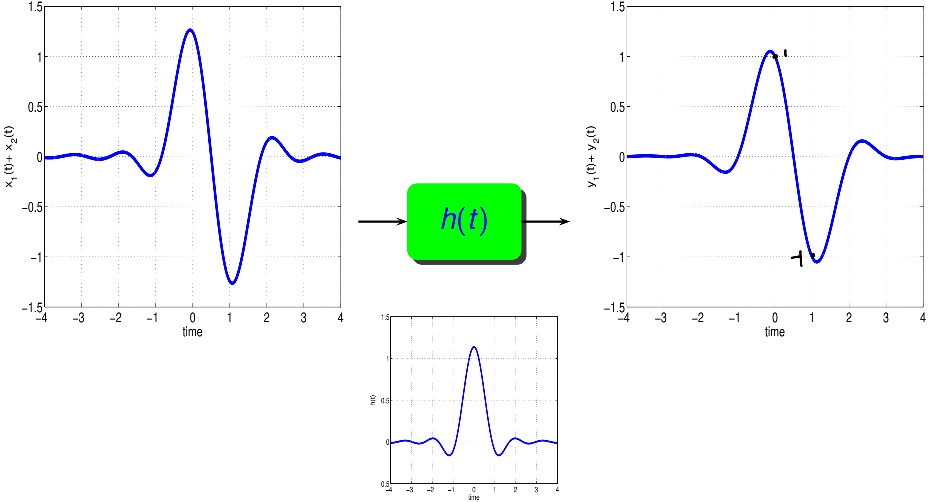




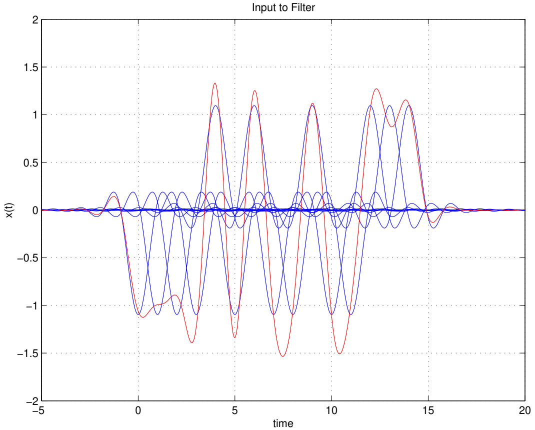


filter

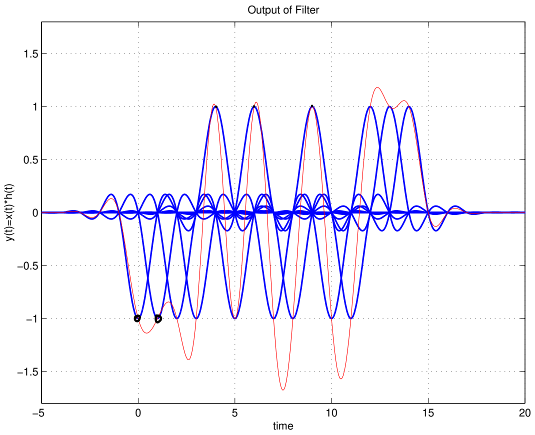
Example of Linearity, Time Invariance



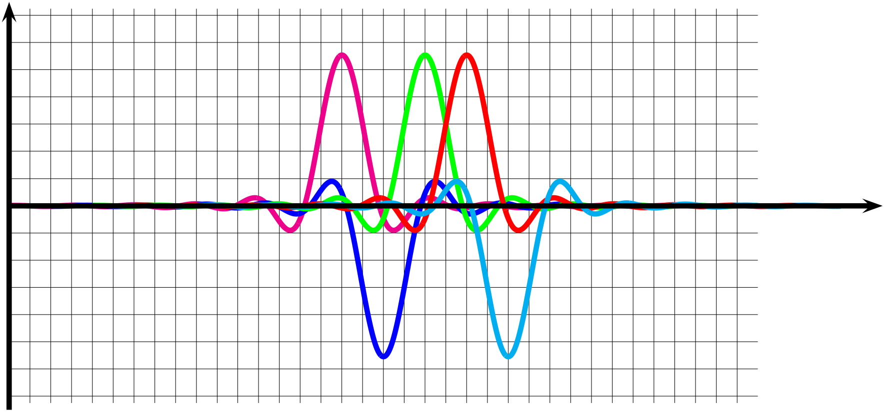
Another Input



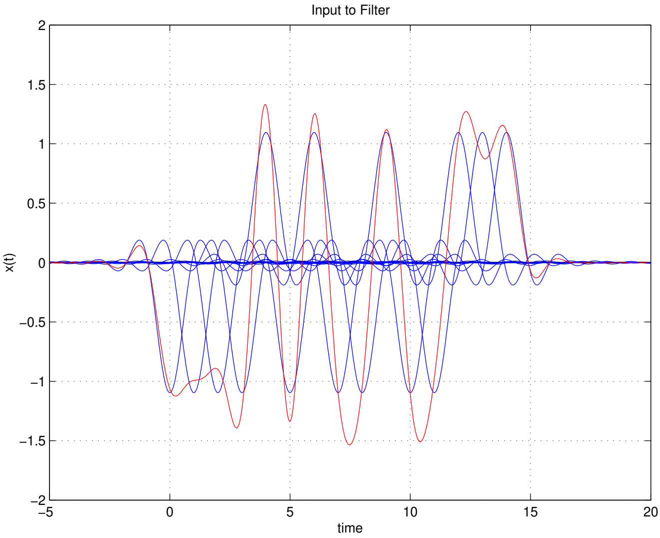
Other Output



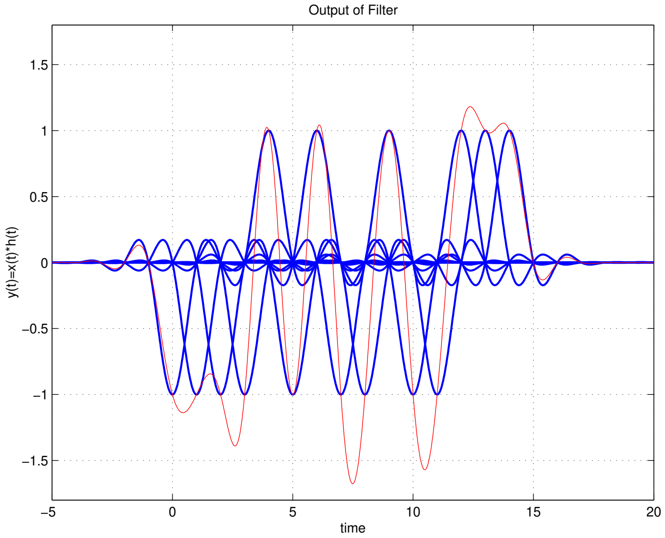
Total Input

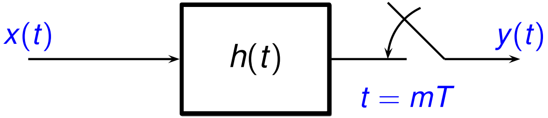


Total Input



Total Output




$x(t)$
$h(t)$
$t = mT$
$y(t)$

Time-Shifted Orthonormal

- Consider the generation of signals from vectors at the transmitter and the generation of vectors from signals at the receiver of a communication system.
- We would like to be able to do each of these without the complexity increasing with N the length of the vector.
- For time-shifted orthonormal signals there is a simple way to do this at the transmitter and the receiver using a single filter each.

Time-Shifted Orthonormal: Transmitter Implementation

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\varphi_n(t) = \varphi_0(t - nT), \quad n = 0, 1, ..., N - 1.$$

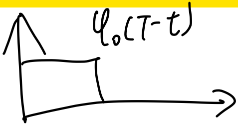
$$\sum_{n=0}^{N-1} s_{m,n}\delta(t - nT) \longrightarrow \boxed{h(t) = \varphi_0(t)} \longrightarrow s_m(t)$$

$$\sum_{n=0}^{N-1} s_{m,n}e^{-j2\pi fnT} \longrightarrow \boxed{H(f)} \longrightarrow S_m(f)$$

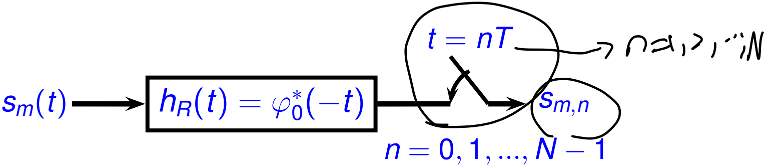
The input to the filter is a sequential (serial) version of the vector. The output of the filter is the desired signal.

Time-Shifted Orthonormal: Transmitter Implementation

$$
\begin{aligned}
\sum_{n=0}^{N-1} s_{m,n}\delta(t-nT) * h(t) &= \int h(t-\tau)\sum_{n=0}^{N-1} s_{m,n}\delta(\tau-nT)d\tau \\
&= \sum_{n=0}^{N-1} s_{m,n}\int \varphi_0(t-\tau)\delta(\tau-nT)d\tau \\
&= \sum_{n=0}^{N-1} s_{m,n}\varphi_0(t-nT) \\
&= \sum_{n=0}^{N-1} s_{m,n}\varphi_n(t) \\
&= s_m(t)
\end{aligned}
$$



Now consider a filter "matched" to $\varphi_0(t)$. That is, $h_R(t) = \varphi_0^*(-t)$.



The filter impulse response is the time flip (and conjugate) of the signal $\varphi_0(t)$.

Recv

The output of the filter is

$$\begin{aligned} y(t) &= \int h_R(t-\tau)s_m(\tau)d\tau \\ &= \int \varphi_0^*(-(t-\tau))s_m(\tau)d\tau \\ &= \int \varphi_0^*(\tau-t)s_m(\tau)d\tau. \end{aligned}$$

Consider sampling the output $y(t)$ at time $t = 0, T, 2T, ..., (N-1)T$.

The sampled output is then

$$
\begin{aligned}
y(nT) &= \int \varphi_0^*(\tau - nT) \underline{s_m(\tau)} d\tau \\
&= \int \varphi_0^*(\tau - nT) \sum_{k=0}^{N-1} s_{m,k}\varphi_k(\tau) d\tau \\
&= \sum_{k=0}^{N-1} s_{m,k} \int \varphi_0^*(\tau - nT)\varphi_k(\tau) d\tau \\
&= \sum_{k=0}^{N-1} s_{m,k} \int \underline{\varphi_n^*(\tau)\varphi_k(\tau)} d\tau \\
&= \sum_{k=0}^{N-1} s_{m,k}\delta_{n,k} \\
&= s_{m,n}.
\end{aligned}
$$

0 if $n \neq k$

1 if $n = k$

- So the samples at times $0, T, 2T, ..., (N-1)T$ correspond to the coefficients of the signal expansion $s_{m,0}, ..., s_{m,N-1}$.
- The samples of the filter output are exactly the coefficients in the signal. Note that the filter with impulse response $h_R(t) = \varphi^*(-t)$ is not a causal filter for the case that $\varphi_0(t) \neq 0$ for $t > 0$ because then $h_R(t) = \varphi(-t) \neq 0$ for $t < 0$.
- As such typically a delay is added to the filter. For example if the signal $\varphi_0(t)$ is only nonzero of $0 \leq t \leq T$ then a delay of T will make the filter $h_R(t) = \varphi^*(T-t)$ causal. such as With $h_R(t) = \varphi_0^*(T-t)$ the outputs at times $T, 2T, ..., NT$ would have corresponded to the coefficients $s_{m,0}, ..., s_{m,N-1}$.
- So matched filtering a signal composed of a linear combination time-shifted orthonormal waveforms will recover the coefficients of the signal. Using signals constructed from time-shifted orthonormal waveforms allows us to recover the coefficients of the signals from samples of a single filter as shown.

Frequency Domain Analysis

- Signals and filtering can also be described in the frequency domain.
- The frequency content of a signal is obtained via the Fourier Transform.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt.$$

- Convolution in the time domain corresponds to multiplication in the frequency domain and thus

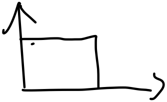
$$y(t) = x(t) * h(t) \Leftrightarrow Y(f) = H(f)X(f).$$

- One useful relation between the frequency domain and time domain is Parseval's Theorem

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \int_{-\infty}^{\infty} X_1(f) X_2^*(f) df.$$

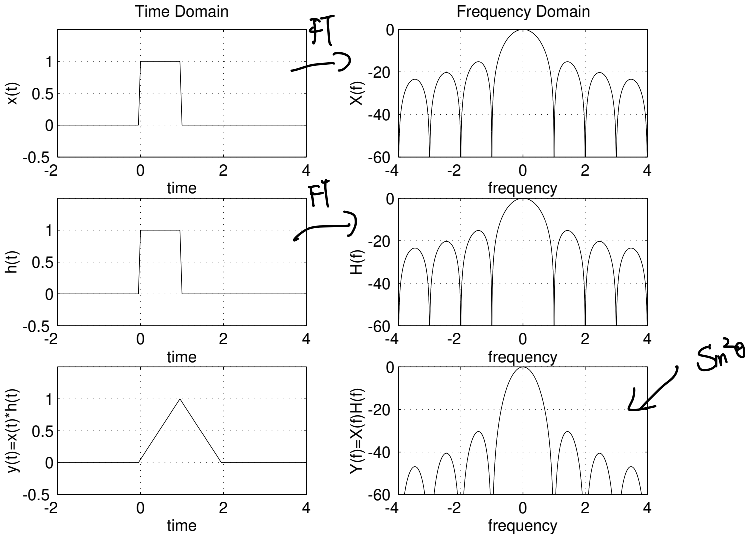
Ex. 1: Rectangular Filtering of Rectangular Pulses

Motivation: This is the simplest form of modulation. A single data bit is transmitted by sending either a positive pulse to represent a 0 or a negative pulse of duration T to represent a 1. The receiver decides which bit was transmitted by filtering the received signal with a filter matched to the transmitted signal and sampling the filter output.



$$
\begin{aligned}
x(t) &= h(t) = p_T(t), \\
y(t) &= h(t) * x(t) = \Lambda_T(t) = \begin{cases} t, & 0 \le t \le T \\ (2 - \frac{t}{T})T, & T \le t \le 2T \\ 0, & \text{otherwise} \end{cases}. \\
X(f) &= H(f) = \underline{T\mathrm{sinc}(fT)e^{-j\pi fT}} = T\frac{\sin(\pi fT)}{\pi fT}e^{-j\pi fT}. \\
Y(f) &= H(f)X(f) = T^2\mathrm{sinc}^2(fT)e^{-j2\pi fT}.
\end{aligned}
$$

Ex. 1: Rectangular Filtering of Rectangular Pulses



Example 2: Square Root Raised Cosine Filtering of Square-Root Raised Cosine Pulses

$$x(t) = h(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T \cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T}.$$

$$X(f) = H(f) = \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

α is called the rolloff.

$0 \le \alpha \le 1$

Example 2: Square Root Raised Cosine Filtering of Square-Root Raised Cosine Pulses=Raised Cosine Pulse

$$y(t) = \frac{\sin(\pi t/T)}{\pi t/T}\frac{\cos(\alpha\pi t/T)}{1-4\alpha^2 t^2/T^2}.$$

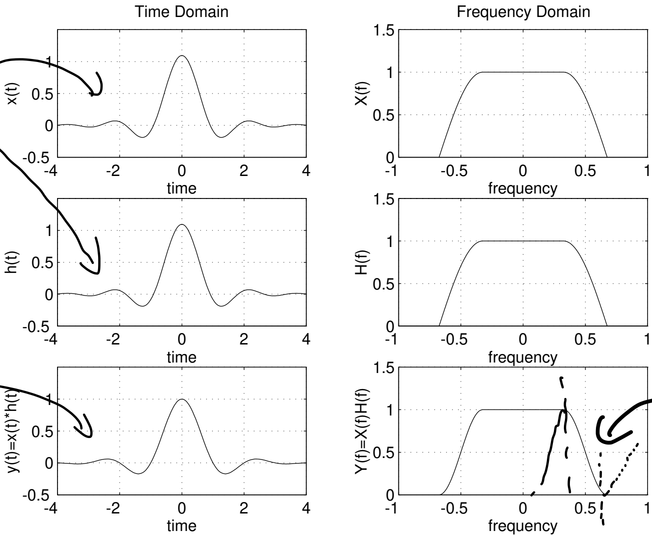
$$Y(f) = \begin{cases} T, & 0 \leq |f| \leq \frac{1-\alpha}{2T} \\ \frac{T}{2}[1-\sin(\pi T(|f|-\frac{1}{2T})/\alpha)], & \frac{1-\alpha}{2T} \leq |f| \leq \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

The parameter α is called the roll-off factor and is between 0 and 1. The (absolute) bandwidth is $W = (1+\alpha)/2T$. Notice that the output is zero at multiples of T except at $t = 0$. Note also that

$$\int_{-\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} Y(f)df = 1$$

Raised Cosine Pulses and Filtering

sinc



rc

rasied cosine

Ex. 3: Gaussian Filtering of a Rectangular Pulse

$$\begin{aligned} x(t) &= p_T(t) \\ h(t) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\{-t^2/(2\sigma^2)\} \end{aligned}$$

The parameter σ is related to the 3dB bandwidth B by

$$\sigma^2 = \frac{\ln(2)}{\pi^2 B}.$$

$$y(t) = \Phi(\frac{t}{\sigma}) - \Phi(\frac{t-T}{\sigma})$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

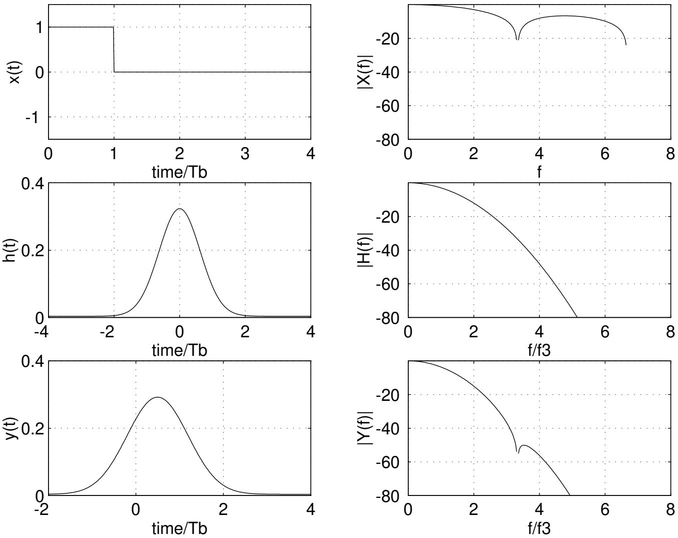
Ex. 3: Gaussian Filtering of a Rectangular Pulse

In the frequency domain

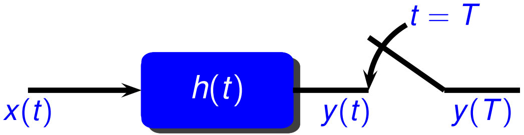
$$\begin{aligned} X(f) &= T\text{sinc}(fT)e^{-j\pi fT} \\ H(f) &= \exp\{-2\pi^2\sigma^2 f^2\} \\ Y(f) &= X(f)H(f). \end{aligned}$$

This is a noncausal filter. In practice a delay must be added to make the filter implementable.

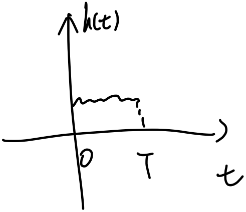
Ex. 3: Gaussian Filtering of a Rectangular Pulse



Sampling a Causal Filter



If we assume that the filter is sampled at time T and that the filter is causal $h(t) = 0, \; t < 0$ ($h(t-\alpha) = 0, \; \alpha > t$) then



$$y(t) = \int_{-\infty}^{t} h(t-\alpha)x(\alpha)d\alpha.$$

If the filter has finite response, say for T seconds then

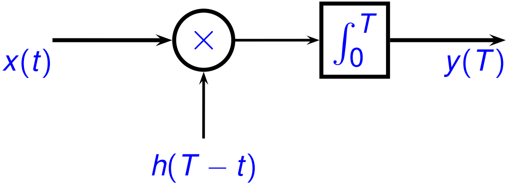
$$y(t) = \int_{t-T}^{t} h(t-\alpha)x(\alpha)d\alpha.$$

If the desired signal is the output sampled at time T.

$$y(T) = \int_{0}^{T} h(T-\alpha)x(\alpha)d\alpha.$$

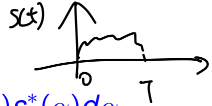
Correlator

This can be implemented with a correlator as shown below



Matched Filters

In a digital communication system it is usual for the received filter to be matched to the transmitted signal. In this case, if $s(t)$ is the transmitted signal and is of duration T beginning at 0, we sample the filter output at time T and $h(t) = s^*(T-t)$. This is called the *matched filter*. The filter output is



$$
\begin{aligned}
y(t) = \int_{-\infty}^{\infty} h(t-\alpha)s^*(\alpha)d\alpha &= \int_{t-T}^{t} \underline{h(t-\alpha)}s^*(\alpha)d\alpha \\
&= \int_{t-T}^{t} \underline{s(T-(t-\alpha))}s^*(\alpha)d\alpha \\
&= \int_{t-T}^{t} s(\alpha-(t-T))s^*(\alpha)d\alpha
\end{aligned}
$$

This is the autocorrelation of the signal $s(t)$.

Matched Filter

The desired filter output is the output sampled at time T.

$$
\begin{aligned}
y(T) = \int_0^T h(T-\alpha)s^*(\alpha)d\alpha &= \int_0^T s(\alpha)s^*(\alpha)d\alpha \\
&= \int_0^T |s(\alpha)|^2 d\alpha \\
&= \text{Energy}
\end{aligned}
$$

Intersymbol Interference Free Pulses

$\varphi_n(0) = \varphi_0(t-nT)$ — time shifted

- If pulses used to transmit data are orthogonal then at the receiver there will be no interference from one symbol (vector component) to another symbol.
- In this case we say the pulses are free of intersymbol interference.
- In this section we derive a condition for the pulses to be intersymbol interference free (i.e. orthogonal) in the frequency domain.

$$\int \varphi_n(t)\varphi_0(t-nT)\,dt = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Intersymbol Interference Free Pulses

Recall that time-shifted orthonormal pulses satisfied

$$\varphi_n(t) = \varphi_0(t - nT).$$

The condition for the signals to be orthonormal in the time domain is

$$\int_t \varphi_0(t)\varphi_n^*(t)dt = \int_t \varphi_0(t)\varphi_0^*(t - nT)dt = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Intersymbol Interference Free Pulses

Let $\Phi_0(f)$ be the Fourier transform of $\varphi_0(t)$. The Fourier transform of $\varphi_0(t-nT)$ is $\Phi_0(f)e^{-j2\pi fnT}$. Parseval's identity (Appendix 2E) then indicates that

$$\begin{aligned}\int_t \varphi_0(t)\varphi_0^*(t-nT)dt &= \int_{f=-\infty}^{\infty} \Phi_0(f)[\Phi_0(f)e^{-j2\pi fnT}]^* df \\ &= \int_{f=-\infty}^{\infty} \Phi_0(f)\Phi_0^*(f)e^{+j2\pi fnT} df. \\ &= \int_{f=-\infty}^{\infty} |\Phi_0(f)|^2 e^{+j2\pi fnT} df.\end{aligned}$$

Intersymbol Interference Free Pulses

Now break the integral into nonoverlapping segments of size $1/T$ and sum the different integrals.

$f = u + m/T$

$u = f - \frac{m}{T}$

$$
\begin{aligned}
\int_t \varphi_0(t)\varphi_0^*(t-nT)dt &= \sum_{m=-\infty}^{\infty} \int_{f=(2m-1)/2T}^{2m+1/2T} |\Phi_0(f)|^2 e^{+j2\pi fnT} df \\
&= \sum_{m=-\infty}^{\infty} \int_{u=-1/2T}^{1/2T} |\Phi_0(u+m/T)|^2 e^{+j2\pi(u+m/T)nT} df \\
&= \int_{u=-1/2T}^{1/2T} [\sum_{m=-\infty}^{\infty} |\Phi_0(u+m/T)|^2] e^{+j2\pi unT} df \\
&= \int_{u=-1/2T}^{1/2T} \tilde{\Phi}(u) e^{+j2\pi unT} df
\end{aligned}
$$

$-3/2T \quad -1/2T \quad 0 \quad \frac{1}{2T} \quad 3/2T \quad f$

where

$$\tilde{\Phi}(f) = \sum_{m=-\infty}^{\infty} |\Phi_0(f+m/T)|^2], \quad -1/(2T) < f < 1/(2T).$$

Intersymbol Interference Free Pulses

Let $\tilde{\varphi}(t)$ be the inverse Fourier transform of $\tilde{\Phi}(f)$. Then

$$\begin{aligned}
\tilde{\varphi}(t) &= \int_{f=-1/(2T)}^{1/(2T)} \tilde{\Phi}(f) e^{+j2\pi ft} df \\
\tilde{\varphi}(nT) &= \int_{f=-1/(2T)}^{1/(2T)} \tilde{\Phi}(f) e^{+j2\pi fnT} df \\
&= \int_t \varphi_0(t)\varphi_0^*(t-nT)dt.
\end{aligned}$$

If $\tilde{\varphi}(nT)$ is 0 for $n \neq 0$ and 1 for $n = 0$ then $\varphi_0(t)$ and $\varphi_n(t)$ will be orthonormal.

Intersymbol Interference Free Pulses

If $\tilde{\Phi}(f)$ satisfies

$$\tilde{\Phi}(f) = \sum_{m=-\infty}^{\infty} |\Phi_0(f + m/T)|^2] = \begin{cases} T, & -1/(2T) \leq f \leq 1/(2T) \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

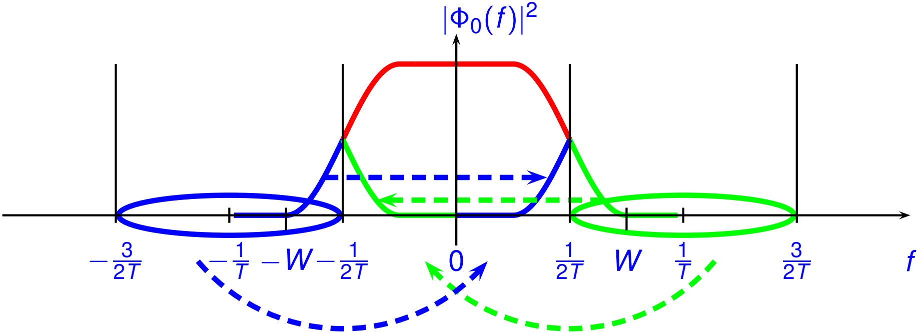
then we have that $\varphi_0(t)$ and $\varphi_n(t)$ will be orthonormal. This is because

$$\begin{aligned} \int_{f=-1/(2T)}^{1/(2T)} Te^{+j2\pi fnT} df &= T \frac{e^{+j2\pi fnT}}{j2\pi nT} \Big|_{f=-1/(2T)}^{1/(2T)} \\ &= T \frac{e^{+j\pi n} - e^{-j\pi n}}{j2\pi nT} \\ &= \frac{e^{j\pi n} - e^{-j\pi n}}{2j\pi n} \\ &= \frac{\sin(\pi n)}{\pi n} \\ &= \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases} \end{aligned}$$

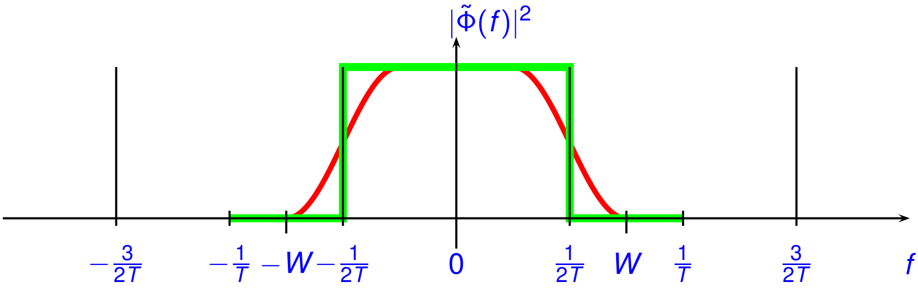
Intersymbol Interference Free Pulses

The condition that the frequency response $\tilde{\Phi}(f)$ is flat for $f \in [-1/(2T), 1/(2T)]$ is the same as the condition that when the frequency content of $|\Phi_0(f)|^2$ in bands outside $[-1/(2T), +1/(2T)]$ are shifted to occupy the frequency range $[-1/(2T), +1/(2T)]$ and summed together the result is a constant 1. That is, the signal in frequency range $[1/(2T), 3/(2T)]$ is shifted left by $1/T$ so as to be in the range $[-1/(2T), 1/(2T)]$. Similarly the signal in frequency range $[3/(2T), 5/(2T)]$ is shifted to the range $[-1/(2T), 1/(2T)]$. Similarly signals with frequencies less than $-1/(2T)$ are shifted to the right. The figure below shows this graphically for one possible frequency pulse shape that has bandwidth that extends past $f = 1/(2T)$ but not past $f = 2/(2T)$. In this case the spectrum from $[1/(2T), 3/(2T)]$ is shifted to $[-1/(2T), 1/(2T)]$ and similarly for negative frequencies.

Intersymbol Interference Free Pulses



Intersymbol Interference Free Pulses



Intersymbol Interference Free Pulses

One pulse shape that satisfy this condition is the "box car" shape.

$$\Phi_0(f) = \begin{cases} \sqrt{T}, & -1/(2T) < f < 1/(2T) \\ 0, & \text{elsewhere.} \end{cases}$$

This corresponds to $\varphi_0(t)$ being a sinc pulse shape. That is

$$\varphi_0(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t/T)}{\pi t/T}.$$

This pulse "dies out" only as $1/t$ which is fairly slow. So when pulses are truncated to make a practical pulse, the duration of this pulse can be quite long.

Intersymbol Interference Free Pulses

- The sinc pulse occupies a bandwidth of $W = 1/(2T)$ Hz.
- Different pulses can be separated by T seconds and not have any intersymbol interference at the receiver.
- Thus each symbol (one dimension) mapped into time-orthogonal pulses separated by T seconds occupies a bandwidth of $W = 1/(2T)$.
- Each pulse can have an arbitrary amplitude and there is no intersymbol interference at the output of the receiver.

Intersymbol Interference Free Pulses

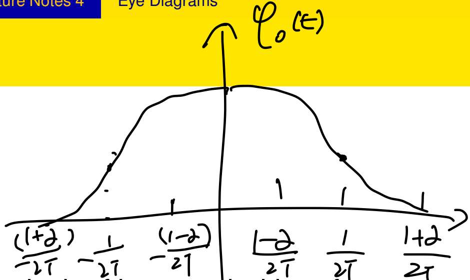
- A sequence of N pulses with each pulse separated in time by T seconds will also occupy bandwidth $1/2T$.
- Now there are N dimensions in time (roughly) NT with bandwidth $W = 1/(2T)$. So $N/(NT)$ (the number of dimensions per second) is $1/T = 2W$.
- This is an example of the relation $N = 2WT$ of Lecture 2 which states that there are $2W$ available (orthogonal) dimensions per second in a bandwidth W.

bps/Hz $R = 2r$ bits/dimensions

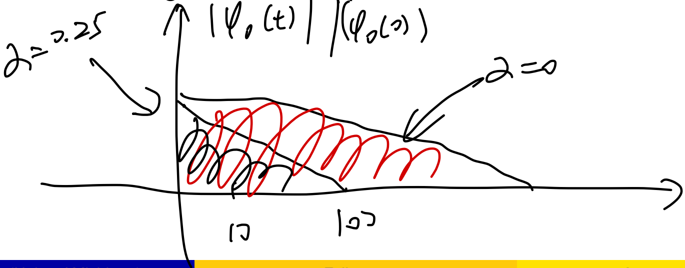
Intersymbol Interference Free Pulses

- Another pulse that satisfies the condition for no intersymbol interference is the square root raised cosine pulse.
- A special case of square-root raised cosine pulse is when $\alpha = 0$ and is the sinc pulse shape discussed above.
- When $\alpha > 0$ the decay in time is $1/t^2$ which is much faster than the case $\alpha = 0$.
- So when truncating the pulse in time, the duration is much shorter than the case $\alpha = 0$.
- However, this pulse will have a larger bandwidth the larger α is
- Any pulse that satisfies the ISI-free condition is called a Nyquist pulse.

Eye Diagrams

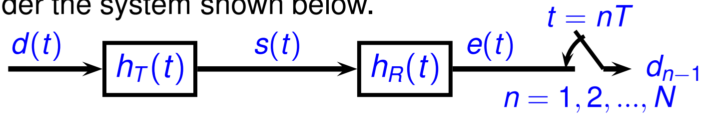


- One way to visualize the lack of intersymbol interference or the amount of intersymbol interference is though an eye diagram.
- The eye diagram is obtained by repeatedly plotting a two symbol interval of the signal at the output of a receiver filter.



Intersymbol Interference Free Pulses

- Consider the system shown below.



$$
\begin{aligned}
d(t) &= \sum_n d_n \delta(t - nT) \\
h_T(t) &= \varphi_0(t) \\
h_R(t) &= \varphi_0^*(T - t)
\end{aligned}
$$

- A sequence of data symbols d_n is the input to the transmitter filter $h_T(t)$ and the output of the receiver filter $h_R(t)$ is $e(t)$.

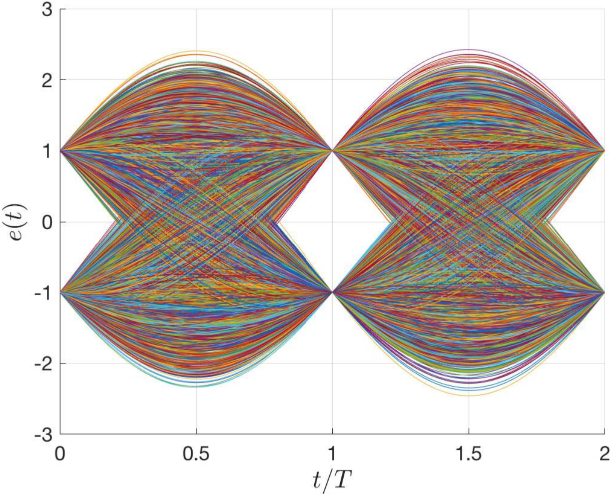
Eye Diagrams

- If the pulses $\varphi_0(t - nT)$ are orthonormal for different n over the time interval of the pulses then at the appropriate sampling time there will be no intersymbol interference at the output of the receiver filter.
- At other times the output of the filter will depend on other data symbols.
- The eye diagram is a way to visualize the filter output for many different possible data sequences.

Eye Diagrams

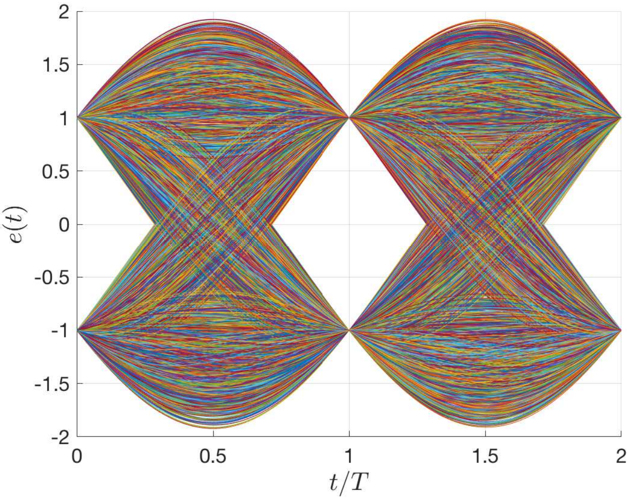
- The eye diagram for a square-root raised cosine pulse used at the transmitter and then filtered by another square-root raised cosine pulse at the receiver is shown in below.
- In this case the data is either +1 or -1, i.e. $d_n \in \{+1, -1\}$ and the symbol duration is 1 second.
- Different sequences of data bits are considered which changes the signal at the output of the receiver.
- Each distinct signal plotted in the eye diagram corresponds to a distinct sequence of data bits.

Eye diagram for square-root raised cosine pulses, $\alpha = 0.05$

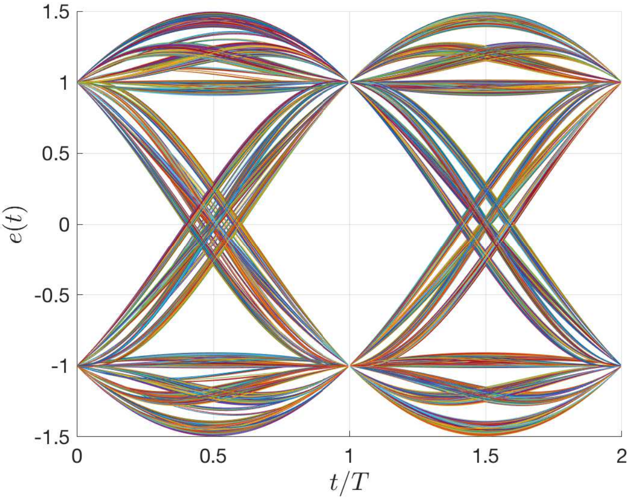


- Similarly the eye diagram for the square-root raised cosine pulse with $\alpha = 0.25, 0.5$ and 0.75 are shown below.
- The eye diagram illustrates the effect of imperfect sampling at the receiver.
- As can be seen in these figures the output for $\alpha = 0.05$ is much more sensitive to not sampling at the correct time compared to $\alpha = 0.5$ or 0.75.

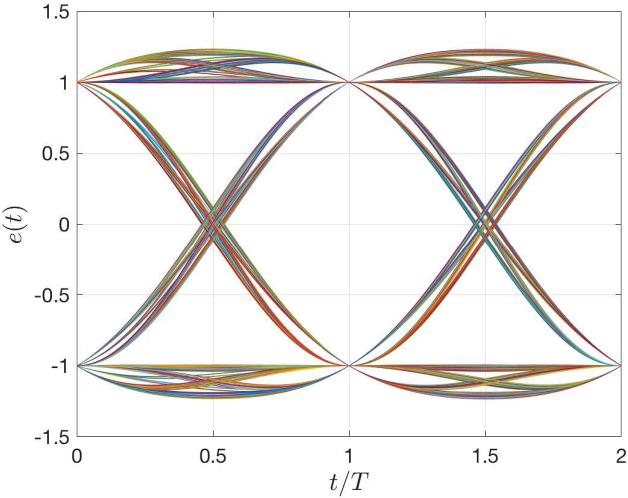
Eye diagram for square-root raised cosine pulses, $\alpha = 0.25$



Eye diagram for square-root raised cosine pulses, $\alpha = 0.5$



Eye diagram for square-root raised cosine pulses, $\alpha = 0.75$



Eye diagram for square-root raised cosine pulses with 4-ary PAM, $\alpha = 0.5$

$d_0 \in \{\pm 1, \pm 3\}$

-3 -1 1 3