University of Michigan Fall 2021

EECS501: Homework 10

Assigned: Nov 16, 2021
Due: Nov 30, 2021 at 11:59PM on gradescope
Text: "Probability and random processes" by J. A. Gubner

1. Markov or Not [10 points]

A die is rolled repeatedly. Determine which of the following processes are Markov chains:

(a) The largest number X_n shown up to the n-th roll.

(b) The number N_n of sixes in n rolls.

(c) At time n the time C_n since the most recent six.

(d) At time n the time B_n until the next six.

2. Markov Chain [20 points]

Consider a gambling game in which you continuously flip a biased coin. For each flip, the head shows up with probability 1/4 and the tail shows up with probability 3/4. You get 2 dollars when both the current flip and the previous flip are heads. Let 1 represent the flip is a head and 0 represent the flip is a tail. Then for 1010, you get 0 dollar; for 11011, you get four dollars; and for the sequence 111, you also get four dollars (you get two dollars in both the second and third flip).

(a) How much you win in the sequence 1101111?

(b) Define a Markov chain whose state is (previous flip, current flip), so the Markov chain has four states $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. Please draw the Markov chain diagram and write down the transition probability matrix.

(c) Calculate the stationary distribution π of the Markov chain and the expected amount of dollars you win per flip at the steady-state.

3. PageRank [10 points]

Consider the following three-node network as shown in Figure 1.

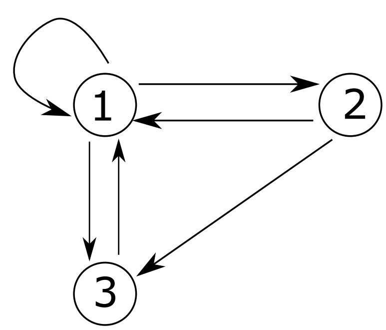

Figure 1: A three-node network

Compute the rank vector r by solving the stationary distribution of the corresponding Markov chain.

4. **Markov Molecules** [10 points]

Two containers A and B are connected through a small aperture and are filled with some ideal gas. A total of m molecules are distributed unequally between them at time 0. Assume some m_A in A and m_B in B, where $m_A + m_B = m$. Assume that at each time slot, one molecule is picked at random from m molecules in the two containers, and is allowed to pass through the aperture to the other. Let X_n denote the number of molecules in A after n time slots. Find the stationary distribution of X_n.

5. **Generalization of Kelly's formula** [20 points]

You are given one million dollars. Each day you bet x dollars on a coin toss. If a head appears, you get Ax dollars ($A > 1$); and if a tail appears, you get Bx dollars ($B < 1$). The head shows up with probability p and the tail shows up with probability $1-p$. Please compute the optimal α such that betting α fraction of your wealth each day maximizes the growth of your wealth.