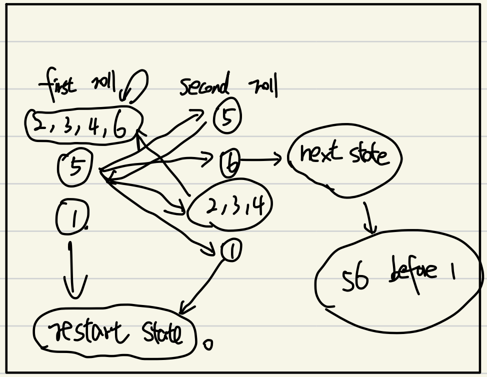
Homework 2 YUZHAN JIANG

P1.



1. let event A_1 = 1 appears in the first roll
   - A_2 = 2, 3, 4, 6 appears in the first roll
   - A_3 = 5 appears in the first roll
   - B_1 = 6 appears in the second roll
   - B_2 = 5 appears in the second roll
   - B_3 = 1 appears in the second roll
   - B_4 = 2, 3, 4 appears in the second roll
   - W = 56 appears before 1

$\therefore$ $P(W^c) = P(\{1 \text{ appears before } 56\}) = 1 - P(W)$

By the low of total probability:

① $P(W) = P(W|A_1)P(A_1) + P(W|A_2)P(A_2) + P(W|A_3)P(A_3)$ where $P(W|A_1) = 0$

$P(W|A_2) = P(W)$ since 2, 3, 4, 6 appears in the first roll we reset the game

② $P(W|A_3) = \overset{(56\ldots)}{P(W|A_3B_1)}P(B_1) + \overset{(55\ldots)}{P(W|A_3B_2)}P(B_2) + \overset{0}{P(W|A_3B_3)}P(B_3) + \overset{(5\,234)}{P(W|A_3B_4)}P(B_4)$

where $P(W|A_3B_3) = 0$, $P(W|A_3B_1) = 1$,

$P(W|A_3B_2) = P(W|A_3)$ since 5 appears in the second roll, we remain in "$P(W|A_3)$ state"

$P(W|A_3B_4) = P(W|A_2) = P(W)$ since 2, 3, 4 appears in the second roll, we continue and reset the state of dice game

$\therefore$ Simplify ①, ② we have,

① $P(W) = P(W)P(A_2) + P(W|A_3)P(A_3)$

② $P(W|A_3) = P(B_1) + P(W|A_3)P(B_2) + P(W)P(B_4)$

$P(A_2) = P(\{2,3,4,6\})$ appears in the first roll

$= \frac{4}{6}$

$= \frac{2}{3}$

$P(A_3) = \frac{1}{6}$ $\quad P(B_1) = \frac{1}{6}$ $\quad P(B_2) = \frac{1}{6}$ $\quad P(B_4) = \frac{1}{2}$

Simplify ①, ② again, we have,

① $P(W) = \frac{2}{3}P(W) + \frac{1}{6}P(W|A_3)$

② $P(W|A_3) = \frac{1}{6} + \frac{1}{6}P(W|A_3) + \frac{1}{2}P(W)$

$$\Rightarrow \begin{cases} \frac{1}{3}P(W) = \frac{1}{6}P(W|A_3) \\ \frac{5}{6}P(W|A_3) = \frac{1}{2}P(W) + \frac{1}{6} \end{cases}$$

$\Rightarrow$ we solve these two equations, we get:

$P(W) = \frac{1}{7}$ $P(W|A_3) = \frac{2}{7}$

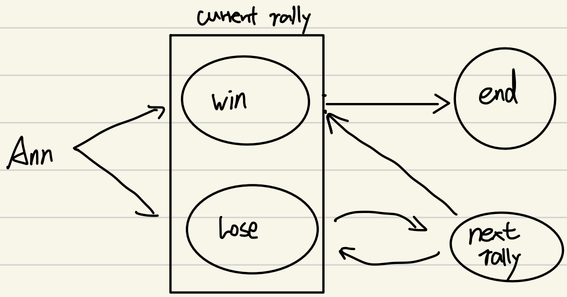
Thus, $P(W^c) = 1 - \frac{1}{7} = \frac{6}{7}$, the probability of observing "1" any time before "56" is $\frac{6}{7}$

P2.

p = Ann wins a rally against Bob

$1-p$ = Ann loses against Bob

Let A = Ann wins the next point, $P(A) = ?$



Let B_1 be the event Ann wins the current rally

B_2 be the event Anns loses current rally and wins next rally

B_3 be the event Anns lose current rally and loses next rally

By the law of total Probability,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

Where $P(B_1) = P$, $P(B_2) = (1-P)P$, $P(B_3) = (1-P)^2$

$$P(A|B_1) = \frac{P(A \cap B_1)}{P(B_1)} = 1, \quad P(A|B_2) = P(A)$$ since Ann loses the current game, Ann goes to next rally ⇒ "stay the same rally state"

$$P(A|B_3) = 0$$

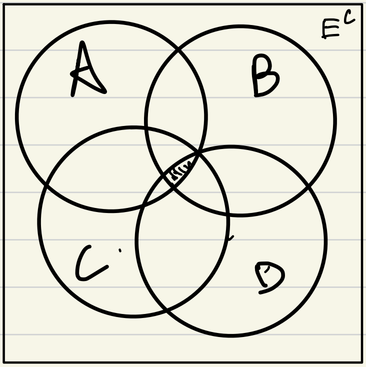
Therefore, we simplify,

$$P(A) = P + (1-P)P \cdot P(A) + 0$$

$$P(A)(1-(P-P^2)) = P$$

$$P(A) = \frac{P}{1-P+P^2}$$

P3.



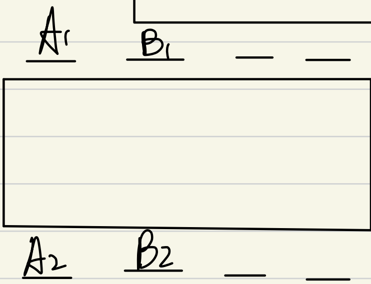
Let A be event that couples A_1, A_2 sit facing each other

Let B be event that couples B_1, B_2 sit facing each other

Let C be event that couples C_1, C_2 sit facing each other

Let D be event that couples D_1, D_2 sit facing each other.

Let E be event that at least one couples are facing each other.



By Inclusion-Exclusion princle:

$$\underbrace{P(A \cup B \cup C \cup D)}_{P(E)} = P(A) + P(B) + P(C) + P(D) - P(A\cap B) - P(A\cap D) - P(C\cap D) - P(B\cap C) - P(B\cap D) - P(A\cap C) + P(A\cap B\cap C) + P(A\cap B\cap D) + P(B\cap C\cap D) + P(A\cap C\cap D) - \underline{P(A\cap B\cap C\cap D)}$$

$$P(A)+P(B)+P(C)+P(D) = \frac{\binom{4}{1}\frac{4!}{(4-1)!}\cdot 2\cdot 6!}{8!} = \frac{4}{7}$$

($\binom{4}{1}$ represents one couple is chosen from 4 couples,

$\frac{4!}{(4-1)!}$ represent that there are 4 positions that couple can sit,

2 represent that the couple can switch their position,

6! represent that cases that the rest people can sit)

$$P(A\cap B)+P(A\cap C)+P(A\cap D)+P(B\cap C)+P(B\cap D)+P(C\cap D)$$

$$= \frac{\binom{4}{2}\cdot\frac{4!}{(4-2)!}\cdot 2^2\cdot 4!}{8!}$$

$$= 6\times 12\times 4\times 24$$

$$= \frac{6}{35}$$

($\binom{4}{2}$ represent two couples are chosen from 4 couples.

$\frac{4!}{(4-2)!}$ represents the cases the # of position the two couples can sit.

....)

$$P(A\cap B\cap C) + P(A\cap B\cap D) + P(A\cap C\cap D) + P(B\cap C\cap D)$$

$$= \frac{\binom{4}{3}\frac{4!}{(4-3)!}\,2^3\cdot 2!}{8!}$$

$$= \frac{4\cdot 24\cdot 8\times 2}{8!}$$

$$= \frac{1}{35}$$

$$P(A\cap B\cap C\cap D) = \frac{\binom{4}{4}\frac{4!}{(4-4)!}\,2^4}{8!}$$

$$= \frac{1\cdot 24\times 16}{8!}$$

$$= \frac{1}{105}$$

$$\therefore P(E) = \frac{4}{7} - \frac{6}{35} + \frac{1}{35} - \frac{1}{105}$$

$$= \frac{3}{7}$$

$\therefore$ At least one couple is facing across from each other w.p. $\frac{3}{7}$

P4:

Let E denote the event that the system works.

Let A_1 be function 1 works, A_2 be function 2 works, A_3 be function 3 works

A_4 be function 4 works, A_5 be functions 5 works

We are given that $P(A_1) = P(A_2) = P(A_3) = P(A_4) = P(A_5) = p = 0.9$

$$P(E) = P(A_1 \cap A_2) \cup P(A_3) \cup P(A_4 \cap A_5)$$

$$= P(A_1 \cap A_2) + P(A_3) + P(A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_3) - P(A_3 \cap A_4 \cap A_5) - P(A_1 \cap A_2 \cap A_4 \cap A_5) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

(by Inclusion-Exclusion Principle)

$$= p^2 + p + p^2 - p^3 - p^3 - p^4 + p^5$$

$$= p + 2p^2 - 2p^3 - p^4 + p^5$$

(a) $$P(A_2 \cup A_3 | E) = \frac{P((A_2 \cup A_3) \cap E)}{P(E)}$$

$$= \frac{P(A_2 \cap E \cup A_3 \cap E)}{P(E)}$$

$$= \frac{P(A_2 \cap E) + P(A_3 \cap E) - P(A_2 \cap A_3 \cap E)}{P(E)}$$ (By Inclusion-Exclusion princle)

$$= P(A_2 | E) + P(A_3 | E) - P(A_2 \cap A_3 | E)$$

$$P(A_2 | E) = \frac{P(E | A_2) P(A_2)}{P(E)}$$ (By Bayes' Theorem)

Where $P(E | A_2) = P(A_2 \cup A_3 \cup (A_4 \cap A_5))$

$$= P(A_2) + P(A_3) + P(A_4 \cap A_5) - P(A_2 \cap A_3) - P(A_2 \cap A_4 \cap A_5) - P(A_3 \cap A_4 \cap A_5) + P(A_2 \cap A_3 \cap A_4 \cap A_5)$$ (By Inclusion-Exclusion princle)

$$= p + p + p^2 - p^2 - p^3 - p^3 + p^4$$

$$= 2p - 2p^3 + p^4$$

$$\therefore \quad P(A_2|E) = \frac{(2p-2p^3+p^4)p}{P(E)}$$

Similarly, $P(A_3|E) = \dfrac{P(E|A_3)\,P(A_3)}{P(E)}$ (Since $P(E|A_3)=1$)

$$= \frac{p}{P(E)}$$

$$P(A_2 \cap A_3 | E) = \frac{P(E|A_2 \cap A_3)\,P(A_2 \cap A_3)}{P(E)}$$

$$= \frac{p^2}{P(E)}$$ (Since $P(E|A_2 \cap A_3) = 1$)

Therefore, $P(A_2 \cup A_3 | E) = \dfrac{2p^2-2p^4+p^5+p-p^2}{P(E)}$

$$= \frac{p+p^2-2p^4+p^5}{p+2p^2-2p^3-p^4+p^5}$$

$$= 0.9919$$

(b) Based on Part (a),

$P(E|A_2 \cup A_3) = \dfrac{P(A_2 \cup A_3|E) \cdot P(E)}{P(A_2 \cup A_3)}$ (By Bayes' Theorem)

where $P(A_2 \cup A_3) = P(A_2) + P(A_3) - P(A_2 \cap A_3)$ (By inclusion-Exclusion Principle)

$$= p+p-p^2$$

$$= 2p-p^2$$

$$\therefore \quad P(E|A_2 \cup A_3) = \frac{p+p^2-2p^4+2p^5}{2p-p^2}$$

$$= 0.9983$$

P5:

Let N denote the number of test conducted.

For each time, the company will continue test under 2 following situation:

① It chooses B-2 battery's assembly line

② It chooses B-1 battery's assembly line, but it tests no fault.

Thus, When $N=1$ denote "the company will only test once,

$$P(X=1) = \frac{1}{2} \cdot P_1$$

when $N=2$:

$$P(X=2) = \left(\frac{1}{2} + \frac{1}{2}(1-P_1)\right) \frac{1}{2} P_1$$
$$= \left(1 - \frac{1}{2}P_1\right) \cdot \frac{1}{2} P_1$$

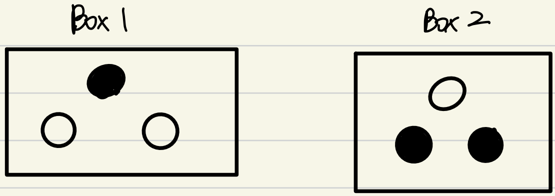
When $N=3$:

$$P(X=3) = \left(1 - \frac{1}{2}P_1\right)^2 \cdot \frac{1}{2} P_1$$

Therefore, $P(X=N) = \left(1 - \frac{1}{2}P_1\right)^{N-1} \cdot \frac{1}{2} P_1$

$$= \frac{P_1}{2} \cdot \left(1 - \frac{P_1}{2}\right)^{N-1}$$

P6.



(a) Let A_1 be the event that box 1 is chosen

A_2 be the event that box 2 is chose

X be the color that the ball is drawn (i.e.: $X = \text{Black}$ or $X = \text{White}$)

$P(A_1) = P$

Recall MAP:

$$P(A_1 | X = \text{white}) = \frac{P(X=W|A_1)\cdot P(A_1)}{P(X=W|A_1)P(A_1) + P(X=W|A_1^c)P(A_1^c)}$$

$$= \frac{\frac{2}{3}\cdot P}{\frac{2}{3}\cdot P + \frac{1}{3}\cdot(1-P)}$$

$$= \frac{2P}{P+1}$$

$$P(A_1|X=B) = \frac{P(X=B|A_1)\cdot P(A_1)}{P(X=B|A_1)P(A_1) + P(X=B|A_1^c)P(A_1^c)}$$

$$= \frac{\frac{1}{3}\cdot P}{\frac{1}{3}P + (1-P)\frac{2}{3}}$$

$$= \frac{P}{2-P}$$

$$P(A_2|X=\text{white}) = \frac{P(X=W|A_2)P(A_2)}{P(X=W|A_2)\cdot P(A_2) + P(X=W|A_2^c)P(A_2^c)}$$

$$= \frac{\frac{1}{3}\cdot(1-P)}{\frac{1}{3}(1-P) + \frac{2}{3}P}$$

$$= \frac{1-P}{1+P}$$

$$P(A_2|X=\text{Black}) = \frac{P(X=B|A_2)P(A_2)}{P(X=B|A_2)P(A_2) + P(X=B|A_2^c)P(A_2^c)}$$

$$= \frac{\frac{2}{3}(1-P)}{\frac{2}{3}(1-P) + \frac{1}{3}P}$$

$$= \frac{2-2P}{2-2P+P}$$

$$= \frac{2-2P}{2-P}$$

If the drawn ball is white, the MAP for choose box 1 satistify:

$$P(A_1|X=\text{white}) > P(A_2|X=\text{white})$$

$$\frac{2P}{P+1} > \frac{1-P}{1+P}$$

$$P > \frac{1}{3}$$

(continue...)

If the drawn ball is black, and MAP for choosing boxing 1 satisfy:

$$P(A_1 | X = \text{Black}) > P(A_2 | X = \text{Black})$$

$$\frac{P}{2-P} > \frac{2-2P}{2-P}$$

$$P > \frac{2}{3}$$

(b) If no ball is drawn, the error probability is $1 - \frac{1}{2} = \frac{1}{2}$

Based on part (A)

Since $P = \frac{1}{2} > \frac{1}{3}$, it means if the drawn ball is white, MAP says that it chooses box 1.

However, if the error happens,

① if the drawn ball is white, actually it is taken from A_2 (incorrect decision)

$$P(A_2 | X = \text{white}) = \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

$$P(A_1 | X = \text{white}) = \frac{2 \cdot \frac{1}{2}}{\frac{1}{2} + 1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

② if the drawn ball is ball, MAP chooses box 2 (but actually from box 1, error!)

$$P(A_2 | X = \text{Black}) = \frac{2 - 1}{2 - \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$P(A_1 | X = \text{Black}) = \frac{1}{3}$$

$$P(\text{error}) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}$$

$$= \frac{1}{3} < \frac{1}{2}$$