1. State TRUE or FALSE

(a) $f_{Y_2,Y_3}(y_2, y_3) = \int_0^{y_2} e^{y_3} dy_1 = y_2 e^{y_3}$

$$f_{Y_1|Y_2,Y_3}(y_1|y_2, y_3) = \frac{f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)}{f_{Y_2,Y_3}(y_2, y_3)} = \frac{1}{y_2}$$

$$E(Y_1|Y_2, Y_3) = \int_0^{y_2} y_1 f_{Y_1|Y_2,Y_3}(y_1|y_2, y_3) dy_1 = \int_0^{y_2} \frac{y_1}{y_2} dy_1 = \frac{Y_2}{2}$$

True

(b) $Var(Z_n) = nVar\left(\frac{1}{\sqrt{n}} X_1\right) = Var(X_1)$

$$Var(X_1) = E(X_1^2) - E(X_1)^2 = \frac{1}{2}$$

$\therefore Var(Z_n) \neq 1$

False

2. Estimation of Gaussian Vector

$\because (X, Y)$ are jointly Gaussian

$$\therefore E(X|Y) = E(X) + Cov(X, Y)Cov(Y)^{-1}\big(Y - E(Y)\big)$$

$$E(X) = E(Y) = E(HX + N) = 0$$

$$Cov(X, Y) = E(XY^T) = E(XX^T H^T + XN^T) = H\Sigma_X + E(XN^T) = H\Sigma_X$$

$$Cov(Y)^{-1} = E\big((HX + N)(X^T H^T + N^T)\big)^{-1} = E\big((HX + N)(X^T H^T + N^T)\big)^{-1}$$

$$= \big(H\Sigma_X H^T + \Sigma_N + 2HE(XN^T)\big)^{-1} = (H\Sigma_X H^T + \Sigma_N)^{-1}$$

$$E(X|Y) = H\Sigma_X (H\Sigma_X H^T + \Sigma_N)^{-1} Y$$

3. Chernoff Inequality

(a) $P(Z \geq p + \delta) = P\big(e^{\theta \sum_{i=1}^n X_i} \geq e^{\theta n(p+\delta)}\big) \leq \frac{E\left(e^{\theta X_1}\right)^n}{e^{\theta n(p+\delta)}} = e^{-n\left(\theta(p+\delta) - \Lambda(\theta)\right)}$

$$\Lambda(\theta) = \log\left(E\left(e^{\theta X_1}\right)\right) = \log\left((1-p) + pe^{\theta}\right)$$

$$f(\theta) = \theta(p + \delta) - \log\left((1-p) + pe^{\theta}\right)$$

$$f'(\theta) = p + \delta - \frac{1}{(1-p) + pe^{\theta}} pe^{\theta}$$

When $f'(\theta) = 0$, $e^{\theta} = \frac{(1-p)(p+\delta)}{p(1-p-\delta)}$

$$\max f(\theta)_{\theta>0} = (p+\delta)\log\frac{(1-p)(p+\delta)}{p(1-p-\delta)} - \log\frac{(1-p)}{1-p-\delta}$$
$$= (p+\delta)\log\frac{p+\delta}{p} + (1-p-\delta)\log\frac{1-p-\delta}{1-p}$$
$$= D((p+\delta)||p)$$

$\therefore P(Z \geq p+\delta) \leq e^{-nD((p+\delta)||p)}$

(b) $P(-Z \geq \delta - p) = P\left(e^{-\theta\sum_{i=1}^{n} X_i} \geq e^{\theta n(\delta-p)}\right) \leq \frac{E\left(e^{-\theta X_1}\right)^n}{e^{-\theta n(p-\delta)}} = e^{-n\left(-\theta(p-\delta)-\Lambda(-\theta)\right)}$

$g(-\theta) = -\theta(p-\delta) - \log\left((1-p)+pe^{-\theta}\right)$

$g'(-\theta) = p - \delta - \frac{1}{(1-p)+pe^{-\theta}}pe^{-\theta}$

When $g'(-\theta) = 0$, $e^{-\theta} = \frac{(1-p)(p-\delta)}{p(1-p+\delta)}$

$$\max g(-\theta)_{\theta<0} = (p-\delta)\log\frac{(1-p)(p-\delta)}{p(1-p+\delta)} - \log\frac{(1-p)}{1-p+\delta}$$
$$= (p-\delta)\log\frac{p-\delta}{p} + (1-p+\delta)\log\frac{1-p+\delta}{1-p}$$
$$= D((p-\delta)||p)$$

$\therefore P(Z \leq p-\delta) \leq e^{-nD((p-\delta)||p)}$

$P(|Z-p| \geq \delta) \leq e^{-nD((p+\delta)||p)} + e^{-nD((p-\delta)||p)}$

When $\delta > 0$, $D((p+\delta)||p)$ and $D((p-\delta)||p)$ are monotonically increasing given p. $D((p+\delta)||p) > 0$, $D((p-\delta)||p) > 0$.

$\therefore P(|Z-p| \geq \delta)$ decays exponentially with n.

4. Concentration Bounds

$$P_{X_i}(x) = \begin{cases} 0.3 & x = 1 \\ 0.2 & x = 0 \\ 0.5 & x = -1 \end{cases}$$

$E(X_i) = -0.2,\ Var(X_i) = 0.8 - 0.04 = 0.76$

$$Z = \sum_{i=1}^{400} X_i$$

$E(Z) = -80,\ Var(Z) = 304$

Chebyshev's inequality:

$$P(Z \geq 0) \leq P(|Z+80| \geq 80) \leq \frac{Var(Z)}{80^2} = 0.0475$$

Central Limit Theorem:

$$P(Z \geq 0) = P\left(\frac{S+80}{\sqrt{304}} \geq \frac{80}{\sqrt{304}}\right) \sim Q(4.588) = 2.238 \times 10^{-6}$$

Chernoff bound:

$$P(Z \geq 0) = P\left(\frac{1}{100}\sum_{i=1}^{400} X_i \geq 0\right)$$

$$\Lambda(\theta) = \log\left(0.5e^{-\theta} + 0.2 + 0.3e^{\theta}\right)$$

$$\frac{d\Lambda(\theta)}{d\theta} = \frac{1}{0.5e^{-\theta} + 0.2 + 0.3e^{\theta}}\left(0.3e^{\theta} - 0.5e^{-\theta}\right) = 0$$

$$e^{\theta} = \sqrt{\frac{5}{3}}$$

$$\sup -\Lambda(\theta) = -\log\left(\sqrt{\frac{3}{5}} + \frac{1}{5}\right)$$

$$P(Z \geq 0) \leq \left(\sqrt{\frac{3}{5}} + \frac{1}{5}\right)^{400} = 3.388 \times 10^{-5}$$

5. Wick's theorem

$$E[X_1 X_3^2 X_4] = C_{13}C_{34} + C_{13}C_{34} + C_{14}C_{33} = 2C_{13}C_{34} + C_{14}C_{33}$$

$$E[X_1^2 X_2^2] = C_{11}C_{22} + C_{12}C_{12} + C_{12}C_{12} = C_{11}C_{22} + 2C_{12}^2$$

$$E[X_1^6] = 5 \times 3 \times C_{11}^3 = 15[E(X_1^2)]^3$$

6. MMSE and LMMSE

$$\hat{E}[X|Y] = E(X) + \frac{Cov(X,Y)}{Var(Y)}\left(Y - E(Y)\right)$$

$$E(X) = \frac{1}{\lambda},\ E(Y) = E(X) + E(N) = \frac{1}{\lambda}$$

$$Cov(X, X+N) = E\left(X(X+N)\right) - E(X)E(X+N) = E(X^2) - E(X)^2 = \frac{1}{\lambda^2}$$

$$Var(X+N) = E(X^2 + 2XN + N^2) - E(X+N)^2 = Var(X) + Var(N) = \frac{1}{\lambda^2} + \sigma^2$$

$$\hat{E}[X|Y] = \frac{1}{\lambda} + \frac{1}{1+\lambda^2\sigma^2}\left(Y - \frac{1}{\lambda}\right) = \frac{Y + \lambda\sigma^2}{1+\lambda^2\sigma^2}$$

$$MSE = E\left(\left(X - \hat{E}[X|Y]\right)^2\right) = E\left(\left(\frac{\lambda^2\sigma^2 X - N - \lambda\sigma^2}{1+\lambda^2\sigma^2}\right)^2\right)$$

$$= \frac{1}{(1+\lambda^2\sigma^2)^2} E(\lambda^4\sigma^4X^2 + N^2 + \lambda^2\sigma^4 - 2\lambda^2\sigma^2XN - 2\lambda^3\sigma^4X + 2\lambda\sigma^2N)$$
$$= \frac{1}{(1+\lambda^2\sigma^2)^2} (2\lambda^2\sigma^4 + \sigma^2 + \lambda^2\sigma^4 - 2\lambda^2\sigma^4)$$
$$= \frac{\sigma^2}{1+\lambda^2\sigma^2}$$