EECS 501
Discussion 11
Solution

1 Review

- Kelly's Formula: A gambler bets x dollars (α fraction of his wealth) each time. He wins with probability p and gets Ax dollars and loses with probability $1-p$ and gets 0 dollars:
  - Define Z_n such that

    $$P(Z_n = 1-\alpha+A\alpha) = p \quad \text{and} \quad P(Z_n = 1-\alpha) = 1-p$$

    Then

    $$W_T = W_0 \prod_n Z_n \quad \text{or} \quad \log W_T = \log W_0 + \sum_n \log Z_n.$$

    From LLN, we use the following approximation:

    $$\frac{\log W_T}{T} \to p\log(1-\alpha+A\alpha) + (1-p)\log(1-\alpha) \quad (a.s.)$$

  - Maximize wealth:

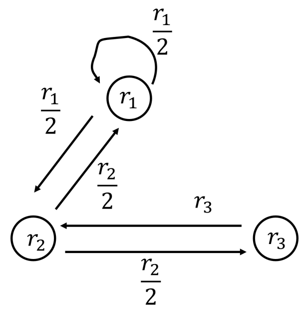
    $$\max_\alpha p\log(1-\alpha+A\alpha) + (1-p)\log(1-\alpha)$$

    $$\frac{p(A-1)}{1-\alpha+A\alpha} - \frac{1-p}{1-\alpha} = 0 \Rightarrow \alpha = \frac{p(A-1)-(1-p)}{A-1}$$

- Page rank:
  - Imagine a random Web surfer.
  - At any time t, surfer is on web i. At time $t+1$ the surfer picks an out-link uniformly at random and goes to the next web.
  - $p_i(t)$: probability that the surfer is at web i at time t.

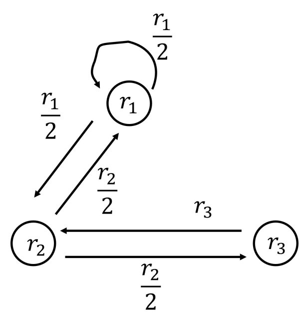
    $$p_i(t+1) = \sum_{j:j\to i} \frac{1}{d_j} p_j(t)$$

  - $r = [r_1, \cdots, r_N]$, where r_i is the importance score of webpage i.
  - r is the stationary distribution of the Markov chain.



2 Practice Problems

Problem 1 Compute the rank vector r of the network below.



Solution:

$$r_i = \sum_{j:j\to i} \frac{1}{d_j} r_j \Rightarrow r = r \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

and we can get $r = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

Problem 2 If the gambler bets x dollars, he wins x with probability p and loses Ax (including the bet) with probability $1-p$. What is the best fraction of wealth to bet at each time.

Solution:

$$P(Z_n = 1+\alpha) = p \quad \text{and} \quad P(Z_n = 1 - A\alpha) = 1-p$$

$$W_T = W_0 \prod_n Z_n \quad \text{or} \quad \log W_T = \log W_0 + \sum_n \log Z_n.$$

$$\frac{\log W_T}{T} \to p \log(1+\alpha) + (1-p)\log(1-A\alpha) \quad (a.s.)$$

Maximize wealth:

$$\max_\alpha p\log(1+\alpha) + (1-p)\log(1-A\alpha)$$

$$\frac{p}{1+\alpha} - \frac{(1-p)A}{1-A\alpha} = 0 \Rightarrow \alpha = \frac{p-(1-p)A}{A}$$

Problem 3 A particle performs a random walk on the vertices of a cube. At each time, it remains where it is with probability 1/4 or moves to one of the three neighboring vertices with probability 1/4. Let v and w be two diametrically opposite vertices. If the walk starts at v find

(a) expected number of time slots until its first return to v.

(b) expected number of time slots until its first visit to w.

Solution:

(a) Besides v and w, the rest 6 vertices can be classified to two groups. The 3 vertices that are adjacent to v will be called state 1 and the rest 3 vertices will be called state 2. v and w are called state 0 and state 3, respectively. Draw the state diagram:

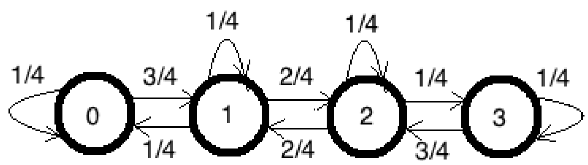


Figure 1: State diagram.

Use one-step analysis:

$$E(0|0) = \frac{1}{4} + \frac{3}{4}(E(0|1) + 1)\,,$$

$$E(0|1) = \frac{1}{4} + \frac{1}{4}(E(0|1) + 1) + \frac{2}{4}(E(0|2) + 1)\,,$$

$$E(0|2) = \frac{2}{4}(E(0|1) + 1) + \frac{1}{4}(E(0|2) + 1) + \frac{1}{4}(E(0|3) + 1)\,,$$

$$E(0|3) = \frac{3}{4}(E(0|2) + 1) + \frac{1}{4}(E(0|3) + 1)\,,$$

$$\Rightarrow E(0|0) = 8,\ E(0|1) = \frac{28}{3},\ E(0|2) = 12,\ E(0|3) = \frac{40}{3}\,.$$

Expected number of time slots until its first return to v is 8.

(b) Use one-step analysis:

$$E(3|0) = \frac{1}{4}(E(3|0) + 1) + \frac{3}{4}(E(3|1) + 1)\,,$$

$$E(3|1) = \frac{1}{4}(E(3|0) + 1) + \frac{1}{4}(E(3|1) + 1) + \frac{2}{4}(E(3|2) + 1)\,,$$

$$E(3|2) = \frac{2}{4}(E(3|1) + 1) + \frac{1}{4}(E(3|2) + 1) + \frac{1}{4}\,,$$

$$E(3|3) = \frac{3}{4}(E(3|2) + 1) + \frac{1}{4}\,,$$

$$\Rightarrow E(3|3) = 8,\ E(3|2) = \frac{28}{3},\ E(3|1) = 12,\ E(3|0) = \frac{40}{3}\,.$$

Expected number of time slots until its first visit to w is $\frac{40}{3}$. Actually we do not need to redo the calculation for part (b). ($E(3|0) = E(0|3)$.)