EECS 551 Homework 9 YUZHAN JIANG

Problem 1

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In [137]:
#Problem 1
n = rand(4:7)
a = randn(n+1) # random vector of polynomial coefficients (a_0,a_1,...,a_n)
b = reverse(a) # reverse the coefficient order
# companion matrix maker:
compan = c-> [-transpose(reverse(c)); [I zeros(length(c)-1)]]
A = compan(a[1:end-1] / a[end])
B = compan(b[1:end-1] / b[end])
[eigvals(A) eigvals(B) 1 ./ eigvals(B)] # study the columns of this array

Out[137]: 4×3 Matrix{ComplexF64}:
 -27.2919+0.0im        -0.912261+0.0im      -1.09618-0.0im
 -1.09618+0.0im       -0.0366409+0.0im      -27.2919-0.0im
 0.477123-0.394317im     1.24532-1.02919im  0.477123+0.394317im
 0.477123+0.394317im     1.24532+1.02919im  0.477123-0.394317im
```

The eigenvalues A are reciprocals of the eigenvalues of B.

Prove:

$$a = [C_0 \;\; C_1 \;\; C_2 \;\cdots\; C_{n-1}] \qquad b = [C_{n-1} \;\; C_{n-2} \;\cdots\cdots\; C_2 \;\; C_1 \;\; C_0]$$

$$(a[1{:}\mathrm{end}-1]/a[\mathrm{end}]) = \left[\frac{C_0}{C_{n-1}} \;\; \frac{C_1}{C_{n-1}} \;\cdots\; \frac{C_{n-2}}{C_{n-1}}\right] \qquad \left[\frac{C_{n-1}}{C_0} \;\; \frac{C_{n-2}}{C_0} \;\cdots\; \frac{C_1}{C_0}\right]$$

A is

$$\begin{bmatrix} -\frac{C_{n-2}}{C_{n-1}} & \cdots & -\frac{C_1}{C_{n-1}} & -\frac{C_0}{C_{n-1}} \\ 1 & 0 & \cdots & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Assume The monic polynomial function for A is

$$P(x) = x^n + \frac{C_{n-2}}{C_{n-1}} x^{n-1} + \cdots + \frac{C_1}{C_{n-1}} x + \frac{C_0}{C_{n-1}}$$

So the monic polynomial function for B is

$$Q(x) = x^n + \frac{C_1}{C_0} x^{n-1} + \cdots + \frac{C_{n-2}}{C_0} x + \frac{C_{n-1}}{C_0}$$

B

$$\begin{bmatrix} -\frac{C_1}{C_0} & -\frac{C_2}{C_0} & \cdots & -\frac{C_{n-2}}{C_0} & -\frac{C_{n-1}}{C_0} \\ 1 & 0 & \cdots & 0 & 0 \\ & 1 & \cdots & 0 & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 1 \end{bmatrix}$$

Divide $Q(x)$ by x^n, we get a new polynomial

$$S(x) = 1 + \frac{C_1}{C_0}\frac{1}{x} + \cdots + \frac{C_{n-2}}{C_0} x^{n-1} + \frac{C_{n-1}}{C_0}\frac{1}{x^n}$$

$$C_0 S(x) = C_0 + C_1 \frac{1}{x} + \cdots + C_{n-2} x^{n-1} + C_{n-1} \frac{1}{x^n}$$

$$\frac{C_0}{C_{n-1}} S(x) = \frac{C_0}{C_{n-1}} + \frac{C_1}{C_{n-1}}\frac{1}{x} + \cdots + \frac{C_{n-2}}{C_{n-1}} x^{n-1} + \frac{1}{x^n}$$

It is easy to see that $P(x) = \frac{C_0}{C_{n-1}} S(\frac{1}{x})$

Let $P(x) = 0$ $\quad \frac{C_0}{C_{n-1}} S(\frac{1}{x}) = 0 \Rightarrow S(\frac{1}{x}) = 0$

The root of $S(\frac{1}{x}) = P(x) = 0$, the roots of $S(x)$ are the reciprocals of the roots of $Q(x)$

$\therefore$ The eigenvalues A are reciprocals of the eigenvalues of B.

P2.

(a) A: $z^N = 1 \quad z^n = e^{-i2\pi k}$

$$z_k = e^{-i2\pi k/N}, \quad \text{for } k = 0, \cdots N-1$$

B: $z_k = e^{-i2\pi(k+1/2)/N} : k=0, \cdots N-1\}$

$$z_k = e^{-i2\pi k/N - i\pi/N}$$

A:
$$\begin{bmatrix} (e^{-i2\pi/N})^{N-1} & (e^{-i4\pi/N})^{N-1} & \cdots & (e^{-i2\pi(N-1)/N})^{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ e^{-i2\pi/N} & e^{-i2\pi 2/N} & \cdots & e^{-i2\pi(N-1)/N} \\ 1 & 1 & & 1 \end{bmatrix}$$

B:
$$\begin{bmatrix} (e^{-i2\pi/N - i\pi/N})^{N-1} & \cdots & (e^{-i2\pi(N-1)/N - i\pi/N})^{N-1} \\ \vdots & \cdots & \vdots \\ e^{-i2\pi/N - i\pi/N} & & e^{-i2\pi(N-1)/N - i\pi/N} \\ 1 & & 1 \end{bmatrix}$$

$B = D \cdot A$ where $D = \begin{bmatrix} (e^{-i\pi/N})^{N-1} & & & \\ & \ddots & & \\ & & e^{-i\pi/N} & \\ & & & 1 \end{bmatrix}$

$$X = [aA \quad b \cdot DA]$$

$$XX' = [aA \quad bDA] \begin{bmatrix} aA' \\ b(DA)' \end{bmatrix} = aA \cdot aA' + bDA \cdot b(DA)'$$

$$= a^2 AA' + b^2 (DA)(DA)'$$

$$= a^2 AA' + b^2 DAA'D$$

$$AA' = \begin{bmatrix} (e^{-i2\pi/N})^{N-1} & \cdots & (e^{-i2\pi(N-1)/N})^{N-1} \\ \vdots & & \vdots \\ 1 & \cdots & e^{-i2\pi(N-1)/N} \\ & & 1 \end{bmatrix} \begin{bmatrix} (e^{+i2\pi/N})^{N-1} & \cdots & (e^{+i2\pi/N}) & 1 \\ \vdots & & & \\ (e^{+i2\pi(N-1)/N})^{N-1} & \cdots & (e^{+i2\pi(N-1)/N}) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1\cdots 1 & & \\ & 1+1\cdots 1 & \\ & & \ddots \\ & & & 1+1+1\cdots 1 \end{bmatrix}$$

$$= NI$$

Similarly,

$$DD' = I$$

$\therefore$ $a^2AA' + b^2DAA'D'$

$$= a^2NI + b^2NI$$

$$= (a^2N + b^2N)I$$

$\therefore$ Matrix X is a tight frame

(b)

The frame bounds is (a^2N+b^2N)

(c) $\hat{x} = \underset{x \in C^{2N}}{\arg\min} \|Xx - y\|_2$

$$\hat{x} = X^{+} y$$

$$= \frac{1}{\alpha} X' y$$

$$= \frac{1}{a^2 N + b^2 N} \begin{bmatrix} aA' \\ bA'D' \end{bmatrix} y$$

$$= \frac{1}{a^2 N + b^2 N} \begin{bmatrix} aA'y \\ bA'D'y \end{bmatrix}$$

P3:

(a) The unitary eigendecomposition of C_1, C_2 is

$$C_1 = Q\Lambda_1 Q'$$
$$C_2 = Q\Lambda_1 Q'$$

All circulant matrices have the same unitary eigenvectors Q

$$\therefore \quad C_1 C_2 = Q\Lambda_1 Q' Q\Lambda_2 Q' = Q\Lambda_1\Lambda_2 Q'$$

and

$$C_2 C_1 = Q\Lambda_2 Q' Q\Lambda_1 Q' = Q\Lambda_2\Lambda_1 Q'$$

$Q\Lambda_1\Lambda_2 Q' = Q\Lambda_2\Lambda_1 Q'$ Since Λ is diagonal matrix

$\therefore$ all circulant matrix commute

(b)

$$C = Q\Lambda Q'$$

$$\therefore C'C = (Q\Lambda Q')'(Q\Lambda Q')$$
$$= Q\Lambda'\Lambda Q'$$

$$CC' = (Q\Lambda Q')(Q\Lambda Q')'$$
$$= Q\Lambda\Lambda' Q'$$

$\Rightarrow C'C = CC'$ Since Λ is diagonal matrix

$\therefore$ all circulant matrices are normal.

3(c)

$$C = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & - & \cdot & \\ \cdot & - & \sim & \smile & - & \\ 1 & 0 & 0 & \cdots 0 & 0 & -1 \end{bmatrix} \in F^{n \times n}$$

$\lambda_j = -1 + w_j$ for $j = 0, 1, \cdots n-1$ where $w_j = \exp(i \frac{2\pi j}{n})$ (By wiki)

$$\lambda_0 = -1 + \exp(0)$$
$$= 0$$
$$\lambda_1 = -1 + \exp(\frac{2\pi i}{n})$$
$$\lambda_2 = -1 + \exp(\frac{4\pi i}{n})$$
$$\vdots$$
$$\lambda_{n-1} = -1 + \exp(i \frac{2\pi(n-1)}{n})$$

3(d)

In [6]:
```
C = [-1 1 0 0
     0 -1 1 0
     0 0 -1 1
     1 0 0 -1]
eigvals(C)
```

Out[6]:
```
4-element Vector{ComplexF64}:
   -2.0000000000000013 + 0.0im
   -1.0000000000000002 - 1.0000000000000002im
   -1.0000000000000002 + 1.0000000000000002im
 -1.8732355726879183e-16 + 0.0im
```

In [13]:
```
for i in 0:3
    @show eigvals_expression = -1 + exp(i *2 * pi * im /4)
end
```

```
eigvals_expression = -1 + exp((i * 2 * pi * im) / 4) = 0.0 + 0.0im
eigvals_expression = -1 + exp((i * 2 * pi * im) / 4) = -0.9999999999999999 + 1.0im
eigvals_expression = -1 + exp((i * 2 * pi * im) / 4) = -2.0 + 1.2246467991473532e-16im
eigvals_expression = -1 + exp((i * 2 * pi * im) / 4) = -1.0000000000000002 - 1.0im
```

3(e) The julia eigenvals expression for $C'C$ is

$$C'C = \begin{bmatrix} 2 & -1 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ -1 & \cdots & \cdots & -1 & 2 \end{bmatrix}$$

$$\lambda_j = 2 + (-1) \cdot w_j + (-1) \cdot w_j^{(n-1)} \quad \text{for } j = 0, 1 \cdots n-1 \quad \text{where } w_j = \exp\left(i \frac{2\pi j}{n}\right) \quad \text{(By wiki)}$$

$$\sigma_i(A) = \sqrt{\text{eig}(A'A)}$$

In [33]:
```julia
function compute_nuclear_norm(C, n)
    norm = 0
    for i in 0:n-1
        eigvals_expression = +2 - exp(i *2 * pi * im /n) - (exp(i *2 * pi * im /n))^(n-1)
        norm = norm + sqrt(eigvals_expression)
    end
    return norm
end
```

Out[33]: compute_nuclear_norm (generic function with 1 method)

Test:

In [34]:
```julia
compute_nuclear_norm(C, 4)
```

Out[34]: 4.82842712474619 - 1.2246467991473532e-16im

In [35]:
```julia
sum(svdvals(C))
```

Out[35]: 4.82842712474619

P4:

(a) $\hat{X} = U_r \hat{\Sigma}_r V_r'$,

$$\hat{\Sigma}_r = \arg\min \frac{1}{2} \| \Sigma_r - S \|_F^2 + \beta \sum_{k=1}^{r} S_k^{\frac{1}{2}}, \quad S = \mathrm{diag}(S_1, \dots S_r)$$

$\Downarrow$

$$\arg\min \left\{ \sum_{k=1}^{r} \left(\frac{1}{2} (\sigma_k - S_k)^2 + \beta S_k^{1/2} \right) \right\}$$

$$\Rightarrow \hat{\sigma}_k = \underset{s \geq 0}{\arg\min} \ \frac{1}{2} (\sigma_k - s)^2 + \beta s^{1/2}$$

It is Homework 8,

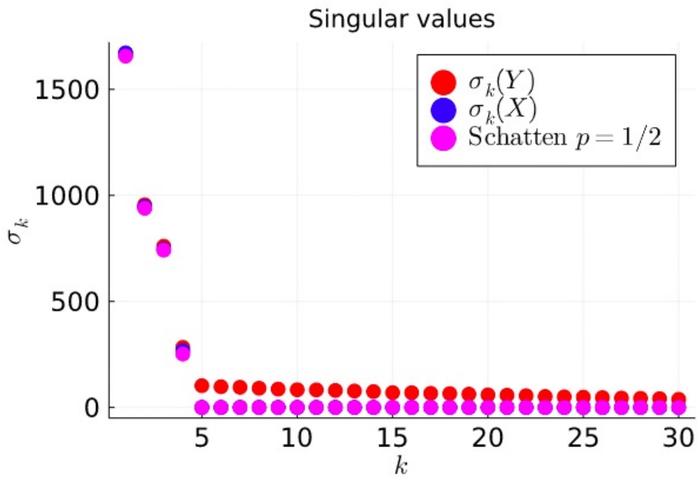
In previous solution:

$$\hat{\sigma}_k = \begin{cases} \frac{4}{3} v \cdot \cos^2 \left(\frac{1}{3} \arccos \left(- \frac{3^{3/2} \beta}{4 v^{3/2}} \right) \right), & v > \frac{3}{2} \beta^{2/3} \\ 0, & \text{otherwise} \end{cases}$$

(c)

```
(l2reg, nrmse(Xr)) = (8.0, 14.127626370282472)
(l2reg, nrmse(Xr)) = (8.5, 11.965607253788333)
(l2reg, nrmse(Xr)) = (9.0, 8.520989739855205)
(l2reg, nrmse(Xr)) = (9.5, 7.088100355507286)
(l2reg, nrmse(Xr)) = (10.0, 7.105809429261925)
(l2reg, nrmse(Xr)) = (10.5, 7.228156064437637)
(l2reg, nrmse(Xr)) = (11.0, 7.615504690985911)
(l2reg, nrmse(Xr)) = (11.5, 14.548670950392834)
(l2reg, nrmse(Xr)) = (12.0, 14.989677301080654)
nrmse(Xs) = 7.102365553759247
```

Out[88]:



5. ① If $m=0$ or $n=0$ $\Rightarrow$ $m+1=1$ or $n+1=1$
P and q don't have a common root.

② Construct the respective companion matices A and B

③ check $A \oplus (-B)$ eigenvalue contain the eigenval of zero

Using $A \oplus (-B) = (A \otimes I_N) + (I_M \otimes B)$

using $\boxed{\text{det}}$ and $\boxed{\text{isapprox}}$