Last (family) name: ____________________

First (given) name: ____________________

I have neither given nor received aid on this examination, nor concealed any violation of the Honor Code.

Signature: ____________________

ID Number: ____________________

EECS 551 Midterm 1, 2021-10-25 6-8PM

- There are 18 problems for a total of 100 points.
- This part of the exam has 8 pages. Make sure your copy is complete.
- Write your name in the upper right of every page!
- You must complete the exam entirely on your own.
- You may use without rederiving any of the results presented in the course notes.
- During the exam you may use two pieces of 8.5×11" paper with notes on both sides, but no electronic devices.
- Clearly $\boxed{\text{box}}$ your final answers. For full credit, show your complete work clearly and legibly in the space provided.
- For full credit, ~~cross out~~ any incorrect intermediate steps.
- For multiple-choice questions, select *all* correct answers.
- To "disprove" any statement, provide a concrete counter-example. For maximum credit, make that counter-example as small and simple as possible, *e.g.*, having the smallest possible matrix dimensions and using the simplest numbers like "0" and "1" as much as possible. For example, to disprove the statement "any square matrix $\boldsymbol{A}$ is invertible," the smallest and simplest counter-example is the 1×1 matrix $\boldsymbol{A} = [0]$.
- For any True-False question, box $\boxed{\text{True}}$ *and* give a short justification if the statement always holds, otherwise box $\boxed{\text{False}}$ *and* give a simple counter-example.
- For multiple-choice questions, $\boxed{\text{box}}$ *all* of the correct given answer choices *and* write a short explanation.
  If none of the answers is always correct, then write in $\boxed{\text{None}}$ and explain briefly.
  If multiple choices are incorrect, it suffices to explain why *one* of them is incorrect.
- You may use the back side of each exam page for scratch work, but that work will not be graded.
  Only the answers in the designated areas will be scanned into gradescope and graded.
- For all JULIA code, assume that the following code has been invoked already:

```
using LinearAlgebra, MIRTjim, FFTW, Random, Plots
```

DO NOT TURN OVER THIS PAGE UNTIL TOLD TO BEGIN!

[4] **1**. For $\boldsymbol{A} = \begin{bmatrix} 3 & -4 \end{bmatrix}$, $\|\boldsymbol{A}\boldsymbol{x}\|_2$ is maximized for unit norm $\boldsymbol{x}$ when $\boldsymbol{x} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

True
False

[4] **2**. If $\boldsymbol{A} \in \mathbb{F}^{9\times 4}$ has 4 nonzero singular values, then $\boldsymbol{A}^{+}\boldsymbol{A} = \boldsymbol{I}$.

True
False

[4] **3**. If $\boldsymbol{A}$ is $M \times N$ with rank N and $M \geq N$, then $\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2 = 0$.

True
False

[4] **4**. If $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}$ then $\min_{\boldsymbol{x}} \|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{y}\|_2 > \min_{\boldsymbol{z}} \|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{y}\|_2$, assuming dimensions match appropriately.

True
False

[4] **5**. When $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \end{bmatrix}$, a unit norm vector $\boldsymbol{x}$ that maximizes $\|\boldsymbol{A}\boldsymbol{x}\|_2$ is:

a: $\begin{bmatrix} 1 & 2 \end{bmatrix}' / \sqrt{5}$
b: $\begin{bmatrix} 6 & 4 & 2 \end{bmatrix}' / \sqrt{14}$
c: $\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}' / \sqrt{14}$
d: $\begin{bmatrix} 2 & 1 \end{bmatrix}' / \sqrt{5}$
e: $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}' / \sqrt{14}$

[4] **6**. A left singular vector of $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 4 & 2 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$, corresponding to its largest singular value is:

a: $\begin{bmatrix} 0 & 2 & 1 \end{bmatrix}'$
b: $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}'$
c: $\begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}'$
d: $\begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}'$

[4] **7**. If $\boldsymbol{A}$ is Hermitian symmetric, then any left or right singular vector of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{A}$.
True
False

[4] **8**. Let $\boldsymbol{A}$ be a $M \times M$ unitary matrix and let $\boldsymbol{B}$ denote the last N columns of $\boldsymbol{A}$, where $1 \leq N \leq M$. What is $\|\boldsymbol{B}\|_2$?

a: 0

b: 1

c: N

d: M

e: $\min(N, M)$

[4] **9**. Nesterov's accelerated gradient method for solving the LS problem $\arg\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2$ will converge if $0 < \mu \leq 1/\sigma_1(\boldsymbol{A}'\boldsymbol{A})$. When $\boldsymbol{A} = \mathbf{1}_2 \mathbf{1}_8'$, where $\mathbf{1}_n =$ `ones(n)`, the maximum value of the step size μ is:

a: 1

b: $1/2$

c: $1/4$

d: $1/8$

e: $1/16$

[4] **10**. $\displaystyle \min_{\boldsymbol{x} \in \mathbb{R}^3} \left\| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\|_2^2 = ?$

a: 0

b: 1

c: 2

d: 3

e: 4

[4] **11**. For $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$, the value of $\|\boldsymbol{A}\|_2$ is:

a: 0
b: 1
c: 5
d: 10
e: 20

[4] **12**. If $\boldsymbol{B}$ is a 200×400 matrix of rank 100, then:

a: $\mathsf{dim}(\mathcal{R}(\boldsymbol{B})) = 100$
b: $\mathsf{dim}\big(\mathcal{R}^\perp(\boldsymbol{B})\big) = 100$
c: $\mathsf{dim}(\mathcal{N}(\boldsymbol{B})) = 100$
d: $\mathsf{dim}\big(\mathcal{N}^\perp(\boldsymbol{B})\big) = 100$
e: The number of distinct singular values is at least 2.

[4] **13**. Let $\boldsymbol{A}$ be a tall matrix having rank $r > 0$ with SVD given by $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}' = \begin{bmatrix} \boldsymbol{U}_r & \boldsymbol{U}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_r \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{V}'$.
Define $\boldsymbol{P}_r^\perp \triangleq \boldsymbol{I} - \boldsymbol{U}_r\boldsymbol{U}_r'$ and $\boldsymbol{P}_0^\perp \triangleq \boldsymbol{I} - \boldsymbol{U}_0\boldsymbol{U}_0'$ and $\boldsymbol{B} \triangleq \boldsymbol{P}_0^\perp \boldsymbol{P}_r^\perp$. Then:

a: $\boldsymbol{B}$ is a unitary matrix
b: $\boldsymbol{B}$ is not a unitary matrix
c: Need more information to assess

[10] **14**. The vectors $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ form a orthonormal basis for a subspace $\mathcal{S}$ of $\mathbb{R}^N$, for $N \geq 5$. Complete the following JULIA function so that, given input vector $\boldsymbol{x} \in \mathbb{R}^N$, it returns the nearest vector in $\mathcal{S}$.
For full credit, your code must use as few floating-point calculations as possible. Explain briefly.

↙ code (answer) | explain ↓

```
function nearest(x, b1, b2, b3)
```

[8] **15**. Complete the following JULIA function so that it returns an orthonormal basis (as a matrix) for the null space of a matrix argument. Assume $\mathsf{dim}(\mathcal{N}(\boldsymbol{A})) > 0$.

↙ code (answer) | explain ↓

```
function nullbasis(A)
    (U,s,V) = svd(A, thin=false)
```

[10] **16**. Let $\boldsymbol{U}$ and $\boldsymbol{V}$ denote unitary $N \times N$ matrices. Complete the following JULIA function so that, given input vector $\boldsymbol{y} \in \mathbb{C}^N$, it returns a linear least-squares solution $\arg\min_{\boldsymbol{x} \in \mathbb{C}^{2N}} \left\| \boldsymbol{y} - \begin{bmatrix} 2\boldsymbol{U} & 3\boldsymbol{V} \end{bmatrix} \boldsymbol{x} \right\|_2$, computed as efficiently as possible.

```julia
function lsuv(y, U, V)
```

↙ code (answer) | explain ↓

[10] **17**. A $M \times N$ matrix having rank 2 has Frobenius norm = 5 and spectral norm = 4.
Determine all of the singular values of this matrix.

[10] **18**.

The following JULIA code produces a scatter plot. Sketch by hand that plot, with labeled values. Add appropriate axis labels relevant to this plot. As always, explain your answer.

```
x = (-1).^(1:8) # vector of alternating 1, -1 entries
y = ones(8)
A = [3x 5y]; B = A * A'
scatter(svdvals(B), label="")
```


end