University of Michigan FALL 2017

EECS 501: Mid Term Examination 1

October 13, 2017

Instructions:

- Time: 3 hours
- Closed book, notes, and homework solutions. You may bring a sheet (both sides) of hand-written notes.
- Total points: 103.
- Electronic devices are NOT permitted
- You may use without rederivation the results derived in the class, the discussion with GSI and the homeworks.
- Write and sign the honor pledge on your exam. ("I have neither given nor received aid on this exam, nor have I concealed any honor code violations")
- There is partial credit for the correct approach unless stated otherwise.
- Please read every question carefully. If you have any doubt, do not hesitate to ask the instructor for a clarification.
- You have to show work to get full credit. Writing the correct answer with no work or with wrong work will earn no credit. Give a justification for every step of your solution.
- Every question has a reasonably short answer. If your solution to any problem appears long, then you may be taking a wrong approach.
- RETURN the exam paper along with your answers.
- There are 3 questions in this exam.

1. **State TRUE or FALSE by giving reasons** [7 points each]

   You must state a correct reason to get credit. No partial credit.

   (a) Consider a sample space $\Omega = \{a, b, c, d, e\}$. Let $E_1 = \{a, b, c\}$, and $E_2 = \{c, d, e\}$ be two events. Then $\mathcal{F}$ is the smallest sigma-algebra that contains both E_1 and E_2, where

   $$\mathcal{F} = \{E_1,\ E_2,\ \phi,\ \Omega,\ (E_1 \cap E_2),\ (E_1 \cap E_2^c),\ (E_1^c \cap E_2)\}.$$

   (b) Consider a sample space $\Omega = \{1, 2, 3, 4\}$. It is given that the outcomes are equally likely. Consider three events $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 1\}$. The three events are mutually independent.

   (c) Let X and Y be independent geometric random variables with parameter p. Let $Z = \max\{X, Y\}$. Then $E(Z) = \frac{3-2p}{p(2-p)}$.

   (d) Consider the system shown in Figure 1. Each component functions with probability p, independent of the other components. The system as a whole functions provided that there is at least one path (from left to right) through functioning elements. Let S denote the event that the system functions, and let E_i denote the event that component i functions.

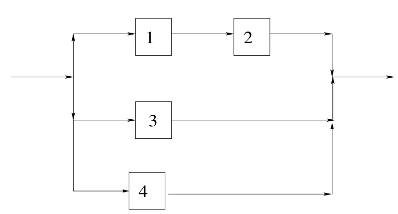

Figure 1:

   Then $P(S \cap (E_1 \cup E_2)) = 5p^2 - 6p^3 + 2p^4$.

   (e) Consider a sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 2, 4, 5\}$. Let $\mathcal{F}$ be the smallest sigma algebra that contains A. It is given that $P(A) = \frac{1}{3}$. Consider a random variable X, given by $X(w) = 1$ if w is a multiple of 3, and $X(w) = 0$ otherwise. Let $\Omega_X = \{0, 1\}$ and let $\mathcal{F}_X = 2^{\Omega_X}$. Then X is measurable with respect to $(\Omega, \mathcal{F}), (\Omega_X, \mathcal{F}_X)$.

   (f) A fair die is rolled 12 times. Let X denote the number of 1's observed and Y denote the number of 6's observed. Then the joint PMF of X and Y is given by

   $$P(X = k, Y = l) = \frac{\binom{12}{k}\binom{12-k}{l}4^{12-k-l}}{6^{12}}$$

for $0 \leq k$, $0 \leq l$, and $k + l \leq 12$.

(g) Consider 3 Bernoulli random variables A, B and C. It is given that A and B are independent. $A \sim$ Be$(\frac{1}{2})$, $B \sim$ Be$(\frac{1}{3})$. Moreover, we also have $C = A$ XOR B. Then B and C are independent.

(h) Let X and Y be independent random variables with $\text{Var}(X) = 1$, and $\text{Var}(Y) = 2$. Let $Z = 3X + 4Y$. Then $\text{Var}(Z) = 41$.

(i) Consider two random variables X and Y, where Y is uniformly distributed over the set $\{1, 2, 3\}$. Moreover, it is given that $E(X|Y = i) = i$ and $E(X^2|Y = i) = i^2 + 1$. Then $\text{Var}(X) = 5/3$.

2. **Coin Tossing Game** [10 points each]

A fair coin is tossed repeatedly until the pattern HTT is found. Let X denote the number of tosses.

(a) Find $E(X)$ using the Law of total expectation. You can leave the answer as a solution to a system of linear equations.

(b) Using the Law of total probability, find a set of difference equations that characterize the PMF of X along with initial conditions.

3. **Indicators** [10 points each]

(a) Use indicator random variables to solve this problem. Consider the following experiment in which cards are chosen at random sequentially one at a time without replacement from a well-shuffled deck of cards until a HEART is found, and then stopped. Let X denote the number of SPADEs selected. Find $E(X)$.

(b) A fair die is rolled 20 times. Let X denote the minimum of the 20 rolls. Using indicator random variables find $E(X)$.