EECS 501 Homework8 YUZHAN JIANG

P1:

(a)

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \begin{cases} e^{-y_3} & \text{if } y_3 \geq y_2 \geq y_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y_2,Y_3}(y_2,y_3) = \int_0^{y_2} e^{-y_3} \cdot dy_1$$

$$= e^{-y_3} y_1 \Big|_0^{y_2}$$

$$= e^{-y_3} y_2$$

$$\therefore f_{Y_1|Y_2,Y_3}(y_1|y_2,y_3) = \frac{f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3)}{f_{Y_2,Y_3}(y_2,y_3)}$$

$$= \frac{e^{-y_3}}{e^{-y_3} \cdot y_2}$$

$$= \frac{1}{y_2}$$

$$\therefore E[Y_1|Y_2,Y_3] = \int y_1 \cdot f_{Y_1|Y_2Y_3}(y_1|y_2,y_3) \cdot dy_1$$

$$= \int_0^{y_2} y_1 \frac{1}{y_2} \cdot dy_1$$

$$= \frac{1}{2} y_1^2 \cdot \frac{1}{y_2} \Big|_0^{y_2}$$

$$= \frac{1}{2} y_2^2 \cdot \frac{1}{y_2}$$

$$= \frac{1}{2} y_2$$

$\therefore$ The MMSE estimate of Y_1 from Y_2 and Y_3 is $\frac{Y_2}{2}$

1(b) False.

We are given that $P(X_i=1) = P(X_i=-1) = \frac{1}{4}$ and $P(X_i=0) = \frac{1}{2}$

$$P(X_i) = \begin{cases} \frac{1}{4}, & X_i = 1 \\ \frac{1}{4}, & X_i = -1 \\ \frac{1}{2}, & X_i = 0 \end{cases}$$

$$E[X_i] = \sum_i x_i P(X_i)$$
$$= 1 \times \frac{1}{4} + (-1) \times \frac{1}{4} + 0 \times \frac{1}{2}$$
$$= 0$$

$$E[X_i^2] = \sum_i x_i^2 P(x_i)$$
$$= 1^2 \times \frac{1}{4} + (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{2}$$
$$= \frac{1}{2}$$

$\therefore$
$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$
$$= \frac{1}{2}$$

And now $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$

$$E[Z_n] = E[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i]$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[X_i] \quad \text{(By Linearity of expecation)}$$
$$= \frac{1}{\sqrt{n}} \cdot \sum_{i=1}^{n} 0$$
$$= 0$$

$$Var[Z_n] = Var(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i)$$
$$= \frac{1}{n} \sum_{i=1}^{n} Var(X_i) \quad \text{(Since } X_1, X_2, \ldots X_n \text{ are i.i.d)}$$
$$= \frac{1}{n} \cdot n \cdot \frac{1}{2}$$
$$= \frac{1}{2}$$

$\therefore$ $$\frac{Z_n - E[Z_n]}{Var(Z_n)} \sim N(0,1)$$

$\therefore$ $$Z_n \sim N(0, \tfrac{1}{2})$$

P2.

We are given that $Y = HX + N$

Since X, Y are jointly Gaussian Vector

$\therefore$ the MMSE of X from Y

$$\hat{E}[X|Y] = \frac{Cov(X,Y)}{Var(Y)}(Y - E[Y]) + E[X]$$

$E[X] = 0$ and $E[Y] = E[HX+N] = HE[X] + E[N] = 0$

$$\begin{aligned} Cov(X,Y) &= E[XY^T] - E[X]E[Y^T] \\ &= E[X \cdot Y^T] \\ &= E[XX^TH^T + XN^T] \quad \text{(By Linearity of Expectation)} \\ &= E[XX^TH^T] + E[XN^T] \\ &= H\, E[XX^T] \\ &= H \cdot \Sigma_X \end{aligned}$$

$$\begin{aligned} Var(Y) &= E[Y^2] - (E[Y])^2 \\ &= E[(HX+N)(HX+N)^T] \\ &= E[(HX+N)(X^TH^T+N^T)] \\ &= E[HXX^TH^T + NN^T + HXN^T + NX^TH^T] \\ &= E[HXX^TH^T] + E[NN^T] + 2H\,E[XN^T] \\ &= H\Sigma_X H^T + \Sigma_N \end{aligned}$$

$$\therefore \quad \hat{E}[X|Y] = \frac{H \cdot \Sigma_X}{H\Sigma_X H^T + \Sigma_N} \cdot Y$$

3(a):

$X_1, X_2, \cdots X_n$ are independent, and $X_i \sim$ Bernouli (P)

$Z = \frac{1}{n}\sum_{i=1}^{n} X_i$ so that $Z \sim$ Binomia (n, P)

$$M_Z(S) = E[e^{sZ}]$$

$$= (Pe^s + q)^n \quad \text{where } q = 1-P$$

Thus, By the definition of chernoff inquality, we have.

$$P(Z \geq P+\delta) = P(nZ \geq n(P+\delta))$$

$$= P(e^{\theta\sum_{i=1}^{n} X_i} \geq e^{\theta n(P+\delta)})$$

$$\leq \frac{E(e^{\theta X_1})^n}{e^{\theta n(P+\delta)}}$$

$$= e^{-n(\theta(P+\delta)-\Lambda(\theta))}$$

where $\Lambda(\theta) = \log(E[e^{\theta X_1}]) = \log((1-P)+Pe^{\theta})$

and let $f(\theta) = \theta(P+\delta) - \log((1-P)+Pe^{\theta})$

Let $f'(\theta) = 0$

$$\Rightarrow (P+\delta) - \frac{1}{(1-P)+Pe^{\theta}}(P \cdot e^{\theta}) = 0$$

$$(P+\delta) = \frac{1}{(1-P)+Pe^{\theta}} P \cdot e^{\theta}$$

$$(P+\delta)((1-P)+Pe^{\theta}) = P \cdot e^{\theta}$$

$$(P+\delta)(1-P) + (P+\delta)P \cdot e^{\theta} = Pe^{\theta}$$

$$Pe^{\theta} - (P+\delta)Pe^{\theta} = (P+\delta)(1-P)$$

$$e^{\theta}(P - P^2 - P\delta) = (P+\delta)(1-P)$$

$$e^{\theta} = \frac{(P+\delta)(1-P)}{P(1-P-\delta)}$$

$$1 + \frac{P+\delta}{P(1-P-\delta)}$$

$$\frac{P(1-P-\delta)+P+\delta}{P(1-P-\delta)}$$

$$\therefore \max_{\theta>0} f(\theta) = (P+\delta)\log\left(\frac{(P+\delta)(1-P)}{P(1-P-\delta)}\right) - \log\left[(1-P) + \frac{(P+\delta)(1-P)}{P(1-P-\delta)}\right]$$

$$= (P+\delta)\log\left(\frac{(P+\delta)(1-P)}{P(1-P-\delta)}\right) - \log\frac{(1-P)}{1-P-\delta}$$

$$= (P+\delta)\log\frac{P+\delta}{P} + (1-P-\delta)\log\left(\frac{1-P-\delta}{1-P}\right)$$

$$= D((P+\delta)||P)$$

$$\therefore \quad P(Z \geq P+\delta) \leq e^{-nD((P+\delta)||P)}$$

3 (b)

$$P(Z \le p-\delta) = P(-Z \ge \delta - p) = P(-nZ \ge n(\delta-p))$$

$$= P(e^{-\theta \sum_{i=1}^{n} x_i} \ge e^{n(\delta-p)})$$

$$\le \frac{E(e^{-\theta x_i})^n}{e^{-n(p-\delta)}}$$

$$= e^{-n(-\theta(p-\delta) - \Lambda(-\theta))}$$

where $\Lambda(-\theta) = \log((1-p) + pe^{-\theta})$

and $f(-\theta) = -\theta(p-\delta) - \log((1-p) + pe^{-\theta})$

and let $f'(-\theta) = 0$

$$\Rightarrow (p-\delta) - \frac{1}{(1-p)+pe^{-\theta}} pe^{-\theta} = 0$$

$$e^{-\theta} = \frac{(1-p)(p-\delta)}{p(1-p+\delta)}$$

$$\therefore \max_{\theta<0} f(-\theta) = (p-\delta)\log\frac{(p-\delta)(1-p)}{p(1-p+\delta)} - \log\left(\frac{1-p}{1-p+\delta}\right)$$

$$= (p-\delta)\log\frac{p-\delta}{p} + (1-p+\delta)\log\left(\frac{1-p+\delta}{1-p}\right)$$

$$= D((p-\delta)\|p)$$

$$\therefore P(Z \le p-\delta) \le e^{-nD((p-\delta)\|p)}$$

Combine it with part (a)

$$P(|Z-p| \ge \delta) \le e^{-nD((p-\delta)\|p)} + e^{-nD((p+\delta)\|p)}$$

Since $\delta > 0$, $D((p-\delta)\|p)$ and $D((p+\delta)\|p)$ are monotonically $\uparrow$

So that $e^{-nD((p-\delta)\|p)}$ and $e^{-nD((p+\delta)\|p)}$ are monotonically $\downarrow$

$\therefore$ $P(|Z-p| \ge \delta)$ decays exponentially in n.

P4.

Let X_i: gain from the ith play.

$$P_{X_i}(x) = \begin{cases} 0.5, & x=-1 \\ 0.2, & x=0 \\ 0.3, & x=1 \end{cases}$$

$$E[X_i] = 0.5\times(-1) + 0.2\times 0 + 0.3\times 1$$
$$= -0.2$$

$$E[X_i^2] = (-1)^2\cdot 0.5 + 0.2\cdot 0^2 + 0.3\cdot 1^2$$
$$= 0.5+0.3$$
$$= 0.8$$

$$\therefore Var(X_i) = E[X_i^2] - (E[X_i])^2$$
$$= 0.8 - (-0.2)^2$$
$$= 0.8 - 0.04$$
$$= 0.76$$

Let $S = X_1 + X_2 + \cdots + X_{400}$

$$E[S] = E[X_1 + \cdots + X_{400}]$$
$$= 400\times(-0.2)$$
$$= -80$$

$$Var[S] = Var(X_1 + X_2 + \cdots + X_{400})$$
$$= Var(X_1) + Var(X_2) + \ldots + Var(X_{400})$$
$$= 400\times 0.76$$
$$= 304$$

① chebyshev's inequality

$$P(S \geq 0) = P(|S-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$= P(|S-(-80)| \geq 80) \leq \frac{Var(S)}{80^2}$$

$$= P(|S+80| \geq 80) \leq \frac{304}{6400} = 0.0475$$

② Central Limit Theorem

$$P(S \geq 0) = P(\underbrace{\frac{S-(-80)}{\sqrt{304}}}_{\sim N(0,1)} \geq \frac{80}{\sqrt{304}}) = 1 - \Phi(4.59)$$

③ The Chernoff Bound

$$P(S \geq 0) = P(\sum_{i=1}^{400} x_i \geq 400 \times 0)$$

The log moment generating function.

$$\Lambda(\theta) = \log(0.5e^{-\theta} + 0.2 + 0.3e^{\theta})$$

In the chernoff Bound, we need to solve.

$$\sup_{\theta > 0} \theta x - \Lambda(\theta) = \sup_{\theta > 0} -\log(0.5e^{-\theta} + 0.2 + 0.3e^{\theta})$$

let $\frac{d\Lambda(\theta)}{d\theta} = 0$

$$\Rightarrow \frac{1}{0.5e^{-\theta}+0.2+0.3e^{\theta}}(-0.5e^{-\theta} + 0.3e^{\theta}) = 0$$

$$-0.5e^{-\theta} + 0.3e^{\theta} = 0$$

$$-0.5 + 0.3e^{2\theta} = 0$$

$$e^{2\theta} = \frac{5}{3}$$

$$e^{\theta} = \sqrt{\frac{5}{3}}$$

$$\Rightarrow \sup 0\theta - \Lambda(\theta)$$

$$= -\log(0.5e^{-\theta} + 0.2 + 0.3e^{\theta})$$

$$= -\log(0.974597)$$

$$\therefore P(S \geq 0) \leq (0.97)^{400} = 3.39 \times 10^{-5}$$

P5:

$$E[x_1 x_3^2 x_4] = C_{13} C_{34} + C_{13} C_{34} + C_{14} C_{33}$$

$$= 2 C_{13} C_{34} + C_{14} C_{33}$$

$$E[x_1^2 x_2^2] = C_{11} C_{22} + C_{12} C_{12} + C_{12} C_{12}$$

$$= C_{11} C_{22} + 2 C_{12} C_{12}$$

$$E[x_1^6] = E[x_1 x_1 x_1 x_1 x_1 x_1] = \underbrace{C_{11} C_{11} C_{11} + C_{11} C_{11} C_{11} + \cdots}_{5 \times 3}$$

$$= 15 \left(E[x_1^2]\right)^3$$

P6.

$Y = X + N$ $\quad$ $X \sim \exp(\lambda)$ so $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$

$N \sim N(0, \sigma^2)$ so $E[N] = 0$ and $Var(N) = \sigma^2$

$$E[Y] = E[X+N]$$

(since X and Y are independent)

$$= E[X] + E[N]$$

$$= \frac{1}{\lambda}$$

$$Var[Y] = Var[X+N]$$

$$= Var(X) + Var(N)$$

$$= \frac{1}{\lambda^2} + \sigma^2$$

By the formula of LMMSE:

$$\hat{X}_L = \frac{Cov(X,Y)}{Var(Y)} (Y - E[Y]) + E[X]$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= E[X(X+N)] - \frac{1}{\lambda} \cdot \frac{1}{\lambda}$$

$$= E[X^2 + XN] - \frac{1}{\lambda^2}$$

$$= E[X^2] + E[XN] - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2} + E[X]E[N] - \frac{1}{\lambda^2}$$

(since X and N are independent)

$$= 0$$

$$\therefore \hat{X}_L = \frac{1}{\lambda}$$

The MSE of the LMMSE:

$$E[(X - \hat{X}_L)^2] = \left(1 - \frac{Cov^2(X,Y)}{Var(X)Var(Y)}\right) Var(X)$$

$$= \frac{1}{\lambda^2}$$