EECS 501
Discussion 4

Review

- Variance
  - $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$
  - $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
  - If X and Y are independent, $Var(X + Y) = Var(X) + Var(Y)$
  - Law of Total Variance $Var(X) = Var(E(X|Y)) + E(Var(X|Y))$
- Cumulative Distribution Function (CDF)
  - The CDF of a random variable is defined as follows,

$$F_X(x) = P\{X \leq x\}$$

  - Properties of CDF:
    * Range of distribution function: $0 \leq F_X(x) \leq 1 \qquad \forall x \in \mathbb{R}$
    * Value at ∞: $F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1$
    * Value at $-\infty$: $F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0$
    * Probability of being in $(a, b]$: $P\{a < X \leq b\} = F_X(b) - F_X(a)$
    * CDF is an increasing function.
- Continuous Random Variable:
  - Definition: A random variable X is said to be continuous if it's CDF is continuous everywhere in the real line,
- Probability Density Function (PDF):
  - For a continuous random variable, PDF is defined as the derivative of CDF

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}$$

  - Properties of density:

    The probability of $\mathbb{R}$ is 1: $\int_{-\infty}^{\infty} f_X(x)dx = 1$
    Probability density function is non-negative:$f_X(x) \geq 0$
    Probability of an event B: $P\{X \in B\} = \int_B f_X(x)dx$
    $P(x \leq X \leq x + \delta) \approx \delta f_X(x) \qquad$ for small δ
- Relation between CDF and PDF,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx \qquad\qquad f_X(x) = \frac{\partial F_X(x)}{\partial x}$$

- Important Remark: for a **continuous** random variable:

$$P(X \leq a) = P(X < a) \qquad \forall a \in \mathbb{R}$$
$$P(X \geq b) = P(X > b) \qquad \forall b \in \mathbb{R}$$

- Measurability
  - Measurable function: A function $f : X \to Y$ is measurable with respect to the measurable spaces (X, Σ_X) and (Y, Σ_Y) if for every $B \in \Sigma_Y$, we have $f^{-1}(B) \in \Sigma_X$.
  - Measurable random variable: a random variable is a function from Ω to Ω_X.

Practice Problems

Problem 1. Calculate the variance of a binomial random variable X with parameters N and p.

Problem 2. Calculate the variance of a geometric random variable X with parameter p.

Problem 3. The number of cars waiting for a traffic light is a geometric random variable with parameter q. You know the occupants of each car with probability p; $0 < p < 1$. Let N be the total number of cars whose occupants you are familiar with.

- What is the conditional pmf of N given the number of cars waiting is k?
- Find the expected value of N.

Problem 4. Assume X is a random variable with mean 1 and Y is a random variable with mean 1 and variance X^2. We have $Z \sim Geometric(p)$. Find $E(YE(Z|X))$.

Problem 5. Assume we have $Y = X_1 X_2 ... X_N$, where $\mathbb{E}[X_i] = \frac{1}{i}$ and $N \sim Geo(p)$. Assume X_i's and N are mutually independent. Find $E(Y)$ (Hint: $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$).

Problem 6. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas trains headed for destination B arrive at 15-minute intervals starting at $7:05$ a.m. If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 a.m. he/she gets on the first train that arrives. What is the probability that he/she goes to destination A?

Problem 7. Let X be a continuous random variable with PDF

$$f_X(x) = x^2(2x + \frac{3}{2}), \quad \text{for } 0 \leq x \leq 1$$

- Find the CDF of X.
- Let $Y = \frac{2}{X} + 3$. Find $Var(Y)$.

Problem 8. Measurability

Consider a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{a, b, \ldots, k, l\}$, $\mathcal{F}$ is the smallest sigma-algebra that contains the events $A = \{a, b, c, d, e, f, g, h\}$ and $B = \{e, f, g, h, i, j, k, l\}$, and $P(A) = \frac{1}{3}$, $P(B) = \frac{4}{5}$. Consider the following random variable

$$X(w) = \begin{cases} 1 & \text{if } w = a, b \\ 2 & \text{if } w = c, d \\ 3 & \text{if } w = e, f, g, h \\ 4 & \text{if } w = i, j, k, l \end{cases}$$

Note that $\Omega_X = \{1, 2, 3, 4\}$.

- Is P a valid probability assignment?
- Show that X is not measurable with respect to $(\Omega, \mathcal{F})$, $(\Omega_X, \mathcal{F}_X)$, where $\mathcal{F}_X = 2^{\Omega_X}$.

- Find the largest sigma-algebra $\mathcal{F}_X^*$ on Ω_X such that X measurable with respect to $(\Omega, \mathcal{F})$, $(\Omega_X, \mathcal{F}_X^*)$.

Problem 9. Measure on the power set

Consider the following measure on the set of all subsets of the unit interval $[0, 1]$: for all $A \subset [0, 1]$,

$$P(A) = \begin{cases} 1 & \text{if } \frac{1}{\pi} \in A \\ 0 & \text{otherwise} \end{cases}$$

Show that this measure satisfies all the three axioms of probability.