EECS 501 Exam 1

Fall 2020

10/8/2020-10/9/2020

Instructions:

- Print your name and sign the pledge in the space provided below. Alternatively, you can write the pledge statement separately in your submission and sign it.
- Open book, open note.
- Calculators, softwares, and the use of online materials are allowed.
- Total points 100.
- You may use, without rederivation, the results derived in the class, and the homeworks.
- Every part of a question has individual points.
- Attempt every question as there is partial credit for the correct approach.
- Please read every question carefully. If you have any doubt, do not hesitate to ask the instructor for a clarification.
- You have to show work to get full credit. Give a justification for every step of your solution.
- There are 4 questions in this exam.

PRINT NAME:

PLEDGE: I have neither given nor received aid during this exam, nor have I concealed any violation of the honor code.

SIGNATURE:

Problem	Points	Score	Problem	Points	Score	Problem	Points	Score	Problem	Points	Score
1.a.)	5		2.a.)	5		3.a.)	5		4.a.)	5	
1.b.)	5		2.b.)	10		3.b.)	10		4.b.)	10	
1.c.)	5		2.c.)	5		3.c.)	10		4.c.)	10	
1.d.)	5										
1.e.)	5										
1.f.)	5										
total	100										

Good Luck!
Hessam Mahdavifar

1. **State TRUE or FALSE by giving reasons** [5 points each]
You must state a valid reason to get full credit.

(a) Let $S = \{1, 2, ..., \}$ be the set of all positive integers. Also, let $\mathcal{F}$ be the set of all subsets of S and $P : \mathcal{F} \to \mathbb{R}^+ \cup \{0\}$ be defined as follows:

$$P(A) = \sum_{i \in A} \frac{1}{2^i}$$

for $A \in \mathcal{F}$ (let $P(\emptyset) = 0$). Then $(S, \mathcal{F}, P)$ is a valid probability space.

(b) Consider three events A, B, and C that are pairwise independent, i.e., A and B are independent and so are A and C as well as B and C. Then A and $B \cup C$ are independent.

(c) Let X be a random variable. Then there exists an $a \in \mathbb{R}$ such that for $Y = X + a$ we have: $Var[Y] = E[Y]^2$.

(d) Let X be a random variable with $P(X \geqslant 0) = 1$. Then for any $a > 0$, we have $P(X \leqslant a) \leqslant \frac{E[X]}{a}$.

(e) Consider a function f given as below:

$$f(x) = \begin{cases} \frac{1}{(n-1)n} & \text{if } n-1 < x \leqslant n, \quad \text{for} \quad n = 2, 3, 4, ... \\ 0 & \text{if } x \leqslant 1 \end{cases}$$

The function f is a valid probability density function (pdf) of some random variable X.

(f) Let X be a Gaussian random variable with mean 0 and variance σ^2. Also, let Y be a uniform random variable over $[0, a]$ for some number $a \in \mathbb{R}^+$. Then there exists some number $b \in \mathbb{R}$ such that

$$P(X \leqslant b) = P(Y \leqslant b).$$

2. **Student Club** [20 points]
A student club of size 30 consists of 10 seniors, 10 juniors, and 10 sophomores. An organizing committee of size 6 is chosen randomly (with all subsets of size 6 equally likely) from the club members.

(a) Find the probability that there are exactly 2 students from each of the senior, junior, and sophomore classes in the committee.
(b) Find the probability that the committee has at least one representative from each of the senior, junior, and sophomore classes.
(c) Given that the event in part (b) occurs, find the probability of the event in (a).

3. **Disease Test** [25 points]
Three persons, Alice, Bob, and Willie show up at a hospital to be tested for a disease. The probability of having the disease by each person is 10^{-3}, which is assumed to be independent from other persons. A sample is collected from each of them and is sent to the lab. Suppose that for some reason the sample labels are lost and when the results are back all they learn is that one of the test results is positive and the other two are negative.

(a) What is the probability that Alice has the disease given the above scenario and assuming that the test is completely accurate?

(b) Suppose that due to errors in the lab, a test result can be altered (from positive to negative or from negative to positive) with probability 10^{-2} independent of other tests. Now, what is the probability that Alice has the disease given the above scenario?

(c) Alice decides to take the test again and this time the result comes back negative. What is now the probability that Alice has the disease given all the information so far and assuming errors in the tests as in part (b)?

4. pdf and cdf [25 points]
   Consider a continuous random variable X with the following pdf:

$$f_X(x) = \begin{cases} |x| & \text{if } |x| \leqslant \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} < x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the cumulative distribution function (cdf) of X?
   (b) Let Y be another random variable with $Y = X^2 - X$. Find the cdf of Y.
   (c) Find the pdf of Y.