University of Michigan Fall 2021

EECS501: Homework 7

Assigned: October 24, 2021
Due: November 2, 2021 at 11:59 PM on gradescope
Text: "Probability and random processes" by J. A. Gubner

Reading assignment: Please read Chapters 4, 5 and 7.

1. Jointly Gaussian [10 points, 5 points, 5 points]

Consider a 3-dimensional Gaussian random vector (X_1, X_2, X_3) with zero mean and covariance matrix given by

$$\begin{bmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix}.$$

(a) Find $E[X_1|X_2, X_3]$ directly from $f_{X_1|X_2,X_3}(x_1|x_2, x_3)$.

(b) Show that the LMMSE of X_1 from X_2 and X_3 is equal to $E[X_1|X_2, X_3]$.

(c) Using part (b), and the fact that $\text{Var}(X_1|X_2, X_3)$ is equal to the minimum mean square error, find $\text{Var}(X_1|X_2, X_3)$.

2. Exponentials [10 points]

Recall that for two independent random variables X and Y with PDFs f_X and f_Y, respectively, the joint PDF of X and Z, where $Z = X + Y$, is given by $f_{XZ}(x, z) = f_X(x)f_Y(z-x)$.

Consider two independent exponential random variables X_1 and X_2 with parameter 1. Let $Y_1 = X_1$, $Y_2 = X_1 + X_2$. Find the MMSE estimate of Y_1 using Y_2.

3. Minimum Mean Absolute Error (MMAE) Estimation [5 points, 5 points]

(a) Consider a continuous random variable with a PDF $f_X(\cdot)$. Show that α^* that achieves the minimization: $\min_\alpha E\left[|X - \alpha|\right]$, is given by $F_X(\alpha^*) = \frac{1}{2}$, i.e., the median of X.
Hint: Break up the integral into $x < \alpha$ and $x \geq \alpha$, and optimize.

(b) Consider two random variables with a joint PDF f_{XY}. It is intended to estimate X by observing Y while minimizing the following objective function

$$\min_{g(\cdot)} E|X - g(Y)|.$$

Show that the best estimate is given by the conditional median of X given Y, i.e.,

$$F_{X|Y}(g^*(Y)|Y) = \frac{1}{2}.$$

4. LMSE [5 point , 10 points]

Consider three random variables X, Y, and Z, with known variances and covariances. Assume that $Var(X) > 0$, $Var(Y) > 0$, and that $Var(X)Var(Y) \neq Cov^2(X, Y)$.

(a) Give a formula for LMSE of Z based on X and Y, assuming X and Y are uncorrelated,.

(b) Give a formula for LMSE of Z based on X and Y in the general case.

5. Linear Innovations Sequence [10 points, 10 points]

Consider zero-mean random variables Y_1, Y_2, Y_3, and X with the following correlation matrix

$$\begin{pmatrix} 1 & 0.5 & 0.5 & 0 \\ 0.5 & 1 & 0.5 & 0.25 \\ 0.5 & 0.5 & 1 & 0.25 \\ 0 & 0.25 & 0.25 & 1 \end{pmatrix}$$

(a) Compute linear innovations sequence for Y_1, Y_2, and Y_3.

(b) Compute the LMSE of X based on the linear innovations sequence.