EECS 501

Solutions to Homework 3

1 Combinatorics

1.1 *Ross 1.10: In how many ways can 8 people be seated in a row if*

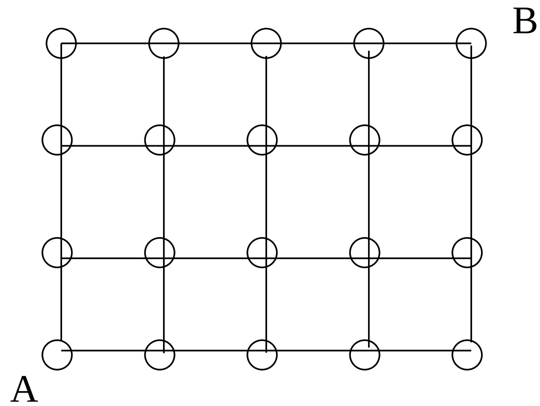
1. *there are no restrictions in the seating arrangements*
2. *persons A and B must sit next to each other*
3. *there are 4 men and 4 women and no 2 men or 2 women can sit next to each other*
4. *there are 5 men and they must sit next to each other*
5. *there are 4 married couples and each couple must sit together*

Solution

1. $8! = 40320$
2. there are 7 positions for the leftmost person A or B, there are 2 ways to arrange them, and there are 6! positions for the remaining 6 persons, for a total of $7 \times 2 \times 6! = 10080$.
3. there are only 2 arrangements of men and women (either MWMWMWMW or WMWMWMWM). For each of those there are 4! ways to arrange the men and 4! to arrange the women, for a total of $2 \times 4! \times 4! = 1152$.
4. there are 4 positions for the leftmost man to sit, and 5! arrangements of men and 3! arrangements of the remaining women, for a total of $4 \times 5! \times 3! = 2880$
5. there are 4! ways to arrange couples and for each such arrangement there are 2 ways for each couple to sit, for a total of $4! \times 2^4 = 384$.

End Solution

1.2 *Ross 1.21: Consider the grid of points shown below. Suppose that starting at point A you can go either one step up or one step to the right at each move. This is continued until point B is reached. How many different paths from A to B are possible?*



Solution

In order to reach B you need to take 4 steps to the right and 3 steps up in any order. Each route can be thought of as a 7-tuple containing either R or U (for right and up, respectively). There are exactly $\binom{7}{4} = \frac{7!}{4!3!} = 35$ ways to pick the positions of the 4 R's in the 7-tuple.

End Solution

1.3 *Ross 1.33: We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimum investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if*

1. *an investment must be made in each opportunity*
2. *investments must be made in at least 3 of the 4 opportunities*

Solution

1. Due to the minimum requirements, there are a total of $20 - (2 + 2 + 3 + 4) = 9$ thousand dollars to split among the 4 investment opportunities. If x_1, x_2, x_3, x_4 are the corresponding amounts (above and beyond the minimum requirements) then the number of ways is exactly the number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 9$. This is equivalent to the number of ways we can arrange 9 (indistinguishable) items and $4-1 = 3$ (indistinguishable) separators which is $\binom{9+3}{3} = \frac{12!}{9!3!} = 220$.

2. If the investments are made to all 4 opportunities, the answer is as above.

   If the investments are made in A,B,C, then the number of ways is equivalent to the number of non-zero integer solutions of the form $x_1 + x_2 + x_3 = 20 - (2 + 2 + 3) = 13$ which is $\binom{13+3-1}{3-1} = \binom{15}{2}$.

   If the investments are made in A,B,D, then the number of ways is equivalent to the number of non-zero integer solutions of the form $x_1 + x_2 + x_4 = 20 - (2 + 2 + 4) = 12$ which is $\binom{12+3-1}{3-1} = \binom{14}{2}$.

   If the investments are made in A,C,D, then the number of ways is equivalent to the number of non-zero integer solutions of the form $x_1 + x_3 + x_4 = 20 - (2 + 3 + 4) = 11$ which is $\binom{11+3-1}{3-1} = \binom{13}{2}$.

   If the investments are made in B,C,D, then the number of ways is equivalent to the number of non-zero integer solutions of the form $x_2 + x_3 + x_4 = 20 - (2 + 3 + 4) = 11$ which is $\binom{11+3-1}{3-1} = \binom{13}{2}$.

   So the total number is $\binom{12}{3} + \binom{15}{2} + \binom{14}{2} + \binom{13}{2} + \binom{13}{2} = 572$.

End Solution

2 Combinatorics and Probability

2.1 *Ross 2.32: A group of individuals containing* b *boys and* g *girls is lined up in a random order - that is, each of the* $(b+g)!$ *permutations is assumed to be equally likely. What is the probability that that the person in the* i*-th position,* $1 \le i \le b+g$ *, is a girl?*

Solution

We have $|\Omega| = (b+g)!$. If $A_i = \{\text{person in the } i\text{-th position is girl}\}$, then $|A_i| = g(b+g-1)!$ which is the number of ways we can choose the girl in the i-th position, times the permutations of the remaining $b+g-1$ persons. So the probability is

$$P(A_i) = \frac{|A_i|}{|\Omega|} = \frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}.$$

End Solution

2.2 *Ross 3.9: Consider 3 urns. Urn A contains 2 white and 4 red balls; Urn B contains 8 white and 4 red balls; Urn C contains 1 white and 3 red balls. If one ball is selected from each urn, what is the probability that the ball chosen from urn A was white, given that exactly 2 white balls were selected?*

Solution

We have

$$\begin{aligned}
P(A = W|2W) &= \frac{P(WWR \text{ or } WRW)}{P(WWR \text{ or } WRW \text{ or } RWW)} \\
&= \frac{(2/6)(8/12)(3/4) + (2/6)(4/12)(1/4)}{(2/6)(8/12)(3/4) + (2/6)(4/12)(1/4) + (4/6)(8/12)(1/4)} \\
&= \frac{48+8}{48+8+32} = \frac{7}{11}.
\end{aligned}$$

End Solution

2.3 *Ross 3.40(c): Consider a sample of size 3 drawn in the following manner: We start with an urn containing 5 white and 7 red balls. At each stage a ball is drawn and its color is noted. The ball is then returned to the urn along with an additional ball of the same color. Find the probability that the sample will contain exactly 3 white balls.*

Solution

Clearly this experiment unfolds over three time periods. Define the events

$$A_i = \{\text{a white ball is picked at time i}\} \qquad i = 1, 2, 3.$$

Then, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_2 \cap A_1) = (5/12)(6/13)(7/14)$.

End Solution

2.4 *The Jones family household includes Mr. and Mrs. Jones, four children, two cats, and three dogs. Every six hours there is a Jones family stroll. The rules for a Jones family stroll are:*

Exactly five living beings (people + dogs + cats) go on each stroll.

Each scroll must include at least one parent and at least one pet.

There can never be a dog and a cat on the same stroll unless both the parents go.

All acceptable stroll groupings are equally likely.

Given that exactly one parent went on the 6 P.M. stroll, what is the probability that Rover, the oldest dog, also went?

Solution ---

Let P the event that *exactly* one parent went and R be the event that Rover, the oldest dog, went. We want to find

$$\begin{aligned}\Pr(R|P) &= \frac{\Pr(R\cap P)}{\Pr(P)} = \frac{|R\cap P|/|\Omega|}{|P|/|\Omega|} = \frac{|R\cap P|}{|P|} \\ &= \frac{\text{Number of ways exactly one parent and Rover can go}}{\text{Number of ways exactly one parent goes}}. \end{aligned} \tag{1}$$

Now if exactly one parent and Rover are going, we can choose the parent in 2 ways and the remaining 3 positions can be filled by any of the remaining 2 dogs or the 4 kids. The number of combinations of this is

$$2\times\binom{6}{3} = 40. \tag{2}$$

If exactly one parent is going, we can choose the parent in 2 ways. Both dogs and cats cannot go together, so we have two cases: either the dogs go or the cats go. Suppose the pet is a dog. So we have to choose four positions with three dogs or four kids such that at least one dog is chosen. This can be done in

$$\underbrace{\binom{4+3}{4}}_{\text{total number of ways}} - \underbrace{1}_{\text{number of ways of not choosing any dog}} \tag{3}$$

Suppose the pet is cat. The number of ways we can fill the remaining four positions is

$$\underbrace{\binom{4+2}{4}}_{\text{total number of ways}} - \underbrace{1}_{\text{number of ways of not choosing any cat}} \tag{4}$$

Thus, the total number of ways exactly one parent can go is

$$2\times\left(\binom{7}{4} - 1 + \binom{6}{4} - 1\right) = 96 \tag{5}$$

Hence,

$$\Pr(R|P) = \frac{40}{96} = \quad 0.4167 \tag{6}$$

--- *End Solution*

2.5 *In the game of bridge the entire deck of 52 cards is dealt out to four players, each getting thirteen. Assume that cards are dealt randomly. What is the probability*

1. *one of the players receives all thirteen spades?*
2. *two of the players receive all thirteen spades and each one has at least one spade?*
3. *each player receives one ace?*
4. *Suppose now we take a reduced deck of 39 cards that consists of thirteen clubs, thirteen hearts, and thirteen diamonds, and we randomly draw a hand of thirteen cards. What is the probability that this hand is void in at least one suit?*

Solution

Number of ways of distributing 52 cards to 4 players, each getting 13 cards is

$$\binom{52}{13;13;13;13} = \frac{52!}{13!\,13!\,13!\,13!} \tag{1}$$

1. Number of ways for one player to receive 13 spades is

$$\underbrace{\binom{4}{1}}_{\text{Choose the player}} \times \underbrace{\binom{39}{13;13;13}}_{\text{Distribute the remaining 39 cards between 3 players}} \tag{2}$$

Thus,

$$\Pr\left(\text{One of the players receives all 13 spades}\right) = \frac{\binom{4}{1}\binom{39}{13;13;13}}{\binom{52}{13;13;13;13}} \tag{3}$$

2. Number of ways for two of players to receive 13 spades such that each one has at least one spade is

$$\underbrace{\binom{4}{2}}_{\text{Choose the 2 players}} \times \underbrace{\binom{39}{13}}_{\text{Choose 13 more cards}} \times \underbrace{\left[\binom{26}{13} - 2\right]}_{\substack{\text{Distribute the 26 cards} \\ \text{(13 spades and 13 others just chosen)} \\ \text{between the two players such that} \\ \text{each player gets at least 1 spade}}} \times \underbrace{\binom{26}{13}}_{\substack{\text{Distribute the remaining 26} \\ \text{cards between the remaining 2 players}}} \tag{4}$$

Thus,

$$\Pr\left(\substack{\text{Two players receive all 13 spades} \\ \text{such that each one has at least one spade}}\right) = \frac{\binom{4}{2}\binom{39}{13}\left[\binom{26}{13}-2\right]\binom{26}{13}}{\binom{52}{13;13;13;13}} \tag{5}$$

3. Number of ways in which each player gets one ace is

$$\underbrace{\binom{48}{12;12;12;12}}_{\substack{\text{Number of ways to distribute the} \\ \text{remaining 48 cards between 4 players}}} \times \underbrace{4!}_{\substack{\text{Number of ways of distributing the 4 aces} \\ \text{between 4 players}}} \tag{6}$$

Thus

$$\Pr\left(\text{Each player gets one spade}\right) = \frac{\binom{48}{12;12;12;12}4!}{\binom{52}{13;13;13;13}}. \tag{7}$$

4.

$$\begin{aligned}
\Pr\left(\substack{\text{The hand drawn is void}\\ \text{in } \textit{at least} \text{ one suit}}\right) &= \Pr\left(\substack{\text{The hand drawn is void}\\ \text{in } \textit{exactly} \text{ one suit}}\right) + \Pr\left(\substack{\text{The hand drawn is void}\\ \text{in } \textit{exactly} \text{ two suits}}\right)\\
&\quad + \Pr\left(\substack{\text{The hand drawn is void}\\ \text{in } \textit{exactly} \text{ three suits}}\right) && (8)\\
&= \Pr\left(\substack{\text{The hand drawn is void}\\ \text{in } \textit{exactly} \text{ one suit}}\right) + \Pr\left(\substack{\text{The hand drawn is void}\\ \text{in } \textit{exactly} \text{ two suits}}\right)\\
&\quad + 0 && (9)
\end{aligned}$$

Further, number of ways to draw hands that are void in exactly 1 suit

$$\underbrace{\binom{3}{1}}_{\substack{\text{Number of ways to}\\ \text{choose the suit}}} \times \underbrace{\left[\binom{26}{13} - 2\right]}_{\substack{\text{Number of ways to draw 13 cards from 26 cards}\\ \text{that is not void in any suit}}} \tag{10}$$

and number of ways to draw hands that are void in exactly 2 suits

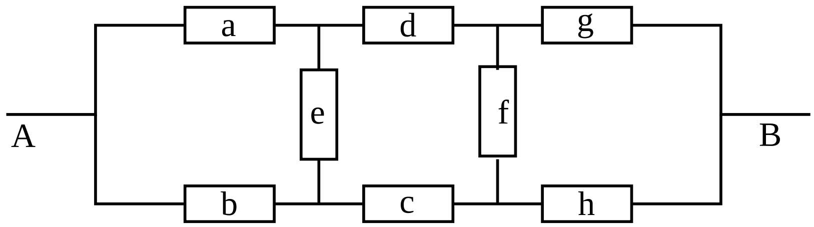
$$\underbrace{\binom{3}{2}}_{\text{Choose the two suits in which we are void}} \times 1 \tag{11}$$

Thus,

$$\Pr\left(\text{The hand drawn is void in } \textit{at least} \text{ one suit}\right) = \frac{\binom{3}{1}\left[\binom{26}{13} - 2\right] + \binom{3}{2}}{\binom{39}{13}} \tag{12}$$

End Solution

2.6 *In the communication network given below, link failures are independent, and each link has a probability of failure p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.*



1. *Given that exactly five links have failed, determine the probability that A can still communicate with B.*

2. *Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.*

3. *Given that a, d and h have failed (but no information about the condition of the other links), determine the probability that A can communicate with B.*

Solution

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1. Let A be the event that "A can communicate with B" and B be the event that "exactly five links have failed". We are asked to find $\Pr(A \mid B)$.

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{1}$$

Now

$$\Pr(A \cap B) = \Pr(\{a\bar{b}\bar{c}d\bar{e}\bar{f}g\bar{h}, \bar{a}bc\bar{d}\bar{e}\bar{f}\bar{g}h\}) = p^5(1-p)^3 + p^5(1-p)^3 = 2p^5(1-p)^3 \tag{2}$$

and

$$\Pr(B) = \binom{8}{5} p^5(1-p)^3 \tag{3}$$

Thus

$$\Pr(A \mid B) = \frac{2p^5(1-p)^3}{\binom{8}{3}p^5(1-p)^3} = \frac{1}{28} \tag{4}$$

2. Let C be the event that either g or h (*but not both*) are working.

$$\Pr(C \mid B) = \frac{\Pr(C \cap B)}{\Pr(B)} \tag{5}$$

To find $\Pr(C \cap B)$ we need to choose one gate out of g and h that is working (and the other is not) and choose four gates out of the remaining six such that these four are not working (and the remaining two are working). Thus,

$$\Pr(C \mid B) = \frac{\binom{2}{1}p(1-p)\binom{6}{4}p^4(1-p)^2}{\binom{8}{5}p^5(1-p)^3} = \frac{15}{28} \tag{6}$$

3. Define the event $D = \{\text{gates } a, d \text{ and } h \text{ have failed}\}$. We want to find $\Pr(A|D) = \Pr(A \cap D)/\Pr(D)$. Given D, the only way A can communicate with B is if gates b, c, f and g are working. Define the event

$$E = A \cap D = \{\text{gates } b,\ c,\ f \text{ and } g \text{ are working and gates } a,\ d \text{ and } h \text{ have failed}\}.$$

We further define the event $F = \{\text{gate e is working}\}$. Using the law of total probability we can write

$$\begin{aligned}\Pr(E) &= \Pr(E \cap F) + \Pr(E \cap F^c) \\ &= \Pr(\bar{a}bc\bar{d}efg\bar{h}) + \Pr(\bar{a}bc\bar{d}\bar{e}fg\bar{h}) \\ &= p^3(1-p)^5 + p^4(1-p)^4 \\ &= p^3(1-p)^4. \end{aligned} \tag{7}$$

Similarly, we can write a series of events describing the operational state of gates b,c,e,f,g and apply the total probability law to find $\Pr(D)$. This will result in

$$\Pr(D) = p^3. \tag{8}$$

Thus

$$\Pr(A \mid D) = (1-p)^4. \tag{9}$$

Note that in this problem $\Pr(A \mid D) = \Pr(A)$, which also shows (as expected) that events A and D are independent.

End Solution

2.7 *You are dealt the following poker hand: Queen of Hearts, Jack of Hearts, 10 of Hearts, 9 of Spades and 9 of Diamonds. You can discard any number of cards (the discarded cards are set aside, i.e., they are not reshuffled into the remaining deck) and draw new cards from the remaining deck.*

You decide to discard the two nines (anticipating a straight or a flush, or a straight flush, or a royal flush). Evaluate the probabilities of getting: a royal flush, a straight flush, four of a kind, full house, flush, straight, three of a kind, two pairs, one pair of Jacks or better.

Solution

There are $\binom{52-5}{2} = 1081$ ways to get the next two cards.

There is only 1 way you can have a royal flush: getting an A and a K of hearts.

There are 2 ways you can have a straight flush (other than royal): getting a K and a 9 of hearts, or a 9 and an 8 of hearts.

There is no way you can get a four of a kind

There is no way you can get a full house

There are $\binom{10}{2} - 3$ ways to get a flush: the ways you can select 2 out of the remaining 10 Hearts minus the above 3 cases of straight and royal flushes.

You can get a straight either with A and K ($4 \times 4 - 1$ ways, ie, excluding royal flush) or with K and 9 ($4 \times 2 - 1$, ie, excluding straight flush) or with 9 and 8 ($4 \times 2 - 1$, ie, excluding straight flush) for a total of 116 ways.

There are $3\binom{3}{2} = 9$ ways to get three of a kind: either with two additional Qs or Js or 10s.

There are $\binom{3}{2}$ ways to pick which of the Q, J, 10 will form the two pairs and then $\binom{3}{1} \times \binom{3}{1}$ ways to pick those cards from the deck, for a total of 27 ways.

There are $\binom{4}{2}$ ways to pick a pair of As, $\binom{4}{2}$ ways to pick a pair of Ks, and 3×38 ways to pick a pair of Qs or J's. The last number is explained as follows (say for Q's): there are 3 ways to pick the second Q and then out of the remaining $52 - 5 - 1 = 46$ cards you can pick any card other than the remaining 3 10's, 3 J's and 2 Q's, ie, $46 - 8 = 38$. These make for a total of 239 ways.

The probabilities of the above events can be now found easily.

End Solution