EECS 455: Problem Set 1

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Due: Wednesday, September 8, 2021, 11pm.

1. A communication system transmits one of 8 equally likely signals. The signals (waveforms) are represented by the vectors shown below.

$$
\begin{aligned}
s_0 &= (+1,+1,+1,+1)\\
s_1 &= (+1,-1,+1,-1)\\
s_2 &= (+1,+1,-1,-1)\\
s_3 &= (+1,-1,-1,+1)\\
s_4 &= (-1,-1,-1,-1)\\
s_5 &= (-1,+1,-1,+1)\\
s_6 &= (-1,-1,+1,+1)\\
s_7 &= (-1,+1,+1,-1)
\end{aligned}
$$

(a) Determine how many information bits can be sent using these signals.

Solution: Since there are $M = 8$ signals, 3 bits can be sent.

(b) Determine the energy of each of the signals and the average energy per information bit.

Solution: The energy of each signal is 4.

(c) Determine the squared Euclidean distance between signal s_0 and all the other signals.

Solution:

$$
\begin{aligned}
d_E^2(s_0,s_1) &= 8\\
d_E^2(s_0,s_2) &= 8\\
d_E^2(s_0,s_3) &= 8\\
d_E^2(s_0,s_4) &= 16\\
d_E^2(s_0,s_5) &= 8\\
d_E^2(s_0,s_6) &= 8\\
d_E^2(s_0,s_7) &= 8\\
d_E^2(s_0,s_8) &= 8
\end{aligned}
$$

(d) Determine the rate of communication in bits/dimension for these signals.

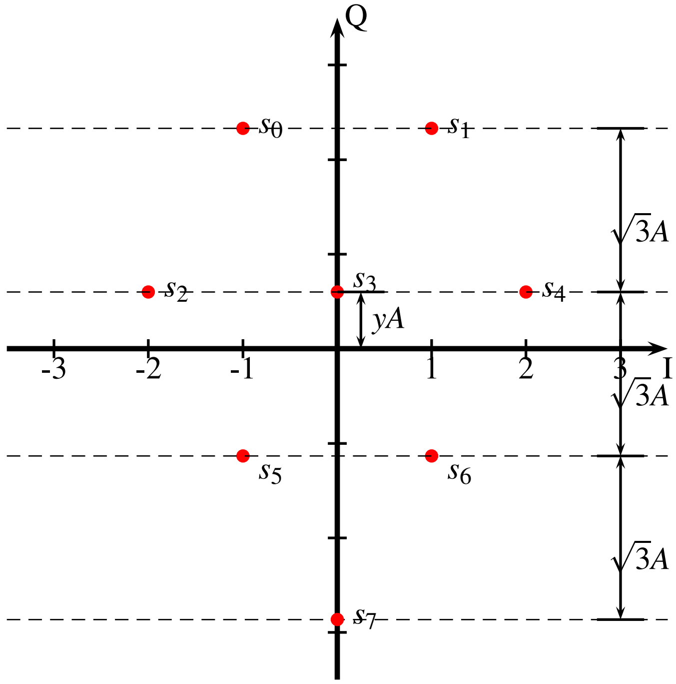
Solution: The rate is $r = 3/4$ bits/dimension.

2. A modulator transmits 3 bits of information using 8 equally likely signals in two dimensions. The signal vectors are given as

$$
s_1 = A(-1, y+\sqrt{3})
$$

$$
\begin{aligned}
s_2 &= A(1, y+\sqrt{3}) \\
s_3 &= A(-2, y) \\
s_4 &= A(0, y) \\
s_5 &= A(2, y) \\
s_6(t) &= A(-1, y-\sqrt{3}) \\
s_7(t) &= A(1, y-\sqrt{3}) \\
s_8(t) &= A(0, y-2\sqrt{3})
\end{aligned}
$$

(a) Determine the optimum value of the parameter y to minimize the average signal energy transmitted.



Solution:

The energy of each signal as a function of y is given below.

$$
\begin{aligned}
E_1 &= A^2(1+(y+\sqrt{3})^2) \\
E_2 &= A^2(1+(y+\sqrt{3})^2) \\
E_3 &= A^2(4+y^2) \\
E_4 &= A^2(y^2) \\
E_5 &= A^2(4+y^2) \\
E_6 &= A^2(1+(y-\sqrt{3})^2) \\
E_6 &= A^2(1+(y-\sqrt{3})^2) \\
E_6 &= A^2(y-2\sqrt{3})^2)
\end{aligned}
$$

The average energy $\bar{E}$ is

$$\bar{E} = \frac{A^2}{8}\left[2(1+(y+\sqrt{3})^2)+2(4+y^2)+y^2+2(1+(y-\sqrt{3})^2)+(y-2\sqrt{3})^2)\right]$$

To minimize the average energy we take the derivative with respect to y.

$$\begin{aligned}\frac{\partial \bar{E}}{\partial y} &= \frac{A^2}{8}\left[4(y+\sqrt{3})+4y+2y+4(y-\sqrt{3})+2(y-2\sqrt{3})\right]\\ &= \frac{A^2}{8}\left[16y-4\sqrt{3}\right]=0\end{aligned}$$

Thus the optimal value for y is $\sqrt{3}/4$ which results in average transmitted energy of

$$\begin{aligned}\bar{E} &= \frac{A^2}{8}\left[2(1+(\frac{5}{4}\sqrt{3})^2)+2(4+\frac{3}{16})+\frac{3}{16}+2(1+(\frac{3}{4}\sqrt{3})^2)+(\frac{7}{4}\sqrt{3})^2)\right]\\ &= \frac{A^2}{8}\left[2(1+\frac{75}{16})+2(4+\frac{3}{16})+\frac{3}{16}+2(1+\frac{27}{16})+(\frac{147}{16})\right]\\ &= \frac{A^2}{8}\left[\frac{182}{16}+\frac{134}{16}+\frac{3}{16}+\frac{86}{16}+\frac{147}{16}\right]\\ &= \frac{A^2}{8}\left[\frac{552}{16}\right]\\ &= 4.3125A^2.\end{aligned}$$

(b) Determine the minimum squared Euclidean distance between any two signals.

Solution:

The minimum squared Euclidean distance between signals is $4A^2$.

(c) Determine the rate of communication in bits/dimension.

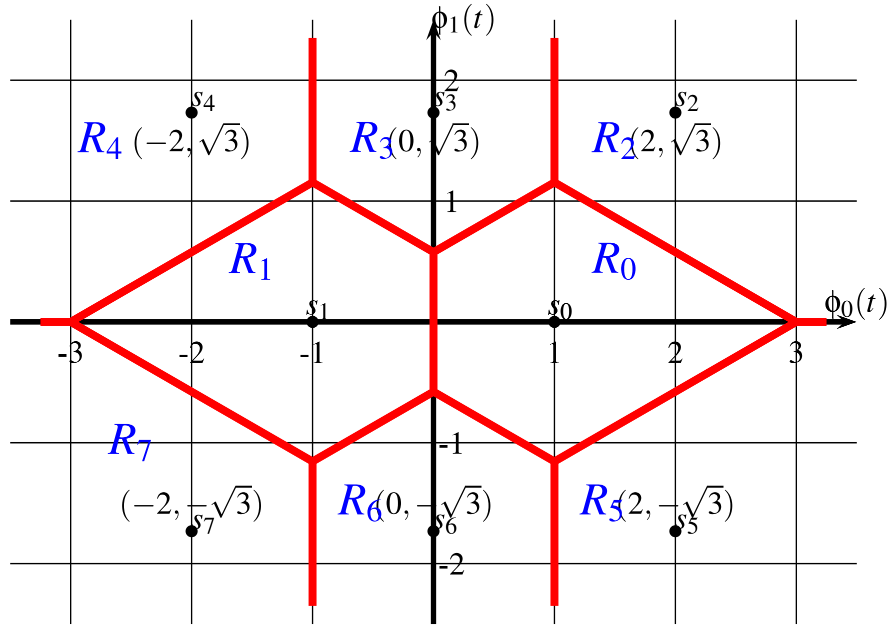
Solution:

The rate of communication is 3 bits/2 dimensions.

3. The eight constellation points for an equal probable signal set received are

$$\begin{array}{cccc} s_0=(1,0), & s_1=(-1,0), & s_2=(2,\sqrt{3}), & s_3=(0,\sqrt{3}) \\ s_4=(-2,\sqrt{3}), & s_5=(2,-\sqrt{3}), & s_6=(0,-\sqrt{3}), & s_7=(-2,-\sqrt{3}) \end{array}$$

shown below.



(a) Determine the average energy of this signal set.

Solution: The average energy is determined as follows.

$$\begin{array}{llll} E_0 = 1 & E_1 = 1 & E_2 = 7 & E_3 = 3 \\ E_4 = 7 & E_5 = 7 & E_2 = 3 & E_3 = 7 \end{array}$$

$$\begin{aligned} E &= \frac{1}{8}\sum_{i=0}^{7} E_i \\ &= \frac{1}{8}36 \\ &= 4.5 \end{aligned}$$

(b) Determine the average energy per bit.

Solution:

$$E_b = 4.5/3 = 1.5$$

(c) Determine the distance between signal s_0 and every other signal.

Solution:

$$\begin{aligned}
d_E^2(s_0,s_1) &= 4+0=4,\ d_E(s_0,s_1)=2 \\
d_E^2(s_0,s_2) &= 1+3=4,\ d_E(s_0,s_2)=2 \\
d_E^2(s_0,s_3) &= 1+3=4,\ d_E(s_0,s_3)=2 \\
d_E^2(s_0,s_4) &= 9+3=12,\ d_E(s_0,s_4)=2\sqrt{3} \\
d_E^2(s_0,s_5) &= 1+3=4,\ d_E(s_0,s_5)=2 \\
d_E^2(s_0,s_6) &= 1+3=4,\ d_E(s_0,s_6)=2 \\
d_E^2(s_0,s_7) &= 9+3=12,\ d_E(s_0,s_7)=2\sqrt{3}
\end{aligned}$$

(d) Determine the rate of communication in bits/dimension.

Solution:

$$R = 3/2\text{bits/dimension}$$

4. Consider the UWB channel which goes from 3.1GHz to 10.6 GHz. Suppose the noise power spectral density is $N_0 = kT = (1.38\times10^{-23})(290) = 4\times10^{-21}$ Watts/Hz. Here k is Boltzman's constant and T is the temperature in Kelvin. A temperature of 290 K corresponds to 62 degree Fahrenheit. The allowed transmitted power *density* is -41.3dBm/MHz =-71.3dB/MHz. (Note 0dBm=1mW, 30dBm=1W, -30dBm=1 μW).

(a) For the given frequency band determine the total power that can be transmitted.

Solution:

The total transmitted power is determined as follows. First determine the power in a 1 MHz bandwidth. Then multiply by 7500 to get the power in 7.5 GHz.

$$\begin{aligned}
P_t|_{1MHz} &= 10^{(-71.3/10)} \\
&= 7.41\times10^{-8}W/MHz
\end{aligned}$$

$$\begin{aligned}
P_t &= 7.41\times10^{-8}(7500) \\
&= .556\times10^{-3}W \\
&= 556\mu\text{Watts.}
\end{aligned}$$

Suppose the received power is related to the transmitted power by

$$P_r = P_t h_t^2 h_r^2/d^4$$

where the d is the distance in meters (independent of frequency), h_t is the height of the transmitting antenna (in meters) and h_r is the height of the receiving antenna (in meters).

(b) Compute the largest possible data rate that can be communicated reliably with both antennas at a height of 1m at a distance of 100 m and 1000 m.

Solution:

$$
\begin{aligned}
P_r &= .556 \times 10^{-3}/d^4 \\
&= \begin{cases} 5.556 \times 10^{-12} & d = 100 \\ 5.556 \times 10^{-16} & d = 1000. \end{cases}
\end{aligned}
$$

The capacity is then

$$
\begin{aligned}
C &= W \log_2 (1 + \frac{P_r}{N_0 W}) \\
&= \begin{cases} 1.8\text{Gbps} & d = 100 \\ 200\text{kbps} & d = 1000. \end{cases}
\end{aligned}
$$

5. (a) A communication system is to be designed. The allocated (absolute) bandwidth is 100kHz. It is desired to communicate 300kbits/sec very reliably (error probability close to zero). What is the smallest value of E_b/N_0 for which this is possible?

Solution: $R/W = 3$. Thus

$$
\begin{aligned}
E_b/N_0 &\geq \frac{2^{R/W} - 1}{R/W} = \frac{2^3 - 1}{3} = 7/3 \\
&= 3.68 dB
\end{aligned}
$$

(b) A channel with absolute bandwidth $W = 100$kHz, power $P = 5$ watts= 5 joules/sec and two sided noise power spectral density $N_0/2 = 1.778 \times 10^{-3}$ watts/ Hz is used. The source is an i.i.d. Gaussian source with mean 0 and variance 1. What is the minimum possible distortion (mean square error) if the source is sampled at rate 4000 samples/sec.

Solution:

$$\begin{aligned}
R &< W\log_2(1+\frac{P}{N_0W}) \\
&= (10^5)\log_2(1+\frac{5}{2(1.778e-3)(10^5)}) \\
&= 2.01\times 10^3
\end{aligned}$$

$$\begin{aligned}
R/R_s &> \frac{1}{2}\log_2\left(\frac{\sigma^2}{D}\right) \\
2^{2R/R_s} &> \frac{\sigma^2}{D} \\
D &> \frac{\sigma^2}{2^{2R/R_s}} \\
&> \frac{\sigma^2}{2^{2(2.01\times 10^3)/(4000)}} \\
&= 0.4975
\end{aligned}$$