Lecture 15

BPSK

$r(t) = b\cos(2\pi f_c t + \theta) + n(t)$

$r(t) = \cos(2\pi f_c t + \theta + a\pi)$

Goals

$a = 0 \quad b = +1$

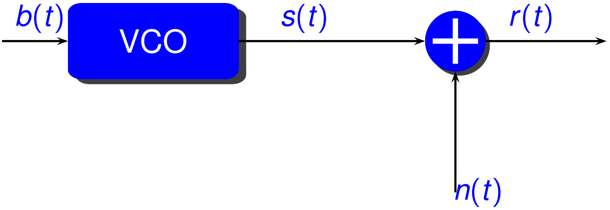
$a = 1 \quad b = -1$

- Be able to demodulate frequency modulated signals without phase reference.
- Be able to demodulate differentially encoded phase modulated signals without phase reference.
- Be able to demodulate orthogonal signals without phase reference.

Frequency Shift Keying (FSK)

- Frequency shift keying communicates information by transmitting different frequencies.
- It can be generated using a voltage controlled oscillator.
- It can be demodulated noncoherently (without knowing the phase of the received signal) by measuring the received energy at the different frequencies.
- It performance is worse than coherently demodulated signals but may be simpler.

Frequency Shift Keying (FSK) Modulator



$$b(t) \quad = \quad \sum_{l=-\infty}^{\infty} b_l\, p_T(t - lT), \qquad b_l \in \{+1, -1\}$$

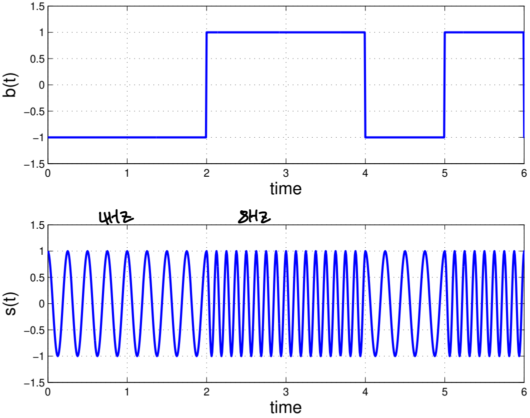
without known

$$s(t) \quad = \quad \sqrt{2P} \sum_{l=-\infty}^{\infty} \cos(2\pi(f_c + b(t)\Delta f)t + \theta)p_T(t - lT)$$

Frequency Shift Keying (FSK)

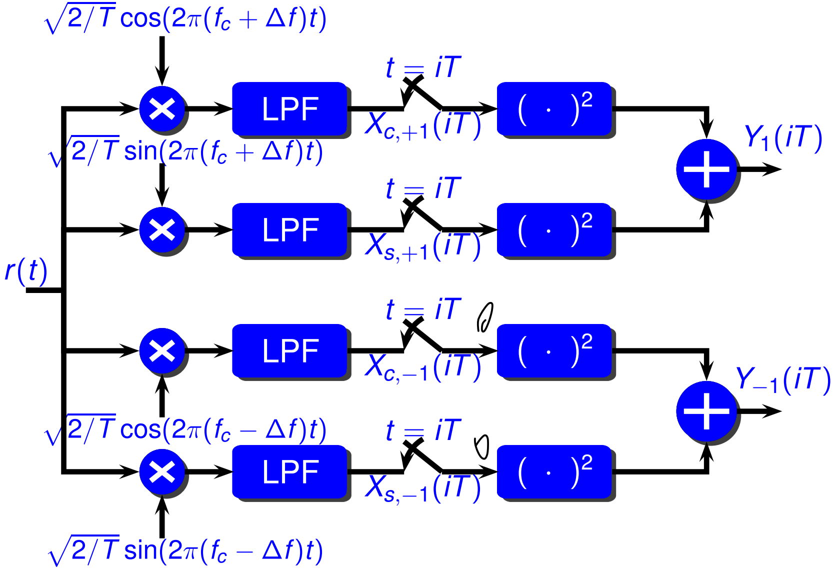
- Δf is half the difference between the two transmitted frequencies and θ is an unknown (to the receiver) phase.
- We let $f_0 = f - \Delta f$ and $f_1 = f + \Delta f$.
- When $b_i = +1$ then a sinusoidal signal at frequency f_1 is transmitted.
- When $b_i = -1$ then a sinusoidal signal at frequency f_0 is transmitted.
- The two frequencies f_0 and f_1 are separated far enough to make the two signals orthogonal.
- Minimum shift keying (MSK) has the minimum separation in order to make the signals orthogonal.

FSK Signals



$f_c = 6$ Hz
$\Delta f = 2$ Hz

Frequency Shift Keying (FSK) Demodulator



$$X_{c,+1} = \int_0^T r(t) \cdot \sqrt{\frac{2}{T}} \cos(2\pi(f_c+\Delta f)t)\, dt$$

with

$b = +1$

$$= \int_0^T \sqrt{2P} \cos(2\pi(f_c+\Delta f)t + \theta) \sqrt{\frac{2}{T}} \cos(2\pi(f_c+\Delta f)t) \cdot dt$$

$$= 2\sqrt{\frac{P}{T}} \int_0^T \frac{1}{2} \cos(\theta)\, dt = \sqrt{\frac{P}{T}} \cos\theta (T)$$

$$= \sqrt{PT} \cos\theta$$

$$= \sqrt{E} \cos\theta$$

$$X_{s,+1} = \int_0^T r(t) \sqrt{\frac{2}{T}} \sin(2\pi(f_c+\Delta f)t \cdot dt$$

$$= \sqrt{E} \sin\theta$$

$$\underline{X_{s,+1}^2 + X_{c,+1}^2 = E} \longleftarrow \text{doesn't depend on } \theta$$

FSK Noncoherent Demodulator

The receiver decides signal -1 was transmitted if $|Y_{-1}| > |Y_1|$ and otherwise decides signal 1. The random variables at the output of the low pass filters are

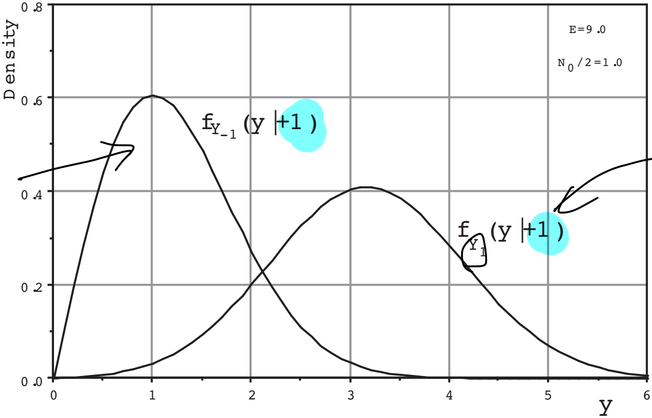
$$
\begin{aligned}
X_{c,1}(iT) &= \sqrt{E}\delta(b_{i-1},1)\cos(\theta) + \eta_{c,1} \\
X_{s,1}(iT) &= \sqrt{E}\delta(b_{i-1},1)\sin(\theta) + \eta_{s,1} \\
X_{c,-1}(iT) &= \sqrt{E}\delta(b_{i-1},-1)\cos(\theta) + \eta_{c,-1} \\
X_{s,-1}(iT) &= \sqrt{E}\delta(b_{i-1},-1)\sin(\theta) + \eta_{s,-1}
\end{aligned}
$$

$=0$ if $a \neq b$

where $\delta(a,b) = 1$ if $a = b$ and is zero otherwise. In the absence of noise ($\eta_{x,i} = 0$) it is easy to see that when $b_{i-1} = +1$ that $Y_1 = \sqrt{E}$ and $Y_{-1} = 0$.

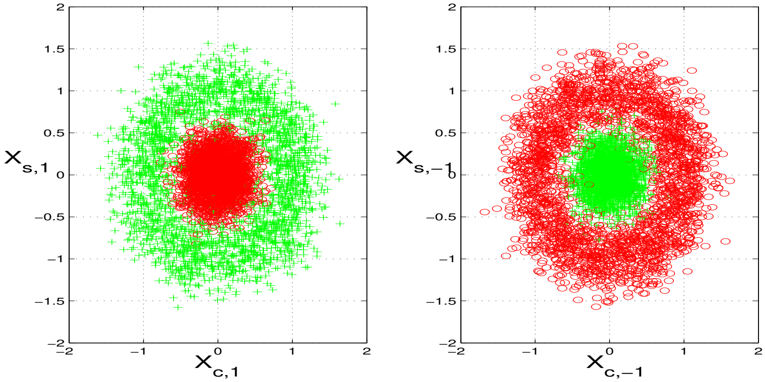
Densities

noise alone

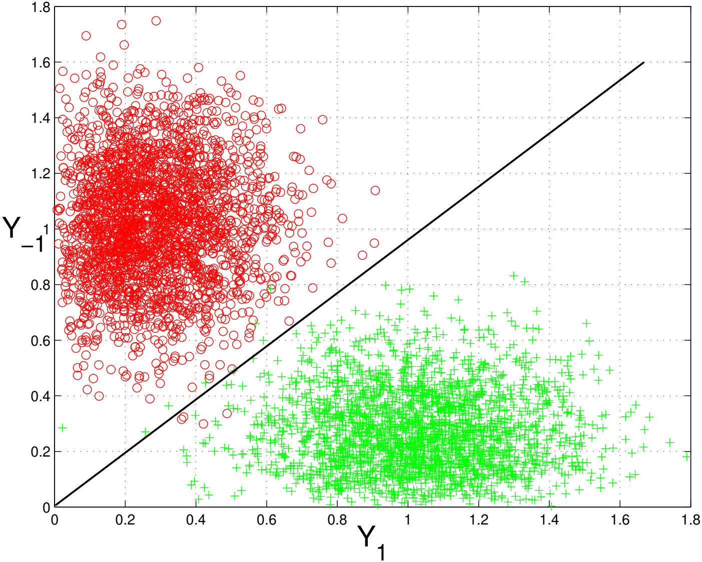


+1 was sent

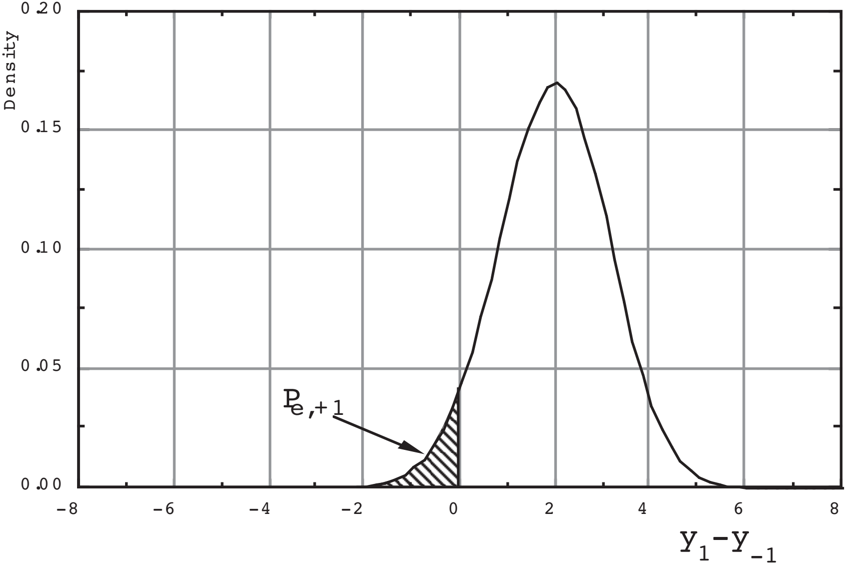
Densities



Densities



Densities



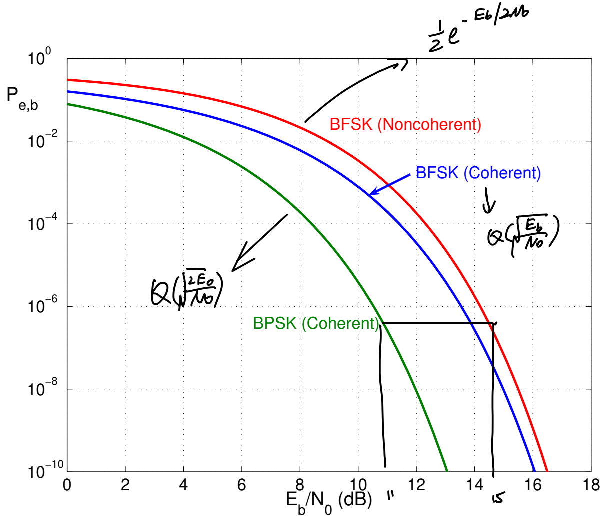
Error probability

The error probability of binary FSK is

$$\begin{aligned} P_{e,b} &= P\{X_{c,1}^2 + X_{s,1}^2 < X_{c,-1}^2 + X_{s,-1}^2 | b = +1\} \\ &= \frac{1}{2} e^{-E_b/2N_0}. \end{aligned}$$

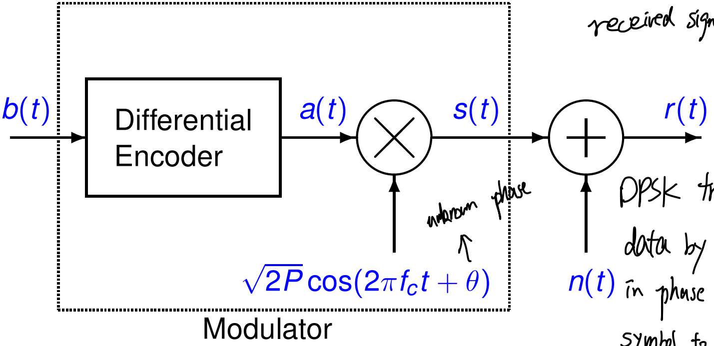
where $E_b = E$.

Error Probability of FSK with Noncoherent Detection



Differential Phase Shift Keying (DPSK)

Also does not need to know the phase of received signal



DPSK transmits data by the change in phase from one symbol to the next

$$b(t) = \sum_{l=-\infty}^{\infty} b_l \, p_T(t - lT), \qquad b_l \in \{+1, -1\}.$$

用前一个 symbol

$$a(t) = \sum_{l=-\infty}^{\infty} a_l \, p_T(t - lT), \qquad a_l \in \{+1, -1\}.$$

Differential Phase Shift Keying (DPSK)

Differential encoder is such that

$$
\begin{aligned}
b_l &= 1 \Rightarrow a_l = a_{l-1} \\
b_l &= -1 \Rightarrow a_l = -a_{l-1}.
\end{aligned}
$$

For example

l	$\dots$	-2	-1	0	1	2	3	$\dots$
b_l		-1	1	1	-1	1	-1	
a_l	-1	1	1	1	-1	-1	1	

初始值 变 不变 变

differentially encoded data

$$
s(t) = \sqrt{2P}a(t)\cos(2\pi f_c t + \theta).
$$

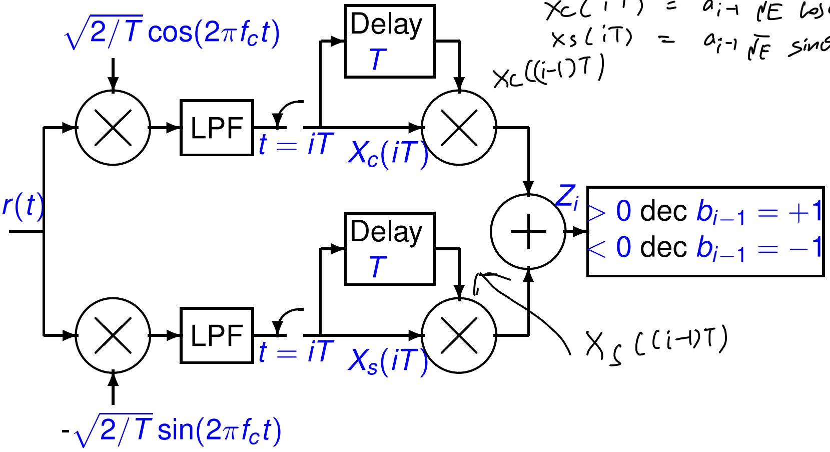
Differential Phase Shift Keying (DPSK) Demodulator

$X_c((i-1)T) = a_{i-2}\sqrt{E}\cos\theta$

$X_s((i-1)T) = a_{i-2}\sqrt{E}\sin\theta$

$X_c(iT) = a_{i-1}\sqrt{E}\cos\theta$

$X_s(iT) = a_{i-1}\sqrt{E}\sin\theta$



Differential Phase Shift Keying (DPSK)

$X_c(iT)\, X_c((i-1)T) = (\sqrt{E}\, a_{i-1} \cos\theta)(\sqrt{E}\, a_{i-2} \cos\theta)$

$= E\, a_{i-1} a_{i-2} \cos^2\theta$

$X_s(iT)\, X_s((i-1)T) = E\, a_{i-1} a_{i-2} \sin^2\theta$

$b_{i-1} = +1$

$\therefore Z_i = E\, a_{i-1}\, a_{i-2} \gtrless 0$

$b_{i-1} = -1$

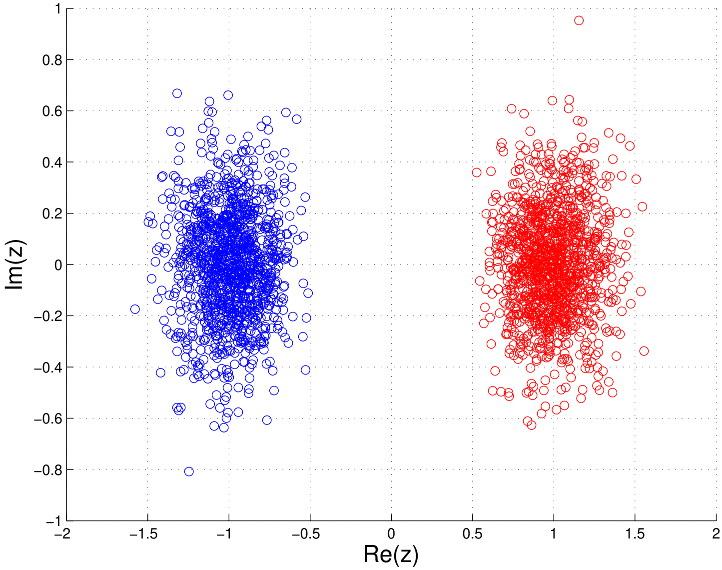
$$
\begin{aligned}
X_c(iT) &= \sqrt{E} a_{i-1} \cos\theta + \eta_{c,i} \\
X_s(iT) &= \sqrt{E} a_{i-1} \sin\theta + \eta_{s,i}.
\end{aligned}
$$

The random variables $\eta_{c,i}$ and $\eta_{s,i}$ are independent identically distributed Gaussian random variables with mean 0 and variance $N_0/2$. Thus

$$
\begin{aligned}
Z_i &= X_c(iT)X_c((i-1)T) + X_s(iT)X_s((i-1)T) \\
Z_i &= \mathrm{Re}[W(iT)W^*((i-1)T)]
\end{aligned}
$$

where $W(iT) = X_c(iT) + jX_s(iT)$.

Differential Phase Shift Keying (DPSK)



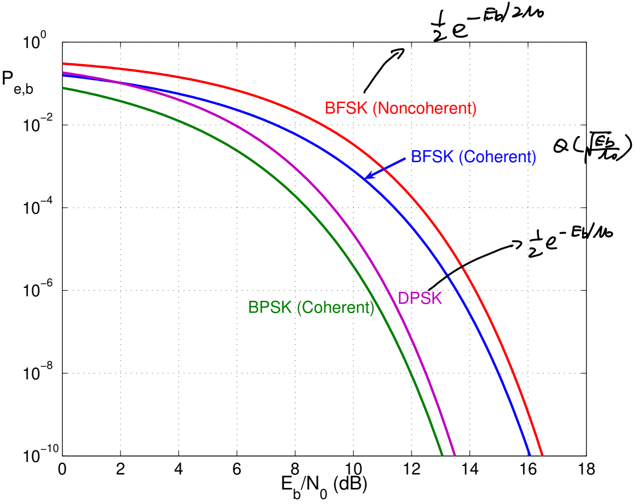
Differential Phase Shift Keying (DPSK)

The error probability for DPSK is

$$P_{e,b} = \frac{1}{2} e^{-E_b/N_0}.$$

Thus differential phase shift keying is 3dB better than FSK with noncoherent detection. However, errors tend to occur in pairs.

DPSK Error Probability



DPSK Error Probability

Modulation	Error Probability
BPSK (Coherent)	$Q(\sqrt{\frac{2E_b}{N_0}})$
FSK (Coherent)	$Q(\sqrt{\frac{E_b}{N_0}})$
FSK (Noncoherent)	$\frac{1}{2}e^{-E_b/2N_0}$
DPSK (Noncoherent)	$\frac{1}{2}e^{-E_b/N_0}$

DPSK Signals

To derive the above expression for DPSK consider the low pass filter with impulse response $h(t) = p_T(t)$. The output of the lowpass filters can be expressed as

$$
\begin{aligned}
X_c(t) &= \int_{-\infty}^{\infty} \sqrt{\frac{2}{T}} \cos(2\pi f_c \tau) h(t-\tau) r(\tau) d\tau \\
X_c(iT) &= \sqrt{\frac{2}{T}} \int_{-\infty}^{\infty} \cos(2\pi f_c \tau) p_T(iT-\tau) r(\tau) d\tau \\
&= \int_{(i-1)T}^{iT} \sqrt{\frac{2}{T}} \cos(2\pi f_c \tau) \left[\sum_{l=-\infty}^{\infty} \sqrt{2P} a_l \cos(2\pi f_c \tau + \theta) p_T(\tau - lT) + n(\tau) \right] d\tau \\
&= \sqrt{\frac{2}{T}} \int_{(i-1)T}^{iT} \sqrt{2P} a_{i-1} \cos(2\pi f_c \tau) \cos(2\pi f_c \tau + \theta) d\tau + \eta_{c,i}
\end{aligned}
$$

DPSK Signals

$n_{c,i}$ is Gaussian random variable, mean 0 variance $N_0/2$. Assuming $2\pi f_c T = 2\pi n$

$$X_c(iT) = \sqrt{E} a_{i-1} \cos\theta + \eta_{c,i}$$

Similarly

$$X_s(iT) = \sqrt{E} a_{i-1} \sin\theta + n_{c,i}$$

Thus

$$Z_i = X_c(iT)X_c((i-1)T) + X_s(iT)X_s((i-1)T)$$

Note that if we write $W(iT) = X_c(iT) + jX_s(iT)$ that $Z_i = \text{Re}[W(iT)W^*((i-1)T)]$. It is clear that this represents the phase difference between two consecutive symbols.

DPSK Signals

$\sqrt{E}\, a_{i-1}\, e^{j\theta}$

$$\begin{aligned} W(iT) &= \sqrt{E}a_{i-1}\cos\theta + \eta_{c,i} + j\sqrt{E}a_{i-1}T\sin\theta + jn_{c,i} \\ W((i-1)T) &= \sqrt{E}a_{i-2}\cos\theta + \eta_{c,i-1} + j\sqrt{E}a_{i-2}\sin\theta + jn_{c,i-1} \end{aligned}$$

$$\begin{aligned} Z_i &= Ea_{i-1}a_{i-2}\cos^2\theta + Ea_{i-1}a_{i-2}\sin^2\theta + \text{noise terms} \\ &= Ea_{i-1}a_{i-2} + \text{noise terms} \\ &= Eb_{i-1} + \text{noise terms} \end{aligned}$$

DPSK Example

$-3T$ $-2T$ $-T$ 0 T

l	$\dots$	-2	-1	0	1	2	
b_l		-1	1	1	-1	1	-1
a_l	-1	1	1	1	-1	-1	1
$X_c(lT)$	$.31$	$-.47$	$-.37$	$-.46$	$.39$	$.38$	$-.42$
$X_s(lT)$	$-.51$	$.37$	$.47$	$.38$	$-.30$	$-.46$	$.42$
$W(lT)$	$.6\angle -59$	$.6\angle 141$	$.6\angle 128$	$.6\angle 141$	$.5\angle -37$	$.6\angle -51$	$.6\angle 45$
$Z(lT)$		-0.34	0.35	0.35	-0.30	0.29	-0.36

DPSK vs. Orthogonal Signals ?

- A DPSK signal over a period of $2T$ represents a single data bit.
- The bit is encoded into the signal from the phase difference between the first T second interval and the second T second interval.
- For a fixed signal over the first T seconds, the two possible signals over the $2T$ second interval are orthogonal.
- The receiver attempts to demodulate the signal over this $2T$ second interval.
- The performance of the receiver depends on the energy over that $2T$ second interval.
- So the error probability is the same as two orthogonal signals demodulated noncoherently but with energy $2E$ where E is the energy per data bit duration.

M-ary Orthogonal Signals

Consider a set of equal energy orthogonal signals with center frequency f_0.

$$u_l(t) = \sqrt{E}\phi_l(t), \quad 0 \le t \le T, \quad l = 0, 1, ..., M-1.$$

$$\int_0^T \phi_l(t)\phi_m(t)dt = \begin{cases} 1, & l = m \\ 0, & l \neq m. \end{cases}$$

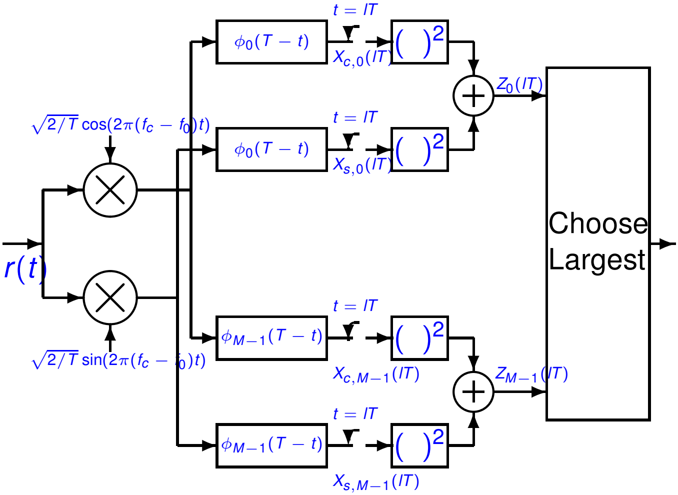
For transmission the signals are mixed up to a carrier frequency f_c (by multiplying by $\cos(2\pi(f_c - f_0)t)$).

$$s_l(t) = u_l(t)\sqrt{\frac{2}{T}}\cos(2\pi(f_c - f_0)t)$$

The received signal has a phase shift and additive white Gaussian noise.

$$r(t) = \sqrt{E}u_l(t)\cos(2\pi(f_c - f_0)t + \theta) + n(t)$$

Optimum Noncoherent Demodulator



Optimum Noncoherent Demodulator

If signal 0 is transmitted during the interval $[(l-1)T, lT)$ then

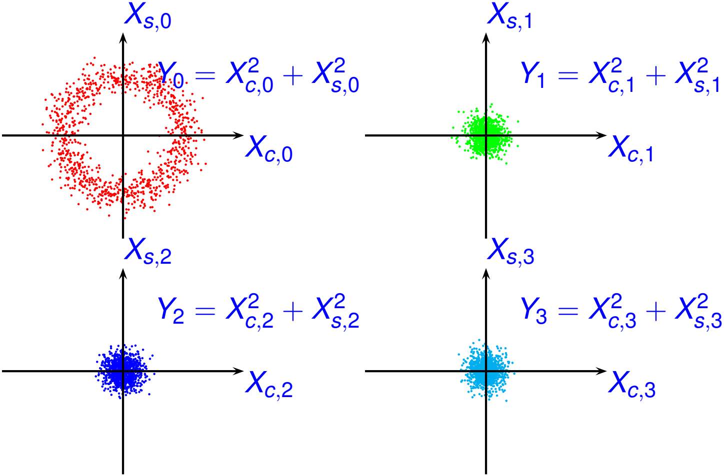
$$
\begin{aligned}
X_{c,m}(lT) &= \begin{cases} \sqrt{E}\cos(\theta) + \eta_{c,0}, & m = 0 \\ \eta_{c,m}, & m \neq 0 \end{cases} \\
X_{s,m}(lT) &= \begin{cases} \sqrt{E}\sin(\theta) + \eta_{s,0}, & m = 0 \\ \eta_{s,m}, & m \neq 0 \end{cases}
\end{aligned}
$$

The decision statistic then (if signal 0 is transmitted) has the form

$$
\begin{aligned}
Z_0(lT) &= E + 2\sqrt{E}(\eta_{c,0}\cos(\theta) + \eta_{s,0}\sin(\theta)) + \eta_{c,0}^2 + \eta_{s,0}^2, \\
Z_1(lT) &= \eta_{c,1}^2 + \eta_{s,1}^2 \\
Z_2(lT) &= \eta_{c,2}^2 + \eta_{s,2}^2 \\
. & \quad\quad . \\
Z_{M-1}(lT) &= \eta_{c,M-1}^2 + \eta_{s,M-1}^2
\end{aligned}
$$

Example: $M = 4$, orthogonal signals ($E_b/N_0 = 10$ dB)

Signal 0 transmitted



Error Probability Analysis

The symbol error probability for noncoherently detection of orthogonal signals is

$$
\begin{aligned}
P_{e,s} &= \frac{1}{M} e^{\{-E_b \log_2(M)/N_0\}} \sum_{m=2}^{M} (-1)^m \binom{M}{m} e^{\{E_b \log_2(M)/(mN_0)\}} \\
&= \frac{1}{M} \sum_{m=2}^{M} (-1)^m \binom{M}{m} e^{\{-\frac{E_b \log_2(M)}{N_0} \frac{m-1}{m}\}}
\end{aligned}
$$

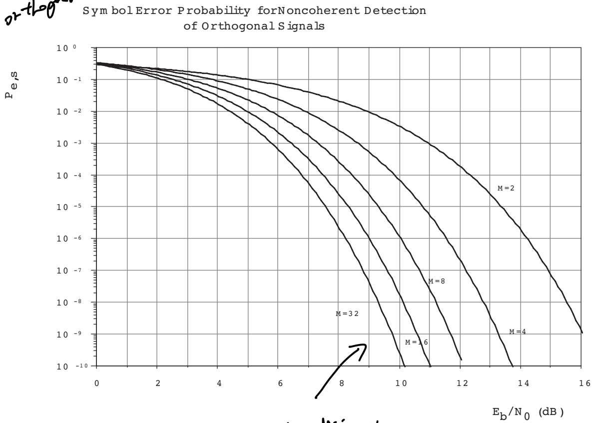
As with coherent demodulation the relation between bit error probability and symbol error probability for noncoherent demodulation of orthogonal signals is

$$
P_{e,b} = \frac{2^{k-1}}{2^k - 1} P_{e,s} = \frac{M}{2(M-1)} P_{e,s}.
$$

Performance of Noncoherent Detection of Orthogonal Signals

$M = N$ for orthogonal

$r = \frac{\log_2 M}{M}$



8psk

$r = \frac{\log_2 8}{2} = \frac{3}{2}$

8ary orth

$r = \frac{\log_2 M}{M} = \frac{3}{8}$

$m \uparrow$ smaller bandwidth efficiency

better energy efficiency

Example

Consider a set of M signals

$$s_l(t) = a_l(t)\cos(2\pi f_c t + \beta_l(t)), \quad 0 \le t \le T, \;\; l = 0, 1, ..., M-1.$$

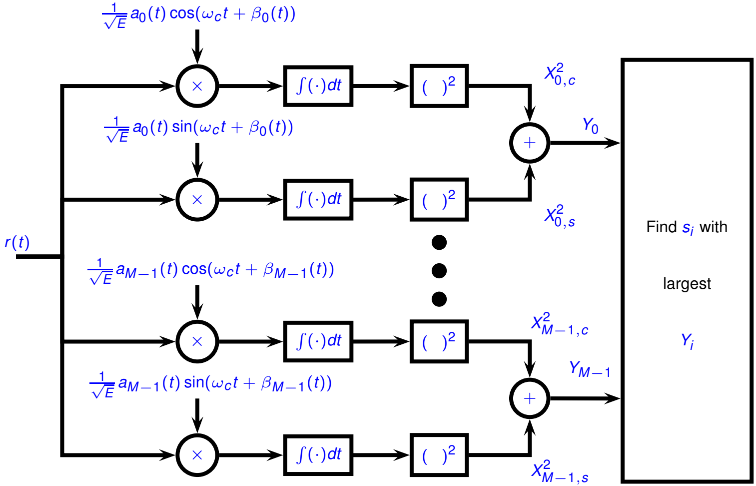
Assumed that

$$\int_0^T s_l(t)s_m(t)dt = \begin{cases} E, & l = m \\ 0, & l \neq m. \end{cases}$$

The received signal is

$$r(t) = a_l(t)\cos(2\pi f_c t + \beta_l(t) + \theta) + n(t), \quad 0 \le t \le T.$$

Receiver Block Diagram



Error Probability for M-orthogonal Signals: Noncoherent reception in AWGN

Preliminaries: First we derive the density for the sum of the squares of two Gaussian random variables. Let

$$\begin{aligned} X_c &\sim N(\mu_c, \sigma^2) \\ X_s &\sim N(\mu_s, \sigma^2) \end{aligned}$$

with X_c, X_s independent. Let $\mu^2 = \mu_c^2 + \mu_s^2$ and

$$Y = X_c^2 + X_s^2.$$

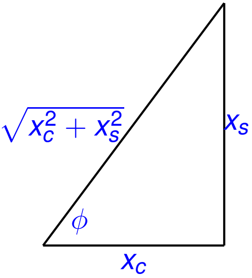
Density

Then

$$
\begin{aligned}
P\{Y \le y\} &= P(X_c^2 + X_s^2 \le y) \\
&= \iint\limits_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(x_c-\mu_c)^2 + (x_s-\mu_s)^2\right]\right\} dx_c dx_s \\
&= \iint \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}[x_c^2 + x_s^2 - 2(x_c\mu_c + x_s\mu_s) + \mu^2]\right\} dx_c dx_s \\
\mu^2 &= \mu_c^2 + \mu_s^2
\end{aligned}
$$

Simple Math

$$\tan(\phi) = \frac{x_s}{x_c}, \quad \phi = \tan^{-1}(\frac{x_s}{x_c})$$
$$\cos\phi = \frac{x_c}{\sqrt{x_c^2 + x_s^2}}$$
$$\sin\phi = \frac{x_s}{\sqrt{x_0^2 + x_s^2}}$$

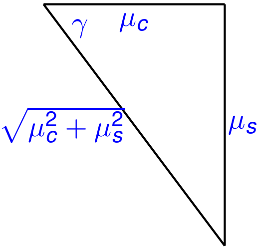


Simple Math

$$\tan\gamma = \frac{-\mu_s}{\mu_c}, \quad \gamma = \tan^{-1}\left(\frac{-\mu_s}{\mu_c}\right)$$

$$\cos\gamma = \frac{\mu_c}{\sqrt{\mu_c^2+\mu_s^2}}$$

$$\sin\gamma = \frac{-\mu_s}{\sqrt{\mu_c^2+\mu_s^2}}$$



More Simple Math

$$
\begin{aligned}
\beta &= \sqrt{\mu_c^2 + \mu_s^2}\,\sqrt{x_c^2 + x_s^2} \\
x_c\mu_c + x_s\mu_s &= \beta\cos(\phi + \gamma) \\
&= \beta\,[\cos\phi\cos\gamma - \sin\phi\sin\gamma]
\end{aligned}
$$

$$
x_c\mu_c + x_s\mu_s = \sqrt{\mu_c^2 + \mu_s^2}\,\sqrt{x_c^2 + x_s^2}\,[\cos(\phi + \gamma)]
$$

Back to Density

$$P\{Y \leq y\} = \iint\limits_{x_c^2+x_s^2 \leq y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[x_c^2 + x_s^2 - 2\mu \sqrt{x_c^2 + x_s^2} \cos(\phi+\gamma) + \mu^2\right]\right\} dx_c dx_s$$

$$= \int_{r^2 \leq \sqrt{y}} \int_{\phi=0}^{2\pi} \frac{r}{2\pi\sigma^2} \exp\left\{-[\frac{r^2}{2\sigma^2} - \frac{\mu r}{\sigma^2}\cos(\phi+\gamma) + \frac{\mu^2}{2\sigma^2}]\right\} dr d\phi$$

$$= \int_{r \leq \sqrt{y}} \frac{r}{\sigma^2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\} e^{-\mu^2/2\sigma^2} \underbrace{\frac{1}{2\pi} \int_{\phi=0}^{2\pi} \exp\left\{\frac{\mu r}{\sigma^2} \cos(\phi+\gamma)\right\} d\phi dr}_{I_0\left(\frac{\mu r}{\sigma^2}\right)}$$

Density

$$P\{Y \le y\} = \int_{r<\sqrt{y}} \frac{r}{\sigma^2} \exp\left\{-\frac{r^2+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu r}{\sigma^2}\right) dr$$

Let $u = r^2$ then $0 \le r \le \sqrt{y}$ is equivalent to $u \le y$. Also $du = 2rdr$.

$$\begin{aligned} P\{Y \le y\} &= \int_{u\le y} \frac{1}{2\sigma^2} \exp\left\{-\frac{u+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu\sqrt{u}}{\sigma^2}\right) du \\ f_Y(y) &= \frac{1}{2\sigma^2} \exp\left\{-\frac{y+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu\sqrt{y}}{\sigma^2}\right), y \ge 0. \end{aligned}$$

Note that since $f_Y(y)$ is a density function

$$\int_0^\infty f_Y(y)dy = 1.$$

Density for Signal and Noise

A change of variables makes for a cleaner expression: Let $W = Y/(2\sigma^2)$. Then $f_W(w) = 2\sigma^2 f_Y(2\sigma^2 w)$. So

$$f_W(w) = \exp\{-(w+\Gamma)\} I_0\left(\sqrt{4\Gamma w}\right)$$

where $\Gamma = \mu^2/(2\sigma^2)$. (If the receiver does this normalization then it must know the power density of the noise). Now let $Z = \sqrt{Y}$. Then

$$\begin{aligned}
P\{Z \le z\} &= P\left\{\sqrt{Y} \le z\right\} = P\left\{Y \le z^2\right\} \\
F_Z(z) &= F_Y(z^2) \\
f_Z(z) &= f_Y(z^2)(2z) \\
&= \frac{z}{\sigma^2} \exp\left\{-\frac{z^2+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu z}{\sigma^2}\right)
\end{aligned}$$

Density for Noise Alone

$$\mu = 0 \quad \Rightarrow \quad f_Y(y) = \frac{1}{2\sigma^2}\, e^{-y/2\sigma^2}$$
$$f_Z(z) = \frac{z}{\sigma^2}\, e^{-z^2/2\sigma^2}$$

Summary

Density of the sum of squares of Gaussian

- Assume X_c and X_s are independent Gaussian random variables with mean μ_c and μ_s respectively ($\mu^2 = \mu_c^2 + \mu_s^2$) and identical variance σ^2. modified Bessel function of order 0
- $Y = X_c^2 + X_s^2$
- $f_Y(y) = \frac{1}{2\sigma^2} \exp\{-\frac{y+\mu^2}{2\sigma^2}\} I_0(\frac{\mu\sqrt{y}}{\sigma^2}), \quad y \geq 0$
- $f_Y(y) = \frac{1}{2\sigma^2} \exp\{-\frac{y}{2\sigma^2}\}, \quad \mu = 0, \quad y \geq 0.$
- $Z = \sqrt{X_c^2 + X_s^2}$
- $f_Z(z) = \frac{z}{\sigma^2} \exp\{-\frac{z^2+\mu^2}{2\sigma^2}\} I_0(\frac{\mu z}{\sigma^2}), \quad z \geq 0$
- $f_Z(z) = \frac{z}{\sigma^2} \exp\{-\frac{z^2}{2\sigma^2}\}, \quad \mu = 0, \quad z \geq 0.$

Error Probability

$$
\begin{aligned}
X_{j,c} &= \sqrt{E}\,\delta_{ij}\cos\theta + n_c \sim N\left(\sqrt{E}\delta_{ij}\cos\theta, \frac{N_0}{2}\right) \\
X_{j,s} &= \sqrt{E}\delta_{ij}\sin\theta + n, \sim N\left(\sqrt{E}\delta_{ij}\sin\theta, \frac{N_0}{2}\right) \\
j &= 0, 1, \ldots, M-1
\end{aligned}
$$

Error Probability

Let $Z_j = \sqrt{X_{jc}^2 + X_{js}^2}$. Then we need to determine the probability that Z_0 which corresponds to nonzero mean random variables is less than $Z_1, Z_2, ..., Z_{M-1}$ which correspond to zero mean random variables.

$$\begin{aligned}
P_{c,i} &= P\{Z_j \leq Z_i, j \neq i \mid s_i \text{ transmitted}\} \\
&= E[P\{Z_j < Z_i, j \neq i \mid Z_i, s_i \text{ trans}\}] \\
&= E[\prod_{j \neq i} P\{Z_j < Z_i \mid s_i \text{ trans}, Z_i\}] \\
&= \int p(z_i)[P\{Z_j \leq z_i\}]^{M-1}\, dz_i
\end{aligned}$$

Error Probability

Doing a change of variables ($\nu = \frac{u^2}{2\sigma^2},\ d\nu = \frac{udu}{\sigma^2}$) we obtain

$$
\begin{aligned}
P\{Z_j \le z_i\} &= \int_{u \le z_i} \frac{u}{\sigma^2}\, e^{-u^2/2\sigma^2} du = \int_{\nu \le z_i^2/2\sigma^2} e^{-\nu} d\nu \\
&= 1 - e^{-z_i^2/2\sigma^2} \\
P_{c,i} &= \int_0^\infty p(z_i)[1 - e^{-z_i^2/2\sigma^2}]^{M-1}\, dz_i \\
&= \int_0^\infty p(z_i) \left(\sum_{l=0}^{M-1} (-1)^l \binom{M-1}{l} e^{-lz_i^2/2\sigma^2} \right) dz_i \\
&= \sum_{l=0}^{M-1} (-1)^l \binom{M-1}{l} \int_0^\infty p(z_i) e^{-lz_i^2/2\sigma^2} dz_i
\end{aligned}
$$

Error Probability

$$\int_0^\infty p(z_i) e^{-lz_i^2/2\sigma^2} dz_i = \int_0^\infty \frac{z_i}{\sigma^2} e^{\frac{-(z_i^2+\mu^2)}{2\sigma^2}} I_0\left(\frac{\mu z_i}{\sigma^2}\right) e^{-lz_i^2/2\sigma^2} dz_i$$
$$= e^{-\mu^2/2\sigma^2} \int_0^\infty \frac{z_i}{\sigma^2} e^{-(l+1)z_i^2/2\sigma^2} I_0\left(\frac{\mu z_i}{\sigma^2}\right) dz_i$$

Do another change of variables, ($w_i = \sqrt{l+1}\, z_i$, $dw_i = \sqrt{l+1}\, dz_i$) we get

$$\int_0^\infty p(z_i) e^{-lz_i^2/2\sigma^2} dz_i = e^{-\mu^2/2\sigma^2} \int_0^\infty \frac{w_i}{\sigma^2\sqrt{l+1}} e^{-w_i^2/(2\sigma^2)} I_0\left(\frac{\mu w_i}{\sqrt{l+1}\,\sigma^2}\right) \frac{dw}{\sqrt{l+}}$$
$$= e^{-\mu^2/2\sigma^2} \frac{1}{(l+1)} \int \frac{w_i}{\sigma^2} e^{-\frac{w_i^2}{2\sigma^2}} I_0\left(\frac{\mu w_i}{\sqrt{l+1}\,\sigma^2}\right) dw_i$$

Error Probability

Let $\hat{\mu} = \frac{\mu}{\sqrt{l+1}}$. Then

$$\begin{aligned}
\int_0^\infty p(z_i) e^{-l z_i^2/2\sigma^2} dz_i &= \frac{e^{-\mu^2/2\sigma^2} e^{\hat{\mu}^2/2\sigma^2}}{l+1} \underbrace{\int \frac{w_i}{\sigma^2}\, e^{-\left(\frac{w_i^2+\hat{\mu}^2}{2\sigma^2}\right)} I_0\left(\frac{\hat{\mu} w_i}{\sigma^2}\right) dw_i}_{=1} \\
&= \frac{\exp\left\{\frac{\hat{\mu}^2-\mu^2}{2\sigma^2}\right\}}{(l+1)} = \frac{\exp\left\{\frac{\frac{\mu^2}{l+1}-\mu^2}{2\sigma^2}\right\}}{l+1} \\
&= \exp\left\{-\frac{l\mu^2}{2(l+1)\sigma^2}\right\} \frac{1}{l+1}.
\end{aligned}$$

Error Probability

Substituting this into the expression for the probability of correct, we obtain

$$P_{c,i} = \sum_{l=0}^{M-1} (-1)^l \binom{M-1}{l} \frac{\exp\left\{\frac{-l\mu^2}{2(l+1)\sigma^2}\right\}}{(l+1)}$$

where

$$\frac{\mu^2}{2\sigma^2} = \frac{E^2}{2\frac{N_0 E}{2}} = \frac{E}{N_0}.$$

Error Probability

Thus

$$P_{c,i} = 1 + \sum_{l=1}^{M-1} \frac{(-1)^l \binom{M-1}{l}}{l+1} \exp\left\{-\frac{l}{(l+1)}\frac{E}{N_0}\right\}$$

$$P_{e,i} = 1 - P_{c,i} = \sum_{l=1}^{M-1} \frac{(-1)^{l+1} \binom{M-1}{l}}{l+1} \exp\left\{-\frac{l}{l+1}\frac{E}{N_0}\right\}$$

$$P_e = \frac{1}{M}\sum_{i=1}^{M-1} P_{e,i} = P_{e,i}$$

$$M \text{ signals} \Rightarrow \log_2 M \text{ bits}$$

Error Probability

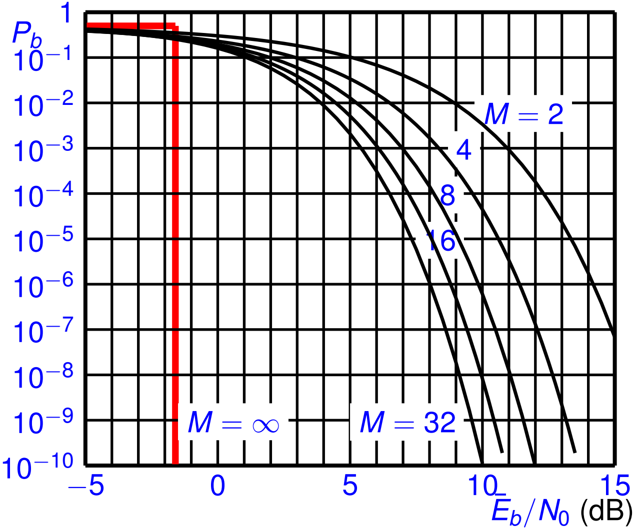
The limiting behavior of the error probability for M-ary orthogonal signals with noncoherent demodulation is the same as the limiting performance of coherent demodulation. If $M = 2^k$

$$P_{e,b} = \frac{1}{2}\frac{M}{M-1}P_{e,i} \quad E_b = E/\log_2 M$$

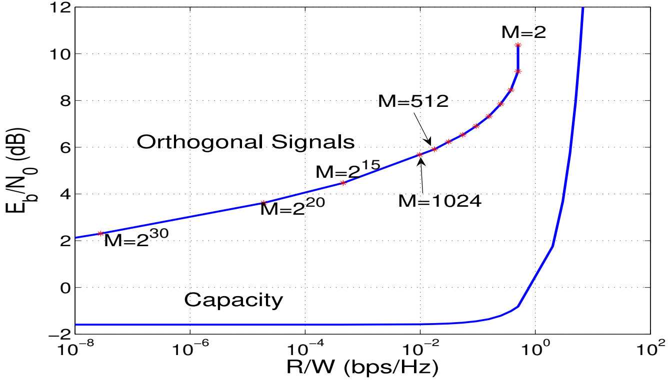
$$M = 2 \;\; P_{e,b} = P_{e,i} = \frac{1}{2}\, e^{-E_b/2N_0}$$

It can be shown that the asymptotic behavior of M orthogonal signals on a additive Gaussian noise channel with noncoherent reception is the same as with coherent reception. That is, for $E_b/N_0 < \ln 2$ the error probability is 1 while for $E_b/N_0 > \ln 2$ the error probability is 0.

Error Probability for M orthogonal signals



Capacity vs. Noncoherent Detection of Orthogonal Signals @ $P_{e,b} = 0.001$



Hadamard-Walsh Orthogonal Signals

Using M different (orthogonal) frequencies to transmit information requires M matched (coherent or noncoherent) filters. This can become complex if M is large. Alternatively we can use the Hadamard-Walsh construction of orthogonal signals. The Hadamard matrix is an N by N matrix with components either +1 or -1 such that every pair of distinct rows are orthogonal. We show how to construct a Hadamard when the number of signals is a power of 2 (which is often the case).

Begin with a two by two matrix

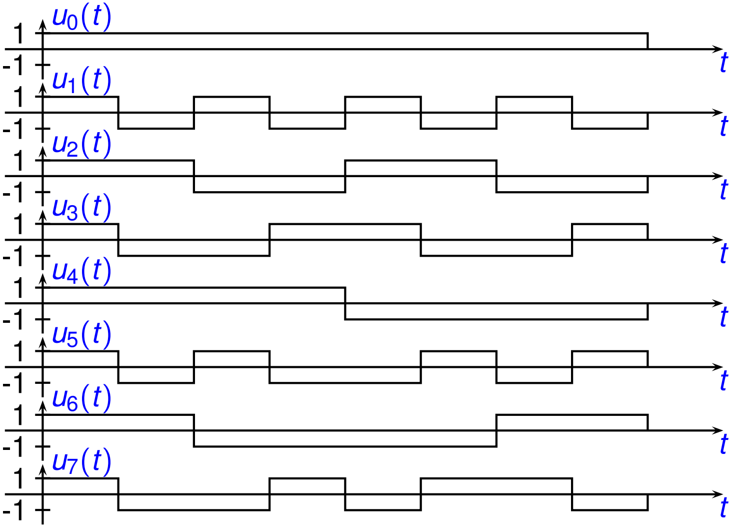
$$H_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}.$$

Then use the recursion

$$H_{2^l} = \begin{bmatrix} +H_{2(l-1)} & +H_{2(l-1)} \\ +H_{2(l-1)} & -H_{2(l-1)} \end{bmatrix}.$$

Now it is easy to check that distinct rows in these matrices are

Waveforms



Signals

Suppose that we use the Hadamard-Walsh signal set of size M. Let $u_i(t)$ $i = 0, 1, 2, ..., M-1$ be the baseband signals. Then the transmitted signal is

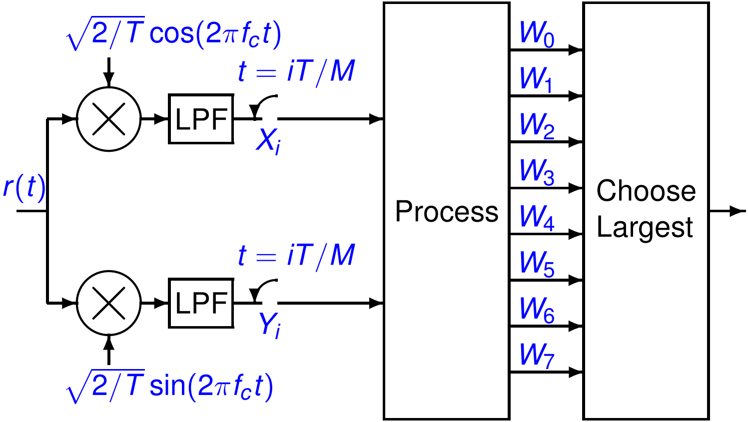
$$s_i(t) = \sqrt{2P} u_i(t) \cos(2\pi f_c t), \quad 0 \leq t \leq T$$

The received signal (assuming signal i is transmitted) is

$$r(t) = \sqrt{2P} u_i(t) \cos(2\pi f_c t + \theta) + n(t), \quad 0 \leq t \leq T$$

where $n(t)$ is white Gaussian noise (power spectral density $N_0/2$) and θ is a phase unknown to the receiver.

Noncoherent Receiver



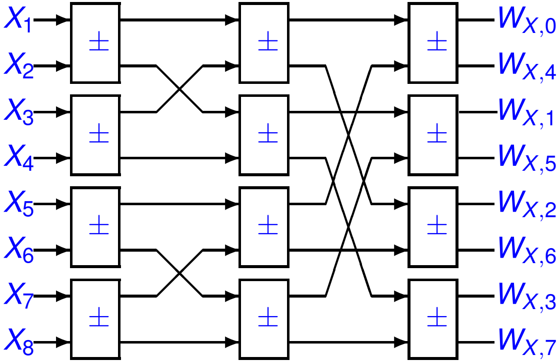
Noncoherent Reception of Hadamard Generated Orthogonal Signals

$$
\begin{aligned}
W_0 &= (W_{X,0}^2 + W_{Y,0}^2) \\
&= (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8)^2 \\
&\quad + (Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8)^2 \\
W_1 &= (X_1 - X_2 + X_3 - X_4 + X_5 - X_6 + X_7 - X_8)^2 \\
&\quad +(Y_1 - Y_2 + Y_3 - Y_4 + Y_5 - Y_6 + Y_7 - Y_8)^2 \\
W_2 &= (X_1 + X_2 - X_3 - X_4 + X_5 + X_6 - X_7 - X_8)^2 \\
&\quad +(Y_1 + Y_2 - Y_3 - Y_4 + Y_5 + Y_6 - Y_7 - Y_8)^2 \\
W_3 &= (X_1 - X_2 - X_3 + X_4 + X_5 - X_6 - X_7 + X_8)^2 \\
&\quad +(Y_1 - Y_2 - Y_3 + Y_4 + Y_5 - Y_6 - Y_7 + Y_8)^2
\end{aligned}
$$

Noncoherent Reception of Hadamard Generated Orthogonal Signals

$$
\begin{aligned}
W_4 &= (X_1 + X_2 + X_3 + X_4 - X_5 - X_6 - X_7 - X_8)^2 \\
&\quad + (Y_1 + Y_2 + Y_3 + Y_4 - Y_5 - Y_6 - Y_7 - Y_8)^2 \\
W_5 &= (X_1 - X_2 + X_3 - X_4 - X_5 + X_6 - X_7 + X_8)^2 \\
&\quad + (Y_1 - Y_2 + Y_3 - Y_4 - Y_5 + Y_6 - Y_7 + Y_8)^2 \\
W_6 &= (X_1 + X_2 - X_3 - X_4 - X_5 - X_6 + X_7 + X_8)^2 \\
&\quad + (Y_1 + Y_2 - Y_3 - Y_4 - Y_5 - Y_6 + Y_7 + Y_8)^2 \\
W_7 &= (X_1 - X_2 - X_3 + X_4 - X_5 + X_6 + X_7 - X_8)^2 \\
&\quad + (Y_1 - Y_2 - Y_3 + Y_4 - Y_5 + Y_6 + Y_7 - Y_8)^2
\end{aligned}
$$

Fast Processing for Hadamard Signals



Fast Processing

$$W_i = W_{X,i}^2 + W_{Y,i}^2, \quad i = 0, 1, ..., 7$$