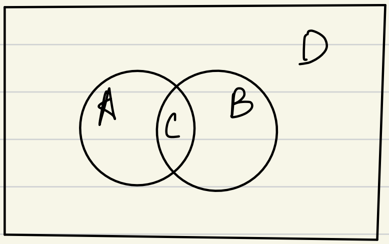
Homework 1   YUZHAN JIANG

1.



Let A be the event: rain on Saturday. Let B be the event: rain on Sunday.
Let C be the event: rain on Both days. Let D be the event: rain on neither day.
Then we have:

$$P(\Omega) = P(A) + P(B) - P(C) + P(D) = 1 \quad (\text{by the normalization axiom})$$

(a) $P(A) + P(B) - P(C) + P(D)$

$= 90\% + 80\% - 70\% + 30\%$

$= 1.1$

$\neq 1$

$\therefore$ These answers are not consistent with the axioms of probability.

(b) $P(A) + P(B) - P(C) + P(D)$

$= 60\% + 80\% - 70\% + 30\%$

$= 1$

However, $P(B) = 80\% > P(C) = 70\%$, $P(\text{rain on sunday}) > P(\text{rain on both day})$

we can get:

$$P(\text{rains on Saturday but not on Sunday}) = 70\% - 80\% = -10\%$$

By the non-negative axiom of probability, it should be non-negative.
Therefore, these answers are not consistent with the axioms of probability.

(c) $P(A) + P(B) - P(C) + P(D)$

$= 80\% + 70\% - 60\% + 10\%$

$= 1$ by the normalization axiom of probability

$\therefore$ These answers are consistent with the axioms of probability.

(d) $P(A) + P(B) - P(C) + P(D)$

$= 70\% + 60\% - 50\% + 90\%$

$\neq 1$

By the normalization axiom of probability,

$\therefore$ These answers are not consistent with the axioms of probability.

2. First, we have $P(A) = 1-\delta$, $P(B) = 1-\delta$

Then by Inclusion-Exclusion principle:

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\therefore P(A\cup B) + P(A\cap B) = P(A) + P(B) > 2-2\delta$$

since $P(A\cup B) \leq 1$

$$\therefore P(A\cap B) > 2-2\delta - P(A\cup B$$

$$P(A\cap B) \geq 2-2\delta-1$$

$$P(A\cap B) \geq 1-2\delta$$

3. (a) proof:

$$P(A\cup B\cup C) = P(A) + P(B\cup C) - P(A\cap(B\cup C))$$ by inclusion exclusion principle

$$= P(A) + P(B) + P(C) - P(B\cap C) - P(A\cap(B\cup C))$$ by inclusion exclusion principle

$$= P(A) + P(B) + P(C) - P(B\cap C) - P((A\cap B)\cup(A\cap C))$$ by distributive laws

$$= P(A) + P(B) + P(C) - P(B\cap C) - (P(A\cap B) + P(A\cap C) - P(A\cap B\cap C))$$ by inclusion exclusion principle

$$= P(A) + P(B) + P(C) - P(B\cap C) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C)$$

$\therefore$ proof is done

(b) Let event A be the first dice shows either 4 or 5
Let event B be the second dice shows either 4 or 5
Let event C be the third dice shows either 4 or 5

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(B\cap C) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C)$$ by part (a)

Where $P(A) = P(B) = P(C) = \frac{2}{6} = \frac{1}{3}$

$P(B\cap C) = P(A\cap B) = P(A\cap C) = \frac{1}{3}\times\frac{1}{3} = \frac{1}{9}$

$P(A\cap B\cap C) = \frac{1}{3}\times\frac{1}{3}\times\frac{1}{3} = \frac{1}{27}$

$$\therefore P(A\cup B\cup C) = 3\times\frac{1}{3} - 3\times\frac{1}{9} + \frac{1}{27}$$

$$= 1 - \frac{1}{3} + \frac{1}{27}$$

$$= \frac{19}{27}$$

4.

(a) P(no more than two heads) = 1 − P(more than 2 heads)

(by Law of Complements)

P(more than 2 heads) = P(3 heads) + P(4 heads)

$$= \frac{\#\text{ of heads}}{\#\text{ of total samples}}$$

$$= \frac{\binom{4}{3}}{16} + \frac{\binom{4}{4}}{16} \quad \text{(by Combinatorics)}$$

$$= \frac{4}{16} + \frac{1}{16}$$

$$= \frac{5}{16}$$

∴ P(no more than two heads) $= 1 - \frac{5}{16}$

$$= \frac{11}{16}$$

(b) The set of two consecutive tails: {TTHH, HTTH, HHTT, TTHT, THTT}

The set of three consecutive tails: {TTTH, HTTT}

The set of four consecutive tails: {TTTT}

∴ P(two or more consecutive tails) = P(2 con tails) + P(3 consecutive tails) + P(4 consecutive tails)

$$= \frac{5}{16} + \frac{2}{16} + \frac{1}{16}$$

$$= \frac{1}{2}$$

5. (a)

Sample space: $\{GGG, GGB, BGG, GBG, GBB, BGB, BBG, BBB\}$

$P(\text{exactly one daughter}) = P(\{GBG, GBB, BBG\})$

$= \frac{3}{8}$

(b) Let A be a girl has no sisters.

Let B be event a family has at least one daughter

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ (by def of conditional probability)

$= \frac{P(\text{exactly one daughter})}{P(\text{one girl}) + P(\text{two girl}) + P(3\text{ girl})}$

$= \frac{\frac{3}{8}}{\frac{7}{8}}$

$= \frac{3}{7}$

6. we have $P(\text{rains on Fri}) = 0.2$

$P(\text{sun shines on fri}) = 0.8$

$P(\text{cloudy} \cap \text{dry on Fri}) = 0.1$

and $P(\text{rains on Fri}) = 1 - P(\text{rains on Fri})$ (by the Laws of complements)

$P(\text{Sunny on Fri}) = 1 - P(\text{cloudy on Fri})$

$\therefore P(\text{cloudy}) = 1 - P(\text{sunny})$ (by law of complements)

$= 0.2$

$$P(\text{dry} \mid \text{cloudy}) = \frac{P(\text{dry} \cap \text{cloudy})}{P(\text{cloudy})}$$

$$= \frac{0.1}{0.2}$$

$$= \frac{1}{2}$$

$\therefore$ the probability of it is dry given it is cloudy on fridays in Ann Arbor is $\frac{1}{2}$

7. $A = \{16, 25, 34, 43, 52, 61\}$

$\therefore P(A) = \frac{6}{36} = \frac{1}{6}$

and $P(B) = P(C) = \frac{1}{6}$

$\therefore A \cap B = \{34\}$

$A \cap C = \{34\}$

$B \cap C = \{34\}$

$\therefore P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{36}$ and we observer that $P(A)P(B) = P(A)P(C) = P(B)P(C) = \frac{1}{36}$

$\therefore$ A, B and C are pairwise independent.

On the other hand,

$A \cap B \cap C = \{34\}$

$P(A \cap B \cap C) = \frac{1}{36} \neq P(A) \cdot P(B) \cdot P(C) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

$\therefore$ A, B and C are not independent as a triplet