EECS 501 HW 11 Name: Moyan Lyu

1.(a) X_t is Poisson Process, $X(y_2)-X(y_1)$ is independent of $X(y_1)$

$$P(X(y_1)<1,\ X(y_2)<2) = P(X(y_1)=0,\ X(y_2)-X(y_1)<2)$$

$$= P(X(y_1)=0,\ X(y_2)-X(y_1)=0) + P(X(y_1)=0,\ X(y_2)-X(y_1)=1)$$

$$P(X(y_1)=k) = \frac{(\lambda y_1)^k e^{-\lambda y_1}}{k!} \quad P(X(y_2)-X(y_1)) = \frac{[\lambda(y_2-y_1)]^k e^{-\lambda(y_2-y_1)}}{k!}$$

$$\therefore P(X(y_1)<1,\ X(y_2)<2) = \frac{(\lambda y_1)^0 e^{-\lambda y_1}}{0!}\cdot\frac{\lambda^0(y_2-y_1)^0 e^{-\lambda(y_2-y_1)}}{0!} + \frac{(\lambda y_1)^0 e^{-\lambda y_1}}{0}\cdot\frac{\lambda(y_2-y_1)e^{-\lambda(y_2-y_1)}}{1!}$$

$$= e^{-\lambda y_1}\cdot e^{-\lambda(y_2-y_1)} + e^{-\lambda y_1}\cdot\lambda(y_2-y_1)\cdot e^{-\lambda(y_2-y_1)}$$

$$= e^{-\lambda y_2} + \lambda(y_2-y_1)\cdot e^{-\lambda y_2}$$

$$= e^{-\lambda y_2}[1+\lambda(y_2-y_1)]$$

(b) $P(Y_1>y_1,\ Y_2>y_2) = P(X(y_1)<1,\ X(y_2)<2) = e^{-\lambda y_2}[1+\lambda(y_2-y_1)]$

$$P(Y_1\le y_1,\ Y_2\le y_2) = 1-P(Y_1>y_1 \cup Y_2>y_2)$$

$$P(Y_1>y_1 \cup Y_2>y_2) = P(Y_1>y_1)+P(Y_2>y_2)-P(Y_1>y_1 \cap Y_2>y_2)$$

$$P(Y_1>y_1) = P(X(y_1)<1) = P(X(y_1)=0) = \frac{(\lambda y_1)^0 e^{-\lambda y_1}}{0!} = e^{-\lambda y_1}$$

$$P(Y_2>y_2) = P(X(y_2)<2) = P(X(y_2)=0)+P(X(y_2)=1) = \frac{(\lambda y_2)^0 e^{-\lambda y_2}}{0!} + \frac{(\lambda y_2)^1 e^{-\lambda y_2}}{1!}$$

$$= e^{-\lambda y_2} + \lambda y_2 e^{-\lambda y_2}$$

$$\Rightarrow F_{Y_1Y_2}(y_1,y_2) = 1-e^{-\lambda y_1} - \cancel{e^{-\lambda y_2}} - \lambda y_2 e^{-\lambda y_2} + e^{-\lambda y_2}[1+\lambda(y_2-y_1)]$$

$$= 1-e^{-\lambda y_1} - \lambda y_1 e^{-\lambda y_2}$$

$$\Rightarrow f_{Y_1Y_2}(y_1,y_2) = \frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2}F_{Y_1Y_2}(y_1,y_2) = \lambda^2 e^{-\lambda y_2}, \text{ for } y_2>y_1>0$$

$\therefore$ The joint PDF of Y_1 and Y_2 is $f_{Y_1Y_2}(y_1,y_2) = \lambda^2 e^{-\lambda y_2}$, for $y_2>y_1>0$, otherwise $=0$.

(c) By the results get from 1(a) & (b), when $k=2$, the result holds true.

By using the induction, assume that when $k=K$, the result holds true.

(d) $P(Y>y)=P(X(y_1)<1,\ X(y_2)<2,\ \cdots\ X(y_k)<k)$

$P(Y\le y)=1-P(Y_1>y_1 \cup Y_2>y_2 \cup \cdots \cup Y_k>y_k)$

$P(Y>y)=e^{-\lambda y_1}\cdot e^{-\lambda(y_2-y_1)}\cdots e^{-\lambda(y_k-y_{k-1})}\cdot[1+\lambda(y_2-y_1)+\lambda^2(y_2-y_1)(y_3-y_2)+$

$\cdots+\underbrace{[\lambda^{k-1}(y_2-y_1)(y_3-y_2)\cdots(y_k-y_{k-1})}_{A}](=A)$

∴ The only term that contains all variables $y_1, y_2, y_3 \cdots y_k$ is A

other term only contain part of the variables $y_1, y_2 \cdots y_k$

∴ The rest of the terms will equal to 0 by the partial derivatives

∴ $f_Y(y)=(-1)^k\frac{\partial^k}{\partial y}P(Y>y)=\lambda^k e^{-\lambda y_k}$, for $y_k \geq y_{k-1} \geq \cdots \geq y_1 \geq 0$

2. A random process $\{X_t\}$: $X_t = A\sin(t+\theta)$

$A \sim$ Bernoulli (mean $=\frac{1}{4}$). $\theta \sim u[0, 2\pi]$

$\sin(t+\theta)=\sin t\cos\theta+\cos t\sin\theta$

Therefore, we obtain

$\mu X_t = E[A](E[\cos\theta]\sin t + E[\sin\theta]\cos t)$

Since θ is uniformly chosen from $[0, 2\pi]$,

$E[\cos\theta]=E[\sin\theta]=0$

$\Rightarrow \mu X_t = 0$

$R_X(s, s+t) = E[X_s X_{s+t}] = E[A\sin(s+\theta)A\sin(s+t+\theta)]$

$= E[A^2]E[\sin(s+\theta)\sin(s+\theta+t)]$

$= E[A^2]\cdot(-\frac{1}{2})E[\cos(2s+2\theta+t)-\cos(-t)]$

$= -\frac{1}{2}E[A^2]E[\cos(2s+2\theta+t)-\cos(t)]$

$= \frac{1}{2}E[A^2]\cos(t) \qquad (E[A^2]=Var(A)+E^2[A]=\frac{3}{4}\times\frac{1}{4}+(\frac{1}{4})^2=\frac{1}{4})$

$= \frac{1}{8}\cos(t)$

$R_X(t,0)=E[X_t X_0]=E[A^2]E[\sin(t+\theta)\sin\theta]=\frac{1}{2}E[A^2]\cos(t)$

$R_X(s, s+t)=R_X(t,0) \Rightarrow \{X_t\}$ is WSS.

3. $X_t = A\sin(t+\theta) + B$, $A \sim \text{Bernoulli}(\text{mean} = \frac{1}{4})$, $B \sim \text{Bernoulli}(\text{mean} = \frac{1}{2})$

$\sin(t+\theta) = \sin t \cos\theta + \cos t \sin\theta$

Therefore, we obtain

$$\mu_{X_t} = E[A](E[\cos\theta]\sin t + E[\sin\theta]\cos t) + E[B]$$

Since θ is uniformly chosen from $[0, 2\pi]$,

$$E[\cos\theta] = E[\sin\theta] = 0$$

$\Rightarrow \mu_{X_t} = \frac{1}{4} \times (0+0) + \frac{1}{2} = \frac{1}{2}$

$$\begin{aligned} R_X(s, s+t) &= E[X_s X_{s+t}] = E[(A\sin(s+\theta)+B)(A\sin(s+t+\theta)+B)] \\ &= E[A^2]E[\sin(s+\theta)\sin(s+\theta+t)] + E[B]E[A\sin(s+\theta)] \\ &\quad + E[B]E[A\sin(s+t+\theta)] + E[B^2] \\ &= \frac{1}{2}E[A^2]\cos(t) + E[B^2] \end{aligned}$$

$\text{Var}(A) = pq = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$, $\text{Var}(B) = pq = \frac{1}{4}$

$\therefore E[A^2] = \text{Var}(A) + E[A]^2 = \frac{1}{4}$, $E[B^2] = \text{Var}(B) + E[B]^2 = \frac{1}{2}$

$\Rightarrow R_X(s, s+t) = \frac{1}{8}\cos(t) + \frac{1}{2}$

$$\begin{aligned} R_X(t, 0) &= E[X_t X_0] = E[(A\sin(t+\theta)+B)(A\sin\theta + B)] \\ &= E[A^2]E[\sin(t+\theta)\sin\theta] + E[B^2] \\ &= \frac{1}{2}E[A^2]\cos(t) + E[B^2] \\ &= \frac{1}{8}\cos(t) + \frac{1}{2} \end{aligned}$$

$\because R_X(s, s+t) = R_X(t, 0)$, $\mu_{X_t} = \frac{1}{2}$

$\therefore \{X_t\}$ is WSS.