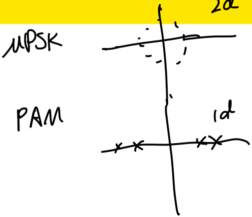
Lecture 12



$P_{e,s} \sim E_b/N_0$

R/N_0

Goals

- Be able to determine bandwidth efficiency and energy efficiency of orthogonal signals.
- Be able to synthesize different types of orthogonal signals.

Orthogonal Signals

- A set of signals $\{\varphi_i(t) : 0 \le t \le T, 0 \le i \le M-1\}$ are said to be orthogonal (over the interval $[0, T]$) if

$$\int_0^T \varphi_i(t)\varphi_j(t)dt = 0, \qquad i \neq j.$$

- In most cases the signals will have the same energy and it is convenient to normalize the signals to unit energy. A set of signals $\{\varphi_i(t) : 0 \le t \le T, 1 \le i \le M\}$ are said to be orthonormal (over the interval $[0, T]$) if

$$\int_0^T \varphi_i(t)\varphi_j(t)dt = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Orthogonal Signals

- The set of orthogonal signals can be described by

$$\begin{aligned} s_0(t) &= \sqrt{E}\varphi_0(t) \\ s_1(t) &= \sqrt{E}\varphi_1(t) \\ s_2(t) &= \sqrt{E}\varphi_2(t) \\ s_{M-1}(t) &= \sqrt{E}\varphi_{M-1}(t) \end{aligned}$$

$\int s_i^2(t) \cdot dt = E$

- Clearly we need $N = M$ orthonormal signals to represent M orthogonal signals.
- Orthonormal signals are also equal energy signals (namely energy 1).

Orthogonal Signals Demodulation

minimize $\rightarrow d_E^2(r(t), s_i(t))$

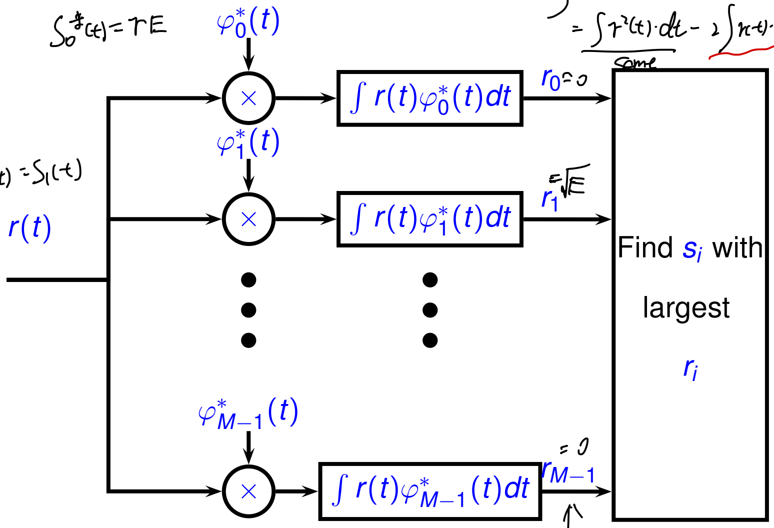
$= \int (r(t) - s_i(t))^2 dt$

$= \underbrace{\int r^2(t)\cdot dt}_{\text{same}} - 2\int r(t)\cdot s_i^*(t)\cdot dt + \underbrace{\int s_i^2(t)dt}_{\text{same}}$

maximum this

$s_0^*(t) = \sqrt{E}$

If $r(t) = s_1(t)$



only one of them is non-zero

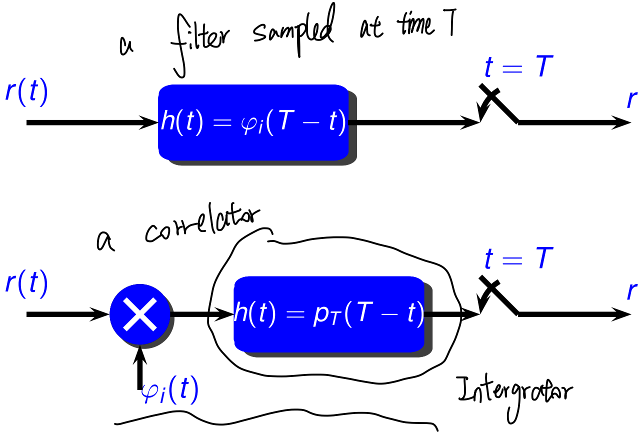
Correlation vs. Filtering

- Assume $\varphi_i^*(t)$ is time limited to the interval $[0, T]$.
- The receiver needs to compute $\int r(t)\varphi_i^*(t)dt$.
- A filter with input $r(t)$ and impulse response $h(t) = \varphi_i^*(T-t)$ sampled at time T has output

$$\begin{aligned} r_i &= \int h(T-t)r(t)dt = \int \varphi_i^*(T-(T-t))r(t)dt \\ &= \int \varphi_i^*(t)r(t)dt \end{aligned}$$

- So either a correlator whereby the received signal is correlated with the orthonormal signal can be used to obtain r_i OR a matched filter with impulse reponse $h(t) = \varphi_i^*(T-t)$ which is sampled at time $t = T$ can be used to obtain r_i.

Correlation vs. Filtering



Orthogonal Modulation Performance

- Suppose signal i is transmitted ($0 \leq i \leq M-1$).
- The output of these filters is given by

noise along

$$r_m = \begin{cases} \eta_m, & m \neq i \\ \sqrt{E} + \eta_m, & m = i \end{cases}$$

signal plus noise

where $\{\eta_m, m = 0, 1, 2, ..., M-1\}$ is a sequence of independent, identically distributed Gaussian random variables with mean zero and variance $N_0/2$.

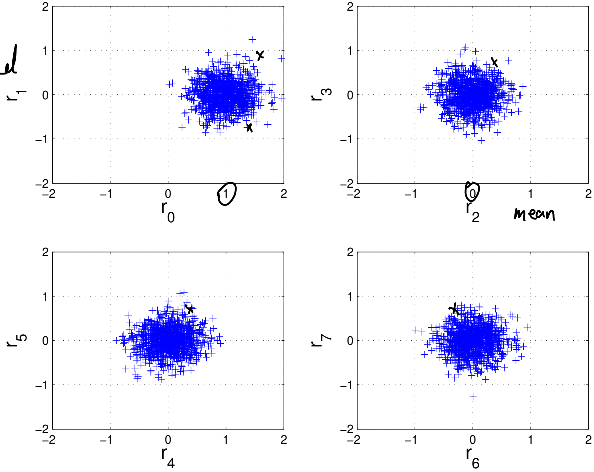
largest

- If $M = 4$ and $r = (r_0, ..., r_3) = (0.6, -.3, 1.2, -.7)$ then the decision is that signal s_2 was sent because $r_2 = \max(r_i, i = 0, ..., 3)$

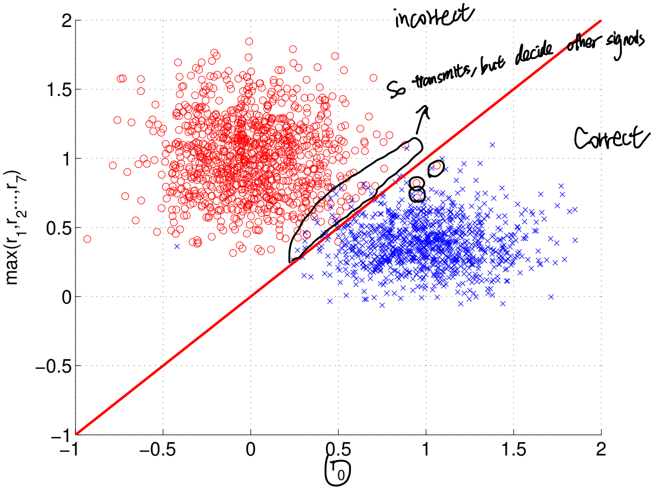
Simulation $M = 8$ (s_0 sent, $E_b/N_0 = 3$dB)

if S_0 is trans, Receiver will be correct if $r_0 > \max(r_1, r_2, r_3, r_4, r_5, r_6, r_7)$

S_0 is transmitted

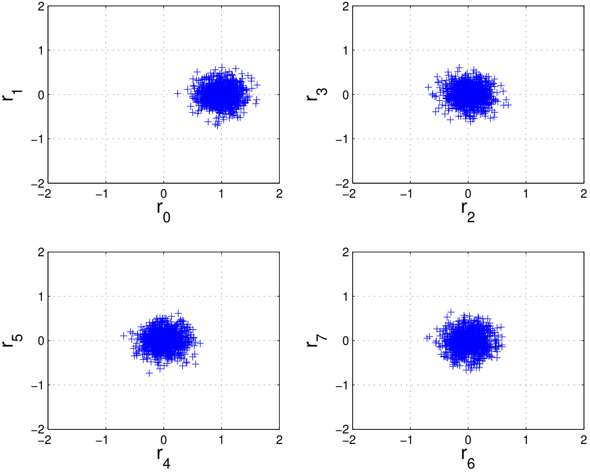


Simulation $M = 8$ (s_0 sent, $E_b/N_0 = 3$dB)

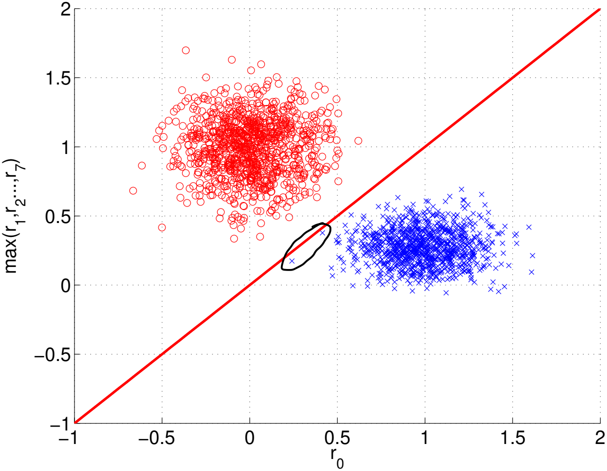


Simulation $M = 8$ (s_0 sent, $E_b/N_0 = 6$dB)

S_0 is transmited

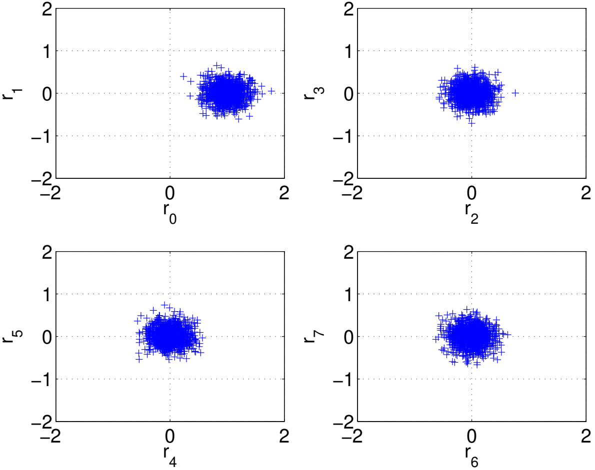


Simulation $M = 8$ (s_0 sent, $E_b/N_0 = 6$dB)

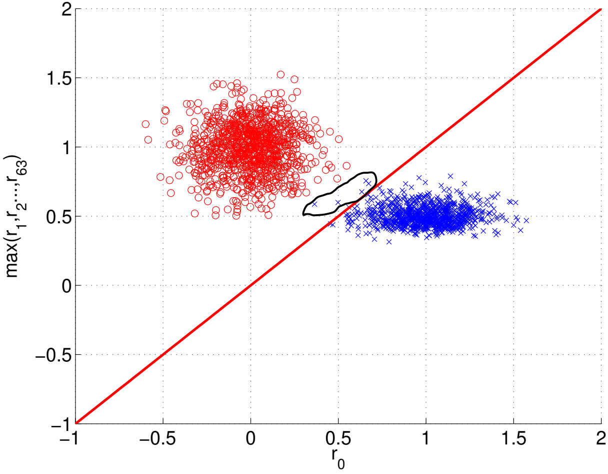


Simulation $M = 64$ (s_0 sent, $E_b/N_0 = 3$dB)

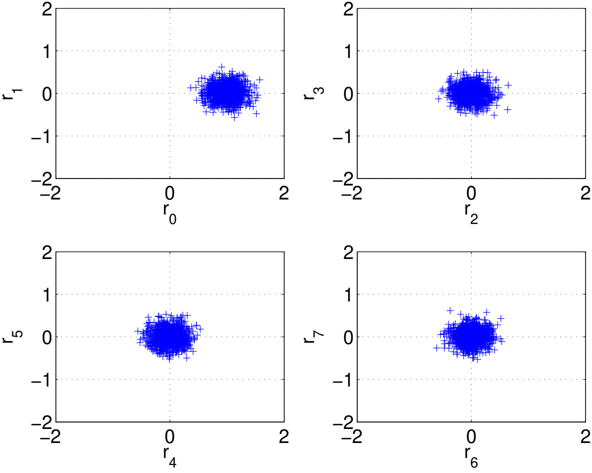
but just show 8 of them



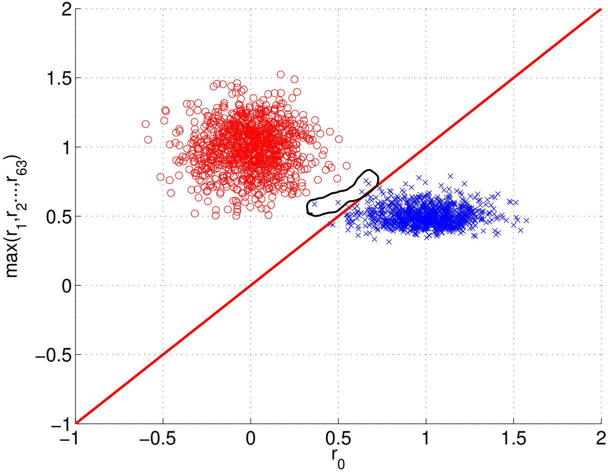
Simulation $M = 64$ (s_0 sent, $E_b/N_0 = 3$dB)



Simulation $M = 256$ (s_0 sent, $E_b/N_0 = 3$dB)



Simulation $M = 256$ (s_0 sent, $E_b/N_0 = 3$dB)



Error Probability

$N = M$

To determine the probability of error we need to determine the probability that the filter output corresponding to the signal present is smaller than one of the other filter outputs.

$r_0 = \sqrt{E} + \eta_0$

$r_1 = \eta_1$

$\vdots$

$r_{M-1} = \eta_{M-1}$

$= 1 - P(r_0 > r_1, r_0 > r_2, r_0 > r_{M-1} | s_0 \text{ transmitted})$

$$
\begin{aligned}
P_{e,0} &= P\{r_0 < \max(r_1, \ldots, r_{M-1}) | s_0 \text{trans}\} \\
&= 1 - P\{r_1 < r_0, r_2 < r_0, r_3 < r_0, \ldots, r_{M-1} < r_0 | s_0 \text{trans}\} \\
&= 1 - P\{\eta_1 < \sqrt{E} + \eta_0, \eta_2 < \sqrt{E} + \eta_0, \ldots \eta_{M-1} < \sqrt{E} + \eta_0\} \\
&= 1 - E[P\{\eta_1 < \sqrt{E} + \eta_0, \eta_2 < \sqrt{E} + \eta_0, \ldots \eta_{M-1} < \sqrt{E} + \eta_0 | \eta_0\}] \quad \text{this event is dependent!} \\
&= 1 - \int P\{\eta_1 < \sqrt{E} + x, \eta_2 < \sqrt{E} + x, \ldots \eta_{M-1} < \sqrt{E} + x | \eta_0 = x\} f_{\eta_0}(x) dx \\
&= 1 - \int P\{\eta_1 < \sqrt{E} + x\} P\{\eta_2 < \sqrt{E} + x\} \cdots P\{\eta_{M-1} < \sqrt{E} + x\} f_{\eta_0}(x) dx
\end{aligned}
$$

event

$\eta_0, \eta_1, \ldots \eta_{M-1}$ are independent

Error Probability

$\sigma = \sqrt{\frac{N_0}{2}}$

$$
\begin{aligned}
P_{e,0} &= 1 - \int_{-\infty}^{\infty} \Phi^{M-1}\left(\frac{\sqrt{E}+x}{\sqrt{N_0/2}}\right) f_{\eta_0}(x)\,dx \\
&= 1 - \int_{-\infty}^{\infty} \Phi^{M-1}\left(\frac{\sqrt{E}+x}{\sqrt{N_0/2}}\right) \frac{1}{\sqrt{\pi N_0}} \exp\{-\frac{x^2}{N_0}\}\,dx \\
&= 1 - \int_{-\infty}^{\infty} \Phi^{M-1}(\sqrt{2E/N_0}+y) \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}\,dy \\
&= (M-1)\int_{-\infty}^{\infty} \Phi^{M-2}(\sqrt{2E/N_0}+y) \frac{1}{\sqrt{2\pi}} e^{-(y+\sqrt{2E/N_0})^2/2} \Phi(y)\,dy \\
&= \frac{M-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi\left(z-\sqrt{\frac{2E}{N_0}}\right) \Phi^{M-2}(z) e^{-z^2/2}\,dz
\end{aligned}
$$

$\frac{y^2}{2} = \frac{x^2}{N_0}$

$y = \frac{x}{\sqrt{\frac{N_0}{2}}}$

integration by parts

Error Probability

$\Phi(u)$ is the distribution function of a zero mean, variance 1, Gaussian random variable given by

$$\Phi(u) = \frac{1}{2\pi}\int_{-\infty}^{u} e^{-x^2/2}dx.$$

The last step in the derivation is obtained by using the integration by parts formula.

$$\int udv = uv - \int vdu$$

where $u = \Phi^{M-1}(\sqrt{2E/N_0} + y)$ and $dv = \frac{1}{\sqrt{2\pi}}\exp\{-y^2/2\}dy$.

Performance of Orthogonal Signals

$$P_{e,s} = \frac{M-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(u - \sqrt{\frac{2E}{N_0}})\Phi^{M-2}(u)e^{-u^2/2}du$$

- The formula for $P_{e,s}$ can be interpreted as the expected value of a conditional probability where
  - we condition on one of the incorrect correlator outputs being a value (say u),
  - the correct output being less than u and
  - $M-2$ incorrect correlator outputs also being less than u.
- Since there are $M-1$ possible incorrect outputs we need to multiply this by $M-1$.

Energy per information bit

Normally a communication engineer is more concerned with the energy transmitted per bit rather than the energy transmitted per signal, E. If we let E_b be the energy transmitted per bit then these are related as follows

$$E_b = \frac{E^{\text{jouls/signal}}}{\log_2 M_{\text{bits/signal}}}.$$

$\log_2 M$ bit/signal

$M = 64 \Rightarrow 6$ bits

$M = 32 \Rightarrow 5$ bits

Symbol Error Probability

$$P_{e,s} = \frac{M-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi\left(u - \sqrt{\frac{2E_b \log_2(M)}{N_0}}\right) \Phi^{M-2}(u) e^{-u^2/2} du$$

use L'Hospital rules

As $M \to \infty$ the symbol error probability of orthogonal signal sets approaches 1 or 0.

$$\lim_{M\to\infty} P_{e,s} = \begin{cases} 1 & E_b/N_0 < \ln(2) \\ 0 & E_b/N_0 > \ln(2) \end{cases}$$

$\Rightarrow$ Arbitrarily small error prob ($\longrightarrow 0$) if we make M large

as long as $\frac{E_b}{N_0} > \ln(2) = 0.69$ (-1.59 dB) bits/dimension $= \frac{\log_2 M}{M} \longrightarrow 0$

The rate of communications (bits/dimension) as $M \to \infty$ goes to 0.

$$\lim_{M\to\infty} \frac{\log_2(M)}{M} = 0$$

That is, orthogonal signals for very large M achieves Shannon's fundamental limit of performance at the point where the rate is zero. That is at a bandwidth efficiency of zero, orthogonal signals have arbitrarily small error probability provided $E_b/N_0 > \ln(2)$.

(−1.59 dB)

Bounds on Performance of Orthogonal Signals

The symbol error probability can be upper bounded as

$$P_{e,s} \leq \begin{cases} 1, & \frac{E}{N_0} \leq \ln M \\ \exp\left\{-\left(\sqrt{\frac{E}{N_0}} - \sqrt{\ln M}\right)^2\right\}, & \ln M \leq \frac{E}{N_0} \leq 4\ln M \\ \exp\{-(\frac{E}{2N_0} - \ln M)\}, & \frac{E}{N_0} \geq 4\ln M. \end{cases}$$

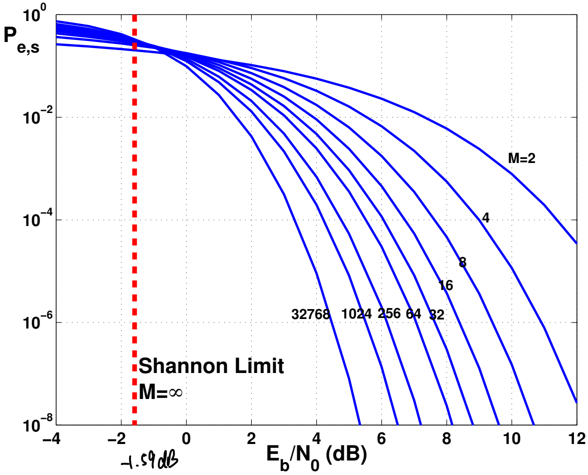
Energy per information bit

The bound on the symbol error probability can be expressed in terms of the energy transmitted per bit as

$$P_{e,s} \leq \begin{cases} 1, & \frac{E_b}{N_0} \leq \ln 2 \\ \exp_2\left\{-\log_2 M \left(\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2}\right)^2\right\}, & \ln 2 \leq \frac{E_b}{N_0} \leq 4\ln 2 \\ \exp_2\left\{-\log_2 M \left(\frac{E_b}{2N_0} - \ln 2\right)\right\}, & \frac{E_b}{N_0} \geq 4\ln 2 \end{cases}$$

where $\exp_2\{x\}$ denotes 2^x. Note that as $M \to \infty$, $P_e \to 0$ if $\frac{E_b}{N_0} > \ln 2$ = -1.59dB.

Symbol Error Probability for Orthogonal Signals



Bit error probability

- So far we have examined the symbol error probability for orthogonal signals; the probability of deciding the wrong symbol or signal.
- Usually the number of such signals is a power of 2, e.g. 4, 8, 16, 32, ...
- If so then each transmission of a signal is carrying $k = \log_2 M$ bits of information.
- In this case a communication engineer is usually interested in the bit error probability as opposed to the symbol error probability.

if S_0 is transmitted, S_0 → S_0, S_1, S_2, S_3, ⋮, S_{M-1} recv

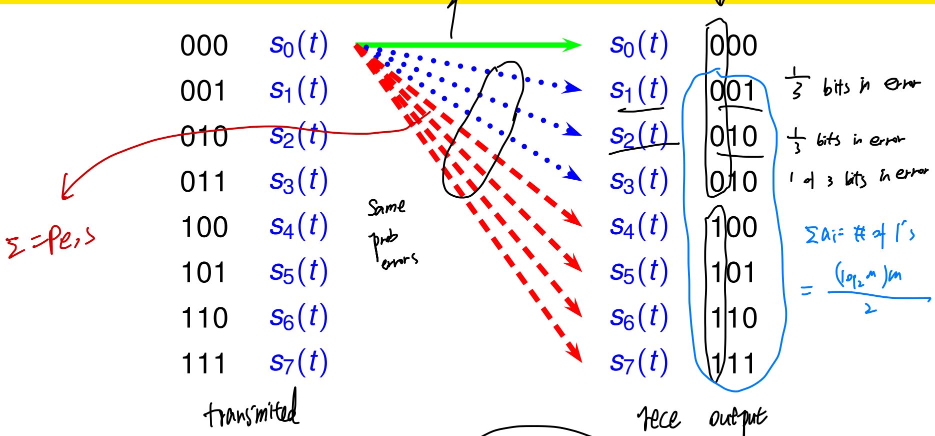
$P(\text{decide } S_j \mid S_0 \text{ trans}) = K \quad j \neq 0$

Bit error probability

- Assume signal 0 is transmitted corresponding to the data bits being (000...00).
- If an error occurs and the demodulator chooses one of the incorrect signals than each of the incorrect signals has the same probability.
- Thus the signal corresponding to data bits being (000...01) has the same probability as an error to a signal corresponding to data bits (111...11).
- If signal 0 is transmitted then there will be $M/2$ other signals that will cause a bit error in any particular bit. Thus

$$P_{e,b} = \frac{M}{2(M-1)} P_{e,s} = \frac{2^{k-1}}{2^k - 1} P_{e,s}.$$

Bit error probability



Red line (dashed) corresponds to a symbol error and a bit error for the first bit, green line (solid) corresponds to no error (symbol or bit). Blue line (dotted) corresponds to a symbol error but not a bit error (for the first bit).

Bit error probability

- Consider the case of 8 signals transmitting 3 bits of information.
- Suppose that the bits 000 are mapped to signal s_0 and 111 are mapped to s_7.
- Then if signal s_0 is transmitted the receiver will either decide correctly or decide incorrectly.
- All the incorrect signals are equally likely to be chosen given an error is made. That is, the probability of the receiver deciding signal s_i given s_0 was transmitted is $P_{e,s}/(M-1)$ for all $i \neq 0$.
- If the receiver decides incorrectly then it has decided one of $s_1, ..., s_7$.
- If it decides s_1 then that will be mapped into bits 001 and one bit error will occur.
- If it decides s_2 then that will be mapped into bits 010 and again one bit error will occur.
- If it decides s_7 then that will be mapped to 111 and three bit errors will occur.

Bit error probability

- Let a_i be the number of bit errors that occur when we decide signal s_i.
- Then the bit error probability is

P(rec dec si)
i=1,2, ..., M-1

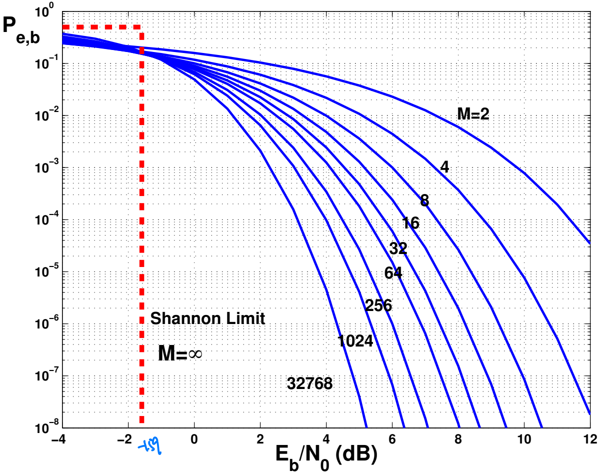
$$\begin{aligned} P_{e,b} &= \sum_{i=1}^{M-1} \frac{a_i}{\log_2(M)} \frac{P_{e,s}}{(M-1)} \\ &= \frac{P_{e,s}}{(M-1)} \frac{1}{\log_2(M)} \sum_{i=1}^{M-1} a_i \\ &= \frac{P_{e,s}}{(M-1)} \frac{1}{\log_2(M)} \log_2(M) M/2 \\ &= \frac{MP_{e,s}}{2(M-1)} \end{aligned}$$

Total # of bits that are 1 (in error)

$(\log_2 8)8/2 = 12$

$\simeq \frac{1}{2} P_{e,s}$ (M large)

Bit Error Probability for Orthogonal Signals



Asymptotic Error Probability of Orthogonal Signal Sets

- As $M \to \infty$ the symbol error probability of orthogonal signal sets approaches 1 or 0.

$$\lim_{M\to\infty} P_{e,s} = \begin{cases} 1 & E_b/N_0 < \ln(2) \\ 0 & E_b/N_0 > \ln(2) \end{cases} \qquad \lim_{M\to\infty} P_{e,b} = \begin{cases} 1/2 & E_b/N_0 < \ln(2) \\ 0 & E_b/N_0 > \ln(2) \end{cases}$$

- The rate of communications (bits/dimension) as $M \to \infty$ goes to 0.

$$\lim_{M\to\infty} \frac{\log_2(M)}{M} = 0$$

- That is, orthogonal signals for very large M achieves Shannon's fundamental limit of performance at the point where the rate is zero. That is at a bandwidth efficiency of zero, orthogonal signals have arbitrarily small error probability provided $E_b/N_0 > \ln(2)$.

Orthogonal Signal Sets

- Below we define several different orthogonal signal sets.
- We will define the bandwidth of a signal set as the minimum difference in carrier frequencies between two such signal sets so that any signal from one set is orthogonal to any signal from the other set.
- We will assume that the double frequency terms are negligible relative to the baseband freqency term.

Review of Integrals of Trig Functions

$$
\begin{aligned}
\rho &= \int_0^T \sin(2\pi f_1 t)\sin(2\pi f_0 t)dt \\
&= \frac{1}{2}\int_0^T \cos(2\pi(f_1 - f_0)t) - \cos(2\pi(f_0 + f_1)t)dt \\
&= \frac{T}{2}\left[\frac{\sin(2\pi(f_1 - f_0)T)}{2\pi(f_1 - f_0)T} - \frac{\sin(2\pi(f_0 + f_1)T)}{2\pi(f_0 + f_1)T}\right] \\
&= \frac{T}{2}\frac{\sin(2\pi(f_1 - f_0)T)}{2\pi(f_1 - f_0)T}
\end{aligned}
$$

This integral will be zero if $2\pi(f_1 - f_0)T = \pi, 2\pi, 3\pi, ...$. The minimum frequency spacing is $f_1 - f_0 = 1/(2T)$.

Review of Integrals of Trig Functions

$$
\begin{aligned}
\rho &= \int_0^T \sin(2\pi f_1 t + \theta)\sin(2\pi f_0 t)dt \\
&= \frac{1}{2}\int_0^T \cos(2\pi(f_1 - f_0)t + \theta) - \cos(2\pi(f_0 + f_1)t + \theta)dt \\
&= \frac{T}{2}[\frac{\sin(2\pi(f_1 - f_0)T + \theta)}{2\pi(f_1 - f_0)T} - \frac{\sin(2\pi(f_0 + f_1)T + \theta)}{2\pi(f_0 + f_1)T}] \\
&= \frac{T}{2}[\frac{\sin(2\pi(f_1 - f_0)T + \theta)}{2\pi(f_1 - f_0)T} - \frac{\sin(\theta)}{2\pi(f_1 - f_0)T}]
\end{aligned}
$$

This integral will be zero if $2\pi(f_1 - f_0)T = 2\pi, 4\pi, 6\pi, ...$. The minimum frequency spacing is $f_1 - f_0 = 1/(T)$.

Review of Integrals of Trig Functions

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\begin{aligned}
\rho &= \int_0^T \cos(2\pi f_1 t)\sin(2\pi f_0 t)dt \\
&= \frac{1}{2}\int_0^T \sin(2\pi(f_1+f_0)t) - \sin(2\pi(f_1-f_0)t)dt \\
&= \frac{T}{2}\left[\frac{\cos(2\pi(f_1-f_0)T)-1}{2\pi(f_1-f_0)T} - \frac{\cos(2\pi(f_0+f_1)T)-1}{2\pi(f_0+f_1)T}\right] \\
&= \frac{T}{2}\left[\frac{\cos(2\pi(f_1-f_0)T)-1}{2\pi(f_1-f_0)T}\right]
\end{aligned}
$$

This integral will be zero if $2\pi(f_1 - f_0)T = 2\pi, 4\pi, 6\pi, ...$. The minimum frequency spacing is $f_1 - f_0 = 1/(T)$.

Review of Integrals of Trig Functions

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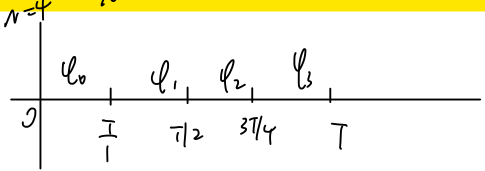
&= \frac{1}{2}\int_0^T \sin(2\pi(f_1+f_0)t+\theta) - \sin(2\pi(f_1-f_0)t+\theta)dt \\
&= \frac{T}{2}[\frac{\cos(2\pi(f_1-f_0)T+\theta)-\cos(\theta)}{2\pi(f_1-f_0)T} - \frac{\cos(2\pi(f_0+f_1)T+\theta)-\cos(\theta}{2\pi(f_0+f_1)T} \\
&= \frac{T}{2}[\frac{\cos(2\pi(f_1-f_0)T+\theta)}{2\pi(f_1-f_0)T} - \frac{\cos(\theta)}{2\pi(f_1-f_0)T}]
\end{aligned}
$$

This integral will be zero if $2\pi(f_1 - f_0)T = 2\pi, 4\pi, 6\pi, ...$. The minimum frequency spacing is $f_1 - f_0 = 1/(T)$.

A. Time-orthogonal

$N = M$ here



$$\varphi_i(t) = \begin{cases} \sqrt{\frac{2N}{T}} \sin(2\pi f_0 t), & \frac{iT}{N} \le t < (i+1)T/N \\ 0, & \text{elsewhere} \end{cases}$$

$$i = 0, 1, \ldots, N,$$

It is clear that $\varphi_i(t)$ and $\varphi_j(t)$ are orthogonal in time (they do not overlap in time) if

N dimension in bandwidth $N/2T$ time T

A. Time-orthogonal: frequency multiplexing

$\varphi_0^{(0)}(t)$, $\varphi_0^{(1)}$ are non-zero in $[0, T/N]$

2 signal sets

$$
\begin{aligned}
\varphi_i^{(0)}(t) &= \begin{cases} \sqrt{\frac{2N}{T}} \sin(2\pi f_0 t), & \frac{iT}{N} \le t < (i+1)T/N \\ 0, & \text{elsewhere} \end{cases} \\
\varphi_i^{(1)}(t) &= \begin{cases} \sqrt{\frac{2N}{T}} \sin(2\pi f_1 t), & \frac{iT}{N} \le t < (i+1)T/N \\ 0, & \text{elsewhere} \end{cases} \\
i &= 0, 1, \ldots, N,
\end{aligned}
$$

orthogonal

The duration of these signals is T/N. For othogonality of $\varphi_i^{(0)}(t)$ and $\varphi_j^{(1)}(t)$ we need

N dimension in bandwidth

$\frac{N}{2T}$ in time T

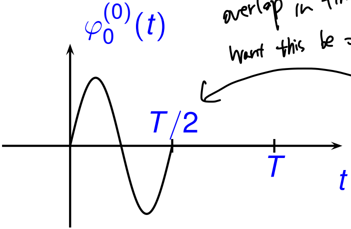
$$f_1 - f_0 = \frac{1}{2(T/N)} = \frac{N}{2T}$$

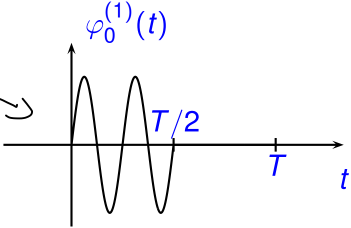
Thus the bandwidth of this signal set is $W = [\frac{(n+1)N}{2T} - \frac{nN}{2T}] = \frac{N}{2T}$

A. Time-orthogonal: frequency multiplexing

overlap in time

want this be orthogonal



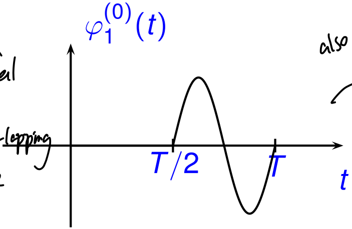


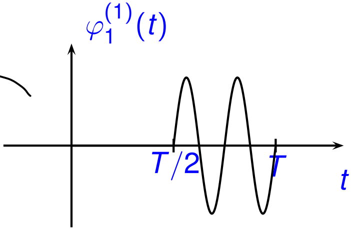
SIGNAL SET 0

SIGNAL SET 1

orthogonal cause not overlapping in time

also want





B. Time-orthogonal quadrature-phase

$$\varphi_{2i}(t) = \begin{cases} \sqrt{\frac{2N}{T}}\sin(2\pi f_0 t), & \frac{2iT}{N} \le t < \frac{2(i+1)T}{N} \\ 0, & \text{elsewhere} \end{cases}$$

$$\varphi_{2i+1}(t) = \begin{cases} \sqrt{\frac{2N}{T}}\cos(2\pi f_0 t), & \frac{2iT}{N} \le t < \frac{2(i+1)T}{N} \\ 0 & \text{elsewhere} \end{cases}$$

$$i = 0, 1, \ldots, \frac{N}{2} - 1, \quad N \text{ even}, \quad f_0 = n\frac{N}{2T},$$

Duration $= T/(2N)$.

$$W = [\frac{(n+1)N}{2T} - \frac{nN}{2T}] = \frac{N}{2T}$$

C. Frequency-orthogonal

$$\begin{aligned}\varphi_i(t) &= \sqrt{\frac{2E}{T}}\sin[2\pi(f_0 + \frac{i}{2T})t], \qquad\qquad 0 \le t \le T \\ i &= 0, 1, \ldots, N-1, \quad f_0 = \frac{nN}{2T}.\end{aligned}$$

$$\begin{aligned}W &= [\frac{(n+1)N}{2T} - \frac{nN}{2T}] \\ &= \frac{N}{2T}\end{aligned}$$

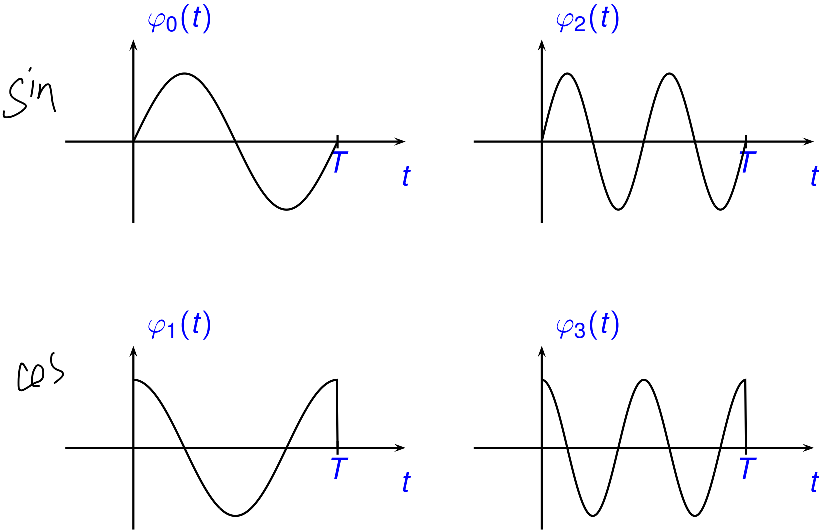
D. Frequency-orthogonal quadrature-phase

$$
\begin{aligned}
\varphi_{2i}(t) &= \sqrt{\frac{2E}{T}} \sin[2\pi(f_0 + \frac{i}{T}), t] && 0 \le t < T \\
\varphi_{2i+1}(t) &= \sqrt{\frac{2E}{T}} \cos[2\pi(f_0 + \frac{i}{T})t], && 0 \le t \le T
\end{aligned}
$$

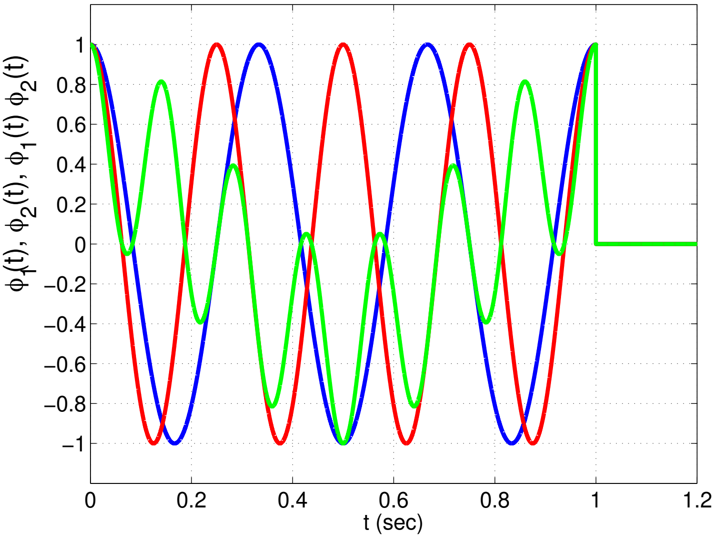
$$f_0 = \frac{nN}{2T}.$$

$$
\begin{aligned}
W &= [\frac{(n+1)N}{2T} - \frac{nN}{2T}] \\
&= \frac{N}{2T}
\end{aligned}
$$

D. Frequency-orthogonal quadrature-phase



Example ($f_c = 1$, $T = 3$)



E. Hadamard-Walsh Construction

The last construction of orthogonal signals is done via the Hadamard Matrix. The Hadamard matrix is an N by N matrix with components either +1 or -1 such that every pair of distinct rows are orthogonal. We show how to construct a Hadamard when the number of signals is a power of 2 (which is often the case).
Begin with a two by two matrix

$$H_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix} \begin{array}{l} \to \varphi_0 \\ \to \varphi_1 \end{array} \qquad \Sigma \varphi_0(n)\varphi_1(n) = 0$$

Then use the recursion

$$H_{2^l} = \begin{bmatrix} +H_{2(l-1)} & +H_{2(l-1)} \\ +H_{2(l-1)} & -H_{2(l-1)} \end{bmatrix}.$$

E. Hadamard-Walsh Construction

Now it is easy to check that distinct rows in these matrices are orthogonal. The i-th modulated signal is then obtained by using a single (arbitrary) waveform N times in nonoverlapping time intervals and multiplying by the $j-th$ repetition of the waveform by the jth component of the i-th row of the matrix.

E. Hadamard-Walsh Construction

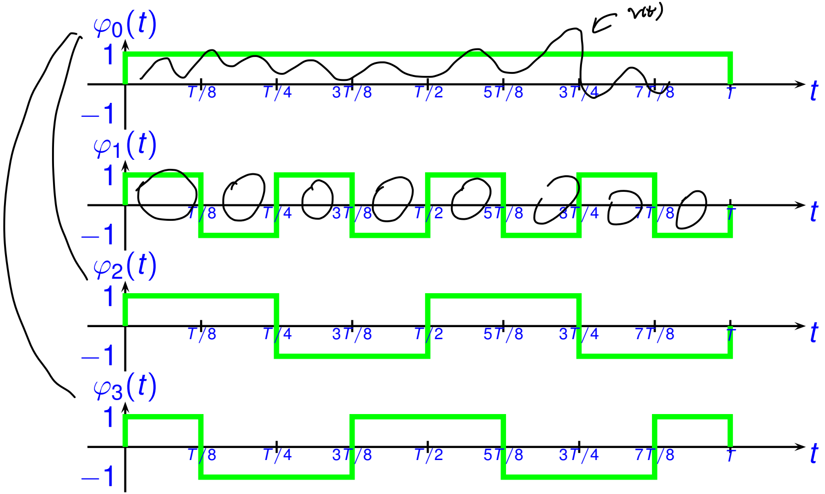
Example ($N = 4$):

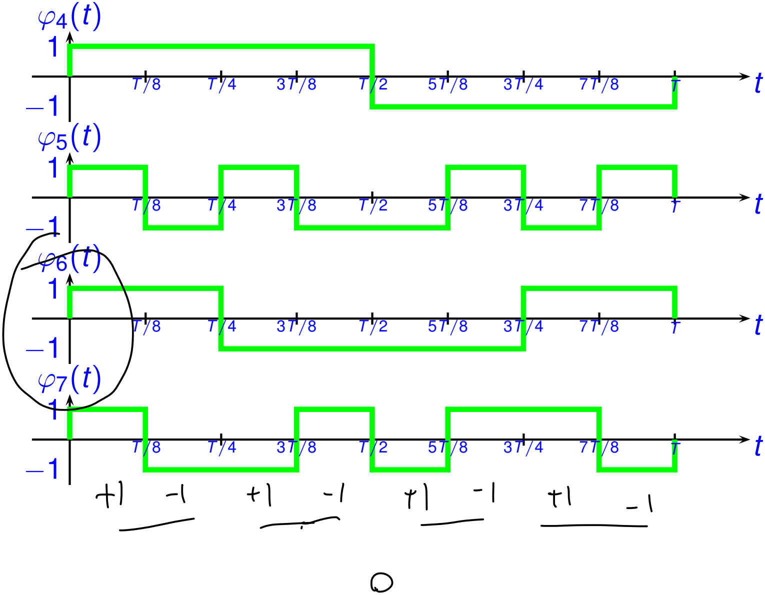
$$
\begin{aligned}
H_4 &= \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} \\
&= \left[\begin{array}{cc|cc} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ \hline +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{array}\right].
\end{aligned}
$$

E. Hadamard-Walsh Construction, $M = 8$

$$
\begin{aligned}
H_8 &= \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} \\
&= \begin{bmatrix} H_2 & H_2 & H_2 & H_2 \\ H_2 & -H_2 & H_2 & -H_2 \\ H_2 & H_2 & -H_2 & -H_2 \\ H_2 & -H_2 & -H_2 & H_2 \end{bmatrix} \\
&= \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{bmatrix}.
\end{aligned}
$$

E. Hadamard-Walsh Construction, $M = 8$




φ4(t)
1
−1
T/8 T/4 3T/8 T/2 5T/8 3T/4 7T/8 T
t
φ5(t)
φ6(t)
φ7(t)
+1 −1 +1 −1 +1 −1 +1 −1
0

Processing of Hadamard Generated Orthogonal Signals

$$\int r(t)\cdot\varphi_i(t)\cdot dt \quad i=0,1,\ldots N-1$$

Let

$$X_i = \int_{iT/N}^{(i+1)T/N} r(t)dt, \quad i = 0, 1, ..., N-1.$$

Then the correlations of $r(t)$ and $\varphi_i(t)$ for $i = 0, 1, ..., 7$ can be calculated from $X_0, X_1, ..., X_7$.

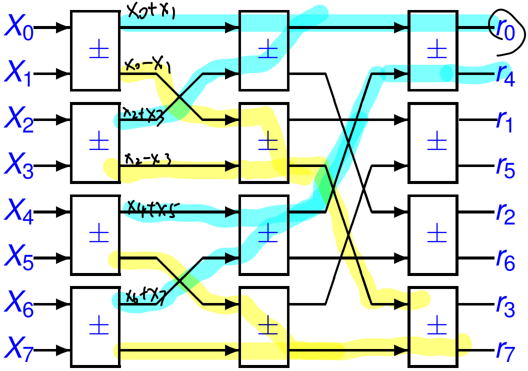
Processing of Hadamard Generated Orthogonal Signals

$$
\begin{aligned}
r_0 &= (X_0 + X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7) \\
r_1 &= (X_0 - X_1 + X_2 - X_3 + X_4 - X_5 + X_6 - X_7) \\
r_2 &= (X_0 + X_1 - X_2 - X_3 + X_4 + X_5 - X_6 - X_7) \\
r_3 &= (X_0 - X_1 - X_2 + X_3 + X_4 - X_5 - X_6 + X_7) \\
r_4 &= (X_0 + X_1 + X_2 + X_3 - X_4 - X_5 - X_6 - X_7) \\
r_5 &= (X_0 - X_1 + X_2 - X_3 - X_4 + X_5 - X_6 + X_7) \\
r_6 &= (X_0 + X_1 - X_2 - X_3 - X_4 - X_5 + X_6 + X_7) \\
r_7 &= (X_0 - X_1 - X_2 + X_3 - X_4 + X_5 + X_6 - X_7)
\end{aligned}
$$

Processing of Hadamard Generated Orthogonal Signals

$M = 8$

using in 2G and 3G system



$r_7 = x_0 - x_1 - (x_2 - x_3) - [(x_4 - x_5) - (x_6 - x_7)]$

Efficient method of correlating for H-W orthogonal signals

Bandwidth of Orthogonal Signals

If we define bandwidth of M signals as minimum frequency separation between two such signal sets such that any signal from one signal set is orthogonal to every signal from a frequency adjacent signal set then for all of these examples of M orthogonal signals the bandwidth is $W = \frac{M}{2T}$.

$\frac{N}{WT} = 2 \quad \frac{\text{dimension}}{\text{Hz}}$

Orthogonal Signals in Bandwidth W, Time T.

There are $N = 2WT$ orthogonal signals in bandwidth W and time duration T.

Equivalently, the bandwidth efficiency of a modulation technique (in bits/second/Hz) can be computed from the rate (in bits/dimension) by multiplying by 2.

$2 \frac{\log_2 M}{M}$

$R/W(\text{bits/second/Hz}) = \mathcal{R}(\text{bits/dimension}) \times (2 \text{ dimensions/second/Hz})$

Capacity

$W = \frac{M}{2T}$

The bandwidth of a set of M orthogonal signals is $W = M/(2T)$. If we transmit $\log_2(M)$ bits in T seconds then the data rate is $R = \log_2(M)/T$. The bandwidth efficiency is

$R = \frac{\log_2 M}{T}$

$$R/W = \frac{\log_2(M)/T}{M/2T} = \frac{2\log_2(M)}{M}$$

M	R/W
2	1.00000000
4	1.00000000
8	0.75000000
16	0.50000000
32	0.31250000
64	0.18750000
128	0.10937500
256	0.06250000
512	0.03515625
1024	0.01953125

Capacity (vs. Modulation at $P_{e,b} = 10^{-5}$)

32 APSK

64QAM - 16QAM

64QAM

more bandwidth effiency

but less energy efficency

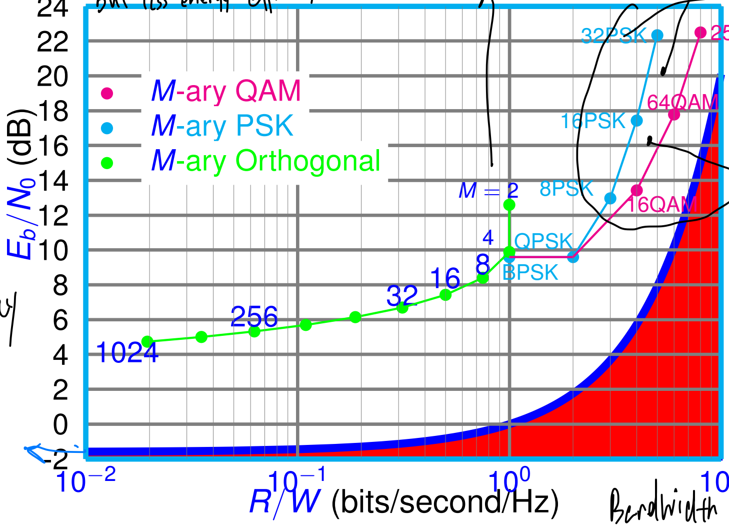
very bad bandwidth efficiency

large bandwidth effiency wened

large energy

noise ratio

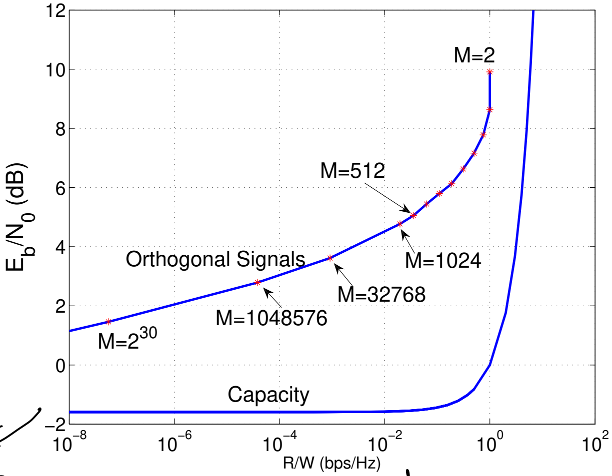
energy efficiency



16 APSK

Bandwidth efficiency

Capacity vs. Orthogonal Signals @ $P_{e,b} = 0.001$



$\frac{R}{W} \to 0$

Poor bandwid efficiency
good energy efficiency

Log scale

- opt rec: Find singal closet to rec signal

  ($\min_i d_E(r, s_i)$)

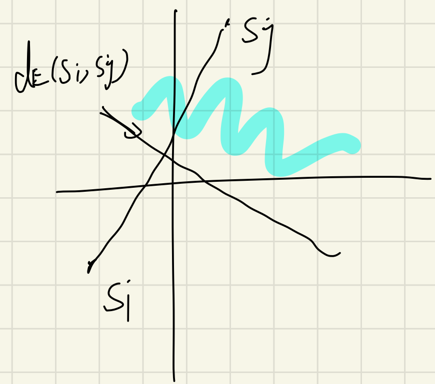
- opt rec if all signal have the same energy is

  fine signal with max correlation $\max (r(t), s_i(t))$

- MPSK, QAM, orthogonal there is a known formula $P_{e,s}$ $P_{e,b}$

- For other modulation scheme $P_{e,s}$, $P_{e,b}$ is

  hard to calculate



Union band:

$$P_{e,s} \le \frac{1}{M} \sum_{i=0}^{N-1} \sum_{j \neq i} P_2(s_i \to s_j)$$

Pairwise error

$$\to P_2(s_i \to s_j) = Q\left(\frac{d_E(s_i, s_j)}{\sigma}\right)$$

- M-ray orthogonal signal have

$P_{e,s}$, $P_{e,b} \longrightarrow 0$ as $M \to \infty$ if $\frac{E_b}{N_0} = \ln(2)$

$= -1.59$ dB

but rate (bit/second) $\to 0$ as $M \to \infty$

- $\frac{R}{W}$ (bits/second) $= 2r$

bits/dimension