Math for Differential-Drive Robot Kinematics

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1. Wheels to Body Twist

Define:

Body Twist $(\dot{\theta}, \dot{x}, \dot{y})$ Wheel velocity $(\dot{u}_R, \dot{u}_L)$ Wheel radius r Wheel base d

Kinematics:

$$\dot{\theta} = \frac{r}{d}(\dot{u}_R - \dot{u}_L)$$

$$\dot{x} = \frac{r}{2}(\dot{u}_R + \dot{u}_L)$$

$$\dot{y} = \frac{r}{2}(\dot{u}_R + \dot{u}_L)$$

In Matrices:

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{r}{d} & -\frac{r}{d} \\ \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_R \\ \dot{u}_L \end{bmatrix}$$

2. Body Twist to Wheels

Define:

Body Twist $(\dot{\theta}, \dot{x}, \dot{y})$ Wheel velocity $(\dot{u}_R, \dot{u}_L)$ Wheel radius r Wheel base d

Inverse Kinematics:

$$\begin{bmatrix} \dot{u}_R \\ \dot{u}_L \end{bmatrix} = \begin{bmatrix} \frac{r}{d} & -\frac{r}{d} \\ \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \end{bmatrix}^{+} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= \left(\begin{bmatrix} \frac{r}{d} & -\frac{r}{d} \\ \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{r}{d} & -\frac{r}{d} \\ \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{r}{d} & \frac{r}{2} & 0 \\ -\frac{r}{d} & \frac{r}{2} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{2r} & \frac{1}{r} & 0 \\ -\frac{d}{2r} & \frac{1}{r} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

3. Integrate Twist to a transform matrix

Define:

Twist $\mathcal{V} = (\boldsymbol{w} \quad \boldsymbol{v}) = (0 \quad 0 \quad wz \quad vx \quad vy \quad 0)$

Exponential:

$$e^{\mathcal{V}} = \begin{bmatrix} e^{w} & (\boldsymbol{I}w + (1 - cos(wz))[\boldsymbol{w}] + (wz - sin(wz))[\boldsymbol{w}]^2)\boldsymbol{v} \\ 0 & 1 \end{bmatrix}$$

We have

$$e^{w} = \begin{bmatrix} \cos(\mathrm{wz}) & -\sin(\mathrm{wz}) & 0 \\ \sin(\mathrm{wz}) & \cos(\mathrm{wz}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(\boldsymbol{I}w + (1 - cos(w\mathrm{z}))[\boldsymbol{w}] + (w\mathrm{z} - sin(w\mathrm{z}))[\boldsymbol{w}]^2)\boldsymbol{v} = \begin{bmatrix} \mathrm{vx}^*\sin(\mathrm{wz}) + \mathrm{vy}^*\cos(\mathrm{wz}) - \mathrm{y} \\ \mathrm{vy}^*\sin(\mathrm{wz}) - \mathrm{vx}^*\cos(\mathrm{wz}) + \mathrm{x} \\ 0 \end{bmatrix}$$

Hence, we get

$$e^{\mathcal{V}} = \begin{bmatrix} \cos(\mathrm{wz}) & -\sin(\mathrm{wz}) & 0 & \mathrm{vx}^*\sin(\mathrm{wz}) + \mathrm{vy}^*\cos(\mathrm{wz}) - \mathrm{y} \\ \sin(\mathrm{wz}) & \cos(\mathrm{wz}) & 0 & \mathrm{vy}^*\sin(\mathrm{wz}) - \mathrm{vx}^*\cos(\mathrm{wz}) + \mathrm{x} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In SE(2), we have:

$$e^{\mathcal{V}} = \begin{bmatrix} \cos(\mathrm{wz}) & -\sin(\mathrm{wz}) & \mathrm{vx}^*\sin(\mathrm{wz}) + \mathrm{vy}^*\cos(\mathrm{wz}) - \mathrm{y} \\ \sin(\mathrm{wz}) & \cos(\mathrm{wz}) & \mathrm{vy}^*\sin(\mathrm{wz}) - \mathrm{vx}^*\cos(\mathrm{wz}) + \mathrm{x} \\ 0 & 0 & 1 \end{bmatrix}$$