Linear Algebra

주재걸
고려대학교 컴퓨터학과

Lecture Overview

- Elements in linear algebra
- Linear system 선형 시스템
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Linear Equation 선형 방정식

- A **linear equation** in the variables $x_1, \cdots, x_n$ is an equation that can be written in the form

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b},$$

where b and the coefficients $a_1, \cdots, a_n$ are real or complex numbers that are usually known in advance.

- The above equation can be written as

$$\mathbf{a}^T\mathbf{x} = b$$

$$[a_1 \cdots a_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$

where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Linear System: Set of Equations

선형 시스템 = 연립방정식

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables - say, $x_1, \cdots, x_n$.

Linear System Example

$$\begin{pmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 66 \\ 74 \\ 78 \end{pmatrix}$$

- Suppose we collected persons' weight, height, and life-span (e.g., how long s/he lived)

output

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

- We want to set up the following linear system:

$$\begin{aligned} 60x_1 + 5.5x_2 + 1 \cdot x_3 &= 66 \\ 65x_1 + 5.0x_2 + 0 \cdot x_3 &= 74 \\ 55x_1 + 6.0x_2 + 1 \cdot x_3 &= 78 \end{aligned}$$

이런 coefficiant를 찾고 싶음

- Once we solve for x_1, x_2, and x_3, given a new person with his/her weight, height, and is_smoking, we can predict his/her life-span.

Linear System Example

- The essential information of a linear system can be written compactly using a **matrix**.
- In the following set of equations,

$$
\begin{aligned}
60x_1 + 5.5x_2 + 1 \cdot x_3 &= 66 \\
65x_1 + 5.0x_2 + 0 \cdot x_3 &= 74 \\
55x_1 + 6.0x_2 + 1 \cdot x_3 &= 78
\end{aligned}
$$

- Let's collect all the coefficients on the left and form a matrix

$$
A = \begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}
$$

- Also, let's form two vectors: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$

From Multiple Equations to Single Matrix Equation

- Multiple equations can be converted into a **single** matrix equations

$$\begin{aligned} 60x_1 + 5.5x_2 + 1 \cdot x_3 &= 66 \\ 65x_1 + 5.0x_2 + 0 \cdot x_3 &= 74 \\ 55x_1 + 6.0x_2 + 1 \cdot x_3 &= 78 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \underbrace{\begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}}_{\mathbf{b}} \Leftarrow \begin{aligned} \mathbf{a}_1^T\mathbf{x} &= 66 \\ \mathbf{a}_2^T\mathbf{x} &= 74 \\ \mathbf{a}_2^T\mathbf{x} &= 78 \end{aligned}$$

- How can we solve for $\mathbf{x}$?

Identity Matrix (항등 행렬)

I or E라 씀

- **Definition**: An identity matrix is a square matrix whose diagonal entries are all 1's, and all the other entries are zeros. Often, we denote it as $I_n \in \mathbb{R}^{n\times n}$.
  - e.g., $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- An identity matrix I_n preserves any vector $\mathbf{x} \in \mathbb{R}^n$ after multiplying $\mathbf{x}$ by I_n :

$$\forall \mathbf{x} \in \mathbb{R}^n, \qquad I_n\mathbf{x} = \mathbf{x}$$

for all

Inverse Matrix (역행렬)

$A^{-1} \cdot A = E(I)$

★ Square Matrix 여야만 Inverse Matrix가 존재함

- **Definition**: For a square matrix $A \in \mathbb{R}^{n\times n}$, its inverse matrix A^{-1} is defined such that

$$A^{-1}A = AA^{-1} = I_n.$$

- For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse matrix A^{-1} is defined as

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$D = ad-bc$

Solving Linear System via Inverse Matrix

- We can now solve $A\mathbf{x} = \mathbf{b}$ as follows:

$$
\begin{aligned}
A\mathbf{x} &= \mathbf{b} \\
A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\
I_n\mathbf{x} &= A^{-1}\mathbf{b} \\
\mathbf{x} &= A^{-1}\mathbf{b}
\end{aligned}
$$

이거 하려면

A^{-1}가 존재해야 함

Solving Linear System via Inverse Matrix

- **Example:**

$$\underbrace{\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}}_{\mathbf{b}} \Longrightarrow A^{-1} = \begin{bmatrix} 0.0870 & 0.0087 & -0.0870 \\ -1.1304 & 0.0870 & 1.1314 \\ 2.0000 & -1.0000 & -1.0000 \end{bmatrix}$$

- One can verify

$$A^{-1}A = AA^{-1} = I_n.$$

- $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 0.0870 & 0.0087 & -0.0870 \\ -1.1304 & 0.0870 & 1.1314 \\ 2.0000 & -1.0000 & -1.0000 \end{bmatrix} \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 20 \\ -20 \end{bmatrix}$

Solving Linear System via Inverse Matrix

- Now, the life-span can be written as

$$\text{(life-span)} = -0.4 \times \text{(weight)} + 20 \times \text{(height)} - 20 \times \text{(is\_smoking)}.$$

Non-Invertible Matrix A for $A\mathbf{x} = \mathbf{b}$

역행렬이 없어

- Note that **if** A is invertible, the solution is uniquely obtained as

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

- **What if** A is non-invertible, i.e., the inverse does not exist?
  - E.g., For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, in $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, the denominator $ad - bc = 0$, so A is not invertible.

- For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc$ is called the determinant of A, or $\det A$.

판별식 D

Does a Matrix Have an Inverse Matrix?

D(A)

- $\det A$ determines whether A is invertible (when $\det A \neq 0$) or not (when $\det A = 0$).

- For more details on how to compute the determinant of a matrix $A \in \mathbb{R}^{n \times n}$ where $n \geq 3$, you can study the following:
  - https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-18-properties-of-determinants/
  - https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-19-determinant-formulas-and-cofactors/

→ numpy. linalg 의 inv 쓰면 됨

Inverse Matrix Larger than 2×2

- If invertible, is there any formula for computing an inverse matrix of a matrix $A \in \mathbb{R}^{n \times n}$ where $n \geq 3$?

(요르단)
가우스 소거법

→ RREF
(reduced row echelon form)

- No, but one can compute it.
- We skip details, but you can study Gaussian elimination in Lay Ch1.2 and then study Lay Ch2.2.

Non-Invertible Matrix A for $A\mathbf{x} = \mathbf{b}$

역행렬이 없다면 linear system은 어쩜?

- Back to the linear system, if A is non-invertible, $A\mathbf{x} = \mathbf{b}$ will have either no solution or infinitely many solutions.

해X

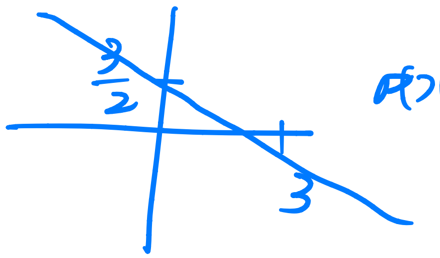
해가 무수히 많음

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases} \quad \xrightarrow{\text{소거}} \quad \begin{array}{r} \cancel{2x} + \cancel{4y} = 6 \\ - \;\; \cancel{2x} + \cancel{4y} = 6 \\ \hline 0x + 0y = 0 \end{array}$$

해가 무수히 많음.

$$y = \frac{1}{2}(-x + 3)$$



여기 모두가 해임

Rectangular Matrix A in $A\mathbf{x} = \mathbf{b}$

(No 정사각)

- **What if** A is a rectangular matrix, e.g., $A \in \mathbb{R}^{m \times n}$, where $m \neq n$?

행 열

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\underbrace{\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}}_{\mathbf{b}}$$

- Recall m = #equations and n = #variables.
- $m < n$: more variables than equations
  - Usually infinitely many solutions exist (under-determined system).
- $m > n$: more equations than variables
  - Usually no solution exists (over-determined system).
- To study how to compute the solution in these general cases, check out Lay Ch1.2 and Lay Ch1.5.

열이 더 많음

행이 더 많음

근사적으로 해를 구해야 함.