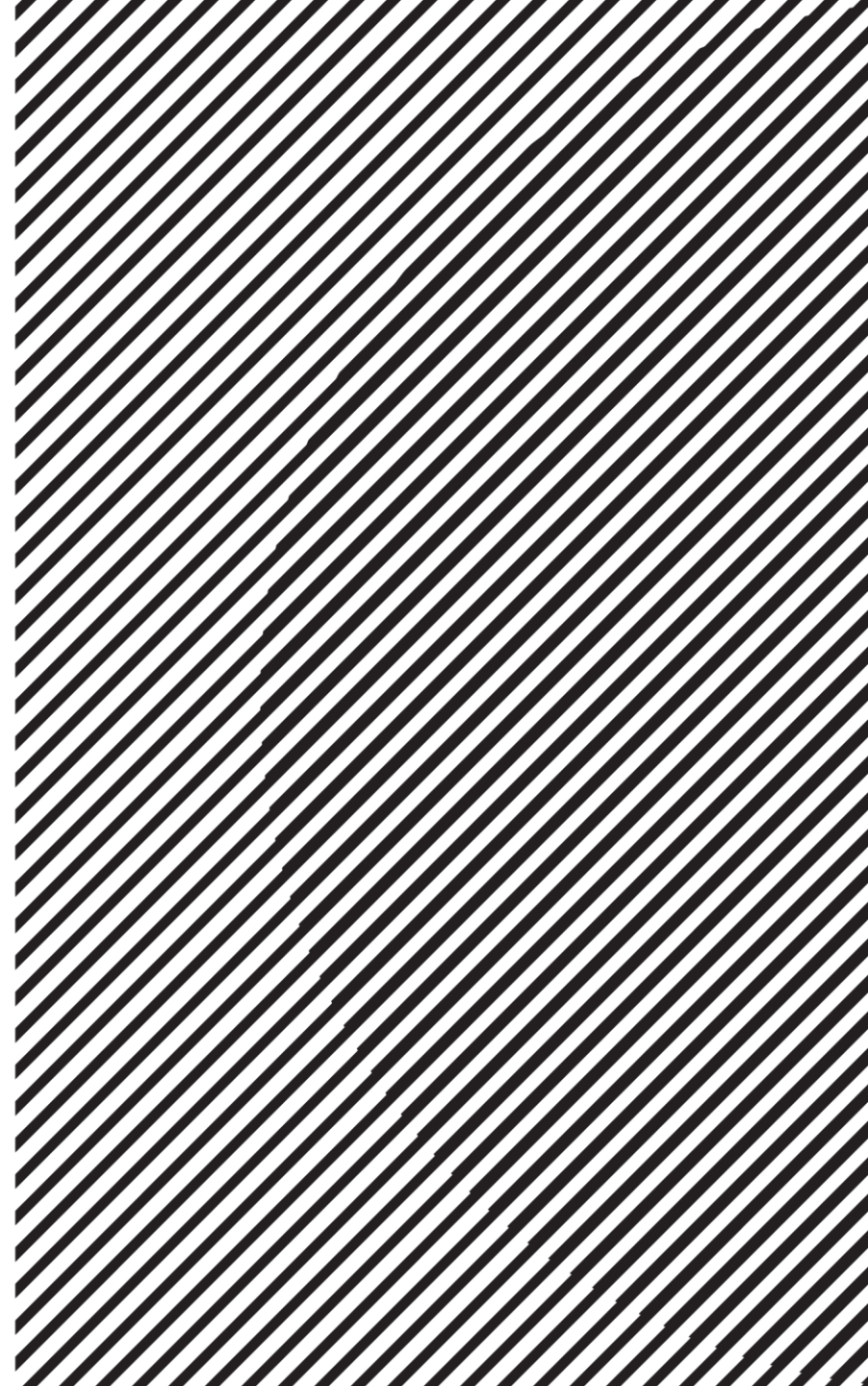


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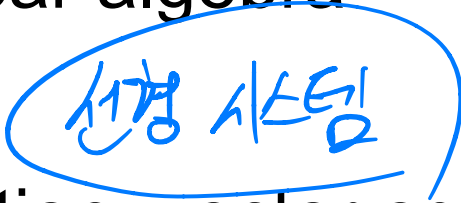
# Linear Algebra

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# Lecture Overview

- Elements in linear algebra
- Linear system 
- Linear combination, vector equation, Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

# Linear Equation

선형 방정식

- A **linear equation** in the variables  $x_1, \dots, x_n$  is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

where  $b$  and the coefficients  $a_1, \dots, a_n$  are real or complex numbers that are usually known in advance.

- The above equation can be written as

$$\mathbf{a}^T \mathbf{x} = b$$

$$[a_1 \dots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$

where  $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$  and  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ .



# Linear System: Set of Equations

선형 방정식 = 연립방정식

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables - say,  $x_1, \dots, x_n$ .

# Linear System Example

$$\begin{pmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 66 \\ 74 \\ 78 \end{pmatrix}$$

- Suppose we collected persons' weight, height, and life-span (e.g., how long s/he lived)

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

- We want to set up the following linear system:

$$60x_1 + 5.5x_2 + 1 \cdot x_3 = 66$$

$$65x_1 + 5.0x_2 + 0 \cdot x_3 = 74$$

$$55x_1 + 6.0x_2 + 1 \cdot x_3 = 78$$

이런 coefficient는  
찾고 싶은

- Once we solve for  $x_1$ ,  $x_2$ , and  $x_3$ , given a new person with his/her weight, height, and is\_smoking, we can predict his/her life-span.



# Linear System Example

- The essential information of a linear system can be written compactly using a **matrix**.
- In the following set of equations,

$$60x_1 + 5.5x_2 + 1 \cdot x_3 = 66$$

$$65x_1 + 5.0x_2 + 0 \cdot x_3 = 74$$

$$55x_1 + 6.0x_2 + 1 \cdot x_3 = 78$$

- Let's collect all the coefficients on the left and form a matrix

$$A = \begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}$$

- Also, let's form two vectors:  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$

# From **Multiple Equations** to **Single Matrix Equation**

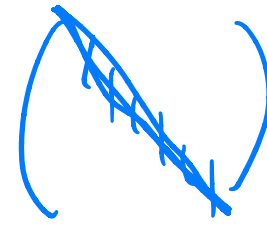
- Multiple equations can be converted into a **single** matrix equations

$$\begin{array}{l} 60x_1 + 5.5x_2 + 1 \cdot x_3 = 66 \\ 65x_1 + 5.0x_2 + 0 \cdot x_3 = 74 \\ 55x_1 + 6.0x_2 + 1 \cdot x_3 = 78 \end{array} \quad \longrightarrow \quad \begin{array}{c} \left[ \begin{array}{ccc} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[ \begin{array}{c} 66 \\ 74 \\ 78 \end{array} \right] \\ \mathbf{A} \qquad \qquad \mathbf{x} = \mathbf{b} \end{array} \quad \longleftarrow \quad \begin{array}{l} \mathbf{a}_1^T \mathbf{x} = 66 \\ \mathbf{a}_2^T \mathbf{x} = 74 \\ \mathbf{a}_3^T \mathbf{x} = 78 \end{array}$$

- How can we solve for  $\mathbf{x}$ ?

# Identity Matrix (항등 행렬)

$I$  or  $E$  or  $\frac{k}{2}$



- **Definition:** An identity matrix is a **square** matrix whose diagonal entries are all 1's, and all the other entries are zeros. Often, we denote it as  $I_n \in \mathbb{R}^{n \times n}$ .

• e.g.,  $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- An identity matrix  $I_n$  preserves any vector  $\mathbf{x} \in \mathbb{R}^n$  after multiplying  $\mathbf{x}$  by  $I_n$ :

$\forall \mathbf{x} \in \mathbb{R}^n$ ,  
for all

$$I_n \mathbf{x} = \mathbf{x}$$



# Inverse Matrix (역행렬)

$$A^{-1} \cdot A = E(I)$$

☆ Square Matrix 여야만  
Inverse Matrix가 존재함

- **Definition:** For a **square** matrix  $A \in \mathbb{R}^{n \times n}$ , its inverse matrix  $A^{-1}$  is defined such that

$$A^{-1}A = AA^{-1} = I_n.$$

- For a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , its inverse matrix  $A^{-1}$  is defined as

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$D = ad - bc$$

# Solving Linear System via Inverse Matrix

- We can now solve  $A\mathbf{x} = \mathbf{b}$  as follows:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I_n\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

이제 하려면

$A^{-1}$ 가 존재해야 함

# Solving Linear System via Inverse Matrix

- **Example:**

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix} \quad \longrightarrow \quad A^{-1} = \begin{bmatrix} 0.0870 & 0.0087 & -0.0870 \\ -1.1304 & 0.0870 & 1.1314 \\ 2.0000 & -1.0000 & -1.0000 \end{bmatrix}$$

$A \quad \quad \mathbf{x} = \mathbf{b}$

- One can verify

$$A^{-1}A = AA^{-1} = I_n.$$

- $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 0.0870 & 0.0087 & -0.0870 \\ -1.1304 & 0.0870 & 1.1314 \\ 2.0000 & -1.0000 & -1.0000 \end{bmatrix} \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 20 \\ -20 \end{bmatrix}$



# Solving Linear System via Inverse Matrix

- Now, the life-span can be written as

$$\begin{aligned}(\text{life-span}) = & -0.4 \times (\text{weight}) + 20 \times (\text{height}) \\ & - 20 \times (\text{is\_smoking}).\end{aligned}$$

# Non-Invertible Matrix $A$ for $A\mathbf{x} = \mathbf{b}$

역행렬이 없어

- Note that **if**  $A$  is invertible, the solution is **uniquely obtained** as  $\mathbf{x} = A^{-1}\mathbf{b}$ .
- **What if**  $A$  is non-invertible, i.e., the inverse does not exist?
  - E.g., For  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ , in  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , the denominator  $ad - bc = 0$ , so  $A$  is not invertible.
- For  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $ad - bc$  is called the **determinant** of  $A$ , or **det  $A$** .

판별식  $D$

# Does a Matrix Have an Inverse Matrix?

$\det(A)$

- $\det A$  determines whether  $A$  is invertible (when  $\det A \neq 0$ ) or not (when  $\det A = 0$ ).
- For more details on how to compute the determinant of a matrix  $A \in \mathbb{R}^{n \times n}$  where  $n \geq 3$ , you can study the following:
  - <https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-18-properties-of-determinants/>
  - <https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-19-determinant-formulas-and-cofactors/>

→ numpy.linalg or inv ~~for~~  $\frac{1}{\det}$

# Inverse Matrix Larger than $2 \times 2$

- If invertible, is there any formula for computing an inverse matrix of a matrix  $A \in \mathbb{R}^{n \times n}$  where  $n \geq 3$ ?
- No, but one can compute it.
- We skip details, but you can study Gaussian elimination in Lay Ch1.2 and then study Lay Ch2.2.

(Gaussian)  
가우스 소거법  
→ RREF  
(reduced row echelon form)

# Non-Invertible Matrix $A$ for $Ax = b$

이항방정식이 아니라면 linear system은 어떻게?

- Back to the linear system, if  $A$  is non-invertible,  $Ax = b$  will have either **no solution** or **infinitely many solutions**.

해가 없음

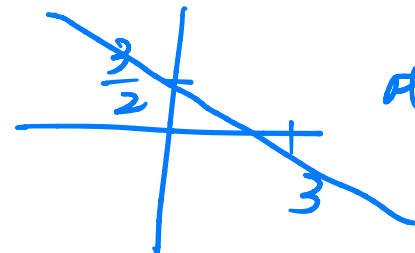
해가 무수히 많음

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases} \rightarrow \begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases} \xrightarrow{\text{2nd} - 2 \times \text{1st}} \begin{cases} x + 2y = 3 \\ 0x + 0y = 0 \end{cases}$$

해가 무수히 많음.

$$y = \frac{1}{2}(-x + 3)$$



이러므로 무수히 많은 해가 있음



# Rectangular Matrix $A$ in $Ax = b$

(No 정사각)

생각하자,  $(m, n) \times (n, 1)$   
→ 같아야 함.  
비스듬스

- What if  $A$  is a rectangular matrix, e.g.,  $A \in \mathbb{R}^{m \times n}$ , where  $m \neq n$ ?

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

- Recall  $m = \# \text{equations}$  and  $n = \# \text{variables}$ .  $A \quad x = b$

- $m < n$  more variables than equations

- Usually infinitely many solutions exist (under-determined system).

- $m > n$  more equations than variables

- Usually no solution exists (over-determined system).

- To study how to compute the solution in these general cases, check out Lay Ch1.2 and Lay Ch1.5.

구해줘요  
해를 구해야 함.