
Linear Algebra

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그람 슈미트 직교화

→ 어떤 부분 공간의 basis를
orthogonal normal set으로
바꾸는 방법

Gram-Schmidt Orthogonalization

★ 대부분 \rightarrow 선형독립 vector 들을 순직화하여 서로 간의 영향을 없애기 위함.

- **Example 1:** Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W .

- **Solution:** Let $\mathbf{v}_1 = \mathbf{x}_1$. Next, Let \mathbf{v}_2 the component of \mathbf{x}_2 orthogonal to \mathbf{x}_1 , i.e.,

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

- The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W .

★ $\mathbf{x}_2^2 = \mathbf{x}_1$
Span(\mathbf{v}_1)에
직교하게 만들어야

다음과 같이 풀기.

이 과정이 번거로우니 \mathbf{x}_1 의 단위벡터로 만들고
계산하면 편함

Exam 1. 풀이

$$\left\{ \underbrace{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{v_2} \right\} \hookrightarrow \{v_1, v_2\} \text{라 할 때 } v_2 \text{를 } v_1 \text{에 직교하게 만들자.}$$

$\hookrightarrow \text{span}(v_1) \text{에}$

i) $\text{span}(v_1)$ 에 v_2 가 직교하게 만들거야.

i-i) v_1 을 마찬가지로 쉽게 단위 벡터화하라

$$u = \vec{v}_1 \times \frac{1}{\|v_1\|} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \times \frac{1}{\sqrt{45}} = \begin{bmatrix} 3/\sqrt{45} \\ 6/\sqrt{45} \\ 0 \end{bmatrix}$$

i-ii) 공식 적용

$$\underbrace{v_2}_{\text{red circle}} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \underbrace{\frac{v_2 \cdot u}{u \cdot u}}_{\substack{\text{red circle} \\ \hookrightarrow 1}} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{\sqrt{45}} \begin{bmatrix} 3/\sqrt{45} \\ 6/\sqrt{45} \\ 0 \end{bmatrix}$$

\downarrow

$\{v_1, v_2\}$ 가 수직 정규집합인 subspace의 basis가 됨.



Gram-Schmidt Orthogonalization

- **Example 2:** Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus is a basis for a subspace W of \mathbb{R}^4 . Construct an orthogonal basis for W .

Gram-Schmidt Orthogonalization

- **Solution:**
- **Step 1.** Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{x}_1\} = \text{Span}\{\mathbf{v}_1\}$.
- **Step 2.** Let \mathbf{v}_2 be the vector produced by subtracting from \mathbf{x}_2 its projection onto the subspace W_1 . That is, let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

- \mathbf{v}_2 is the component of \mathbf{x}_2 orthogonal to \mathbf{x}_1 , and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the subspace W_2 spanned by \mathbf{x}_1 and \mathbf{x}_2 .



Gram-Schmidt Orthogonalization

- **Step 2' (optional).** If appropriate, scale \mathbf{v}_2 to simplify later computations, e.g.,

$$\mathbf{v}_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \rightarrow \mathbf{v}'_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Gram-Schmidt Orthogonalization

- **Step 3.** Let \mathbf{v}_3 be the vector produced by subtracting from \mathbf{x}_3 its projection onto the subspace W_2 . Use the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$ to compute this projection onto W_2 :

$$\text{proj}_{W_2} \mathbf{x}_3 = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_3 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_3 \cdot \mathbf{v}_2'} \mathbf{v}_2' = \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

- Then \mathbf{v}_3 is the component of \mathbf{x}_3 orthogonal to W_2 , namely,

$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Gram-Schmidt Orthogonalization

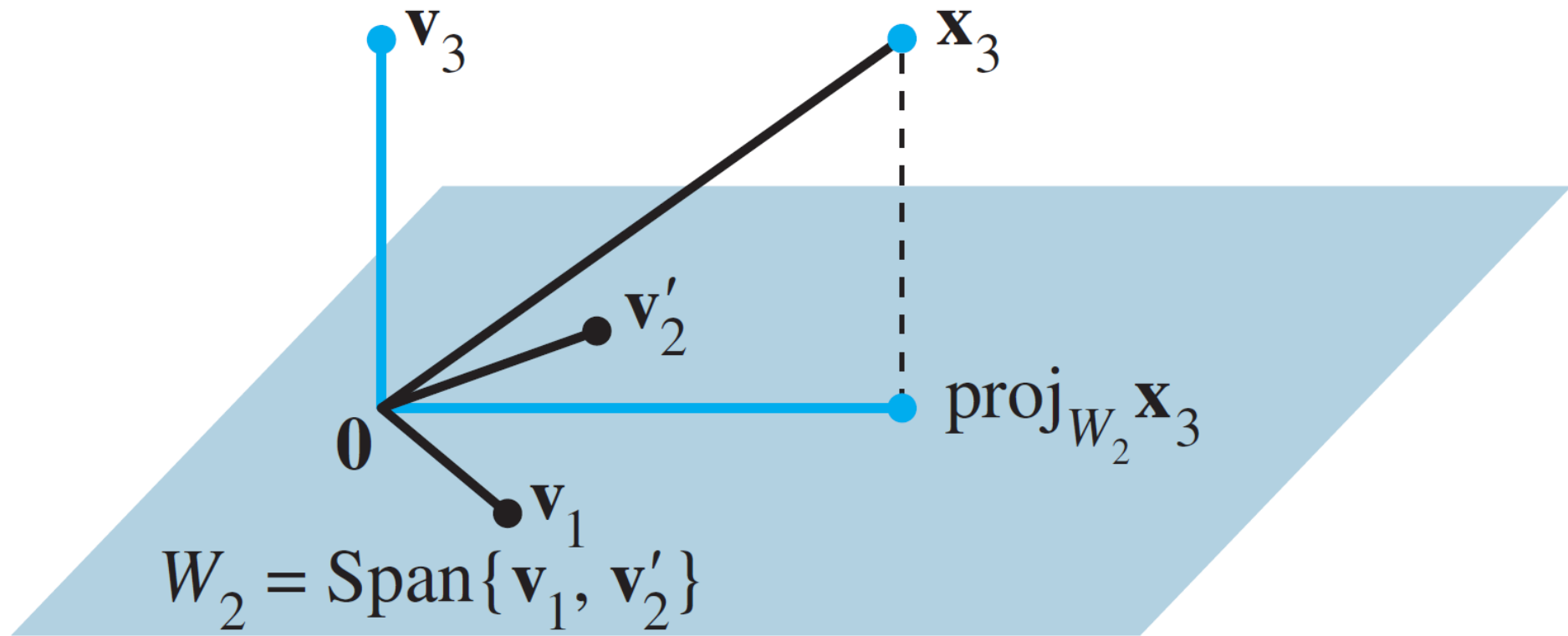


FIGURE 2 The construction of \mathbf{v}_3 from \mathbf{x}_3 and W_2 .

Figure from Lay Ch6.4

QR Factorization

↳ 분해.

- If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{Col } A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

$A = Q \times R$
↳ 직교 변환.
→ 이런 행렬 A 는 $Q \times R$ 로 분해될 수 있음



Computing QR Factorization

- **Step 1 (Construction of Q):** The columns of A form a basis for $\text{Col } A$ since they are linearly independent. Let these columns be $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. Then, we can construct the orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ for $\text{Col } A$ by the Gram-Schmidt process described by Theorem 11. Using this basis, we can construct Q as

$$Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

Computing QR Factorization

- **Step 2 (Construction of R):** From (1) in Theorem 11, for $k = 1, \dots, n$, \mathbf{x}_k is in $\text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$. Therefore, there exist constants r_{1k}, \dots, r_{kk} such that

$$\mathbf{x}_k = r_{1k}\mathbf{u}_1 + \dots + r_{kk}\mathbf{u}_k + 0 \cdot \mathbf{u}_{k+1} + \dots + 0 \cdot \mathbf{u}_n$$

- We can always make $r_{kk} \geq 0$ because if $r_{kk} < 0$, then we can multiply both r_{kk} and \mathbf{u}_k by -1. Using this linear combination representation, we can construct \mathbf{r}_k , the k -th column of R , as

$$\mathbf{r}_k = \begin{bmatrix} r_{1k} \\ \vdots \\ r_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$



Computing QR Factorization

- That is, $\mathbf{x}_k = Q\mathbf{r}_k$ for $k = 1, \dots, n$. Let $R = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]$. Then,
$$A = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_n] = [Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_n] = QR$$
- The fact that R is invertible follows easily from the fact that the columns of A are linearly independent (Exercise 19). Since R is clearly upper triangular (from the previous slide) and invertible, the diagonal entries r_{kk} 's should be nonzero. By combining this with the fact that $r_{kk} \geq 0$, r_{kk} 's must be positive.

Example: QR Factorization

- **Example 4:** Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- **Solution:** Let $A = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$. We first obtain $\mathbf{v}_1 = \mathbf{x}_1$ and its normalized vector is $\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$.
- Thus, $\mathbf{x}_1 = 2\mathbf{u}_1$, which gives us $\mathbf{r}_{11} = 2$, i.e., $\mathbf{r}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Example: QR Factorization

- Next, we obtain \mathbf{v}_3 as $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 -$

$$\frac{\mathbf{x}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \frac{2}{\sqrt{12}} \begin{bmatrix} -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \text{ and its}$$

normalized vector \mathbf{u}_2 as $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$

- Thus, $\mathbf{x}_3 = 1\mathbf{u}_1 + \frac{2}{\sqrt{12}}\mathbf{u}_2 + \frac{2}{\sqrt{6}}\mathbf{u}_3$, i.e., $\mathbf{r}_3 = \begin{bmatrix} 1 \\ 2/\sqrt{12} \\ 2/\sqrt{6} \end{bmatrix}.$



Example: QR Factorization

- In conclusion, $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$

and $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3] = \begin{bmatrix} 2 & -3/2 & 1 \\ 0 & -3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$