Linear Algebra

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Diagonalization (대각화)

- We want to change a given square matrix $A \in \mathbb{R}^{n \times n}$ into a diagonal matrix via the following form:

$$D = V^{-1}AV$$

where $V \in \mathbb{R}^{n \times n}$ is an invertible matrix and $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix. This is called a diagonalization of A.

- It is not always possible to diagonalize A. For A to be diagonalizable, an invertible V should exist such that $V^{-1}AV$ becomes a diagonal matrix.

Finding V and D

- How can we find an invertible P and the resulting diagonal matrix $D = V^{-1}AV$?
- $D = V^{-1}AV \Longrightarrow VD = AV$ → V^{-1}이 존재해야만 가능함.
- Let us represent the following:
- $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$ where $\mathbf{v}_i$'s are column vectors of V
- $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

Finding V and D

- $AV = A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n]$
- $VD = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

  $= [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n]$
- $AV = VD \Leftrightarrow [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n]$

Finding V and D

- Equating columns, we obtain

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \ldots,\ A\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

- Thus, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ should be eigenvectors and $\lambda_1, \lambda_2, \ldots, \lambda_n$ should be eigenvalues.
- Then, For $VD = AV \Longrightarrow D = V^{-1}AV$ to be true, V should invertible.
- In this case, the resulting diagonal matrix D has eigenvalues as diagonal entries.

Diagonalizable Matrix

$D = V^{-1}AV$

- For V to be invertible,
  V should be a square matrix in $\mathbb{R}^{n \times n}$, and
  V should have n linearly independent columns.

다 선형 독립이어야 함.

- Recall columns of V are eigenvectors.
  Hence, A should have n linearly independent eigenvectors.

- It is not always the case, but if it is, A is diagonalizable.

대각화 가능함.

Eigendecomposition

고유값 분해

- If A is diagonalizable, we can write $\underline{D = V^{-1}AV}$.

- We can also write $A = VDV^{-1}$.

  which we call eigendecomposition of A.

- A being diagonalizable is equivalent to A having eigendecomposition.

Linear Transformation via Eigendecomposition

- Suppose A is diagonalizable, thus having eigendecomposition

$$A = VDV^{-1}$$

- Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
- $T(\mathbf{x}) = A\mathbf{x} = VDV^{-1}\mathbf{x} = V\left(D(V^{-1}\mathbf{x})\right)$.

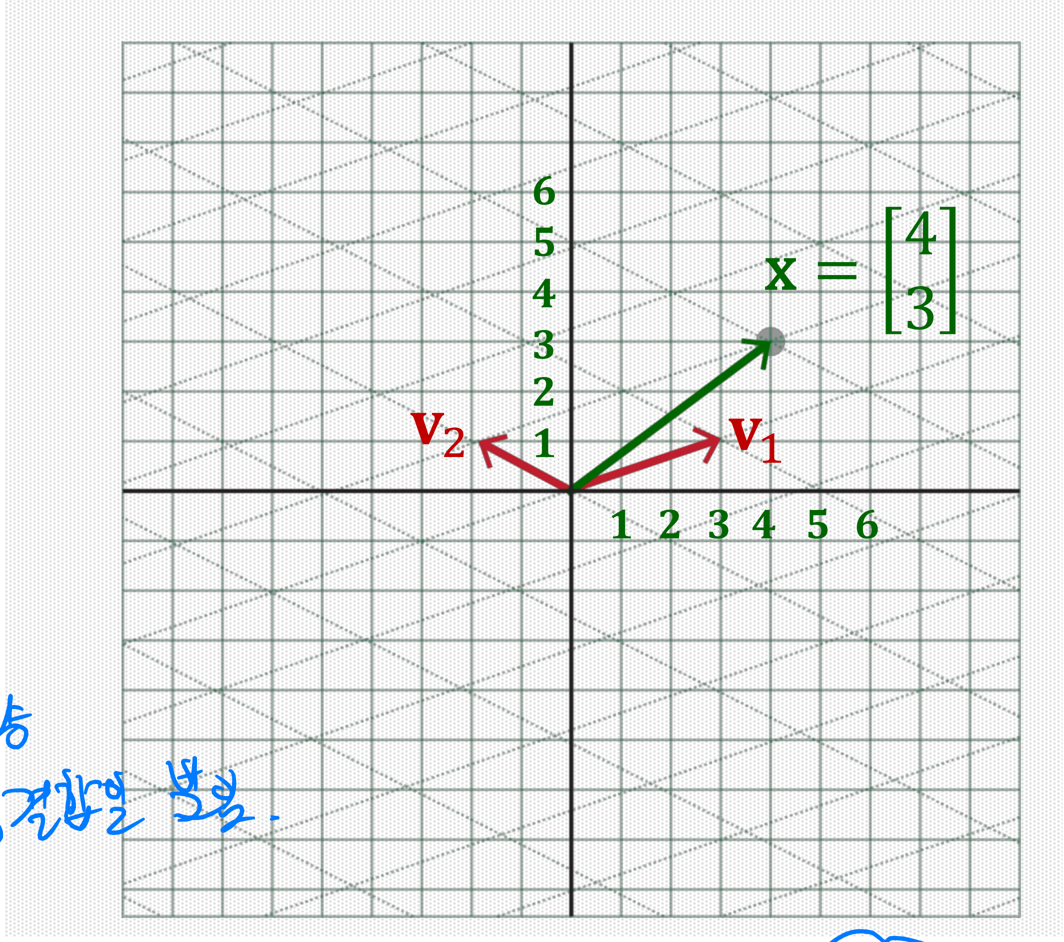
Change of Basis

- Suppose $A\mathbf{v}_1 = -1\mathbf{v}_1$ and $A\mathbf{v}_2 = 2\mathbf{v}_2$.
- $T(\mathbf{x}) = A\mathbf{x} = VDV^{-1}\mathbf{x} = V\left(D\left(\boxed{V^{-1}\mathbf{x}}\right)\right)$
- Let $\boxed{\mathbf{y} = V^{-1}\mathbf{x}.}$ Then,

$$V\mathbf{y} = \mathbf{x}$$

- $\mathbf{y}$ is a new coordinate of $\mathbf{x}$ with respect to a new basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$V^{-1}x$는 고유
선형결합의 본질.



$$\mathbf{x} = \begin{bmatrix}4\\3\end{bmatrix} = 4\overset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}} + 3\overset{v_2}{\begin{bmatrix}0\\1\end{bmatrix}} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix} = P\mathbf{y} = [\mathbf{v}_1 \quad \mathbf{v}_2]\begin{bmatrix}y_1\\y_2\end{bmatrix} = 2\mathbf{v}_1 + 1\mathbf{v}_2 \Rightarrow \mathbf{y} = \begin{bmatrix}2\\1\end{bmatrix}$$

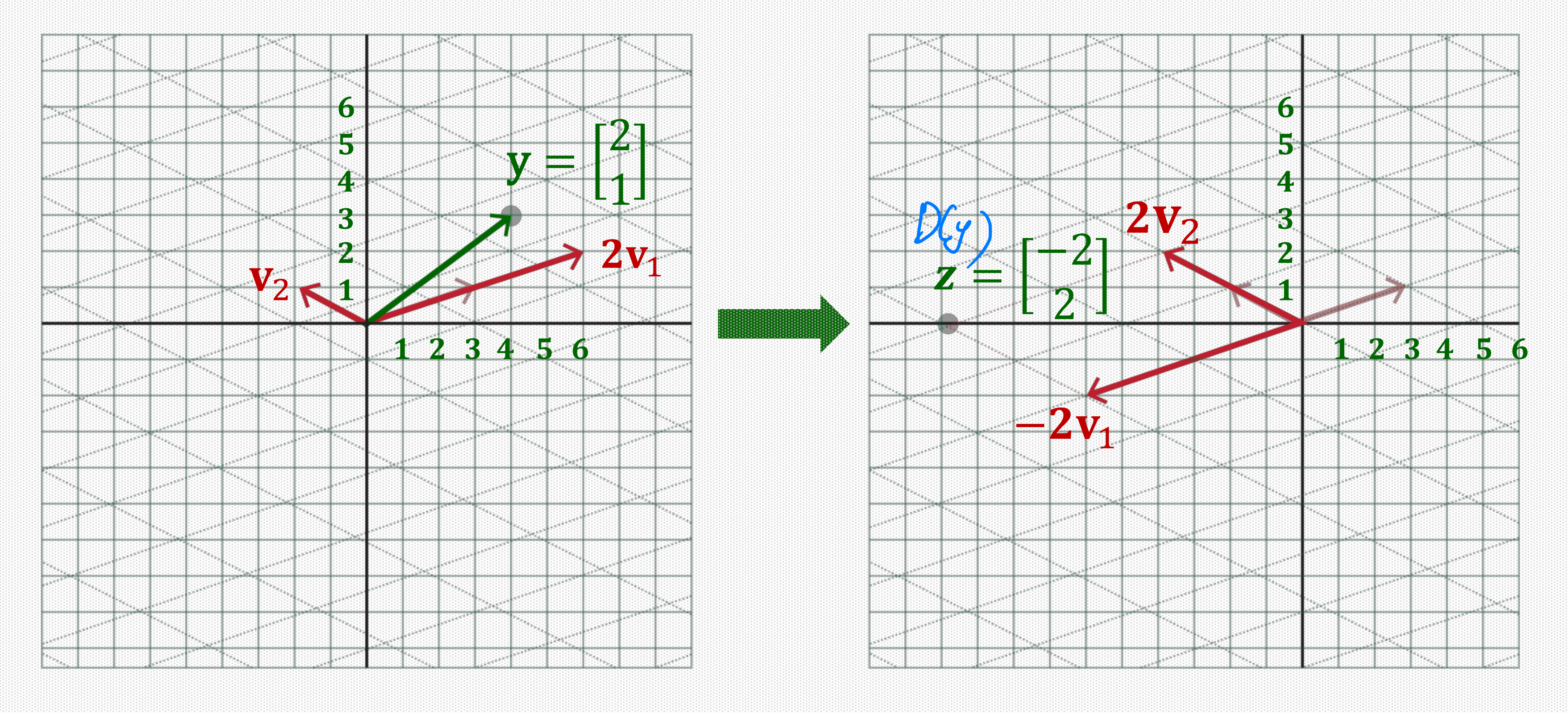
Element-wise Scaling

D가 대각행렬 이므로
각 원소에 대한 scaling인 것이다.
상식적임.

- $T(\mathbf{x}) = V\big(D(V^{-1}\mathbf{x})\big) = V(D\mathbf{y})$
- Let $z = D\mathbf{y}$. This computation is a simple Element-wise scaling of $\mathbf{y}$.
- **Example**: Suppose $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$z = D\mathbf{y} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1) \times 2 \\ 2 \times 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Dimension-wise Scaling



Back to Original Basis

$D(V^{-1}x))$ 이니까 당연 OO

- $T(\mathbf{x}) = V(D\mathbf{y}) = Vz$
- z is still a coordinate based on the new basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- Vz converts z to another coordinates based on the original standard basis.
- That is, Vz is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ using the coefficient vector z.
- That is,

$$Vz = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{v}_1 z_1 + \mathbf{v}_2 z_2$$

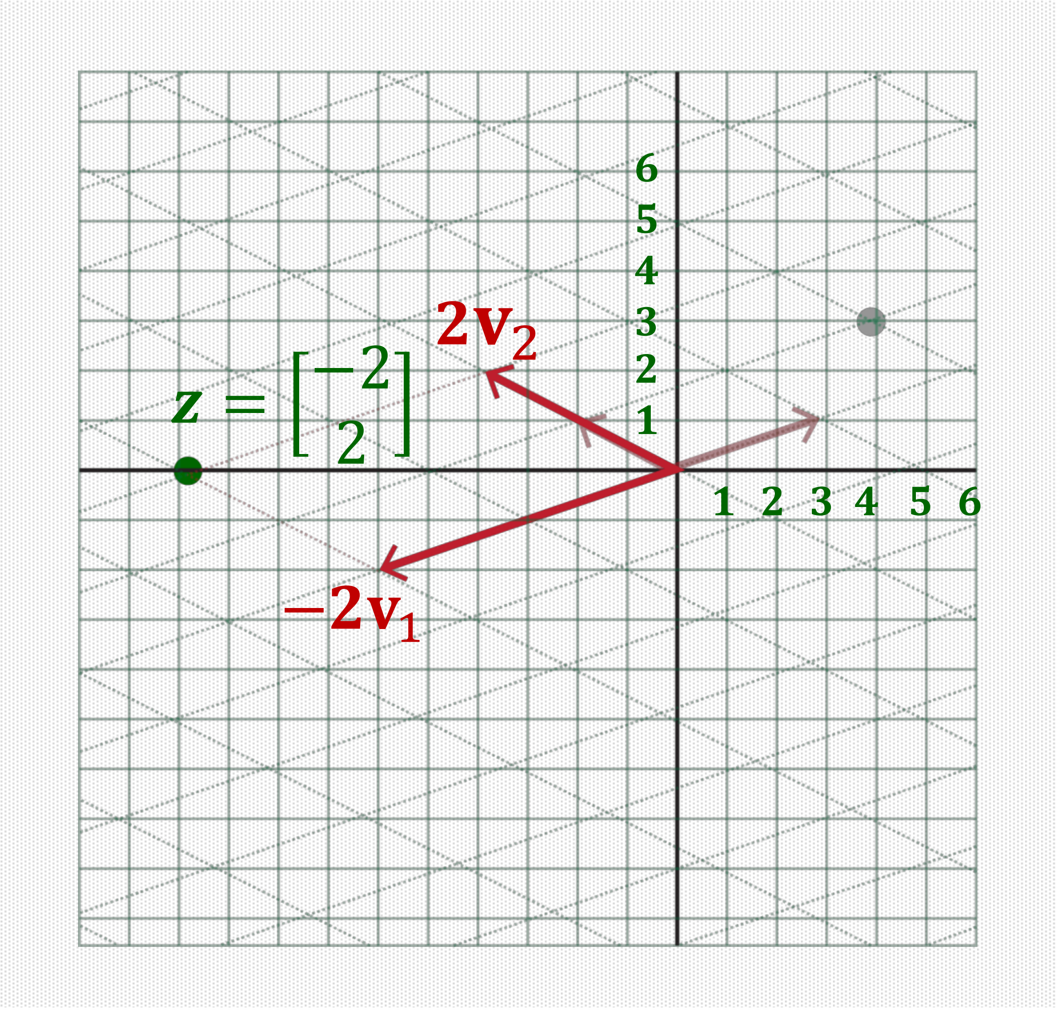
Back to Original Basis

- $T(\mathbf{x}) = V{\color{red}z} = [\mathbf{v}_1 \quad \mathbf{v}_2]\begin{bmatrix}-2\\2\end{bmatrix}$

$$= -2\mathbf{v}_1 + 2\mathbf{v}_2$$

$$= -2\begin{bmatrix}3\\1\end{bmatrix} + 2\begin{bmatrix}-2\\1\end{bmatrix}$$

$$= \begin{bmatrix}-10\\0\end{bmatrix}$$



Overview of Transformation using Eigendecomposition

$$\begin{tikzcd} x \arrow[r, "A"] \arrow[d, "V^{-1}"'] & T(x) = Ax = V\big(D(V^{-1}x)\big) \\ y \arrow[r, "D"] & z \arrow[u, "V"'] \end{tikzcd}$$

$V^{-1}x$ $\qquad\qquad$ $D(V^{-1}(x))$

Linear Transformation via A^k

- Now, consider recursive transformation $A \times A \times \cdots \times A\mathbf{x} = A^k\mathbf{x}$.
- If A is diagonalizable, A has eigendecomposition

$$A = VDV^{-1}$$

- $A^k = (VDV^{-1})(VDV^{-1})\cdots(VDV^{-1}) = \underline{VD^kV^{-1}}$

↳ 계산이 간편해짐!

- D^k is simply computed as

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$$

Linear Transformation via A^k

- $A^k\mathbf{x} = VD^kV^{-1}\mathbf{x}$ can be computed in the similar manner to the previous example.
- It is much faster to compute $V\left(D^k(V^{-1}\mathbf{x})\right)$ than to compute $A^k\mathbf{x}$.

Further Study

- Determining whether a matrix $A \in \mathbb{R}^{n\times n}$ diagonalizable.
  - Geometric multiplicity should be equal to algebraic multiplicity.
  - As a special case, A has n distinct eigenvalues, A is diagonalizable.
  - Lay Ch5.3

- Solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for a given eigenvalue λ
  - Lay Ch1.5