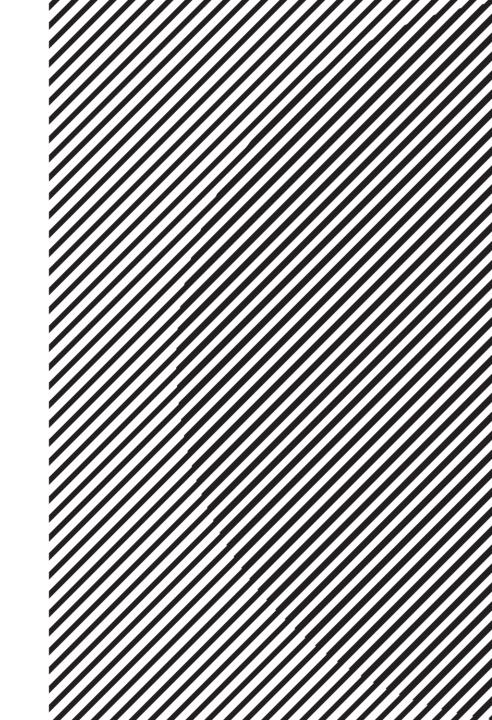# Linear Algebra

주재걸
고려대학교 컴퓨터학과
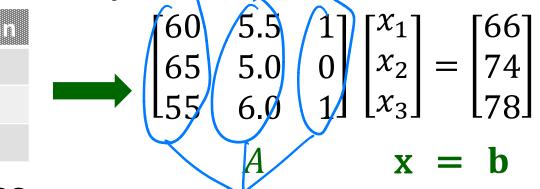
Lecture Overview

- Elements in linear algebra

- Linear system

- Linear combination, vector equation,
  Four views of matrix multiplication

- <u>Linear independence, span, and subspace</u>

- Linear transformation 선형독립, span, 부분공간

- Least squares

- Eigendecomposition

- Singular value decomposition

Recall: Linear System = 명령방정식. or vector 방정식.

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

➡

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

A $\mathbf{x} = \mathbf{b}$

- Or, a vector equation is written as

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a_1} x_1 + \mathbf{a_2} x_2 + \mathbf{a_3} x_3 = \mathbf{b}$$

재료 vector
v_1, v_2, v_3

$b \in$ Span(v_1, v_2, v_3)면
해가 존재

Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when $\mathbf{b} \in \text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- If the solution exists for $A\mathbf{x} = \mathbf{b}$, when is it unique?
- It is unique when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly independent.
- Infinitely many solutions exist when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent.

Linear Independence

(Practical) Definition:

... of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, check if $\mathbf{v}_j$ can be ... as a linear combination of the previous vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{j-1}\}$ for $j = 1, ..., p$, e.g.,

$$\mathbf{v}_j \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{j-1}\} \text{ for some } j = 1, ..., p?$$

- If at least one such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly dependent.
- If no such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly independent.

선형독립 → 어떠한 vector의 set이
해당 간의 선형 결합으로
표현되지 아니함.

Span$\{v_1, \cdots\} \in \mathbb{R}^{\textcircled{2}}$ 이고
($v_1 \cdots v_3$은 선형 독립.
선형독립이오면
Span을 vector 개수만큼
 확장 받아야

직관적으로
V_j가 이전의 vector의

선형결합으로 표현되면
Span의 확장이 없어지
(하나라도 발견되면 선형 종속임.

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Linear Independence

homogeneous equation

(Formal) Definition:

- Consider $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots + x_p\mathbf{v}_p = \mathbf{0}$.

$$v_p = -\frac{x_1}{x_p}v_1 - \frac{x_2}{x_p}v_2 \cdots$$

- Obviously, one solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

  which we call a trivial solution.

다른 것으로 표현될 수 있다면

선형종속임

non-trivial solution

- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly independent if this is the only solution.
- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent if this system also has other nontrivial solutions, e.g., at least one x_i being nonzero.

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Two Definitions are Equivalent

- If $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent, consider a nontrivial solution.

- In the solution, let's denote j as the last index such that $x_j \neq 0$.

- Then, one can write $x_j \mathbf{v}_j = -x_1 \mathbf{v}_1 - \cdots - x_{j-1} \mathbf{v}_{j-1}$, and safely divide it by x_j, resulting in

$$\mathbf{v}_j = -\frac{x_1}{x_j} \mathbf{v}_1 - \cdots - \frac{x_{j-1}}{x_j} \mathbf{v}_{j-1} \in \text{Span} \left\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1} \right\}$$

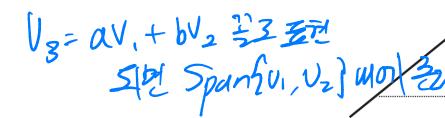which means $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors.
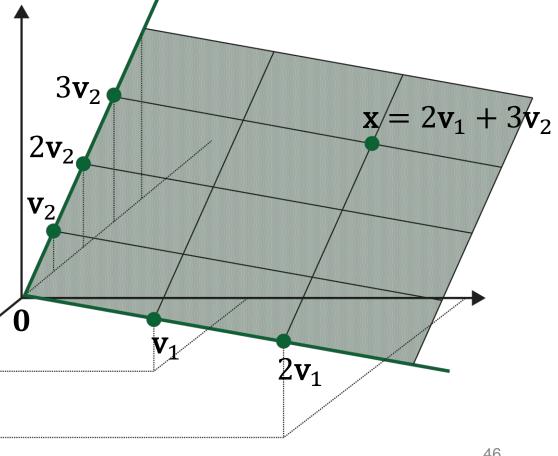
→ 전 페이지에서 설명했음. formal definition의 관계일 뿐임

Geometric Understanding of Linear Dependence

- Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, Suppose Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane on the right.

- When is the third vector $\mathbf{v}_3$ linearly dependent of $\mathbf{v}_1$ and $\mathbf{v}_2$?

- That is, $\mathbf{v}_3 \in$ Span $\{\mathbf{v}_1, \mathbf{v}_2\}$?

$V_3 = aV_1 + bV_2$ 할 수 있는 조건

되면 $Span\{v_1, v_2\}$ 에 있게 됨

$3\mathbf{v}_2$

$2\mathbf{v}_2$

$\mathbf{v}_2$

$\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2$

$\mathbf{0}$

$\mathbf{v}_1$

$2\mathbf{v}_1$

Linear Dependence

- <mark>A linearly dependent vector does not increase Span</mark>!
- If $\mathbf{v}_3 \in$ Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, then

$$\text{Span } \{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\},$$

- Why?
- Suppose $\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$, then the linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can be written as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (c_1 + d_1)\mathbf{v}_1 + (c_1 + d_1)\mathbf{v}_2$$

which is also a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.

Linear Dependence and Linear System Solution

$\{v_1, v_2, v_3\}$ 가 선형종속이라면
이 linear system은
해가 무수히 많아짐.

해들이 많아
unique하
해를 가짐

- Also, a linearly dependent set produces multiple possible linear combinations of a given vector.

- Given a vector equation $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{b}$, suppose

  the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, i.e., $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{b}$.

- Suppose also $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$, a linearly dependent case.

- Then, $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_1 + 3\mathbf{v}_2) = 5\mathbf{v}_1 + 5\mathbf{v}_2$,

  so $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is another solution. Many more solutions exist.

Linear Dependence and Linear System Solution

- Actually, many more solutions exist.
- e.g., $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_3 - 1\mathbf{v}_3)$
$$= 3\mathbf{v}_1 + 2\mathbf{v}_2 + 2(2\mathbf{v}_1 + 3\mathbf{v}_2) - 1\mathbf{v}_3 = 7\mathbf{v}_1 + 8\mathbf{v}_2 - 1\mathbf{v}_3,$$

thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}$ is another solution.

Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

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