Linear Algebra

주재걸
고려대학교 컴퓨터학과

Orthogonal Projection Perspective

직교 투영
(사영)

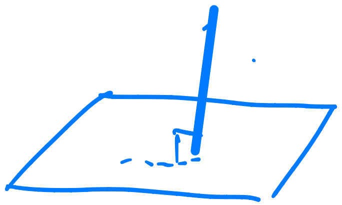
- Back to the case of invertible $C = A^TA$, consider the orthogonal projection of $\mathbf{b}$ onto Col A as

b 벡터를 $\hat{b}$로 선형 변환함.

$$\hat{\mathbf{b}} = f(\mathbf{b}) = A\hat{\mathbf{x}} = A(A^TA)^{-1}A^T\mathbf{b}$$

Col A에 수선의 발을 내린

$$\hat{b} = A\hat{x}$$
$$= A \cdot ((A^TA)^{-1}A^T \cdot b)$$



Orthogonal and Orthonormal Sets

직교 집합과 정규(normal) 직교집합

아무거나 2개 잡고 내적하면 "직교" 함.

- **Definition**: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors from the set is orthogonal 직교집합. That is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

정규 직교집합

- **Definition**: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors.

$\vec{v} \times \frac{1}{\|\vec{v}\|}$ 로 길이 1로 맞춰야

- Is an orthogonal (or orthonormal) set also a linearly independent set? What about its converse?

Yes.

직교집합은 선형 독립임.

상식적으로

Orthogonal and Orthonormal Basis

- Consider basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of a p-dimensional subspace W in $\mathbb{R}^n$.

- Can we make it as an orthogonal (or orthonormal) basis?

  그람 슈미츠 수직화(직교화)

  - Yes, it can be done by Gram–Schmidt process. → QR factorization.

  (orthognalization)

- Given the orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ of W,
  let's compute the orthogonal projection of $\mathbf{y} \in \mathbb{R}^n$ onto W.

Orthogonal Projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Line

$\hat{y} = Ax = A\cdot(A^TA)^{-1}A^Tb$ 인데, 이를 기하학적으로 이해해보자

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto one-dimensional subspace L.

L공간에 $\vec{y}$를 직교시켜

- $\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

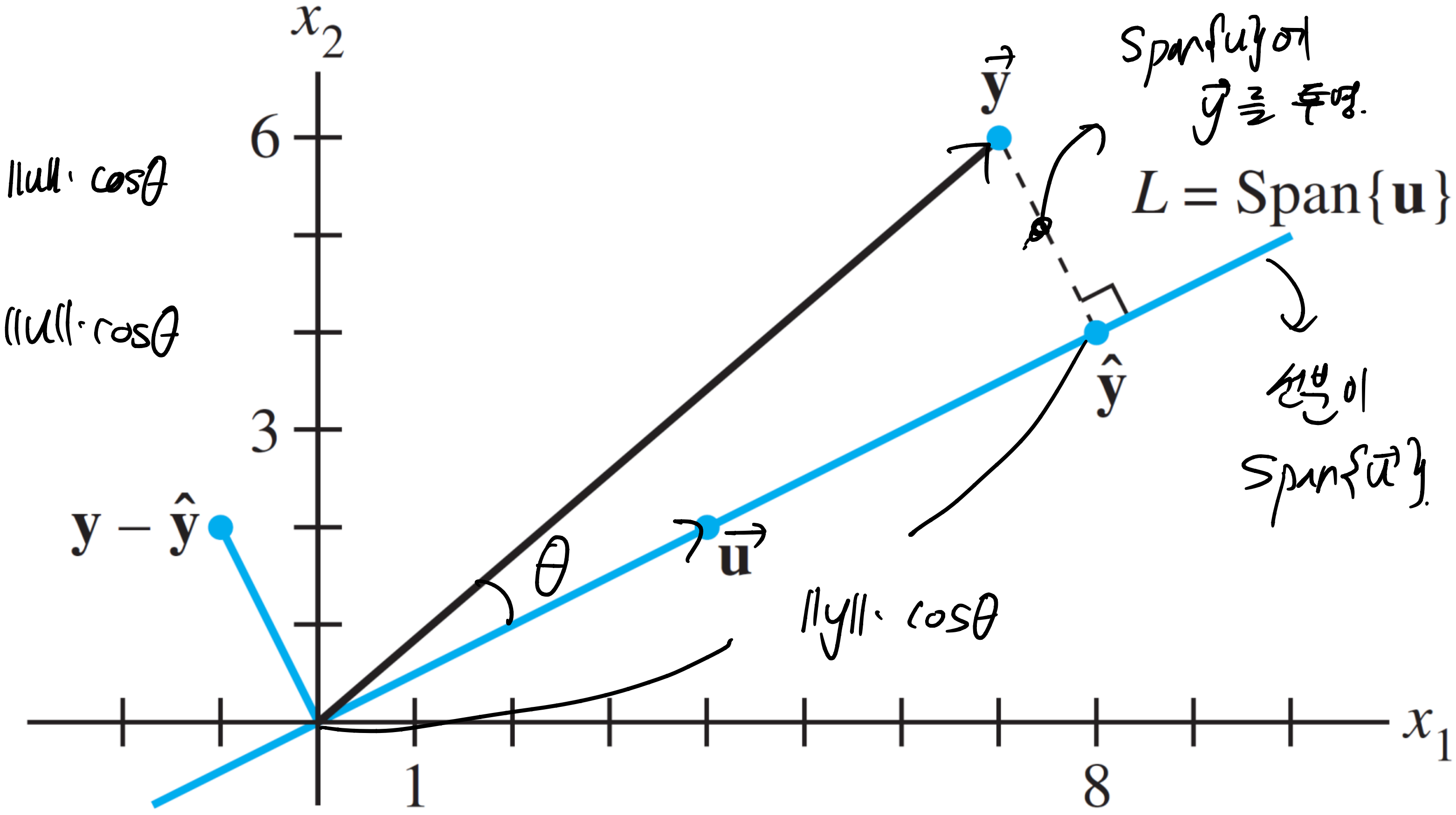
$\|y\|\cdot\|u\|\cdot\cos\theta$

$\|u\|\cdot\|u\|\cdot\cos\theta$

- If $\mathbf{u}$ is a unit vector,

  $\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = (\mathbf{y}\cdot\mathbf{u})\mathbf{u}$

$u\cdot u = 1$



$y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $L = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$, L에 proj된 $\hat{y}$을 구하라.

$$\hat{y} = \frac{y \cdot u}{u \cdot u} \cdot u$$

Span이므로 basis 길이를 늘리거나 줄여도 같음.

i) L은 $\text{span}\left\{ \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$으로 바꿀 수 있음.

ii) $y \cdot u = \frac{1-2+6}{\sqrt{6}} = \frac{5}{\sqrt{6}}$

$u \cdot u = \frac{1+1+4}{6} = 1$

iii) $\hat{y} = \frac{5}{\sqrt{6}} \cdot \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{-5}{6} \\ \frac{10}{6} \end{bmatrix}$

Orthogonal Projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Plane

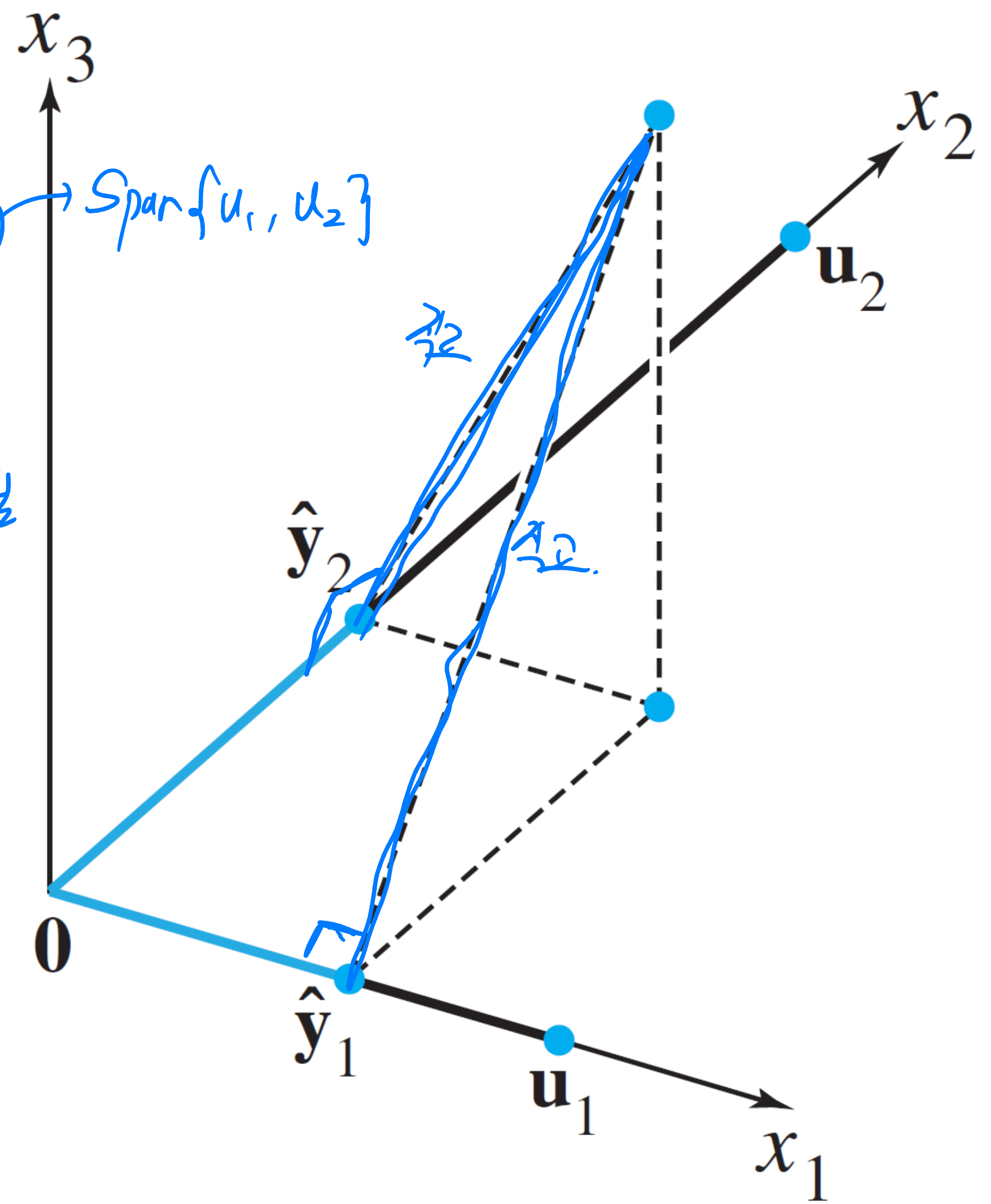
- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto two-dimensional subspace W → Span{u_1, u_2}

- $\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = \dfrac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \dfrac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2$

그냥 각각에 대해서 사영하면 됨

- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors,
  $\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2$

- Projection is done independently on each orthogonal basis vector.



Orthogonal Projection when $\mathbf{y} \in W$

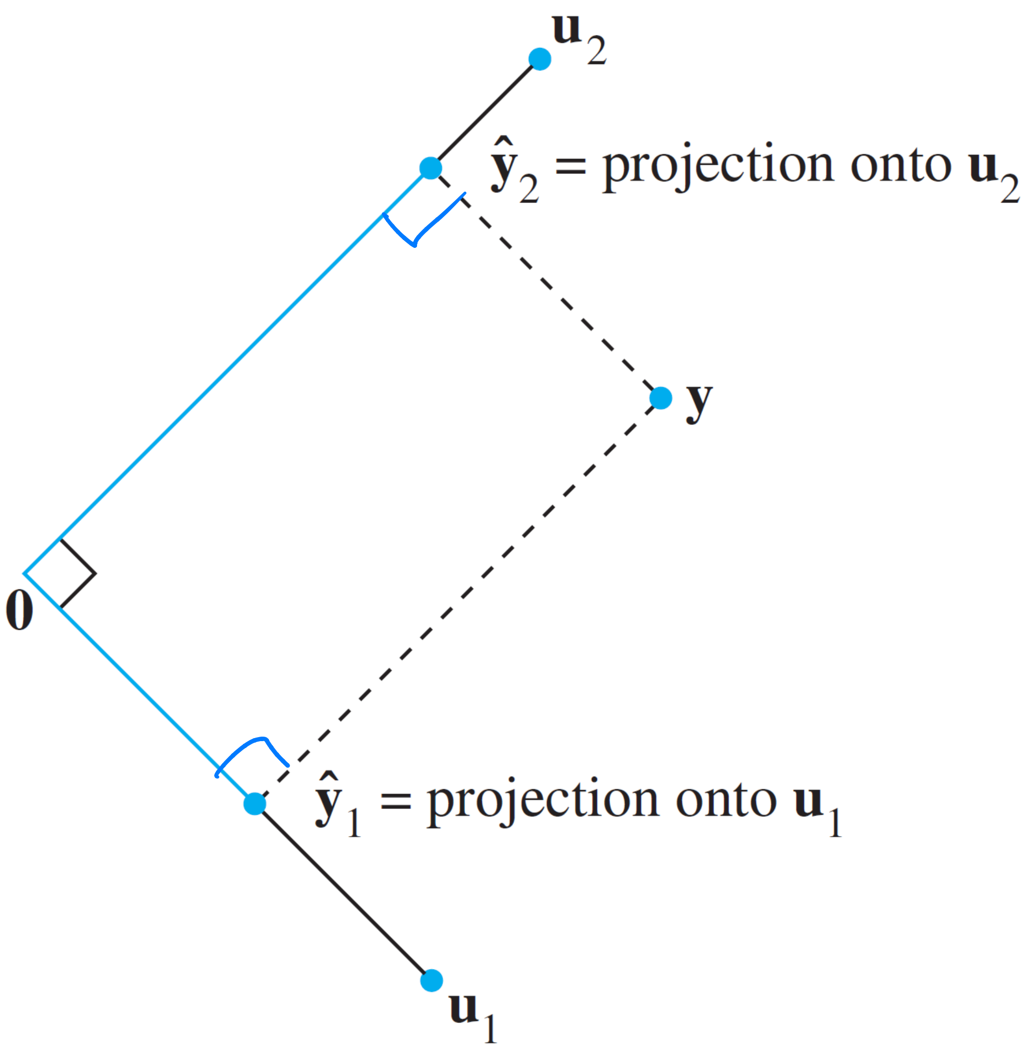
전 페이지를 조강뷰

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto two-dimensional subspace W, where $\mathbf{y} \in W$

- $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$

- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors,
  $\hat{\mathbf{y}} = \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2$

- The solution is the same as before.
  Why?



Transformation: Orthogonal Projection

- Consider a transformation of orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$, given orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of a subspace W:

W의 정규직교집합 basis가 있다면....

$$\hat{\mathbf{b}} = f(\mathbf{b}) = (\mathbf{b} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{b} \cdot \mathbf{u}_2)\mathbf{u}_2$$

→ u_1, u_2는 $\boxed{\text{유닛벡터}}$니까

$\frac{b \cdot u_1}{u_1 \cdot u_1} u_1 = b \cdot u_1 u_1$ 임

$$= (\mathbf{u}_1^T \mathbf{b})\mathbf{u}_1 + (\mathbf{u}_2^T \mathbf{b})\mathbf{u}_2$$

$u_1^T b$, $u_2^T b$는 스칼라니까 교환법칙

$$= \mathbf{u}_1(\mathbf{u}_1^T \mathbf{b}) + \mathbf{u}_2(\mathbf{u}_2^T \mathbf{b})$$

$$= (\mathbf{u}_1\mathbf{u}_1^T)\mathbf{b} + (\mathbf{u}_2\mathbf{u}_2^T)\mathbf{b}$$

$$= (\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T)\mathbf{b}$$

$$= [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} \mathbf{b} = UU^T\mathbf{b} \quad \Rightarrow \text{linear transformation!}$$

Orthogonal Projection Perspective

- Let's verify the following, when $A = U = [\mathbf{u}_1 \quad \mathbf{u}_2]$ has orthonormal columns:

  Back to the case of invertible $C = A^T A$, consider the orthogonal projection of $\mathbf{b}$ onto Col A as

$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b} = f(\mathbf{b})$$

- $C = A^T A = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} [\mathbf{u}_1 \quad \mathbf{u}_2] = I$. Thus,

$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b} = A(I)^{-1} A^T \mathbf{b} = AA^T \mathbf{b} = UU^T \mathbf{b}$$