Linear Algebra

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Lecture Overview

① vector → Span{v_1 ⋯ v_p}.

② H → vector
(subspace)
non-unique.

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace 부분공간!
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Span and Subspace

$au_1 + bu_2$가 어떤공간 H에 닫혀있다.

→ 야 이거 span{v_1, v_2} 아님? = 맞다!

"닫혀있다" 선형조합에 대해서

- **Definition**: A **subspace** H is defined as a subset of $\mathbb{R}^n$ closed under linear combination:
  - For any two vectors, $\mathbf{u}_1, \mathbf{u}_2 \in H$, and any two scalars c and d, $c\mathbf{u}_1 + d\mathbf{u}_2 \in H$.
- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is always a subspace. Why?
  - $\mathbf{u}_1 = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$, $\mathbf{u}_2 = b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p$
  - $c\mathbf{u}_1 + d\mathbf{u}_2 = c(a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p) + d(b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p)$
    $= (ca_1 + db_1)\mathbf{v}_1 + \cdots + (ca_p + db_p)\mathbf{v}_p$

V_1과 V_2의 선형 결합에 닫혀있음 그 공간이 Subspace임

- In fact, a subspace is always represented as Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

Basis of a Subspace

"기저" 구성함

non-unique

이제, span이 주어지면 span을 구성하는 vector를 찾아야 함. "발상의 전환"

- **Definition**: A **basis** of a subspace H is a set of vectors that satisfies both of the following:
  - Fully spans the given subspace H

    = H를 전부 덮을 수 있어야 함
  - Linearly independent (i.e., no redundancy)

    = 선형 독립이어야 함

$v_3 = 2v_1 + 3v_2$ 이므로 v_3은 빼야함.

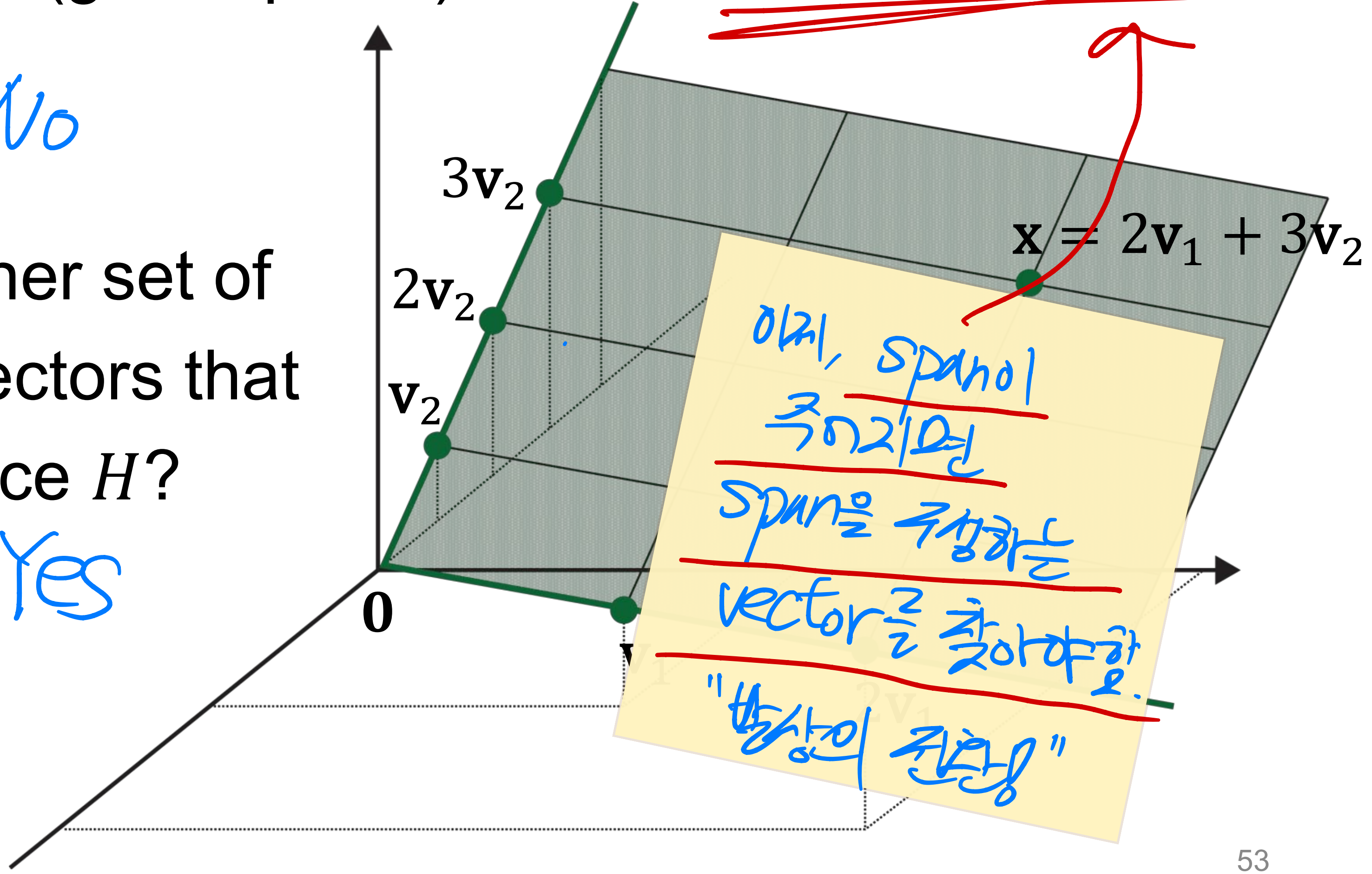
- In the previous example, where $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a plane, but $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H, but not $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1\}$ is a basis.

Non-Uniqueness of Basis → subspace를 구성하는 방법은 아주 다양함.

basis는 unique하지 않음

- Consider a subspace H (green plane).
- Is a basis unique? ⇒ No
- That is, is there any other set of linearly independent vectors that span the same subspace H?

⇒ Yes



이제, span이 주어지면 span을 구성하는 vector를 찾아야 함. "발상의 전환!"

Dimension of Subspace

dim H는 결정적이야

basis vector의 갯수

- What is then unique, given a particular subspace H?
- Even though different bases exist for H, the number of vectors in any basis for H will be unique.
- We call this number as the dimension of H, denoted as $\dim H$.
- In the previous example, the dimension of the plane is 2, meaning any basis for this subspace contains exactly two vectors.

Column Space of Matrix

- **Definition**: The **column space** of a matrix A is the subspace spanned by the columns of A. We call the column space of A as Col A.

$$A = \begin{bmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix} \quad\Longrightarrow\quad \text{Col } A = \text{Span}\left\{ \overset{v_1}{\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}} \right\}$$

- What is dim Col A? 2

Matrix with Linearly Dependent Columns

↳ basis 조건에 안 맞으니 빼야지

- Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, note that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

  i.e., the third column is a linear combination of the first two.

$$\text{Col } A = \text{Span}\left\{ \begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v_2 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} v_3 \\ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \end{array} \right\} \Longrightarrow \text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- What is dim Col A?

2

$v_3 = v_1 + v_2$ 이므로 선형종속임.

$\Rightarrow$ basis에서 빼야 함.

Rank of Matrix

- **Definition**: The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A:
  - rank A = dim Col A

rank A = dim span $\left(\square \; \square \; \cdots \; \square \right)$

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Summary So Far

- Scalars, vectors, matrices, and their operations such as addition, scalar multiple, matrix multiplication, transpose

$Ax=b$

$x=A^{-1}b$

- Linear system: solving using inverse matrix
- Matrix equation and vector equation

$Ax=b \rightarrow \left(\;\right)\left(x\right)=\left(b\right)$

- Linear combination and Span
  - When does the solution of a linear system exist?
- Four views of matrix multiplication: inner product, column combination, row combination, sum of rank-1 outer products
- Linear independence 선형 독립
  - If the solution of a linear system exists, when is it unique or many?

선형독립이어야

$v_1 x_1 + v_2 x_2 = b$

$b \in \mathrm{span}\{v_1, v_2\}$

이어야 해 존재

+

선형 독립이어야 unique

- Subspace
  - Subset of vectors in $\mathbb{R}^n$ closed under linear combination
  - Basis and dimension
  - Column space and rank of a matrix

$\mathrm{Rank}\, A = \dim(\mathrm{Col}\, A)$