Solution for Assignment 04

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PROBLEM 1. For each of the following statements, say whether true or false. For false statements,give the correct version of the statement.

(i) $P(A \cap B) = P(A) * P(B)$ if $A,\ B$ are independent.

(ii) $P(A \cup B) = P(A) + P(B)$ if $A,\ B$ are independent.

(iii) In a sequence of n independent identical trials, each of which results in either "success" or "failure", with probability θ of success, the number of successes follows a Bernoulli distribution.

SOLUTION.

(i) True.The equation is the definition of independent subsets.

(ii) False. As we know, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A) * P(B)$.

(iii) True. We have proved it on class, part 3.3.2 of lecture notes.

PROBLEM 2. In five independent tosses of an unbiased coin, find

(i) the probability that the total number of heads is even;

(ii) the probability that there are exactly five heads.

SOLUTION. Let H_i denotes the i-th coin is head up, T_i denotes the i-th coin is tail up, where $1 \leq i \leq 5$.
By the independent, we have $P(\cap_{i=1}^5 H_i) = \prod_{i=1}^5 P(H_i) = \frac{1}{2^5}$, and any replace of H_i to T_i remains the equation.
Now let k be the total number of heads.

(i)

$$\begin{aligned}
P(k=2) = & \sum_{1 \leq i_3 \leq i_4 \leq i_5 \leq 5, i_k \neq i_1, i_2} \sum_{1 \leq i_1 \leq i_2 \leq 5} P(H_{i_1} \cap H_{i_2} \cap T_{i_3} \cap T_{i_4} \cap T_{i_5}) \\
= & C_5^2 * \frac{1}{2^5} \\
= & \frac{5}{16}
\end{aligned}$$

By the same argument, $P(k=4) = C_5^1 * \frac{1}{2^5} = \frac{5}{32}$, and

$$P(kiseven) = P(k=2) + P(k=4) = \frac{15}{32}$$

(ii) As the first argument of proof, $P(k=5) = P(\cap_{i=1}^5 H_i) = \frac{1}{32}$.

PROBLEM 3. A discrete random variable X has possible values -2, 1, 3, 4 with probabilities satisfying

$$P(X=2) = P(X=1) = 2P(X=3) = 3P(X=4).$$

Find the probability mass function and the (cumulative) distribution function of X, and graph them both.

SOLUTION. Let $k = P(X=4)$, by the equation $P(X=-2) + P(X=1) + P(X=3) + P(X=4) = 1$ and the assumption of the problem, the equation can change to

$$3k + 3k + 1.5k + k = 1.$$

So $k = \frac{2}{17}$, $P(X = -2) = \frac{6}{17}, P(X = 1) = \frac{6}{17}, P(X = 3) = \frac{3}{17}, P(X = 4) = \frac{2}{17}$.

Now let F be the cumulative distribution function of X, then for $x < -2, F(x) = P(X \leq x < -2) = 0$,

$$F(x) = P(X \leq x) = P(X = -2) = \frac{6}{17}, \; -2 \leq x < 1,$$

$$F(x) = P(X \leq x) = P(X = -2) + P(X = 1) = \frac{12}{17}, \; 1 \leq x < 3,$$

$$F(x) = P(X \leq x) = P(X = -2) + P(X = 1) + P(X = 3) = \frac{15}{17}, \; 3 \leq x < 4,$$

$$F(x) = P(X \leq x) = P(X = -2)+P(X = 1)+P(X = 3)+P(X = 4) = 1, \; x \geq 4,$$

Graph is skipped here.

PROBLEM 4. The following table shows the probability mass function of a discrete random variable X. Plot the (cumulative) distribution function of this random variable.

k	1	2	3	4	5
$P(X = k)$	0.1	0.2	0.4	0.1	0.2

SOLUTION. Skip.

PROBLEM 5. Suppose $F(x)$ is the c.d.f. of a random variable X. Show that $F(x)$ has the following properties:

(i) $0 \leq F(x) \leq 1$;

(ii) $F(x)$ is an increasing function of x, i.e., $F(x) \leq F(y)$for any $x < y$;

(iii) $\lim_{x \to +\infty} F(x) = 1$; $\lim_{x \to -\infty} F(x) = 0$;

(iv) Show that $F(x)$ is a right-continuous function of $x \in \mathbb{R}$: (Just show that if a sequence of real numbers $x_n \downarrow x$, then $\lim_{n \to \infty} F(x_n) = F(x)$).

SOLUTION.

(i) $F(x) = P(X \leq x)$,so we have the inequality by property of probability measure;

(ii) for any $x < y$,

$$
\begin{aligned}
F(y) - F(x) =& P(X \leq y) - P(X \leq x) \\
=& P(x < X \leq y) \\
\geq& 0.
\end{aligned}
$$

so $F(y) \geq F(x)$.

(iii) Consider any $x_{n n \geq 1}$, s.t. $\lim_{n \to \infty} x_n = \infty$.We have

$$
\bigcup_{n=1}^{\infty} \{X \geq x_n\} = \Omega.
$$

So by the continuity of probability measure,

$$
\begin{aligned}
\lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} & P(X \geq x_n) \\
=& P(\bigcup_{n=1}^{\infty} \{X \geq x_n\}) \\
=& P(\Omega) \\
=& 1
\end{aligned}
$$

By the same argument, we get $\lim_{x \to -\infty} F(x) = 0$.

(iv) Just same as argument in (iii).