Suggested Solutions of Homework 5 MA327

Ex 1. By calculation,

$$\begin{aligned}
\mathbf{N} &= \left(-\frac{c\sin u}{\sqrt{c^2+v^2}}, \frac{c\cos u}{\sqrt{c^2+v^2}}, -\frac{v}{\sqrt{c^2+v^2}}\right) \\
E &= v^2+c^2, F=0, G=1; \\
e &= 0, f=\frac{c}{\sqrt{c^2+v^2}}, g=0.
\end{aligned}$$

We may assume $c \neq 0$. Then the asymptotic curve is

$$\frac{c}{\sqrt{c^2+v^2}}u'v' = 0 \quad \Rightarrow \quad u'=0 \quad or \quad v'=0.$$

so u = const, v = v(t) or v = const, u = u(t).

The lines of curvature is

$$(v^2+c^2)(u')^2 - (v')^2 = 0.$$

$$u(t) = \pm\log(\sqrt{v(t)^2+c^2}+v(t)) + const.$$

The mean curvature is 0 since e, F, g are all zero.

Ex 2. Since

$$\begin{aligned}
&\mathbf{x}_{uu} = (-\cosh v\cos u, -\cosh v\sin u, 0), \quad \mathbf{x}_{uv} = (-\sinh v\sin u, \sinh v\cos u, 0), \\
&\mathbf{x}_{vv} = (\cosh v\cos u, \cosh v\sin u, 0) \quad \mathbf{N} = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v). \\
&\Rightarrow \quad e=-1, \quad f=0, \quad g=1 \\
&\Rightarrow \quad u+v = Const \quad or \quad u-v = Const.
\end{aligned}$$

Ex 3. (a), (b) are pure calculation.

(c) The coefficients a_{ij} of dN_p with basis $\mathbf{x}_u, \mathbf{x}_v$ are

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = -\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{2}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{2}{(1+u^2+v^2)^2} \end{bmatrix}$$

Thus, the principal curvature are $\frac{2}{(1+u^2+v^2)^2}$ and $-\frac{2}{(1+u^2+v^2)^2}$.

(d)Since a_{ij} has been a diagonal matrix, the lines of curvatures are the coordinate curves.

(e)Take e, f, g into the differential equation of asymptotic curves, there are $u'+v'=0$ and $u'-v'=0$. We get the conclusion after integration.

Ex 4. We translate the curve by $(x,y) \mapsto (x-1,y)$. Then the parametrization of revolution surface is $\mathbf{x}(u,v) = (v\cos u, v\sin u, (v+1)^3)$. It suffices to show $d\mathbf{N}_{(0,-1)} = 0$.

By calculation

$$\begin{aligned} \mathbf{x}_u(0,-1) &= (0,-1,0) \quad \mathbf{x}_v(0,-1) = (1,0,0) \\ \mathbf{x}_{uu}(0,-1) &= (1,0,0) \quad \mathbf{x}_{uv}(0,-1) = (0,1,0) \\ \mathbf{x}_{vv}(0,-1) &= (0,0,0) \quad \mathbf{N}(0,-1) = (0,0,1). \end{aligned}$$

$$\begin{aligned} \Rightarrow &\quad E=1 \quad F=0 \quad G=1 \quad e=f=g=0 \\ \Rightarrow &\quad d\mathbf{N}_{(0,-1)} = 0 \end{aligned}$$

Ex 5.
The proof could be found on textbook Lemma 2 page327 (2nd edition).

Proof. . *Since S is compact, S is bounded. Therefore, there are spheres of $\mathbb{R}^3$, centered in a fixed point $O \in \mathbb{R}^3$, such that S is contained in the interior of the region bounded by any of them. Consider the set of all such spheres. Let r be the infimum of their radii and let $\Sigma \subset \mathbb{R}^3$ be a sphere of radius r centered in O. It is clear that Σ and S have at least one common point, say p. The tangent plane to Σ at p has only the common point p with S, in a neighborhood of p. Therefore, Σ and S are tangent at p. By observing the normal sections at p, it is easy to conclude that any normal curvature of S at p is greater than or equal to the corresponding curvature of Σ at p. Therefore, $K_S(p) \geq K_\Sigma(p) > 0$, and p is an elliptic point, as we wished.*

(One can also see more details in Lecture 15.)

Ex 6. In Question 5 above, we prove every compact surface has an elliptic point. That means the mean curvature H at this point is not zero. i.e. not a minimal surface.

Ex 7.
Helicoid: $\mathbf{x}(u,v) = (a \sinh v \cos u, a \sinh v \sin u, au)$.
Catenoid:$\mathbf{y}(u,v) = (-a \cosh v \sin u, a \cosh v \cos u, av)$.

$$\begin{aligned} \mathbf{x}_u &= (-a \sinh v \sin u, a \sinh v \cos u, a), \\ \mathbf{x}_v &= (a \cosh v \cos u, a \cosh v \sin u, 0), \\ \mathbf{y}_u &= (-a \cosh v \cos u, -a \cosh v \sin u, 0), \\ \mathbf{y}_v &= (-a \sinh v \sin u, a \sinh v \cos u, a). \end{aligned}$$

(a) Since $\mathbf{x}_u = \mathbf{y}_v$, $\mathbf{x}_v = -\mathbf{y}_u$, they are conjugate minimal surfaces.
(b) Let E_1, F_1, G_1 be the first fundamental form of $\mathbf{x}$ and E_2, F_2, G_2 be the first fundamental form of $\mathbf{y}$. Since $\mathbf{x}$, $\mathbf{y}$ are conjugate minimal surface, and isothermal respectively, $E_1 = G_1 = E_2 = G_2, F_1 = F_2 = 0$.

$$\begin{aligned} &\mathbf{z}_u = \cos t \mathbf{x}_u + \sin t \mathbf{y}_u, \\ &\mathbf{z}_v = \cos t \mathbf{x}_v + \sin t \mathbf{y}_v; \\ &E = \langle \mathbf{z}_u, \mathbf{z}_u \rangle = (\cos t)^2 E_1 + (\sin t)^2 E_2 = E_1. \\ &F = \langle \mathbf{z}_u, \mathbf{z}_v \rangle = \cos t \sin t (\langle \mathbf{x}_u, \mathbf{y}_v \rangle + \langle \mathbf{y}_u, \mathbf{x}_v \rangle) = 0 \\ &G = G_1. \end{aligned}$$

Then we can see $\mathbf{z}$ is an isothermal parametrization. Furthermore, the linear combination of harmonic functions is still harmonic, so $\mathbf{z}$ is minimal for all $t \in \mathbb{R}$.
(c) It has been proven in (b).