Solution for Assignment 13

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December 1, 2022

PROBLEM 1.

(a) Suppose X is an continuous r.v. with p.d.f. $f_X(x)$. For any real value $-\infty < t < +\infty$, define a real-valued function, denoted by $M_X(t)$,as $M_X(t) = E(e^{tX})$.

Further assume that $M_X(t)$ is well-defined for any $-\infty < t < +\infty$.

(i) Write down the integration form of $M_X(t)$.

(ii) If X is non-negative, show that $M_X(t)$ is a nondecreasing function of t.

(iii) If X is non-negative, show that

$$ift < 0then0 \leq M_X(t) \leq 1andM_X(0) = 1.$$

(iv) If $Y = aX + b$ where a and b are two constants. Show that

$$M_Y(t) = e^{bt} M_X(at).$$

(v) Suppose X and Y are two independent continuous r.v.s. Show that

$$M_X + Y(t) = M_X(t) M_Y(t).$$

(b) Suppose X is a discrete r.v. with p.m.f. $p_k = P(X = x_k), k > 1$. For any real value $-\infty < t < +\infty$, define a real-valued function,denoted by $M_X(t)$, as $M_X(t) = E(e^{tX})$. Further assume that $M_X(t)$ is well-defined for any $t \in \mathbb{R}$.

(i) Write down the integration form of $M_X(t)$.

(ii) If X is non-negative, show that $M_X(t)$ is a nondecreasing function of t.

(iii) If X is non-negative, show that

$$if\ t < 0\ then\ 0 \leq M_X(t) \leq 1\ and\ M_X(0) = 1.$$

(iv) If $Y = aX + b$ where a and b are two constants. Show that

$$M_Y(t) = e^{bt} M_X(at).$$

(v) Suppose X and Y are two independent continuous r.v.s. Show that

$$M_X + Y(t) = M_X(t) M_Y(t).$$

SOLUTION.

(a) (i) $M_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx} p(x) dx$.

(ii) Assume $t < s$, since X is non-negative, $p_X(x) = 0$ for any $x < 0$, so

$$\begin{aligned} M_X(t) = E(e^{tX}) =& \int_{\mathbb{R}} e^{tx} p(x) dx \\ =& \int_0^{\infty} e^{tx} p(x) dx \\ \leq& \int_0^{\infty} e^{sx} p(x) dx \\ =& M_X(s). \end{aligned}$$

So $M_X(t)$ is nondecreasing about t.

(iii) $M_X(t) \leq 0$ is trivial, for the other side, we obverse $e^{tx} \leq e^0 = 1$ for any $t < 0$, so

$$\begin{aligned} M_X(t) = E(e^{tX}) = & \int_{\mathbb{R}} e^{tx} p(x) dx \\ \leq & \int_0^{\infty} p(x) dx \\ = & 1. \end{aligned}$$

Or we can just use the conclusion of (ii) and

$$M_X(t) \leq M_X(0) = \int_0^{\infty} e^0 p(x) dx = \int_0^{\infty} p(x) dx = 1.$$

(iv)

$$\begin{aligned} M_Y(t) = E(e^{t(aX+b)}) = & \int_{\mathbb{R}} e^{t(ax+b)} p(x) dx \\ = & e^{bt} \int_0^{\infty} e^{(at)x} p(x) dx \\ = & e^{bt} M_X(at). \end{aligned}$$

(v) By the independence of X and Y, we have $p_{(X,Y)}(x, y) = p_X(x) * p_Y(y)$, so

$$\begin{aligned} M_{X+Y}(t) = E(e^{t(X+Y)}) = & \int_{\mathbb{R}} \int_{\mathbb{R}} e^{t(x+y)} p_{(X,Y)}(x, y) dy dx \\ = & \int_{\mathbb{R}} e^{tx} \int_{\mathbb{R}} e^{ty} p_X(x) * p_Y(y) dy dx \\ = & int_{\mathbb{R}} e^{tx} p_X(x) M_Y(t) dx \\ = & M_X(t) * M_Y(t) \end{aligned}$$

(b) (i) $M_X(t) = E(e^{tX}) = \sum_{k \geq 1} p_k e^{tx_k}$.

(ii) Assume $t < s$, since X is non-negative, $x_k \leq 0$ for any $k \geq 1$, so

$$\begin{aligned} M_X(t) = E(e^{tX}) = & \sum_{k \geq 1} p_k e^{tx_k} \\ \leq & \sum_{k \geq 1} p_k e^{sx_k} \\ = & M_X(s). \end{aligned}$$

So $M_X(t)$ is nondecreasing about t.

(iii) $M_X(t) \leq 0$ is trivial, for the other side, we obverse $e^{tx_k} \leq e^0 = 1$ for any $t < 0$, so

$$\begin{aligned} M_X(t) = E(e^{tX}) =& \sum_{k\geq 1} p_k e^{tx_k} \\ \leq& \sum_{k\geq 1} p_k \\ =&1. \end{aligned}$$

Or we can just use the conclusion of (ii) and

$$M_X(t) \leq M_X(0) = \sum_{k\geq 1} p_k e^0 = \sum_{k\geq 1} p_k = 1.$$

(iv)

$$\begin{aligned} M_Y(t) = E(e^{t(aX+b)}) =& \sum_{k\geq 1} p_k exp[t(ax_k + b)] \\ =&e^{bt} \sum_{k\geq 1} p_k exp[(at)x_k] \\ =&e^{bt} M_X(at). \end{aligned}$$

(v) By the independence of X and Y, we have $p_{(X,Y)}(x_k, y_l) = p_X(x_k) * p_Y(y_l)$, so

$$\begin{aligned} M_{X+Y}(t) = E(e^{t(X+Y)}) =& \sum_{k\geq 1} \sum_{l\geq 1} p_{(X,Y)}(x_k, y_l) e^{t(x_k+y_l)} \\ =& \sum_{k\geq 1} e^{tx_k} \sum_{l\geq 1} p_X(x_k) \cdot p_Y(y_l) e^{ty_l} \\ =& \sum_{k\geq 1} p_X(x_k) e^{tx_k} M_Y(t) \\ =&M_X(t) \cdot M_Y(t) \end{aligned}$$

PROBLEM 2. Find the m.g.f of

(i) the discrete random variable X with $P(X = 4) = 1$;

(ii) the Bernoulli random variable with parameter $p(0 < p < 1)$, and then applying the properties of m.g.f. to find the m.g.f. of the Binomial random variable with parameter $p(0 < p < 1)$ and n where n is a positive integer;

(iii) the Poisson random variable with parameter $\lambda > 0$;

(iv) the Geometric random variable with parameter $p(0 < p < 1)$, and then applying the properties of m.g.f. to find the m.g.f. of the Negative Binomial random variable with parameter p and r where r is a positive integer.

(v) the continuous random variable Y with probability density function

$$f_Y(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & otherwise. \end{cases} \tag{1}$$

(vi) the random variable $X \sim U[a, b](-\infty < a < b < +\infty)$.

(vii) the exponential random variable with parameter $\lambda > 0$, and then applying the properties of m.g.f. to find the m.g.f. of the Gamma random variable with parameter $\lambda > 0$ and m where m is a positive integer;

(viii) the general Gamma random variable with parameter $\lambda > 0$ and α, where $\alpha > 0$ may NOT be a positive integer;

(ix) the standard normal random variable $Z \sim N(0, 1)$; Define $X = \mu + \sigma Z$ for real numbers μ, σ with $\sigma > 0$, use the properties of m.g.f. $M_Z(t)$ to find the m.g.f. $M_X(t)$ of X.

SOLUTION.

(i) We see X is a discrete r.v., so

$$M_X(t) = e^{4t} P(X = 4) = e^{4t}.$$

And $M_X(t)$ is well-defined for $t \in \mathbb{R}$.

(ii) Assume $X \sim B(p, 1)$, then $P(X = 1) = p, P(X = 0) = 1 - p$, and

$$M_X(t) = e^t P(X = 1) + e^0 P(X = 0) = p(e^t - 1) + 1.$$

For $Y \sim B(p, n)$, we can write $Y = \sum_{k=1}^{n} X_k$, with X_k independent and $X_k \sim B(p, 1)$, so by the property 1(b)(v), we have

$$M_Y(t) = \prod_{k=1}^{n} M_{X_k}(t) = [1 + p(e^t - 1)]^n.$$

And $M_X(t)$ is well-defined for $t \in \mathbb{R}$.

(iii) Suppose $X \sim Poisson(\lambda)$, then we have $p_k = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, for any $k \leq 0, k \in \mathbb{Z}$. So

$$\begin{aligned} M_X(t) =& \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\ =& e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \\ =& e^{\lambda(e^t - 1)}. \end{aligned}$$

The last eqution is from the Taylor expansion of $e^{\lambda e^t}$.

And $M_X(t)$ is well-defined for $t \in \mathbb{R}$.

(iv) Assume $X \sim G(p)$, then $p(k) = p \cdot (1 - p)^{k-1}$ for any $k \geq 1$. So

$$\begin{aligned} M_X(t) =& \sum_{k=1}^{\infty} e^{tk} p \cdot (1 - p)^{k-1} \\ =& p \cdot e^t \sum_{k=1}^{\infty} [(1 - p)e^t]^{k-1} \\ =& \frac{pe^t}{1 - (1 - p)e^t} \end{aligned}$$

And for $Y \sim NB(p,r)$, we can write $Y = \sum\limits_{k=1}^{r} X_k$, with $X_k \sim G(p)$ and X_k are independent. So the same as 2(ii), we can compute

$$M_Y(t) = \prod_{k=1}^{r} M_{X_k}(t) = [\frac{pe^t}{1-(1-p)e^t}]^r$$

And $M_X(t)$ is well-defined for $t < -\ln(1-p)$.

(v) For $t = 0$, $M_X(t) = 1$, for $t \neq 0$, we have

$$\begin{aligned}
M_Y(t) &= \int_0^1 e^{ty} 2y dy \\
&= \int_0^t 2y d\frac{e^{ty}}{t} \\
&= 2\frac{ye^{ty}}{t}\Big|_0^1 - \frac{2}{t}\int_0^1 e^{ty} dy \\
&= 2\frac{e^t}{t} - 2\frac{e^{ty}}{t^2}\Big|_0^t \\
&= -2\frac{e^t}{t^2} + 2\frac{e^t}{t} + 2\frac{1}{t^2}.
\end{aligned}$$

(vi) We see the p.d.f. of X is

$$f_X(y) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b, \\ 0, & otherwise. \end{cases} \tag{2}$$

So for $t = 0$, $M_X(t) = 1$, for $t \neq 0$, we have

$$\begin{aligned}
M_X(t) &= \int_a^b \frac{e^{xt}}{b-a} dx \\
&= \int_a^b \frac{1}{b-a} d\frac{e^{xt}}{t} \\
&= \frac{e^{xt}}{t(b-a)}\Big|_a^b \\
&= \frac{e^{bt} - e^{at}}{t(b-a)}.
\end{aligned}$$

(vii) Assume $X \sim exp(\lambda)$, then the p.d.f. of X is

$$f_X(y) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise. \end{cases} \tag{3}$$

So

$$\begin{aligned} M_X(t) = & \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx \\ = & \lambda \int_0^\infty e^{-(\lambda - t)x} dx \\ = & \lambda \cdot [-\frac{e^{-(\lambda-t)x}}{\lambda - t}]|_0^\infty \\ = & \frac{\lambda}{\lambda - t}. \end{aligned}$$

And $M_X(t)$ is well-defined for $t < \lambda$. If $Y \sim \Gamma(\lambda, m)$, where m is a positive integer, we can write $Y = \sum_{k=1}^{m} X_k$, with $X_k \sim exp(\lambda)$ and X_k are independent. So the same as 2(ii), we can compute

$$M_Y(t) = \prod_{k=1}^{m} M_{X_k}(t) = [\frac{\lambda}{\lambda - t}]^m$$

And $M_Y(t)$ is well-defined for $t < \lambda$.

(viii) Assume $X \sim \Gamma(\lambda, \alpha)$, the p.d.f. of X is

$$f_X(y) = \begin{cases} \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, & x > 0 \\ 0, & otherwise. \end{cases} \tag{4}$$

So

$$\begin{aligned} M_X(t) = & \int_0^\infty e^{tx} \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx \\ = & [\frac{\lambda}{\lambda - t}]^\alpha \int_0^\infty \frac{(\lambda - t) e^{-(\lambda - t)x} \cdot [(\lambda - t)x]^{\alpha - 1}}{\Gamma(\alpha)} dx \\ = & [\frac{\lambda}{\lambda - t}]^\alpha \end{aligned}$$

The last equation is from the integral part is the p.d.f. of $\Gamma(\lambda - t, \alpha)$.And $M_X(t)$ is well-defined for $t < \lambda$.

(ix) For $Z \sim N(0,1)$, the p.d.f $f_Z(x)$ is

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

And $M_Z(t)$ is well-defined for $t \in \mathbb{R}$.

For $x \in \mathbb{R}$, so

$$\begin{aligned} M_Z(t) = & \int_{\mathbb{R}} e^{xt} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ = & e^{\frac{t^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\frac{t}{2})^2}{2}} dx \\ = & e^{\frac{t^2}{2}}. \end{aligned}$$

So for $X = \mu + \sigma Z$, by 1(a)(iv), we have

$$M_X(t) = M_{\mu+\sigma Z}(t) = e^{\mu t} \cdot M_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

And $M_X(t)$ is well-defined for $t \in \mathbb{R}$.

PROBLEM 3. Suppose that the m.g.f. of a r.v. X is given by $M_X(t) = e^{3(e^t-1)}$. What is the probability $P(X=0)$? Also, find $E(X)$ and $Var(X)$. (Hint: You do not need to do any detailed calculations. Just find what the distribution of the r.v. X is and then use the known results to answer this question.)

SOLUTION.

$$\begin{aligned} M_X(t) = & e^{3(e^t-1)} \\ = & \frac{1}{e^3}[1 + 3e^t + \frac{(3e^t)^2}{2!} + \cdots] \\ = & \frac{1}{e^3} + \frac{3}{e^3} e^t + \frac{3^2}{e^3 2!} e^{2t} + \cdots \end{aligned}$$

So compare the series form of $M_X(t) = E(e^{tX}) = \sum_{k \geq 0} P(X=k) e^{tk}$, we konw $P(X=0)$ is the const term of $M_X(t)$, which is

$$P(X=0) = \frac{1}{e^3}.$$

And we also have

$$P(X=k)=\frac{3^k}{e^3\cdot k!}.$$

So

$$\begin{aligned}
E(X)=&\sum_{k=0}^{\infty}k\frac{3^k}{e^3\cdot k!}\\
=&\sum_{k=1}^{\infty}k\frac{3^k}{e^3\cdot k!}\\
=&3\sum_{k=1}^{\infty}\frac{3^{k-1}}{e^3\cdot (k-1)!}\\
=&3\sum_{k=0}^{\infty}k\frac{3^k}{e^3\cdot k!}\\
=&3.
\end{aligned}$$

$$\begin{aligned}
E(X^2)=&\sum_{k=0}^{\infty}k^2\frac{3^k}{e^3\cdot k!}\\
=&\sum_{k=1}^{\infty}k^2\frac{3^k}{e^3\cdot k!}\\
=&3\sum_{k=1}^{\infty}k\frac{3^{k-1}}{e^3\cdot (k-1)!}\\
=&3\sum_{k=1}^{\infty}(k-1)\frac{3^{k-1}}{e^3\cdot (k-1)!}+3\sum_{k=1}^{\infty}\frac{3^{k-1}}{e^3\cdot (k-1)!}\\
=&3^2\sum_{k=2}^{\infty}\frac{3^{k-2}}{e^3\cdot (k-2)!}+3\\
=&9\sum_{k=0}^{\infty}k\frac{3^k}{e^3\cdot k!}+3\\
=&12
\end{aligned}$$

So $Var(X)=E(X^2)-[E(X)]^2=3$.

Or we can use prop of m.g.f. and get

$$E(X)=M_X^{(1)}(0)=3e^xe^{3(e^t-1)}\big|_{x=0}=3,$$

$$E(X^2)=M_X^{(2)}(0)=[3e^xe^{3(e^t-1)}+9e^{2x}e^{3(e^t-1)}]\big|_{x=0}=12.$$