Step-1

To find the closest straight line to the parabola $y = x^2$ over $-1 \le x \le 1$

Let the closest straight line to the parabola $y = x^2$ over $-1 \le x \le 1$ be $y = C + Dx$

Then by the least squares, the equation $A^T A\hat{x} = A^T b$ is

$$\begin{bmatrix} (1,1) & (1,x) \\ (x,1) & (x,x) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} (1,x^2) \\ (x,x^2) \end{bmatrix}$$ â€¦â€¦ (1)

Step-2

Now, calculate inner products that are used in formula;

$$\begin{aligned} (1,1) &= \int_{-1}^{1} 1.1\,dx \\ &= [x]_{-1}^{1} \\ &= 1+1 \\ &= 2 \end{aligned}$$

Step-3

Again, calculate other inner product

$$\begin{aligned} (x,1) &= \int_{-1}^{1} x.1\,dx \\ &= \left[\frac{x^2}{2}\right]_{-1}^{1} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

Similarly $(1,x) = 0$

$$(x,x)=\int_{-1}^{1} x^2 dx$$
$$=\left[\frac{x^3}{3}\right]_{-1}^{1}$$
$$=\frac{1}{3}+\frac{1}{3}$$
$$=\frac{2}{3}$$

Step-4

Now, calculate inner product on R.H.S;

$$(1,x^2)=\int_{-1}^{1} 1.x^2 dx$$
$$=\left[\frac{x^3}{3}\right]_{-1}^{1}$$
$$=\frac{1}{3}+\frac{1}{3}$$
$$=\frac{2}{3}$$

$$(x,x^2)=\int_{-1}^{1} x^3 dx$$
$$=\left[\frac{x^4}{4}\right]_{-1}^{1}$$
$$=\frac{1}{4}-\frac{1}{4}$$
$$=0$$

Step-5

Substitute inner product values obtained above in (1), and get;

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}=\begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

This implies;

$$2C = \frac{2}{3}$$
$$\frac{2}{3}D = 0$$

Thus,

$$C = \frac{1}{3}$$
$$D = 0$$

Step-6

Therefore,

$$\begin{aligned} y &= C + Dx \\ &= \frac{1}{3} + 0x \\ &= \frac{1}{3} \end{aligned}$$

Hence the closest straight line to the parabola is $\boxed{y = \frac{1}{3}}$