

Step-1

Given that $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

We need to compute the eigenvalues and the eigenvectors of the above two matrices.

Step-2

Now $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

Step-3

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} \\ &= (-\lambda)(3-\lambda) - 4 \\ &= -3\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 3\lambda - 4 \end{aligned}$$

Step-4

we know that $|A - \lambda I| = 0$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda^2 - 4\lambda + \lambda - 4 = 0$$

$$\lambda(\lambda - 4) + 1(\lambda - 4) = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, -1$$

$$\lambda = 4, -1$$

Hence the eigenvalues of A are 4, -1

Step-5

Case(i) Let $\lambda = 4$

Eigenvectors X corresponding to the eigenvalue 4 are given by

$$(A - 4I)X = 0$$

$$\text{That is } \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-6

$$\text{By } 2R_2 + R_1 = R_2$$

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4x_1 + 2x_2 = 0$$

$$\text{Let } x_1 = k (\text{say})$$

$$\text{Therefore } x_2 = 2k$$

Therefore eigenvectors corresponding to eigenvalue 4 are given by $k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where k is a non-zero parameter.

Step-7

Case(ii) Let $\lambda = -1$

Eigenvectors X corresponding to the eigenvalue -1 are given by

$$(A + I)X = 0$$

$$\text{That is } \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-8

$$\text{By } R_2 - 2R_1 = R_2$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 = 0$$

Step-9

$$\text{Let } x_1 = k (\text{say})$$

$$\text{Therefore } 2x_2 = -k$$

$$x_2 = -k/2$$

Therefore eigenvectors corresponding to eigenvalue -1 are given by $k \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ where k

is a non-zero parameter

$$\text{Now } \boxed{A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}}$$

$$(A^{-1}) - \lambda I = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 - \lambda & 1/2 \\ 1/2 & -\lambda \end{bmatrix}$$

$$= (-3/4 - \lambda)(-\lambda) - 1/4$$

$$= \frac{3}{4}\lambda + \lambda^2 - \frac{1}{4}$$

$$= \lambda^2 + \frac{3}{4}\lambda - \frac{1}{4}$$

Step-10

We know that $|A - \lambda I| = 0$

$$\lambda^2 + \frac{3}{4}\lambda - \frac{1}{4} = 0$$

$$4\lambda^2 + 3\lambda - 1 = 0$$

Step-11

Compare this equation with $ax^2 + bx + c = 0$

$$\lambda = \frac{-3 \pm \sqrt{9+16}}{8}$$

$$= \frac{-3 \pm \sqrt{25}}{8}$$

$$= \frac{-3 \pm 5}{8}$$

$$= \frac{-3+5}{8}, \frac{-3-5}{8}$$

$$= \frac{1}{4}, -1$$

Step-12

Case(i) Let $\lambda = \frac{1}{4}$

Eigenvectors X corresponding to the eigenvalue $\frac{1}{4}$ are given by

$$\left((A^{-1}) - \frac{1}{4}I \right) X = 0$$

$$\text{That is } \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-13

$$\text{By } R_2 + \frac{R_1}{2} = R_2$$

$$\begin{bmatrix} -1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + \frac{1}{2}x_2 = 0$$

Step-14

Let $x_1 = k$ (say)

$$-k + \frac{1}{2}x_2 = 0$$

Therefore $x_2 = 2k$

Therefore eigenvectors corresponding to eigenvalue $\frac{1}{4}$ are given by $k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where k

is a non-zero parameter

Step-15

Case(ii) Let $\lambda = -1$

Eigenvectors X corresponding to the eigenvalue -1 are given by

$$\left((A^{-1}) + I \right) X = 0$$

$$\text{That is } \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-16

$$\text{By } R_2 / 2 - R_1 = R_2$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{4}x_1 + \frac{1}{2}x_2 = 0$$

$$\Rightarrow x_1 + 2x_2 = 0$$

Step-17

Let $x_1 = k$ (say)

Therefore $x_2 = -k / 2$

Therefore eigenvectors corresponding to eigenvalue -1 are given by $k \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ where k is a non-zero parameter.

A^{-1} has the same eigenvectors as A. When A has eigenvalues λ_1 and λ_2 , its inverse has eigenvalues $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$.