Step-1

Given matrix is $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

We have to compute $\|A\|=\lambda_{\max}$ and $\|A^{-1}\|=\dfrac{1}{\lambda_{\min}}$

Step-2

We know that

1) For a symmetric matrix A, $\|A\|=\lambda_{\max}$ and $\|A^{-1}\|=\dfrac{1}{\lambda_{\min}}$

2) The conditional number of the A is $c=\|A\|\|A^{-1}\|$

3) The system $Ax = b$ and $A^{-1}\delta b=\delta x$ gives $\|b\|\le\|A\|\|x\|$ and $\|\delta x\|\le\|A^{-1}\|\|\delta b\|$

Step-3

First we find the eigenvalues of the given matrix.

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$\Rightarrow\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}=0$$
$$\Rightarrow(2-\lambda)(2-\lambda)-1=0$$
$$\Rightarrow\lambda^2-4\lambda+3=0$$
$$\Rightarrow(\lambda-1)(\lambda-3)=0$$
$$\Rightarrow\lambda=1,3$$

So, the eigenvalues of the given matrix are $\lambda_{\min}=1,$ and $\lambda_{\max}=3$.

Step-4

Since both the eigenvalues are positive, we follow that the given matrix is positive definite.

By (1), we get $\|A\|=\lambda_{\max}=3$

And

$$\left\|A^{-1}\right\| = \frac{1}{\lambda_{\min}}$$
$$= \frac{1}{1}$$
$$= 1$$

Therefore, $\boxed{\|A\| = 3 \text{ and } \left\|A^{-1}\right\| = 1}$

Step-5

By (2), we have

$$c = \|A\|\left\|A^{-1}\right\|$$
$$\Rightarrow c = 3(1)$$
$$= 3$$

Therefore, $\boxed{c = 3}$

Step-6

By (3), we get $\|b\| \le 3\|x\|$ and $\|\delta x\| \le 1 \times \|\delta b\|$

From these, we can write $\frac{\|\delta x\|}{\|x\|} \le \frac{3}{1} \times \frac{\|\delta b\|}{\|b\|}$

Or, precisely, we can write $\frac{\|\delta x\|}{\|x\|} \le c\frac{\|\delta b\|}{\|b\|}$