Step-1

The objective is to find the condition on b_1, b_2, b_3 for the following system to be solvable:

$$\begin{aligned} x+2y-2z&=b_1 \\ 2x+5y-4z&=b_2 \\ 4x+9y-8z&=b_3 \end{aligned}$$

Rewrite the system in $AX=b$ form:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Step-2

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right].$$

Apply row reduced echelon form to the augmented matrix.

Add -2 times row 1 to row 2.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 4 & 9 & -8 & b_3 \end{array}\right].$$

Step-3

Add -4 times row 1 to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 0 & 1 & 0 & b_3-4b_1 \end{array}\right].$$

Add -1 times row 2 to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-2b_1-b_2 \end{array}\right].$$

The system is consistent if $-2b_1 - b_2 + b_3 = 0$.

Thus, the condition for the system to be solvable is $\boxed{-2b_1 - b_2 + b_3 = 0}$.

Step-4

Now determine the solutions for the system by using the obtained condition.

Take $b_1 = 1, b_2 = 1,$ then $b_3 = 3$.

The vector b is:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

The system in matrix form is:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Step-5

The augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 5 & -4 & 1 \\ 4 & 9 & -8 & 3 \end{array}\right].$$

Apply row reduced echelon to get,

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Step-6

From the matrix observe that z is a free variable, and the system can be written as:

$$\begin{aligned} x - 2z &= 3 \\ y &= -1 \end{aligned}$$

Take $z = t$ then $x = 3 + 2t$.

On substitution,

$$X = \begin{bmatrix} 3+2t \\ -1 \\ t \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the solution is $\boxed{X = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$, here t is an arbitrary constant.