Step-1

$A=\begin{bmatrix}\varepsilon & 1\\ 1 & 1\end{bmatrix}, \varepsilon=10^{-3},10^{-6},10^{-9},10^{-12},10^{-5}$

(1) $A=\begin{bmatrix}10^{-3} & 1\\ 1 & 1\end{bmatrix}$

$R_2 \to R_2 - 10^3 R_1 \Rightarrow \begin{bmatrix}10^{-3} & 1\\ 0 & -999\end{bmatrix}$

The resultant is the upper triangular matrix *U*.

Applying the same elementary operation on the identity matrix, we get

$E=\begin{bmatrix}1 & 0\\ -1000 & 1\end{bmatrix}$

We see that $EA = U$.

So, $A = E^{-1}U$

The inverse of an elementary matrix is elementary.

The inverse of *E* is obtained by applying $R_2 \to R_2 + 10^3 R_1$ on the identity matrix.

i.e., $E^{-1}=\begin{bmatrix}1 & 0\\ 1000 & 1\end{bmatrix}=L$ the lower triangular matrix such that $A=LU$

Step-2

(2) Repeating the above procedure, when $A=\begin{bmatrix}10^{-6} & 1\\ 1 & 1\end{bmatrix}$, we can write it as the product of lower and upper triangular matrices $LU=\begin{bmatrix}10^{-6} & 1\\ 0 & -10^{6}+1\end{bmatrix}\begin{bmatrix}1 & 0\\ 10^{6} & 1\end{bmatrix}$

(3) When $A=\begin{bmatrix}10^{-9} & 1\\ 1 & 1\end{bmatrix}$, we get $LU=\begin{bmatrix}10^{-9} & 1\\ 0 & -10^{9}+1\end{bmatrix}\begin{bmatrix}1 & 0\\ 10^{9} & 1\end{bmatrix}$

(4) When $A=\begin{bmatrix}10^{-12} & 1\\ 1 & 1\end{bmatrix}$, we get $LU=\begin{bmatrix}10^{-12} & 1\\ 0 & -10^{12}+1\end{bmatrix}\begin{bmatrix}1 & 0\\ 10^{12} & 1\end{bmatrix}$

(5) When $A=\begin{bmatrix}10^{-15} & 1\\ 1 & 1\end{bmatrix}$, we get $LU=\begin{bmatrix}10^{-15} & 1\\ 0 & -10^{15}+1\end{bmatrix}\begin{bmatrix}1 & 0\\ 10^{15} & 1\end{bmatrix}$

Step-3

$Ax = b$ is given by $\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+\varepsilon \\ 1 \end{bmatrix}$

This can conveniently be written as $x = A^{-1}b$

$A^{-1} = \frac{1}{\varepsilon - 1}\begin{bmatrix} 1 & -1 \\ -1 & \varepsilon \end{bmatrix}$ and so, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\varepsilon - 1}\begin{bmatrix} 1 & -1 \\ -1 & \varepsilon \end{bmatrix}\begin{bmatrix} 1+\varepsilon \\ 1 \end{bmatrix}$

$= \frac{1}{\varepsilon - 1}\begin{bmatrix} \varepsilon \\ -1 \end{bmatrix}$

$= \begin{bmatrix} \frac{\varepsilon}{\varepsilon - 1} \\ \frac{1}{1-\varepsilon} \end{bmatrix}$ â€¦â€¦ (6)

Step-4

Using $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-5}$ in (6), we get the solution of the system $Ax = b$ as

(7) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{10^{-3}}{10^{-3}-1} \\ \frac{1}{1-10^{-3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{-999} \\ \frac{1000}{999} \end{bmatrix} = \begin{bmatrix} -0.001 \\ 1.001 \end{bmatrix}$

(8) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{10^{-6}}{10^{-6}-1} \\ \frac{1}{1-10^{-6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{-999999} \\ \frac{1000}{999999} \end{bmatrix} = \begin{bmatrix} -10^{-6} \\ 1+10^{-6} \end{bmatrix}$

(9) Similarly, when $\varepsilon = 10^{-9}$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10^{-9} \\ 1+10^{-9} \end{bmatrix}$

(10) When $\varepsilon = 10^{-12}$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10^{-12} \\ 1+10^{-12} \end{bmatrix}$

(11) When $\varepsilon = 10^{-15}$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10^{-15} \\ 1+10^{-15} \end{bmatrix}$

Step-5

The true solution of the system is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so, the error in each case is

$\varepsilon =$	10^{-3}	10^{-6}	10^{-9}	10^{-12}	10^{-15}
$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} =$	$\begin{bmatrix} -1-10^{-3} \\ 10^{-3} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-6} \\ 10^{-6} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-9} \\ 10^{-9} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-12} \\ 10^{-12} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-15} \\ 10^{-15} \end{bmatrix}$