Step-1

Consider the second order equation,

$\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix} u$ with $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The objective is to find the solution of it.

Step-2

By comparing the equation $\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix} u$ with $\frac{d^2u}{dt^2} = Au$ get $A = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix}$

Find eigenvalues and eigenvectors of matrix A.

Characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -5-\lambda & -1 \\ -1 & -5-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-5-\lambda) - (-1)\cdot(-1) = 0$$

$$(-5-\lambda)^2 - 1 = 0$$

$$\lambda^2 + 10\lambda + 25 - 1 = 0$$

$$\lambda^2 + 10\lambda + 24 = 0$$

$$\lambda^2 + 6\lambda + 4\lambda + 24 = 0$$

$$\lambda(\lambda+6) + 4(\lambda+6) = 0$$

$$(\lambda+4)(\lambda+6) = 0$$

$$\lambda = -4, -6$$

So, eigenvalues of matrix A are $\lambda = -4, -6$

Step-3

Let $\mathbf{x}_1 = (m_1, m_2)^T$ be the eigenvector corresponding to $\lambda_1 = -4$. Then,

$$
\begin{aligned}
&(A-(-4)\cdot I)\mathbf{x}_1 = 0\\
&\begin{pmatrix}-5+4 & -1\\ -1 & -5+4\end{pmatrix}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\\
&\underbrace{\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}}_{\mathbf{P}}\begin{pmatrix}m_1\\ m_2\end{pmatrix}=\underbrace{\begin{pmatrix}0\\0\end{pmatrix}}_{\mathbf{0}}
\end{aligned}
$$

Step-4

Augmented matrix associated with the above notation is,

$$
\begin{aligned}
[\mathbf{P} \mid \mathbf{0}] &= \left[\begin{array}{cc|c}-1 & -1 & 0\\ -1 & -1 & 0\end{array}\right]\\
&R_2 \to R_2 - R_1\\
&\approx \left[\begin{array}{cc|c}-1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\\
&R_1 \to (-1)R_1\\
&\approx \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]
\end{aligned}
$$

From the last matrix, get the equation,

$$m_1 + m_2 = 0$$

Note that we have two unknowns (m_1, m_2) and one equation. So, there must be 2-1=1 free variable.

Step-5

Let $m_2 = s,\ s \in \mathbb{R}$

Then from the equation $m_1 + m_2 = 0$ get $m_1 = -s$

Therefore, eigenvector corresponding to the eigenvalue $\lambda_1 = -4$ is,

$$
\begin{aligned}
\mathbf{x}_1 &= (m_1, m_2)^{\mathrm{T}}\\
&= \left\{\begin{pmatrix}-s\\ s\end{pmatrix} : s \in \mathbb{R}\right\}\\
&= \left\{s\begin{pmatrix}-1\\ 1\end{pmatrix} : s \in \mathbb{R}\right\}
\end{aligned}
$$

Step-6

Let $\mathbf{x}_2 = (n_1, n_2)^{\mathrm{T}}$ be the eigenvector corresponding to $\lambda_2 = -6$. Then,

$$\left(A - (-6) \cdot I\right) \mathbf{x}_2 = 0$$
$$\begin{pmatrix} -5+6 & -1 \\ -1 & -5+6 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\mathbf{Q}} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{0}}$$

Augmented matrix associated with the above notation is,

$$[\mathbf{Q} \mid \mathbf{0}] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$
$$R_2 \to R_2 + R_1$$
$$\approx \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

From the last matrix, get the equation,

$$n_1 - n_2 = 0$$

Note that we have two unknowns (n_1, n_2) and one equation. So, there must be 2-1=1 free variable.

Step-7

Let $n_2 = t,\ t \in \mathbb{R}$

Then from the equation $n_1 - n_2 = 0$ get $n_1 = t$

Therefore, eigenvector corresponding to the eigenvalue $\lambda_2 = -6$ is,

$$\mathbf{x}_2 = (n_1, n_2)^{\mathrm{T}}$$
$$= \left\{ \begin{pmatrix} t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$
$$= \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Step-8

Hence, eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix}$ are,

$\lambda_1 = -4$; $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -6$; $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Step-9

If the matrix A has negative eigenvalues $\lambda_1, \ldots, \lambda_n$ and if $\omega_j = \sqrt{-\lambda_j}$, then general solution of $\frac{d^2u}{dt^2} = Au$ is,

$$u(t) = (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t)\mathbf{x}_1 + \cdots + (a_n \cos \omega_n t + b_n \sin \omega_n t)\mathbf{x}_n$$

Step-10

Substitute -4 for λ_1 in the equation $\omega_1 = \sqrt{-\lambda_1}$

$$\begin{aligned} \omega_1 &= \sqrt{-(-4)} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Substitute -6 for λ_2 in the equation $\omega_2 = \sqrt{-\lambda_2}$

$$\begin{aligned} \omega_2 &= \sqrt{-(-6)} \\ &= \sqrt{6} \end{aligned}$$

Therefore, the frequencies are $\omega_1 = 2$ and $\omega_2 = \sqrt{6}$.

Step-11

Put the values $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\omega_1 = 2$, $\omega_2 = \sqrt{6}$ in the following equation to get the general solution.

$$\begin{aligned} u(t) &= (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t)\mathbf{x}_1 + (a_2 \cos \omega_2 t + b_2 \sin \omega_2 t)\mathbf{x}_2 \\ &= (a_1 \cos 2t + b_1 \sin 2t)\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \left(a_2 \cos \sqrt{6}t + b_2 \sin \sqrt{6}t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Step-12

Apply initial conditions $u(0)=\begin{bmatrix}1\\0\end{bmatrix}$ and $u'(0)=\begin{bmatrix}0\\0\end{bmatrix}$ to obtain the values of $a_1, b_1, a_2,$ and b_2 in the solution $u(t)$.

As $u(t)=\left(a_1\cos 2t+b_1\sin 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cos\sqrt{6}t+b_2\sin\sqrt{6}t\right)\begin{bmatrix}1\\1\end{bmatrix}$ get,

$$u'(t)=\left(-2a_1\sin 2t+2b_1\cos 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\sin\sqrt{6}t+\sqrt{6}b_2\cos\sqrt{6}t\right)\begin{bmatrix}1\\1\end{bmatrix}$$

Step-13

From the condition $u(0)=\begin{bmatrix}1\\0\end{bmatrix}$ get,

$$\begin{bmatrix}1\\0\end{bmatrix}=\left(a_1\cos 2(0)+b_1\sin 2(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cos\sqrt{6}(0)+b_2\sin\sqrt{6}(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\0\end{bmatrix}=\left(a_1\cos(0)+b_1\sin(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cos(0)+b_2\sin(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\0\end{bmatrix}=\left(a_1\cdot 1+b_1\cdot 0\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cdot 1+b_2\cdot 0\right)\begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-a_1+a_2\\a_1+a_2\end{bmatrix}$$

Equating the corresponding positions get,

$$\begin{aligned}-a_1+a_2&=1\\a_1+a_2&=0\end{aligned}$$

By solving the system, get, $a_1=-\frac{1}{2}, a_2=\frac{1}{2}$

Step-14

From the condition $u'(0)=\begin{bmatrix}0\\0\end{bmatrix}$ get,

$$\begin{bmatrix}0\\0\end{bmatrix}=\left(-2a_1\sin 2(0)+2b_1\cos 2(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\sin\sqrt{6}(0)+\sqrt{6}b_2\cos\sqrt{6}(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\left(-2a_1\sin(0)+2b_1\cos(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\sin(0)+\sqrt{6}b_2\cos(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\left(-2a_1\cdot 0+2b_1\cdot 1\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\cdot 0+\sqrt{6}b_2\cdot 1\right)\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}-2b_1+\sqrt{6}b_2\\2b_1+\sqrt{6}b_2\end{bmatrix}$$

Equating the corresponding positions get,

$$-2b_1+\sqrt{6}b_2=0$$
$$2b_1+\sqrt{6}b_2=0$$

By solving the system, get, $b_1=0, b_2=0$

Step-15

Finally, solution of the given problem is,

$$\begin{aligned}u(t)&=\left(\left(-\frac{1}{2}\right)\cos 2t+(0)\sin 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(\left(\frac{1}{2}\right)\cos\sqrt{6}t+(0)\sin\sqrt{6}t\right)\begin{bmatrix}1\\1\end{bmatrix}\\&=\left(-\frac{1}{2}\cos 2t+0\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(\frac{1}{2}\cos\sqrt{6}t+0\right)\begin{bmatrix}1\\1\end{bmatrix}\\&=\boxed{\frac{1}{2}\cos 2t\begin{bmatrix}1\\-1\end{bmatrix}+\frac{1}{2}\cos\sqrt{6}t\begin{bmatrix}1\\1\end{bmatrix}}\end{aligned}$$