MA327 Homework 7

1. Compute the Christoffel symbols for an open set of the plane: (a) In cartesian coordinates. (b) In polar coordinates. Use the Gauss formula to compute K in both cases.

2. Justify why the surfaces below are not pairwise locally isometric: (a) Sphere. (b) Cylinders. (c) Saddle: $z = x^2 - y^2$.

3. Show that if a regular curve C (with nowhere vanishing curvature) in a regular surface is both a line of curvature and a geodesic, then C is a plane curve. (Hint: Consider the torsion of C.)

4. Show that the straight lines are the only geodesics of a plane.

5. Let v and w be vector fields along a curve $\alpha : I \to S$. Prove that

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \langle \frac{Dv}{dt}, w(t)\rangle + \langle \frac{Dw}{dt}, v(t)\rangle.$$

6. Prove that a curve $C \subset S$ is both an asymptotic curve and a geodesic if and only if C is a (segment of a) straight line.

7. Consider the torus of revolution generated by rotating the circle

$$(x-a)^2 + z^2 = r^2, y = 0,$$

about the z axis ($a > r > 0$). The parallels generated by the points $(a+r, 0), (a-r, 0), (a, r)$ are called the maximum parallel, the minimum parallel, and the upper parallel, respectively. Check which of these parallels is (a) a geodesic, (b) an asymptotic curve, (c) a line of curvature.

8. Compute the geodesic curvature of the upper parallel of the torus in Question 7.

9. Show that the geodesic curvature of an oriented curve $C \subset S$ at a point $p \in C$ is equal to the curvature of the plane curve obtained by projecting C onto the tangent plane T_pS along the normal to the surface at p.

10. Let S be an oriented regular surface and let $\alpha : I \to S$ be a curve parametrized by arc length. At the point $p = \alpha(s)$ consider the three unit vector (the Darboux trihedron) $T(s) = \alpha'(s), N(s) =$ the normal vector to S at p, $V(s) = N(s) \wedge T(s)$. Show that

$$\frac{dT}{ds} = 0 + aV + bN,$$
$$\frac{dV}{ds} = -aT + 0 + cN,$$
$$\frac{dN}{ds} = -bT - cV + 0,$$

where $a = a(s), b = b(s), c = c(s), s \in I$. The above formulas are the analogue of Frenet's formulas for the trihedron T, V, N. To establish the geometrical meaning of the coefficients, prove that

(a) $c = -\langle dN/ds, V\rangle$. Conclude from this that $\alpha(I) \subset S$ is a line of curvature if and only if $c \equiv 0$.

(b) b is the normal curvature of $\alpha(I) \subset S$ at p.

(c) a is the geodesic curvature of $\alpha(I) \subset S$ at p.

11. Let $S \subset \mathbb{R}^3$ be a regular, compact, connected, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

12. Compute the Euler-Poincaré characteristic of an ellipsoid.