Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1. Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (1) If $T : V \to W$ is a linear map and $v_1, v_2, \cdots, v_n$ are vectors in V, and $Tv_1, Tv_2, \cdots, Tv_n$ is linearly independent, then $v_1, v_2, \cdots, v_n$ is linearly independent.

   (2) Suppose that $p_0, p_1, \cdots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(-1) = 0$ for all j, then $p_0, p_1, \cdots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$.

   (3) Let T be a linear operator defined on a 3 dimensional real vector space, then T always has an eigenvalue.

   (4) If v_1, v_2, and v_3 are eigenvectors of T such that $v_3 = v_1 + v_2$, then all three vectors have the same eigenvalue.

   (5) Let V be a complex vector space and U_1, U_2, U_3 be its subspaces with intersection $U_1 \cap U_2 \cap U_3 = \{\mathbf{0}\}$. Then $U_1 + U_2 + U_3$ is a direct sum.

2. If U_1 and U_2 are subspaces of a finite-dimensional vector space. Prove that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

3. Define $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

   for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

4. Define V^4 by

$$V^4 = V \times V \times V \times V.$$

   Prove that V^4 and $\mathcal{L}(\mathbb{F}^4, V)$ are isomorphic vector spaces.

5. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

6. Consider the standard basis $1, x, x^2$ of $V = \mathcal{P}_2(\mathbb{R})$. Let $\varphi_1, \varphi_2, \varphi_3$ be corresponding the dual basis of $\mathcal{P}_2(\mathbb{R})'$. Let $\varphi : V \to \mathbb{R}$ be the linear function

$$f(x) \mapsto f(2) + \int_0^1 f(x)dx.$$

Find the coefficients a, b, c for which $\varphi = a\varphi_1 + b\varphi_2 + c\varphi_3$.

7. Let V be a vector space that is generated by $v_1, v_2, \cdots, v_n$, and let $u_1, u_2, \cdots, u_m$ be a linearly independent list of vectors in V. Show that $m \leq n$ and there exists a subset H of $v_1, v_2, \cdots, v_n$ containing exactly $n - m$ vectors such that $H \cup \{u_1, u_2, \cdots, u_m\}$ generates V.

8. Let V be a finite-dimensional complex vector space and let S and T be linear operators on V such that $ST = TS$. Prove that if S and T can each be diagonalized, then there is a basis for V which simultaneously diagonalizes S and T.