MA327 Homework 2

Due on 18th March

1. Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a regular surface, and find parametrizations whose coordinate neighborhoods cover it.

2. Is the set $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 \leq 1\}$ a regular surface? Is the set $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 < 1\}$ a regular surface?

3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2-z^2 = 0\}$, is not a a regular surface.

4. Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value of f and yet that $f^{-1}(0)$ is a regular regular.

5. Let $P = \{(x, y, z) \in \mathbb{R}^3 \mid x = y\}$ (a plane) and let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\mathbf{x}(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in \mathbb{R}^2 \mid u > v\}$. Clearly, $\mathbf{x}(U) \subset P$. Is $\mathbf{x}$ a parametrization of P?

6. Let $f(x, y, z) = (x + y + z - 1)^2$. (a) Locate the critical points and critical values of f. (b) For what values of c is the set $f(x, y, z) = c$ is a regular surface? (c) Answer the questions of parts (a) and (b) for the function $f(x, y, z) = xyz^2$.

7. Let $\mathbf{x}(u, v)$ be as in the definition of regular surfaces. Verify that $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one if and only if

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

8. Show that $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), \quad a, b, c \neq 0,$$

where $0 < u < \pi, 0 < v < 2\pi$, is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u = \textit{constant}$ on the ellipsoid.

9. One way to define a system of coordinates for the sphere S^2, given by $x^2 + y^2 + (z - 1)^2 = 1$, is to consider the so-called **stereographic projection** $\pi : S^2 - \{N\} \to \mathbb{R}^2$ which carries a point $p = (x, y, z)$ of the sphere S^2 minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy plane with the straight line which connects N

to p (see Figure-1 in the file "HW2-Figures"). Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 - \{N\}$ and $(u, v) \in xy$ plane.

(a) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2 - \{N\}$ is given by

$$\begin{cases} x = \frac{4u}{u^2+v^2+4}, \\ y = \frac{4v}{u^2+v^2+4}, \\ z = \frac{2(u^2+v^2)}{u^2+v^2+4}. \end{cases}$$

(b) Show that it is possible, using stereographic projection, to cover the sphere two coordinate neighborhoods.

10. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A : S^2 \to S^2$ be the antipodal map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

11. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

12. Construct a diffeomorphism between the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the sphere $x^2 + y^2 + z^2 = 1$.

13. Let $S \subset \mathbb{R}^3$ be a regular surface, and let $d : S \to \mathbb{R}$ be given by $d(p) = |p - p_0|$, where $p \in S, p_0 \in \mathbb{R}^3, p_0 \notin S$, that is d is the distance from p to a fixed point p_0 not in S. Prove that d is differentiable.

14. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

15. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and $H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$. Denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and south poles of S^2, respectively, and let $F : S^2 - \{N\} - \{S\} \to H$ be defined as follows: For each $p \in S^2 - \{N\} - \{S\}$ let the perpendicular from p to the z axis meet $0z$ at q. Consider the half-line l starting at q and containing p. Then $F(p) = l \cap H$ (see Figure-2 in the file "HW2-Figures"). Prove that F is differentiable.