Step-1

Since A is assumed to be uv^{T} and that the rank of A is 1, it is clear that A is not the zero matrix.

Therefore, neither u not v^{T} can be a zero matrix.

Therefore, the requirement is that the matrix $v^{\mathrm{T}}u$ be the zero matrix.

Step-2

Let $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Therefore,

$$\begin{aligned} A &= uv^{\mathrm{T}} \\ &= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} (b_1, b_2, \ldots, b_n) \\ &= \begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 & a_nb_2 & \ldots & a_nb_n \end{bmatrix} \end{aligned}$$

Step-3

Similarly, consider

$$\begin{aligned} v^{\mathrm{T}}u &= (b_1, b_2, \ldots, b_n) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\ &= [b_1a_1 + b_2a_2 + \ldots + b_na_n] \end{aligned}$$

Thus, if $b_1a_1 + b_2a_2 + \ldots + b_na_n = 0$, the matrix A will have the rank 1 and $A^2 = 0$.

Step-4

For example, let $a_1 = 2$, $a_2 = 3$, and $a_3 = -4$. Let $b_1 = 1$, $b_2 = 6$, $b_3 = 5$.

Then

$$\begin{aligned} b_1a_1 + b_2a_2 + b_3a_3 &= 1\cdot 2 + 6\cdot 3 + 5\cdot(-4) \\ &= 2 + 18 - 20 \\ &= 0 \end{aligned}$$

Consider

$$\begin{aligned} A &= \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}(2,3,-4) \\ &= \begin{bmatrix} 2 & 3 & -4 \\ 12 & 18 & -24 \\ 10 & 15 & -20 \end{bmatrix} \end{aligned}$$

Step-5

Note that there is only one independent row in the matrix A. Therefore, rank of A is 1. As per the above discussion, A^2 should be zero.

$$\begin{aligned} A^2 &= \begin{bmatrix} 2 & 3 & -4 \\ 12 & 18 & -24 \\ 10 & 15 & -20 \end{bmatrix}\begin{bmatrix} 2 & 3 & -4 \\ 12 & 18 & -24 \\ 10 & 15 & -20 \end{bmatrix} \\ &= \begin{bmatrix} 4+36-40 & 6+54-60 & -8-72+80 \\ 24+216-240 & 36+324-360 & -48-432+480 \\ 20+180-200 & 30+270-300 & -40-360+400 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$