

Step-1

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given that

We have to find the weighted least-squares solution \hat{x}_w to $Ax = b$.

Step-2

We know that the least squares solution to $WAx = Wb$ is \hat{x}_w .

We know that the weighted normal equations are obtained by $A^T W^T W A \hat{x}_w = A^T W^T W b$.

Now

$$\begin{aligned} A^T W^T W A \hat{x}_w &= A^T W^T W b \\ \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x}_w &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x}_w &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x}_w &= \begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

Step-3

Applying $R_2 \rightarrow 2R_2 - R_1$, we get

$$\begin{aligned} \begin{bmatrix} 6 & 3 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ \Rightarrow 7x_2 = 4 \text{ and } 6x_1 + 3x_2 &= 2 \end{aligned}$$

$$\Rightarrow x_2 = \frac{4}{7} \text{ and } 6x_1 = 2 - 3x_2 = \frac{2}{7}$$

$$\Rightarrow x_2 = \frac{4}{7} \text{ and } x_1 = \frac{1}{21}$$

$$\hat{x}_w = \begin{bmatrix} \frac{1}{21} \\ \frac{4}{7} \end{bmatrix}$$

Hence the weighted least-squares solution to $Ax = b$ is \hat{x}_w .

Step-4

We have to check that the projection $A\hat{x}_w$ is perpendicular to the error $b - A\hat{x}_w$.

Now

$$\begin{aligned} A\hat{x}_w &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{21} \\ \frac{4}{7} \end{bmatrix} \\ &= \begin{bmatrix} 1\left(\frac{1}{21}\right) + 0\left(\frac{4}{7}\right) \\ 1\left(\frac{1}{21}\right) + 1\left(\frac{4}{7}\right) \\ 1\left(\frac{1}{21}\right) + 2\left(\frac{4}{7}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \end{aligned}$$

Step-5

And

$$b - A\hat{x}_w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

Now

$$A\hat{x}_w \cdot \left[W^T W (b - A\hat{x}_w) \right] = \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

Step-6

Continuation to the above

$$= \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

$$= \frac{1}{21^2} [-4 + 104 - 100]$$

$$= 0$$

Hence $A\hat{x}_w$ is perpendicular to $W^T W (b - A\hat{x}_w)$.

