MA323 Topology Midterm Exam

2:00–3:50 pm

November 15, 2022

1. (20 points) List all the possible topologies on the set with 3 points, up to homeomorphism. For each, identify whether it is
   (i) T_1,
   (ii) T_2,
   (iii) connected.

2. (15 points) Identify the values of n for which the following statement is true:

   > Suppose X is a space and $A_1, A_2, \ldots, A_n \subset X$ are connected subspaces such that $\bigcup A_i = X$ and $A_i \cap A_j \neq \emptyset$ for all i, j. Then X is connected.

3. (10 points) Give an example of a metric space X with a closed, bounded subset $K \subset X$ which is not compact.

4. (15 points) Suppose X is compact, and that $X = \bigcup A_i$ for some subsets A_i. Suppose that for every point p there exists an i such that A_i is a *neighborhood* of p (but A_i is not necessarily open). Show that this cover of X has a finite subcover.

5. (20 points) Suppose X is a compact metric space and $f: X \to X$ is a function that *strictly decreases* distance: $d(f(x), f(y)) < d(x, y)$ for any $x \neq y$. Given any $x_0 \in X$, we can inductively define a sequence $\{x_n\}$ by $x_{n+1} = f(x_n)$. Show that this sequence has a limit x, and that x is the unique point of X satisfying $f(x) = x$. (Hint: Start by showing that it has a convergent subsequence.)

6. (20 points) Suppose X and Y are spaces and that $f: X \to Y$ is a continuous bijection. Suppose further that

- X is *locally compact*: every point of X has a compact neighborhood (not necessarily open).
- Y is *compactly generated*: a subset $C \subset Y$ is closed if and only if, for any compact subspace $K \subset Y$, $C \cap K$ is closed in K.

Is f necessarily a homeomorphism? Prove or give a counterexample.