Step-1

Since we want these columns to be linearly independent, the values of c and d should be such that $\boxed{3d - 5c \neq 12}$.

Step-2

(a) We need to find the values of d and c, such that the matrix A has real eigenvalues and orthogonal eigenvectors.

We know that a real symmetric matrix has real eigenvalues and orthogonal eigenvectors.

For the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & d & c \\ 0 & 5 & 3 \end{bmatrix}$ to be symmetric, c should be 5and d could be any real number.

Thus, $\boxed{c = 5}$ and $\boxed{d \in \mathbf{R}}$.

Step-3

(b) In general, for finding three orthonormal vectors, we can start with three linearly independent vectors and then we can apply Gram Schmidt Method of Orthogonalization. Thus, we want the columns of the matrix A to be linearly independent.

Consider $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & d & c \\ 0 & 5 & 3 \end{bmatrix}$.

Let us start with assuming that there exists a linear combination of three columns of A, which produces zero vector. Then, the vector $\begin{bmatrix} 2 \\ d \\ 5 \end{bmatrix}$ can be expressed as a linear combination of the other vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ c \\ 3 \end{bmatrix}$.

Step-4

Consider the following:

$$\begin{aligned} \begin{bmatrix} 2 \\ d \\ 5 \end{bmatrix} &= p \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ c \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} p \\ 2p + cq \\ 3q \end{bmatrix} \end{aligned}$$

Therefore, we get the following equations:

$$\begin{aligned} p &= 2 \\ 3q &= 5 \\ 2p + cq &= d \end{aligned}$$

Step-5

Substituting $p = 2$ and $q = \frac{5}{3}$ in the equation $2p + cq = d$, we get the following:

$$\begin{aligned} 2(2) + c\left(\frac{5}{3}\right) &= d \\ 4 + \frac{5c}{3} &= d \\ 12 + 5c &= 3d \\ 3d - 5c &= 12 \end{aligned}$$

Therefore, when c and d are such that $3d - 5c = 12$, the three vectors $\begin{bmatrix} 2 \\ d \\ 5 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ c \\ 3 \end{bmatrix}$.