Step-1

The objective is to construct a system with more unknowns than equations with no solution and also find all solutions x_n by changing right-hand side to zero.

Consider the following system:

$$x + y + z = 1$$
$$x + y + z = 0$$

Step-2

This is the system with 3 unknowns of 2 equations.

Let,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-3

Then, the augmented matrix is,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Apply the row operation.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

This can be written as,

$$x_1 + x_2 + x_3 = 1$$
$$\text{and} \qquad 0 = -1$$

The second one is impossible.

Therefore it has no solution. The system is inconsistent.

Step-4

Consider the following system:

$$Ax = 0$$

Find the all solutions x_n of the above system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply the row operation.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step-5

This is in the form $Rx = 0$, where *R* is in the reduced form *x* is pivot variable, *y, z* are free variables.

Therefore,

$$\begin{aligned} x + y + z &= 0 \\ x &= -y - z \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -y - z \\ y \\ z \end{bmatrix}$$

$$x_n = \boxed{y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

Hence, the solution set is,

$$\boxed{(x, y, z) = c(-1,1,0) + d(-1,0,1)}.$$

Where, *c* and *d* are constants.