MIDTERM EXAM 2023 FALL

Write your answers with **detailed steps** in the provided answer sheets. Partial answers can get partial credits.

Question 1 (20 points). *Let $A, B \subset \mathbb{R}^n$. Suppose that $A \subset B$, and A is measurable. If $m(A) = m_*(B) < \infty$, then show that B is also measurable.*

Question 2 (20 points). *Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that $\sum_{k=1}^{\infty} m(E_k) < \infty$. Let*

$$E = \{x \in \mathbb{R}^d \mid x \in E_k \text{ for infinitely many } k\}.$$

Show that $m(E) = 0$.

Question 3 (20 points). *Suppose that f is integrable on $\mathbb{R}^d$. Then for $\epsilon > 0$:*

(1) *There exists a set of finite measure B such that $\int_{B^c} |f| < \epsilon$.*

(2) *There is a $\delta > 0$ such that $\int_E |f| < \epsilon$ whenever $m(E) < \delta$. (Hint: for (2), consider $E_N = \{x \mid f(x) < N\}$ and $f \cdot \chi_{E_N}$.)*

Question 4 (20 points). *Let f_i be measurable functions and $E \subset \mathbb{R}^n$ is a measurable set. Assume that* $\text{Supp} f_i \subset E$ *and* $\lim_{i\to\infty} f_i = f$ *a.e.. Prove the following claims or give counterexamples:*

(1) *If $m(E) < \infty$, then $\lim_{i\to\infty} \int f_i = \int f$.*

(2) *If there exists $M \in \mathbb{N}$ such that $|f_i| < M$, then $\lim_{i\to\infty} \int f_i = \int f$.*

Question 5 (20 points). *Show that if f is integrable on $\mathbb{R}^d$ and $\int_E f \geq 0$ for every measurable set E, then $f \geq 0$ a.e. x.*