Step-1

Given that for any A and b, one and only one of the following systems has a solution:

(i) $Ax = b$

(ii) $A^T y = 0, y^T b \neq 0$

And b in the column space $\mathbf{C}(A)$ or there is a y in $\mathbf{N}(A^T)$ such that $y^T b \neq 0$

We have to show that it is contradictory for (i) and (ii) both to have solution.

Step-2

Given b is in the column space $\mathbf{C}(A)$ implies $Ax = b$ â€¦â€¦ (1)

And y belongs to $\mathbf{N}(A^T)$ means $y^T A = 0$ â€¦â€¦. (2)

Then we have

$$\begin{aligned} y^T b &= y^T (Ax) \quad (\text{by } (1)) \\ &= (y^T A) x \\ &= 0x \quad (\text{by } (2)) \\ &= 0 \end{aligned}$$

This contradicts to $y^T b \neq 0$.

$\boxed{\text{Therefore only one of the systems (i) and (ii) has a solution}}$.