## Step-1

Let $p_1$ is the projection of $b$ onto the line through $a_1 = \frac{a_1^T b}{a_1^T a_1} a_1$ â€¦â€¦ (1)

$$a_1^T b = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$= 0 + 2 + 0$$
$$= 2$$

$$a_1^T a_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

$$= \frac{4}{9} + \frac{4}{9} + \frac{1}{9}$$
$$= 1$$

Use these in (1), to get $p_1 = \frac{2}{1} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$

$$= \begin{pmatrix} 4/3 \\ 4/3 \\ -2/3 \end{pmatrix}$$

## Step-2

Let $p_2$ is the projection of $b$ onto the line through $a_2 = \frac{a_2^T b}{a_2^T a_2} a_2$ â€¦â€¦ (2)

$$a_2^T b = \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$= 0 + 2 + 0$$
$$= 2$$

$$a_2^T a_2 = \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$
$$= \frac{1}{9} + \frac{4}{9} + \frac{4}{9}$$
$$= 1$$

Use these in (2), and get;

$$P_2 = \frac{2}{1} \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$
$$= \begin{pmatrix} -2/3 \\ 4/3 \\ 4/3 \end{pmatrix}$$

## Step-3

Let $p_3$ is the projection of b onto the line through $a_3 = \frac{a_3^T b}{a_3^T a_3} a_3$ â€¦â€¦ (3)

$$a_3^T b = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$
$$= 0 - 1 + 0$$
$$= -1$$

$$a_3^T a_3 = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$
$$= \frac{4}{9} + \frac{1}{9} + \frac{4}{9}$$
$$= 1$$

Use these in (3), and get;

$$P_3 = \frac{-1}{1} \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$
$$= \begin{pmatrix} -2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

## Step-4

Using all the results above, and get;

$$
\begin{aligned}
p &= p_1 + p_2 + p_3 \\
&= \left(\frac{4}{3}, \ \frac{4}{3}, \ \frac{-2}{3}\right) + \left(\frac{-2}{3}, \ \frac{4}{3}, \ \frac{4}{3}\right) + \left(\frac{-2}{3}, \ \frac{1}{3}, \ \frac{-2}{3}\right) \\
&= \left(\frac{4-2-2}{3}, \ \frac{4+4+1}{3}, \ \frac{-2+4-2}{3}\right) \\
&= (0,3,0) \\
&= b
\end{aligned}
$$

## Step-5

Further, consider;

$$
\begin{aligned}
a_1 a_1^T &= \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{pmatrix} \\
&= \begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
a_2 a_2^T &= \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\
&= \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ -2/9 & 4/9 & 4/9 \\ -2/9 & 4/9 & 4/9 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
a_3 a_3^T &= \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \\
&= \begin{bmatrix} 4/9 & -2/9 & 4/9 \\ -2/9 & 1/9 & -2/9 \\ 4/9 & -2/9 & 4/9 \end{bmatrix}
\end{aligned}
$$

## Step-6

$$a_1a_1^T + a_2a_2^T + a_3a_3^T = \begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix} + \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ -2/9 & 4/9 & 4/9 \\ -2/9 & 4/9 & 4/9 \end{bmatrix} + \begin{bmatrix} 4/9 & -2/9 & 4/9 \\ -2/9 & 1/9 & -2/9 \\ 4/9 & -2/9 & 4/9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4+1+4}{9} & \frac{4-2-2}{9} & \frac{-2-2+4}{9} \\ \frac{4-2-2}{9} & \frac{4+4+1}{9} & \frac{-2+4-2}{9} \\ \frac{-2-2+4}{9} & \frac{-2+4-2}{9} & \frac{1+4+4}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Follow;

$$P = a_1a_1^T + a_2a_2^T + a_3a_3^T$$
$$= I$$

while $\{a_1, a_2, a_3\}$ form an orthonormal basis.