## Step-1

Given that add the extra column $b$ and reduce $A$ to echelon form, A combination of the rows $A$ has produced the zero row.

$$[A \quad b] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

We need to find the combination, need to find the null space vector and left null space vector.

Let $R_1, R_2, R_3$ be the three rows of A and apply the row operation $R_3 \to R_3 - 2R_2 + R_1$

Then a combination of the rows of $A$ has produced the zero rows and the rows $R_1, R_2, R_3$ are linearly dependent.

The combination is $\text{Row3-2(Row2)+Row1=zero row}$

Therefore, the combination is $\boxed{\text{Row3-2(Row2)+Row1=zero row}}$.

Therefore, the vectors $\boxed{C\{(1,-2,1)\}}$ are in null space.

## Step-2

In other way, we write the homogeneous equations using this.

$$Ax = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 2R_2 + R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow x_1 + 2x_2 + 3x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \\ \Rightarrow x_2 &= -2x_3 \\ x_1 &= -2x_2 - 3x_3 \qquad (\text{Substitue } x_2 \text{ value}) \\ &= x_3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of the given matrix is spanned by $\{(x_3, -2x_3, x_3)\}$

In other words, the basis of the null space of $A$ is $N(A) = \{(1,-2,1)\}$

Therefore, the vectors $\boxed{C\{(1,-2,1)\}}$ are in null space.

## Step-3

We need to find the null space vector of $A^T$.

The transpose of the matrix $A$ is $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

We write the homogeneous equations using this.

$A^T x = 0$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to \frac{R_3}{3}$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to \frac{R_2}{-3}$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

# Step-4

$$\begin{aligned} &\Rightarrow x_1 + 4x_2 + 7x_3 = 0 \\ &\qquad x_2 + 2x_3 = 0 \\ &\Rightarrow x_2 = -2x_3 \\ &\quad x_1 = -4x_2 - 7x_3 \quad (\text{Substitue } x_2 \text{ value}) \\ &\quad\ \ = x_3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of $A^T$ spanned by $\{(x_3, -2x_3, x_3)\}$

In other words, the basis of the null space of $A^T$ is $N(A^T) = \{(1,-2,1)\}$

Therefore, the vectors $\boxed{C\{(1,-2,1)\}}$ are in null space of $A^T$.