## Step-1

We have to find the special solutions to $Rx = 0$ and $R^T y = 0$ for the following $R$.

$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

## Step-2

Given $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $x = \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix}$

## Step-3

$$Rx = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u + 2w + 3t = 0$$
$$v + 4w + 5t = 0$$

## Step-4

First, second columns are pivot columns.

$u, v$ are pivot variables.

$w, t$ are free variables.

$$u = -2w - 3t$$
$$v = -4w - 5t$$

Therefore

$$\begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} -2w-3t \\ -4w-5t \\ w \\ t \end{bmatrix} = w\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The columns in the above solution are the special solutions for $Rx = 0$

## Step-5

Now $R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

## Step-6

Therefore

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 - 3R_1]{R_3 - 2R_1,} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 - 5R_2]{R_3 - 4R_2,} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## Step-7

Now

$R^T y = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 0, y_2 = 0$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The column in the above solution is the special solution for $R^T y = 0$

Given

$$R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $Rx = 0$, where $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$v + 2w = 0$
$\Rightarrow v = -2w$

Second column is pivot column; $v$ is pivot variable and $w$ is free variable.

# Step-8

Therefore

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ -2w \\ w \end{bmatrix}$$
$$= u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

The columns in the above solution are the special solutions for $Rx = 0$

## Step-9

Now

$$R^T y = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 = 0$$

## Step-10

Therefore

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The columns in the above solution are the special solutions for $R^T y = 0$