Step-1

Consider the two Jordan matrices J and K given by:

$$J=\left[\begin{array}{cc|cc}0&1&0&0\\1&0&0&0\\ \hline 0&0&0&1\\0&0&0&0\end{array}\right],\ K=\left[\begin{array}{ccc|c}0&1&0&0\\0&0&1&0\\0&0&0&0\\ \hline 0&0&0&0\end{array}\right].$$

Since block sizes for both the matrices are different, so J and K are not similar matrices.

Objective is to compare the product matrix JM with MK for any matrix M. If $JM = MK$ then prove that matrix M is not invertible.

Step-2

Consider the condition that $JM = MK$. Use the method of contrary and assume that M is some invertible matrix. Then the inverse of M will exist, say M^{-1}.

Pre-multiply by M^{-1} in the condition $JM = MK$ gives

$$\begin{aligned}M^{-1}JM &= M^{-1}MK\\ M^{-1}JM &= IK\\ M^{-1}JM &= K\end{aligned}$$

Square both the sides and get,

$$\begin{aligned}\left(M^{-1}JM\right)^2 &= K^2\\ \left(M^{-1}JM\right)\left(M^{-1}JM\right) &= K^2\\ M^{-1}J\left(M\cdot M^{-1}\right)JM &= K^2\\ M^{-1}J^2M &= K^2 \quad \left[M\cdot M^{-1}=I\right]\end{aligned}$$

Step-3

Note that the block matrix J is nilpotent matrix with index 2. Therefore, $J^2 = 0$. Then the equation $M^{-1}J^2M = K^2$ implies that

$K^2 = 0$.

But

$$K^2 = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\neq 0$$

This is a contradiction. So, assumed assumption was wrong and M is not invertible. Thus, M^{-1} does not exist and $M^{-1}JM = K$ is not possible.

Step-4

Hence, matrices J and K are not similar.