

Step-1

(a) Let us consider the two vectors $(1,0)$ and $(0,1)$. Note that every vector x lies in between these two vectors.

Therefore, every vector Ax will lie in between $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Step-2

We have assumed that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

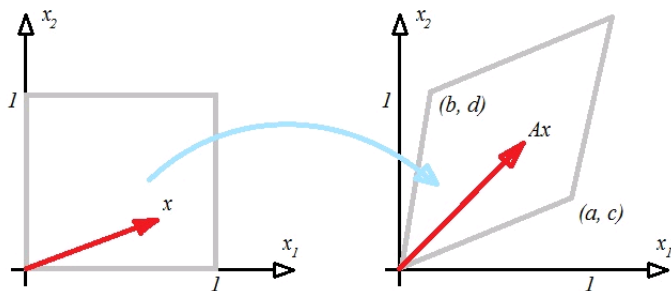
Thus, we get

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Step-3

Thus, every vector Ax lies in between (a,c) and (b,d) . Observe the figure drawn below:



The shape of the transformed region will depend on the matrix A . But, it has to be a parallelogram. It might be a rectangle or a square or in the degenerate case, it might be a straight line also. Since every rectangle and every square is a parallelogram and the straight line can be considered as the limiting case of a parallelogram, only, we can say that the shape of the transformed region will be a parallelogram.

Step-4

(b) We have the following:

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

If we want the transformed region to be a square, then the vectors (a, c) and (b, d) should be perpendicular.

Therefore, $ab + cd = 0$.

Thus, the transformed region will be a square if the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is such that $ab + cd = 0$.

Step-5

(c) Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$.

Now if $b = 0$ and $d = 0$, then $\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ cx_1 \end{bmatrix}$, which is equal to $x_1 \begin{bmatrix} a \\ c \end{bmatrix}$. In this case, every vector x will be transformed to a vector along (a, c) . Therefore, in this case the transformed region will be a straight line.

Similarly, if $a = 0$ and $c = 0$, then $\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} bx_2 \\ dx_2 \end{bmatrix}$, which is equal to $x_2 \begin{bmatrix} b \\ d \end{bmatrix}$. In this case, every vector x will be transformed to a vector along (b, d) . Therefore, here also the transformed region will be a straight line.

Step-6

Thus, the transformed region will be a straight line if the matrix A has $a = 0$ and $c = 0$ or $b = 0$ and $d = 0$.

Step-7

(d) We have

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

The area of the region due to the transformation will be 1 if the cross product of $(a, c)(b, d)$ has magnitude equal to 1.

Therefore, $ad - bc$ must be equal to 1.

Step-8

Thus, the area of the transformed region will be 1, provided $\boxed{ad - bc = 1}$.