(B) $r_1 \le r_3 \le r_2$.

(C) $r_3 \le r_1 \le r_2$.

(D) $r_2 \le r_1 \le r_3$.

multiple choice:

A D C B B

2. (25 points, 5 points each) Fill in the blanks.

(共 25 分, 每小题 5 分) 填空题.

(1) Let $u, v \in \mathbb{R}^n$ with $||u|| = 2$, $||v|| = 4$ and $u^T v = 6$. Then $||3u - v|| =$ 4.

设 $u, v \in \mathbb{R}^n$ 且 $||u|| = 2$, $||v|| = 4$ 以及 $u^T v = 6$. 则 $||3u - v|| =$__________.

(2) Let A be an $n \times n$ matrix with $A^2 = -A$ and let I be the $n \times n$ identity matrix. Then $(A - I)^{-1} =$ $-\frac{1}{2}A - I$.

设 A 为一个 n 阶矩阵, 且 $A^2 = -A$, I 表示 n 阶单位矩阵. 则 $(A - I)^{-1} =$__________.

(3) Let $A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$ with $\text{rank}(A) = 1$. Then $a =$ 1.

设 $A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$ 且 $\text{rank}(A) = 1$. 则 $a =$__________.

(4) Let α be a nonzero 3-dimensional real column vector in $\mathbb{R}^3$ with $\alpha^T \alpha \neq 1$, and I_3 be the 3×3 identity matrix. Then $\text{rank}(I_3 - \alpha\alpha^T) =$ 3.

设 $\alpha \in \mathbb{R}^3$ 为一个非零列向量且 $\alpha^T \alpha \neq 1$, I_3 为 3×3 单位矩阵. 则 $\text{rank}(I_3 - \alpha\alpha^T) =$ __________.

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\hat{x} =$ $\begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$.

$\alpha \neq 0,1,-3.$

(a) By applying row operations, determine for which values of α is the matrix A_α invertible.

(b) Find the values of α such that the nullspace of A_α, $N(A_\alpha)$, has dimension 1? $0,1,-3$

(c) Let $\alpha = 2$. Write down the matrix inverse of A_α.

设 α 为实数, A_α 为

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

$$A_\alpha^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{20} & -\frac{4}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{2} & \frac{3}{20} & \frac{2}{5} \end{bmatrix}$$
$\alpha = 2$

(a) 对矩阵 A_α 作初等行变换, α 取何值时, A_α 为可逆矩阵?

(b) α 取何值时, 矩阵 A_α 的零空间的维数等于 1?

(c) 设 $\alpha = 2$. 求矩阵 A_α 的逆矩阵.

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ -1 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Find an LU factorization of A.

设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

求 A 的一个 LU 分解.

5. (10 points) Consider the following system of linear equations:

$$(I): \begin{cases} x_1 + x_2 = 0, \\ x_2 - x_4 = 0. \end{cases}$$

Note that the above system (I) has four variables x_1, x_2, x_3, x_4. Suppose another homogeneous system of linear equations (II) has special solutions

$$u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

$k \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ k \neq 0.$

Find the common nonzero solutions of systems (I) and (II).

6. (8 points) Let $\mathbb{R}^{2\times 2}$ be the vector space consisting of all 2×2 real matrices. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and

$$E = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) Show that E is a basis for $\mathbb{R}^{2\times 2}$.

- linear independence
- E spans $\mathbb{R}^{2\times 2}$.

(b) Show that $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, X \mapsto XA$ is a linear transformation.

- $T(X+Y) = T(X) + T(Y)$,
- $T(\lambda X) = \lambda T(X)$.

(c) Find the matrix representation of T with respect to the ordered basis $E_{11}, E_{12}, E_{21}, E_{22}$.

设 $\mathbb{R}^{2\times 2}$ 为所有 2×2 实矩阵构成的向量空间. 设 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$M = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}.$$

$$E = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) 证明: E 为 $\mathbb{R}^{2\times 2}$ 的一组基.

(b) 证明: $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, X \mapsto XA$ 为线性变换.

(c) 求 T 在有序基 $E_{11}, E_{12}, E_{21}, E_{22}$ 下的矩阵表示.

7. (6 points) Let A, B be two $m \times n$ matrices. Prove

(a) rank $(A+B) \le \text{rank}A + \text{rank}B$.

(b) rank $(A+B) \ge \text{rank}A - \text{rank}B$.

设 A, B 都为 $m\times n$ 矩阵. 证明:

(a) rank $(A+B) \le \text{rank}A + \text{rank}B$.

(b) rank $(A+B) \ge \text{rank}A - \text{rank}B$.

7 (a) Pivot columns of A: $a_1, a_2, \cdots, a_r$

Pivot columns of B: $b_1, b_2, \cdots, b_s$

$\text{rank} A = r$, $\text{rank} B = s$.

$V = \text{span}(a_1, \cdots, a_r, b_1, \cdots, b_s)$. $\dim V \le r+s$.

$\text{span}(a_1, \cdots, a_r, b_1, \cdots, b_s) \supseteq C(A+B)$

$\Rightarrow \dim C(A+B) \le \dim V$

$\Rightarrow \text{rank}(A+B) \le \text{rank}A + \text{rank}B$.

8. (6 points) Let A be an $m\times n$ matrix with rank r. Show that there exist an $m \times r$ matrix B and an $r\times n$ matrix C such that $A = BC$ and both B and C have rank r.

设 A 为一个秩为 r 的 $m\times n$ 矩阵. 证明: 存在一个 $m\times r$ 矩阵 B 和一个 $r\times n$ 矩阵 C, 使得 $A = BC$, 其中 B, C 的秩都为 r.

8. $A = P_1 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_1$ P_1, Q_1 invertible

$= \underbrace{P_1 \begin{bmatrix} I_r \\ 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} I_r & 0 \end{bmatrix} Q_1}_{C}$.

7 (b) $A+B-B = A$

$\text{rank}(A+B-B) \overset{\text{by (a)}}{\le} \text{rank}(A+B) + \underbrace{\text{rank}(-B)}_{=\,\text{rank}(B)}$

$\Rightarrow$

$\text{rank}A - \text{rank}B \le \text{rank}(A+B)$.