

## Step-1

Consider the following matrix:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$$

Then,

$$A^T = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$$

Now,

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1+4 & 4+16 \\ 4+16 & 16+64 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix} \end{aligned}$$

## Step-2

Now the characteristic equation is,

$$\begin{aligned} |A - I\lambda| &= 0 \\ \begin{vmatrix} 5-\lambda & 20 \\ 20 & 80-\lambda \end{vmatrix} &= 0 \\ (5-\lambda)(80-\lambda) - (20)(20) &= 0 \\ 400 - 85\lambda + \lambda^2 - 400 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda^2 - 85\lambda &= 0 \\ \lambda(\lambda - 85) &= 0 \\ \lambda &= 0 \text{ or } 85 \end{aligned}$$

Therefore, the Eigen values are 0 and 85.

Now,

$$\begin{aligned} \sigma_1^2 &= 85 \\ \sigma_1 &= \sqrt{85} \end{aligned}$$

Hence,  $A^T A = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix}$  has only  $\boxed{\sigma_1^2 = 85}$ .

### Step-3

Let  $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$  be Eigen vector corresponding to  $\lambda_2 = 85$ .

Then,

$$\begin{pmatrix} 5-85 & 20 \\ 20 & 80-85 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -80 & 20 \\ 20 & -5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -80 & 20 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\text{Apply } R_2 \rightarrow R_1 + 4R_2)$$

$$-4y_1 + y_2 = 0$$

Hence, the Eigen vector is  $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ .

Now the unit Eigen vector corresponding to  $\lambda_2 = 85$  is  $\boxed{v_2 = \begin{pmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{pmatrix}}$ .

### Step-4

Let  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  be Eigen vector corresponding to Eigen value  $\lambda_1 = 0$ .

So,

$$\begin{pmatrix} 5-0 & 20 \\ 20 & 80-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 20 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\text{Apply } R_2 \rightarrow R_2 - 4R_1)$$

$$5x_1 + 20x_2 = 0$$

$$x_1 + 4x_2 = 0$$

Therefore, the Eigen vector is  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ .

## Step-5

Thus, the unit Eigen vector corresponding to  $\lambda_1 = 0$  is  $v_1 = \begin{pmatrix} \frac{4}{\sqrt{17}} \\ \frac{-1}{\sqrt{17}} \end{pmatrix}$ .

Hence,  $A^T A = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix}$  has only  $\sigma_1^2 = 85$  with  $v_2 = \begin{pmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{pmatrix}$ , so  $v_2 = \begin{pmatrix} \frac{4}{\sqrt{17}} \\ \frac{-1}{\sqrt{17}} \end{pmatrix}$ .