Step-1

Let A be a matrix.

We have to prove that $A^H A$ is always Hermitian.

Step-2

We know that a matrix A is Hermitian if $A^H = A$.

Let $P = A^H A$

Now

$$P^H = \left(A^H A\right)^H$$

$$= A^H \left(A^H\right)^H$$
$$= A^H A \qquad \left(\text{Since } \left(A^H\right)^H = A\right)$$
$$= P$$

Since $P^H = P \Rightarrow P$ is a hermitatian matrix

Hence $A^H A$ is always Hermitian.

Step-3

Given that $A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$

We have to compute $A^H A$ and AA^H.

Step-4

We have $A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} -i & 1 & -i \\ 1 & -i & -i \end{bmatrix}$

Therefore, $A^H = \overline{A}^T = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$

Step-5

Now

$$A^H A = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix} \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2+1 & -i+i & -i^2+i \\ i-i & 1-i^2 & i-i^2 \\ -i^2-i & -i-i^2 & -i^2-i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -(-1)+1 & -i+i & -(-1)+i \\ i-i & 1-(-1) & i-(-1) \\ -(-1)-i & -i-(-1) & -(-1)-(-1) \end{bmatrix} \qquad \left(\text{Since } i^2=-1\right)$$

$$= \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$$

Therefore, $\boxed{A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}}$

Step-6

Now we compute AA^H

$$AA^H = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 + 1 - i^2 & i - i - i^2 \\ -i + i - i^2 & 1 - i^2 - i^2 \end{bmatrix}$$

$$= \begin{bmatrix} -(-1) + 1 - (-1) & i - i - (-1) \\ -i + i - (-1) & 1 - (-1) - (-1) \end{bmatrix} \qquad (\text{Since } i^2 = -1)$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Hence $\boxed{AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}}$