

## Step-1

By using Gram-Schmidt, we have to construct an orthonormal pair  $q_1$  and  $q_2$  from  $a_1 = (4, 5, 2, 2)$  and  $a_2 = (1, 2, 0, 0)$ . We have to express  $a_1$  and  $a_2$  as linear combination of  $q_1$  and  $q_2$ , and we have to find the triangular  $R$  in  $A = QR$

## Step-2

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \text{ where } a_1 = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Let

$$\begin{aligned} \|a_1\| &= \sqrt{4^2 + 5^2 + 2^2 + 2^2} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

## Step-3

$$\begin{aligned} q_1 &= \frac{a_1}{\|a_1\|} \\ &= \frac{1}{7} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix} \end{aligned}$$

## Step-4

$$q_2 = \frac{\beta}{\|\beta\|} \text{ where } \beta = a_2 - (q_1^T a_2) q_1$$

$$q_1^T a_2 = \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{2}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{4+10+0+0}{7}$$

$$= 2$$

### Step-5

$$(q_1^T a_2) q_1 = 2 \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix}$$

$$= \begin{bmatrix} 8/7 \\ 10/7 \\ 4/7 \\ 4/7 \end{bmatrix}$$

### Step-6

$$\beta = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 8/7 \\ 10/7 \\ 4/7 \\ 4/7 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}$$

### Step-7

$$\|\beta\| = \sqrt{\frac{1}{49} + \frac{16}{49} + \frac{16}{49} + \frac{16}{49}}$$

$$= 1$$

$$q_2 = \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}$$

Therefore

$$= \left\{ \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix}, \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix} \right\}$$

Hence an orthonormal basis for the column space of  $A$

$$\begin{aligned} a_1 &= \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} \\ &= 7 \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix} + 0 \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix} \\ &= \boxed{7q_1 + 0q_2} \end{aligned}$$

Therefore  $a_1$  is a linear combination of  $q_1$  and  $q_2$

## Step-8

$$\begin{aligned} a_2 &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \\ &= 2 \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix} + 1 \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix} \\ &= \boxed{2q_1 + 1q_2} \end{aligned}$$

Therefore  $a_2$  is a linear combination of  $q_1$  and  $q_2$

## Step-9

$$\begin{aligned}
 q_1^T a_1 &= \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{2}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} \\
 &= \frac{16+25+4+4}{7} \\
 &= 7
 \end{aligned}$$

### Step-10

$$\begin{aligned}
 q_1^T a_2 &= \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{2}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \\
 &= \frac{4+10+0+0}{7} \\
 &= 2
 \end{aligned}$$

### Step-11

$$\begin{aligned}
 q_2^T a_2 &= \begin{bmatrix} \frac{-1}{7} & \frac{4}{7} & \frac{-4}{7} & \frac{-4}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \\
 &= \frac{-1+8+0+0}{7} \\
 &= 1
 \end{aligned}$$

### Step-12

Therefore

$$\begin{aligned}
 A &= [a_1 \quad a_2] \\
 &= [q_1 \quad q_2] \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix} \\
 &= \begin{bmatrix} 4/7 & -1/7 \\ 5/7 & 4/7 \\ 2/7 & -4/7 \\ 2/7 & -4/7 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 0 & 1 \end{bmatrix} = QR
 \end{aligned}$$