## Step-1

**(a)**

To determine $A^{-1}$ of the matrix:

$$A=\begin{pmatrix}1&2&0\\0&3&0\\0&4&1\end{pmatrix}$$

First to find the cofactor matrix *C*, determine the various cofactors of *A*:

Cofactor of $A$ as $C_{11}$ as shown below,

$$\begin{aligned}C_{11}&=(-1)^{1+1}\det A_{11}\\&=\begin{vmatrix}3&0\\4&1\end{vmatrix}\\&=3\end{aligned}$$

Cofactor of $A$ as $C_{12}$ as shown below,

$$\begin{aligned}C_{12}&=(-1)^{1+2}\det A_{12}\\&=-\begin{vmatrix}0&0\\0&1\end{vmatrix}\\&=0\end{aligned}$$

Cofactor of $A$ as $C_{13}$ as shown below,

$$\begin{aligned}C_{13}&=(-1)^{1+3}\det A_{13}\\&=\begin{vmatrix}0&3\\0&4\end{vmatrix}\\&=0\end{aligned}$$

## Step-2

Cofactor of $A$ as $C_{21}$ as shown below,

$$\begin{aligned}C_{21}&=(-1)^{2+1}\det A_{21}\\&=-\begin{vmatrix}2&0\\4&1\end{vmatrix}\\&=-2\end{aligned}$$

Cofactor of $A$ as $C_{22}$ as shown below,

$$\begin{aligned}C_{22} &= (-1)^{2+2}\det A_{22}\\ &= \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\\ &= 1\end{aligned}$$

Cofactor of $A$ as $C_{23}$ as shown below,

$$\begin{aligned}C_{23} &= (-1)^{2+3}\det A_{23}\\ &= -\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix}\\ &= -4\end{aligned}$$

## Step-3

Cofactor of $A$ as $C_{31}$ as shown below,

$$\begin{aligned}C_{31} &= (-1)^{3+1}\det A_{31}\\ &= \begin{vmatrix}2 & 0\\ 3 & 0\end{vmatrix}\\ &= 0\end{aligned}$$

Cofactor of $A$ as $C_{32}$ as shown below,

$$\begin{aligned}C_{32} &= (-1)^{3+2}\det A_{32}\\ &= -\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}\\ &= 0\end{aligned}$$

Cofactor of $A$ as $C_{33}$ as shown below,

$$\begin{aligned}C_{33} &= (-1)^{3+3}\det A_{33}\\ &= \begin{vmatrix}1 & 2\\ 0 & 3\end{vmatrix}\\ &= 3\end{aligned}$$

## Step-4

Therefore, cofactor matrix $C$ is:

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{pmatrix}$$

Determinant of $A$ comes out to be,

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1 \cdot 3 + 2 \cdot 0 + 0 \cdot 0 \\ &= 3 \end{aligned}$$

# Step-5

Therefore,

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} C^T \\ &= \frac{1}{3} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{-4}{3} & 1 \end{pmatrix} \end{aligned}$$

Hence, the inverse of the given matrix is $\boxed{\begin{pmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{-4}{3} & 1 \end{pmatrix}}$.

# Step-6

**(b)**

Consider the following matrix $C$:

$$A=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

To determine the various cofactors of *A* to find the matrix *C*, but now use the fact that *A* is symmetric, which implies that $A_{ij}=A_{ij}^T$ and so

$$\begin{aligned} C_{ij} &= (-1)^{i+j}\det A_{ij} \\ &= (-1)^{i+j}\det A_{ji}^T \\ &= (-1)^{i+j}\det A_{ji} \\ &= C_{ji} \end{aligned}$$ s

Thus, Compute $C_{ij}$ for which $i\le j$ only.

# Step-7

Value of $C_{11}$ as shown below,

$$\begin{aligned} C_{11} &= (-1)^{1+1}\det A_{11} \\ &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 3 \end{aligned}$$

Value of $C_{12}$ as shown below,

$$\begin{aligned} C_{12} &= (-1)^{1+2}\det A_{12} \\ &= -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\ &= 2 \end{aligned}$$

Value of $C_{13}$ as shown below,

$$\begin{aligned} C_{13} &= (-1)^{1+3}\det A_{13} \\ &= \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} \\ &= 1 \end{aligned}$$

# Step-8

Value of $C_{21}$ as shown below,

$$\begin{aligned} C_{21} &= (-1)^{2+1} \det A_{21} \\ &= -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} \\ &= 2 \end{aligned}$$

Value of $C_{22}$ as shown below,

$$\begin{aligned} C_{22} &= (-1)^{2+2} \det A_{22} \\ &= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \\ &= 4 \end{aligned}$$

Value of $C_{23}$ as shown below,

$$\begin{aligned} C_{23} &= (-1)^{2+3} \det A_{23} \\ &= -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \\ &= 2 \end{aligned}$$

## Step-9

Value of $C_{31}$ as shown below,

$$\begin{aligned} C_{31} &= (-1)^{3+1} \det A_{31} \\ &= \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 1 \end{aligned}$$

## Step-10

Value of $C_{32}$ as shown below,

$$\begin{aligned} C_{32} &= (-1)^{3+2} \det A_{32} \\ &= -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} \\ &= 2 \end{aligned}$$

Value of $C_{33}$ as shown below,

$$
\begin{aligned}
C_{33} &= (-1)^{3+3}\det A_{33} \\
&= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\
&= 3
\end{aligned}
$$

# Step-11

Therefore, cofactor matrix $C$ is:

$$
\begin{aligned}
C &= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \\
&= \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}
\end{aligned}
$$

# Step-12

Determinant of $A$ comes out to be,

$$
\begin{aligned}
\det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= 2\cdot 3 + (-1)\cdot 2 + 0\cdot 1 \\
&= 4
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det A}C^{T} \\
&= \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}
\end{aligned}
$$

Hence, the inverse of the given matrix is $\begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$.