Step-1

Consider the mat1rix $A=\begin{bmatrix}1&0&0\\6&1&0\\9&8&1\end{bmatrix}\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\end{bmatrix}$

Let $B=\begin{bmatrix}1&0&0\\6&1&0\\9&8&1\end{bmatrix}_{3\times 3}$ and $C=\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\end{bmatrix}_{3\times 4}$

The objective is to find bases for the four fundamental subspaces without computing product *A*.

Note that the order of the matrix *A* is 3 by 4 and matrix *B* is invertible because it is non-singular.

Step-2

Column space of *A*= column space of *C*

Reduce the matrix *C* into reduced echelon form as follows:

$$\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\end{bmatrix}$$

$2R_2-R_1$

$$\sim\begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&1&2\end{bmatrix}$$

$R_1+R_2\ ,\quad 2R_3-R_2$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&1\\0&0&1&2\end{bmatrix}$$

$C_4-C_2\ ,\ C_4-2C_3$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\qquad\qquad ......(1)$$

The rank of *C* is 3.

Thus, the basis for column space of *A* is $\{(1,0,0),(0,1,0),(0,0,1)\}$

Step-3

And, row space of A= row space of C

A basis forÂ row space of C consists of the nonzero rows in the reduced matrix (1)

Thus, the basis for row space of $A=\{(1,2,3,4),(0,1,2,3),(0,0,1,2)\}$

Step-4

Null space of A= Null space of C

In order to find the null space, set $C\mathbf{x}=0$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1+2x_2+3x_3+4x_4=0 \\ \quad x_2+2x_3+3x_4=0 \\ \quad\quad x_3+2x_4=0 \end{cases}$$

$$\begin{cases} x_3=-2x_4 \\ x_2=-2x_3-3x_4 \\ \quad =4x_4-3x_4 \\ \quad =x_4 \\ x_1=-2x_2-3x_3-4x_4 \\ \quad =-2x_4+6x_4-4x_4 \\ \quad =0 \end{cases}$$

Thus, the solution is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ x_4 \\ -2x_4 \\ x_4 \end{bmatrix}$$

$$=x_4\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \quad , \ x_4 \text{ is arbitrary}$$

So, basis for null space $=(0,1,-2,1)$

Left Null space of A= left Null space of C^T

In order to find the left null space, set $C^T x = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 = 0 \\ 2x_1 + x_2 = 0 \\ 3x_1 + 2x_2 + x_3 = 0 \\ 4x_1 + 3x_2 + 2x_3 = 0 \end{cases}$$

$$\Rightarrow x_1 = x_2 = x_3 = 0$$

So, basis for left null space is empty.

Therefore, the required bases are as follows:

$$\boxed{\begin{aligned} &C\left(A^T\right) = \left\{(1,2,3,4),(0,1,2,3),(0,0,1,2)\right\}, \\ &N(A) = (0,1,-2,1), \\ &C(A) = \left\{(1,0,0),(0,1,0),(0,0,1)\right\}, \\ &N\left(A^T\right) = \text{empty} \end{aligned}}$$