Step-1

We have to find R for each of the following matrices, and the special solutions:

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} A & A \end{bmatrix}$, and $C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$

Step-2

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2,\ \frac{1}{2}R_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_{13}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_1} \boxed{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R}$$

This is row reduced echelon form.

Step-3

Let $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

Then

$$Ax = 0$$
$$\Rightarrow Rx = 0$$

Step-4

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u + 2v = 0$$
$$w = 0$$

R has two pivot columns, u, w are pivot variables, v is free variable.

$$u = -2v$$
$$w = 0$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2v \\ v \\ 0 \end{bmatrix} = v \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Step-5

Therefore the special solution is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Or the null spaces of A is $\left\{ v \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} : v \in R \right\}$

Step-6

Given

$$B = \begin{bmatrix} A & A \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \end{bmatrix}$$

By using same operations of A we have B has reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore B has rank â€˜2â€™, there are two pivot columns.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$Rx = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-7

$$\begin{aligned} \Rightarrow x_1 + 2x_2 + x_4 + 2x_5 &= 0 \\ x_3 + x_6 &= 0 \end{aligned}$$

x_1, x_3 are pivot variables, remaining are free variables.

$$x_3 = -x_6$$

$$x_1 = -2x_2 - x_4 - 2x_5$$

$$x_3 = -x_6$$

Step-8

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - 2x_2 - x_4 - 2x_5 \\ x_2 \\ -x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Step-9

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

Therefore the special solutions of *B* are

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Step-10

Given

$$C=\begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3,\frac{1}{2}R_6]{\frac{1}{3}R_2,\frac{1}{3}R_5}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Step-11

$$\xrightarrow[R_3-R_6]{R_2-R_5,}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_{16},R_{25}}\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_{54}}\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3-3R_4]{R_1-3R_2,} \boxed{\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R}$$

This is a row reduced echelon form.

Step-12

There are 1,3,4,5 columns are pivot columns.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$, x_1, x_3, x_4, x_5 are pivot variables x_2, x_6 are free variables.

$$\begin{aligned} &Rx = 0 \\ &\Rightarrow x_1 + 2x_2 = 0 \\ &x_3 = 0 \end{aligned}$$

$$\begin{aligned} &x_4 + 2x_5 = 0 \\ &x_6 = 0 \end{aligned}$$

Step-13

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ -2x_5 \\ x_5 \\ 0 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Step-14

Therefore special solutions are given by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$