SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

1. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

   (a) List the sample space S.

   (b) List the elements that make up the following events:

   (1) $A =$ the sum of the two values is at least 5;

   (2) $B =$ the value for the first die is higher than the value of the second;

   (3) $C =$ the first value is 4.

   (c) List the elements of the following events:

   (1) $A \cap C$;

   (2) $B \cup C$;

   (3) $A \cap (B \cup C)$.

   Proof:

   (a)

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

   (b)

$$A = \left\{ \begin{array}{cccccc} & & & (1,4) & (1,5) & (1,6) \\ & & (2,3) & (2,4) & (2,5) & (2,6) \\ & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

$$B = \left\{ \begin{array}{lllll} (2,1) & & & & \\ (3,1) & (3,2) & & & \\ (4,1) & (4,2) & (4,3) & & \\ (5,1) & (5,2) & (5,3) & (5,4) & \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) \end{array} \right\}.$$

$$C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}.$$

(c)

$$A \cap C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}.$$

$$B \cup C = \left\{ \begin{array}{llllll} (2,1) & & & & & \\ (3,1) & (3,2) & & & & \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & & \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & \end{array} \right\}.$$

$$A \cap (B \cup C) = \left\{ \begin{array}{llllll} & (3,2) & & & & \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & & \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & \end{array} \right\}.$$

2. Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

   Proof: $C = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c)$.

3. Suppose A and B are two events such that $A \subset B$. show that

$$P(B \setminus A) = P(B) - P(A).$$

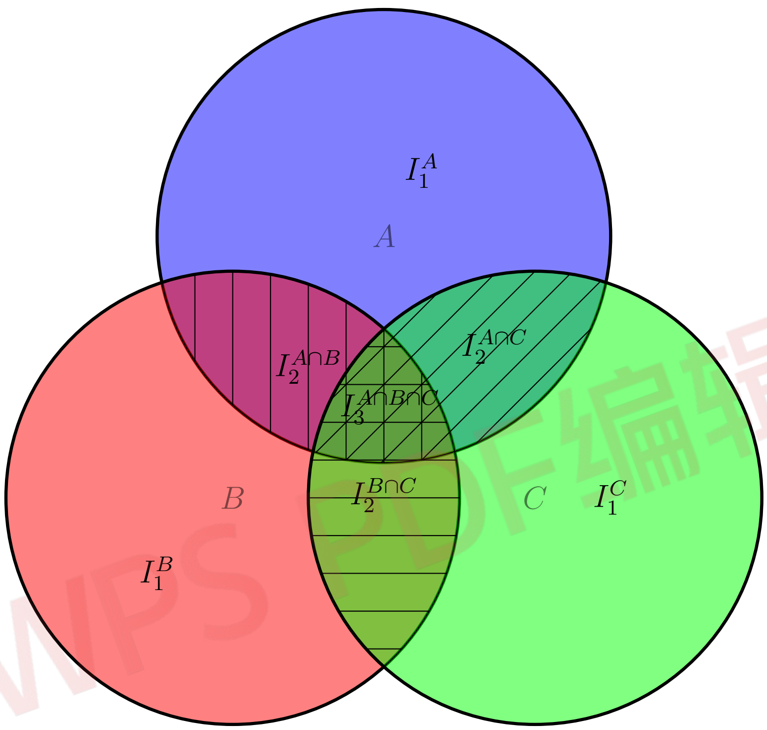
   Proof: Suppose A and B are two events such that $A \subset B$. Then we can obtain $B = A \uplus (B \setminus A)^*$, so $P(B) = P(A) + P(B \setminus A), \Longrightarrow$

$$P(B \setminus A) = P(B) - P(A).$$

*Symbolic $\uplus$ represent disjoint union

4. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions in the first chapter.

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$



Proof:

(a) See the above figure. we have

$$A \cup B \cup C = \left[I_1^A \dot{\cup} I_1^B \dot{\cup} I_1^C\right] \dot{\cup} \left[I_2^{A\cap B} \dot{\cup} I_2^{A\cap C} \dot{\cup} I_2^{B\cap C}\right] \dot{\cup} I_3^{A\cap B\cap C}.$$

Hence

$$\begin{aligned} P(A \cup B \cup C) = & \boxed{P(I_1^A) + P(I_1^B) + P(I_1^C)} \\ & + \boxed{P(I_2^{A\cap B}) + P(I_2^{A\cap C}) + P(I_2^{B\cap C})} \\ & + P(I_3^{A\cap B\cap C}) \\ = & \boxed{P\left(A \setminus \left(I_2^{A\cap B} \dot{\cup} I_2^{A\cap C} \dot{\cup} I_3^{A\cap B\cap C}\right)\right) + P\left(B \setminus \left(I_2^{A\cap B} \dot{\cup} I_2^{B\cap C} \dot{\cup} I_3^{A\cap B\cap C}\right)\right) + P\left(C \setminus \left(I_2^{A\cap C} \dot{\cup} I_2^{B\cap C} \dot{\cup} I_3^{A\cap B\cap C}\right)\right)} \\ & + \boxed{P(I_2^{A\cap B}) + P(I_2^{A\cap C}) + P(I_2^{B\cap C})} \\ & + P(I_3^{A\cap B\cap C}) \\ = & \boxed{P(A) - P\left(I_2^{A\cap B} \dot{\cup} I_2^{A\cap C} \dot{\cup} I_3^{A\cap B\cap C}\right) + P(B) - P\left(I_2^{A\cap B} \dot{\cup} I_2^{B\cap C} \dot{\cup} I_3^{A\cap B\cap C}\right) + P(C) - P\left(I_2^{A\cap C} \dot{\cup} I_2^{B\cap C} \dot{\cup} I_3^{A\cap B\cap C}\right)} \end{aligned}$$

$$
\begin{aligned}
&+ \boxed{P(I_2^{A\cap B}) + P(I_2^{A\cap C}) + P(I_2^{B\cap C})} \\
&+ P(I_3^{A\cap B\cap C}) \\
= & \boxed{P(A) - [P(I_2^{A\cap B}) + \cancel{P(I_2^{A\cap C})} + P(I_3^{A\cap B\cap C})] + P(B) - [\cancel{P(I_2^{A\cap B})} + P(I_2^{B\cap C}) + P(I_3^{A\cap B\cap C})] + P(C) - [P(I_2^{A\cap C}) + \cancel{P(I_2^{B\cap C})} + P(I_3^{A\cap B\cap C})]} \\
&+ \boxed{\cancel{P(I_2^{A\cap B})} + \cancel{P(I_2^{A\cap C})} + \cancel{P(I_2^{B\cap C})}} \\
&+ P(I_3^{A\cap B\cap C}) \\
= & P(A) + P(B) + P(C) \\
&- \boxed{[P(I_2^{A\cap B}) + P(I_3^{A\cap B\cap C})] - [P(I_2^{B\cap C}) + P(I_3^{A\cap B\cap C})] - [P(I_2^{A\cap C}) + P(I_3^{A\cap B\cap C})]} \\
&+ P(I_3^{A\cap B\cap C}) \\
= & P(A) + P(B) + P(C) \\
&- P(A\cap B) - P(A\cap C) - P(B\cap C) \\
&+ P(A\cap B\cap C).
\end{aligned}
$$

(b) Suppose E and F are any two events, Note that

$$
E \cup F = (E \setminus F) \uplus (F \setminus E) \uplus (E \cap F).
$$

Then, we can obtain

$$
\begin{aligned}
P(E\cup F) &= P\left((E\setminus F)\uplus(F\setminus E)\uplus(E\cap F)\right) \\
&= P(E\setminus F) + P(F\setminus E) + P(E\cap F) \\
&= [P(E\setminus F) + P(E\cap F)] + [P(F\setminus E) + P(E\cap F)] - P(E\cap F) \\
&= P(E) + P(F) - P(E\cap F).
\end{aligned}
$$

$$
\begin{aligned}
& P(A\cup B\cup C) \\
= & P\left((A\cup B)\cup C\right) \\
= & P(A\cup B) + P(C) - P\left((A\cup B)\cap C\right) \\
= & P(A\cup B) + P(C) - P\left((A\cap C)\cup(B\cap C)\right) \\
= & [P(A) + P(B) - P(A\cap B)] + P(C) - P\left((A\cap C)\cup(B\cap C)\right) \\
= & [P(A) + P(B) - P(A\cap B)] + P(C) - [P(A\cap C) + P(B\cap C) - P\left((A\cap C)\cap(B\cap C)\right)] \\
= & [P(A) + P(B) - P(A\cap B)] + P(C) - [P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)] \\
= & P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) + P(B\cap C) - P(A\cap B\cap C).
\end{aligned}
$$

5. Suppose $\{A_i; 1 \leq i \leq n\}$ are events.

   (i) Show that **inclusion-exclusion formula:**

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1\leq i<j\leq n} P(A_i \cap A_j) + \sum\sum\sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k)$$
$$- \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

   (ii) Write this formula for cases of $n = 2, n = 3, n = 4$ and $n = 5$ clearly.

Proof:

(i) (**Use the Mathematical induction**) For $n = 2$, we have

$$\begin{aligned}
P(A_1 \cup A_2) &= P\left((A_1 \setminus A_2) \dot{\cup} (A_2 \setminus A_1) \dot{\cup} (A_1 \cap A_2)\right) \\
&= P(A_1 \setminus A_2) + P(A_2 \setminus A_1) + P(A_1 \cap A_2) \\
&= [P(A_1 \setminus A_2) + P(A_1 \cap A_2)] + [P(A_2 \setminus A_1) + P(A_1 \cap A_2)] - P(A_1 \cap A_2) \\
&= P(A_1) + P(A_2) - P(A_1 \cap A_2).
\end{aligned}$$

Assume, for $n = m$, we have

$$P(\bigcup_{i=1}^{m} A_i) = \sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \sum\sum\sum_{1\leq i<j<k\leq m} P(A_i \cap A_j \cap A_k)$$
$$- \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m).$$

Then, for $n = m + 1$,

$$\begin{aligned}
P(\bigcup_{i=1}^{m+1} A_i) &= P\left((\bigcup_{i=1}^{m} A_i) \cup A_{m+1}\right) \\
&= \boxed{P(\bigcup_{i=1}^{m} A_i)} + P(A_{m+1}) - P\left((\bigcup_{i=1}^{m} A_i) \cap A_{m+1}\right) \\
&= \boxed{\sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m)} \\
&\quad + P(A_{m+1}) - \boxed{P\left((\bigcup_{i=1}^{m} A_i) \cap A_{m+1}\right)} \\
&= \sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m) \\
&\quad + P(A_{m+1}) - \boxed{P\left(\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right)} \\
&= \sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m)
\end{aligned}$$

$$
\begin{aligned}
&+ P(A_{m+1}) \\
&- \boxed{\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum\sum_{1 \le i < j \le m} P\left((A_i \cap A_{m+1}) \cap (A_j \cap A_{m+1})\right) + \ldots + (-1)^{m-1} P\left((A_1 \cap A_{m+1}) \cap \ldots \cap (A_m \cap A_{m+1})\right)} \\
= &\sum_{i=1}^{m} P(A_i) - \sum\sum_{1 \le i < j \le m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m) \\
&+ P(A_{m+1}) \\
&- \boxed{\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum\sum_{1 \le i < j \le m} P\left(A_i \cap A_j \cap A_{m+1}\right) + \ldots + (-1)^{m-1} P\left(A_1 \cap \cap \ldots \cap A_m \cap A_{m+1}\right)} \\
= &\sum_{i=1}^{m+1} P(A_i) \\
&- \boxed{\sum\sum_{1 \le i < j \le m} P(A_i \cap A_j) + \sum_{i=1}^{m} P(A_i \cap A_{m+1})} \\
&+ \boxed{\sum\sum\sum_{1 \le i < j < k \le m} P(A_i \cap A_j \cap A_k) + \sum\sum_{1 \le i < j \le m} P\left(A_i \cap A_j \cap A_{m+1}\right)} \\
&+ \ldots + (-1)(-1)^{m-1} P\left(A_1 \cap \cap \ldots \cap A_m \cap A_{m+1}\right) \\
= &\sum_{i=1}^{m+1} P(A_i) \\
&- \boxed{\sum\sum_{1 \le i < j \le m+1} P(A_i \cap A_j)} \\
&+ \boxed{\sum\sum\sum_{1 \le i < j < k \le m+1} P(A_i \cap A_j \cap A_k)} \\
&+ \ldots + (-1)^{(m+1)-1} P\left(A_1 \cap \cap \ldots \cap A_m \cap A_{m+1}\right).
\end{aligned}
$$

Altogether, we get the inclusion-exclusion formula

$$
\begin{aligned}
P(\bigcup_{i=1}^{n} A_i) = &\sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) \\
&- \ldots + (-1)^{n-1} P(A_1 \cap A_2 \ldots A_n).
\end{aligned}
$$

(ii) - if $n = 2$, then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

- if $n = 3$, then

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) = &P(A_1) + P(A_2) + P(A_3) \\
&- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&+ P(A_1 \cap A_2 \cap A_3).
\end{aligned}
$$

- if $n = 4$, then

$$P(\overset{4}{\underset{i=1}{\cup}} A_i) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$$

$$-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4)$$

$$+P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4)$$

$$- P(A_1 \cap A_2 \cap A_3 \cap A_4).$$

- if $n = 5$, then

$$P(\overset{5}{\underset{i=1}{\cup}} A_i) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)$$

$$-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_1 \cap A_5) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_2 \cap A_5) - P(A_3 \cap A_4) - P(A_3 \cap A_5) - P(A_4 \cap A_5)$$

$$+P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_2 \cap A_5) + P(A_1 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_5) + P(A_1 \cap A_4 \cap A_5) + P(A_2 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_5) + P(A_2 \cap A_4 \cap A_5) + P(A_3 \cap A_4 \cap A_5)$$

$$+P(A_1 \cap A_2 \cap A_3 \cap A_4) + +P(A_1 \cap A_2 \cap A_3 \cap A_5) + P(A_1 \cap A_2 \cap A_4 \cap A_5) + P(A_1 \cap A_3 \cap A_4 \cap A_5) + P(A_2 \cap A_3 \cap A_4 \cap A_5)$$

$$- P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5).$$

6. (i) If $\{A_n; n \geq 1\}$ is an increasing sequence of events, i.e. for all $n \geq 1, A_n \subset A_{n+1}$, then $\lim\limits_{n\to\infty} P(A_n) = P(\overset{\infty}{\underset{n=1}{\cup}} A_n)$.

   (ii) If $\{A_n; n \geq 1\}$ is a decreasing sequence of events, i.e. for all $n \geq 1, A_n \supset A_{n+1}$, then $\lim\limits_{n\to\infty} P(A_n) = P(\overset{\infty}{\underset{n=1}{\cap}} A_n)$.

Proof:

(i) Let $A_0 = \emptyset, B_n = A_n \setminus A_{n-1}, n \geq 1$. Then (from the Tutorial 01.07) we know

$\{B_n, n \geq 1\}$ are (pairwise) adjoint and $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty} B_n$. Therefore

$$\begin{aligned}
P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) &= \sum_{n=1}^{\infty} P(B_n) \\
&= \sum_{n=1}^{\infty} P(A_n \setminus A_{n-1}) \\
&= \sum_{n=1}^{\infty} [P(A_n) - P(A_{n-1})] \text{ since } A_{n-1} \subset A_n, n \geq 1 \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} [P(A_n) - P(A_{n-1})] \\
&= \lim_{k\to\infty} (P(A_k) - P(A_0)) \\
&= \lim_{k\to\infty} (P(A_k) - P(\emptyset)) = \lim_{k\to\infty} (P(A_k) - 0) \\
&= \lim_{k\to\infty} P(A_k) = \lim_{n\to\infty} P(A_n).
\end{aligned}$$

i.e., $\lim\limits_{n\to\infty} P(A_n) = P(\bigcup\limits_{n=1}^{\infty} A_n)$.

(ii) Let $B_n = A_1 \setminus A_n, n \geq 1$, then $B_n \uparrow$.[†] Using the above result of (i), we obtain

$$\lim_{n\to\infty} P(B_n) = P(\bigcup_{n=1}^{\infty} B_n).$$

Then, we can get

$$\begin{aligned}
P(A_1) - P(\bigcap_{n=1}^{\infty} A_n) &= P\left(A_1 \setminus (\bigcap_{n=1}^{\infty} A_n)\right) = P\left(A_1 \cap (\bigcap_{n=1}^{\infty} A_n)^c\right) \\
&= P\left(A_1 \cap (\bigcup_{n=1}^{\infty} A_n^c)\right) = P\left(\bigcup_{n=1}^{\infty} (A_1 \cap A_n^c)\right) \\
&= P\left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\right) \\
&= \boxed{P(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n)} \\
&= \lim_{n\to\infty} P(A_1 \setminus A_n) \\
&= \lim_{n\to\infty} [P(A_1) - P(A_n)] \text{ note that } A_n \subset A_1, n \geq 1 \\
&= P(A_1) - \lim_{n\to\infty} P(A_n).
\end{aligned}$$

$\Longrightarrow \lim\limits_{n\to\infty} P(A_n) = P(\bigcap\limits_{n=1}^{\infty} A_n)$.

Method 2: Assume $(\Omega, \mathscr{F}, P)$ is a probability space,let $B_n = A_n^c = \Omega \setminus A_n, n \geq 1$, then $B_n \uparrow$. Using the above result of (i), we obtain

$$\lim_{n\to\infty} P(B_n) = P(\bigcup_{n=1}^{\infty} B_n).$$

[†] i.e., $\{B_n; n \geq 1\}$ is an increasing sequence of events

Then, we can get

$$
\begin{aligned}
1 - P(\bigcap_{n=1}^{\infty} A_n) &= P\left((\bigcap_{n=1}^{\infty} A_n)^c\right) \\
&= P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \\
&= \boxed{P(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n)} \\
&= \lim_{n\to\infty} P(A_n^c) \\
&= \lim_{n\to\infty} [1 - P(A_n)] \text{ note that } P(A_n^c) = 1 - P(A_n), n \geq 1 \\
&= 1 - \lim_{n\to\infty} P(A_n).
\end{aligned}
$$

$\Longrightarrow \lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n)$.

7. Suppose that $\{A_n; n \geq 1\}$ is a sequence of events which may not be disjoint. Show that the **sub-additive property:**

$$P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

Also for any $k \geq 2$, we have

$$P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n).$$

In particular, for any two events A and B, we have $P(A \cup B) \leq P(A) + P(B)$.

Proof: Let $B_1 = A_1, B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i), n \geq 2$. Then for any $k \geq 1$,

$$\{B_n, n \geq 1\} \text{ are (pairwise) disjoint }, \bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n \text{ and } \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Firstly, for any $m, n \in \mathbb{N}_+$, we show that $\{B_n; n \geq 1\}$ are (pairwise) disjoint.

Indeed, without loss of generality, we can suppose $m > n$, note that $m, n \in \mathbb{N}_+$, so

$$m - 1 \geq n \geq 1.$$

then $\bigcup_{i=1}^{m-1} A_i \supset A_n$, yields $(\bigcup_{i=1}^{m-1} A_i)^c \subset A_n^c$. However,

$$B_m = A_m \setminus (\bigcup_{i=1}^{m-1} A_i) = A_m \cap (\bigcup_{i=1}^{m-1} A_i)^c \subset A_n^c \subset A_n^c \cup (\bigcup_{i=1}^{n-1} A_i) = \left(A_n \setminus (\bigcup_{i=1}^{n-1} A_i)\right)^c = B_n^c.$$

Altogether, we have for any $m > n \geq 1$,

$$B_m \subset B_n^c,$$

i.e., $B_n \cap B_m = B_m \cap B_n = \emptyset$.

Hence, $\{B_n; n \geq 1\}$ are (pairwise) disjoint.

Secondly, we show that $\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n$.

Indeed, for $n = 1$,

$$\bigcup_{n=1}^{1} A_n = A_1 = B_1 = \bigcup_{n=1}^{1} B_n.$$

Assume for $k = m$, we have $\bigcup_{n=1}^{m} A_n = \bigcup_{n=1}^{m} B_n$, then

$$\begin{aligned}
\bigcup_{n=1}^{m+1} B_n &= B_{m+1} \cup (\bigcup_{n=1}^{m} B_n) \xlongequal{asd} (B_{m+1}) \cup (\bigcup_{n=1}^{m} A_n) \\
&= \left(A_{m+1} \setminus (\bigcup_{i=1}^{(m+1)-1} A_i)\right) \cup (\bigcup_{n=1}^{m} A_n) \\
&= \left(A_{m+1} \setminus (\bigcup_{i=1}^{m} A_i)\right) \cup (\bigcup_{n=1}^{m} A_n) \\
&= \left(A_{m+1} \setminus (\bigcup_{n=1}^{m} A_n)\right) \cup (\bigcup_{n=1}^{m} A_n) \\
&= \left(A_{m+1} \cap (\bigcup_{n=1}^{m} A_n)^c\right) \cup (\bigcup_{n=1}^{m} A_n) \\
&= A_{m+1} \cup (\bigcup_{n=1}^{m} A_n) = \bigcup_{n=1}^{m+1} A_n.
\end{aligned}$$

Finally, we have $\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n$.

Or **Method 2: direct proof**

For any fixed $k \geq 2$, note that $B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i) \subset A_n, n \geq 1$, then

$$\bigcup_{n=1}^{k} A_n \supset \bigcup_{n=1}^{k} B_n.$$

Assume $x \in \bigcup_{n=1}^{k} A_n$,

- if $x \in A_1$, then $x \in A_1 = B_1 \subset \bigcup_{n=1}^{k} B_n$, i.e. $x \in \bigcup_{n=1}^{k} B_n$;
- if $x \notin A_1$, then
  - if $x \in A_2$, then $x \in A_2 \setminus A_1 = B_2 \subset \bigcup_{n=1}^{k} B_n$, i.e. $x \in \bigcup_{n=1}^{k} B_n$;
  - if $x \notin A_2$, then

    after $k-1$ step $\cdots$
    - if $x \in A_{k-1}$, then $x \in A_{k-1} \setminus (\bigcup_{i=1}^{k-2} A_i) = B_{k-1} \subset \bigcup_{n=1}^{k} B_n$, i.e. $x \in \bigcup_{n=1}^{k} B_n$.

- if $x \notin A_{k-1}$, then
  - Notice that $x \in \bigcup_{n=1}^{k} A_n$, so $x \in A_k$, then

$$x \in A_k \setminus (\bigcup_{i=1}^{k-1} A_i) = B_k \subset \bigcup_{n=1}^{k} B_n, \text{ i.e. } x \in \bigcup_{n=1}^{k} B_n.$$

Finally, we obtain $x \in \bigcup_{n=1}^{k} B_n$, i.e. $\bigcup_{n=1}^{k} A_n \subset \bigcup_{n=1}^{k} B_n$. Hence for any fixed $k \geq 2$,

$$\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n.$$

Thirdly, we will show that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

In fact, since $A_k \subset \bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n \subset \bigcup_{n=1}^{\infty} B_n$, then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} B_n$. Note that $B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i) \subset A_n, n \geq 1, \Rightarrow \bigcup_{n=1}^{\infty} A_n \supset \bigcup_{n=1}^{\infty} B_n$. Hence, we obtain

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Or sending $n \to \infty$, then

$$\begin{aligned}
\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} A_k &= \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n \\
&= \boxed{\lim_{k\to\infty} \bigcup_{n=1}^{k} A_n = \lim_{k\to\infty} \bigcup_{n=1}^{k} B_n} \\
&= \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} B_n \\
&= \bigcup_{k=1}^{\infty} B_k = \bigcup_{n=1}^{\infty} B_n.
\end{aligned}$$

1°. Note that $B_n \subset A_n, n \geq 1$, so $P(B_n) \leq P(A_n), n \geq 1$. Therefore

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

2°. **Method 1:** Note that $B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i) \subset A_n, 1 \leq n \leq k$, so

$$P(B_n) \leq P(A_n), 1 \leq n \leq k.$$

Therefore

$$P(\bigcup_{n=1}^{k} A_n) = P(\bigcup_{n=1}^{k} B_n) = \sum_{n=1}^{k} P(B_n) \leq \sum_{n=1}^{k} P(A_n).$$

i.e. for any $k \geq 2$, we have $P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n)$.

Method 2: For any $k \geq 2$, let $A_{k+1} = A_{k+2} = A_{k+3} = \cdots = \emptyset$, thus

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{k} A_n, \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{k} P(A_n).$$

Indeed, it is easy to see $\bigcup_{n=1}^{\infty} A_n \supset \bigcup_{n=1}^{k} A_n$. On the other hand, if $x \in \bigcup_{n=1}^{\infty} A_n$, then $\exists n_0 \geq 1, s.t.\ x \in A_{n_0}$. Since $A_m = \emptyset, m \geq k+1$, so $1 \leq n_0 \leq k$, hence $x \in \bigcup_{n=1}^{k} A_n$, i.e. $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{k} A_n$. Altogether we have $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{k} A_n$.

Assume $k (k \geq 2)$ is fixed, for any $\varepsilon > 0$, let $m = k$[‡], then

$$\left| \sum_{n=1}^{m} P(A_n) - \sum_{n=1}^{k} P(A_n) \right| = 0 < \varepsilon.$$

Therefore, $\sum_{n=1}^{\infty} P(A_n) = \lim_{m \to \infty} \sum_{n=1}^{m} P(A_n) = \sum_{n=1}^{k} P(A_n)$. Hence,

$$P(\bigcup_{n=1}^{k} A_n) = P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{k} P(A_n).$$

i.e. for any $k \geq 2$, we have $P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n)$.

In particular, let $k = 2, A_1 = A, A_2 = B$, then we have

$$\begin{aligned} P(A \cup B) &= P(A_1 \cup A_2) \\ &= \boxed{P(\bigcup_{n=1}^{2} A_n) \leq \sum_{n=1}^{2} P(A_n)} \\ &= P(A_1) + P(A_2) = P(A) + P(B). \end{aligned}$$

i.e. $P(A \cup B) \leq P(A) + P(B)$.

Remark: we even have a fact, assume $(\Omega, \mathscr{F}, P)$ is a probability space:

Prop: $P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$ is equivalent to for any $k \geq 2, P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n)$.[§]

[‡]Only need to take any a number of N s.t. $N \geq k$

[§]Moreover, the count additivity is not equivalent to the finite additivity. Obviously, count additivity must be finite additivity, but conversely it is not true, need to add additional conditions, generally add continuity conditions.

we have proven necessity, now just prove sufficiency. Suppose, for any $k \geq 2$,

$$P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n).$$

Firstly, we show that

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n.$$

Indeed, since $A_k \subset \bigcup_{n=1}^{k} A_n, \forall k \geq 1$, then $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n$. On the other hand, if $x \in \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n$, then $\exists k_0 \geq 1$, s.t. $x \in \bigcup_{n=1}^{k_0} A_n$, therefore, $\exists n_{k_0}, 1 \leq n_{k_0} \leq k_0$, s.t. $x \in A_{n_{k_0}}$, so $x \in \bigcup_{k=1}^{\infty} A_k$. Hence $\bigcup_{k=1}^{\infty} A_k \supset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n$. Finally, we get

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n.$$

Note that $\{\bigcup_{n=1}^{k} A_n\}_{k=1}^{\infty}$ is a increasing sequence, sending $k \to \infty$, then

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{k=1}^{\infty} A_k) = P(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n) = P(\lim_{k\to\infty} \bigcup_{n=1}^{k} A_n)$$

$$= \boxed{\lim_{k\to\infty} P(\bigcup_{n=1}^{k} A_n) \leq \lim_{k\to\infty} \sum_{n=1}^{k} P(A_n)} = \sum_{n=1}^{\infty} P(A_n).$$