Step-1

a) Given that a skew symmetric matrix satisfies $K^T = K$, as in

$$K = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

In the 3 by 3 case, we have to find why is $\det(-K) = (-1)^3 \det(K)$ and on the other hand, $\det(K^T) = \det(K)$, and we have to deduce that the determinant must be zero.

Step-2

$$K^T = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$
$$= -K$$
$$= (-1)^3 K$$

Step-3

So we get

$$\det K^T = \det(-K)$$
$$= \det K$$

Step-4

For any $n \times n$ matrix A , we have

$$\det(tA) = t^n \det A$$

So for a 3×3 matrix K we have

$$\det(-K) = (-1)^3 \det K$$
$$= -\det K$$

Step-5

Therefore for skew symmetric matrix K we get

$\det K = -\det K$ and hence $2\det K = 0$

Giving that $\det K = 0$

Step-6

b) We have to write down a 4 by 4 skew symmetric matrix with $\det K \neq 0$

Consider

$$K = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 3 & -1 \\ 0 & -3 & 0 & 2 \\ -2 & 1 & -2 & 0 \end{bmatrix}$$, where K is a skew symmetric matrix of order 4×4.

Step-7

$$\det K = -\begin{vmatrix} -1 & 3 & -1 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 & 3 \\ 0 & -3 & 0 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= -\left[-2(2+6)\right] - 2\left[(-3)(2+6)\right]$$

$$= 64$$
$$\neq 0$$

Step-8

Thus $$k = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 3 & -1 \\ 0 & -3 & 0 & 2 \\ -2 & 1 & -2 & 0 \end{bmatrix}$$ is an example of a 4 by skew symmetric matrix with $\det K \neq 0$.