Step-1

Assume that the homogeneous system of equation $Ax = 0$ has a nonzero solution. Objective is to prove that for any function *f*, $A^T y = f$ does not have any solution. Also write an example in the support of the proof.

Let nonzero number *x* is the solution of $Ax = 0$ and $f = x$ then

$A^T y = x$.

Multiply both the sides by x^T and get,

$$\begin{aligned} x^T A^T y &= x^T x \\ (Ax)^T y &= \|x\| \quad \left[(AB)^T = B^T A^T\right] \\ 0 &= \|x\| \end{aligned}$$

since $Ax = 0$.

Step-2

But norm of *x* is zero if and only if $x = 0$. That is,

$\|x\| = 0$ implies $x = 0$.

This cannot be possible because *x* is assumed to be nonzero.

Thus, there does not exist any function *f* such that $A^T y = f$ possesses any solution.

Step-3

Consider the following example:

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $f = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Note that system $Ax = 0$ has a nonzero solution $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Solve $A^T y = f$ as:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

gives

$$x+2y=1$$
$$2x+4y=5$$

Second equation is $x+2y=2.5$. Since $1\neq 2.5$, therefore $A^T y = f$ has no solution.