Step-1

We need to construct a matrix with the required property and need to explain why we canâ€™t.

(a)

Given a column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ row space contains $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\5\end{bmatrix}$

The required matrix is $A=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$

The dimension of this matrix is,

Apply $R_2 \to R_2 - R_1$

$$\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$$

Apply $R_2 \leftrightarrow R_1$

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

The matrix has two pivots of columns, so dim $(\text{col } A)=2$.

So, basis for column space is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Step-2

The transpose of matrix is $A^T=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$

So, 1 and 3columns are independent and the dimension is,

$$\begin{aligned}\dim\left(col\ A^T\right)&=\dim\ \left(row\ \left(A\right)\right)\\&=2\end{aligned}$$

The basis for row space is,

$$\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}2\\5\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}+5\begin{bmatrix}0\\1\end{bmatrix}$$

Therefore, row space contains $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\5\end{bmatrix}$

Therefore, the required matrix is $\boxed{A=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}}$.

Step-3

(b)

Given a column space has basis $\begin{bmatrix}1\\2\\3\end{bmatrix}$, null space has basis $\begin{bmatrix}3\\2\\1\end{bmatrix}$

We know that $\dim\left(N\left(A\right)\right)+\dim\left(\text{col}\left(A\right)\right)$ = number of columns

i.e. $\dim\left(col\ A\right)=1$, $\dim\left(N\left(A\right)\right)=1$ and $\text{the number of columns}=3$

So,

$$\begin{aligned}1+1&\neq 3\\2&\neq 3\end{aligned}$$

Therefore, itâ€™s $\boxed{\text{impossible}}$.

Step-4

(c)

Given the dimension of null space = 1 + dimension of left null space.

The required matrix is $A=\begin{bmatrix}1&1\end{bmatrix}$

Letâ€™s write in $AX=0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$
$$\Rightarrow x_1 + x_2 = 0$$
$$\Rightarrow x_1 = -x_2$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$
$$= x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the basis for null space is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and the dimension of null space is 1.

i.e. $\dim(\mathrm{Nul}(A)) = 1$

Step-5

The transpose of the matrix is $A^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Letâ€™s write in $A^T X = 0$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow x_1 = 0$$

The dimension of left null space is 0.

i.e. $\dim(\mathrm{Nul}(A^T)) = 0$

Therefore,

$$\dim(\mathrm{Nul}(A)) = 1 + \dim(\mathrm{Nul}(A^T))$$
$$1 = 1 + 0$$
$$1 = 1$$

Therefore, the dimension of null space = 1 + dimension of left null space.

Therefore, the required matrix is $\boxed{A = \begin{bmatrix} 1 & 1 \end{bmatrix}}$

Step-6

(d)

Given left null space contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

The required matrix is $A=\begin{bmatrix}-9 & -3\\3 & 1\end{bmatrix}$

The transpose of the matrix is $A^T=\begin{bmatrix}-9 & 3\\-3 & 1\end{bmatrix}$

Letâ€™s write in $A^T X=0$

$$\Rightarrow \begin{aligned}-9x_1+3x_2&=0\\-3x_1+x_2&=0\end{aligned}$$

$$\Rightarrow 3x_1=x_2$$

So,

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\3x_1\end{bmatrix}$$
$$=x_1\begin{bmatrix}1\\3\end{bmatrix}$$

So, left null space contains $\begin{bmatrix}1\\3\end{bmatrix}$ and $\dim\left(N\left(A^T\right)\right)=1$.

The second row is $\begin{bmatrix}3\\1\end{bmatrix}$. So, the row space contains $\begin{bmatrix}3\\1\end{bmatrix}$

Therefore, the required matrix is $\boxed{A=\begin{bmatrix}-9 & -3\\3 & 1\end{bmatrix}}$.

Step-7

(e)

Given row space = column space, $\text{nullspace} \neq \text{leftnullspace}$

Letâ€™s consider m by n matrix.

We need to construct the matrix with these properties.

Given row space = column space

The dimension of row space is r and dimension of column space is r.

It requires $m = n$.

Then $m - r = n - r$

Step-8

We know that $\dim(N(A)) + \dim(\mathrm{col}(A))$ = number of columns and $\dim(N(A^T)) + \dim(\mathrm{col}(A^T))$ = number of columns

i.e. $\dim(N(A)) + \dim(C(A)) = \dim(N(A^T)) + \dim(C(A^T))$

$$\Rightarrow \dim(N(A)) + r = \dim(N(A^T)) + r \qquad (\text{Since} \dim c(A) = \dim c(A^T) = r)$$
$$\Rightarrow \dim(N(A)) = \dim(N(A^T))$$

But given that $\text{nullspace} \neq \text{leftnullspace}$

Therefore, itâ€™s $\boxed{\text{impossible}}$.