MIDTERM EXAM 2022 FALL (H)

Write your answers with **detailed steps** in the provided answer sheets. Partial answers can get partial credits.

Question 1 (20 points)**.** *Let $A, B \subset \mathbb{R}^n$ with $m_*(A), m_*(B) < \infty$. Then $m_*(A \cup B) = m_*(A) + m_*(B)$ if and only if there exist measurable sets E, F suh that $E \supset A, F \supset B$, and $m(E \cap F) = 0$.*

Proof. ($\Leftarrow$) For $\epsilon > 0$, take open set $U \supset A \cup B$ such that $m_*(A \cup B) + \epsilon > m(U)$. Then

$$\begin{aligned} m_*(A) + m_*(B) \leq m(E \cup U) + m(F \cup U) &\leq m((E \cup F) \cap U) + m(E \cap F) \\ &\leq m(U) < m_*(A \cup B) + \epsilon. \end{aligned}$$

Let $\epsilon \to 0$, we have the desired claim.

($\Rightarrow$) There exist G_δ sets U, V such that $U \supset A, V \supset B$ and $m_*(A) = m(U), m_*(B) = m(V)$.

Then $m_*(A \cup B) \leq m(U \cap V) + m(U \cup V) = m(U) + m(V) = m_*(A) + m_*(B)$. Thus $m(U \cap V) = 0$. □

Question 2 (20 points)**.** *Show the following claims. Let $E \subset [0, 1]$ be a measurable set.*

(1) *If $m(E) = 1$, then $\bar{E} = [0, 1]$ (here $\bar{E}$ is the closure of E).*
(2) *If $m(E) = 0$, then $E^\circ = \emptyset$ (here E° is the interior of E).*

Proof. (1) If $[0, 1] - \bar{E} \neq \emptyset$, then there exists $(a, b) \subset [0, 1] - \bar{E} \neq \emptyset$. Hence $m([0, 1] - E) \geq b - a > 0$. A contradiction.

(2) If $E^\circ \neq \emptyset$, then there exists $(a, b) \subset E^\circ$. Hence $m(E) \geq b - a > 0$, a contradiction. □

Question 3 (20 points)**.** *Suppose $\{E_k\}_{k=1}^\infty$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that $\sum_{k=1}^\infty m(E_k) < \infty$. Let*

$$E = \{x \in \mathbb{R}^d \mid x \in E_k, \text{ for infinitely many } k\}.$$

(1) *Directly show that E is measurable.*
(2) *Prove $m(E) = 0$.*

Answer. (1) $E = \cap_n \cup_{k>n} E_k$, hence measurable.

(2) By monotone converges (or its corollary),

$$\int \sum \chi_{E_i} = \sum \int \chi_{E_i} = \sum m(E_i) < \infty.$$

Hence $\sum \chi_{E_i} < \infty$ a.e. x. That is $m(E) = 0$. (one can also get (1) from $m(E) = 0$.)

$\square$

Question 4 (20 points). *Let f be an integrable function on $\mathbb{R}^d$, then $\lim_{\delta\to 1} \|f(\delta x) - f(x)\|_{L^1(\mathbb{R}^d)} = 0$. (Hint: use continuous functions of compact support are dense in $L^1(\mathbb{R}^d)$.)*

Answer. Let ϕ_n be continuous functions with compact support such that

$$\lim_{n\to\infty} \|\phi_n - f(x)\|_{L^1(\mathbb{R}^d)} = 0.$$

Then

(1)

$$\begin{aligned}\|f(\delta x) - f(x)\| &\leq \|f(\delta x) - \phi_n(\delta x)\| + \|\phi_n(\delta x) - \phi_n(x)\| + \|\phi_n(x) - f(x)\| \\ &= \frac{1}{\delta^d}\|f(x) - \phi_n(x)\| + \|\phi_n(\delta x) - \phi_n(x)\| + \|\phi_n(x) - f(x)\|.\end{aligned}$$

For ϵ, fix an n such that $(1 + 2^d)\|f(x) - \phi_n(x)\| < \epsilon/2$. As ϕ_n is continuous supporting in a compact set B, ϕ_n is uniformly continuous on B. Let $m(B) = M$, for $\frac{\epsilon}{2M}$, there exists $\delta' > 0$ such that when $\|x_1 - x_2\| < \delta'$, we have $|\phi_n(x_1) - \phi_n(x_2)| < \frac{\epsilon}{2M}$. Thus there exists $\delta < 1$ (so that $\|x - \delta x\| < \delta'$ for $x \in B$) such that $\|\phi_n(x) - \phi_n(\delta x)\| \leq \frac{\epsilon}{2M} \cdot M = \epsilon/2$. Thus take $\delta > 1/2$, by (1)

$$\|f(\delta x) - f(x)\| \leq \epsilon/2 + \epsilon/2.$$

$\square$

Question 5 (20 points). *Let $E \subset \mathbb{R}^1$ be a measurable set and $L^1(E)$ be the set of Lebesgue integrable functions.*

(1) *Let $f \in L^1(E)$, show that*

$$\lim_{k\to\infty} \int_{\{x\in E \mid |f(x)|<\frac{1}{k}\}} |f(x)| dx = 0.$$

(2) *Let f be a continuous function on $[0, \infty]$. Let $a \in \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = a$, then for any $m > 0$, we have*

$$\lim_{k\to\infty} \int_{[0,m]} f(kx) dx = am.$$

Proof. 1. Let $E_k = \{x \in E \mid |f(x)| < \frac{1}{k}\}$, then $|f| \geq |f|\chi_{E_k}$. By the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\{x\in E \mid |f(x)|<\frac{1}{k}\}} |f(x)| dx = \int \lim_{k\to\infty} |f|\chi_{E_k} = \int 0 = 0.$$

2. By the assumption, there exists M such that $|f| < M$. Hence by the bounded convergence theorem,

$$\lim_{k\to\infty} \int_{[0,m]} f(kx)dx = \int_{[0,m]} \lim_{k\to\infty} f(kx) = \int_{[0,m]} a = ma.$$

$\square$