Step-1

(a).

Consider the matrix $A=\begin{pmatrix}1 & 3\\ 3 & 5\end{pmatrix}$.

The objective is to decide the positive definiteness of these matrices and write out the corresponding $f=x^TAx$

Step-2

Now the corresponding $f=x^TAx$ is,

$$\begin{aligned}f&=x^TAx\\&=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\&=\begin{pmatrix}x+3y & 3x+5y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\&=(x+3y)x+(3x+5y)y\end{aligned}$$

$$=x^2+3xy+3xy+5y^2$$

$$=x^2+6xy+5y^2$$

Step-3

Compare quadratic expression $f=x^2+6xy+5y^2$ with $f=ax^2+2bxy+cy^2$

So, $a=1,\ 2b=6\Rightarrow b=3,\ c=5.$

Now $a>0$ and $ac-b^2=(1)(5)-(3)^2=-4<0.$

This implies that $a>0$ and $ac<b^2$

Notice that if $ax^2+2bxy+cy^2$ stays positive, then it necessary $ac>b^2$

Hence, here $\boxed{f \textbf{ is not positive definite.}}$

Step-4

(b).

Consider the matrix $A=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

Now the corresponding $f=x^TAx$ is,

$$f=x^TAx$$
$$=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}x-y & -x+y\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=(x-y)x+(-x+y)y$$

$$=x^2-xy-xy+y^2$$

$$=x^2-2xy+y^2$$

Step-5

Compare the quadratic expression $f=x^2-2xy+y^2$ with $f=ax^2+2bxy+cy^2$

We get $a=1,\ 2b=-2 \Rightarrow b=-1,\ c=1.$

Now $a>0$ and $ac-b^2=1-(-1)^2=0.$

Notice that $a>0$ and $ac=b^2$ then it is **positive semidefinite**.

Therefore, $\boxed{f \textbf{ is positive semi definite.}}$

Step-6

(c).

Consider the matrix $A=\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}$

The corresponding $f=x^TAx$ is,

$$f=x^TAx$$
$$=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$= \begin{pmatrix} 2x+3y & 3x+5y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= (2x+3y)x + (3x+5y)y$

$= 2x^2 + 3xy + 3xy + 5y^2$

$= 2x^2 + 6xy + 5y^2$

Step-7

Compare the quadratic expression $f = 2x^2 + 6xy + 5y^2$ with $f = ax^2 + 2bxy + cy^2$

We get $a = 2,\ 2b = 6,\ c = 5.$

Now $a > 0$ and $ac - b^2 = 1 > 0.$

Notice that if $ax^2 + 2bxy + cy^2$ stays positive, then it necessary $ac > b^2$

Therefore, $\boxed{f \textbf{ is positive definite.}}$

Step-8

(d).

Consider the matrix $A = \begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix}$

The corresponding $f = x^T Ax$ is,

$f = x^T Ax$

$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} -x+2y & 2x-8y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= (-x+2y)x + (2x-8y)y$

$= -x^2 + 2xy + 2xy - 8y^2$

$= -x^2 + 4xy - 8y^2$

Step-9

Compare the quadratic expression $-x^2+4xy-8y^2$ with $f=ax^2+2bxy+cy^2$

We get $a=-1,\ 2b=4 \Rightarrow b=2$ and $c=-8$.

Now $a<0$ and $ac-b^2=4>0$

This implies that $a<0$ and $ac>b^2$

The quadratic form is negative definite if and only if $a<0$ and $ac>b^2$

Therefore, $\boxed{f \textbf{ is negative definite}.}$