Step-1

Given that

$$A=\begin{bmatrix}1 & 2\\ 1 & 3\\ 1 & 2\end{bmatrix}, \text{and } B=\begin{bmatrix}5 & 4\\ 6 & 3\\ 5 & 1\end{bmatrix}$$

We have to find a vector in both column spaces $\mathbf{C}(A)$ and $\mathbf{C}(B)$

Step-2

Let $x=\begin{bmatrix}a\\ b\end{bmatrix}, \hat{x}=\begin{bmatrix}c\\ d\end{bmatrix}$

Given $Ax=B\hat{x}$

$Ax-B\hat{x}=0$

Step-3

This means that

$$\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x\\ -\hat{x}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 5 & 4\\ 1 & 3 & 6 & 3\\ 1 & 2 & 5 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Apply $R_2 \to R_2-R_1,\ R_3 \to R_3-R_1$

$$\begin{bmatrix}1 & 2 & 5 & 4\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & -3\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Step-4

Therefore

$a+2b+5c+4d=0$, â€¦â€¦ (1)

$b+c-d=0$, â€¦â€¦ (2)

$$-3d=0$$
$$\Rightarrow d=0$$

Step-5

By (2),

$$b+c=0$$
$$\Rightarrow b=-c$$

If $c=k$ then $b=-k$

Step-6

By (1),

$$a=-2b-5c-4d$$
$$=2k-5k$$
$$=-3k$$

Step-7

Therefore

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} -3k \\ -k \\ k \\ 0 \end{bmatrix}$$
$$=-k\begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$
$$=\begin{bmatrix} x \\ -\hat{x} \end{bmatrix}$$

Step-8

Hence

$$x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, -\hat{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-9

Therefore

$$Ax = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$$

$$B\hat{x} = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$$

Hence $Ax = B\hat{x} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$ is a vector in both column spaces $\mathbf{C}(A)$ and $\mathbf{C}(B)$