Step-1

Assume that the matrix A has three eigenvalues which are 0,3, and 5 with independent eigenvectors u, v, and w respectively.

Then, the following equations are obtained,

$$\left.\begin{aligned} Au &= 0u \\ &= 0, \\ Av &= 3v, \\ Aw &= 5w. \end{aligned}\right\} \text{â€¦â€¦ (1)}$$

Step-2

(a)

The objective is to determine a basis for the null space and a basis for column space.

Step-3

The null space of A consists of all those vectors x such that $Ax = 0$.

That is, a basis for the null space is a vector corresponding to the eigenvalue 0.

From the data, the zero eigenvalue is 0.

From (1),

$$Au = 0$$

This is obtained from the eigenvalue 0.

The eigenvector corresponding to eigenvalue 0 is u.

Hence, a basis for the null space is, $\boxed{\{u\}}$.

Step-4

A basis for the column space are vectors corresponding to the non-zero eigenvalues.

From the data, the non-zero eigenvalues are 3 and 5.

From (1),

$$\begin{aligned} Av &= 3v, \\ Aw &= 5w. \end{aligned}$$

These are obtained from the eigenvalues 3 and 5.

The eigenvector corresponding to eigenvalues 3 and 5 are v and w respectively.

Hence, a basis for the column space is, $\boxed{\{v, w\}}$.

Step-5

(b)

The objective is to determine a particular solution and find all solutions to $Ax = v + w$.

Step-6

Consider the equation,

$$Ax = v + w$$

Here $v + w$ is the linear combination of the basis for the column space.

So,

$$\begin{aligned} Ax &= A(av + bw) \\ &= a(Av) + b(Aw) \\ &= a(3v) + b(5w) && \text{From } (1), Av = 3v, Aw = 5w. \\ &= (3a)v + (5b)w \end{aligned}$$

To obtain $Ax = v + w$, choose $a = \frac{1}{3}$ and $b = \frac{1}{5}$.

Thus, the particular solution is,

$$\begin{aligned} x &= av + bw \\ &= \frac{1}{3}v + \frac{1}{5}w \end{aligned}$$

There is no term for u in this particular solution, so u term becomes 0.

Hence, the particular solution is, $x = \boxed{\left(0, \frac{1}{3}, \frac{1}{5}\right)}$.

All solution of the equation $Ax = v + w$ is,

$$\{cx \mid c \in R\} = \boxed{\left\{c\left(0, \frac{1}{3}, \frac{1}{5}\right) \mid c \in R\right\}}$$

Step-7

(c)

The objective is to prove that $Ax = u$ has no solution.

Step-8

Consider the equation,

$$Ax = u$$

Since, u, v, and w are independent vectors, any vector x can be expressed as a linear combination of these three vectors.

Assume that,

$$x = au + bv + cw.$$

Then,

$$\begin{aligned} u &= Ax \\ &= A(au + bv + cw) \\ &= a(Au) + b(Av) + c(Aw) \\ &= a(0) + b(3v) + c(5w) \qquad \text{From } (1),\ Au = 0,\ Av = 3v,\ Aw = 5w. \end{aligned}$$

$$u = 3bv + 5cw$$

Since, u, v, and w are independent vectors none of these three can be expressed as a linear combination of the others. Therefore, $u = 3bv + 5cw$ is impossible.

Hence, the equation $Ax = u$ has no solution. It may be noted that if this equation had some solution, then u would be in the column space of A.