Step-1

By choosing the correct vector in b in the Schwarz inequality, we have to prove that

$\left(a_1+a_2+\cdots+a_n\right)^2 \le n\left(a_1{}^2+a_2{}^2+\cdots+a_n{}^2\right)$, and also we have to find that, when does the equality holds.

Step-2

Let $a=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Choose $b=(1,1,...,1)$

Schwarz inequality is $\left|a^T b\right| \le \|a\|\|b\|$ â€¦â€¦ (1)

Step-3

$$a^T b=\left(a_1,a_2,...,a_n\right)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$
$$=a_1+a_2+\cdots+a_n$$
$$\Rightarrow \left|a^T b\right|=\left|a_1+a_2+\cdots+a_n\right|$$
$$\Rightarrow \left|a^T b\right|^2=\left(a_1+a_2+\cdots+a_n\right)^2$$

Step-4

And $\|a\|=\sqrt{a_1^2+a_2^2+\cdots+a_n^2}$

$$\|b\|=\sqrt{1^2+1^2+\cdots+1^2\,(n\text{ times})}$$
$$=\sqrt{n.1}$$

Step-5

$$\Rightarrow \|a\|^2=a_1^2+a_2^2+\cdots+a_n^2 \text{ and } \|b\|^2=n$$

By (1), $|a^T b|^2 \le \|a\|^2 \, \|b\|^2$

Hence $\boxed{(a_1 + a_2 + \cdots + a_n)^2 \le n\left(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2\right)}$

Step-6

If $a_1 = a_2 = \cdots = a_n$ then

$$
\begin{aligned}
&(a_1 + a_2 + \cdots + a_n)^2 \\
&= (na_1)^2 \\
&= n^2 a_1^2
\end{aligned}
$$

Step-7

And

$$
\begin{aligned}
&n\left(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2\right) \\
&= n\left(na_1{}^2\right) \\
&= n^2 a_1^2
\end{aligned}
$$

Therefore $(a_1 + a_2 + \cdots + a_n)^2 = n\left(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2\right)$

Hence the equality holds if $a_1 = a_2 = \cdots = a_n$