

Step-1

Consider the boundary value problem,

$$-\frac{d^2u}{dx^2} + u = x, \quad u(0) = u(1) = 0$$

Approximate second derivative by,

$$\begin{aligned} \frac{d^2u}{dx^2} &\approx \frac{\Delta^2 u}{\Delta x^2} \\ &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \end{aligned}$$

With the above substitution given differential equation becomes,

$$-\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + u(x) = x$$

Step-2

Measure u at the mesh points $x = jh$ by substituting x by jh .

$$\begin{aligned} &-\frac{u(jh+h) - 2u(jh) + u(jh-h)}{h^2} + u(jh) = jh \\ &-\left[u(jh+h) - 2u(jh) + u(jh-h)\right] + h^2 u(jh) = h^2 jh \\ &-u(jh+h) + 2u(jh) - u(jh-h) + h^2 u(jh) = h^3 j \\ &-u(j+1)h + 2u(jh) - u(j-1)h + h^2 u(jh) = h^3 j \end{aligned}$$

Use the notation $u_k = u(kh)$

$$-u_{j+1} + 2u_j - u_{j-1} + h^2 u_j = h^3 j$$

$$-u_{j+1} + \left(2 + h^2\right)u_j - u_{j-1} = h^3 j$$

$$\text{Let } h = \frac{1}{4}.$$

$$-u_{j+1} + \left(2 + \left(\frac{1}{4}\right)^2\right)u_j - u_{j-1} = \left(\frac{1}{4}\right)^3 j$$

$$-u_{j+1} + \left(2 + \frac{1}{16}\right)u_j - u_{j-1} = \frac{j}{64}$$

$$-u_{j+1} + \left(\frac{33}{16}\right)u_j - u_{j-1} = \frac{j}{64} \quad (1)$$

Step-3

For $j=1$ difference equation (1) becomes,

$$\begin{aligned} -u_{1+1} + \left(\frac{33}{16}\right)u_1 - u_{1-1} &= \frac{1}{64} \\ -u_2 + \left(\frac{33}{16}\right)u_1 - u_0 &= \frac{1}{64} \end{aligned}$$

From the data $u_0 = u(0) = 0$, this simplifies to,

$$\begin{aligned} -u_2 + \left(\frac{33}{16}\right)u_1 - 0 &= \frac{1}{64} \\ -u_2 + \frac{33}{16}u_1 &= \frac{1}{64} \quad (i) \end{aligned}$$

Step-4

For $j=2$ difference equation (1) becomes,

$$\begin{aligned} -u_{2+1} + \left(\frac{33}{16}\right)u_2 - u_{2-1} &= \frac{2}{64} \\ -u_3 + \frac{33}{16}u_2 - u_1 &= \frac{2}{64} \quad (ii) \end{aligned}$$

Step-5

For $j=3$ difference equation (1) becomes,

$$\begin{aligned} -u_{3+1} + \left(\frac{33}{16}\right)u_3 - u_{3-1} &= \frac{3}{64} \\ -u_4 + \frac{33}{16}u_3 - u_2 &= \frac{3}{64} \end{aligned}$$

Since $u_4 = u\left(4 \times \frac{1}{4}\right) = u(1) = 0$, the above equation simplified to,

$$-0 + \frac{33}{16}u_3 - u_2 = \frac{3}{64}$$

$$\frac{33}{16}u_3 - u_2 = \frac{3}{64} \quad \text{â€œâ€œâ€œ (iii)}$$

Step-6

From equations (i), (ii), and (iii) get the system,

$$\begin{aligned} -u_2 + \frac{33}{16}u_1 &= \frac{1}{64} \\ -u_3 + \frac{33}{16}u_2 - u_1 &= \frac{2}{64} \\ \frac{33}{16}u_3 - u_2 &= \frac{3}{64} \end{aligned}$$

Rearrange it as shown.

$$\begin{aligned} \frac{33}{16}u_1 - u_2 + 0 \cdot u_3 &= \frac{1}{64} \\ -u_1 + \frac{33}{16}u_2 - u_3 &= \frac{2}{64} \\ 0 \cdot u_1 - u_2 + \frac{33}{16}u_3 &= \frac{3}{64} \end{aligned}$$

Step-7

Write it in matrix notation to get the required matrix equation.

$$\begin{bmatrix} \frac{33}{16} & -1 & 0 \\ -1 & \frac{33}{16} & -1 \\ 0 & -1 & \frac{33}{16} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{64} \\ \frac{2}{64} \\ \frac{3}{64} \end{bmatrix}$$

It is of the form $Au = b$, where A is 3×3 matrix.