Step-1

Consider A is a symmetric 3 by 3 matrix with eigenvalues $0,1,2$.

The objective is to solve the given properties.

Since A is symmetric $\Rightarrow A^T = A$

a)

We have to explain the properties of the corresponding unit eigenvectors u, v, w.

Since A is symmetric.

So the eigenvectors corresponding to different eigenvalues are orthogonal.

Therefore here u, v, w are unit eigenvectors corresponding to eigenvalues $0,1,2$.

Hence u, v, w are orthogonal to each other.

Step-2

b)

We have to describe the Nullspace, left Nullspace, row space and column space of A in terms of u, v, w.

Since u is the eigenvector corresponding to the eigenvalue $\lambda = 0$.

Therefore the null space is spanned by u.

The left null space is also spanned by u.

The remaining two eigenvectors v, w corresponding to the eigenvalues $\lambda = 1,2$ spans the row space as well as column space.

Step-3

c)

Given that A is symmetric 3×3 matrix with eigen values 0,1,2 and respect unit eigen vectors are $u, v, w.$

By the definition of eigen vector $Ax = \lambda x,$ where λ is eigen value and x is eigen vector.

We have,

$$\left.\begin{aligned} Au &= 0\cdot u = 0 \\ Av &= 1.v = v \\ Aw &= 2(w) = 2w \end{aligned}\right\} \qquad ......(1)$$

Since the basis for the solution space is (u, v, w).

Let x be in the solution space,

$$\begin{aligned} x &= au + bv + cw \\ Ax &= A(au + bv + cw) \\ &= aAu + bAv + cAw \\ &= 0 + bv + 2cw \qquad \text{from}(1), \end{aligned}$$

Put $a = 0, b = 1$ and $c = \frac{1}{2}$ then the value of x is

$\boxed{x = v + \frac{1}{2}w}$.

Check:

R.H.S,

$$\begin{aligned} v + w &= Av + \frac{1}{2}Aw \qquad \text{from}(1), \\ &= Av + A\left(\frac{1}{2}w\right) \\ &= A\left(v + \frac{1}{2}w\right) \\ &= Ax \quad \text{where } x = \text{v} + \frac{1}{2}w. \end{aligned}$$

$= L.H.S.$

Therefore, the value of x is $\boxed{x = v + \frac{1}{2}w}$.

Step-4

d)

We have to find the conditions on b such that the solution to $Ax = b$ have a solution.

Since A is symmetric

So $A = A^T$

Since λ is an eigenvalue and b is a solution.

So λ and b are real.

We have the eigenvectors are perpendicular.

Therefore, $xb^T \cdot u = 0$

Therefore if $b^T u = 0$ then $Ax = b$ has solution.

Step-5

e)

We have to find S^{-1} and $S^{-1}AS$ if u, v, w are the columns of S.

Since A is real and symmetric.

So the eigenvectors are also real.

These orthonormal eigenvectors form an orthogonal matrix S with

$$S^T S = I$$
$$\Rightarrow S^T = S^{-1}$$

And $S^{-1}AS = \wedge = diag(0,1,2)$

Therefore, $\boxed{S^T = S^{-1} \text{ and } S^{-1}AS = diag(0,1,2)}$.