Applied stochastic processes
Review for midterm

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Math 208, Spring 2023

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1. If X has pdf f_X and g is 1-to-1, then $Y = g(X)$ has pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

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2. If $X = (X_1, X_2)$ has joint pdf $f_X(x)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is 1-to-1, then $(Y_1, Y_2) = Y = g(X)$ has joint pdf f_Y given by

$$f_Y(y) = f_X(g^{-1}(y))|J|^{-1}$$

where J is the Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

3. The moment generating function (m.g.f.) of X is

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4. A stochastic process is a family of random variables $\{X_t : t \in \mathbb{T}\}$ with time set $\mathbb{T}$ and state S.

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- If X and Y are independent, then $E(X|Y) = E(X)$.

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9. Calculating probability by conditioning

$$P(A) = E(P(A|Y)).$$

10. $\{X_n\}$ is a *Markov chain* if

$$\mathbb{P}(X_{n+1} = j | X_n = i,\ X_{n-1} = i_{n-1},\ \cdots,\ X_0 = i_0) = p(i,j),$$

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- The m-step transition matrix is P^m, namely, $P^{(m)} = P^m$.

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- $$\mathbb{P}_y(T_y^k < \infty) = \rho_{yy}^k.$$

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- If x is recurrent and $\rho_{xy} > 0$, then $\rho_{yx} = 1$.

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- A set A is *closed* if it is impossible to get out.
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- If C is a finite closed irreducible set, then all states in C are recurrent.
- If the state space S is finite, then S can be written as a disjoint union

$$S = T \cup R_1 \cup \cdots \cup R_k,$$

where T is a set of transient states and R_i, $1 \le i \le k$, are closed irreducible sets of recurrent states.

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- If x is recurrent and $x \to y$, then y is recurrent.
- In a finite closed set there has to be at lease one recurrent state.

16. Vector π is a *stationary distribution* if $\pi P = \pi$.

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- Suppose that the $k \times k$ transition matrix is irreducible. Then, there is a unique solution to $\pi P = \pi$ with $\sum_{i=1}^{k} \pi_i = 1$. Further, $\pi_i > 0$, $\forall\ i$.

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- If P is double stochastic with N states, then $\pi(x) = \frac{1}{N}$, $\forall x$ is a stationary distribution.

18. A distribution π satisfies the *detailed balance condition* if

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- The MC is *reversible* if $\hat{p}(i,j) = p(i,j)$, $\forall\ i,j \in S$.
- (X_n) is reversible iff DBC holds.

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- Under (I) and (S), we have

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22. Under (I), (S) and $\sum_x |f(x)|\pi(x) < \infty$, we have

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Namely, long term average equals spatial average, i.e., ergodicity.

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For any irreducible chain the following are equivalent:

1. There is a positive recurrent state.
2. There is a stationary distribution π.
3. All states are positive recurrent.

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$$\phi(\theta) = \sum_{k=0}^{\infty} p_k \theta^k.$$

Let $\rho = \mathbb{P}_1(\text{extinction})$. ρ is the smallest root of $x = \phi(x)$, $0 \le x \le 1$.