Step-1

Therefore, we have

$$P = \frac{aa^{\mathrm{T}}}{a^{\mathrm{T}}a}$$
$$= \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}}$$

Step-2

(b) We have, $P = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}$. To obtain the eigenvalues of *P*, we solve $\det(P - \lambda I) = 0$.

Consider

$$0 = \begin{vmatrix} \frac{4}{9} - \lambda & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} - \lambda & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} - \lambda \end{vmatrix}$$
$$= \left(\frac{4}{9} - \lambda\right)\left(\left(\frac{1}{9} - \lambda\right)\left(\frac{4}{9} - \lambda\right) - \frac{4}{81}\right) - \frac{2}{9}\left(\left(\frac{2}{9}\right)\left(\frac{4}{9} - \lambda\right) - \frac{8}{81}\right) + \frac{4}{9}\left(\frac{4}{81} - \left(\frac{4}{9}\right)\left(\frac{1}{9} - \lambda\right)\right)$$
$$= \left(\frac{4}{9} - \lambda\right)^2\left(\frac{1}{9} - \lambda\right) - \frac{4}{81}\left(\frac{4}{9} - \lambda\right) - \frac{4}{81}\left(\frac{4}{9} - \lambda\right) + \frac{16}{729} + \frac{16}{729} - \frac{16}{81}\left(\frac{1}{9} - \lambda\right)$$

$$\begin{aligned}
0 &= \left(\frac{4}{9}-\lambda\right)^2\left(\frac{1}{9}-\lambda\right)-\frac{16}{729}+\frac{4\lambda}{81}-\frac{16}{729}+\frac{4\lambda}{81}+\frac{16}{729}+\frac{16}{729}-\frac{16}{729}+\frac{16\lambda}{81} \\
&= \left(\frac{16}{81}-\frac{8\lambda}{9}+\lambda^2\right)\left(\frac{1}{9}-\lambda\right)+\frac{24\lambda}{81}-\frac{16}{729} \\
&= \frac{16}{729}-\frac{16\lambda}{81}-\frac{8\lambda}{81}+\frac{8\lambda^2}{9}+\frac{\lambda^2}{9}-\lambda^3+\frac{24\lambda}{81}-\frac{16}{729} \\
&= \lambda^2-\lambda^3
\end{aligned}$$

Step-3

Thus, the only positive eigenvalue of P is $\boxed{1}$.

Note that we obtained the eigenvalue by carrying out the long calculations. However, the eigenvalue can be easily guessed by taking into consideration the fact that P projects any vector on the straight line through $a=(2,1,2)$. If x is a vector, not along $a=(2,1,2)$, then $Px \neq \lambda x$ except when $Px=0$, that is, when x is perpendicular to the vector $a(2,1,2)$. On the other hand, if x is along the vector $a=(2,1,2)$, then $Px=x$ itself. Thus, the eigenvalues of P must be 0 and 1. Therefore, the only positive eigenvalue of P must be 1.

Corresponding eigenvector to eigenvalue 1 is, obviously, $a=(2,1,2)$, or any multiple of $\boxed{a=(2,1,2)}$.

Step-4

(c) Consider the equation $u_{k+1}=Pu_k$, where $u_0=(9,9,0)$. Thus, we have

$$\begin{aligned}
u_1 &= Pu_0 \\
&= \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}
\end{aligned}$$

Further, we have

$$
\begin{aligned}
u_2 &= Pu_1 \\
&= \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}
\end{aligned}
$$

It is clear that $\boxed{u_{k+1} = (6,3,6)}$.