Chapter 2. Conditional Expectation

2.1. Discrete case

Recall: For two events A and B, if $P(B) > 0$, then

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

Now we consider two discrete random variables X and Y.Suppose $P(Y = y) > 0$, then

$$p_{X|Y}(x|y) \equiv P(X = x|Y = y) = \frac{P(X = x,\ Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)},$$

is the conditional probability mass function of X given $Y = y$.

Similarly, we can define conditional distribution function

$$F_{X|Y}(x|y) \equiv P(X \leq x|Y = y) = \sum_{a \leq x} p_{X|Y}(a|y).$$

Finally, we define the conditional expectation of X given $Y = y$:

$$E(X|Y = y) \equiv \sum_{x} x p_{X|Y}(x|y).$$

Note that it is a function of y.Sometimes we omit y and write

$$E(X|Y) = \sum_{x} x p_{X|Y}(x|Y).$$

This is called the conditional expectation of X given Y.

Remark

If X and Y are independent, then $E(X|Y) = E(X)$.

Proof: Do it on board!

Example

Suppose that $p(x,y)$ *is given by*

$$p(0,1)=.1,\quad p(0,2)=.2,\quad p(1,1)=.3,\quad p(1,2)=.4$$

Find $E(X|Y)$ *and* $E(Y|X)$.

Solution: Do it on board!

Example

Take N free-throws and denote by X the number of points you get. Suppose that $N \sim P(\lambda)$ and your free-throw rate is p. Find $E(X|N)$ and $E(X)$.

Solution: Do it on board!

Note that $E(pN) = p\lambda = E(X)$.We have the famous "tower rule":

$$E(E(X|N)) = E(X).$$

Example

Suppose that X and Y are independent Poisson random variables with parameters λ and μ, respectively. Find $E(X|X+Y)$.

Solution: Do it on board!

2.2. Continuous case

Suppose X and Y has joint pdf $f(x,y)$ and $f_Y(y)>0$. Then,

$$
\begin{aligned}
P(X\le x|Y=y) &\equiv \lim_{h\to 0} P(X\le x|y<Y<y+h)\\
&= \lim_{h\to 0}\frac{P(X\le x, y<Y<y+h)}{P(y<Y<y+h)}\\
&= \lim_{h\to 0}\frac{\int_y^{y+h}\int_{-\infty}^x f(u,v)dudv}{\int_y^{y+h} f_Y(v)dv}\\
&= \lim_{h\to 0}\frac{h^{-1}\int_y^{y+h}\int_{-\infty}^x f(u,v)dudv}{h^{-1}\int_y^{y+h} f_Y(v)dv}\\
&= \frac{\int_{-\infty}^x f(u,y)du}{f_Y(y)}
\end{aligned}
$$

So, we define the conditional density of X given $Y = y$ as

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Next, we define the conditional expectation of X given $Y = y$ as

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx.$$

Since it is a function of y,we omit it and define the conditional expectation of X given Y as:

$$E(X|Y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|Y) dx.$$

Example

Suppose that X and Y has joint pdf

$$f(x,y)=\begin{cases}\frac{2}{\pi}(x^2+y^2) & x^2+y^2\le 1\\ 0 & otherwise\end{cases}$$

Find $f_{X|Y}(x|y)$ and $E(X|Y)$.

Solution: Do it on board!

Example

Suppose that X and Y have joint pdf

$$f(x, y) = c(x + y), \qquad 0 \leq y \leq x \leq 1,$$

where c is a constant. Find $E(X|Y)$.

Solution: Do it on board!

Example

Suppose that X and Y are independent exponential random variables with parameters λ and μ, respectively. Calculate $E(X|X+Y)$.

Solution: Do it on board!

2.3. Calculating expectation by conditioning

Recall

$$E(X) = E(E(X|Y)).$$

Example

Suppose that Tom chooses to read a probability book or a history book with equal probability. The number of typos in the probability book is a Poisson random variable with mean 2, and that for the other book is Poisson with mean 5. Let X be the number of typos Tom will find. What is $E(X)$?

Solution: Do it on board!

Example

Suppose that the number of accidents per month at a busy intersection has a mean 3, and in average, about two persons got injured per accident. What is the average number of personal injures at this intersection per month?

Solution: Do it on board!

Example

Suppose that your free-throw rate is p. Keep trying until you get 1 point. Let X be the # of tries. Find $E(X)$.

Solution: Do it on board!

Example

In last example, if you will stop until k consecutive points, what is the average # of tries.

Solution: Do it on board!

Finally, we calculate the variance of X by conditioning. Recall

$$V(X) = E(X^2) - (EX)^2.$$

Theorem

$$V(X) = E(V(X|Y)) + V(E(X|Y)).$$

Proof: Do it on board!

Example

Suppose X_1, X_2, $\cdots$ *i.i.d. with mean* μ *and variance* σ^2, *and* N *(indep. of* X_i*'s) take integer values. Let*

$$S = \sum_{i=1}^{N} X_i.$$

Find $V(S)$. *What if* $N \sim P(\lambda)$*?*

Solution: Do it on board!

Example

P140. 40. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.
(a) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?
(b) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)
(c) For part (a) find the variance of the number of days until the prisoner reaches freedom.

Solution: Do it on board!

2.4. Calculating probability by conditioning

Let A be an event and define

$$X = 1_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Then, X is a r.v. and $P(A) = E(X)$.Therefore, this section is indeed a special case of the last one.

Example

Let X and Y be indep. r.v.'s with pdf's f_X and f_Y, resp. Calculate $P(X < Y)$.

Solution: Do it on board!

Example

At a party n men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. What is the probability of no matches? What is the probability of exactly k matches?

Solution: Do it on board!

Example

A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have won two more games than the other.

(a) Find the probability that A is the overall winner.

(b) Find the expected number of games played.

Solution: Do it on board!

HW: Ch3, 1, 5, 7, 10, 12, 14, 17, 24, 27, 29, 37, 38, 41, 49.