Step-1

We have to explain, why all the following statements are false.

a) The complete solution is any linear combination of x_p and x_n.

The statement is false.

Since the particular solution x_p is always multiplied by 1, and x_n is multiplied by any real number.

Step-2

b) A system $Ax = b$ has at most one particular solution.

The statement is false. Since, there are infinitely many particular solutions for the system. In fact, any solution is itself a particular solution.

Step-3

c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$).

Let $A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

Now consider $Ax = b$, where $b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

Step-4

Therefore

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 + 3x_2 = 6$$
$$\Rightarrow x_1 + x_2 = 2$$
$$\Rightarrow x_1 = 2 - x_2$$

Step-5

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 - x_2 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now $x_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, x_n = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Step-6

$$\|x_p\| = \sqrt{2^2 + 0^2}$$
$$= 2$$

$$\|x_n\| = \sqrt{(-1)^2 + (1)^2}$$
$$= \sqrt{2}$$

Therefore x_n has shorter length than x_p

Hence the statement is false.

Step-7

d) If A is invertible there is no solution x_n in the nullspace.

Consider the invertible matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Then consider the homogeneous system

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-8

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_1-2R_2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow x_1=0, x_2=0$$

Therefore $\begin{bmatrix}0\\0\end{bmatrix}$ is a solution in the null space of A.

Hence the given system is false.