Step-1

(a)

Consider that $f = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right)$.

The objective is to find 3×3 the symmetric matrix A which produce the function $f = x^T Ax$ and check and check whether the symmetric matrix A is positive definite or not.

Step-2

Given quadratic is,

$$\begin{aligned} f &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right) \\ &= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \end{aligned}$$

Find the 3×3 symmetric matrix which produce $f = x^T Ax$ as,

i) $a_{ii} = \text{coefficient of } x_i^{\,2}$

ii) $a_{ij} = \frac{1}{2}\left(\text{coefficient of } x_{ij}\right) \quad \text{where } i \neq j$

Step-3

Therefore, the function $f = x^T Ax$ can be written as,

$$f = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Where $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Step-4

Recall: the real symmetric matrix A to be positive definite if and only if all pivots satisfy $d_k > 0$ where d_k is ratio of $\det\left(A_k\right)$ to $\det\left(A_{k-1}\right)$.

Step-5

Now the upper left determinants are,

$A_1 = 2 > 0$

$$\begin{aligned} A_2 &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 4 - 1 \\ &= 3 > 0 \end{aligned}$$

$$\begin{aligned} A_3 &= |A| \\ &= 2(4-1) + 1(-2) + 0 \\ &= 6 - 2 \\ &= 4 > 0 \end{aligned}$$

The pivots are $\frac{A_1}{1}, \frac{A_2}{A_1}$ and $\frac{A_3}{A_2}$.

That is, $\frac{2}{1}, \frac{3}{2}$ and $\frac{4}{3}$ all are positive.

Therefore, $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ have pivots $\boxed{2, \frac{3}{2} \text{ and } \frac{4}{3}}$.

Therefore, A is positive definite.

Step-6

(b)

Consider that $f = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right)$.

The objective is to find 3×3 the symmetric matrix A which produce the function $f = x^T Ax$ and check and check whether the symmetric matrix A is positive definite or not.

Step-7

Given quadratic is,

$$\begin{aligned} f &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right) \\ &= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 \end{aligned}$$

Therefore, the function $f = x^T Ax$ can be written as,

$$f=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

Where the matrix $A=\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}$

Step-8

Now the upper left determinants are,

$$A_1=2>0$$

$$\begin{aligned}A_2&=\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}\\ &=3>0\end{aligned}$$

Continuing the previous steps,

$$\begin{aligned}A_3&=|A|\\ &=2(4-1)+1(-2-1)-1(1+2)\\ &=6-3-3\\ &=0\end{aligned}$$

Therefore, $A=\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}$ is singular.

Step-9

Let $x=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$.

Consider,

$$Ax = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ this means the symmetric matrix A is **positive semidefinite** but not positive definite.