MA202 Complex Analysis, Final Exam

Name:

ID:

Problem 1. [15 pts]

(i) What is the definition of a meromorphic function on a domain $\Omega \subseteq \mathbb{C}$?

(ii) State the Riemann mapping theorem.

Problem 2. [10 pts] Calculate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x$, $a \in \mathbb{R}^{\times}$.

Problem 3. [10 pts] Calculate the integral $\int_{0}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$.

Problem 4. [15 pts] Calculate the integral

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|\,\mathrm{d}\theta, \quad |a| \neq 1$$

Problem 5. [15 pts] Find an infinite product formula of the function $\cos z$ in terms of its zeros.

Problem 6. [15 pts] Let $f(z)$ be an entire function and n a non-negative integer. Show that if $\lim\limits_{z\to\infty} \frac{f(z)}{z^n}$ exists and is nonzero, then $f(z)$ is a polynomial of degree n.

Problem 7. [10 pts] Let $\mathbb{D}$ denote the open unit disk and f be a holomorphic function on $\mathbb{D}$. Show that the diameter

$$d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$$

satisfies the inequality $d \geqslant 2|f'(0)|$.

Problem 8. Suppose f and g are holomorphic in a region containing the disc $|z| \leqslant 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leqslant 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show the following:

(i) [5pts] $f_\epsilon(z)$ has a unique zero z_ϵ in $|z| \leqslant 1$ if ϵ is sufficiently small.

(ii) [5pts] The map $\epsilon \to z_\epsilon$ is holomorphic when ϵ is sufficiently smally.