Midterm I for Calculu II (solutions)

1. (1) $a_n = \frac{2^n+4^n}{3^n+4^n} = \frac{(\frac{1}{2})^n+1}{(\frac{3}{4})^n+1} \to 1$ as $n\to\infty$

$\therefore a_n \not\to 0 \quad \therefore \sum a_n$ div.

(2) $f(x) = \frac{1}{x(\ln x)^2}$, $f(x) > 0$ & $\searrow$ for $x \geqslant 2$.

$a_n = f(n)$ for $n \geqslant 2$.

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\ln x)^2}\,dx \overset{u=\ln x}{\underset{du=\frac{1}{x}dx}{=\!=}} \int_{\ln 2}^\infty \frac{1}{u^2}\,du \quad \text{conv. } (p=2>1)$$

$\therefore \sum a_n$ conv. & A.C. $(a_n > 0)$

(3) $a_n = \frac{1}{n\sqrt[n]{n}}$, $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \Rightarrow$ both conv. or both div.

$\because \sum b_n$ div. $\therefore \sum a_n$ div.

(4) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{(n+1)!}\,\cancel{(n+2)!}\,(n+3)!}{(3n+3)!} \cdot \frac{(3n)!}{n!\,\cancel{(n+1)!}\,\cancel{(n+2)!}} = \frac{(n+3)(n+2)(n+1)}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27} < 1$ as $n\to\infty$

$\Rightarrow \sum a_n$ A.C.

(5) $\sum_{n=1}^\infty (-1)^n u_n$, $u_n = \sqrt{n^2+1} - n > 0$,

$u_n = \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \searrow$ in n.

$\lim_{n\to\infty} u_n = 0$. $\therefore$ it is conv. by alternating series test.

$\sum_{n=1}^\infty u_n = \sum_{n=1}^\infty \frac{1}{\sqrt{n^2+1}+n}$, $b_n = \frac{1}{n}$

$\frac{u_n}{b_n} = \frac{n}{\sqrt{n^2+1}+n} \to \frac{1}{2}$ $\therefore \sum u_n$ & $\sum b_n$ both conv. or both div.

Since $\sum \frac{1}{n}$ div., $\sum u_n$ div. In summary, $\sum (-1)^n u_n$ is C.C.

2. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{|x|^n} \longrightarrow |x|$

$\therefore$ radius of conv. $r = 1$.

$x = 1$: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+3}}$ conv. by alternating series test

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ div. Since $\frac{\frac{1}{\sqrt{n^2+3}}}{\frac{1}{n}} \to 1$ & $\sum \frac{1}{n}$ div.

$\therefore$ C.C. at $x=1$.

$x = -1$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ div.

In summary, radius of conv. is 1 ;
interval of conv. is $(-1, 1]$;
the series is A.C. in $(-1, 1)$ & C.C. at $x=1$.

3. $f(x) = (x+1)e^x = (x+1)\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$

$= 1 + 2x + \left(\frac{1}{2!} + 1\right)x^2 + \left(\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \cdots$

$= 1 + \sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!} + \frac{1}{n!}\right)x^n = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$

4. $\left.\begin{array}{l} \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \\ [\ln(1+x)]' = \frac{1}{1+x} \end{array}\right\}$

$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\Rightarrow \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\Rightarrow 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$

4 (Continued)

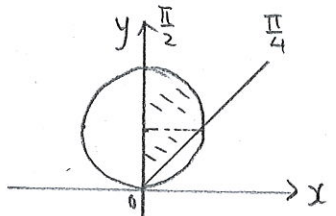
$$\lim_{x\to 0}\frac{\ln(1+x^2)}{1-\cos x} = \lim_{x\to 0}\frac{x^2-\frac{x^4}{2}+\frac{x^6}{3}\cdots}{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}\cdots} = \boxed{2}.$$

5. $$L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \int_0^{2\pi}\frac{3}{2}\,|\sin 2t|\,dt = (4)\left(\frac{3}{2}\right)\int_0^{\frac{\pi}{2}}\sin 2t\,dt$$

$$= -3\cos 2t\Big]_0^{\frac{\pi}{2}} = \boxed{6}.$$

6. $r = 2\sin\theta$, $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$.

$r^2 = 2r\sin\theta$, $\quad x^2+y^2=2y$, $\quad x^2+(y-1)^2=1$



$$A=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}[1-\cos 2\theta]\,d\theta = \left[\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \boxed{\frac{\pi}{4}+\frac{1}{2}}.$$

Alternative method: $A=\frac{1}{4}$(area of disc) + area of the triangle

$$= \frac{\pi}{4}+\frac{1}{2}.$$

7. $(1+x)^m = 1+\sum_{k=1}^{\infty}\binom{m}{k}x^k$, $\quad \binom{m}{k}=\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$ for $|x|<1$.

$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots$ for $|x|<1$.

8. First assume $\{x_n\}$ conv. to L. Then $\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_n=L$.
Thus $L=\frac{L}{2}+\frac{1}{L} \Rightarrow L=\sqrt{2}$ (we observe $L\ge 0$).
Now we show $\{x_n\}$ conv. indeed:

- Since $x_n>0$, we see $x_{n+1}=\frac{x_n}{2}+\frac{1}{x_n}\ge\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$ for all n.
- $x_{n+1}-x_n=\frac{2-x_n^2}{2x_n}\le 0$

Thus, $\{x_n\}$ is nonincreasing and bounded from below by $\frac{\sqrt{2}}{2}$, and therefore it must converge. $\square$.