Step-1

Given that $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$

We have to solve the normal equations $A^T A\hat{x} = A^T b$.

Step-2

First we write the equations that would hold if a line could go through the all four points.

Then every $C + Dt$ would agree exactly with *b*.

Now $Ax = b$ is

$$C + D(0) = 0$$
$$C + D(1) = 8$$
$$C + D(3) = 8$$
$$C + D(4) = 20$$

Or $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

Where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $x = \begin{bmatrix} C \\ D \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

Step-3

We know that least-square solution is

$A^T A\hat{x} = A^T b$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \begin{pmatrix} 1(1)+1(1) \\ +1(1)+1(1) \end{pmatrix} & \begin{pmatrix} 1(0)+1(1) \\ +1(3)+1(4) \end{pmatrix} \\ \begin{pmatrix} 0(1)+1(1) \\ +3(1)+4(1) \end{pmatrix} & \begin{pmatrix} 0(0)+1(1) \\ +3(3)+4(4) \end{pmatrix} \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1(0)+1(8) \\ +1(8)+1(20) \end{pmatrix} \\ \begin{pmatrix} 0(0)+1(8) \\ +3(8)+4(20) \end{pmatrix} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

Step-4

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 4 & 8 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 36 \\ 40 \end{bmatrix}$$

$\Rightarrow 4\hat{C} + 8\hat{D} = 36$ and $10\hat{D} = 40$

$\Rightarrow \hat{D} = 4$ and $\hat{C} = \dfrac{36-8(4)}{4} = 1$

Therefore $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Hence the best line is $\boxed{b = 1 + 4t}$.

Step-5

We have to find the four heights p_i.

Now

$$
\begin{aligned}
p_1 &= a_1 \hat{x} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= 1+0 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
p_2 &= a_2 \hat{x} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= 1+4 \\
&= 5
\end{aligned}
$$

Step-6

And

$$
\begin{aligned}
p_3 &= a_3 \hat{x} \\
&= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= 1+12 \\
&= 13
\end{aligned}
$$

$$
\begin{aligned}
p_4 &= a_4 \hat{x} \\
&= \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= 1+16 \\
&= 17
\end{aligned}
$$

Where a_1, a_2, a_3, a_4 are rows of A

Now we have to find the four errors e_i

Now

$$
\begin{aligned}
e_1 &= a_1 \hat{x} - b_1 \\
&= 1-0 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
e_2 &= a_2 \hat{x} - b_2 \\
&= 5-8 \\
&= -3
\end{aligned}
$$

Step-7

And

$$\begin{aligned}
e_3 &= a_3\hat{x} - b_3 \\
&= 13 - 8 \\
&= 5
\end{aligned}$$

$$\begin{aligned}
e_4 &= a_4\hat{x} - b_4 \\
&= 17 - 20 \\
&= -3
\end{aligned}$$

Step-8

Now

$$\begin{aligned}
E^2 &= e_1^2 + e_2^2 + e_3^2 + e_4^2 \\
&= (1)^2 + (-3)^2 + 5^2 + (-3)^2 \\
&= 1 + 9 + 25 + 9 \\
&= 44
\end{aligned}$$

Hence the minimum value is $\boxed{E^2 = 44}$.