Step-1

Consider the matrix as follows:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Find the column space and null space of A and the solution to $Ax = b$.

Step-2

Step1:

Consider the augmented matrix as;

$$[A:b] = \begin{bmatrix} 2 & 4 & 6 & 4 & : & 4 \\ 2 & 5 & 7 & 6 & : & 3 \\ 2 & 3 & 5 & 2 & : & 5 \end{bmatrix}$$

Apply row operation: $R_2 - R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & 1 \end{bmatrix}$$

Step-3

Apply row operation: $R_1 \to \frac{1}{2} R_1$ and $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply row operation: $R_3 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-4

Step2:

The last equation shows the solvability condition $0 = 0$.

Step-5

Step3:

The column space of A is the plane containing all combination of the pivot columns that is column first and second.

Therefore, the column space of matrix A is $\boxed{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}}$.

Step-6

Step 4:

To find null space of matrix A;

Since $Ax = 0$ is same as $Ux = 0$, so

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has infinitely many solutions:

$$\begin{aligned} x_1 &= -x_3 + 2x_4 \\ x_2 &= x_3 - 2x_4 \\ x_3 &= \text{Free variable} \\ x_4 &= \text{Free variable} \end{aligned}$$

Then, the solution can be written in the vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 + 2x_4 \\ x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the null space has a basis formed by the set $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}}$.

Step-7

Step5:

First find particular solution.

Elimination takes $Ax = b$ to $Ux = c$.

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

The system has infinitely many solutions:

$$x_1 = 4 - x_3 + 2x_4$$
$$x_2 = -1 + x_3 - 2x_4$$
$$x_3 = \text{Free variable}$$
$$x_4 = \text{Free variable}$$

Substitute $x_3 = 0, x_4 = 0$, to get

$$x_1 = 4$$
$$x_2 = -1$$
$$x_3 = 0$$
$$x_4 = 0$$

Therefore, the particular integral that is x_p is $\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

Hence, the complete solution to $Ax = b$ is

$$x_p + \text{all } x_n = \boxed{\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$$