Step-1

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

To find the eigen values of A, we consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} \frac{8}{10} - \lambda & \frac{3}{10} \\ \frac{2}{10} & \frac{7}{10} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (.8 - \lambda)(.7 - \lambda) - (.3)(.2) = 0$$
$$\Rightarrow 0.56 - 1.5\lambda + \lambda^2 - 0.6 = 0$$
$$\Rightarrow \lambda^2 - 1.5\lambda + 0.5 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 0.5\lambda + 0.5 = 0$$
$$\Rightarrow \lambda(\lambda - 1) - 0.5(\lambda - 1) = 0$$
$$\Rightarrow (\lambda - 1)(\lambda - 0.5) = 0$$

$\Rightarrow \lambda = 1, \lambda = 0.5$ are the eigen values of the given matrix

Step-2

To find the eigen vector corresponding to $\lambda = 1$, we consider $|A - \lambda I|x = 0$

$$\begin{bmatrix} .8 - 1 & .3 \\ .2 & .7 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying row operation $R_2 \to R_2 + R_1$, we get $\begin{bmatrix} -0.2 & 0.3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix.

So, back substituting the equations, we get $x_1 = 1.5x_2$

Putting $x_2 = 2$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

Step-3

Similarly, to find the eigen vector corresponding to $\lambda = 0.5$, we consider $|A - \lambda I| x = 0$

$$\begin{bmatrix} .8-0.5 & .3 \\ .2 & .7-0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} .3 & .3 \\ .2 & .2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying row operation $R_2 \to R_2 - \frac{2}{3}R_1$, we get $\begin{bmatrix} 0.3 & 0.3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix.

So, back substituting the equations, we get $x_1 = -x_2$

Putting $x_1 = 1$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 0.5$

Step-4

The matrix whose columns are the eigen vectors of A is $P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

$$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$$

So, by the diagonalizability, we have $P^{-1}AP = D$ where D is the diagonal matrix.

$$\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & .5 \\ 2 & -.5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

This is the diagonal matrix whose diagonal entries are nothing but the eigen values of A.

Step-5

The above equation can otherwise be written as $A = PDP^{-1}$

Raising this equation to the power n, we easily see

$$\begin{aligned} A^n &= \left(PDP^{-1}\right)\left(PDP^{-1}\right)\ldots\left(PDP^{-1}\right) \\ &= PD^nP^{-1} \end{aligned}$$

So, $A^2 = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1^2 & 0 \\ 0 & .5^2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ â€¦â€¦ (1)

$$= \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0.5 & -0.75 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3.5 & 2.25 \\ 1.5 & 2.75 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

Step-6

$$\begin{aligned} \lim_{n\to\infty} A^n &= P\left\{\lim_{n\to\infty} D^n\right\}P^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\left\{\lim_{n\to\infty}\begin{bmatrix} 1^n & 0 \\ 0 & 0.5^n \end{bmatrix}\right\}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

$= \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$ â€¦â€¦ (2)

Step-7

We have the equation $A^n = PD^nP^{-1}$

So, the eigen values of A^n are nothing but the n^{th} powers of the diagonal matrix of A.

That is $1^n, 0.5^n$

We know that $1^2 = 1, \lim_{n\to\infty} 1^n = 1$

The average of $1, \lim_{n\to\infty} 1^n$ is 1^2 â€¦â€¦ (3)

But $0.5^2 = \frac{1}{2}(0.5), \lim_{n\to\infty} 0.5^n = 0$

The average of 0.5 and 0 is 0.5^2 â€¦â€¦ (4)

Putting (3) and (4) together, we can say that the eigen values of A^2 are nothing but the averages of the eigen values of A, and $\lim_{n\to\infty} A^n$.