SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 06 Solutions

1. Suppose the probability density function of the random variable X is:

$$f(x) = \begin{cases} cx^3, & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

   (a) Find the value of the constant c.

   (b) Sketch $f(x)$.

   (c) Obtain the cumulative distribution function $F(x)$.

   (d) Find P(0.25 < X < 0.75).

Solution:

(a) The p.d.f must integrate to 1 and thus in this case

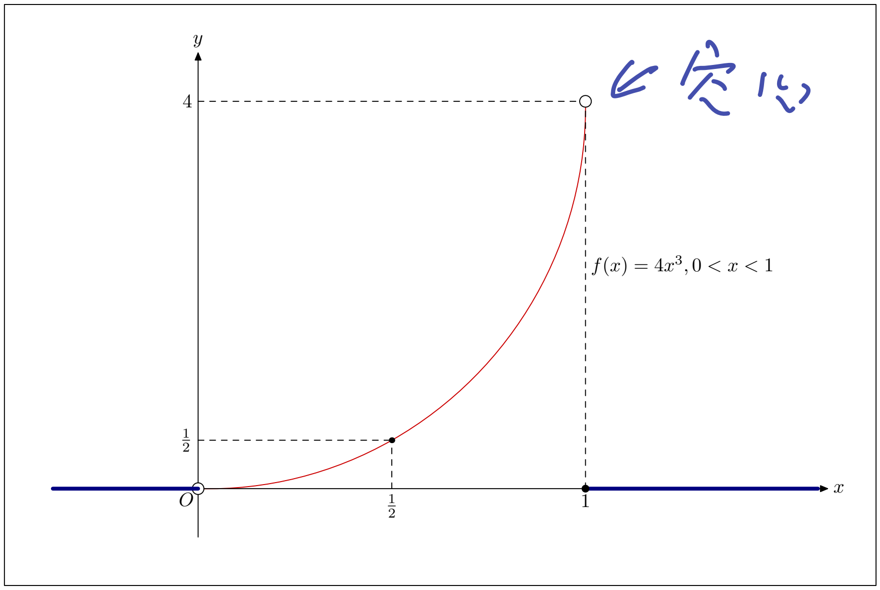
$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \int_{-\infty}^{0} f(x)\mathrm{d}x + \int_{0}^{1} f(x)\mathrm{d}x + \int_{1}^{\infty} f(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} 0\mathrm{d}x + \int_{0}^{1} cx^3\mathrm{d}x + \int_{1}^{\infty} 0\mathrm{d}x \\ &= \int_{0}^{1} cx^3\mathrm{d}x = \frac{cx^4}{4}\Big|_0^1 = \frac{c}{4}. \end{aligned}$$

So, $c = 4$.

(b) Notice that the probability density function is:

$$f(x) = \begin{cases} 0, & x \le 0, \\ 4x^3, & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

Hence, the sketch of the probability density function $f(x)$ is:



(c) 1° if $x < 0$, then $F(x) = \int_{-\infty}^{x} f(u)\mathrm{d}u = \int_{-\infty}^{x} 0 = 0$.
2° if $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{x} f(u)\mathrm{d}u = \int_{0}^{x} 4u^3\mathrm{d}u = u^4\Big|_0^x = x^4.$$

3° if $x > 1$, $F(x) = \int_{-\infty}^{x} f(u)\mathrm{d}u = \int_0^1 4u^3\mathrm{d}u = 1$.
So the c.d.f. is

$$F(x) = \begin{cases} 0, & x < 0, \\ x^4, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

(d) Finally,

$$\mathrm{P}\left(0.25 < \mathrm{X} < 0.75\right) = \mathrm{F}(0.75) - \mathrm{F}(0.25) = 0.75^4 - 0.25^4 = \frac{5}{16}.$$

Remark: For any continuous random variable X, $a, b \in \mathbb{R}$,

$$\mathrm{P}(\mathrm{a} < \mathrm{X} < \mathrm{b}) = \mathrm{P}(\mathrm{a} \le \mathrm{X} < \mathrm{b}) = \mathrm{P}(\mathrm{a} < \mathrm{X} \le \mathrm{b}) = \mathrm{P}(\mathrm{a} \le \mathrm{X} \le \mathrm{b}) = \mathrm{F}(\mathrm{b}) - \mathrm{F}(\mathrm{a}).$$

2. A non-negative-valued continuous random variable X is said to have a memoryless property if

$$\mathrm{P}(\mathrm{X} > \mathrm{s} + \mathrm{t} | \mathrm{X} > \mathrm{t}) = \mathrm{P}(\mathrm{X} > \mathrm{s})$$

is true for all $s > 0$ and $t > 0$.

Show that any exponential random variable has the memoryless property.

r.v. 连续
补、无记忆性 $\Rightarrow$. 指数分布.

Proof: Assume X obeys the exponential distribution with parameter λ, i.e.

$$X \sim \mathrm{Exp}(\lambda).$$

Then the c.d.f. of X takes the form

$$F(x) = \begin{cases} 1 - \mathrm{e}^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Therefore for any $t > 0$ and $s > 0$, we have

$$\begin{aligned}
&\mathrm{P(X > s + t | X > t)} \\
&= \frac{\mathrm{P(X > s + t, X > t)}}{\mathrm{P(X > t)}} \\
&= \frac{\mathrm{P(X > s + t)}}{\mathrm{P(X > t)}} \quad \text{since } \{X > s + t\} \subset \{X > t\}, s > 0, t > 0 \\
&= \frac{1 - \mathrm{P(X \leq s + t)}}{1 - \mathrm{P(X \leq t)}} = \frac{1 - \left(1 - e^{\lambda(s+t)}\right)}{1 - (1 - e^{\lambda t})} \\
&= \frac{\mathrm{e}^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} \\
&= 1 - (1 - e^{-\lambda s}) = 1 - \mathrm{P(X \leq s)} \\
&= \mathrm{P(X > s)}.
\end{aligned}$$

Hence any exponential random variable has the memoryless property.

3. For a certain type of electrical component, the lifetime X (the unit is every 1000 hours) has an Exponential distribution with rate parameter $\lambda = 0.5$.

   (a) What is the probability that a new component will last longer than 1000 hours?

   (b) If a component has already lasted 1000 hours, what is the probability that it will last at least 1000 hours more?

   Solution:

   (a) Since $X \sim \mathrm{Exp}(\lambda)$, the c.d.f. of X takes the form

   $$F(x) = \begin{cases} 1 - \mathrm{e}^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

   Note that in our case, measuring time is in units of 1000 hours and $\lambda = 0.5$, and thus

   $$\{X > 1\} = \{\text{a new component will last longer than 1000 hours}\}.$$

   Hence we only need to find the probability $\mathrm{P(X > 1)}$. Thus,

   $$\begin{aligned}
   \mathrm{P(X > 1)} = 1 - \mathrm{P(X \leq 1)} = 1 - F(1)& \\
   &= 1 - \left(1 - \mathrm{e}^{-0.5\times 1}\right) = \mathrm{e}^{-0.5}.
   \end{aligned}$$

(b) Notice that

$$\{X > 1\} = \{\text{a component has already lasted 1000 hours}\};$$
$$\{X \geq 2\} = \{\text{the component will last at least 1000 hours more}\}.$$

We now want to obtain $\mathrm{P}(\mathrm{X} \geq 2 \mid \mathrm{X} > 1)$. But by the memoryless property of the Exponential distribution, we have

$$\mathrm{P}(\mathrm{X} \geq 2 \mid \mathrm{X} > 1) = \mathrm{P}(\mathrm{X} \geq 1) = \mathrm{P}(\mathrm{X} > 1) = \mathrm{e}^{-0.5}.$$

4. The number of phone calls received at a certain residence in any period of t hours is a Poisson random variable with parameter $\lambda = \mu t$ for some $\mu > 0$.

   (a) What is the probability that no calls are received during a period of t hours?

   (b) Let T be the time (the unit is hourly) at which the first call after time zero is received.

   1° Write down an expression for $\mathrm{P}(\mathrm{T} \leq \mathrm{t})$.

   2° What is the name of the distribution of the random variable T?

Solution:

(a) Let X_t be the number of phone calls received at a certain residence in any period of t hours, then

$$X_t \sim \text{Poisson}(\mu t).$$

So,

$$\mathrm{P}(\mathrm{X_t} = 0) = \frac{(\mu \mathrm{t})^0}{0!} \cdot \mathrm{e}^{-\mu \mathrm{t}} = \mathrm{e}^{-\mu \mathrm{t}}.$$

(b) Notice that

$$\{T \leq t\} = \{X_t \geq 1\} = \{X_t > 0\}.$$

1° Then for $t > 0$,

$$\mathrm{P}(\mathrm{T} \leq \mathrm{t}) = \mathrm{P}(\mathrm{X_t} > 0) = 1 - \mathrm{P}(\mathrm{X_t} \leq 0) = 1 - \mathrm{P}(\mathrm{X_t} = 0) = 1 - \mathrm{e}^{-\mu \mathrm{t}}.$$

2° That is, T obeys the exponentially distributed with (rate) parameter μ.

5. The Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ has (cumulative) distribution function

$$F(x) = 1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^\beta} \qquad x \geq 0.$$

   (a) Find the median of the distribution in terms of the parameters α, β(The median of a random variable X is the value m such that $Pr(X \leq m) = 0.5$) .

注: m. median. $P(X \geq m) \geq 1/2$, $P(X \leq m) \geq 1/2$.

若 $I=[a,b]$. 满足 $P(I)>\frac{1}{2}$. 证 $m\in I$.

(b) From the Weibull distribution function given above, derive an expression for the corresponding probability density function.

Solution:

(a) Notice that the median of a random variable X is the value m such that

$$\mathrm{P(X \leq m) = 0.5},$$

i.e.,

$$\begin{aligned}
\mathrm{P(X \leq m) = F(m)} &= 0.5 \\
\Longrightarrow 1 - \mathrm{e}^{-\left(\frac{m}{\alpha}\right)^{\beta}} &= 0.5 \\
\Longrightarrow \mathrm{e}^{-\left(\frac{m}{\alpha}\right)^{\beta}} &= 0.5 \\
\Longrightarrow -\left(\frac{m}{\alpha}\right)^{\beta} &= \ln(0.5) = \ln\left(\frac{1}{2}\right) \\
\Longrightarrow \left(\frac{m}{\alpha}\right)^{\beta} &= -\ln\left(\frac{1}{2}\right) = \ln 2 \\
\Longrightarrow \frac{m}{\alpha} &= (\ln 2)^{\frac{1}{\beta}} \\
\Longrightarrow m &= \alpha (\ln 2)^{\frac{1}{\beta}}.
\end{aligned}$$

(b) Notice that $F(0) = 0$ and the c.d.f. $F(x)$ is an increasing function, so for $x \leq 0$, we have

$$0 \leq F(x) \leq F(0) = 0, \quad \Rightarrow F(x) = 0.$$

Therefore the Weibull distribution function is:

$$F(x) = \begin{cases} 1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then a probability density function $f(x)$ is:

1° For $x > 0$,

$$\begin{aligned}
f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right) \\
&= \left(-\mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right) \times \left(-\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\right) \times \left(\frac{1}{\alpha}\right) \\
&= \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1} \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}} \\
&= \frac{\beta}{\alpha^{\beta}} x^{\beta-1} \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}.
\end{aligned}$$

2° For $x < 0$, $f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) = \frac{\mathrm{d}}{\mathrm{d}x}(0) = 0$.

3° As to $x = 0$, we may let $f(x) = f(0) = 0$.

Hence a probability density function $f(x)$ is:

$$f(x) = \begin{cases} \dfrac{\beta}{\alpha^\beta} x^{\beta-1} \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^\beta}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$