

Step-1

Given that $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$

Now $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix} \end{aligned}$$

Step-2

Now

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} \\ &= (-1-\lambda)(-\lambda) - 6 \\ &= \lambda + \lambda^2 - 6 \\ &= \lambda^2 + \lambda - 6 \end{aligned}$$

Step-3

We know that

$$\begin{aligned} |A - \lambda I| &= 0 \\ \lambda^2 + \lambda - 6 &= \lambda^2 + 3\lambda - 2\lambda - 6 \\ &= \lambda(\lambda + 3) - 2(\lambda + 3) \\ &= (\lambda + 3)(\lambda - 2) \\ (\lambda + 3)(\lambda - 2) &= 0 \\ \lambda &= -3, 2 \end{aligned}$$

Hence the Eigen values of A are -3, 2

Step-4

Case (i) Let $\lambda = -3$

Eigen vectors X corresponding to Eigen value -3 are given by

$$(A - (-3)I)X = 0$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add -1 times second row to the first row to the second row

$$\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$2x_1 + 3x_2 = 0$$

Step-5

Let $x_2 = k$

Therefore

$$x_1 = -3k / 2$$

Therefore eigenvectors corresponding to eigenvalue -3 are given by $k \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ where k is a non-zero parameter.

Step-6

Case (ii) Let $\lambda = 2$

Eigen vectors X corresponding to the Eigen value 2 are given by

$$(A - 2I)X = 0$$

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add +1 times first row to the second row

$$\begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-7

Add 3 times second row and -1 times first row to second row

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-3x_1 + 3x_2 = 0$$

Let $x_2 = k (\neq 0)$

Therefore $x_1 = k$

Therefore Eigen vectors corresponding to Eigen value 2 are given by $k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Where k is a non-zero parameter

Step-8

$$\text{Now } A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7 - \lambda & -3 \\ -2 & 6 - \lambda \end{bmatrix}$$

Step-9

$$|A - \lambda I| = \begin{vmatrix} 7 - \lambda & -3 \\ -2 & 6 - \lambda \end{vmatrix}$$
$$= (7 - \lambda)(6 - \lambda) - 6$$
$$= 42 - 7\lambda - 6\lambda + \lambda^2 - 6$$
$$= \lambda^2 - 13\lambda + 36$$

We know that

$$|A - \lambda I| = 0$$
$$\lambda^2 - 13\lambda + 36 = 0$$

Step-10

We have a formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{13 \pm \sqrt{(13)^2 - 4 \times 1 \times 36}}{2 \times 1}$$

$$= \frac{13 \pm \sqrt{169 - 144}}{2}$$

$$= \frac{13 \pm \sqrt{25}}{2}$$

$$= \frac{13 \pm 5}{2}$$

$$= \frac{18}{2}, \frac{8}{2}$$

$$= 9, 4$$

$$\lambda = 9, 4$$

Hence the Eigen value of A^2 are 9, 4

Step-11

Case (i) Let $\lambda = 9$

Eigen vectors X corresponding to Eigen value 9 are given by

$$(A - 9I)X = 0$$

$$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add -1 times second row and +1 times the first row to the second row

$$\begin{bmatrix} -2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 3x_2 = 0$$

Step-12

Let $x_2 = k$

Therefore $x_1 = -3k / 2$

Therefore eigenvectors corresponding to eigenvalue 9 are given by $k \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ where k

is a non-zero parameter.

Step-13

Case (ii) Let $\lambda = 4$

Eigen vectors X corresponding to Eigen value 4 are given by

$$(A - 4I)X = 0$$

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add +1 times the first row to the second row

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-14

Add 3 times second row and -1 times first row to second row

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$3x_1 - 3x_2 = 0$$

Let $x_2 = k$

Therefore $x_1 = k$

Therefore Eigen vectors corresponding to Eigen value 4 are given by $k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Where k is a non-zero parameter

A^2 has the same square as A

When A has an Eigen value λ_1, λ_2 then A^2 has Eigen values λ_1^2, λ_2^2