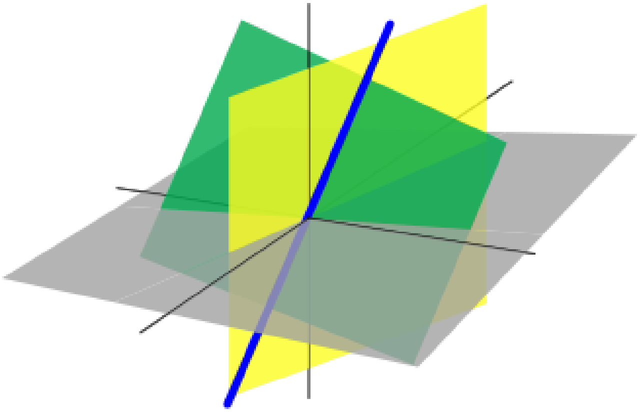
2 Vector Spaces (向量空间)

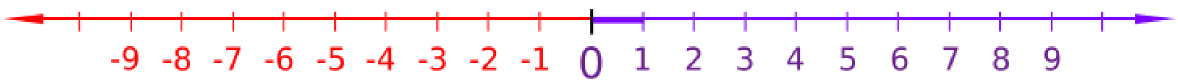
2.1

VECTOR SPACES AND SUBSPACES (向量空间与子空间)

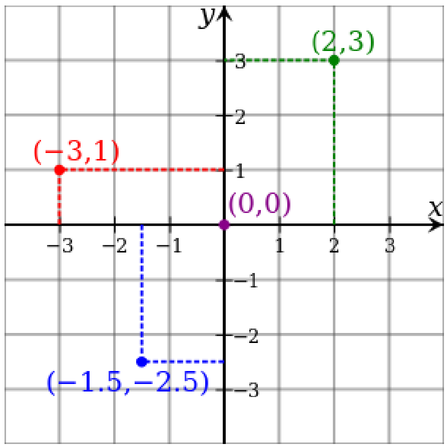
- Vector spaces
- Subspaces
- Spanned subspaces
- Column spaces and nullspaces



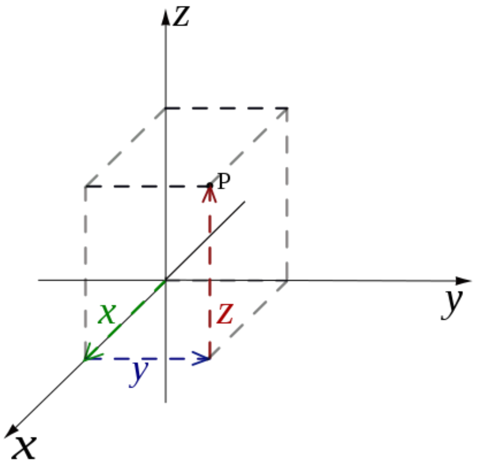
Let **R** (实数域) be the set of all real numbers. Then



• $\mathbf{R}^1 = \mathbf{R}$ can be represented as a line, called the **number line**.



• Let $\mathbf{R}^2$ be the collection of all pairs (x, y) of real numbers. Then each element of $\mathbf{R}^2$ can be represented as a point on an xy-plane, called **Euclidean plane**.



• Let $\mathbf{R}^3$ be the collection of all triples (x, y, z) of real numbers. Then each element of $\mathbf{R}^3$ can be represented as a point on an xyz-space, representing the **3-dimensional space** we live in. The location we are at can be described by a triple of numbers (x, y, z).

引例1

在第一章中,我们讨论了$\mathbf{R}$上的n维向量空间$\mathbf{R}^n$,

定义了两个向量的加法和数量乘法(数乘):

$$(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)$$

$$k(a_1, a_2, \cdots, a_n) = (ka_1, ka_2, \cdots, ka_n), \quad k \in \mathbf{R}$$

而且这两种运算满足一些重要的规律, 如

加法
$$\left\{\begin{array}{l} \boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{\beta} + \boldsymbol{\alpha} \\ (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \boldsymbol{\gamma} = \boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma}) \\ \boldsymbol{\alpha} + \boldsymbol{0} = \boldsymbol{\alpha} \\ \boldsymbol{\alpha} + (-\boldsymbol{\alpha}) = \boldsymbol{0} \end{array}\right.$$

$$\left.\begin{array}{l} 1\boldsymbol{\alpha} = \boldsymbol{\alpha} \\ k(l\boldsymbol{\alpha}) = (kl)\boldsymbol{\alpha} \\ (k+l)\boldsymbol{\alpha} = k\boldsymbol{\alpha} + l\boldsymbol{\alpha} \\ k(\boldsymbol{\alpha} + \boldsymbol{\beta}) = k\boldsymbol{\alpha} + k\boldsymbol{\beta} \end{array}\right\}$$
数乘

$$\forall \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbf{R}^n, \quad \forall k, l \in \mathbf{R}$$

引例2

R上的一元多项式全体**R**[x]中, 定义了两个多项式的加法和数与多项式的乘法, 而且这两种运算同样满足上述这些重要的规律, 即

$$
\text{加法}\begin{cases} f(x)+g(x)=g(x)+f(x) \\ (f(x)+g(x))+h(x)=f(x)+(g(x)+h(x)) \\ f(x)+0=f(x) \\ f(x)+(-f(x))=0 \end{cases} \qquad \forall f(x),g(x),h(x)\in\mathbf{R}[x],
$$

$$
\text{数乘}\begin{cases} 1f(x)=f(x) \\ k(lf(x))=(kl)f(x) \\ (k+l)f(x)=kf(x)+lf(x) \\ k(f(x)+g(x))=kf(x)+kg(x) \end{cases} \qquad \forall k,l\in\mathbf{R}
$$

推广　　**抽象**

三维向量空间$\mathbf{R}^3$

- 加法和数乘运算
- 可以描述一些几何和力学问题的有关属性

n维向量空间$\mathbf{R}^n$

- 定义n维向量的加法和数乘
- 讨论向量的线性相关性
- 阐明一般线性方程组的解的理论

一般向量空间V

- 加法和数乘?
- 封闭？规则?

撇开集合的具体对象和两种运算的具体含义

把集合对这两种运算的封闭性及运算满足的规则抽象出来，

就形成了抽象的向量空间(又称作线性空间)的概念

这种抽象将使得我们进一步研究的向量空间的理论可以在相当广泛的领域内得到应用

数学是研究抽象结构的理论。

——布尔巴基学派

不管数学的任一分支是多么抽象，总有一天会应用在这实际世界上。

——罗巴切夫斯基

数学是一项工具，特别适合于处理任何一类抽象概念，而且，它在这方面的作用是无止境的。因此，一本论述新物理学的书，如果不是单纯地描述实验工作的，其本质上，必定是一本数学书。

——狄拉克

The beauty and power of linear algebra will be seen more clearly when you view $\mathbf{R}^n$ *as only one of a variety of vector spaces that arise naturally in applied problems.*

Actually, a study of vector spaces is not much different from a study of $\mathbf{R}^n$ *itself, because you can use your geometric experience with* $\mathbf{R}^2$ *and* $\mathbf{R}^3$ *to visualize many general concepts.*

In fact, many other mathematical systems have the same properties. The specific properties of interest are listed in the following definition.

I. Vector Spaces with Examples

- **Definition 1** A **vector space (向量空间)** V is a nonempty set (非空集合) of objects, called *vectors* (向量), on which are defined two operations, called *addition* (加法) *and scalar multiplication*(数乘)…
- These two operations must produce vectors in the space (*i.e.*, *a vector space V is a set that is closed* (封闭) *under vector addition and scalar multiplication*), and they must satisfy the following eight conditions:
- [设 V 是一个非空集合，P 是一个数域(e.g., 实数域**R**, 复数域**C**等), 在 V 中有两种运算: 一是加法, 记作 $\alpha+\beta$; 二是数量乘法 (简称数乘, 是P中数λ和V中元素α 相乘), 记作$\lambda\alpha$. 且V 对两种运算封闭 (运算结果仍属于V), 并满足8条运算规则:]
- $\forall \alpha, \beta, \gamma \in V, \forall k, l \in P,$

$\forall \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in V, \forall k, l \in P,$

这里的向量不一定是有序数组

Vector Addition

(1) $\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\beta}+\boldsymbol{\alpha}$; *Commutativity*

(2) $(\boldsymbol{\alpha}+\boldsymbol{\beta})+\boldsymbol{\gamma}=\boldsymbol{\alpha}+(\boldsymbol{\beta}+\boldsymbol{\gamma})$; *Associativity of vector addition*

(3) There is a zero vector $\boldsymbol{0} \in V$, s.t. $\boldsymbol{\alpha}+\boldsymbol{0}=\boldsymbol{\alpha}$; *Additive identity*

(4) For each $\boldsymbol{\alpha} \in V$, there is a vector $-\boldsymbol{\alpha} \in V$, s.t. $\boldsymbol{\alpha}+(-\boldsymbol{\alpha})=\boldsymbol{0}$, where $-\boldsymbol{\alpha}$ is called the negative of $\boldsymbol{\alpha}$; *Existence of additive inverse*

Scalar Multiplication

(5) $1\boldsymbol{\alpha}=\boldsymbol{\alpha}$; *Scalar multiplication identity*

(6) $k(l\boldsymbol{\alpha})=(kl)\boldsymbol{\alpha}$; *Associativity of scalar multiplication*

(7) $(k+l)\boldsymbol{\alpha}=k\boldsymbol{\alpha}+l\boldsymbol{\alpha}$; *Distributivity of scalar sums*

(8) $k(\boldsymbol{\alpha}+\boldsymbol{\beta})=k\boldsymbol{\alpha}+k\boldsymbol{\beta}$. *Distributivity of vector sums*

Notes: Using the rules above, we can show that:

1. The zero vector is unique (零元是唯一的).

Proof Let $\boldsymbol{0}_1$, $\boldsymbol{0}_2$ be zero vectors of V, then

$$\boldsymbol{0}_1 = \boldsymbol{0}_1 + \boldsymbol{0}_2 = \boldsymbol{0}_2 + \boldsymbol{0}_1 = \boldsymbol{0}_2.$$

Zero vectors

2. the vector $-\alpha$ is unique for each α in V (每一个元的负元是唯一的).

Proof Let $\alpha + \beta_1 = \alpha + \beta_2 = \boldsymbol{0}$, then

$$\beta_1 = \beta_1 + \boldsymbol{0} = \beta_1 + (\alpha + \beta_2) = (\beta_1 + \alpha) + \beta_2 = \boldsymbol{0} + \beta_2 = \beta_2.$$

We can define: $\beta - \alpha = \beta + (-\alpha)$.

3. $k\,\boldsymbol{0} = \boldsymbol{0}$; $k(-\boldsymbol{\beta}) = -(k\boldsymbol{\beta})$; In particular,

$(-1)\,\boldsymbol{\alpha} = -\,\boldsymbol{\alpha}$, $0\,\boldsymbol{\alpha} = \boldsymbol{0}$; $(-l)\,\boldsymbol{\alpha} = -(l\,\boldsymbol{\alpha})$ (denoted by $-l\,\boldsymbol{\alpha}$)

4. If $\lambda\,\boldsymbol{\alpha} = \boldsymbol{0}$, then $\lambda = 0$ or $\boldsymbol{\alpha} = \boldsymbol{0}$.

If $\lambda \neq 0$, then

$$\boldsymbol{\alpha} = 1\,\boldsymbol{\alpha} = (\lambda^{-1}\lambda)\,\boldsymbol{\alpha} = \lambda^{-1}(\lambda\,\boldsymbol{\alpha}) = \lambda^{-1}\,\boldsymbol{0} = \boldsymbol{0}\,.$$

We can also show that for the equation

$$\lambda\,\boldsymbol{\beta} + \lambda_1\,\boldsymbol{\alpha}_1 + \lambda_2\,\boldsymbol{\alpha}_2 + \cdots + \lambda_r\,\boldsymbol{\alpha}_r = \boldsymbol{0}\,,$$

the solution is

$$\boldsymbol{\beta} = -\,\lambda^{-1}\lambda_1\,\boldsymbol{\alpha}_1 - \lambda^{-1}\lambda_2\,\boldsymbol{\alpha}_2 - \cdots - \lambda^{-1}\lambda_r\,\boldsymbol{\alpha}_r\,,$$

if $\lambda \neq 0$.

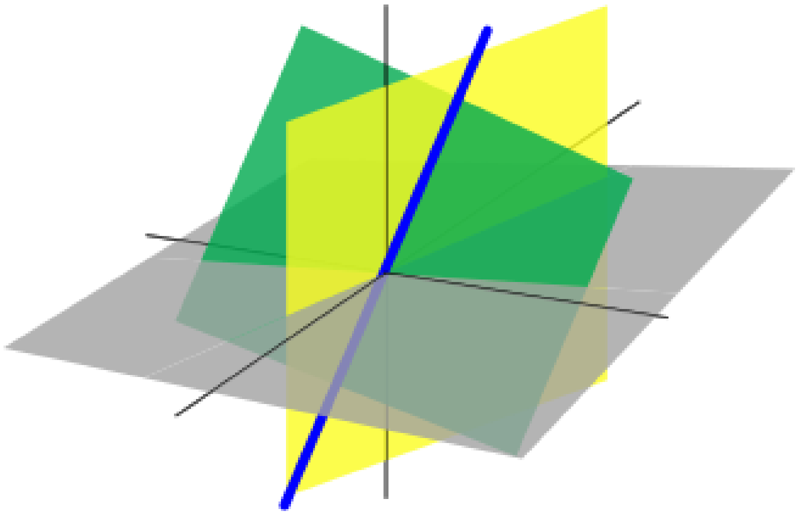
Example 1 The space of m by n matrices.

全体$m\times n$实矩阵对矩阵加法与数乘运算构成实数域**R**上的向量空间, 记为$\mathbf{R}^{m\times n}$（或 $M_{m\times n}(\mathbf{R})$).

其零元素是$m\times n$零矩阵; 任一元素A的负元是$(-1)A$.

In this case, the "vectors" are matrices!

Example 2 证明：**R**上的向量空间 V 若含有一个非零向量, 则 V 一定含有无穷多个向量.



Geometrically, think of the usual three-dimensional $\mathbf{R}^3$ and choose any plane through the origin.

在三维几何空间$\mathbf{R}^3$中，考虑一个过原点的平面

该平面上的所有向量对于加法和数乘组成一个二维的向量空间 (***That plane is a vector space in its own right.***)

双重身份:

- ✓ 一方面是$\mathbf{R}^3$的一部分
- ✓ 另一方面其自身对于原来的运算也构成向量空间

It is a ***subspace*** of the original space $\mathbf{R}^3$.

II. Subspaces with Examples

Definition 2 A ***subspace*** (子空间) of a vector space is a nonempty subset S that satisfies the requirements for a vector space:

(i) If we add any vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in the subspace, $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is *in the subspace*.

(ii) If we multiply any vector $\boldsymbol{\alpha}$ in the subspace by any scalar k, $k\boldsymbol{\alpha}$ is *in the subspace*.

Notes:

1. A sub*space* is a subset that is "closed" (**封闭的**) under addition and scalar multiplication, that is:

   Linear combinations stay in the subspace.

   (The distinction between a subset and a subspace)

Notes:

2. Notice in particular that ***the zero vector will belong to every subspace***.

3. Every subspace is a vector space.

4. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

5. The smallest subspace contains only one vector, the zero vector.

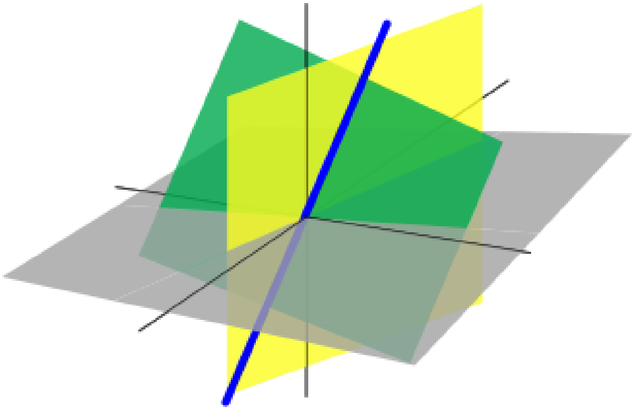
   At the other extreme, the largest subspace is the whole of the original space.

   If V is a vector space, then the subsets $\{\mathbf{0}\}$, and V are subspaces, called trivial subspaces (平凡子空间).

Example 3

If the original space is $\mathbf{R}^3$, then the possible subspaces are easy to describe:

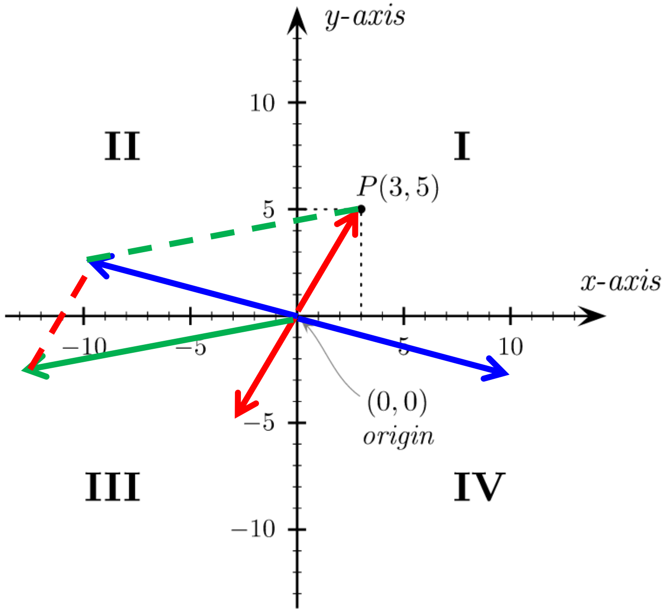
$\mathbf{R}^3$ *itself, any plane through the origin, any line through the origin, or the origin (the zero vector) alone*.



Example 4

- Start from the vector space of 3 by 3 matrices.
- One possible subspace is the set of *lower triangular matrices*.
- Another is the set of *symmetric matrices*.

The distinction between a subset and a subspace



Example 5

In $\mathbf{R}^2$, the *smallest* subspace containing the first quadrant (第一象限) is:

<u>the whole space $\mathbf{R}^2$</u>.

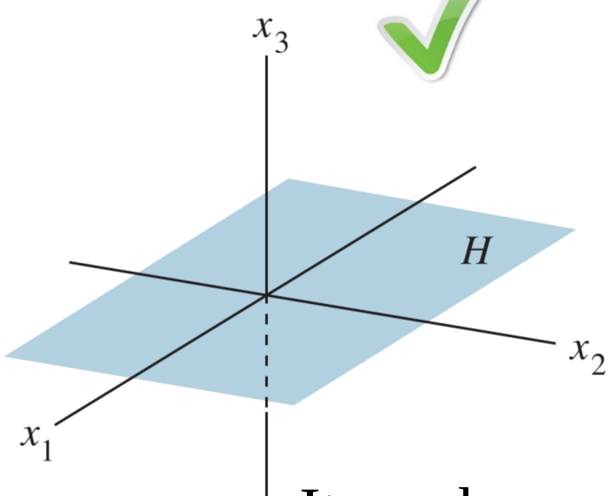
Example 7

The vector space $\mathbf{R}^2$ is *not* a subspace of $\mathbf{R}^3$ because $\mathbf{R}^2$ is *not* even a subset of $\mathbf{R}^3$.

The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

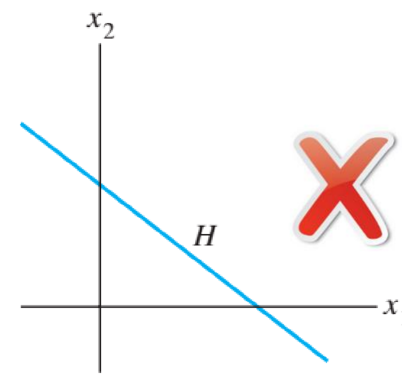
is a subset of $\mathbf{R}^3$ that "looks" and "acts" like $\mathbf{R}^2$, although it is logically distinct from $\mathbf{R}^2$.



It can be proved that H in the Figure is a subspace of $\mathbf{R}^3$.

FIGURE

The x_1x_2-plane as a subspace of $\mathbb{R}^3$.



FIGURE

A line that is not a vector space.

A line in $\mathbf{R}^2$ *not* through the origin, is *not* a subspace of $\mathbf{R}^2$.

III. Spanned Subspaces

Example 8 Let $\boldsymbol{v}$ be a vector of $\mathbf{R}^n$. Let V be a small subspace of $\mathbf{R}^n$ containing $\boldsymbol{v}$. Then V contains

$$2\boldsymbol{v},\ 3\boldsymbol{v},\ 4\boldsymbol{v},\ \ldots,\ 100\boldsymbol{v},\ \ldots$$

and generally, V contains

$$k\boldsymbol{v},\ \text{where}\ \ k \in \mathbf{R}.$$

Claim. $V = \{k\boldsymbol{v} \mid k \in \mathbf{R}\}$, and V is a subspace, because

• the addition of two vectors $k_1\boldsymbol{v}$ and $k_2\boldsymbol{v}$ equals $(k_1 + k_2)\boldsymbol{v}$, belonging to V ;

• for each number $m \in \mathbf{R}$, the scalar multiplication $m(k\boldsymbol{v})$ equals $(mk)\boldsymbol{v}$, belonging to V .

<u>This subspace V is uniquely determined by the vector $\boldsymbol{v}$.</u>

Geometrically, V is a line containing the vector $\boldsymbol{v}$.

Example 9 Let $\boldsymbol{u}$, $\boldsymbol{v}$ be two vectors of $\mathbf{R}^n$, and let V be the smallest subspace of $\mathbf{R}^n$ containing $\boldsymbol{u}$, $\boldsymbol{v}$. Then V contains all scalar multiplications

$$\boldsymbol{u}, 2\boldsymbol{u}, 3\boldsymbol{u}, \ldots, k\boldsymbol{u}, \ldots$$
$$\boldsymbol{v}, 2\boldsymbol{v}, 3\boldsymbol{v}, \ldots, l\boldsymbol{v}, \ldots$$

and further contains additions of all scalar multiplications:

$$k\boldsymbol{u} + l\boldsymbol{v}, \text{ where } k, l \text{ are numbers.}$$

We can show that $V = \{k\boldsymbol{u} + l\boldsymbol{v} \mid k, l \in \mathbf{R}\}$, and V is a subspace.
This subspace is uniquely determined by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, leading to some *key concepts*.

linear combination — ***one of the central ideas of linear algebra***

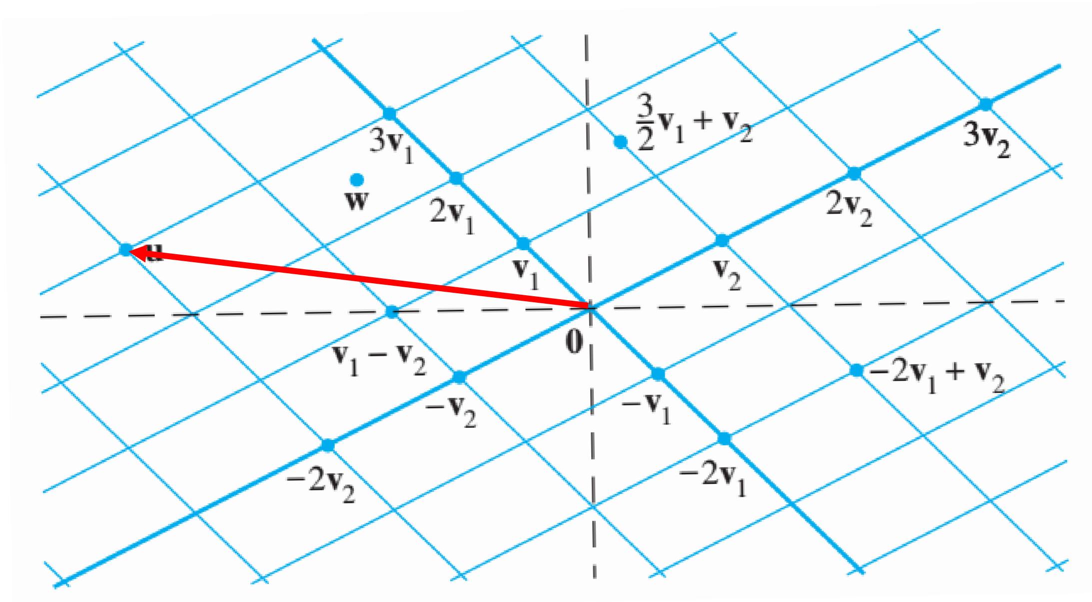
Definition 3 (*Review*) Let $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a subset of $\mathbf{R}^n$. A vector with the form

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \ldots + k_r\boldsymbol{v}_r, \text{ where } k_1, k_2, \ldots, k_r \in \mathbf{R}$$

is called a **linear combination(线性组合)** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$.

Note:

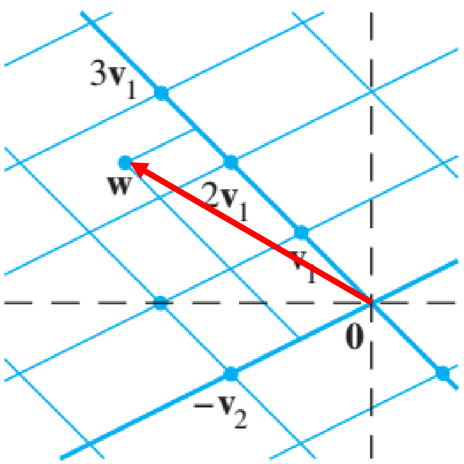
$k_1, k_2, \ldots, k_r$: weights(权), can be any real numbers, including zero.



$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This figure identifies selected *linear combinations* of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

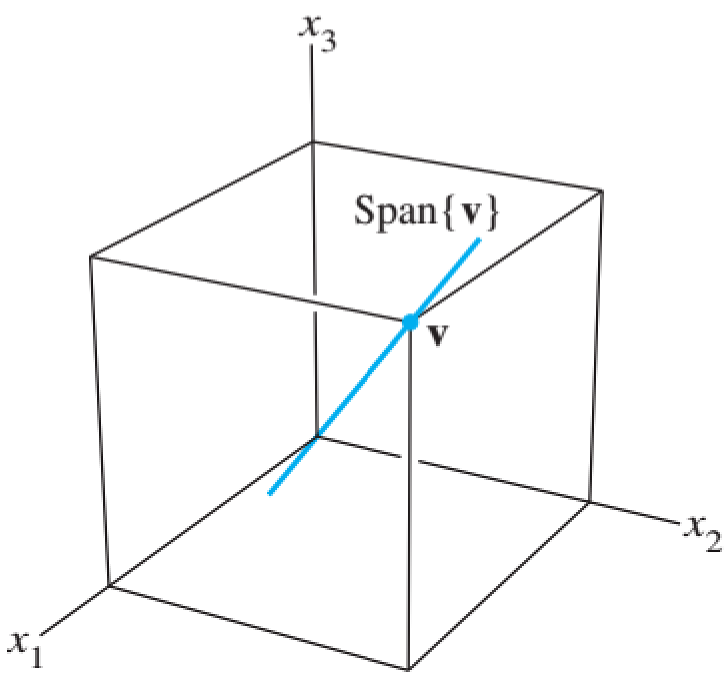
$$\boldsymbol{u} = 3\boldsymbol{v}_1 - 2\boldsymbol{v}_2, \qquad \boldsymbol{w} = \frac{5}{2}\boldsymbol{v}_1 - \frac{1}{2}\boldsymbol{v}_2.$$



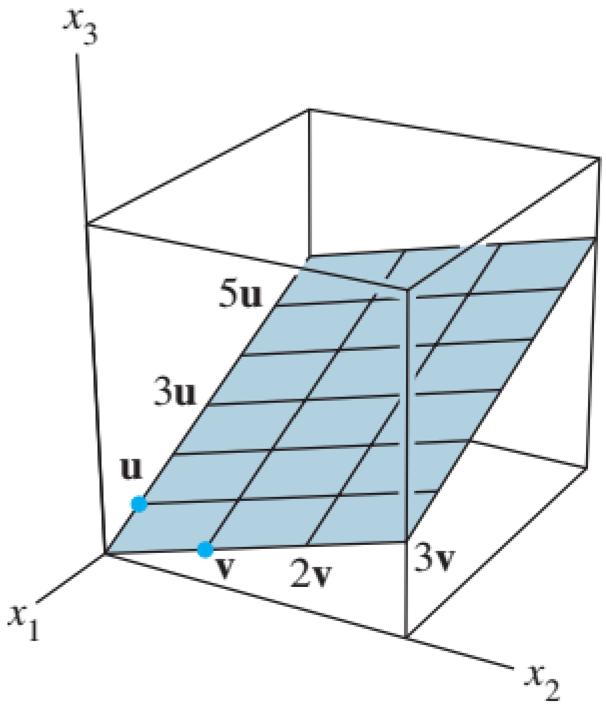
*A Geometric Description of Span{**v**} and Span{**u**,**v**}*

Let ***v*** be a nonzero vector in $\mathbf{R}^3$.
Then Span{***v***} is the set of all scalar multiples of ***v***.

If ***u*** and ***v*** are nonzero vectors in $\mathbf{R}^3$, with ***v*** not a multiple of ***u***.
Then Span{***u***, ***v***} is the plane in $\mathbf{R}^3$ that contains ***u***, ***v***, and ***0***.



Span{***v***} as a line through the origin.



Span{***u***,***v***} as a plane through the origin.

Let $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a subset of $\mathbf{R}^n$. Let

$$V = \{k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \ldots + k_r\boldsymbol{v}_r \mid k_1, k_2, \ldots, k_r \in \mathbf{R}\},$$

consisting of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$.

Theorem 1 *The subset V defined above is a subspace of $\mathbf{R}^n$, and is the smallest subspace of $\mathbf{R}^n$ which contains A.*

Definition 4 This subspace V is called the **span** of A(由A张成的子空间), denoted by Span(A). In symbols,

$$\text{Span}(A) = \{k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \ldots + k_r\boldsymbol{v}_r \mid k_i \in \mathbf{R}, 1 \le i \le r\},$$

consisting of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$.

A is called the spanning set of V.

Some important questions we concern:

- Given a subset A of vectors of $\mathbf{R}^n$, does A span the whole space $\mathbf{R}^n$?
- Let A be a subset of $\mathbf{R}^n$ and $\boldsymbol{w} \in \mathbf{R}^n$. Is $\boldsymbol{w}$ in Span(A)?

Example 10 Let $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

Then Span{$\boldsymbol{a}_1$, $\boldsymbol{a}_2$} is a plane through the origin in $\mathbf{R}^3$.
Is $\boldsymbol{b}$ in that plane?

Solution Does the equation $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = \boldsymbol{b}$ have a solution?

To answer this, row reduce the augmented matrix $[\boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \boldsymbol{b}]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The third equation is $0x_1 + 0x_2 = -2$, which shows that the system has no solution.
The equation $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = \boldsymbol{b}$ has no solution, and so $\boldsymbol{b}$ is not in Span{$\boldsymbol{a}_1$, $\boldsymbol{a}_2$}.

IV. Vector Spaces Defined from a Matrix

Let $\boldsymbol{A}$ be an $(m\times n)$-matrix.

Let R_i with $1 \le i \le m$ be the row vectors of $\boldsymbol{A}$,
and C_j with $1 \le j \le n$ be the column vectors of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & & C_j & & C_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{c} \\ R_1 \\ R_2 \\ \\ R_i \\ \\ R_m \end{array}$$

Definition 5

The subspace of $\mathbf{R}^n$ spanned by $R_1, R_2, \ldots, R_m$ is called the **row space (行空间)** of $\boldsymbol{A}$.

The subspace of $\mathbf{R}^m$ spanned by $C_1, C_2, \ldots, C_n$ is called the **column space (列空间)** of $\boldsymbol{A}$, denoted by $\mathrm{Col}(A)$ or $C(A)$.

For example, let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$

Then the row space of $\boldsymbol{A}$ is spanned by

$$(1,1,1) \text{ and } (1,2,3),$$

And the column space is spanned by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Theorem 2

The system $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable *if and only if* the vector $\boldsymbol{b}$ can be expressed as a combination of the columns of $\boldsymbol{A}$. *Then* $\boldsymbol{b}$ *is in the column space of* $\boldsymbol{A}$, *i.e.*, $\boldsymbol{b} \in C(\boldsymbol{A})$.

$$\begin{cases} \boxed{a_{11}x_1} + \boxed{a_{12}x_2} + \cdots + \boxed{a_{1n}x_n} = \boxed{b_1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\boxed{\boldsymbol{\alpha}_1 x_1} + \boxed{\boldsymbol{\alpha}_2 x_2} + \cdots + \boxed{\boldsymbol{\alpha}_n x_n} = \boxed{\boldsymbol{b}}$$

What if -- $\boldsymbol{b}=\boldsymbol{0}$?

$\boldsymbol{Ax} = \boldsymbol{0}$: always allows the solution $\boldsymbol{x} = \boldsymbol{0}$.
But there may be infinitely many other solutions.
(There always are, if there are more unknowns than equations, $n > m$.)
The solutions to Ax = 0 form a vector space—the nullspace of A.

Definition 6 Let

$$\boldsymbol{Ax} = \boldsymbol{0}$$

be a system of linear equations with coefficient matrix $\boldsymbol{A}$. Then all solutions of this system of linear equations form a vector space, called the **solution space (解空间)**. The solution space of this system is also called the **nullspace** (零空间) of $\boldsymbol{A}$, denoted by null($\boldsymbol{A}$) or Nul($\boldsymbol{A}$) or $N(\boldsymbol{A})$.

$$\mathrm{N}(\boldsymbol{A}) = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbf{R}^n \text{and } \boldsymbol{Ax} = \boldsymbol{0}\}$$

*The term **space** in **nullspace** is appropriate.*

Note:

The nullspace is a subspace of $\mathbf{R}^n$.

N($\boldsymbol{A}$) is a subset of $\mathbf{R}^n$ because $\boldsymbol{A}$ has n columns,

and $\boldsymbol{0} \in$ N($\boldsymbol{A}$).

Requirement (i) holds:

If $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{Ax'} = \boldsymbol{0}$, then $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{x'}) = \boldsymbol{0}$.

Requirement (ii) also holds: If $\boldsymbol{Ax} = \boldsymbol{0}$ then $\boldsymbol{A}(c\boldsymbol{x}) = \boldsymbol{0}$.

Both requirements fail if the right-hand side is not zero!

Only the solutions to a *homogeneous* equations ($\boldsymbol{b} = \boldsymbol{0}$) form a subspace.

(只有齐次线性方程组的解集能形成子空间)

Example 11 Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix},$$

Determine if $\boldsymbol{u}$ belongs to the nullspace of $\boldsymbol{A}$.

Solution To test if $\boldsymbol{u}$ satisfies $\boldsymbol{Au}=\boldsymbol{0}$, simply compute

$$\boldsymbol{Au} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Thus $\boldsymbol{u}$ is in N($\boldsymbol{A}$).

An Explicit Description of N(A)

- There is no obvious relation between vectors in N($\boldsymbol{A}$) and the entries in $\boldsymbol{A}$.
- Solving the equation $\boldsymbol{Ax} = \boldsymbol{0}$ amounts to producing an *explicit* description of N($\boldsymbol{A}$).

为了得到N($\boldsymbol{A}$)的显式表达：
解方程 $\boldsymbol{Ax} = \boldsymbol{0}$.

Example 12 For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix},$$

the nullspace of $\boldsymbol{A}$ is the space spanned by the solutions of $\boldsymbol{Ax} = \boldsymbol{0}$, which is

$$x = z, y = -2z.$$

So a solution vector is of the form

$$(x, y, z)^{\mathrm{T}} = (z, -2z, z)^{\mathrm{T}} = (1, -2, 1)^{\mathrm{T}} z,$$

where $z \in \mathbf{R}$.

The Contrast (Connection) between N(A) and C(A) for an $m \times n$ matrix A

N(A) Nullspace of A A的零空间	C(A) Column space of A A的列空间
N(A) is a subspace of $\mathbf{R}^n$. N(A)是$\mathbf{R}^n$的一个子空间.	C(A) is a subspace of $\mathbf{R}^m$. C(A)是$\mathbf{R}^m$的一个子空间.
N(A) is implicitly defined; *i.e.*, you are given only a condition that vectors in N(A) must satisfy. N(A)是隐式定义的, 即仅给出了一个N(A)中向量必须满足的条件($A\boldsymbol{x}=\boldsymbol{0}$).	C(A) is explicitly defined; *i.e.*, you are told how to build vectors in C(A). C(A)是显式定义, 即明确指出如何建立C(A)中的向量.

N(A) Nullspace of A A的零空间	C(A) Column space of A A的列空间
It takes time to find vectors in N(***A***). Row operations on [***A 0***] are required. 求N(***A***)中的向量需要时间, 需要对[***A 0***]做行变换.	It is easy to find vectors in C(***A***). The columns of ***A*** are displayed; others are formed from them. 容易求出C(***A***)中的向量, ***A***中的列就是C(***A***)中向量, 其余的可由***A***的列表示出来.
There is no obvious relation between N(***A***) and the entries in ***A***. N(***A***) 与***A***的数值之间没有明显的关系.	There is an obvious relation between C(***A***) and the entries in ***A***, since each column of ***A*** is in C(***A***). C(***A***)与***A***的数值之间有明显的关系, 因为***A***的列就在C(***A***)中.

N(A) Nullspace of A A的零空间	C(A) Column space of A A的列空间
A typical vector $\boldsymbol{v}$ in N($\boldsymbol{A}$) has the property that $\boldsymbol{Av} = \boldsymbol{0}$. N($\boldsymbol{A}$)中的一个典型向量$\mathbf{v}$具有$\boldsymbol{Av} = \boldsymbol{0}$的性质.	A typical vector $\boldsymbol{v}$ in C($\boldsymbol{A}$) has the property that the equation $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent. C($\boldsymbol{A}$)中一个典型向量$\mathbf{v}$具有方程$\boldsymbol{Ax} = \mathbf{v}$是相容的性质.
Given a specific vector $\boldsymbol{v}$, it is easy to tell if $\boldsymbol{v}$ is in N($\boldsymbol{A}$). Just compute $\boldsymbol{Av}$. 给一个特定向量$\mathbf{v}$, 容易判断$\mathbf{v}$是否在N($\boldsymbol{A}$)中, 仅需计算$\boldsymbol{Av}$.	Given a specific vector $\boldsymbol{v}$, it may take time to tell if $\boldsymbol{v}$ is in C($\boldsymbol{A}$). Row operations on $[\boldsymbol{A}\ \ \boldsymbol{v}]$ are required. 给一个特定向量$\mathbf{v}$, 弄清$\mathbf{v}$是否在C($\boldsymbol{A}$)中需要时间, 需要对$[\boldsymbol{A}\ \ \mathbf{v}]$做行变换.
N($\boldsymbol{A}$) = {$\boldsymbol{0}$} if and only if the equation $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution. N($\boldsymbol{A}$) = {$\boldsymbol{0}$} 当且仅当$\boldsymbol{Ax} = \boldsymbol{0}$仅有一个平凡解.	C($\boldsymbol{A}$)=$\mathbf{R}^m$ if and only if the equation $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$ in $\mathbf{R}^m$. C($\boldsymbol{A}$)=$\mathbf{R}^m$ 当且仅当方程$\boldsymbol{Ax} = \boldsymbol{b}$对每一个$\boldsymbol{b} \in \mathbf{R}^m$有一个解.

This section is about basic concepts regarding
***vector spaces and subspaces*.**
Spanned subspaces
Column spaces and nullspaces

Homework

See Blackboard

Deadline: next tutorial