Step-1

Let us use Induction on the order of the matrices *A* and *M* to prove this.

Let both the matrices be of order 1. Thus, $A = [a_{11}]$, $M = [m_{11}]$

We have $Ax = \lambda Mx$. This gives,

$$\begin{aligned} Ax &= \lambda Mx \\ [a_{11}]x &= \lambda [m_{11}]x \\ x &= \lambda \frac{[m_{11}]}{[a_{11}]} x \\ &= \lambda \frac{m_{11}}{a_{11}} x \end{aligned}$$

Therefore, $\lambda_1 = \frac{a_{11}}{m_{11}}$. Thus, Î» is not greater than $\frac{a_{11}}{m_{11}}$. In case of one by one matrix, there is only one eigenvalue and thus, that itself is the smallest eigenvalue.

Step-2

Assume that when *A* and *M* are *n* by *n* matrices, we get the smallest eigenvalue Î»1 is not greater than $\frac{a_{11}}{m_{11}}$.

Now let *A* and *M* be *n*+1 by *n*+1 matrices. From these two matrices, we can throw their last row and last column and can obtain new matrices *Aâ€™* and *Mâ€™*. Both these are of the order *n* by *n*.

Suppose λ_1' be the smallest eigenvalue of $A'x = \lambda M'x$.

Step-3

Suppose λ_1 be the smallest eigenvalue of $Ax = \lambda Mx$.

If *P* is any square matrix and if *Q* is a matrix obtained by throwing any row and column of *P*, then we know that $\lambda_1(P) \le \lambda_1(Q)$.

By using this, we can say that $\lambda_1 \le \lambda_1'$

But by induction hypothesis, $\lambda_1' \le \frac{a_{11}}{m_{11}}$

Therefore, $\lambda_1 \le \frac{a_{11}}{m_{11}}$.

Step-4

Thus, we have shown that the smallest eigenvalue $\lambda_1 \leq \frac{a_{11}}{m_{11}}$.