Step-1

a) Find a basis for the subspace $\mathbf{S}$ in $\mathbf{R}^4$ spanned by all solutions of $x_1 + x_2 + x_3 - x_4 = 0$.

From the data, $x_1 + x_2 + x_3 - x_4 = 0$

Put $x_4 = k, x_3 = r, x_2 = t$ then,

$$\begin{aligned} x_1 &= -x_2 - x_3 + x_4 \\ &= -t - r + k \end{aligned}$$

So, the subspace $\mathbf{S}$ is generated as,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-r+k \\ t \\ r \\ k \end{bmatrix}$$
$$= t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the required basis is $\boxed{\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$.

Step-2

b) Find a basis for the orthogonal complement $\mathbf{S}^\perp$.

Since $\mathbf{S} = \text{null}(A)$, it must be the case that $\mathbf{S}^\perp$ is the row space of A.

Therefore, one row of A gives a basis for $\mathbf{S}^\perp$.

Hence a basis for the orthogonal complement $\mathbf{S}^\perp$ is $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\right\}$

Step-3

c) Find b_1 in $\mathbf{S}$ and b_2 in $\mathbf{S}^{\perp}$ so that $b_1 + b_2 = b = (1,1,1,1)$.

Here, b_1 in $\mathbf{S}$ means b_1 is a linear combination of vectors of $\mathbf{S}$.

Therefore

$$b_1 = a\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$, where a,b,c are scalars.

Step-4

And b_2 in $\mathbf{S}^{\perp}$ means b_2 is a linear combination of vectors of $\mathbf{S}^{\perp}$

Therefore

$$b_2 = d\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$, where d is scalar

Step-5

Given that $b_1 + b_2 = b = (1,1,1,1)$

$$a\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + d\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

The matrix form of the system is,

$$\begin{bmatrix}-1 & -1 & 1 & 1\\1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Step-6

Now, apply row operations to solve the system as follow.

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

Step-7

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$$

Now, form the above system, $-4d = -2 \Rightarrow d = \frac{1}{2}$.

$$\begin{aligned} c + 3d &= 3 \\ c + 3\left(\frac{1}{2}\right) &= 3 \\ c + \frac{3}{2} &= 3 \\ c &= \frac{3}{2} \end{aligned}$$

Substitute $c = \frac{3}{2}$, $d = \frac{1}{2}$ in the equation $-b + c + 2d = 2$.

$$-b+\frac{3}{2}+2\left(\frac{1}{2}\right)=2$$
$$-b+\frac{3}{2}+1=2$$
$$-b+\frac{5}{2}=2$$
$$b=\frac{1}{2}$$

And now, substitute $b=\frac{1}{2}$, $c=\frac{3}{2}$, $d=\frac{1}{2}$ in the equation.

$$-a-b+c+d=1$$
$$-a-\frac{1}{2}+\frac{3}{2}+\frac{1}{2}=1$$
$$-a+\frac{3}{2}=1$$
$$a=\frac{1}{2}$$

Step-8

Now, the linear combination is,

$$b_1 = a\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{2}-\frac{1}{2}+\frac{3}{2}\\ \frac{1}{2}+0+0\\ 0+\frac{1}{2}+0\\ 0+0+\frac{3}{2}\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}}$$

Step-9

And also,

$$b_2 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}$$

Hence, the required vectors are $b_1 = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}, b_2 = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}$.

Step-10

Verification:

$$
\begin{aligned}
b_1 + b_2 &= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= b
\end{aligned}
$$