Step-1

Suppose we diagonalize the symmetric matrix A using an orthogonal matrix Q. That is, if Î› denotes the diagonal matrix, which has all the eigenvalues of A along its diagonal, then we get the following:

$$Q^{\mathrm{T}} AQ = \Lambda$$

Let $x = Qy$. Thus, we get

$$\begin{aligned} R(x) &= \frac{x^{\mathrm{T}} Ax}{x^{\mathrm{T}} x} \\ &= \frac{(Qy)^{\mathrm{T}} A(Qy)}{(Qy)^{\mathrm{T}} (Qy)} \\ &= \frac{y^{\mathrm{T}} \Lambda y}{y^{\mathrm{T}} y} \\ &= \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \ldots + y_n^2} \end{aligned}$$

Step-2

Since Î»$_{\text{n}}$ is the largest eigenvalue of the matrix A, we have $\lambda_k \le \lambda_n$, where Î»$_{\text{k}}$ is any eigenvalue of A.

Thus, we have

$$\begin{aligned} R(x) &= \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \ldots + y_n^2} \\ &\le \frac{\lambda_n y_1^2 + \lambda_n y_2^2 + \ldots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \ldots + y_n^2} \\ &= \frac{\lambda_n \left(y_1^2 + y_2^2 + \ldots + y_n^2\right)}{y_1^2 + y_2^2 + \ldots + y_n^2} \\ &= \lambda_n \end{aligned}$$

Step-3

Thus, we have shown that $\boxed{R(x) \le \lambda_n}$.