Step-1

Given that the system $Ax = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} = b$ has no solution.

We have to sketch and solve a straight line fit that leads to the minimization of the quadratic $(C-D-4)^2 + (C-5)^2 + (C+D-9)^2$.

Step-2

Given $Ax = b$ is $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$

Where $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, x = \begin{bmatrix} C \\ D \end{bmatrix}, \ b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$

We know that the least-square solution to a problem is $A^T A\hat{x} = A^T b$

Step-3

Now $A^T A\hat{x} = A^T b$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1(1)+1(1)+1(1) & 1(-1)+1(0)+1(1) \\ -1(1)+0(1)+1(1) & -1(-1)+0(0)+1(1) \end{bmatrix}\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1(4)+1(5)+1(9) \\ -1(4)+0(5)+1(9) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \end{bmatrix}$$

$$\Rightarrow 3\hat{C} = 18, 2\hat{D} = 5$$

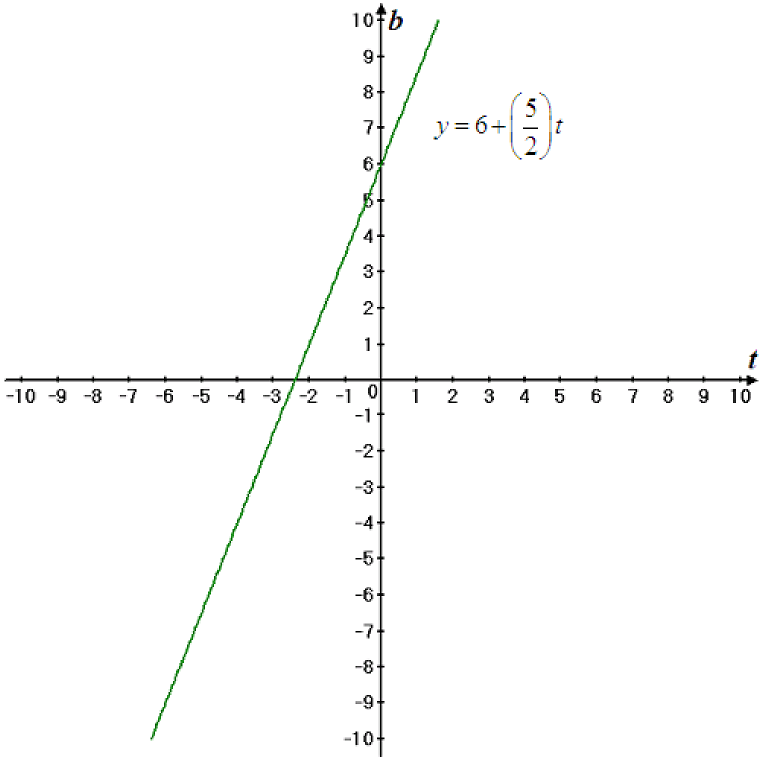
$$\Rightarrow \hat{C} = 6, \hat{D} = \frac{5}{2}$$

Hence the best line is $\boxed{y = 6 + \left(\frac{5}{2}\right)t}$

Step-4

The sketch of the straight line that fit that leads to the minimization is shown below.

Step-5



Step-6

Now we have to find the projection of b onto the column space of A.

We know that the projection $p = A\hat{x}$

Now

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(0)+1(1) & 1(0)+0(1)+1(1) \\ 0(1)+1(0)+1(1) & 0(0)+1(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & 0+0+1 \\ 0+0+1 & 0+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step-7

We now find the inverse of $A^T A$.

Now

$$\left(A^T A\right)^{-1} = \frac{1}{4-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \left(\begin{array}{l} \text{Since if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{then } A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{array} \right)$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Step-8

Now

$$\left(A^T A\right)^{-1} A^T = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Step-9

Hence

$$\begin{aligned}\hat{x} &= \left(A^T A\right)^{-1} A^T b\\ &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}\\ &= \begin{bmatrix} \frac{2}{3}(4)-\frac{1}{3}(5)+\frac{1}{3}(9) \\ -\frac{1}{3}(4)+\frac{2}{3}(5)+\frac{1}{3}(9) \end{bmatrix}\\ &= \begin{bmatrix} 6 \\ \frac{5}{2} \end{bmatrix}\end{aligned}$$

Step-10

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Therefore, the projection is

$$\begin{aligned}p &= A\hat{x}\\ &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ \frac{5}{2} \end{bmatrix}\\ &= \begin{bmatrix} \frac{7}{2} \\ 6 \\ \frac{17}{2} \end{bmatrix}\end{aligned}$$

Hence the projection of b onto the column space of A is $\boxed{p = \begin{bmatrix} \frac{7}{2} \\ 6 \\ \frac{17}{2} \end{bmatrix}}$.