

Step-1

Consider the space of 2 by 2 matrices.

Suppose that the transformation T on this space is defined as $T(A) = A^T$.

That is, when the linear transformation T is applied to any matrix, the resultant matrix is the transpose of the original matrix.

Thus, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T(A) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

The objective is to find the eigenvalues and eigenmatrices for $A^T = \lambda A$.

Step-2

Let us suppose $A^T = \lambda A$.

Then the equation becomes $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

From this, the obtained equation are as follows:

$$a = \lambda a \quad \dots\dots(1)$$

$$c = \lambda b \quad \dots\dots(2)$$

$$b = \lambda c \quad \dots\dots(3)$$

$$d = \lambda d \quad \dots\dots(4)$$

The equation (1) can be solved as follows:

$$a = \lambda a$$

$$a - \lambda a = 0$$

$$a(1 - \lambda) = 0$$

$$a = 0 \text{ or } 1 - \lambda = 0$$

$$a = 0 \text{ or } \lambda = 1$$

Step-3

Substitute $c = \lambda b$ in $b = \lambda c$.

$$b = \lambda c$$

$$b = \lambda(\lambda b)$$

$$b = \lambda^2 b$$

Solve the equation $b = \lambda^2 b$.

$$b = \lambda^2 b$$

$$b - \lambda^2 b = 0$$

$$b(1 - \lambda^2) = 0$$

$$b = 0 \text{ or } 1 - \lambda^2 = 0$$

$$b = 0 \text{ or } \lambda^2 = 1$$

$$b = 0 \text{ or } \lambda = \pm 1$$

Step-4

The equation (4) can be solved as follows:

$$d = \lambda d$$

$$d - \lambda d = 0$$

$$d(1 - \lambda) = 0$$

$$d = 0 \text{ or } 1 - \lambda = 0$$

$$d = 0 \text{ or } \lambda = 1$$

Therefore, the eigenvalues for $A^T = \lambda A$ are 1, 1, 1, $\hat{\infty}$.

Step-5

When $\lambda = 1$, then from equation (2) obtained that $b = c$.

Then the matrix A becomes as follows:

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad (\text{Use } b = c) \\ &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

When $\lambda = -1$, then from equation (2) obtained that $b = -c$.

Then the matrix A becomes as follows:

$$\begin{aligned}
A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ -b & d \end{bmatrix} \quad (\text{ Use } c = -b) \\
&= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Hence, the eigenmatrices of for $A^T = \lambda A$ are $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.