2020－2021 期中 高数上

1.(1) B

$$f(x)=|x|\sin x=\begin{cases} x\sin x, & x\geq 0 \\ -x\sin x, & x<0 \end{cases} \qquad f'(x)=\begin{cases} \sin x+x\cos x, & x\geq 0 \\ -\sin x - x\cos x, & x<0 \end{cases}$$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{|x|}{x}\sin x=0$$

$$\Rightarrow f''_{+}(0)=\lim_{x\to 0^+}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0}\frac{\sin x+x\cos x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}+\cos x\right)=2$$

$$f''_{-}(0)=\lim_{x\to 0^-}\left(-\frac{\sin x}{x}-\cos x\right)=-2$$

$\Rightarrow f^{(2)}(0)$ does not exist.

$\Rightarrow f^{(n)}(0)$ exists, n 最大是 $n=1$.

(2) A.

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=\frac{1}{2}$$

$\Rightarrow$ 方法一：$y=ax+b+\frac{1}{2}$ 是 $y=\frac{x^2+1}{x+1}=\underline{\underline{x-1}}+\frac{2}{x+1}$ 的渐近线 $\Rightarrow ax+b+\frac{1}{2}=x-1$

$$\therefore \begin{cases} a=1 \\ b+\frac{1}{2}=-1 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=\frac{3}{-2} \end{cases}$$

方法二：

$$a=\lim_{x\to\infty}\frac{\frac{x^2+1}{x+1}}{x}=1$$

$$b=\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-x-\frac{1}{2}\right)=\lim_{x\to\infty}\left(\frac{1-x}{x+1}-\frac{1}{2}\right)=-1-\frac{1}{2}=\frac{3}{-2}.$$

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(3) B

$$g(c) = \frac{\int_0^6 (x^2+6)\,dx}{6} = \frac{\left.\frac{1}{3}x^3+6x\right]_0^6}{6} = \frac{1}{3}\times 36+6 = 18$$

(4). D

$$f(x)=|x|\sin|x| \Rightarrow f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{|x|}{x}\underbrace{\sin|x|}_{\to 0} = 0.$$

$$f(x)=|x|\sin\sqrt{|x|} \Rightarrow f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{|x|}{x}\underbrace{\sin\sqrt{|x|}}_{\to 0} = 0.$$

$$f(x)=\cos|x| \Rightarrow f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{\cos|x|-1}{x} = \lim_{x\to 0}\frac{-\sin^2|x|}{x}\cdot\underbrace{\frac{1}{\cos|x|+1}}_{\to\frac{1}{2}}$$

$$= \frac{-1}{2}\lim_{x\to 0}\left(\frac{\sin|x|}{|x|}\right)^2\cdot\underbrace{\frac{|x|^2}{x}}_{\to 0} = 0.$$

$$f(x)=\cos\sqrt{|x|} \Rightarrow f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{\cos\sqrt{|x|}-1}{x} = \lim_{x\to 0}\frac{-\sin^2\sqrt{|x|}}{x}\cdot\underbrace{\frac{1}{\cos\sqrt{|x|}+1}}_{\to\frac{1}{2}}$$

$$= \frac{-1}{2}\lim_{x\to 0}\left(\frac{\sin\sqrt{|x|}}{\sqrt{|x|}}\right)^2\cdot\underbrace{\frac{|x|}{x}}_{\to\pm 1} = \begin{cases} -\frac{1}{2} = f'_+(0) \\ \frac{1}{2} = f'_-(0) \end{cases}$$

$\Rightarrow f'(0)$ does not exist.

(5) C

$$f(x) = \frac{1-\sin x}{1+\sin x} \Rightarrow f'(x) = \frac{-\cos x(1+\sin x)-\cos x(1-\sin x)}{(1+\sin x)^2} = \frac{-2\cos x}{(1+\sin x)^2}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{-2\times\frac{\sqrt{3}}{2}}{\left(\frac{3}{2}\right)^2} = -\frac{4\sqrt{3}}{9} = -\frac{4}{3\sqrt{3}}$$

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2(1) $\underline{0}$. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sin^5 x \cos^3 x}_{\text{奇函数}}\, dx = 0$

(2) $\underline{\frac{1}{12}}$. $\int_0^{x^3-1} f(t)\,dt = x \Rightarrow 3x^2 f(x^3-1) = 1$ let $x=2 \Rightarrow f(7) = \frac{1}{12}$

(3) $\underline{\frac{1}{3}x^3 + 2x - \frac{1}{x} - \frac{4}{3}}$, $f'(x) = (x+\frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2}$

$\Rightarrow f(x) = \frac{1}{3}x^3 + 2x - \frac{1}{x} + C$, $f(1)=1 \Rightarrow C = -\frac{4}{3}$

(4) $\underline{-2}$. $x^2+y^2+z^2 = 13^2 \Rightarrow 2x\cdot x' + 2y\cdot y' + 2z\cdot z' = 0$.

let $t = x_0$. $x(t_0)=3$, $x'(t_0)=4$. $y(t_0)=4$, $y'(t_0)=3$

$z(t_0)=12$

$\Rightarrow 24+24+24z'(t_0) = 0 \Rightarrow z'(t_0) = -2$.

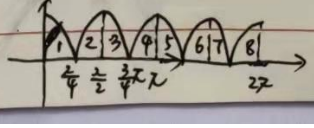
5. $\underline{\frac{a}{\sqrt{a^2+1}}}$. $\lim_{s\to a} \frac{\sqrt{s^2+1} - \sqrt{a^2+1}}{s-a} = \frac{d}{ds}(\sqrt{s^2+1})\Big|_{s=a} = \frac{a}{\sqrt{a^2+1}}$

3. (1) $\lim_{x\to\infty} \frac{\sqrt{3+x} - \sqrt{x+1}}{x^2+x-2} = \lim_{x\to\infty} \frac{\frac{\sqrt{3+x}}{x^2} - \frac{\sqrt{x+1}}{x^2}}{1+\frac{1}{x} - \frac{2}{x^2}} = \underline{0}$

(2) $\lim_{x\to 0} \frac{\cos x - \sec^2 x}{x\sin x} = \lim_{x\to 0} \frac{\cos x - \frac{1}{\cos^2 x}}{x \sin x} = \lim_{x\to 0} \frac{\cos^3 x - 1}{x\sin x} = \lim_{x\to 0} \frac{(\cos x - 1)}{x\sin x}\cdot \underbrace{(\cos^2 x + \cos x + 1)}_{\downarrow\ 3}$

$\Rightarrow 3\lim_{x\to 0} \frac{\cos x - 1}{x \sin x} = 3\lim_{x\to 0} \frac{-\sin^2 x}{x\sin x}\cdot\frac{1}{\cos x + 1} = -\frac{3}{2}\lim_{x\to 0}\frac{\sin x}{x} = \underline{-\frac{3}{2}}$

4. (1) $\int_0^{2\pi} |\sin^2 x - \cos^2 x|\,dx = \int_0^{2\pi} |\cos 2x|\,dx = 8\int_0^{\frac{\pi}{4}} \cos 2x\,dx = 4\sin 2x\Big]_0^{\frac{\pi}{4}} = \underline{4}$.



$0 < x < \frac{\pi}{4}$ → 8个 2π.

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(2) $\int_0^1 (x+2)\sqrt{1-x^2}\,dx = \int_0^1 x\sqrt{1-x^2}\,dx + 2\int_0^1 \sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}\Big]_0^1 + 2\times\frac{\pi}{4} = \frac{1}{3}+\frac{\pi}{2}$

5. $f(x) = \dfrac{x^3+x-2}{x-x^2}$

(1) $f(x) = \dfrac{x^3+x-2}{x-x^2} = \dfrac{(x-1)(x^2+x+2)}{x(1-x)} = \dfrac{x^2+x+2}{-x} = -x-1-\dfrac{2}{x}$ domain: $x\neq 0$, $x\neq 1$.

$f'(x) = -1+\dfrac{2}{x^2}$, $f''(x) = -\dfrac{4}{x^3}$

let $f'(x)=0 \Rightarrow x=\pm\sqrt{2}$, critical points at $x=\pm\sqrt{2}$

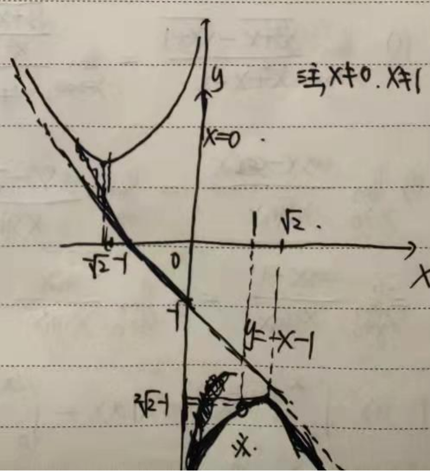
$x<-\sqrt{2} \Rightarrow f'(x)<0$ ↓
$-\sqrt{2}<x<0 \Rightarrow f'(x)>0$ ↑
$0<x<1 \Rightarrow f'(x)>0$ ↑
$1<x<\sqrt{2} \Rightarrow f'(x)>0$ ↑
$x>\sqrt{2} \Rightarrow f'(x)<0$ ↓

$f(\sqrt{2}) = -\sqrt{2}-1-\frac{2}{\sqrt{2}} = -2\sqrt{2}-1$
$f(-\sqrt{2}) = \sqrt{2}-1+\frac{2}{\sqrt{2}} = 2\sqrt{2}-1$
$\Rightarrow$ local max at $x=\sqrt{2}$
Local minimum at $x=-\sqrt{2}$.
~~No absoluel extreme~~.

$f''(x) = \dfrac{4}{-x^3} \Rightarrow x<0$, $f''(x)>0$ concave up
$x>0$, $f''(x)<0$ concave down.
but $x\neq 0 \Rightarrow$ has no inflection point.

(2) horizontal asymptote: no.
vertical asymptote: $x=0$.
oblique asymptote: $y=-x-1$

(3) 注: $x\neq 0$, $x\neq 1$.

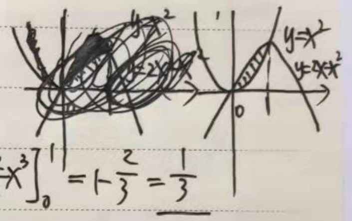


6. (1) $y=\int_1^{1+2x}\sqrt{t^2-1}\,dt \Rightarrow \frac{dy}{dx}=2\sqrt{(1+2x)^2-1}=\underline{2\sqrt{4x^2+4x}}$

(2) $x^2-y^2=9 \Rightarrow 2x-2y\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{x}{y}=-\frac{5}{4}.$

$\underline{y-(-4)=-\frac{5}{4}(x-5)}$

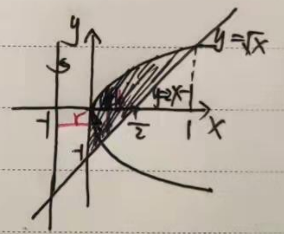
7. $\begin{cases} y=x^2 \\ y=2x-x^2 \end{cases} \Rightarrow 2x^2=2x \Rightarrow x=0,\ x=1.$



$S=\int_0^1[(2x-x^2)-x^2]dx=\int_0^1(2x-2x^2)dx=x^2-\frac{2}{3}x^3\Big]_0^1=1-\frac{2}{3}=\underline{\frac{1}{3}}$

8. $\begin{cases} y=\sqrt{x} \\ y=2x-1 \end{cases} \Rightarrow x^2=4x^2-4x+1 \quad (3x-1)(x-1)=0$

$3x^2-4x+1=0 \quad x=\frac{1}{3},\ x=1$



$V=2\pi\int_0^1(x+1)(\sqrt{x}-(2x-1))dx$

$=2\pi\int_0^1(x^{\frac{3}{2}}-2x^2+x+\sqrt{x}-2x+1)dx$

$=2\pi(\frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^3-\frac{1}{2}x^2+\frac{2}{3}x^{\frac{3}{2}}+x)\Big]_0^1$

$=2\pi(\frac{2}{5}-\frac{2}{3}-\frac{1}{2}+\frac{2}{3}+1)=\underline{\frac{9\pi}{5}}.$

9. $f(x)$ is continuous on $[0,1] \Rightarrow f(x)$ has max, min., $f(\alpha)=M=f(x)_{max}$. $f(\beta)=m=f(x)_{min}$. if $\alpha\le\beta$

$f(\beta)\le\underbrace{\int_0^1 f(x)dx}_{y_0}\le f(\alpha)$

By the Intermediate Value Theorem, $\exists c\in[\alpha,\beta]\subset[0,1]$ s.t. $f(c)=\int_0^1 f(x)dx$.