

Step-1

Orthogonal Matrix: If matrix A is skew-symmetric then e^{At} is an orthogonal matrix.

Skew symmetric: If transpose of matrix A (A^T) is equal to negative of matrix A , then matrix A is skew-symmetric.

In a conservative system following are observed:

$$\begin{aligned}A^T &= -A \\(e^{At})^T &= e^{-At} \\\|e^{At}u(0)\| &= \|u(0)\|\end{aligned}$$

Step-2

Consider the skew-symmetric equation:

$$\begin{aligned}\frac{du}{dt} &= Au \\&= \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\end{aligned}$$

Step-3

(a) Compute u_1', u_2', u_3' and confirm the following:

$$u_1u_1' + u_2u_2' + u_3u_3' = 0$$

Calculate the following from the skew-symmetric equation

$$\begin{aligned}u_1' &= cu_2 - bu_3 \\u_2' &= -cu_1 + au_3 \\u_3' &= bu_1 - au_2\end{aligned}$$

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Now,

$$u_1 u_1' = cu_2 u_1 - bu_3 u_1$$

$$u_2 u_2' = -cu_1 u_2 + au_3 u_2$$

$$u_3 u_3' = bu_1 u_3 - au_2 u_3$$

$$u_1 u_1' + u_2 u_2' + u_3 u_3' = 0$$

Therefore, $\boxed{u_1 u_1' + u_2 u_2' + u_3 u_3' = 0}$.

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(b) Evaluate the length of $u_1^2 + u_2^2 + u_3^2$.

Length of $u_1^2 + u_2^2 + u_3^2$ can be written as $\|u(t)\|^2$. Now,

$$\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$$

Differentiate the following:

$$\frac{d\|u(t)\|^2}{dt} = 2u_1 u_1' + 2u_2 u_2' + 2u_3 u_3'$$

Substitute the above result, to get derivate of $\|u(t)\|^2$ equal to zero. This shows that $\|u(t)\|^2$ is constant.

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Recall that $u(t) = e^{At}u(0)$, then

$$\begin{aligned}\|u(t)\|^2 &= \|e^{At}u(0)\|^2 \\ &= \|u(0)\|^2 \\ &= \text{constant}\end{aligned}$$

Therefore, $\boxed{\|u(t)\|^2 = \|u(0)\|^2}$.

Step-7

(c) Find the Eigen values of matrix A .

To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 0 - \lambda & c & -b \\ -c & 0 - \lambda & a \\ b & -a & 0 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(-\lambda)^3 - \lambda(a^2 + b^2 + c^2) = 0$$

$$-\lambda(\lambda^2 + (a^2 + b^2 + c^2)) = 0$$

After solving following values are obtained:

$$\lambda_1 = 0$$

$$\lambda_2 = \sqrt{(a^2 + b^2 + c^2)}$$

$$\lambda_3 = -\sqrt{(a^2 + b^2 + c^2)}$$

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Therefore, Eigen values are $\boxed{0, \pm \sqrt{(a^2 + b^2 + c^2)}}$.