SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Homework 10

1. Suppose $Y = \mathrm{e}^X$ where X is normally distributed with parameters μ and σ^2. Use the following two methods to obtain E(Y).

   (i) First obtain the p.d.f of Y, denoted by $f_Y(y)$ and then find E(Y) by using $f_Y(y)$.

   (ii) Find E(Y) directly by viewing Y as a function of X and then using the formula of getting the expected value of a function of the random variable X.

Solution:

(i) Let the cdf of Y be $F_Y(y)$. Then

$$F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}\{e^X \le y\}.$$

Now,

1° if $y < 0$, then $\{\omega \in \Omega : \mathrm{e}^{X(\omega)} \le y\} = \{\mathrm{e}^X \le y\} = \emptyset$ and thus

$$F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}\{e^X \le y\} = \mathrm{P}(\emptyset) = 0.$$

Hence, the probability density function $f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = 0$.

2° if $y = 0$, then $\{\omega \in \Omega : \mathrm{e}^{X(\omega)} \le 0\} = \{\mathrm{e}^X \le 0\} = \emptyset$ and thus

$$F_Y(0) = \mathrm{P}(Y \le 0) = \mathrm{P}\{e^X \le 0\} = \mathrm{P}(\emptyset) = 0.$$

Since the probability density function in a point modify the value does not affect the distribution function, for $y = 0$, let $f_Y(y) = f_Y(0) = 0$.

3° if $y > 0$, then by the increasing property of the function $\ln(\cdot)$, we know that

$$F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}(e^X \le y) = \mathrm{P}\{X \le \ln y\} = F_X(\ln y),$$

where $F_X(x)$ is the c.d.f of X.
Differentiating $F_Y(y) = F_X(\ln y)$, we obtain

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_X(\ln y) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} F_Y(u)\big|_{u=\ln y} \cdot \frac{\mathrm{d}}{\mathrm{d}y}(\ln y) \\ &= f_X(u)\big|_{u=\ln y} \cdot \frac{1}{y} \\ &= f_X(\ln y) \cdot \frac{1}{y}. \end{aligned}$$

Since $X \sim \mathrm{N}(\mu, \sigma^2)$, then $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ and so

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} \cdot \frac{1}{y} = \frac{1}{y\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}.$$

To sum up, we obtain

$$f_Y(y) = \begin{cases} \dfrac{1}{y\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}, & \text{if} \quad y > 0, \\ 0, & \text{if} \quad y \le 0. \end{cases}$$

So,

$$
\begin{aligned}
\mathrm{E(Y)} &= \int_{-\infty}^{+\infty} y f_Y(y)\mathrm{d}y = \int_{-\infty}^{0} y f_Y(y)\mathrm{d}y + \int_{0}^{+\infty} y f_Y(y)\mathrm{d}y \\
&= \int_{-\infty}^{0} y \cdot 0\mathrm{d}y + \int_{0}^{+\infty} y \cdot \frac{1}{\sqrt{2\pi}\sigma y}\, \mathrm{e}^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}\, \mathrm{d}y \\
&= \int_{0}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}\, \mathrm{d}y \\
&\xlongequal[y=\mathrm{e}^x]{x=\ln y} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\, \mathrm{e}^x\, \mathrm{d}x \\
&\xlongequal[x=\mu+\sigma t]{t=\frac{x-\mu}{\sigma}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mu+\sigma t} \cdot \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{t^2}{2}}\, \sigma \mathrm{d}x \\
&= \mathrm{e}^{\mu} \int_{-\infty}^{+\infty} \cdot \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{t^2-2\sigma t}{2}}\, \mathrm{d}x = \mathrm{e}^{\mu} \int_{-\infty}^{+\infty} \cdot \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{(t-\sigma)^2-\sigma^2}{2}}\, \mathrm{d}x \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{(t-\sigma)^2}{2}}\, \mathrm{d}x \xlongequal{s=t-\sigma} \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{s^2}{2}}\, \mathrm{d}s \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \times 1 = \mathrm{e}^{\mu+\frac{\sigma^2}{2}}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\mathrm{E(Y)} &= \int_{-\infty}^{+\infty} \mathrm{e}^x\, f_X(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \mathrm{e}^x \cdot \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\, \mathrm{d}x \\
&\xlongequal[x=\mu+\sigma t]{t=\frac{x-\mu}{\sigma}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mu+\sigma t} \cdot \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{t^2}{2}}\, \sigma \mathrm{d}x \\
&= \mathrm{e}^{\mu} \int_{-\infty}^{+\infty} \cdot \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{t^2-2\sigma t}{2}}\, \mathrm{d}x = \mathrm{e}^{\mu} \int_{-\infty}^{+\infty} \cdot \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{(t-\sigma)^2-\sigma^2}{2}}\, \mathrm{d}x \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{(t-\sigma)^2}{2}}\, \mathrm{d}x \xlongequal{s=t-\sigma} \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{s^2}{2}}\, \mathrm{d}s \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \times 1 = \mathrm{e}^{\mu+\frac{\sigma^2}{2}}.
\end{aligned}
$$

Remark: Assume a continuous random vector X have joint p.d.f $f_X(x)$, $g\colon \mathbb{R}^n \to \mathbb{R}$ is a (measure) function, Let $Y = g(X)$, then we have the following formula:

$$
\mathrm{E[Y]} = \mathrm{E[g(X)]} = \int_{\mathbb{R}^n} \mathrm{g(x)f_X(x)dx}.
$$

2. (a) Suppose the random variable X obeys the uniformly distribution over interval $[a, b]$. Find $\mathrm{E(X^2)}$ and then obtain the value of $\mathrm{E(X^2)} - (\mathrm{E(X)})^2$.
   (b) Suppose X is normally distributed random variable with parameters μ and σ^2. Find $\mathrm{E(X^2)}$ and then obtain the value of $\mathrm{E(X^2)} - (\mathrm{E(X)})^2$.

Solution:

(a) Recall the p.d.f of random variable X the uniformly distribution is:

$$
f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b, \\ 0 & \textit{otherwise}. \end{cases}
$$

so,

$$
\mathrm{E(X)} = \int_{-\infty}^{+\infty} x f_X(x)\mathrm{d}x
$$

$$
\begin{aligned}
&= \int_{-\infty}^{a} x f_X(x)\mathrm{d}x + \int_{a}^{b} x f_X(x)\mathrm{d}x + \int_{b}^{+\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{a} x \cdot 0\mathrm{d}x + \int_{a}^{b} x \cdot \frac{1}{b-a}\mathrm{d}x + \int_{b}^{+\infty} x \cdot 0\mathrm{d}x \\
&= \frac{1}{b-a}\int_{a}^{b} x\mathrm{d}x = \frac{1}{b-a}\frac{x^2}{2}\Big|_a^b \\
&= \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} = \frac{a+b}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\mathrm{X}^2) &= \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{a} x^2 f_X(x)\mathrm{d}x + \int_{a}^{b} x^2 f_X(x)\mathrm{d}x + \int_{b}^{+\infty} x^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{a} x^2 \cdot 0\mathrm{d}x + \int_{a}^{b} x^2 \cdot \frac{1}{b-a}\mathrm{d}x + \int_{b}^{+\infty} x^2 \cdot 0\mathrm{d}x \\
&= \frac{1}{b-a}\int_{a}^{b} x^2\mathrm{d}x = \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b \\
&= \frac{1}{b-a}\cdot\frac{b^3-a^3}{3} = \frac{a^2+ab+b^3}{3}.
\end{aligned}
$$

Hence, $\mathrm{E}(\mathrm{X}^2) - (\mathrm{E}(\mathrm{X}))^2 = \frac{\mathrm{a}^2+\mathrm{ab}+\mathrm{b}^3}{3} - (\frac{\mathrm{a}+\mathrm{b}}{2})^2 = \frac{(\mathrm{b}-\mathrm{a})^2}{12}$

(b) Recall the p.d.f of normally distributed random variable with parameters μ and σ^2 is:

$$
f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty.
$$

Hence,

$$
\begin{aligned}
\mathrm{E}(\mathrm{X}) &= \int_{-\infty}^{+\infty} x f_X(x)\mathrm{d}x = \int_{-\infty}^{+\infty} x\cdot\frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}y \\
&\overset{y=\frac{x-\mu}{\sigma}}{=\!=\!=} \int_{-\infty}^{+\infty} (\mu+\sigma y)\frac{1}{\sqrt{2\pi}\sigma}\cdot\mathrm{e}^{-\frac{y^2}{2}}\cdot\sigma\mathrm{d}y \\
&= \mu\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y + \sigma\int_{-\infty}^{+\infty} y\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y \\
&= \mu\times 1 + \sigma\times 0 = \mu.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\mathrm{X}^2) &= \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{d}x = \int_{-\infty}^{+\infty} x^2\cdot\frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x \\
&\overset{y=\frac{x-\mu}{\sigma}}{=\!=\!=} \int_{-\infty}^{+\infty} (\mu+\sigma y)^2\cdot\frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{y^2}{2}}\cdot\sigma\mathrm{d}y = \int_{-\infty}^{+\infty}(\sigma^2y^2+2\sigma\mu y+\mu^2)\cdot\frac{1}{\sqrt{2\pi}}\cdot\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y \\
&= \sigma^2\int_{-\infty}^{+\infty} y^2\cdot\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y + 2\sigma\mu\int_{-\infty}^{+\infty} y\cdot\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y + \mu^2\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y \\
&= \sigma^2\int_{-\infty}^{+\infty} y\cdot\frac{1}{\sqrt{2\pi}}\mathrm{d}(-\,\mathrm{e}^{-\frac{y^2}{2}}) + 2\sigma\mu\times 0 + \mu^2\times 1 \\
&= \sigma^2\left[y(-\,\mathrm{e}^{-\frac{y^2}{2}})\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}(-\,\mathrm{e}^{-\frac{y^2}{2}})\mathrm{d}y\right] + \mu^2 = \sigma^2[0+1]+\mu^2 \\
&= \mu^2+\sigma^2.
\end{aligned}
$$

So, $\mathrm{E}(\mathrm{X}^2) - (\mathrm{E}(\mathrm{X}))^2 = \sigma^2+\mu^2-(\mu)^2 = \sigma^2$.

3. (a) If the probability density function of an (absolutely) continuous random variable X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{x(\ln 3)} & 1 < x < 3, \\ 0 & otherwise. \end{cases}$$

Find $\mathrm{E(X)}$, $\mathrm{E(X^2)}$ and $\mathrm{E(X^3)}$.

(b) Use the results of part (a) to determine $\mathrm{E(X^3 + 2X^2 - 3X + 1)}$.

Solution:

(a)

$$\begin{aligned}
\mathrm{E(X)} &= \int_{-\infty}^{+\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{1} x f_X(x)\mathrm{d}x + \int_{1}^{3} x f_X(x)\mathrm{d}x + \int_{3}^{+\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{1} x \cdot 0\mathrm{d}x + \int_{1}^{3} x \cdot \frac{1}{x(\ln 3)}\mathrm{d}x + \int_{3}^{+\infty} x \cdot 0\mathrm{d}x \\
&= \frac{1}{\ln 3}\int_{1}^{3} \mathrm{d}x = \frac{2}{\ln 3}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E(X^2)} &= \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{1} x^2 f_X(x)\mathrm{d}x + \int_{1}^{3} x^2 f_X(x)\mathrm{d}x + \int_{3}^{+\infty} x^2 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{1} x^2 \cdot 0\mathrm{d}x + \int_{1}^{3} x^2 \cdot \frac{1}{x(\ln 3)}\mathrm{d}x + \int_{3}^{+\infty} x^2 \cdot 0\mathrm{d}x \\
&= \frac{1}{\ln 3}\int_{1}^{3} x\mathrm{d}x = \frac{1}{\ln 3}\frac{x^2}{2}\Big|_1^3 \\
&= \frac{1}{\ln 3} \cdot \frac{3^2 - 1^2}{2} = \frac{4}{\ln 3}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E(X^3)} &= \int_{-\infty}^{+\infty} x^3 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{1} x^3 f_X(x)\mathrm{d}x + \int_{1}^{3} x^3 f_X(x)\mathrm{d}x + \int_{3}^{+\infty} x^3 f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{1} x^3 \cdot 0\mathrm{d}x + \int_{1}^{3} x^3 \cdot \frac{1}{x(\ln 3)}\mathrm{d}x + \int_{3}^{+\infty} x^3 \cdot 0\mathrm{d}x \\
&= \frac{1}{\ln 3}\int_{1}^{3} x^2\mathrm{d}x = \frac{1}{\ln 3}\frac{x^3}{3}\Big|_1^3 \\
&= \frac{1}{\ln 3} \cdot \frac{3^3 - 1^3}{3} = \frac{26}{3\ln 3}.
\end{aligned}$$

(b) According to the linear property of expectations, we have

$$\begin{aligned}
\mathrm{E(X^3 + 2X^2 - 3X + 1)} &= \mathrm{E(X^3) + 2E(X^2) - 3E(X) + E(1)} \\
&= \frac{26}{3\ln 3} + 2 \times \frac{4}{\ln 3} - 3 \times \frac{2}{\ln 3} + 1 \\
&= \frac{32}{3\ln 3} + 1.
\end{aligned}$$

4. If the probability density function of an (absolutely) continuous random variable X is given by

$$f_X(x) = \begin{cases} \dfrac{x}{2} & 0 < x \leq 1, \\ \dfrac{1}{2} & 1 < x \leq 2, \\ \dfrac{3-x}{2} & 2 < x \leq 3, \\ 0 & otherwise. \end{cases}$$

Find the expectation of $g(X) = X^2 - 5X + 3$.

Solution:

$$\begin{aligned}
\mathrm{E[g(X)]} &= \int_{-\infty}^{+\infty} g(x) f_X(x) \mathrm{d}x \\
&= \int_0^1 (x^2 - 5x + 3) \cdot (\frac{x}{2}) \mathrm{d}x + \int_1^2 (x^2 - 5x + 3) \cdot \frac{1}{2} \mathrm{d}x + \int_2^3 (x^2 - 5x + 3) \cdot (\frac{3-x}{2}) \mathrm{d}x \\
&= \frac{\frac{x^4}{4} - \frac{5}{3}x^3 + \frac{3}{2}x^2}{2}\Big|_0^1 + \frac{\frac{x^3}{3} - \frac{5}{2}x^2 + 3x}{2}\Big|_1^2 + \frac{x^3 - \frac{15}{2}x^2 + 9x - \frac{x^4}{4} + \frac{5}{3}x^3 - \frac{3}{2}x^2}{2}\Big|_2^3 \\
&= \frac{1}{24} - \frac{13}{12} - \frac{19}{24} \\
&= -\frac{11}{6}.
\end{aligned}$$

<u>**Method 2:**</u>

$$\begin{aligned}
\mathrm{E[X^2]} &= \int_{-\infty}^{+\infty} x^2 f_X(x) \mathrm{d}x \\
&= \int_0^1 x^2 \cdot (\frac{x}{2}) \mathrm{d}x + \int_1^2 x^2 \cdot \frac{1}{2} \mathrm{d}x + \int_2^3 x^2 \cdot (\frac{3-x}{2}) \mathrm{d}x \\
&= \frac{\frac{x^4}{4}}{2}\Big|_0^1 + \frac{\frac{x^3}{3}}{2}\Big|_1^2 + \frac{x^3 - \frac{x^4}{4}}{2}\Big|_2^3 \\
&= \frac{1}{8} + \frac{7}{6} + \frac{11}{8} \\
&= \frac{64}{24} = \frac{8}{3}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E[X]} &= \int_{-\infty}^{+\infty} x \cdot f_X(x) \mathrm{d}x \\
&= \int_0^1 x \cdot (\frac{x}{2}) \mathrm{d}x + \int_1^2 x \cdot \frac{1}{2} \mathrm{d}x + \int_2^3 x \cdot (\frac{3-x}{2}) \mathrm{d}x \\
&= \frac{\frac{1}{3}x^3}{2}\Big|_0^1 + \frac{\frac{x^2}{2}}{2}\Big|_1^2 + \frac{3\frac{x^2}{2} - \frac{x^3}{3}}{2}\Big|_2^3 \\
&= \frac{1}{6} + \frac{3}{4} + \frac{7}{12} \\
&= \frac{18}{12} = \frac{3}{2}.
\end{aligned}$$

According to the linear property of expectations, we have

$$\mathrm{E[g(X)] = E(X^2 - 5X + 3) = E(X^2) - 5E(X) + E(3)}$$

$$
\begin{aligned}
&= \frac{8}{3} - 5 \times \frac{3}{2} + 3 \\
&= -\frac{11}{6}
\end{aligned}
$$

5. The two continuous random variables X and Y have joint p.d.f

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & otherwise. \end{cases}$$

Find $\mathrm{E}[(\mathrm{X}+\mathrm{Y})^2]$.

Solution:

$$
\begin{aligned}
\mathrm{E}[(\mathrm{X}+\mathrm{Y})^2] &= \iint_{\mathbb{R}^2} (x+y)^2 f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} (x+y)^2 f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} (x+y)^2 (x+y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} (x+y)^3\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} (x^3+3x^2y+3xy^2+y^3)\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 (x^3+3x^2y+3xy^2+y^3)\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 (\frac{x^4}{4} + x^3y + \frac{3}{2}x^2y^2 + xy^3)\Big|_{x=0}^{x=1}\mathrm{d}y \\
&= \int_0^1 (\frac{1}{4} + y + \frac{3}{2}y^2 + y^3)\mathrm{d}y = (\frac{1}{4}y + \frac{y^2}{2} + \frac{3}{2}\cdot\frac{y^3}{3} + \frac{y^4}{4})\Big|_0^1 \\
&= \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} \\
&= \frac{3}{2}.
\end{aligned}
$$