Step-1

(a)

The objective is to construct 2 by 2 matrices such that the eigen values of AB are not the products of the eigen values of A and B, and the eigenvalues of $A+B$ are not the sums of the individual eigen values.

Assume that, $A=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$.

The number λ is an eigen values of A if and if

$$\begin{aligned}|A-\lambda I|&=0\\ \begin{vmatrix}3-\lambda & 2\\ 1 & 4-\lambda\end{vmatrix}&=0\\ (3-\lambda)(4-\lambda)-2&=0\\ 12+\lambda^2-7\lambda-2&=0\\ \lambda^2-7\lambda+10&=0\\ (\lambda-5)(\lambda-2)&=0\\ \lambda=2 \text{ or } \lambda&=5\end{aligned}$$

So, the eigen values of the matrix are $\lambda=2$ or $\lambda=5$.

Step-2

Let $B=\begin{bmatrix}-6 & -1\\ 2 & 3\end{bmatrix}$

The number λ is an eigen values of A if and only if

$$\begin{aligned}|B-\lambda I|&=0\\ \begin{vmatrix}-6-\lambda & -1\\ 2 & 3-\lambda\end{vmatrix}&=0\\ (-6-\lambda)(3-\lambda)+2&=0\\ -18+6\lambda-3\lambda+\lambda^2+2&=0\\ 3\lambda+\lambda^2-16&=0\end{aligned}$$

Use the quadratic formula $\lambda=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, then the values of λ are

$$\lambda = \frac{-(3) \pm \sqrt{9+4(16)}}{2(1)}$$
$$= \frac{-(3) \pm \sqrt{73}}{2}$$
$$\lambda = \frac{1}{2}\left(-3-\sqrt{73}\right) \text{ or} \lambda = \frac{1}{2}\left(-3+\sqrt{73}\right)$$

So, the eigen values of the matrix are $\lambda = \frac{1}{2}\left(-3-\sqrt{73}\right) \text{ or} \lambda = \frac{1}{2}\left(-3+\sqrt{73}\right)$.

Step-3

The product of individual eigenvalue of A and B is,

$$\lambda_1 = 2 \times \frac{1}{2}\left(-3-\sqrt{73}\right)$$
$$= -3-\sqrt{73}$$

$$\lambda_2 = 5 \times \frac{1}{2}\left(-3-\sqrt{73}\right)$$
$$= \frac{5}{2}\left(-3-\sqrt{73}\right)$$

While the sum of the individual eigenvalue of A and B is,

$$\lambda_1 = 2 + \frac{1}{2}\left(-3+\sqrt{73}\right)$$
$$= \frac{1}{2}\left(1+\sqrt{73}\right)$$

And,

$$\lambda_2 = 2 + \frac{1}{2}\left(-3-\sqrt{73}\right)$$
$$= \frac{1}{2}\left(1-\sqrt{73}\right)$$

Now consider,

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -6 & -1 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3\times -6 + 2\times 2 & 3\times -1 + 2\times 3 \\ 1\times -6 + 4\times 2 & 1\times -1 + 4\times 3 \end{bmatrix} \\ &= \begin{bmatrix} -14 & 3 \\ 2 & 11 \end{bmatrix} \end{aligned}$$

Step-4

The eigen values of the matrix AB is,

$$\begin{aligned} |AB - \lambda I| &= 0 \\ \begin{vmatrix} -14-\lambda & 3 \\ 2 & 11-\lambda \end{vmatrix} &= 0 \\ (-14-\lambda)(11-\lambda) - 6 &= 0 \\ -154 + 14\lambda - 11\lambda + \lambda^2 - 6 &= 0 \\ \lambda^2 + 3\lambda - 160 &= 0 \end{aligned}$$

Use the quadratic formula $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then the values of λ are

$$\begin{aligned} \lambda &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-160)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{649}}{2} \end{aligned}$$

Clearly, the eigenvalues of AB are not equal with product of individual eigenvalue of A and B.

Step-5

Write the sum of the two matrices A and B and this is equal to,

$$\begin{aligned} A + B &= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -6 & -1 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 3 & 7 \end{bmatrix} \end{aligned}$$

The eigen values of the matrix $A + B$ is,

$$\left|(A+B)-\lambda I\right|=0$$

$$\begin{vmatrix} -3-\lambda & 1 \\ 3 & 7-\lambda \end{vmatrix}=0$$

$$\lambda^2-4\lambda-24=0$$

$$(\lambda-2)^2-28=0$$

$$\lambda=2\pm2\sqrt{7}$$

Therefore, the eigen values of the matrix $A+B$ are not the sum of the individual eigenvalues of A and B.

Step-6

(b)

The sum of the eigen values $A+B$ is, $2+2\sqrt{7}+2-2\sqrt{7}=4$.

Sum of all eigenvalue of A is, $2+5=7$.

While the sum of all eigenvalue of B is, $\frac{1}{2}\left(-3-\sqrt{73}\right)+\frac{1}{2}\left(-3+\sqrt{73}\right)=-3$.

Sum of all eigenvalue of A and B is, $7-3=4$.

Hence, Sum of all eigen values of $A+B=$ Sum of the eigenvalue of all individual eigenvalues of A and B.

Also, this generally true because,

sum of all eigenvalues of $A+B=tr(A+B)=tr(A)+tr(B)=$ all eigenvalue of $A+$ sum of all eigenvalues of B .

Step-7

The product of the actual eigen values of AB is,

$$\begin{aligned}\lambda_1\lambda_2&=\left(\frac{-3}{2}+\frac{\sqrt{649}}{2}\right)\left(\frac{-3}{2}-\frac{\sqrt{649}}{2}\right)\\&=\frac{9}{4}-\frac{649}{4}\\&=\frac{-640}{4}\\&=-160\end{aligned}$$

Product of all eigenvalues of A is, $2\times5=10$.

While, product of all eigenvalues of *B* is, $\left[\frac{1}{2}\left(-3-\sqrt{73}\right)\right]\left[\frac{1}{2}\left(-3+\sqrt{73}\right)\right]=\frac{1}{4}(9-73)$

$=-16$

Product of all eigenvalues of *A* and *B* is, -160.

Hence, product of all eigen values of $AB=$Product of the eigenvalue of all individual eigenvalues of *A* and *B*.