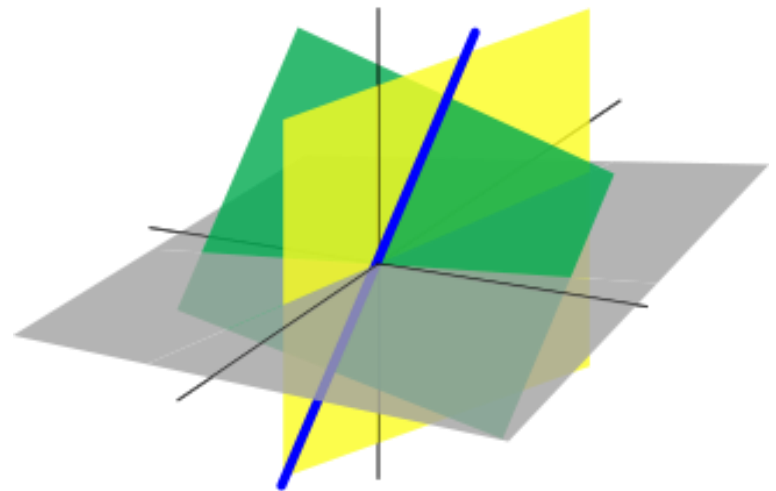# 2

Vector Spaces
(向量空间)

2.3

LINEAR INDEPENDENCE, BASIS AND DIMENSION
(线性无关性、基和维数)

Linear Independence

Basis

Coordinates (坐标)

Dimension

I. Introduction

在三维几何向量空间 $\mathbf{R}^3$ 中,

$$\boldsymbol{i} = (1,0,0)^{\mathrm{T}}, \; \boldsymbol{j} = (0,1,0)^{\mathrm{T}}, \; \boldsymbol{k} = (0,0,1)^{\mathrm{T}}.$$

$$\boldsymbol{\alpha} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}.$$

向量$(a_1, a_2, a_3)^{\mathrm{T}}$是 $\boldsymbol{\alpha}$ 关于一组基 $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ 的坐标.

- 向量组$\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$线性无关

- 向量组$\{\boldsymbol{\alpha}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$线性相关

- $\mathbf{R}^3$ 中的任何一个向量可以由$\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$线性表示, 但不可以仅由它的真子集表示:

  $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$是$\mathbf{R}^3$ 的一组基

- $\mathbf{R}^3$的维数是3 (基含有的向量个数)

- 系数$(a_1, a_2, a_3)^{\mathrm{T}}$是向量$\boldsymbol{\alpha}$在这组基下的坐标
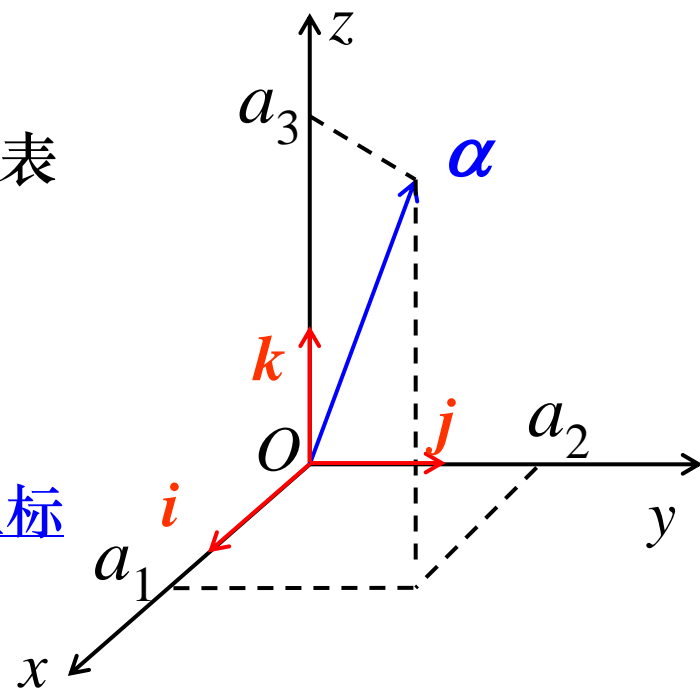
Use your geometric experience with $\mathbf{R}^2$ and $\mathbf{R}^3$ to visualize general concepts

In $\mathbf{R}^n$, let

$$e_1 = (1,0,0, ..., 0,0)^{\mathrm{T}},$$
$$e_2 = (0,1,0, ..., 0,0)^{\mathrm{T}},$$
$$...$$
$$e_n = (0,0,0, ..., 0,1)^{\mathrm{T}}.$$

Then $\{e_1, e_2, ..., e_n\}$ is a spanning set of $\mathbf{R}^n$, since each vector $v = (x_1, x_2, ..., x_n)^{\mathrm{T}}$ is a linear combination of them:

$$v = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

It is obvious that no *proper subset* (真子集) of $\{e_1, e_2, ..., e_n\}$ is a spanning set of $\mathbf{R}^n$.

Let V be a subspace of $\mathbf{R}^n$.

A set of vectors $\{v_1, v_2, ..., v_r\}$ is called a ***basis*** (一组基) if it is a *spanning set* of V but any proper subset is not.

Bases are important in the study of vector spaces.

Recall: **Spanning Sets**

Recall that, if V is a subspace of $\mathbf{R}^n$ and A is a subset of $\mathbf{R}^n$ such that

$$V = \text{span}(A)$$

then V is the **span** of A, and A is a **spanning set** for V.

Example 1 Let $A = \{\boldsymbol{u}, \boldsymbol{v}\}$, defined below.
(1) If $\boldsymbol{u} = (1,1)^{\text{T}}$ and $\boldsymbol{v} = (2,2)^{\text{T}}$, then

$$\text{span}(A) = \{(\alpha, \alpha)^{\text{T}} \,|\, \alpha \in \mathbf{R}\}.$$

(2) For $\boldsymbol{u} = (1,1)^{\text{T}}$ and $\boldsymbol{v} = (1,2)^{\text{T}}$, then

$$\text{span}(A) = \{(\alpha, \beta)^{\text{T}} \,|\, \alpha, \beta \in \mathbf{R}\} = \mathbf{R}^2.$$

Example 2 Let $A = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$, defined below.
(1) If $\boldsymbol{u} = (1,0,0)^{\text{T}}$, $\boldsymbol{v} = (0,1,0)^{\text{T}}$, and $\boldsymbol{w} = (2,3,0)^{\text{T}}$, then

$$\text{span}(A) = \{(\alpha, \beta,0)^{\text{T}} \,|\, \alpha, \beta \in \mathbf{R}\}.$$

(2) Let $\boldsymbol{u} = (1,0,0)^{\text{T}}$, $\boldsymbol{v} = (0,1,0)^{\text{T}}$ and $\boldsymbol{w} = (0,0,1)^{\text{T}}$, then

$$\text{span}(A) = \mathbf{R}^3.$$

A natural question is how to determine whether a given set of vectors is a spanning set of a vector space?

Example 3 Let $A = \{(1,1,1)^{\mathrm{T}}, (1,3,5)^{\mathrm{T}}, (1,2,3)^{\mathrm{T}}\}$. Is A a spanning set for $\mathbf{R}^3$?

We need to see if *any* vector $\boldsymbol{v} = (a, b, c)^{\mathrm{T}} \in \mathbf{R}^3$ is a linear combination of A. Let

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

for some scalars x_1, x_2, x_3. This forms a system of linear equations.

If this system has solutions for *any* $(a, b, c)^{\mathrm{T}}$, then A is a spanning set; otherwise, A is not a spanning set .

Answer?

What if -- Let $A = \{(1,1,1)^{\mathrm{T}}, (1,3,5)^{\mathrm{T}}, (1,2,4)^{\mathrm{T}}\}$?

II. Linear Independence

实例

某调料公司用6种成分制造了6 种调味品

每包调味品所需各成分的量

成 分 \ 调味品	A	B	C	D	E	F
红辣椒	3	1.5	4.5	7.5	9	4.5
姜黄	2	4	0	8	1	6
胡椒	1	2	0	4	2	3
大蒜粉	0.5	1	0	2	2	1.5
盐	0.5	1	0	2	2	1.5
丁香油	0.25	0.5	0	2	1	0.75

- 顾客是否可只买其中部分调味品，并配出其余几种？
- 最少要购买几种调味品？哪几种？
- 能否配制出下列新调味品？

红辣椒:18; 姜黄:18; 胡椒:9; 大蒜粉:4.8; 盐:4.5; 丁香油: 3.25.

II. Linear Independence

Definition 1 (*Linear dependence*). Let $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ ($k \geq 2$) be a set of vectors of $\mathbf{R}^n$. Then

(i) A is called **linearly dependent** (线性相关) if *one of* the vectors can be expressed as a linear combination of the others,

i.e., there exists $\boldsymbol{v}_i \in A$ such that

$$\boldsymbol{v}_i = \sum_{j \neq i} \lambda_j \boldsymbol{v}_j$$

where λ_j's are numbers;

(ii) A is called **linearly independent** (线性无关) if *no* vector in A is a linear combination of the others.

A set containing a single vector $\boldsymbol{v}$ ($k=1$) is linearly independent if and only if $\boldsymbol{v} \neq \boldsymbol{0}$. (仅有一个向量$\boldsymbol{v}$的集合线性无关当且仅当$\boldsymbol{v} \neq \boldsymbol{0}$)

For example,

- $\{(1,1)^T, (2, 2)^T\}$ is linearly dependent.
- $\{(1, 1)^T, (1, 2)^T\}$ is linearly independent.
- $\{(1, 0, 0)^T, (0, 1, 0)^T, (2, 3, 0)^T\}$ is linearly dependent.
- $\{(1, 0, 0)^T, (0, 1, 0)^T, (2, 3, 1)^T\}$ is linearly independent.

The next theorem provides us with a method for deciding whether a set of vectors is linearly independent.

Theorem 1 *Let $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of vectors of $\mathbf{R}^n$. Then*
(1) *A is* linearly independent *if and only if*
$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_k\boldsymbol{v}_k = \boldsymbol{0}$$
holds **only** *for $\boldsymbol{\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0}$; equivalently*
(2) *A is* linearly dependent *if and only if*
$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_k\boldsymbol{v}_k = \boldsymbol{0}$$
holds **for some scalars $\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_k}$ which are** **not all zeros**.

Note: A set of vectors containing the zero vector must be linearly dependent. (含零向量的向量组必定线性相关.)

A Method: Given $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subset \mathbf{R}^n$, the following process decides if A is linearly independent.

Step 1. Set up a vector equation

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_k\boldsymbol{v}_k = \boldsymbol{0}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are unknowns.

Step 2. Write this vector equation into a system of linear equations with unknowns $\lambda_1, \lambda_2, \ldots, \lambda_k$ (关于$\lambda_1, \lambda_2, \ldots, \lambda_k$的线性方程组).

Step 3. Solve the system of linear equations.

Step 4. Discussion: If $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$ is the only solution (该线性方程组只有零解, trivial), then A is linearly independent (线性无关).

If there is a solution such that some $\lambda_i \neq 0$ (该线性方程组有非零解, 即某些λ_i的取值非零, nontrivial), then A is linearly dependent (线性相关).

Example 4 Let
$$\boldsymbol{u} = (1, 0, -1, 0)^{\mathrm{T}}, \boldsymbol{v} = (1, 1, 0, 2)^{\mathrm{T}}, \boldsymbol{w} = (0, 3, 1, -2)^{\mathrm{T}}.$$
Is $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ linearly independent?

Solution The key to answer this question is to solve the equation
$$\lambda_1 \boldsymbol{u} + \lambda_2 \boldsymbol{v} + \lambda_3 \boldsymbol{w} = \boldsymbol{0}.$$

In terms of components, we have a system of linear equations

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this system of equations, we reduce the augmented matrix to row echelon form

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ -1 & 0 & 1 & | & 0 \\ 0 & 2 & -2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This system only has solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and so $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent.

Example 5 Let

$$u = (1, 0, -1, 0)^{\text{T}}, v = (1, 1, 0, 2)^{\text{T}}, w = (1, 3, 2, 6)^{\text{T}}.$$

Is $\{u, v, w\}$ linearly independent?

Solution The key to answer this question is to solve the equation

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0.$$

In terms of components, we have a system of linear equations. To solve this system of equations, we reduce the augmented matrix to row echelon form

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ -1 & 0 & 2 & | & 0 \\ 0 & 2 & 6 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Now, the system of equations has non-trivial solutions, so u, v, w are linearly dependent.

Two independent rows;
Two independent columns

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

For instance, $(\lambda_1, \lambda_2, \lambda_3)$ $= (2, -3, 1)$ is a solution, and hence

$$2u - 3v + w = 0.$$

Example 6   The columns of the following triangular matrix are *linearly independent*. It has **no zeros on the diagonal**.

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Look for a combination of the columns that makes zero:
Solve $Ac = 0$:

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The only combination to produce the zero vector is the trivial combination ($c_1 = c_2 = c_3 = 0$).
The nullspace of A contains only the zero vector.

The columns of A are independent exactly when $N(A) = \{0\}$.

III. Basis

Definition 2 (*Basis*) Let V be a subspace of $\mathbf{R}^n$, and let A be a set of vectors of V. Then A is called a **basis (基)** for V if

• A is a spanning set for V, i.e., $V = \mathrm{span}(A)$, and *(not too few vectors)*

• A is linearly independent. *(not too many vectors)*

A basis is: 最小的生成集和最大的线性无关组

a **minimal** spanning set *(It cannot be made smaller and still span the space),* and a **maximal** linearly independent set *(It cannot be made larger without losing independence).*

For example,

• $\{(1,0)^{\mathrm{T}}, (0,1)^{\mathrm{T}}\}$ is a basis for $\mathbf{R}^2$, and so is $\{(1,0)^{\mathrm{T}}, (1,1)^{\mathrm{T}}\}$.

• $\{(1, 0, 0)^{\mathrm{T}}, (0, 1, 0)^{\mathrm{T}}, (0, 0, 1)^{\mathrm{T}}\}$ is a basis for $\mathbf{R}^3$, and so is $\{(1, 0, 0)^{\mathrm{T}}, (0, 2, 0)^{\mathrm{T}}, (1, 1, 1)^{\mathrm{T}}\}$.

• In $\mathbf{R}^n$, the set $\{e_1, e_2, ..., e_n\}$ is a basis for $\mathbf{R}^n$, called the **standard basis** (标准基, 或自然基) of $\mathbf{R}^n$.

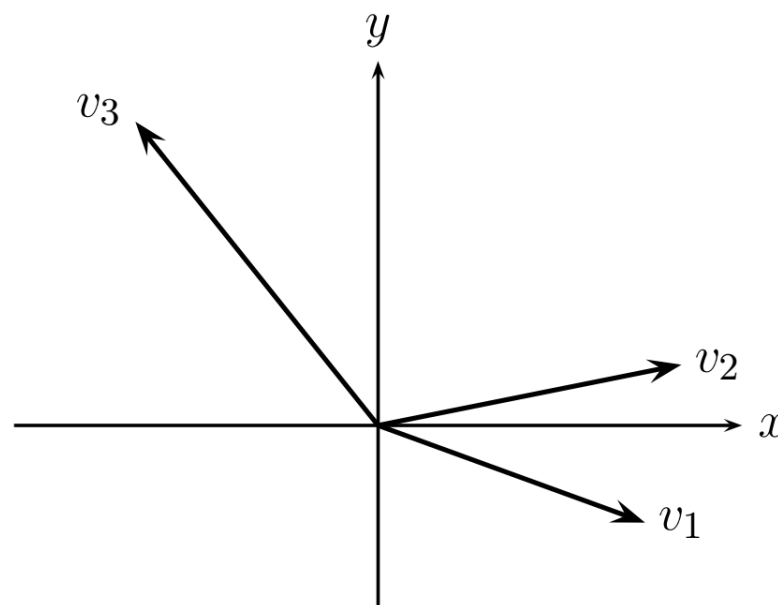The *x-y* plane in the figure is just $\mathbf{R}^2$. The vector $\boldsymbol{v}_1$ by itself is linearly independent, but it fails to span $\mathbf{R}^2$. The three vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ certainly span $\mathbf{R}^2$, but are not independent. *Any two* of these vectors, say $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, have both properties—they span, and they are independent. So they form a basis.

Notice again that *a vector space does not have a unique basis*. (向量空间的基不唯一)

A spanning set: $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.
Bases: $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.

A Method for deciding if *A* is a basis:

Show that

(1) each vector in *A* lies in *V*,

(2) *A* is linearly independent, i.e., none of the vectors in *A* is a linear combination of the others,

(3) *A* is a spanning set for *V*, i.e., every vector of *V* is a linear combination of *A*.

Example 7 Let $V = \{(x, y, z)^{\mathrm{T}} | x + y + z = 0\}$. Show that V is a subspace of $\mathbf{R}^3$, and find a basis for V.

(1) Show that V is a subspace of $\mathbf{R}^3$.

1) $(0,0,0)^{\mathrm{T}} \in V$ since $0 + 0 + 0 = 0$.

2) $\forall \boldsymbol{u} = (x_1, y_1, z_1)^{\mathrm{T}}, \boldsymbol{v} = (x_2, y_2, z_2)^{\mathrm{T}} \in V$, then $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$, and therefore
$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)^{\mathrm{T}} \in V$$

since
$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2)$$
$$= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$$

3) $\forall k \in \mathbf{R}, \forall \boldsymbol{u} = (x, y, z)^{\mathrm{T}} \in V$,
$$k\boldsymbol{u} = (kx, ky, kz)^{\mathrm{T}} \in V$$

since $kx + ky + kz = k(x + y + z) = 0$.

So V is a subspace of $\mathbf{R}^3$.

Example 7 Let $V = \{(x, y, z)^{\mathrm{T}} | x + y + z = 0\}$. Show that V is a subspace of $\mathbf{R}^3$, and find a basis for V.

(2) Find a basis for V.

$\forall \boldsymbol{u} = (x, y, z)^{\mathrm{T}} \in V$, then $x + y + z = 0$ and
$$\boldsymbol{u} = (-y - z, y, z)^{\mathrm{T}} = y(-1,1,0)^{\mathrm{T}} + z(-1,0,1)^{\mathrm{T}}.$$

We claim that $\{(-1,1,0)^{\mathrm{T}}, (-1,0,1)^{\mathrm{T}}\}$ is a basis for V since

1) Clearly $(-1,1,0)^{\mathrm{T}}, (-1,0,1)^{\mathrm{T}} \in V$.

2) $(-1,1,0)^{\mathrm{T}}, (-1,0,1)^{\mathrm{T}}$ are linearly independent since
$$c_1(-1,1,0)^{\mathrm{T}} + c_2(-1,0,1)^{\mathrm{T}} = (0,0,0)^{\mathrm{T}}$$
$$\Longrightarrow (-c_1 - c_2, c_1, c_2)^{\mathrm{T}} = (0,0,0)^{\mathrm{T}} \Longrightarrow c_1 = c_2 = 0.$$

3) $\forall \boldsymbol{u} = (x, y, z)^{\mathrm{T}} \in V$,
$$\boldsymbol{u} = (-y - z, y, z)^{\mathrm{T}} = y(-1,1,0)^{\mathrm{T}} + z(-1,0,1)^{\mathrm{T}}$$

Every vector of V is a linear combination of $(-1,1,0)^{\mathrm{T}}$ and $(-1,0,1)^{\mathrm{T}}$.

IV. Coordinates

An important property of a basis is that it provides a unique representation for each vector.

Theorem 2 *Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a basis for a space V. Then for each vector $\boldsymbol{w} \in V$, there is a **unique** choice of scalars $a_1, a_2, \ldots, a_k$ such that*

$$\boldsymbol{w} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k.$$

We call the scalars $a_1, a_2, \ldots, a_k$ the **coordinates (坐标)** of $\boldsymbol{w}$ in the basis B, denoted by $[\boldsymbol{w}]_B$.

Why -- unique?

Hint:
$$\boldsymbol{w} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k$$
$$\boldsymbol{w} = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + \cdots + b_k\boldsymbol{v}_k$$

$$\boldsymbol{0} = (a_1 - b_1)\boldsymbol{v}_1 + (a_2 - b_2)\boldsymbol{v}_2 + \cdots + (a_k - b_k)\boldsymbol{v}_k$$

Example 8

- $A = \{(1, 0)^T, (0, 1)^T\}$ is a basis for $\mathbf{R}^2$, and so is $B = \{(2, 1)^T, (-1, 1)^T\}$.
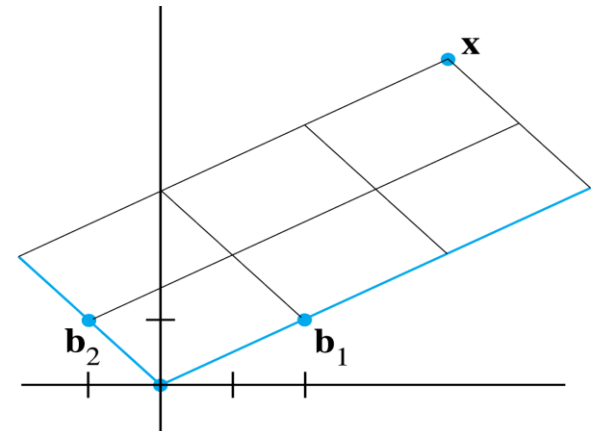
The vector $\boldsymbol{x} = (4, 5)^T$ has coordinates

$$[\boldsymbol{x}]_A = (4, 5)^T,$$
$$[\boldsymbol{x}]_B = (3, 2)^T.$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ i.e., } \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This equation can be solved by row operations on an augmented matrix or by using the inverse of the coefficient matrix on the left.

The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)^T$.

https://www.geogebra.org/m/nxjzra8u

Example 9 The standard basis $B_1 = \{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbf{R}^n$, and so is $B_2 = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$,

where $\boldsymbol{\beta}_1 = (1, -1, 0, \cdots, 0)^{\mathrm{T}}$, $\boldsymbol{\beta}_2 = (0, 1, -1, 0, \cdots, 0)^{\mathrm{T}}$, $\ldots$,

$\boldsymbol{\beta}_{n\text{-}1} = (0, \cdots, 0, 1, -1)^{\mathrm{T}}$, $\boldsymbol{\beta}_n = (0, \cdots, 0, 1)^{\mathrm{T}}$.

Determine the coordinates of the vector $\boldsymbol{\alpha} = (a_1, a_2, \cdots, a_n)^{\mathrm{T}}$ in B_1 and B_2.

Solution It is easy to see that

$$\boldsymbol{\alpha} = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n,$$

so

$$[\boldsymbol{\alpha}]_{B_1} = (a_1, a_2, \ldots, a_n)^{\mathrm{T}}.$$

For the basis $B_2 = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$, we suppose that

$$\boldsymbol{\alpha} = x_1 \boldsymbol{\beta}_1 + x_2 \boldsymbol{\beta}_2 + \cdots + x_n \boldsymbol{\beta}_n$$

$$\beta_1 = (1,-1,0,\cdots,0)^{\mathrm{T}}, \quad \beta_2 = (0,1,-1,0,\cdots,0)^{\mathrm{T}}, \ldots,$$
$$\beta_{n-1} = (0,\cdots,0,1,-1)^{\mathrm{T}}, \quad \beta_n = (0,\cdots,0,1)^{\mathrm{T}},$$

$$\alpha = x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n = (\beta_1, \beta_2, \ldots, \beta_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

Place the vectors α, β_1, β_2, $\cdots$, β_n into column forms of the system, then

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}.$$

By solving the system, we can get

$$[\alpha]_{B_2} = \left(x_1, x_2, \ldots, x_n \right)^{\mathrm{T}}$$

$$= \begin{pmatrix} a_1 \\ a_1 + a_2 \\ \vdots \\ a_1 + a_2 + \ldots + a_{n-1} + a_n \end{pmatrix}.$$

V. Dimension

The most fundamental parameter for a vector space would be its dimension.

Theorem 3 *Every basis for a subspace V of $\mathbf{R}^n$ contains the* *same number of vectors.*

(*Proof: see next slide or G. Strang: LA and its applications, p97*)

Definition 3 (*Dimension*) The number of vectors in a basis for a vector space *V* is called the **dimension (维数)** of *V* .

For example,

• $B = \{(1, 1)^{\mathrm{T}}, (1, 2)^{\mathrm{T}}\}$ is a basis for $\mathbf{R}^2$, and so is $C = \{(1, 0)^{\mathrm{T}}, (1, 1)^{\mathrm{T}}\}$. $\mathbf{R}^2$ is of dimension 2.

• $B = \{(2, 0, 0)^{\mathrm{T}}, (1, 1, 0)^{\mathrm{T}}, (1, 1, 1)^{\mathrm{T}}\}$ is a basis for $\mathbf{R}^3$, and so is $C = \{(1, 0, 0)^{\mathrm{T}}, (0, 2, 0)^{\mathrm{T}}, (1, 1, 1)^{\mathrm{T}}\}$. $\mathbf{R}^3$ is of dimension 3.

Proof of Theorem 3 (If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both bases for the same vector space, then $m = n$.)

Proof. Suppose there are more w's than v's ($n > m$).

Since the v's form a basis, they must span the space.

Every w_j can be written as a combination of the v's:

$$w_1 = a_{11}v_1 + \cdots + a_{m1}v_m, \ \ldots, \ w_n = a_{1n}v_1 + \cdots + a_{mn}v_m.$$

So we have

$$[w_1 \quad w_2 \quad \cdots \quad w_n] = [v_1 \quad v_2 \quad \cdots \quad v_m] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

which can be rewritten as

$$W = VA. \quad (A \text{ is } m \text{ by } n)$$

There is a nonzero solution to $Ax = 0$ (since $n > m$),

then $VAx = 0$,

which is $Wx = 0$. (*A combination of the w's gives zero!*)

The w's could not be a basis. So we cannot have $n > m$.

Similarly, we cannot have $m > n$.

The only way to avoid a contradiction is to have $m = n$.
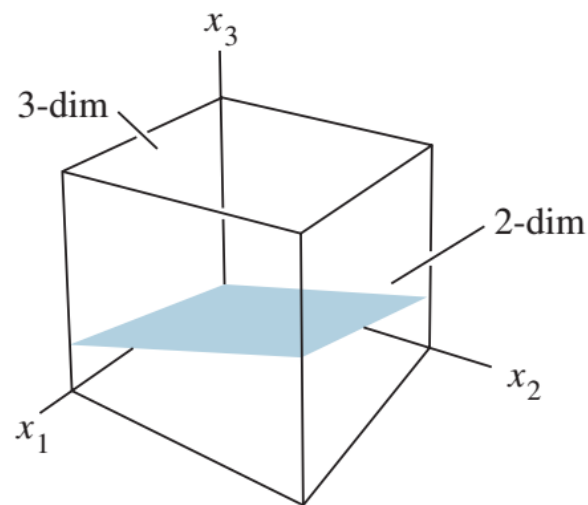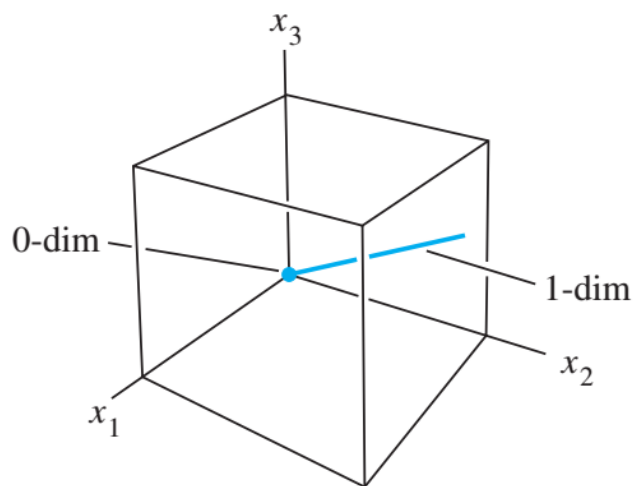
$$r \leq m < n$$

$$n - r > 0$$

Corollary (推论): *Let V be a space of dimension k. Then*
(a) *any set of more than k vectors is linearly dependent,*
(b) *any set of less than k vectors is not a spanning set,*
(c) *a proper subspace of* $\mathbf{R}^n$ *has dimension less than n.*

For example,

• Any set of 4 vectors of $\mathbf{R}^3$ is linearly dependent.
• Any set of 2 vectors of $\mathbf{R}^3$ is not a spanning set.
• A proper subspace of $\mathbf{R}^3$ has dimension at most 2.

The results in the following theorem are important and very useful.

Theorem 4 *Let V be a space. Then*

(1) *any linearly independent set of V can be extended to be a basis;*

(2) *any spanning set for V contains a basis.*

Example 10 (*Basis from a spanning set*) Let

$$A = \{(1, 1, -2)^T, (-2, -2, 4)^T, (-1, -2, 3)^T, (1, -1, 0)^T\},$$

and let $V = \text{span}(A)$. Find a basis contained in A.

Solution
$$\begin{bmatrix} 1 & -2 & -1 & 1 \\ 1 & -2 & -2 & -1 \\ -2 & 4 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$b_1 \quad b_2 \quad b_3 \quad b_4$ $\qquad\qquad$ $c_1 \quad c_2 \quad c_3 \quad c_4$

$b_2 = -2b_1 \qquad b_4 = 3b_1 + 2b_3$ $\qquad$ $c_2 = -2c_1 \qquad c_4 = 3c_1 + 2c_3$

初等行变换不改变列向量之间的线性相关性！Why?

Basis: $\{b_1, b_3\}$, or $\{b_2, b_3\}$, or $\{b_1, b_4\}$, or $\{b_2, b_4\}$, or $\{b_3, b_4\}$.

Example 11 Find the dimension and a basis for the vector space $\mathbf{R}^{2 \times 2}$.

Solution. Let

$$\boldsymbol{K}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{K}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\boldsymbol{K}_{11}, \boldsymbol{K}_{12}, \boldsymbol{K}_{21}, \boldsymbol{K}_{22}$ are linearly independent.

In fact, by $a\boldsymbol{K}_{11} + b\boldsymbol{K}_{12} + c\boldsymbol{K}_{21} + d\boldsymbol{K}_{22} = \boldsymbol{0}$, i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{0}$, we have $a = b = c = d = 0$.

And $\forall \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbf{R}^{2 \times 2}$, it can be expressed as

$$\boldsymbol{A} = a_{11}\boldsymbol{K}_{11} + a_{12}\boldsymbol{K}_{12} + a_{21}\boldsymbol{K}_{21} + a_{22}\boldsymbol{K}_{22},$$

so $\boldsymbol{K}_{11}, \boldsymbol{K}_{12}, \boldsymbol{K}_{21}, \boldsymbol{K}_{22}$ is a basis for $\mathbf{R}^{2 \times 2}$, and the dimension of space $\mathbf{R}^{2 \times 2}$ is 4.

Remark. The coordinates of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

in this coordinate system are $(a_{11}, a_{12}, a_{21}, a_{22})$.

In general, the vector space $\mathbf{R}^{m \times n}$ is of dimension $m \times n$,

$$K_{ij} = \begin{bmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & \ddots \\ 0 & & & & & 0 \end{bmatrix} \begin{matrix} \\ \\ \text{Row } i \\ \\ \\ \\ \\ \end{matrix}$$

$$\begin{matrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{matrix}$$

is a basis for $\mathbf{R}^{m \times n}$.

column j

$$\forall A = [a_{ij}] \in \mathbf{R}^{m \times n}, \quad A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} K_{ij}.$$

Key words:

Linearly dependent, linearly independent, spanning set, basis, dimension, coordinate

Homework

See Blackboard

Note: 20(c): If $u^{\mathrm{T}}v = 0$, then u, v are called **perpendicular** (垂直).