Differential Geometry MA327 2020 Spring Final Test

Open Book! Hand in before 23:59 26th August by Email: huangsc@sustech.edu.cn

Total=100 points

1.*(5 points)* Show that every regular parametrized differentiable curve can be re-parametrized by arc length.

2.*(5 points)* Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curve and let $[a, b] \subset I$. Show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|dt.$$

3.*(10 points)* Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized (by arc length s) curve with nowhere vanishing curvature. Show that α is a plane curve (i.e. $\alpha(I)$ is contained in a plane) if and only if its torsion τ is identically equal to 0.

4.*(10 points)* (a) Write down the definition of a regular surface. (b) Write down an example of some "two dimensional" object which is not a regular surface and explain why your example is not a regular surface without proof. (c) Write down an example of a regular surface and prove your example is a regular surface (you may use some facts in the textbook or lecture notes without proof).

5.*(5 points)* Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

6.*(5 points)* Compute the first fundamental forms of the following parametrized surfaces where they are regular:

(a) $\mathbf{x}(u, v) = (u, v)$; plane.
(b) $\mathbf{x}(\rho, \theta) = (\rho \cos\theta, \rho \sin\theta)$; plane.
(c) $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid.
(d) $\mathbf{x}(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u)$; torus.
(e) $\mathbf{x}(u, v) = (u, v, v^2 - u^2)$; hyperbolic paraboloid.

7. *(5 points)* Compute the area of the whole torus. Here the torus means the surface of revolution generated by rotating the circle $(x - a)^2 + z^2 = r^2, y = 0$, about the z axis $(a > r > 0)$. (You may use the parametrization in Question 6(d).)

8. *(10 points)* Compute the Gaussian curvature of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

9. *(10 points)* Show that a surface which is compact (i.e. it is bounded and closed in $\mathbb{R}^3$) has an elliptic point.

10.*(15 points)* (a) What are the normal curvature and geodesic curvature for a regular oriented curve on an oriented surface? (Write down their definitions.)

(b) Derive the following relation for a regular oriented curve C on an oriented surface S:

$$k^2 = k_n^2 + k_g^2.$$

Here k_n is the normal curvature of C, k_g is the geodesic curvature of C and k is the curvature of C.

(c) Prove that a curve $C \subset S$ is both an asymptotic curve and a geodesic if and only if C is a (segment of a) straight line.

11. *(5 points)* Describe all geodesics on the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

12. *(15 points)* (a) Write down an example of a regular surface with negative Gaussian curvature everywhere. (You do not need to compute the Gaussian curvature for your example, just write down the example.)

(b) Write down one version of global Gauss-Bonnet theorem without proof.

(c) Use the global Gauss-Bonnet theorem you write down to show that there are points on the torus where the Gaussian curvature is positive, negative, and zero. Here the torus as in Question 7 means the surface of revolution generated by rotating the circle $(x - a)^2 + z^2 = r^2, y = 0$, about the z axis ($a > r > 0$).