Step-1

Let A be a matrix.

We know that

Def: 1: The ***conditional number*** of A is defined to be $c = \|A\|\|A^{-1}\|$

Def: 2: The ***norm*** of A is the number $\|A\|$ defined by $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

Step-2

a) We have to verify that A and A^{-1} have same condition number or not.

We know that the norm is a non negative number given by $\|kV\| = |k|\|V\|$, where k is a scalar and V is a vector (matrix is also a vector).

So, the product of *norms* is commutative.

Consequently, we follow that $c = \|A^{-1}\|\|A\|$

Further, by the algebraic properties of matrices, we have $\left(A^{-1}\right)^{-1} = A$

Step-3

So, replacing A with $\left(A^{-1}\right)^{-1}$ in the above equation, we get

$$\begin{aligned} c &= \|A^{-1}\|\left\|\left(A^{-1}\right)^{-1}\right\| \\ &= \|A^{-1}\|\|A\| \qquad \left(\text{Since} \left(A^{-1}\right)^{-1} = A\right) \end{aligned}$$

Comparing this with the definition of the conditional number, we get c is the conditional number of A^{-1}.

Therefore, it follows that A and A^{-1} have the same conditional numbers.

Step-4

b) Suppose the non homogeneous system is $Ax = b$ and $\delta x = A^{-1}\delta b$

We know that $\|b\| \leq \|A\|\|x\|$ and $\|\delta x\| \leq \|A^{-1}\|\|\delta b\|$

Conveniently, we can write $\frac{\|b\|}{\|x\|} \leq \|A\|$ and $\frac{\|\delta x\|}{\|\delta b\|} \leq \|A^{-1}\|$ â€¦â€¦ (1)

$$\Rightarrow \frac{\|b\|}{\|x\|}\frac{\|\delta x\|}{\|\delta b\|} \le \|A\|\|A^{-1}\|$$

$$\Rightarrow \frac{\|b\|}{\|x\|}\frac{\|\delta x\|}{\|\delta b\|} \le c \qquad \left(\text{Since } \|A\|\|A^{-1}\| = c\right)$$

Precisely, $\frac{\|\delta x\|}{\|x\|} \le c\frac{\|\delta b\|}{\|b\|}$

Step-5

The equation (1) can otherwise be written as $\frac{\|x\|}{\|b\|} \le \frac{1}{\|A\|}$ and $\frac{\|\delta b\|}{\|\delta x\|} \le \frac{1}{\|A^{-1}\|}$

Multiplying the respective sides, we get

$$\frac{\|x\|}{\|b\|}\frac{\|\delta b\|}{\|\delta x\|} \le \frac{1}{\|A\|}\frac{1}{\|A^{-1}\|}$$

$$\Rightarrow \frac{\|x\|}{\|b\|}\frac{\|\delta b\|}{\|\delta x\|} \le \frac{1}{c} \qquad \left(\text{Since } \|A\|\|A^{-1}\| = c\right)$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \le \frac{1}{c}\frac{\|b\|}{\|\delta b\|}$$

Hence $\boxed{\frac{\|\delta x\|}{\|x\|} \le \frac{1}{c}\frac{\|b\|}{\|\delta b\|}}$