Step-1

This tells us that k must be equal to 1. There is a condition $z(0)=-2$.

Thus,

$$\begin{aligned} z &= e^{kt}+C_1 \\ -2 &= e^{k(0)}+C_1 \\ &= 1+C_1 \\ C_1 &= -3 \end{aligned}$$

Step-2

Now consider the differential equation $\frac{dy}{dt}=4y+3z$, where $y(0)=-5$.

Let $y=e^{mt}+e^{kt}+A$ be the solution of the above differential equation.

Then,

$$\begin{aligned} 4\left(e^{mt}+e^{kt}+A\right)+3\left(e^{kt}+C_1\right) &= \frac{d}{dt}\left(e^{mt}+e^{kt}+A\right) \\ 4e^{mt}+7e^{kt}+4A+3C_1 &= me^{mt}+ke^{kt} \end{aligned}$$

Therefore, $m=4$.

We have $y(0)=-5$. Also, we have $C_1=-3$. This gives $A=1$.

Step-3

Thus, the solution of the differential equation is $\boxed{y=e^{4t}+e^{t}}$.