Step-1

The objective is to find the smallest subspace of 3×3 matrices that contains all symmetric matrices and all lower triangular matrices and need to find the largest subspace that is contained in both those subspaces.

Step-2

A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space: Linear combinations stay in the subspace.

(i) If add any vectors x and y in the subspace, then $x+y$ is in that subspace.

(ii) If multiply any vector x in the subspace be any scalar c, then cx is in that subspace.

Step-3

Assume that $$S=\left\{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} / a,b,c\in \text{ú}\right\}$$.

Then,

$$A=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix}, B=\begin{bmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{bmatrix}$$

Suppose that $A,B\in S$ then,

$$\begin{aligned} A+B &= \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix}+\begin{bmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & b_1+b_2 & 0 \\ 0 & 0 & c_1+c_2 \end{bmatrix} \\ &\in S \end{aligned}$$

Suppose that $C\in \text{ú}, A\in S$ then,

$$CA = C\begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix}$$
$$= \begin{bmatrix} Ca_1 & 0 & 0 \\ 0 & Cb_1 & 0 \\ 0 & 0 & Cc_1 \end{bmatrix}$$
$$\in S$$

Therefore, S is a subspace of all 3×3 matrices and S contains all symmetric and lower triangular matrices.

Step-4

Assume that $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is a symmetric and lower triangular matrix.

Since A is lower triangular matrix. So, the values of a_2, a_3 and b_3 should be zero.

i.e. $a_2 = a_3 = b_3 = 0$

i.e. $A = \begin{bmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Since A is symmetric matrix. So, the values of b_1, c_1 and c_2 should be zero. i.e. $b_1 = 0, c_1 = 0$ and $c_2 = 0$

i.e. $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$

Therefore, the matrix $A \in S$.

Therefore, the matrix $\boxed{S}$ is the smallest sub space of 3×3 matrices that contains all symmetric matrices, all lower triangular matrices and diagonal matrices.

Step-5

â€œ S â€ is also the largest subspace that is contained in both the subspace of symmetric matrices, the subspace of lower triangular matrices where $\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$ is the smallest subspace contained in both subspaces.