Chapter 5. Poisson processes

5.1. Exponential distributions

Definition

A r.v. T has *exponential distribution with rate* λ,denote $T \sim \text{expo.}(\lambda)$ if

$$\mathbb{P}(T \leq t) = 1 - e^{-\lambda t}, \qquad \forall\, t \geq 0.$$

Note that the density function is

$$f(t) = \lambda e^{-\lambda t}, \qquad t \geq 0,$$

and the tail is

$$\mathbb{P}(T > t) = e^{-\lambda t}, \qquad t \geq 0.$$

Proposition

- $\mathbb{E}T = \frac{1}{\lambda},\ Var(T) = \frac{1}{\lambda^2}.$
- *If* $T \sim expo.(\lambda)$, *then* $\lambda T \sim expo.(1)$.

Proposition (Lack of memory)

$$\mathbb{P}(T > t + s | T > t) = \mathbb{P}(T > s).$$

Proof: Do it on board!

Example

Consider a post office that is run by two clerks.Suppose that when Mr. Smith enters the system he discovers that Mr. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Jones or Brown leaves.If the amount of time that a clerk spends with a customer is exponentially distributed with mean $\frac{1}{\lambda}$, what is the probability that, of the three customers, Mr. Smith is the last to leave the post office?

Solution: Do it on board!

Proposition (Exponential races)

Let $T_1, \cdots, T_n$ *be indep.* $T_i \sim expo.(\lambda_i)$, $i = 1, \cdots, n$*. Then, i)*

$$V \equiv \min(T_1, \cdots, T_n) \sim expo.(\lambda_1 + \cdots + \lambda_n).$$

ii) Let I *be the index* i *s.t.* $T_i = V$*. Then,* I *and* V *indep.*
iii)

$$\mathbb{P}\left(T_i = V\right) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}.$$

Proof: Do it on board!

Example

Suppose you arrive at a post office having two clerks at a moment when both are busy but there is no one else waiting in line. You will enter service when either clerk becomes free. If service times for clerk i are exponential with rate λ_i, $i = 1, 2$, find $E[T]$, where T is the amount of time that you spend in the post office.

Solution: Do it on board!

Example

There are n cells in the body, of which cells $1, \ldots, k$ are target cells. Associated with each cell is a weight, with w_i being the weight associated with cell i, $i = 1, \ldots, n$. The cells are destroyed one at a time in a random order, which is such that if S is the current set of surviving cells then, independent of the order in which the cells not in S have been destroyed, the next cell killed is i, $i \in S$, with probability $\frac{w_i}{\sum_{j \in S} w_j}$. Let A denote the total number of cells that are still alive at the moment when all the cells $1, 2, \ldots, k$ have been killed, and find $E[A]$.

Solution: Do it on board!

Example

Suppose that customers are in line to receive service that is provided sequentially by a server; whenever a service is completed, the next person in line enters the service facility.However, each waiting customer will only wait an exponentially distributed time with rate θ; if its service has not yet begun by this time then it will immediately depart the system.These exponential times, one for each waiting customer, are independent.In addition, the service times are independent exponential random variables with rate μ.Suppose that someone is presently being served and consider the person who is nth in line.

(a) Find P_n, the probability that this customer is eventually served.

(b) Find W_n, the conditional expected amount of time this person spends waiting in line given that she is eventually served.

Solution: Do it on board!

Theorem

Let $\tau_1, \ \tau_2, \ \cdots$ be i.i.d. expo.(λ). Then, $T_n \equiv \tau_1 + \cdots + \tau_n$ has a gamma(n, λ) distribution, namely, it has the pdf

$$f_{T_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geq 0. \tag{1.1}$$

Proof: Do it on board!

5.2 Defining the Poisson process

Definition

A stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that occur by time t.

Some examples:

1. If we let $N(t)$ equal the number of persons who enter a store at or prior to time t, then $\{N(t), t \geq 0\}$ is a counting process.
   Note that if we had let $N(t)$ equal the number of persons in the store at time t, then $\{N(t), t \geq 0\}$ would not be a counting process.
2. If we say that an event occurs whenever a child is born, then $\{N(t), t \geq 0\}$ is a counting process when $N(t)$ equals the total number of people who were born by time t.
3. If $N(t)$ equals the number of goals that a given soccer player scores by time t, then $\{N(t), t \geq 0\}$ is a counting process.

Some properties for counting processes:

1. $N(t) \geq 0$.
2. $N(t)$ is integer valued.
3. If $s < t$, then $N(s) \leq N(t)$.
4. If $s < t$, $N(t) - N(s)$ is the number of events occur in $(s, t]$.

Definition

A counting process is said to possess *independent increments* if the numbers of events that occur in disjoint time intervals are independent.

In other wors, $N(t) - N(s)$ and $N(s)$ are independent.
The independent increments assumption for (a) and (c), not reasonable for (b).

Definition

A counting process is said to possess *stationary increments* if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval.

In other words, the distribution of $N(t) - N(s)$ depends on $t - s$ only.

Definition

The counting process $\{N(t), t \geq 0\}$ is said to be a *Poisson process* with rate $\lambda > 0$ if the following axioms hold:

i) $N(0) = 0$.

ii) $\{N(t), t \geq 0\}$ has independent increments.

iii) As $h \to 0$,

$$P(N(t+h) - N(t) = 1) = \lambda h + o(h).$$

iv)

$$P(N(t+h) - N(t) \geq 2) = o(h).$$

Theorem

If $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, then $N(s+t) - N(s) \sim P(\lambda t)$.

Proof: Do it on board!

Let S_i, $i = 1, 2, \cdots$ be the arrival time of the ith event with $S_0 = 0$.Let $T_i = S_i - S_{i-1}$ be the interarrival times, $i = 1, 2, \cdots$.Then,

$$S_n = T_1 + T_2 + \cdots + T_n.$$

Proposition

T_1, T_2, $\cdots$ are i.i.d. with expo(λ) distribution.

Proof: Do it on board!

We can also consider the case that the rate λ is time-dependent.

Definition

$\{N(s),\ s \geq 0\}$ is a (nonhomogeneous) Poisson process with rate $\lambda(s)$ if

i) $N(0) = 0$.

ii) $N(t) - N(s) \sim Poisson(\int_s^t \lambda(u)du)$.

iii) $N(t)$ has indep. increments.

Remark

The inter-arrival times are no-longer exponential or indep.

Proof: Do it on board!

5.3. Compound Poisson processes

Example

Suppose that the cars arrive at a McDonald's as a Poisson process $N(t)$ with rate λ. The # of people in cars are i.i.d. r.v's $Y_1, Y_2, \cdots$. The, the # of customers by time t is

$$X(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}.$$

Such a process is a compound Poisson process.

Recall:

Theorem

Let $Y_1,\ Y_2,\ \cdots$ be i.i.d. r.v.'s, let N be a $\mathbb{Z}_+$-valued r.v. and $X = Y_1 + \cdots + Y_N$. Then,

i) $\mathbb{E}X = \mathbb{E}N\mathbb{E}Y_1$.

ii)

$$Var(X) = \mathbb{E}NVar(Y_1) + Var(N)(\mathbb{E}Y_1)^2.$$

If $N \sim P(\lambda)$, then $Var(S) = \lambda\mathbb{E}(Y_1^2)$.

Corollary

For a compound Poisson process

$$\mathbb{E}X(t) = \lambda t\mathbb{E}Y_1,$$

$$Var(X(t)) = \lambda t\mathbb{E}(Y_1^2).$$

Example

Consider a single-server service station in which customers arrive according to a Poisson process having rate λ. An arriving customer is immediately served if the server is free; if not, the customer waits in line (that is, he or she joins the queue). The successive service times are independent with a common distribution.
Such a system will alternate between idle periods and busy periods.Let B denote the length of a busy period. Find its mean and variance.

Solution: Do it on board!

Example

Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ *per week.If the number of people in each family is independent and takes on the values 1, 2, 3, 4 with respective probabilities* $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, *what is the expected value and variance of the number of individuals migrating to this area during a fixed 50-week period? Find the approximate probability that at least 240 people migrate to the area within the next 50 weeks.*

Solution: Do it on board!

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5.4. Transformations

a) Thinning

Let $Y_1,\ Y_2,\ \cdots$ be i.i.d.

Let $N_j(t)$ be the times before $N(t)$ that $Y_i = j$,i.e.

$$N_j(t) = \sum_{i=1}^{N(t)} 1_{Y_i=j}.$$

Then, $N_j(t)$ should be a PP.

Theorem

$N_j(t)$ is a Poisson process with rate $\lambda\mathbb{P}(Y_1 = j)$.

Proof: Do it on board!

Remark

The PPs $N_1(t), N_2(t), \cdots$ *are independent.*

The last theorem can be generalized.

Theorem

Suppose that in a PP with rate λ*, we keep the point that lands at* s *(one of* T_i*) with prob.* $p(s)$*. Then, the result is a non-homogeneous PP with rate* $\lambda p(s)$*.*

Example (An Infinite ServerQueue)

Suppose that customers arrive at a service station in accordance with a Poisson process with rate λ. Upon arrival the customer is immediately served by one of an infinite number of possible servers, and the service times are assumed to be independent with a common distribution G. What is the distribution of $X(t)$, the number of customers that have completed service by time t? What is the distribution of $Y(t)$, the number of customers that are being served at time t?

Solution: Do it on board!

Example

Suppose offers to buy an item that you want to sell arrive according to a Poisson process with rate λ.Assume that each offer is the value of a r.v. with pdf $f(x)$. Once the offer is presented to you, you must either accept it or reject it and wait for the next offer.
Suppose that you incur costs at a rate c per unit time until the item is sold, and that your objective is to maximize your expected total return.
Suppose you employ the policy of accepting the first offer that is greater than some specified value y.
What is the best value of y?What is the maximal expected net return?

Solution: Do it on board!

b) Superposition.

Opposite to thinning(keep a Poisson point with certain probability, or split into a few indep. PP)is the so-called superposition.

Theorem

Suppose that $N_1(t), \cdots, N_k(t)$ *are indep. PP with rates* $\lambda_1, \cdots, \lambda_k$, *then* $N(t) \equiv N_1(t) + \cdots + N_k(t)$ *is a PP with rate* $\lambda \equiv \lambda_1 + \cdots + \lambda_k$.

Proof: Do it on board!

Example (A Poisson race)

Given a PP of red arrivals with rate λ, and an indep. PP of green arrivals with rate μ, what is the probability that we will get 6 red arrivals before a total of 4 green ones?

Solution: Do it on board!

c) Conditioning.
Let $S_1, S_2, \cdots$ be arrival times of PP with rate λ. Let $U_1, U_2, \cdots$ i.i.d. unif$(0, t)$ and $V_1 < V_2 < \cdots < V_n$ be a rearrangement of $U_1, \cdots, U_n$.

Theorem

Conditioning on $N(t) = n$, the law of $(S_1, \cdots, S_n)$ coincides with the law of $(V_1, \cdots, V_n)$, namely,

$$\mathcal{L}((S_1, \cdots, S_n)|N(t) = n) = \mathcal{L}(V_1, \cdots, V_n).$$

Proof: Do it on board!

Theorem

If $s < t$ *and* $0 \leq m \leq n$, *then*

$$\mathbb{P}(N(s) = m | N(t) = n) = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}. \qquad (1.2)$$

Proof: Do it on board!

Example

Customers arrive at a bank at a Poisson rate λ.Suppose two customers arrived during the first hour. What is the probability that
(a) both arrived during the first 20 minutes?
(b) at least one arrived during the first 20 minutes?

Solution: Do it on board!
HW: Ch5, 1, 2, 9, 28, 42, 43, 50, 52, 53, 59, 62, 70.