

Step-1

$$\text{Given } a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$

We have to solve the least squares problem $Ax = b$ by using $A = QR$

Step-2

$$\begin{aligned} q_1 &= \frac{a_1}{\|a_1\|} \\ &= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \end{aligned}$$

Step-3

$$\begin{aligned} q_2 &= \frac{\beta}{\|\beta\|} \text{ where} \\ \beta &= a_2 - (q_1^T a_2) q_1 \end{aligned}$$

Step-4

$$\begin{aligned} q_1^T a_2 &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ &= \frac{1+6+2}{3} \\ &= 3 \end{aligned}$$

Step-5

$$\begin{aligned}(q_1^T a_2) q_1 &= 3 \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\end{aligned}$$

Step-6

$$\begin{aligned}\beta &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\|\beta\| &= \sqrt{0+1+1} \\ &= \sqrt{2}\end{aligned}$$

Step-7

Therefore

$$\begin{aligned}q_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\end{aligned}$$

Step-8

$$\begin{aligned}q_1^T a_1 &= \begin{bmatrix} 1 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \frac{1+4+4}{3} \\ &= 3\end{aligned}$$

Step-9

$$\begin{aligned}
 q_1^T a_2 &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\
 &= \frac{1+6+2}{3} \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 q_2^T a_2 &= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\
 &= \frac{0+3-1}{\sqrt{2}} \\
 &= \frac{2}{\sqrt{2}} \\
 &= \sqrt{2}
 \end{aligned}$$

Step-10

$$\begin{aligned}
 A &= [a_1 \quad a_2] \\
 &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \\
 &= [q_1 \quad q_2] \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix} \\
 &= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \\
 &= QR
 \end{aligned}$$

$$A = QR = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

Therefore

Step-11

By the method of least squares,

$$\hat{R}x = Q^T b \text{ where } b = [1 \quad 1 \quad 1]^T$$

And $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$

Step-12

Now

$$R \hat{x} = Q^T b$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3C + 3D \\ \sqrt{2}D \end{bmatrix} = \begin{bmatrix} 5/3 \\ 0 \end{bmatrix}$$

Step-13

$$\Rightarrow 3C + 3D = \frac{5}{3} \text{ and } \sqrt{2}D = 0$$

$$\Rightarrow D = 0, 3C = \frac{5}{3}$$

Therefore $C = \frac{5}{9}, D = 0$

Hence the solution of the system $Ax = b$ is $\hat{x} = \begin{bmatrix} 5/9 \\ 0 \end{bmatrix}$