

Step-1

(a)

Find a 2 by 3 system $Ax = b$ whose complete solution is;

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Suppose A be a 2×3 matrix such that the complete solution of $Ax = b$ is;

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Step-2

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

If $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ then $Ax = b$ reduces to $Rx = C$ where R is row reduced echelon form.

$$u = 1 + w$$

$$v = 2 + 3w$$

$$u - w = 1$$

$$v - 3w = 2$$

Step-3

And w is an independent vowel.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Then the system whose solution is;

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Step-4

(b)

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Find a 3 by 3 system with the solutions exactly when $b_1 + b_2 = b_3$.

Let A be a 3×3 matrix such that ;

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Is the complete solution of $Ax = B$

Step-5

Then,

$$\begin{array}{l} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ \xrightarrow{R_1 \rightarrow R_1 + R_2} \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ \xrightarrow{R_{13}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array}$$

Step-6

Let,

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$b_1 = 1$$

$$b_2 = 2$$

$$b_3 = 3$$

So it is clear that $b_1 + b_2 = b_3$.

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, the system whose solution