Suggested Solutions of Homework 3 MA327

Ex 1.

Proof. WLOG, we may assume the revolution axis is z-axis. Then the map $\mathbf{x}$: $S \to S$ is a restriction of linear map $R_{z,\theta}$:

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

so $\mathbf{x}$ is differentiable. In the same way, we can define the inverse map x^{-1} as the restriction of $R_{z,-\theta}$, which is also differentiable. Hence $\mathbf{x}$ is a diffeomorphism.

In fact, this method is from the textbook, you can check it in Section 2.3 Example 3.2. □

Ex 2.

(a) The surface $\mathbf{x} : (a, b) \times I \to \mathbb{R}^3$ where I = [0, 1] is defined as $\mathbf{x}(s, t) = \alpha(s)t$.

(b) At first, if there are (s_1, t_1), (s_2, t_2) where $s_1 \neq s_2$ such that $t_1\alpha(s_1) = t_2\alpha(s_2)$, then for this the point $\mathbf{x}(s_1, t_1)$ is a self-intersection point, which is not regular. Second, tangent space should be span by two partial derivatives $x_s= t\alpha'(s)$ and $x_t= \alpha(s)$. α is a regular curve which dose not pass through the origin, so α, $\alpha' \neq 0$. Then x_s, x_t are linearly independent if and only if $t \neq 0$. Third, the curve C is the boundary of the surface, which is not convenient to define the differential d$\mathbf{x}$, so it is not regular. To sum up, we know the points which is self-intersected, origin and curve C are not regular.

(c) Remove origin, self-intersections and curve C.

Ex 3. Note: this question has been illustrated in Lecture 7.

Proof. ($\Longrightarrow$) If F is defined on an open set of $\mathbb{R}^3$ containing p such that $f = F\big|_V$. Then any parametrization $\mathbf{x} : U \cap \mathbb{R}^2 \to V$ such that $p \in \mathbf{x}(U)$, $F \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable. Hence f is differentiable at $p \in V$.

($\Longleftarrow$) Let $\mathbf{x} : U \to \mathbb{R}^3$ be a parametrization of S in p. Extend $\mathbf{x}$ to $F : U \times \mathbb{R} \to \mathbb{R}^3$ as follows:

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t),$$

where $\mathbf{x}$(u,v)=(x(u,v), y(u,v), z(u,v)). We may assume $\frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0$.

By inverse function theorem, let W be a neighborhood of $p \in \mathbb{R}^3$ on which F^{-1} is a diffeomorphism. Define $g : U \times \mathbb{R} \to U$ by $g(q) = f \circ \mathbf{x} \circ \pi \circ F^{-1}(q)$, $q \in W$, where $\pi : U \times \mathbb{R} \to U$ is the natural projection. Then g is differentiable and $f = g\big|_{W\cap S}$. $\Longrightarrow$ f is differentiable at $p \in U$. □

Ex 4.

(a) Suppose $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$; $p = \alpha(t_0)$; since the curve C is regular, we may assume $\alpha_1'(t_0) \neq 0$. Define $x : I \times \mathbb{R}^2 \to \mathbb{R}^3$ as $x(t, u, v) = (\alpha_1(t), \alpha_2(t) + u, \alpha_3(t) + v)$. Then

$$dx = \begin{bmatrix} \alpha_1' & 0 & 0 \\ \alpha_2' & 1 & 0 \\ \alpha_3' & 0 & 1 \end{bmatrix}.$$

, which is invertible, hence x^{-1} exists. Furthermore, $\alpha^{-1} = x^{-1}\big|_{u=v=0}$ exists and differentiable. Then we prove $\alpha^{-1} \circ \beta$ is differentiable. In similar, $\beta^{-1} \circ \alpha$ exists and differentiable. Hence h is a diffeomorphism.

(b) Since $\beta = \alpha \circ h$, $\beta'(\tau) = \alpha'(h(\tau)h'(\tau) = \alpha'(t)h'(\tau)$. Besides, $h'(\tau) = (x^{-1} \circ \beta)'(\tau) \neq 0$, so h' would not change sign. Hence

$$
\begin{aligned}
|\int_{t_0}^{t} \big|\alpha'(t)\big|dt| &= |\int_{\tau_0}^{\tau} \big|\alpha'(h(\tau))\big|h'(\tau)d\tau| \\
&= |\int_{\tau_0}^{\tau} \big|\alpha'(h(\tau))h'(\tau)\big|d\tau| \\
&= |\int_{\tau_0}^{\tau} \big|\beta'(\tau)\big|d\tau|
\end{aligned}
$$

Ex 5. It suffices to prove $(f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0))$ is the normal vector of tangent place. Suppose $\mathbf{x}(u, v)$ is a parametrization of a neighbourhood of (x_0, y_0, z_0). Then $f \circ \mathbf{x} = 0$. Then

$$
\begin{aligned}
f_u = df(x_0, y_0, z_0) \cdot x_u = 0 \\
f_v = df(x_0, y_0, z_0) \cdot x_v = 0,
\end{aligned}
$$

which means $df(x_0, y_0, z_0)$ is a normal vector, so we know that equation is the tangent place at (x_0, y_0, z_0).

Ex 6. Adopt the conclusion in Ex 5, the tangent place is $x(a - x) + y(b - y) = 0, a, b \in \mathbb{R}^2$. If let a = b = 0 in this formula, we get $-x^2 - y^2 = 0$. Meanwhile, (x, y, 0) $\in$ the regular surface, so $x^2 + y^2 = 1$. We get a contradiction now.

Ex 7. In this surface,

$$
\begin{aligned}
\mathbf{x}_t &= \alpha'(t) + v\alpha''(t), \\
\mathbf{x}_v &= \alpha'(t),
\end{aligned}
$$

then

$$
\mathbf{N} = \frac{\mathbf{x}_t \wedge \mathbf{x}_v}{\big|\mathbf{x}_t \wedge \mathbf{x}_v\big|} = sgn(v)\frac{\alpha'(t) \wedge \alpha''(t)}{\big|\alpha'(t) \wedge \alpha''(t)\big|}.
$$

When t is Const., the normal vector $\mathbf{N}$ is parallel, so the tangent planes are equal.

Ex 8. For any $w \in T_p(S)$, there is a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$. Then

$$
df_p(w) = (f \circ \alpha)'(0) = 2w \cdot (p - p_0).
$$

Ex 9. Suppose $w = (x, y, z)$, write L(w) as

$$
L(w) = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_{11}x + l_{12}y + l_{13}z \\ l_{21}x + l_{22}y + l_{23}z \\ l_{31}x + l_{32}y + l_{33}z \end{bmatrix}.
$$

There is no doubt that linear map is differential, so its restriction $L\big|S$ is differential. For any $w \in T_p(S)$, there is a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$, then

$$
\begin{aligned}
dL_p(w) &= (L \circ \alpha)'(0) \\
&= (l_{11}x'(0) + l_{12}y'(0) + l_{13}z'(0), l_{21}x'(0) + l_{22}y'(0) + l_{23}z'(0), l_{31}x'(0) + l_{32}y'(0) + l_{33}z'(0)) \\
&= L(w)
\end{aligned}
$$

Ex 10 For any (u, v)

$$\begin{aligned}
\mathbf{x}_u &= (f'(u)\cos v, f'(u)\sin v, g'(u)),\\
\mathbf{x}_v &= (-f(u)\sin v, f(u)\cos v, 0),\\
\mathbf{x}_u \wedge \mathbf{x}_v &= (-g'(u)f(u)\cos v, -g'(u)f(u)\sin v, f'(u)f(u)).
\end{aligned}$$

so the line pass through the normal vector l(t) is

$$l(t) = (f(u)\cos v - tg'(u)f(u)\cos v, f(u)\sin v - g'(u)f(u)\sin v, g(u) + tf'(u)f(u)),$$

when t = 1/g'(u), the x-component and y-component become 0, so l(t) passes z-axis.

Ex 11.

(a)For any $w \in T_p(S)$, there is a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = (x(0), y(0), z(0)) = p$ and $\alpha'(0) = (x'(0), y'(0), z'(0)) = w$, then

$$df_p(w) = (f \circ \alpha)'(0) = \begin{bmatrix} \frac{x-x_0}{|p-p_0|} & \frac{y-y_0}{|p-p_0|} & \frac{z-z_0}{|p-p_0|} \end{bmatrix} \begin{bmatrix} x'(0) \\ y'(0) \\ z'(0) \end{bmatrix},$$

hence $df_p = \frac{p-p_0}{|p-p_0|}$. Thus p is critical $\Leftrightarrow$ $df_p = 0 \Leftrightarrow p - p_0$ is a normal vector at p $\Leftrightarrow$ the line joining p to p_0 is normal to S at p.

(b)Suppose $v = (v_1, v_2, v_3)$. Since $h(p) = p \cdot v$ is a linear map, by the method similar to Ex 9, one can prove $dh_p = v^T$. Hence $dh_p = 0 \Leftrightarrow v^T w = 0$ for any $w \in T_pS \Leftrightarrow$ v is a normal vector.

Ex 12. Since

$$\mathbf{x}(u, v) = \overline{\mathbf{x}}(\overline{u}(u, v), \overline{v}(u, v)),$$

we have

$$\begin{aligned}
\mathbf{x}_u &= \overline{\mathbf{x}}_{\overline{u}}\frac{\partial \overline{u}}{\partial u} + \overline{\mathbf{x}}_{\overline{v}}\frac{\partial \overline{v}}{\partial u};\\
\mathbf{x}_v &= \overline{\mathbf{x}}_{\overline{u}}\frac{\partial \overline{u}}{\partial v} + \overline{\mathbf{x}}_{\overline{v}}\frac{\partial \overline{v}}{\partial v};
\end{aligned}$$

Take the result into $w = a_1\mathbf{x}_u + a_2\mathbf{x}_v$, we get the conclusion.

Ex 13. For any $w \in T_p(S)$, there is a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$, then

$$\begin{aligned}
d(\psi \circ \phi)_p(w) &= (\psi \circ \phi \circ \alpha)'(0)\\
&= d\psi_{\phi(p)}((\phi \circ \alpha)'(0)\\
&= d\psi_{\phi(p)}(d\phi_p(w))\\
&= d\psi_{\phi(p)} \circ d\phi_p(w)
\end{aligned}$$

then we prove $d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$.

Ex 14. (a)

$$\begin{aligned}
\mathbf{x}_u &= (a\cos u\cos v, b\cos u\sin v, -c\sin u)\\
\mathbf{x}_v &= (-a\sin u\sin v, b\sin u\cos v, 0)\\
E &= a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u\\
F &= (b^2 - a^2)\sin u\cos u\sin v\cos v\\
G &= a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{x}_u &= (a\cos v, b\sin v, 2u)\\
\mathbf{x}_v &= (-au\sin v, bu\cos v, 0)\\
E &= a^2\cos^2 v + b^2\sin^2 v + 4u^2\\
F &= (b^2 - a^2)u\sin v\cos v\\
G &= a^2u^2\sin^2 v + b^2u^2\cos^2 v
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{x}_u &= (a\cosh v, b\sinh v, 2u)\\
\mathbf{x}_v &= (au\sinh v, bu\cosh v, 0)\\
E &= a^2\cosh^2 v + b^2\sinh^2 v + 4u^2\\
F &= (a^2 + b^2)u\sinh v\cosh v\\
G &= a^2u^2\sinh^2 v + b^2u^2\cosh^2 v
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{x}_u &= (a\cosh u\cos v, b\cosh u\sin v, c\sinh u)\\
\mathbf{x}_v &= (-a\sinh u\sin v, b\sinh u\cos v, 0)\\
E &= a^2\cosh^2 u\cos^2 v + b^2\cosh^2 u\sin^2 v + c^2\sinh^2 u\\
F &= (b^2 - a^2)\sinh u\cosh u\sin v\cos v\\
G &= a^2\sinh^2 u\sin^2 v + b^2\sinh^2 u\cos^2 v
\end{aligned}$$

Ex 15.

$$\begin{aligned}
\mathbf{x} &= (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4})\\
\mathbf{x}_u &= (\frac{4(-u^2+v^2+4)}{(u^2+v^2+4)^2}, \frac{-8uv}{(u^2+v^2+4)^2}, \frac{16u}{(u^2+v^2+4)^2})\\
\mathbf{x}_v &= (\frac{-8uv}{(u^2+v^2+4)^2}, \frac{4(u^2-v^2+4)}{(u^2+v^2+4)^2}, \frac{16v}{(u^2+v^2+4)^2})\\
E &= \frac{16}{(u^2+v^2+4)^2}\\
F &= 0\\
G &= \frac{16}{(u^2+v^2+4)^2}
\end{aligned}$$

Ex 16 (1) Show $\mathbf{x}$ is a parametrization. i)Obviously, $\mathbf{x}$ is differentiable. ii)From the graph we see $\mathbf{x}^{-1}$ exists. iii)$|\mathbf{x}_u \wedge \mathbf{x}_v| = u\sin\alpha \neq 0$. Since the cone minus origin is regular, $\mathbf{x}$ is a parametrization of the cone.

(2)

$$\begin{aligned}
\mathbf{x}_u &= (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha) \\
\mathbf{x}_v &= (-u\sin\alpha\sin v, u\sin\alpha\cos v, 0) \\
E &= 1 \\
F &= 0 \\
G &= u^2\sin^2\alpha
\end{aligned}$$

For this curve: $u = c\cdot\exp(v\sin\alpha\cot\beta)$. We define this curve as $l(v)$.

$$\begin{aligned}
& & l'(v) &= \mathbf{x}_u(u,v)\sin\alpha\cot\beta\cdot u + \mathbf{x}_v(u,v) \\
&\implies & <l', l'> &= \sin^2\alpha\cot^2\beta\cdot u^2 + u^2\sin^2\alpha \\
&\implies & \frac{<l', \mathbf{x}_u>}{|l'||\sqrt{E}|} &= \cos\beta,
\end{aligned}$$

hence the curve intersects the generators of the cone under the constant angle β.