Step-1

(a) Consider the following:

$$\begin{aligned}A &= uv^{\mathrm{T}}\\ Au &= uv^{\mathrm{T}}u\\ &= u\left(v^{\mathrm{T}}u\right)\\ &= \left(v^{\mathrm{T}}u\right)u\end{aligned}$$

Thus, we have shown that $Au = \left(v^{\mathrm{T}}u\right)u$.

Step-2

When u is a column and v is a row, $v^{\mathrm{T}}u$ is simply a number.

Thus, we have shown that Au is equal to a number multiplied by u. Therefore, u is an eigenvector.

Since, $Au = \left(v^{\mathrm{T}}u\right)u$, the corresponding eigenvalue is $\boxed{\lambda = v^{\mathrm{T}}u}$.

Step-3

(b) Since $A = uv^{\mathrm{T}}$ is a rank 1 matrix, it has only one independent row. Therefore, it has only one nonzero pivot, rest all are zero pivots.

Therefore, the other eigenvalues of A are zeros.

Step-4

(c) Let u and v be as follows: $u = \left(u_1, u_2, \ldots, u_n\right)^{\mathrm{T}}$ and $v = \left(v_1, v_2, \ldots, v_n\right)^{\mathrm{T}}$.

Therefore, we get

$$\begin{aligned}A &= uv^{\mathrm{T}}\\ &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\left(v_1, v_2, \ldots, v_n\right)^{\mathrm{T}}\\ &= \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \cdots & \cdots & \cdots & \cdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{bmatrix}\end{aligned}$$

Step-5

Therefore, the trace $(A) = u_1v_1 + u_2v_2 + ... + u_nv_n$. Now consider $v^{\mathrm{T}}u$.

$$\begin{aligned} v^{\mathrm{T}}u &= \left(v_1, v_2, ..., v_n\right)^{\mathrm{T}} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\ &= v_1u_1 + v_2u_2 + ... + v_nu_n \\ &= u_1v_1 + u_2v_2 + ... + u_nv_n \end{aligned}$$

This is equal to sum of the eigenvalues.