Step-1

Given that every matrix Z can be split into a Hermitian part A and a skew-Hermitian part K, that is $Z = A + K$.

Also given that the real part of Z is half of $Z + Z^H$.

Given $Z = \begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix}$ and $z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

We have to find a formula for the imaginary part K and split the given matrices into $Z = A + K$.

Step-2

Here A is Hermitian and K is skew-Hermitian.

We have

$$A = \frac{1}{2}\left(Z + Z^H\right)$$
$$K = \frac{1}{2}\left(Z - Z^H\right)$$

Now we find the matrices A and K.

We have $Z = \begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix}$

Then $Z^H = \begin{bmatrix} 3-i & 0 \\ 4-2i & 5 \end{bmatrix}$

Step-3

Now

$$\begin{aligned} A &= \frac{1}{2}\left(Z + Z^H\right) \\ &= \frac{1}{2}\left\{\begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 3-i & 0 \\ 4-2i & 5 \end{bmatrix}\right\} \\ &= \frac{1}{2}\begin{bmatrix} 3+i+3-i & 4+2i+0 \\ 0+4-2i & 5+5 \end{bmatrix} \end{aligned}$$

$$=\frac{1}{2}\begin{bmatrix} 6 & 4+2i \\ 4-2i & 10 \end{bmatrix}$$
$$=\begin{bmatrix} 3 & 2+i \\ 2-i & 5 \end{bmatrix}$$

Step-4

Now $\overline{A}=\begin{bmatrix} 3 & 2-i \\ 2+i & 5 \end{bmatrix}$

And $A^{H}=\overline{A}^{T}=\begin{bmatrix} 3 & 2+i \\ 2-i & 5 \end{bmatrix}$

Therefore, $A^{H}=A$

Hence A is Hermitian.

Step-5

Now

$$K=\frac{1}{2}\left(Z-Z^{H}\right)$$
$$=\frac{1}{2}\left\{\begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix}-\begin{bmatrix} 3-i & 0 \\ 4-2i & 5 \end{bmatrix}\right\}$$
$$=\frac{1}{2}\begin{bmatrix} 3+i-3+i & 4+2i-0 \\ 0-4+2i & 5-5 \end{bmatrix}$$
$$=\frac{1}{2}\begin{bmatrix} 2i & 4+2i \\ -4+2i & 0 \end{bmatrix}$$
$$=\begin{bmatrix} i & 2+i \\ -2+i & 0 \end{bmatrix}$$

Step-6

Now $\overline{K}=\begin{bmatrix} -i & 2-i \\ -2-i & 0 \end{bmatrix}$

And $K^{H}=\overline{K}^{T}=\begin{bmatrix} -i & -2-i \\ 2-i & 0 \end{bmatrix}$

Now

$$-K^H = -\begin{bmatrix} -i & -2-i \\ 2-i & 0 \end{bmatrix}$$
$$= \begin{bmatrix} i & 2+i \\ -2+i & 0 \end{bmatrix}$$

Since $K = -K^H$

So K is a skew-symmetric matrix.

Therefore, $\boxed{Z = \begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2+i \\ 2-i & 5 \end{bmatrix} + \begin{bmatrix} i & 2+i \\ -2+i & 0 \end{bmatrix}}$

Step-7

Now we have to write $Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$ as a sum of Hermitian and skew-Hermitian matrices.

Given $Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

Z can be split into $Z = A + K$

Here A is Hermitian and K is a skew-Hermitian matrices.

Step-8

Since $Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

So $Z^H = \begin{bmatrix} -i & i \\ -i & -i \end{bmatrix}$

$$A = \frac{1}{2}\left\{Z + Z^H\right\}$$
$$= \frac{1}{2}\left\{\begin{bmatrix} i & i \\ -i & i \end{bmatrix} + \begin{bmatrix} -i & i \\ -i & -i \end{bmatrix}\right\}$$
$$= \frac{1}{2}\begin{bmatrix} i-i & i+i \\ -i-i & i-i \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix}$$

$$=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$$

Step-9

Now $\overline{A}=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$

And $A^H=\overline{A}^T=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$

Therefore, $A^H=A$

Hence A is Hermitian.

Step-10

Now

$$K=\frac{1}{2}\{Z-Z^H\}$$

$$=\frac{1}{2}\left\{\begin{bmatrix}i & i\\ -i & i\end{bmatrix}-\begin{bmatrix}-i & i\\ -i & -i\end{bmatrix}\right\}$$
$$=\frac{1}{2}\begin{bmatrix}i+i & i-i\\ -i+i & i+i\end{bmatrix}$$
$$=\frac{1}{2}\begin{bmatrix}2i & 0\\ 0 & 2i\end{bmatrix}$$
$$=\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}$$

Step-11

Now $\overline{K}=\begin{bmatrix}-i & 0\\ 0 & -i\end{bmatrix}$

And $K^H=\overline{K}^T=\begin{bmatrix}-i & 0\\ 0 & -i\end{bmatrix}$

Now

$$-K^H = -\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

Since $K = -K^H$

So K is a skew-symmetric matrix.

Therefore, $\boxed{Z = \begin{bmatrix} i & i \\ -i & i \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}}$