

Step-1

We have to find the fourth Legendre polynomial; it is a cubic $x^3 + ax^2 + bx + c$ that is orthogonal to 1, x , and $x^2 - \frac{1}{3}$ over the interval $-1 \leq x \leq 1$

Step-2

The fourth Legendre polynomial

$$u_4 = x^3 - \frac{(1, x^3)}{(1, 1)}1 - \frac{(x, x^3)}{(x, x)}x - \frac{\left(x^2 - \frac{1}{3}, x^3\right)}{\left(x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right)}\left(x^2 - \frac{1}{3}\right) \in (1)$$

Now compute the each of the inner products,

$$\begin{aligned}(1, x^3) &= \int_{-1}^1 x^3 \cdot 1 dx \\ &= 0 \text{ (Since } x^3 \text{ is odd) }\end{aligned}$$

Step-3

$$\begin{aligned}(1, 1) &= \int_{-1}^1 1 \cdot 1 dx \\ &= [x]_{-1}^1 \\ &= 2\end{aligned}$$

Step-4

$$\begin{aligned}(x, x^3) &= \int_{-1}^1 x^4 \cdot 1 dx \\ &= 2 \int_0^1 x^4 dx \\ &= 2 \left[\frac{x^5}{5} \right]_0^1 \\ &= \frac{2}{5}\end{aligned}$$

Step-5

$$\begin{aligned}
 (x, x) &= \int_{-1}^1 x x dx \\
 &= 2 \int_0^1 x^2 dx \\
 &= 2 \left[\frac{x^3}{3} \right]_0^1 \\
 &= \frac{2}{3}
 \end{aligned}$$

Step-6

$$\begin{aligned}
 \left(x^2 - \frac{1}{3}, x^3 \right) &= \int_{-1}^1 \left(x^2 - \frac{1}{3} \right) \cdot x^3 dx \\
 &= 0 \left(\text{since } \left(x^2 - \frac{1}{3} \right) x^3 \text{ is odd} \right)
 \end{aligned}$$

Step-7

$$\begin{aligned}
 \left(x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \right) &= \int_{-1}^1 \left(x^2 - \frac{1}{3} \right) \cdot \left(x^2 - \frac{1}{3} \right) dx \\
 &= 2 \int_0^1 \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9} \right) dx \\
 &= 2 \left[\frac{x^5}{5} - \frac{2}{3} \left(\frac{x^3}{3} \right) + \frac{1}{9} x \right]_0^1 \\
 &= 2 \left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9} \right) \\
 &= \frac{8}{45}
 \end{aligned}$$

Step-8

Hence (1) becomes

$$\begin{aligned}
 u_4 &= x^3 - \frac{0}{2} \cdot 1 - \frac{2/5}{2/3} x - \frac{0}{8/45} \left(x^2 - \frac{1}{3} \right) \\
 &= x^3 - \frac{3}{5} x
 \end{aligned}$$

Therefore the fourth Legendre polynomial is

$$u_4 = x^3 - \frac{3}{5}x$$