1. Label the following statements as **True** or **False**.

   (a) Let V and W be two inner product spaces and $T \in \mathcal{L}(V, W)$. Then T is injective if and only if T^* is surjective.

   (b) The operator $T \in \mathcal{L}(\mathbb{R}^2)$ defined by $T(w, z) = (-z, w)$ is diagonalizable.

   (c) Suppose V is a 7 dimensional real vector space and $T \in \mathcal{L}(V)$. Then T has an eigenvalue.

   (d) Let $T \in \mathcal{L}(\mathbb{C}^3)$ with eigenvalues $0, 1, 2$, then $T^3 - 3T^2 + 2T = 0$.

   (e) Let V be a finite-dimensional inner product space. Suppose $T \in \mathcal{L}(V)$, then $T^2 + 4T + I$ is invertible.

   Solution. TFTTF

2. (15 points) Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all real polynomials of degree at most 2. Define a linear operator T on $\mathcal{P}_2(\mathbb{R})$ by

$$T(p(x)) = xp'(x) - p(1).$$

   (a) Find the matrix representation of T with respect to the basis $1, x, x^2$ of $\mathcal{P}_2(\mathbb{R})$.

   (b) Find all the eigenvalues of T.

   (c) Determine whether T is diagonalizable. If so, find a basis with respect to which T has a diagonal matrix.

   Solution.

   (a)

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

   (b) $-1, 1, 2$.

   (c) $1, 1 - 2x, 1 - 3x^2$.

3. (20 points) Let

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{pmatrix}.$$

(a) Show that A and B are similar.

(b) Find an invertible matrix P such that $B = P^{-1}AP$.

Solution.

(a) A and B have the same Jordan form:

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) Suppose $P_1^{-1}AP_1 = J, P_2^{-1}BP_2 = J$, then $P^{-1}AP = B$, where $P = P_1P_2^{-1}$.

$$P_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

Therefore,

$$P = \begin{pmatrix} -1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 1 \end{pmatrix}.$$

4. Suppose V is finite dimensional complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the distinct eigenvalues of T.

(a) Let $G(\lambda_i, T)$ be the generalized eigenspace corresponding to λ_i. Show that

$$V = G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_m, T).$$

(b) Show that $G(\lambda_1, T)$ is invariant under T and $T|_{G(\lambda_1,T)}$ is nilpotent.

Solution. See 8.21.

(a) Induction on the dimension of V.

(b) Check the definitions.

5. Let $V = \mathcal{P}_3(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx \text{ for all } f(x), g(t) \in V.$$

(a) Find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

(b) Find the orthogonal projection $f_1(x)$ of $f(x) = x^3$ on $\mathcal{P}_2(\mathbb{R})$.

Solution.

(a)
$$\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}).$$

(b) $\frac{3}{5}x$.

6. Suppose V is a finite dimensional inner product space and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if there exists a unique isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$.

Solution. If T is invertible, then S is unique. Actually, $S = S_1$:

$$S_1 : \text{range}\sqrt{T^*T} \to \text{range}\sqrt{T}$$

$$S_1\sqrt{T^*T}v = Tv.$$

Therefore, $S = S_1$ and $T = S\sqrt{T^*T}$.

To prove the other direction, we suppose T is not invertible, then it is easy to show that S can be chosen differently, which contradicts the uniqueness of S. Therefore, T has to be invertible.

7. Suppose V is a finite dimensional real vector space and $T \in \mathcal{L}(V)$. Suppose there exist $b, c \in \mathbb{R}$ such that $T^2 + bT + cI = 0$. Prove that T has an eigenvalue if and only if $b^2 \geq 4c$.

Solution. Suppose T has an eigenvalue λ, then $\lambda^2 + b\lambda + c = 0$. If $b^2 < 4c$, then

$$\lambda^2 + b\lambda + c = 0$$

will not have a real solution. It is a contradiction!

If $b^2 \geq 4c$, then $\lambda^2 + b\lambda + c = 0$ will have a real solution λ. This λ is an eigenvalue of $T_{\mathbb{C}}$. Which is also an eigenvalue of T. Therefore, T has an eigenvalue.

8. Suppose V is a complex vector space. Prove that every invertible operator on V has a cube root R, namely, $R^3 = T$.

Solution. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}(G(\lambda_j, T))$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j(I + \frac{N_j}{\lambda_j})$$

for each j. $(I + \frac{N_j}{\lambda_j})$ has a cute root. Multiplying a cute root of the complex number λ_j by a cute root of $(I + \frac{N_j}{\lambda_j})$, we obtain a cute root R_j of $T|_{G(\lambda_j,T)}$.

A typical vector $v \in V$ can be written uniquely in the form

$$v = u_1 + u_2 + \cdots + u_m,$$

where each u_j is in $G(\lambda_j, T)$. Using this decomposition, define an operator $R \in \mathcal{L}(V)$ by

$$Rv = R_1 u_1 + R_2 u_2 + \cdots + R_m u_m.$$

You should verify that this operator R is a cute root of T, completing the proof.

9. Let T be a linear operator on a finite-dimensional vector space V and $\lambda_1, \lambda_2, \cdots, \lambda_k$ be its distinct eigenvalues of T. Then T is diagonalizable if and only if the minimal polynomial of T is of the form

$$p(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k).$$

Proof. Suppose that T is diagonalizable. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of T, and define

$$p(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k),$$

$p(t)$ divides the minimal polynomial of T. Let $v_1, v_2, \cdots, v_n$ be a basis for V consisting of eigenvectors of T, and consider any v_i in the list, we have $(T - \lambda_j I)(v_i) = 0$ for some eigenvalue λ_j. Since $(t - \lambda_j)$ divides $p(t)$, there is a polynomial $q_j(t)$ such that $p(t) = q_j(t)(t - \lambda_j)$. Hence

$$p(T)(v_i) = q_j(T)(T - \lambda_j I)(v_i) = 0.$$

It follows that $p(T) = 0$, since $p(T)$ takes each vector in a basis for V into 0. Therefore $p(t)$ is the minimal polynomial of T.

Conversely, suppose that there are distinct scalars $\lambda_1, \cdots, \lambda_k$ such that the minimal polynomial $p(t)$ of T factors as

$$p(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k),$$

the λ_i's are eigenvalues of T. We apply mathematical induction on $n = \dim(V)$. Clearly T is diagonalizable for $n = 1$. Now assume that T is diagonalizable whenever $\dim(V) < n$ for some $n > 1$, and let $\dim(V) = n$ and $W = \operatorname{range}(T - \lambda_k I)$. Obviously $W \neq V$, because λ_k is an eigenvalue of T. If $W = \{0\}$, then $T = \lambda_k I$, which is clearly diagonalizable. So suppose that $0 < \dim(W) < n$. Then W is invariant under T, and for any $x \in W$,

$$(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_{k-1} I)(x) = 0.$$

It follows that the minimal polynomial of $T|_W$ divides the polynomial $(t - \lambda_1) \cdots (t - \lambda_{k-1})$. Hence by the induction hypothesis, $T|_W$ is diagonalizable. Furthermore, λ_k is not an eigenvalue of $T|_W$. Therefore $W \bigcap \operatorname{null}(T - \lambda_k I) = \{0\}$. Now let $v_1, \cdots, v_m$ be a basis for W consisting of eigenvectors of $T|_W$, and let $w_1, \cdots, w_p$ be a basis for $\operatorname{null}(T - \lambda_k I)$, the eigenspace of T corresponding to λ_k. $m + p = n$ by the fundamental theorem of linear maps applied to $T - \lambda_k I$. We show that $v_1, \cdots, v_m, w_1, \cdots, w_p$ is linear independent. Consider scalars $a_1, \cdots, a_m$ and $b_1, \cdots, b_p$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m + b_1 w_1 + b_2 w_2 + \cdots + b_p w_p = 0.$$

Let

$$x = \sum_{i=1}^{m} a_i v_i \text{ and } y = \sum_{i=1}^{p} b_i w_i.$$

Then $x \in W, y \in \operatorname{null}(T - \lambda_k I)$, and $x + y = 0$. It follows that

$$x = -y \in W \cap \operatorname{null}(T - \lambda_k I) = \{0\},$$

and therefore $x = 0$. Since $v_1, \cdots, v_m$ is linearly independent, we have that $a_1 = a_2 = \cdots = a_m = 0$. Similarly, $b_1 = b_2 = \cdots = b_p = 0$, we conclude that $v_1, \cdots, v_m, w_1, \cdots, w_p$ is linear independent subset of V consisting of n eigenvectors. It follows that $v_1, \cdots, v_m, w_1, \cdots, w_p$ is a basis for V consisting of eigenvectors of T, and consequently T is diagonalizable. $\square$