

## Step-1

(a)

The objective is to comment on the nature of the sum of a complex number and its conjugate.

Let  $a + ib$  be a complex number.

Then the conjugate of  $a + ib = \overline{(a + ib)}$   
 $= a - ib$

Now the sum of  $a + ib$  and  $a - ib$  is;

$$\begin{aligned} &= (a + ib) + (a - ib) \\ &= (a + a) + i(b - b) \\ &= 2a \end{aligned}$$

Therefore the sum of the complex number and its conjugate is **twice the real part** of the given complex number.

## Step-2

(b)

The objective is to comment on the nature of the conjugate of a complex number on the unit circle

Let  $a + ib$  be a complex number on the unit circle.

Then  $r = \sqrt{a^2 + b^2} = 1$

The conjugate of  $a + ib = a - ib$

And the radius of the complex number  $r' = \sqrt{a^2 + b^2} = 1$

Therefore, the conjugate of a complex number is also **lies on the unit circle**.

## Step-3

(c)

The objective is to comment on the nature of the product of two complex numbers on the unit circle

Let  $(a + ib), (c + id)$  are the complex numbers on unit circle that is,

$$a^2 + b^2 = 1 \text{ and } c^2 + d^2 = 1$$

Then the product

$$\begin{aligned}(a+ib)(c+id) &= ac+ibc+iad+i^2bd \\ &= ac+ibc+iad-bd \\ &= (ac-bd)+i(ad+bc)\end{aligned}$$

The radius  $r = \sqrt{(ac-bd)^2 + (ad+bc)^2}$

Without loss of generality consider  $a = \cos \theta_1, b = \sin \theta_1; c = \cos \theta_2, d = \sin \theta_2$  where  $\theta_1, \theta_2 \in [0, 2\pi)$ .

$$\begin{aligned}r &= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= \sqrt{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)^2 + (\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1)^2} \\ &= \sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)} \\ &= 1\end{aligned}$$

Radius  $\boxed{r=1}$  thus the product lies **on the same unit circle**.

## Step-4

(d)

The objective is to comment on the nature of the sum of two complex numbers on the unit circle

Let  $(a+ib), (c+id)$  are the complex numbers on unit circle that is,

$$a^2 + b^2 = 1 \text{ and } c^2 + d^2 = 1$$

Then the sum of two complex numbers

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

The radius  $r = \sqrt{(a+c)^2 + (b+d)^2}$

Without loss of generality consider  $a = \cos \theta_1, b = \sin \theta_1; c = \cos \theta_2, d = \sin \theta_2$  where  $\theta_1, \theta_2 \in [0, 2\pi)$ .

$$\begin{aligned}r &= \sqrt{(a+c)^2 + (b+d)^2} \\ &= \sqrt{a^2 + c^2 + 2ac + b^2 + d^2 + 2bd} \\ &= \sqrt{(a^2 + b^2) + (c^2 + d^2) + 2(ac + bd)} \\ &= \sqrt{2(1 + ac + bd)}\end{aligned}$$

Substituting  $a$ ,  $b$ ,  $c$  and  $d$  in the above equation gives:

$$\begin{aligned} r &= \sqrt{2(1 + \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)} \\ &= \sqrt{2(1 + \cos(\theta_1 - \theta_2))} \end{aligned}$$

As,

$$\begin{aligned} -1 &\leq \cos(\theta_1 - \theta_2) \leq 1 \\ 1 - 1 &\leq 1 + \cos(\theta_1 - \theta_2) \leq 1 + 1 \\ 0 &\leq 2(1 + \cos(\theta_1 - \theta_2)) \leq 4 \\ 0 &\leq \sqrt{2(1 + \cos(\theta_1 - \theta_2))} \leq 2 \end{aligned}$$

Thus,

$$0 \leq r \leq 2$$

## Step-5

It implies the sum of two complex number on a unit circle lies in a region of  $\boxed{r \leq 2}$ .