Step-1

Consider the system,

$$u+v+2w=2$$
$$2u+3v-w=5$$
$$3u+4v+w=c$$

The objective is to determine the value of c that makes to solve the system and solve it.

Step-2

The above system can be written as,

$$\begin{bmatrix}1 & 1 & 2\\2 & 3 & -1\\3 & 4 & 1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}2\\5\\c\end{bmatrix}$$

This is in the form of $Ax=b$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1,}\begin{bmatrix}1 & 1 & 2\\0 & 1 & -5\\0 & 1 & -5\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}2\\1\\c-6\end{bmatrix}$$

$$\xrightarrow{R_3-R_2}\begin{bmatrix}1 & 1 & 2\\0 & 1 & -5\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}2\\1\\c-7\end{bmatrix}$$

$$\xrightarrow{R_1-R_2}\begin{bmatrix}1 & 0 & 7\\0 & 1 & -5\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\1\\c-7\end{bmatrix}$$

The system is consistent if $c-7=0$

That is, $\boxed{c=7}$

Step-3

If $c=7$, then $Ax=b$ becomes,

$$\begin{bmatrix}1 & 0 & 7\\0 & 1 & -5\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix},$$

Here, w is a free variable.

$$u+7w=1$$
$$v-5w=1$$

It follows that,

$$u=1-7w$$
$$v=1+5w$$

Step-4

Therefore,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1-7w \\ 1+5w \\ w \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$$

Hence, the solution is, $\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$.

The column space is a plane.