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Chapter 6. Continuous time Markov chains

6.1. Definitions and examples

Recall: $\{X_n: \ n \in \mathbb{Z}_+\}$ is a MC if

$$\begin{aligned} & \mathbb{P}(X_{n+1} = j | X_n = i, \ X_{n-1} = i_{n-1}, \ \cdots, \ X_0 = i_0) \\ = \ & \mathbb{P}(X_{n+1} = j | X_n = i). \end{aligned}$$

In continuous time case, how to define the history and the conditional probability (and expectation) given the history?
Answer: We pick arbitrary finite number of times in the past.

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Definition

A continuous time countable state process $\{X_t : \ t \geq 0\}$ is a *Markov chain* if$\forall \ t \geq 0, \ \forall \ s > s_n > \cdots > s_0 \geq 0$, and $\forall \ i_0, \ i_1, \ \cdots, \ i_n, \ i, \ j \in S$,

$$\begin{aligned} & \mathbb{P}(X_{t+s} = j | X_s = i, \ X_{s_n} = i_n, \cdots, \ X_{s_0} = i_0) \\ = \ & \mathbb{P}(X_{t+s} = j | X_s = i). \end{aligned}$$

Example (1)

Let $N(t)$ be a Poisson process and let Y_n, indep. of N, be a MC.Then, $X_t \equiv Y_{N(t)}$ is a continuous time MC.

Proof: Do it on board!

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We first study a special class of Markov chain:birth-death processes.
Consider a system whose state at any time is represented by the number of people in the system at that time.Suppose that whenever there are n people in the system, then
(i) new arrivals enter the system at an exponential rate λ_n,
(ii) people leave the system at an exponential rate μ_n.
(iii) The arrival and departure are independent.

Such a system is called a birth and death process. The parameters $\{\lambda_n\}$ and $\{\mu_n\}$ are called, respectively, the arrival (or birth) and departure (or death) rates.
Thus, a birth and death process is a continuous-time Markov chain with states $\{0, 1, ...\}$ for which transitions from state n may go only to either state $n - 1$ or state $n + 1$.

This chain can be visualized as follows:Let

$$v_n = \lambda_n + \mu_n, \quad (\mu_0 = 0).$$

When at state $n \geq 0$, it waits an exponential time with rate $v_n$and then jumps to either $n+1$ (birth event) or $n-1$ (death event) with the following transition probabilities:

$$r(n, n+1) = \frac{\lambda_n}{\lambda_n + \mu_n},$$

$$r(n, n-1) = \frac{\mu_n}{\lambda_n + \mu_n}.$$

Example (Poisson Process)

It is a birth-death process with death rate $\mu_n = 0$ *and birth rate* $\lambda_n = \lambda$.

Example (A Linear Growth Model with Immigration)

Each individual in the population gives birth at an exponential rate λ;in addition, there is an exponential rate of increase θ of the population due to an external source such as immigration.Deaths are assumed to occur at an exponential rate μ for each member of the population.

In this model:

$$\mu_n = n\mu, \qquad n \geq 1$$

$$\lambda_n = n\lambda + \theta, \qquad n \geq 0.$$

It is called a linear growth process with immigration.Such processes occur naturally in the study of biological reproduction and population growth.

Let $X(t)$ denote the population size at time t.Suppose that $X(0) = i$ and let

$$m(t) = E(X(t)).$$

Find $m(t)$.

Solution: Do it on board!

Example (Queueing System M/M/1)

Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate λ.
Upon arrival, each customer goes directly into service if the server is free;if not, then the customer joins the queue (that is, he waits in line).
When the server finishes serving a customer, the customer leaves the system and the next customer in line, if there are any waiting, enters the service.The successive service times are assumed to be independent exponential random variables with parameter μ.

Discussion:

The preceding is known as the M/M/1 queueing system.The first M refers to the fact that the interarrival process is Markovian (since it is a Poisson process) and the second to the fact that the service distribution is exponential (and, hence, Markovian).The 1 refers to the fact that there is a single server. If we let $X(t)$ denote the number of customers in the system at time t,then $\{X(t), t \geq 0\}$ is a birth and death process with

$$\mu_n = \mu, \quad n \geq 1; \quad \lambda_n = \lambda, \quad n \geq 0.$$

□

Example (Multiserver Exponential Queueing System)

Consider an exponential queueing system in which there are s servers available, each serving at rate μ. An entering customer first waits in line and then goes to the first free server. This is known as an $M/M/s$ queueing model. Let $X(t)$ denote the number of customers in the system at time t. Let T_i be the time it takes from state i to state $i+1$. Find $E(T_i)$.

Discussion: This is a birth and death process with parameters

$$\mu_i = (i \wedge s)\mu,\ i \geq 1; \qquad \lambda_i = \lambda,\ i \geq 0.$$

Do it on board!

6.2. C-K equations

Definition

$\mathbb{P}(X_{t+s} = j | X_s = i)$ is the *transition probability* of the MC, denoted by $p_t(i,j)$.

Example 1 (conti). $N(t)$ Poisson process, $X(t) = Y_{N(t)}$.Let $u^n(i,j)$ be the transition probability of the MC Y_n.

$$\begin{aligned} p_t(i,j) &= \mathbb{P}(Y_{N(t+s)} = j | Y_{N(s)} = i) \\ &= \mathbb{E}\left(u^{N(t+s)-N(s)}(i,j)\right) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} u^n(i,j). \end{aligned}$$

□

Recall: $P^{n+m} = P^n P^m$ (matrix multiplication).

Theorem (Chapman-Kolmogorov equation)

$$\sum_k p_s(i,k)p_t(k,j) = p_{t+s}(i,j).$$

Proof: Do it on board!

Suppose that $\forall\ i \neq j \in S$, the derivative of $p_t(i,j)$ at 0 exists,denote

$$q(i,j) \equiv \lim_{h\to 0} \frac{1}{h}(p_h(i,j) - 0) = \lim_{h\to 0} \frac{1}{h} p_h(i,j) \geq 0.$$

For $i = j$,

$$\begin{aligned} q(j,j) &= \lim_{h\to 0} h^{-1}\left(p_h(j,j) - 1\right) \\ &= -\lim_{h\to 0} h^{-1} \sum_{k\neq j} p_h(j,k) \\ &= -\sum_{k\neq j} q(j,k) \leq 0. \end{aligned}$$

Then,

$$
\begin{aligned}
\partial_t p_t(i,j) &= \lim_{h\to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h} \\
&= \lim_{h\to 0} h^{-1} \left(\sum_k p_t(i,k) p_h(k,j) - \sum_k p_t(i,k)\delta_{kj} \right) \\
&= \sum_k p_t(i,k) \lim_{h\to 0} h^{-1} \left(p_h(k,j) - \delta_{kj} \right) \\
&= \sum_k p_t(i,k) q(k,j).
\end{aligned}
$$

So, p_t can be obtained from the matrix equation

$$\partial_t P_t = P_t Q.$$

$q(i,j)$ is the **jump rate** from i to j.
Note that we also can prove

$$\partial_t P_t = Q P_t.$$

Example 2 (conti). $X(t) = Y_{N(t)}$. Suppose that $u(i,i) = 0$. Recall

$$p_t(i,j) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} u^n(i,j).$$

Then,

$$\begin{aligned} \partial_t p_t(i,j) &= \sum_{n=1}^{\infty} \frac{u^n(i,j)\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} \\ &\quad -\lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} u^n(i,j) \\ &= \lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \left(u^{n+1}(i,j) - u^n(i,j)\right) \end{aligned}$$

$$q(i,j) = \lambda(u(i,j) - \delta_{ij}).$$

Note that

$$q(i,i) = -\lambda$$

and for $j \neq i$,

$$u(i,j) = \frac{q(i,j)}{-q(i,i)}.$$

It waits an expo.(λ) time, then jumps according to $u(i,j)$. $\square$

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Example (Birth-death process)

Let

$$v_n = \lambda_n + \mu_n, \quad (\mu_0 = 0).$$

When at state $n \geq 0$, it waits an exponential time with rate $v_n$and then jumps to either $n+1$ (birth event) or $n-1$ (death event) with the following transition probabilities:

$$r(n, n+1) = \frac{\lambda_n}{\lambda_n + \mu_n},$$

$$r(n, n-1) = \frac{\mu_n}{\lambda_n + \mu_n}.$$

Now we construct general MC if the rate is given. Suppose $v(i) = v_i = -q(i,i) \in (0, \infty)$.Let

$$r(i,j) = \frac{q(i,j)}{v_i}, \qquad j \neq i.$$

Let (Y_n) be a discrete time MC with transition probability $r(i,j)$.
When to jump?For initial Y_0, it will wait $t_1 \sim expo(v(Y_0))$ until it jumps to Y_1.Let $\tau_0 \sim expo.(1)$. Then,

$$t_1 \equiv \frac{\tau_0}{v(Y_0)} \sim expo.(v(Y_0)).$$

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Let $\tau_0,\ \tau_1,\ \cdots$ be i.i.d. $expo.(1)$.Let

$$t_n = \frac{\tau_{n-1}}{v(Y_{n-1})}, \qquad n = 1, 2, \cdots .$$

Then, $t_n \sim expo.(v(Y_{n-1}))$. Let

$$T_n = t_1 + \cdots + t_n.$$

Theorem

Define

$$X_t = Y_n, \qquad T_n \leq t < T_{n+1}.$$

Then, X_t is a continuous time MC with rate $q(i,j)$.

Proof: Do it on board!

6.3. Computing the transition probability

Recall:

$$\partial_t p_t(i,j) = \sum_k p_t(i,k)q(k,j).$$

In matrix form

$$\frac{d}{dt}P_t = P_t Q. \tag{1.1}$$

We can also prove that

$$\frac{d}{dt}P_t = QP_t. \tag{1.2}$$

(1.1) is *Kolmogorov's forward equation.*
(1.2) is *Kolmogorov's backward equation.*

From (1.1) & (1.2), $P_t Q = Q P_t$.
For a square matrix A, we define

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Then,

$$\begin{aligned}\frac{d}{dt} e^{Qt} &= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{Q^n}{n!} t^n = \sum_{n=1}^{\infty} \frac{Q^n}{n!} n t^{n-1} \\ &= Q \sum_{n=1}^{\infty} \frac{Q^{n-1}}{(n-1)!} t^{n-1} = Q e^{Qt}.\end{aligned}$$

So, $P_t = e^{Qt}$ is a sol.Note $P_0 = I$.

Example (Poisson process w/ rate λ)

$$q(i, i+1) = \lambda, \qquad q(i, i) = -\lambda.$$

Find the transition probability.

Sol: Do it on board!

Example (Two state chains)

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

Sol: Do it on board!

6.4. Limiting behavior

Definition

(X_t) is *irreducible* if $\forall i, j \in S$, $\exists k_0 = i,\ k_1, \cdots, k_n = j$ s.t.

$$q(k_{m-1}, k_m) > 0, \qquad 1 \le m \le n.$$

Lemma

If (X_t) is irreducible and $t > 0$, then $p_t(i, j) > 0$.

Proof: Do it on board!

Definition

$\{\pi_j,\ j \in S\}$ is a *stationary distribution* if $\pi P_t = \pi,\ \forall t > 0$.

Lemma

π is stationary iff $\pi Q = 0$.

Proof: Do it on board!

Note that for any $h > 0$, P_h is irreducible,

$$\lim_{n\to\infty} p_{nh}(i,j) = \pi_j.$$

Theorem

If (X_t) is irreducible and has a stationary distribution π, then

$$\lim_{t\to\infty} p_t(i,j) = \pi_j.$$

Proof: Do it on board!

Example (Duke-UNC basketball)

MC with 4 states:

$$1 = \text{Duke on offense}, \qquad 2 = \text{Duke scores},$$

$$3 = \text{UNC on offense}, \qquad 4 = \text{UNC scores},$$

and

$$Q = \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & -5 & 5 & 0 \\ 1 & 0 & -2.5 & 1.5 \\ 6 & 0 & 0 & -6 \end{bmatrix}.$$

Find its stationary distribution.

Sol: Do it on board!

Detailed balance condition

$$\pi_k q(k,j) = \pi_j q(j,k).$$

Theorem

DBC implies stationary.

Proof:

$$\sum_k \pi_k q(k,j) = \sum_k \pi_j q(j,k) = 0.$$

$\square$

Example (Birth and death chains, $S = \mathbb{Z}_+$)

$$q(n, n+1) = \lambda_n > 0, \qquad n \geq 0,$$
$$q(n, n-1) = \mu_n > 0, \qquad n > 0.$$

Is there a stationary distribution?

Sol: Do it on board!

HW: Ch6, 1, 5, 8, 10, 12, 13, 32, 36, 37, 38