Step-1

Given that $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$

We have to write four equations $Ax = b$.

Step-2

First we write the equations that would hold if a line could go through all four points.

Then every $C + Dt$ would agree exactly with *b*.

Now $Ax = b$ is

$$C + D(0) = 0$$
$$C + D(1) = 8$$
$$C + D(3) = 8$$
$$C + D(4) = 20$$

Or $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

Where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $x = \begin{bmatrix} C \\ D \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

Step-3

We know that the least-square solution to a problem is $A^T A\hat{x} = A^T b$.

Now

$$A^T A\hat{x} = A^T b$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \begin{pmatrix} 1(1)+1(1) \\ +1(1)+1(1) \end{pmatrix} & \begin{pmatrix} 1(0)+1(1) \\ +1(3)+1(4) \end{pmatrix} \\ \begin{pmatrix} 0(1)+1(1) \\ +3(1)+4(1) \end{pmatrix} & \begin{pmatrix} 0(0)+1(1) \\ +3(3)+4(4) \end{pmatrix} \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1(0)+1(8) \\ +1(8)+1(20) \end{pmatrix} \\ \begin{pmatrix} 0(0)+1(8) \\ +3(8)+4(20) \end{pmatrix} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

Step-4

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 4 & 8 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 36 \\ 40 \end{bmatrix}$$

$$\Rightarrow 4\hat{C} + 8\hat{D} = 36 \text{ and } 10\hat{D} = 40$$

$$\Rightarrow \hat{D} = 4 \text{ and } \hat{C} = \frac{36-8(4)}{4} = 1$$

Therefore $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Step-5

Now we have to find the exact solution to $A\hat{x} = p$ by changing the measurements to $p = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$.

Now $p = A\hat{x}$

$$\Rightarrow \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ x+3y \\ x+4y \end{bmatrix}$$

Step-6

From this we get the equations

$x = 1, x + y = 5, x + 3y = 13, x + 4y = 17$

Solving these equations, we get

$x = 1\ , y = 4$

Hence the exact solution to $A\hat{x} = p$ is $\boxed{\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}}$