

Step-1

The matrix S is a diagonal matrix. Hence, its inverse is obtained by taking reciprocals of its diagonal elements and keeping remaining zeros as they are.

$$S^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Therefore,

This gives,

$$\begin{aligned} S^{-1}T &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \end{aligned}$$

Step-2

To obtain the eigenvalues of $S^{-1}T$, we solve the equation $\det(S^{-1}T - \lambda I) = 0$, where λ is an eigenvalue of $S^{-1}T$.

Therefore,

$$\begin{aligned} 0 &= \begin{vmatrix} 0-\lambda & \frac{1}{3} \\ \frac{1}{3} & 0-\lambda \end{vmatrix} \\ &= \begin{vmatrix} -\lambda & \frac{1}{3} \\ \frac{1}{3} & -\lambda \end{vmatrix} \\ &= \lambda^2 - \frac{1}{9} \end{aligned}$$

Thus, we have $\lambda^2 - \frac{1}{9} = 0$. Therefore, $\lambda = \pm \frac{1}{3}$.

Step-3

For the Gauss-Seidel method, consider the following equation:

$$\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + b$$

Thus, here we have

$$S = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step-4

Let us obtain the inverse of S . The determinant of S is 9.

$$S^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{9} & \frac{1}{3} \end{bmatrix}$$

Therefore,

This gives,

$$\begin{aligned} S^{-1}T &= \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{9} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix} \end{aligned}$$

To obtain the eigenvalues of $S^{-1}T$, we solve the equation $\det(S^{-1}T - \mu I) = 0$, where μ is an eigenvalue of $S^{-1}T$.

Therefore,

$$\begin{aligned} 0 &= \begin{vmatrix} 0 - \mu & \frac{1}{3} \\ 0 & \frac{1}{9} - \mu \end{vmatrix} \\ &= \begin{vmatrix} -\mu & \frac{1}{3} \\ 0 & \frac{1}{9} - \mu \end{vmatrix} \\ &= \mu^2 - \frac{\mu}{9} \end{aligned}$$

Thus, we have $\mu^2 - \frac{\mu}{9} = 0$. Therefore, $\boxed{\mu = 0 \text{ or } \frac{1}{9}}$.

Step-5

The maximum eigenvalue in Gauss-Seidel method is $\frac{1}{9}$ and in Jacobi method, it is $\frac{1}{3}$. Since, $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$, we can say that $|\lambda|_{\max}$ for Gauss-Seidel is equal to the $|\lambda|_{\max}^2$ for Jacobi.