Step-1

If Q_1, Q_2 are orthogonal, then $Q_1^T Q_1 = I$ and $Q_2^T Q_2 = I$ â€¦â€¦ (1)

We consider $(Q_1Q_2)^T (Q_1Q_2)$

$= (Q_2^T Q_1^T) Q_1 Q_2$ By the properties of transposing matrices

$= Q_2^T (Q_1^T Q_1) Q_2$ By the associativity of multiplication

$= Q_2^T (I Q_2)$ by (1)

$= Q_2^T Q_2$
$= I$

Therefore $(Q_1Q_2)^T (Q_1Q_2) = I$

Hence Q_1Q_2 is orthogonal

Step-2

Given that Q_1 is the rotation through θ

So, $Q_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Similarly, Q_2 is a rotation matrix through ϕ

So, $Q_2 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

Let us consider $Q_1Q_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$
$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

This is nothing but the rotation matrix through an angle α where $\alpha = \theta + \phi$

Therefore, $Q_1 Q_2$ is a rotation matrix through an angle $\theta + \phi$.