Step-1

In order to obtain the minimum value of $R(x)=\frac{x_1^2-x_1x_2+x_2^2}{x_1^2+x_2^2}$, let us write $R(x)=\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x}$ such that *A* is a positive definite symmetric matrix.

Let $A=\begin{bmatrix} a & c \\ c & b \end{bmatrix}$

We have

$$\begin{aligned}(x_1,x_2)\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}&=(x_1,x_2)\begin{bmatrix} ax_1+cx_2 \\ cx_1+dx_2 \end{bmatrix}\\ &=ax_1^2+cx_1x_2+cx_1x_2+dx_2^2\\ &=ax_1^2+2cx_1x_2+dx_2^2\end{aligned}$$

Step-2

Since, $ax_1^2+2cx_1x_2+dx_2^2=x_1^2-x_1x_2+x_2^2$, we get,

$$\begin{aligned}a&=1\\ d&=1\\ c&=-\frac{1}{2}\end{aligned}$$

Step-3

Therefore, $A=\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$. It is clear that the smallest eigenvalue will give the smallest value of $R(x)=\frac{x_1^2-x_1x_2+x_2^2}{x_1^2+x_2^2}$. To obtain the eigenvalues of *A*, solve $\det(A-\lambda I)=0$.

$$\begin{aligned}0&=\det(A-\lambda I)\\ &=\begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix}\\ &=(1-\lambda)^2-\frac{1}{4}\\ &=\lambda^2-2\lambda+\frac{3}{4}\end{aligned}$$

Step-4

Obtain the roots of $\lambda^2 - 2\lambda + \frac{3}{4} = 0$.

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{4-3}}{2} \\ &= \frac{2 \pm 1}{2} \\ &= \frac{3}{2} \text{ or } \frac{1}{2}\end{aligned}$$

The smallest eigenvalue of A is $\frac{1}{2}$. Therefore, the smallest value of $R(x) = \frac{x_1^2 - x_1x_2 + x_2^2}{x_1^2 + x_2^2}$ is $\boxed{\frac{1}{2}}$.

Step-5

In order to find the smallest value of $R(x) = \frac{x_1^2 - x_1x_2 + x_2^2}{2x_1^2 + x_2^2}$, let us write $R(x) = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}Mx}$, where M is also positive definite symmetric matrix.

Let $M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$

We have

$$\begin{aligned}(x_1, x_2)\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= (x_1, x_2)\begin{bmatrix} ax_1 + cx_2 \\ cx_1 + dx_2 \end{bmatrix} \\ &= ax_1^2 + 2cx_1x_2 + dx_2^2\end{aligned}$$

Since $ax_1^2 + 2cx_1x_2 + dx_2^2 = 2x_1^2 + x_2^2$, we get

$$\begin{aligned}a &= 2 \\ c &= 0 \\ d &= 1\end{aligned}$$

Thus, $M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Step-6

Consider the matrix $M^{-1}A$. Since, M is a diagonal matrix, its inverse is obtained by replacing the diagonal entries by their reciprocals.

Thus, $M^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. This gives the following:

$$M^{-1}A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Step-7

To obtain the eigenvalues of $M^{-1}A$, solve $\det\left(M^{-1}A - \lambda I\right) = 0$. This gives

$$0 = \begin{vmatrix} \frac{1}{2} - \lambda & -\frac{1}{4} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix}$$
$$= \left(\frac{1}{2} - \lambda\right)(1-\lambda) - \frac{1}{8}$$
$$= \lambda^2 - \frac{3}{2}\lambda + \frac{3}{8}$$
$$= 8\lambda^2 - 12\lambda + 3$$

Step-8

Obtain the roots of $8\lambda^2 - 12\lambda + 3 = 0$. We have

$$\lambda = \frac{12 \pm \sqrt{144-96}}{16}$$
$$= \frac{12 \pm \sqrt{48}}{16}$$
$$= \frac{12 \pm 4\sqrt{3}}{16}$$
$$= \frac{3 \pm \sqrt{3}}{4}$$

Step-9

The smallest eigenvalue of $M^{-1}A$ is $\frac{3-\sqrt{3}}{4}$. This gives that, the smallest value of $R(x)=\frac{x_1^2-x_1x_2+x_2^2}{2x_1^2+x_2^2}$ is also $\boxed{\frac{3-\sqrt{3}}{4}}$.