

Step-1

The objective is to prove that $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

Step-2

Consider matrix A and B , each column of matrix AB be the combination of the column of matrix A .

Let A be $m \times n$ matrix, B matrix be $n \times p$ then, the combination of matrix AB be $m \times p$ matrix.

The j^{th} column of matrix AB is:

$$\begin{bmatrix} (AB)_{1j} \\ (AB)_{2j} \\ \vdots \\ (AB)_{mj} \end{bmatrix}$$

Since,
$$(AB)_{1j} = \sum_{k=1}^n A_{1k} \cdot B_{kj}$$

Now, the j^{th} column of matrix AB is:

$$\begin{aligned} \begin{bmatrix} \sum_{k=1}^n A_{1k} \cdot B_{kj} \\ \sum_{k=1}^n A_{2k} \cdot B_{kj} \\ \vdots \\ \sum_{k=1}^n A_{mk} \cdot B_{kj} \end{bmatrix} &= \sum_{k=1}^n \begin{bmatrix} A_{1k} B_{kj} \\ A_{2k} B_{kj} \\ \vdots \\ A_{mk} B_{kj} \end{bmatrix} \\ &= \sum_{k=1}^n \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{mk} \end{bmatrix} [B_{kj}] \\ &= \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} [B_{1j}] + \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} [B_{2j}] + \cdots + \begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} [B_{nj}] \end{aligned}$$

Therefore, this is the linear combination of matrix A .

Step-3

Now, each column of matrix AB be the combination of the column of matrix A .

Any column of matrix (AB) is:

$$\begin{aligned}(AB) &\subseteq \text{span}\{\text{columns of } A\} \\ \text{span}\{\text{columns of } (AB)\} &\subseteq \text{span}\{\text{columns of } A\} \\ \dim\{\text{span}\{\text{columns of } (AB)\}\} &\leq \dim\{\text{span}\{\text{columns of } A\}\}\end{aligned}$$

Thus, $\text{rank}(AB) \leq \text{rank}(A)$.

Hence, **Proved.**

Step-4

Now, prove that $\text{rank}(AB) \leq \text{rank}(B)$. Let each column of matrix AB be the combination of the column of matrix B .

The i^{th} row of matrix AB is:

$$\left[(AB)_{i1} \quad (AB)_{i2} \quad \cdots \quad (AB)_{ip} \right]$$

Since,

$$\left[\sum_{k=1}^n (A_{ik} B_{k1}) \quad \sum_{k=1}^n (A_{ik} B_{k2}) \quad \cdots \quad \sum_{k=1}^n (A_{ik} B_{kp}) \right] = \sum_{k=1}^n A_{ik} \left[B_{k1} \quad B_{k2} \quad \cdots \quad B_{kp} \right]$$

Thus, the i^{th} row of matrix (AB) be a linear combination of rows of matrix B .

Each row of matrix (AB) is:

$$\begin{aligned}(AB) &\subseteq \text{span}\{\text{rows of } B\} \\ \text{span}\{\text{rows of } (AB)\} &\subseteq \text{span}\{\text{rows of } B\} \\ \dim\{\text{span}\{\text{rows of } (AB)\}\} &\leq \dim\{\text{span}\{\text{rows of } B\}\}\end{aligned}$$

Thus, $\text{rank}(AB) \leq \text{rank}(B)$.

Hence, **Proved.**