Step-1

Consider the matrix $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$.

The objective is to find all eigenvalues and eigenvectors of *A* and write two different diagonalizing matrices *S*.

Step-2

Find the eigenvalues for the matrix *A* as,

For that, the characteristic polynomial of A is,

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{vmatrix} R_1\to R_1+R_2+R_3$$

$$=\begin{vmatrix}3-\lambda&3-\lambda&3-\lambda\\1&1-\lambda&1\\1&1&1-\lambda\end{vmatrix}$$

$$=(3-\lambda)\begin{vmatrix}1&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{vmatrix}\begin{matrix}R_2\to R_2-R_1,\\R_3\to R_3-R_1\end{matrix}$$

$$=(3-\lambda)\begin{vmatrix}1&1&1\\0&-\lambda&0\\0&0&-\lambda\end{vmatrix}$$

$$=(3-\lambda)(1)(-\lambda)(-\lambda)$$

Find the eigenvalues by equating the characteristic polynomial to zero.

That is,

$$(3-\lambda)(1)(-\lambda)(-\lambda)=0$$

Therefore, the eigenvalues are $\lambda_1=0, \lambda_2=0,$ and $\lambda_3=3$.

Step-3

Find the eigenvector *X* corresponding to the eigenvalue $\lambda_1=0$ as,

$$
\begin{aligned}
(A-\lambda_1 I)x &= 0 \\
(A-0(I))x &= 0 \quad \text{since } \lambda_1 = 0 \\
Ax &= 0
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0
$$

By applying the row operation on the coefficient matrix, the system reduces to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

From the above obtain the system,

$x_1 + x_2 + x_3 = 0$ â€¦â€¦ (1)

Supposing x_2, x_3 are free variables and letting $x_2 = k_1, x_3 = k_2$, obtain $x_1 = -k_1 - k_2$

So, the solution set of $(A-\lambda_1 I)x = 0$ is

$$
\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} \\
&= k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

So, the eigen vectors corresponding to the repeated eigen values $\lambda_1 = 0, \lambda_2 = 0$ are

$$
\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
$$

Step-4

Find the eigenvector *X* corresponding to the eigenvalue $\lambda_3 = 3$ as,

$$(A-\lambda_3 I)x=0$$
$$(A-3(I))x=0 \qquad \text{since } \lambda_3=3$$
$$\left(\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$
$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

Applying the row operations on the coefficient matrix,

$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\sim\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\begin{matrix}R_2\to 2R_2+R_1,\\R_3\to 2R_3+R_1\\ \end{matrix}$$
$$\sim\begin{bmatrix}-2&1&1\\0&-3&3\\0&3&-3\end{bmatrix}\begin{matrix}R_3\to R_3+R_2\\R_2/-3\\ \end{matrix}$$
$$\sim\begin{bmatrix}-2&1&1\\0&1&-1\\0&0&0\end{bmatrix}$$

Step-5

This is the reduced matrix and so, we rewrite the homogeneous equations from this as,

$$\left.\begin{aligned}-2x_1+x_2+x_3&=0\\x_2-x_3&=0\end{aligned}\right\}$$ â€¦â€¦ (2)

$$\Rightarrow x_2=x_3,$$
$$\Rightarrow x_1=x_3$$

Using $x_3=1$, obtain the solution set $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ is the eigen vector corresponding to $\lambda_3=3$

Step-6

While the number of eigen values is equal to the number of eigen vectors of A, we say that A is diagonalizable.

Using the eigen vectors as the columns of the matrix S, we see that the diagonalization of the given matrix is $A = S\Lambda S^{-1}$ where $S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, S^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ â€¦â€¦ (3)

Step-7

On the other hand, we try for the other diagonalization.

Let us consider (1) and write $x_3 = -x_1 - x_2$

Then the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = m\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + n\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ where $x_1 = m, x_2 = n$ are the parameters.

So, the other possible eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 0$ are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Similarly, considering (2), the eigen vector corresponding to $\lambda_3 = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Using these eigen vectors as the columns of S, we get $A = S\Lambda S^{-1}$ where $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ â€¦â€¦ (4)

Observe that (3) and (4) are the different diagonalizations for the same matrix.