Step-1

In case of eigenvalues, we know that the sum of the eigenvalues is equal to the trace of the matrix and the product of eigenvalues is equal to the determinant of the matrix.

The trace of the matrix is n, which is positive. The determinant of A is zero, except when $n = 2$. Thus, A should have at least one positive eigenvalue and whenever the order of A is greater than 2, it should have zero as one of the eigenvalues.

Step-2

Let B be another matrix, obtained from A by deleting the last row and the last column of the matrix A.

Thus, $B = \begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & . & 0 \end{bmatrix}$.

It is clear that each eigenvalue of B is equal to zero.

Step-3

When a matrix B is obtained from A by deleting some k^{th} row and k^{th} column from A, then we get the following inequality with their eigenvalues:

$$\lambda_1(A) \le \lambda_1(B) \le \lambda_2(A) \le \ldots \le \lambda_{n-1}(A) \le \lambda_{n-1}(B) \le \lambda_n(A)$$

Now in this case, we have $A = \begin{bmatrix} 0 & . & 0 & 1 \\ . & . & 0 & 2 \\ 0 & 0 & 0 & . \\ 1 & 2 & . & n \end{bmatrix}$ and $B = \begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & . & 0 \end{bmatrix}$.

Step-4

We have shown that all the eigenvalues of B are zero. Thus, we get

$$\lambda_1(A) \le 0 \le \lambda_2(A) \le \ldots \le \lambda_{n-1}(A) \le 0 \le \lambda_n(A)$$

We have also shown that one of the eigenvalues of A should be positive. From the above inequality it is clear that it is the only positive eigenvalue A can have. That will be the largest eigenvalue of A.

Also, from the inequality, it is clear that $\lambda_2(A) = 0$, $\lambda_3(A) = 0$,..., $\lambda_{n-1}(A) = 0$.

The smallest eigenvalue $\lambda_1(A)$ may be zero or may be negative.

Step-5

We claim that the smallest eigenvalue $\lambda_1(A)$ must be negative.

Let $\lambda \neq 0$ be an eigenvalue of A. Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be the corresponding eigenvector. Therefore, we have

$$Ax = \lambda x$$

$$\begin{bmatrix} 0 & . & 0 & 1 \\ . & . & 0 & 2 \\ 0 & 0 & 0 & . \\ 1 & 2 & . & n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

Step-6

Thus, we get the following equations:

$$\begin{aligned} x_n &= \lambda x_1 \\ 2x_n &= \lambda x_2 \\ 3x_n &= \lambda x_3 \\ (n-1)x_n &= \lambda x_{n-1} \\ x_1 + 2x_2 + \ldots + nx_n &= \lambda x_n \end{aligned}$$

Thus, we get

$$x_1 = \frac{x_n}{\lambda}$$

$$x_2 = \frac{2x_n}{\lambda}$$

$$x_3 = \frac{3x_n}{\lambda}$$

And so on!

Step-7

Therefore,

$$\frac{x_n}{\lambda}+\frac{4x_n}{\lambda}+\frac{9x_n}{\lambda}+\ldots+\frac{(n-1)^2 x_n}{\lambda}+nx_n=\lambda x_n$$

$$\left(\frac{1}{\lambda}+\frac{4}{\lambda}+\frac{9}{\lambda}+\ldots+\frac{(n-1)^2}{\lambda}+n\right)x_n=\lambda x_n$$

$$\left(1+4+9+\ldots+\left(n-1^2\right)+n\lambda\right)=\lambda^2$$

$$\lambda^2-n\lambda-\left(1+4+9+\ldots+\left(n-1^2\right)\right)=0$$

The sum of the roots of the last equation is â€“*n*, which is negative and thus, this equation must have a negative root.

Step-8

Therefore, the smallest eigenvalue of A must be negative.

Therefore, when A is an n by n matrix, it has $\boxed{\text{one positive eigenvalue}}$, $\boxed{\text{one negative eigenvalue}}$ and $n$$\boxed{\text{â€“2 number of zero eigenvalues}}$.