

Linear Algebra and Its Applications

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1 Matrices and Gaussian Elimination

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Matrices and Gaussian Elimination

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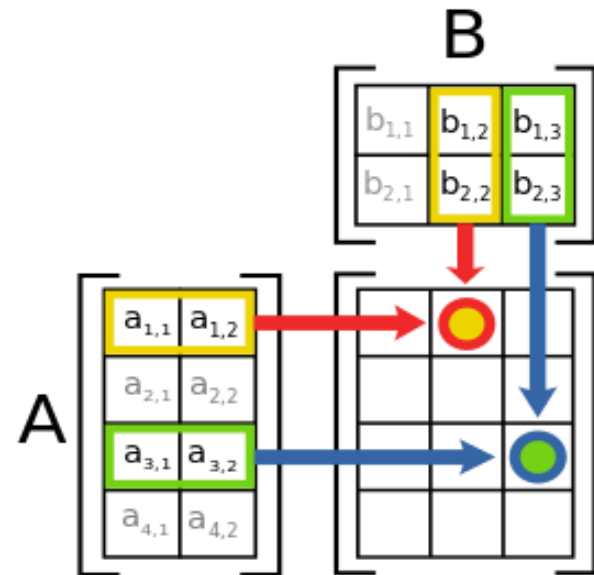
INVERSES AND TRANSPOSES

(矩阵的逆和转置)

Definitions

Properties

Algorithms

*** Textbook: Section 1.6**

I. Transpose of a Matrix (转置矩阵)

1. Definition (定义)

Given an $m \times n$ matrix A , the **transpose** of A is the $n \times m$ matrix, denoted by A^T , whose columns are formed from the corresponding rows of A : the i th row of A becomes the i th column of A^T . (把 $m \times n$ 矩阵 A 的行换成同序数的列所得到的 $n \times m$ 矩阵称为矩阵 A 的**转置矩阵**, 记作 A^T (或 A') .)

Let $A = [a_{ij}]_{m \times n}$, $A^T = [b_{ij}]_{n \times m}$, then $a_{ij} = b_{ji}$.

For example,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 5 & 8 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 2 & 8 \end{bmatrix};$$

$$B = (18, 6), \quad B^T = \begin{bmatrix} 18 \\ 6 \end{bmatrix}.$$

2、Rules (转置矩阵的运算性质)

Let A , B , A_1, \dots , and A_n denote matrices whose sizes are appropriate for the following sums and products.

$$(1) \quad (A^T)^T = A;$$

$$(2) \quad (A + B)^T = A^T + B^T;$$

$$(A_1 + A_2 + \dots + A_n)^T = A_1^T + A_2^T + \dots + A_n^T;$$

$$(3) \quad (kA)^T = kA^T;$$

$$(4) \quad \underline{(AB)^T = B^T A^T};$$

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

$$(A_1 A_2 \dots A_n)^T = A_n^T \dots A_2^T A_1^T.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 15 \end{bmatrix}, \quad \mathbf{B}^T \mathbf{A}^T = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 7 & 15 \end{bmatrix}.$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T.$$

“Proof”.

$$\begin{aligned} (j, i)\text{-entry of } (\mathbf{AB})^T &= (\mathbf{AB})_{ij} = (\text{row } i \text{ of } \mathbf{A}) \text{ times } (\text{column } j \text{ of } \mathbf{B}) \\ &= [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \end{aligned}$$

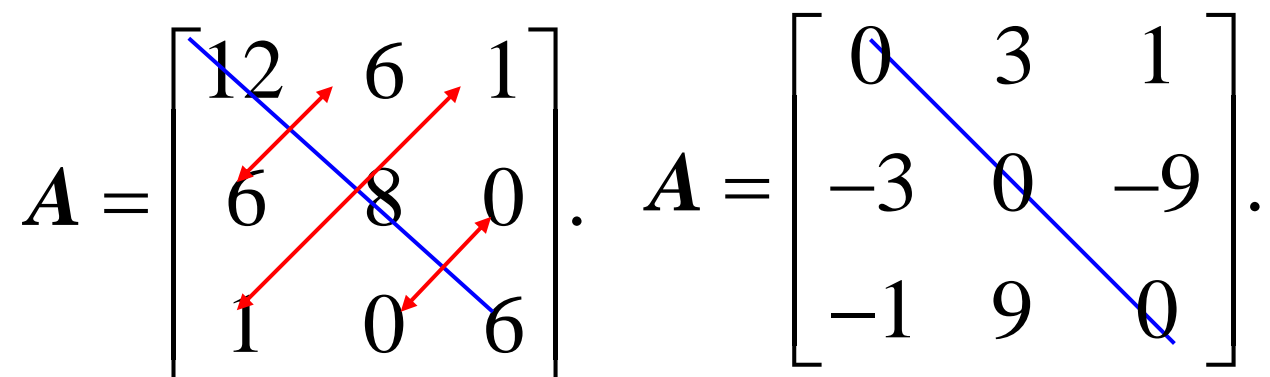
$$\begin{aligned} (j, i)\text{-entry of } \mathbf{B}^T \mathbf{A}^T &= (\text{row } j \text{ of } \mathbf{B}^T) \text{ times } (\text{column } i \text{ of } \mathbf{A}^T) \\ &= [b_{1j} \quad b_{2j} \quad \cdots \quad b_{nj}] \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{nj}a_{in} \end{aligned}$$

对称(symmetric)与反对称(skew-symmetric)

Definition Let A be a square matrix of order n . If $A^T = A$, i.e.,

$$a_{ij} = a_{ji} \quad (i, j = 1, 2, \dots, n),$$

then A is called a **symmetric matrix** (对称矩阵).



$$A = \begin{bmatrix} 12 & 6 & 1 \\ 6 & 8 & 0 \\ 1 & 0 & 6 \end{bmatrix}. \quad A = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -9 \\ -1 & 9 & 0 \end{bmatrix}.$$

If $A^T = -A$, then A is called a **skew-symmetric matrix** (反对称矩阵), i.e.,

$$a_{ij} = -a_{ji} \quad (i, j = 1, 2, \dots, n).$$

For a skew-symmetric matrix A , we have $a_{ii} = 0$.

Example 1 Suppose A, B are n by n matrices, try to verify that $AB^T + BA^T$ is symmetric.

Proof Since

$$\begin{aligned}(AB^T + BA^T)^T &= (AB^T)^T + (BA^T)^T \\ &= (B^T)^T A^T + (A^T)^T B^T \\ &= BA^T + AB^T = AB^T + BA^T,\end{aligned}$$

therefore $AB^T + BA^T$ is symmetric.

Similary, if A is an $m \times n$ matrix, then it is easy to show that $AA^T, A^T A$ are both symmetric matrices.

e.g., $A = [1 \ 2]$. What are AA^T and $A^T A$?

不一定相等，但都是对称的

Inverse

引例 *Coding & Decoding; Encrypting & Decrypting*

A	B	C	D	E	F	G	H	I	J	K	L	M	Space 0
1	2	3	4	5	6	7	8	9	10	11	12	13	
N	O	P	Q	R	S	T	U	V	W	X	Y	Z	
14	15	16	17	18	19	20	21	22	23	24	25	26	

考虑加解密方案：明文信息经编码后分成三个一组（空格也是一种明文信息，不足3个时可加空格）。对明文 $(p_1, p_2, p_3)^T$ ，相应的密文为：

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \mathbf{b}, \quad \text{其中 } A = \begin{bmatrix} 0 & 0 & -2 \\ -1 & -4 & -3 \\ -1 & -3 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}.$$

已知明文DIG,求密文； 已知密文 $(-8, -103, -86)^T$,求明文。

II. Inverse of a Matrix (矩阵的逆)-- Definition

- An $n \times n$ matrix A is said to be **invertible** (可逆的) if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I ,$$

where $I = I_n$, the $n \times n$ identity matrix.

- In this case, C is an **inverse of A** (A 的逆).
- In fact, C is uniquely determined by A , because if B were another inverse of A , then

$$B = BI = B(AC) = (BA)C = IC = C .$$

- This unique inverse is denoted by A^{-1} (' A inverse'), so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I .$$

For example, $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, M = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix},$

Since $AM = MA = I$, M is the inverse of A .

- A matrix that is *not* invertible is also called a **singular matrix (奇异矩阵)**, and an invertible matrix is called a **nonsingular matrix (非奇异矩阵)**.

Not all matrices have inverses. (并非所有矩阵都可逆)

An inverse is impossible when $A\mathbf{x}$ is zero and \mathbf{x} is nonzero.

当 $A\mathbf{x}=0$, $\mathbf{x} \neq 0$ 时, A 一定不可逆。
反证, 两边同乘 A 的逆

Application:

If A is invertible, the *one and only solution* to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

当 A 可逆(即: 非奇异)时, $A\mathbf{x} = \mathbf{0}$ 只有零解 $\mathbf{x} = \mathbf{0}$.

- **Theorem 1** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then

A is invertible and

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

If $a_{11}a_{22} - a_{12}a_{21} = 0$, then A is not invertible.

- The quantity $a_{11}a_{22} - a_{12}a_{21}$ is called the **determinant** (行列式) of A , and we write $\det A = a_{11}a_{22} - a_{12}a_{21}$.
- This theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Proof: (by definition, or 待定系数法)

Let $\mathbf{M} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, which satisfies that $\mathbf{A}\mathbf{M} = \mathbf{I}$,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_3 & a_{11}x_2 + a_{12}x_4 \\ a_{21}x_1 + a_{22}x_3 & a_{21}x_2 + a_{22}x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_3 = 1, \\ a_{21}x_1 + a_{22}x_3 = 0, \\ a_{11}x_2 + a_{12}x_4 = 0, \\ a_{21}x_2 + a_{22}x_4 = 1, \end{cases} \Rightarrow \begin{cases} x_1 = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_2 = \frac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_3 = \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_4 = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}}. \end{cases} \Rightarrow \mathbf{M} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$

We can also check that $\mathbf{A}\mathbf{M} = \mathbf{M}\mathbf{A} = \mathbf{I}$. Therefore

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

求二阶可逆矩阵 A 的逆的“**两调一除**”方法：

先将矩阵 A 的主对角元素互换位置, 再将次对角元素反号, 最后用 $\det A$ 去除 A 的每一个元素.

For example,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{\begin{matrix} 0 & -3 \\ 3 & 2 \end{matrix}} = \begin{bmatrix} 0 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Example 2 A diagonal matrix has an inverse provided no diagonal entries are zero. (主对角元都是非零数的对角阵是可逆的.)

$$\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}^{-1} = \begin{bmatrix} a_1^{-1} & & & \\ & a_2^{-1} & & \\ & & \ddots & \\ & & & a_n^{-1} \end{bmatrix}$$

Note:

If $ab \neq 0$, then $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Note: $A+B$ is not necessarily invertible for invertible matrices A, B .

And we can easily give an example that shows

$$(A+B)^{-1} \neq A^{-1} + B^{-1}, \text{ even if } A+B \text{ is invertible.}$$

For example, $A = \text{diag}(2, -1)$, $B = I_2$, $C = \text{diag}(1, 2)$.

$$A+B = \text{diag}(3, 0) : \text{ not invertible}$$

$$A+C = \text{diag}(3, 1) : \text{ invertible}$$

$$(A+C)^{-1} = \text{diag}(1/3, 1) \neq A^{-1} + C^{-1} = \text{diag}(3/2, -1/2).$$

Example 3 Let a square matrix B be idempotent (幂等, i.e., $B^2=B$), and $A=I+B$. Show that A is invertible, and $A^{-1}=(3I-A)/2$.

Proof By $B=A-I$, $B^2=(A-I)^2=A^2-2A+I$,
and $B^2=B$, we can get

$$A^2-2A+I=A-I,$$

$$A^2-3A=A(A-3I)=-2I,$$

i.e., $A[(3I-A)/2]=I.$

Similarly, $[(3I-A)/2]A=I.$

Therefore, A is invertible, and $A^{-1}=(3I-A)/2$.

III. Inverse -- Properties

- **Theorem 2** If A is an invertible $n \times n$ matrix, then for each \mathbf{b} in \mathbb{R}^n , the equation $\mathbf{A} \mathbf{x} = \mathbf{b}$ has the unique (one and only one) solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

- **Proof:** Take any \mathbf{b} in \mathbb{R}^n .

A solution exists because if $\mathbf{A}^{-1}\mathbf{b}$ is substituted for \mathbf{x} , then $\mathbf{A}\mathbf{x} = \mathbf{A}(\mathbf{A}^{-1}\mathbf{b}) = (\mathbf{A}\mathbf{A}^{-1})\mathbf{b} = \mathbf{I} \mathbf{b} = \mathbf{b}$.

So $\mathbf{A}^{-1}\mathbf{b}$ is a solution.

To prove that the solution is unique, show that if \mathbf{u} is any solution, then \mathbf{u} must be $\mathbf{A}^{-1}\mathbf{b}$.

If $\mathbf{A} \mathbf{u} = \mathbf{b}$, we can multiply both sides by \mathbf{A}^{-1} and obtain $\mathbf{A}^{-1} \mathbf{A} \mathbf{u} = \mathbf{A}^{-1} \mathbf{b}$, $\mathbf{I} \mathbf{u} = \mathbf{A}^{-1} \mathbf{b}$, and $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$.

■ Theorem 3

- a. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$. (若 A 可逆, 则 A^{-1} 亦可逆)
- b. If A and B are $n \times n$ invertible matrices, then so is AB (若 A, B 为同阶可逆方阵, 则 AB 也可逆), and the inverse of AB is the product of the inverses of A and B in the *reverse* order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (A^k)^{-1} = (A^{-1})^k = A^{-k}$$

- c. If A is an invertible matrix, then so is A^T (若 A 可逆, 则 A^T 亦可逆), and the inverse of A^T is the transpose of A^{-1} . That is,

$$(A^T)^{-1} = (A^{-1})^T.$$

- **Proof:** To verify statement (a), find a matrix C such that

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I.$$

These equations are satisfied with A in place of C .

Hence A^{-1} is invertible, and A is its inverse.

- Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$.

- For statement (c), use Theorem of transpose matrix, read from right to left,

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

Similarly, $A^T(A^{-1})^T = I^T = I$.

The inverses of elementary matrices (初等矩阵的逆矩阵)

Recall: (1)初等对换矩阵:

将单位矩阵的第 i, j 行(或列)对换

$$P_{ij} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 & & & \\ & & 0 & \cdots & & 1 & & \\ & & \vdots & & \ddots & \vdots & & \\ & & & & & 1 & & \\ & & 1 & \cdots & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

第 i 列 第 j 列

← 第 i 行
← 第 j 行

(2)初等倍乘矩阵:

将单位矩阵第 i 行(或列)乘 $k \neq 0$

$$D_i(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & k & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{第 } i \text{ 行} \\ \text{第 } i \text{ 列} \end{array}$$

(3)初等倍加矩阵:

将单位矩阵第 i 行乘 k 加到第 j 行,
 或将第 j 列乘 k 加到第 i 列

$$E_{ij}(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & k & \cdots & 1 \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

← 第 i 行

← 第 j 行

第 i 列 第 j 列

The inverses of elementary matrices (初等矩阵的逆矩阵)

变换 $r_i \leftrightarrow r_j$ 的逆变换是其本身, 则

$$\mathbf{P}_{ij}^{-1} = \mathbf{P}_{ij};$$

变换 kr_i ($k \neq 0$) 的逆变换是 $\frac{1}{k}r_i$, 则

$$\mathbf{D}_i^{-1}(k) = \mathbf{D}_i\left(\frac{1}{k}\right);$$

变换 $r_j + kr_i$ 的逆变换是 $r_j + (-k)r_i$, 则

$$\mathbf{E}_{ij}^{-1}(k) = \mathbf{E}_{ij}(-k).$$

初等矩阵的逆矩阵仍为同类型的初等矩阵.

Each elementary matrix \mathbf{E} is invertible. The inverse of \mathbf{E} is the elementary matrix of the same type that transforms \mathbf{E} back into \mathbf{I} .

IV. Algorithm (初等变换法求逆矩阵)

定理4 可逆矩阵可以经过若干次初等变换化为单位矩阵.

析 任何矩阵 A , 都可经初等行变换将其化为行简化阶梯形矩阵.

任何方阵 A , 都可经初等行变换将其化为上三角形矩阵.

任何可逆矩阵 A , 都可经初等行变换将其化为单位矩阵 I .

即 $P_s \dots P_2 P_1 A = I$. (P_1, \dots, P_s 均为初等矩阵)

Hint: Suppose that A is invertible.

Then, since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} (Theorem 2), A has a pivot position in every row.

Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n .

Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices P_1, \dots, P_s such that $P_s \dots P_2 P_1 A = I$.

由 $\underline{P_s \cdots P_2 P_1} A = I$ 得 $A = P_1^{-1} P_2^{-1} \cdots P_s^{-1} I$ 和

$$A^{-1} = P_s \cdots P_2 P_1 = \underline{P_s \cdots P_2 P_1} I$$

初等矩阵的逆矩阵
仍然是初等矩阵

(A^{-1} results from applying P_1, \dots, P_s successively to I .)

This is the same sequence that reduced A to I .

- 结论:** (1) 可逆矩阵可以表示为若干初等矩阵的乘积;
(2) 对 A 作若干初等变换, 将 A 化为单位矩阵 I 时,
同样的这些初等变换将单位矩阵 I 化为 A^{-1} .

$$P_s \cdots P_2 P_1 [A \quad I] = [I \quad A^{-1}]$$

Row reduce the matrix $[A \quad I]$. If A is row equivalent to I , then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$.

Otherwise, A does not have an inverse.

用初等行变换求 A 的逆矩阵

(the Gauss-Jordan method for calculating A^{-1})

即对 $n \times 2n$ 矩阵 $[A \ I_n]$ 实施一系列初等行变换，把矩阵 A 变成 I_n 时，原来的 I_n 就变成了 A^{-1} 。

$$[A, I_n] \xrightarrow{\text{ERO}} [I_n, A^{-1}]$$

- **Theorem 5** An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n , and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1} .

Example 4 Write the matrix A as the product of elementary matrices, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution The matrix A can be obtained from the 3 by 3 identity matrix I by 4 elementary operations

$$r_2 \leftrightarrow r_3, \quad c_1 + 2c_3, \quad (-1)r_3, \quad (-1)c_3$$

therefore $A = P_3 P_1 I P_2 P_4 = P_3 P_1 P_2 P_4,$

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Example 5 Use ERO to find the inverse of $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.

Solution

$$[A, I] = \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r1 \leftrightarrow r2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{r3+r1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} r1+r3 \times (-2) \\ r2+r3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} r2 \times \frac{1}{2} \\ r1+r2 \times (-1) \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}.$$

Remark: 1. *Can we use elementary **column** operations on A to find its inverse?*

也可以对 $\begin{bmatrix} A \\ I \end{bmatrix}$ 施行初等**列**变换, 当 A 变成单位矩阵时, I 被化为了 A^{-1} .



Remark: 2. *Can we use elementary operations to solve system of linear equations?*

初等行变换求逆矩阵的方法, 还可用于求矩阵 $A^{-1}b$.

$$A^{-1}[A \quad b] = [I \quad A^{-1}b]$$

$[A \quad b] \xrightarrow{\text{(ERO)}} [I \quad x]$



$[A \quad b]$



I



$A^{-1}b$

初等**行**变换

Example 6 Find the matrix X , such that $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}.$$

Solution If A is invertible, then $X = A^{-1}B$.

$$[A \quad B] = \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 4 & 3 & 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & -2 & -6 & -2 & -12 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & -2 & -6 & -2 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & -4 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & 0 & -1 & -1 & -3 \end{bmatrix}$$

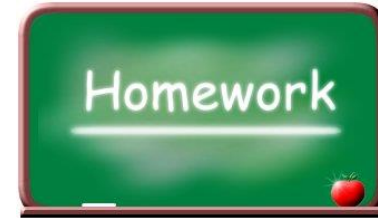
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & -2 & 0 & 4 & 6 \\ 0 & 0 & -1 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix},$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 3 & 2 \\ -2 & -3 \\ 1 & 3 \end{bmatrix}.$$

What if -- we want
to solve $\mathbf{XA}=\mathbf{B}$ for \mathbf{X} ?

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{I} \\ \mathbf{x} \end{bmatrix}$$

Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

Deadline (DDL):

- Next tutorial class

