Step-1

Consider the following four fundamental subspaces:

The row space is $C(A^T)$, a subspace of $\mathbb{R}^n$.

The column space $C(A)$, a subspace of $\mathbb{R}^m$.

The nullspace is $N(A)$, a subspace of $\mathbb{R}^n$.

The left nullspace is $N(A^T)$, a subspace of $\mathbb{R}^m$.

Find the system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to which of the given four fundamental subspaces.

Step-2

Suppose the system $A\mathbf{x} = \mathbf{b}$ has a solution.

Then the vector $\mathbf{b}$ is lie in the column space $C(A)$.

Thus, $\mathbf{b}$ is orthogonal to the left nullspace $N(A^T)$.

Hence, if the system $A\mathbf{x} = \mathbf{b}$ has a solution, then the vector $\mathbf{b}$ is orthogonal to $N(A^T)$.

Step-3

Conversely,

Suppose the vector $\mathbf{b}$ is orthogonal to the left nullspace $N(A^T)$.

Then the vector $\mathbf{b}$ is lie in the column space $C(A)$.

Thus, the system $A\mathbf{x} = \mathbf{b}$ has a solution.

Hence, if the vector $\mathbf{b}$ is orthogonal to $N(A^T)$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution.