SUSTC

Solutions for Final of Calculus II in Spring Semester, 2018

1. (15 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

   (1) If $f(x, y)$ has both partial derivatives $f_x(x, y)$, $f_y(x, y)$ at point (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

   (2) The curvature of a circle is the radius of the circle.

   (3) If both $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ converge, then $\sum_{n=1}^{+\infty} a_n b_n$ must also converge.

   (4) Let $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + xy\mathbf{k}$ represent the velocity of a gas flowing in space. The gas is neither expanding nor compressing at any point.

   (5) If $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is defined on an open region, and its component functions have continuous first partial derivatives and satisfy
   $$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
   Then $\mathbf{F}$ is conservative.

   Solution: (1) F; (2) F; (3) F; (4) T; (5) F.

2. (12 pts) Please fill in the blank for the questions below.

   (1) If $\mathbf{r}$ is a differentiable vector function of t of constant length, then $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} =$ ______________.

   (2) The direction in which $f(x, y) = x^2 y + e^{xy} \sin y$ decreases most rapidly at the point $(1, 0)$ is ______________.

   (3) The equation for the tangent plane at the point $(1, -1, 3)$ on the surface $x^2 + 2xy - y^2 + z^2 = 7$ is ______________.

   (4) Suppose that $f(x, y)$ and its first and second partial derivatives are continuous, and $f(0, 0) = 1$, $f_x(0, 0) = 2$, $f_y(0, 0) = 3$, $f_{xx}(0, 0) = 2$, $f_{xy}(0, 0) = -1$, $f_{yy}(0, 0) = 4$. Then $f(x, y) \approx$ ______________ when both x and y are small (using Taylor's formula for $f(x, y)$ at $(0, 0)$) to find the quadratic approximation of f.

   Solution: (1) 0; (2) $(0, -1)$; (3) $2y + 3z = 7$; (4) $1 + 2x + 3y + x^2 - xy + 2y^2$.

3. (3pts) Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$, $f_{xx}(a, b) = -2$, $f_{xy}(a, b) = 1$, $f_{yy}(a, b) = 2$. Then

(A) f has a local maximum at (a, b); (B) f has a local minimum at (a, b);
(C) f has a saddle point at (a, b); (D) the test is inconclusive.

Solution: C.

4. (20 pts) Which of the following series converge absolutely, which converge conditionally, and which diverge? Give reasons for your answer.

$$(1) \sum_{n=1}^{+\infty} (-1)^n \frac{1}{\sqrt{n(n+1)}}; \quad (2) \sum_{n=2}^{+\infty} (-1)^n \frac{1}{n(\ln n)^3};$$
$$(3) \sum_{n=1}^{+\infty} (-1)^n \frac{n^2+1}{2n^2+n-1}; \quad (4) \sum_{n=1}^{+\infty} \frac{(-3)^n}{n!}.$$

Solution:

(1) Converge conditionally. Alternating series test + Comparison test.
(2) Converge absolutely. Integral test.
(3) Diverge. The nth term test.
(4) Converge absolutely. Ratio test.

5. (10 pts) Find the Maclaurin series for the function $f(x) = \frac{1}{(2-x)^2}$.

Solution:

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{+\infty} \frac{x^n}{2^n}$$
$$\frac{1}{(2-x)^2} = \left(\frac{1}{2-x}\right)' = \sum_{n=0}^{+\infty} \frac{(n+1)x^n}{2^{n+2}}.$$

6. (10 pts) Find the length of the astroid

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi.$$

Solution:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3|\cos t \sin t|.$$
$$4\int_0^{\frac{\pi}{2}} 3\cos t \sin t \, dt = 6.$$

7. (10 pts) Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$. Show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = (f_x)^2 + (f_y)^2.$$

Solution:

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= \cos\theta f_x + \sin\theta f_y\end{aligned}$$
$$\frac{\partial w}{\partial \theta} = -r\sin\theta f_x + r\cos\theta f_y.$$

8. (10 pts) Find the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$, and the curvature κ for the plane curve

$$\mathbf{r}(t) = (2t+3)\mathbf{i} + (5-t^2)\mathbf{j}.$$

Solution:

$$\begin{aligned}
\mathbf{v}(t) &= (2, -2t) \\
|\mathbf{v}(t)| &= 2\sqrt{1+t^2} \\
\mathbf{T}(t) &= (\frac{1}{\sqrt{1+t^2}}, \frac{-t}{\sqrt{1+t^2}}) \\
\frac{d\mathbf{T}}{dt} &= \left(\frac{-t}{(1+t^2)^{\frac{3}{2}}}, \frac{-1}{(1+t^2)^{\frac{3}{2}}}\right) \\
\left|\frac{d\mathbf{T}}{dt}\right| &= \frac{1}{1+t^2} \\
\mathbf{N}(t) &= \left(\frac{-t}{\sqrt{1+t^2}}, \frac{-1}{\sqrt{1+t^2}}\right) \\
\kappa(t) &= \frac{1}{2(1+t^2)^{\frac{3}{2}}}
\end{aligned}$$

9. (15 pts) Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(1) Show that $f(x,y)$ is continuous at $(0,0)$.

(2) Compute $f_y(0,0)$.

(3) Compute $f_{yx}(0,0)$.

Solution:

(1) Because

$$\lim_{(x,y)\to(0,0)} \frac{xy \cdot x^2}{x^2+y^2} = 0,$$

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0.$$

(2)

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

(3) When $(x,y) \neq (0,0)$,

$$f_y(x,y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2}.$$

$$f_{yx}(0,0) \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1.$$

10. (10 pts) Use the Lagrange multipliers to find the minimal and maximal value of $f(x, y, z) = x^4 + y^4 + z^4$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

Solution: Use the Lagrange Multiplier, we have $\nabla f = \lambda \nabla g$, i.e.,

$$\begin{aligned} 4x^3 &= 2\lambda x \\ 4y^3 &= 2\lambda y \\ 4z^3 &= 2\lambda z \end{aligned}$$

If $x, y, z \neq 0$, we have $x^2 = y^2 = z^2 = 1/3$, $f(x, y, z) = 1/3$.

If there are one 0 in x, y, z. Without lost of generality, let's say $x = 0$. Then $y^2 = z^2 = 1/2$, $f(x, y, z) = 1/2$.

If there are two 0s in x, y, z, let's say $x = y = 0$, then $z^2 = 1$, $f(x, y, z) = 1$.

Therefore, the minimal value is $1/3$ and the maximal value is 1.

One can also use elementary inequality to show this results. We have $1 = (x^2+y^2+z^2)^2 \geq (x^4+y^4+z^4)$ and $(x^4+y^4+z^4)(1+1+1) \geq (x^2+y^2+z^2)^2 = 1$ (Cauchy inequality). We get $1 \geq x^4 + y^4 + z^4 \geq 1/3$ as before.

11. (10 pts) Consider

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx.$$

(1) Sketch the region of integration.

(2) Reverse the order of integration, and evaluate the integral.

Solution:

$$\begin{aligned} \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dxdy \\ &= \int_0^4 \frac{1}{2} e^{2y}\, dy \\ &= \frac{1}{4}\left(e^8 - 1\right). \end{aligned}$$

12. (10 pts) Set up a triple integral in spherical coordinates that gives the volume of the solid bounded below by the xy-plane, on the sides by the sphere $x^2 + y^2 + z^2 = 4$, and above by the cone $z = \sqrt{x^2 + y^2}$, and then evaluate the integral.

Solution:

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\varphi\, d\rho d\varphi d\theta = \frac{8\sqrt{2}\pi}{3}$$

13. (10 pts) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the **substitution in double integral** (please find the transformation by yourself) to evaluate the integral

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy.$$

Solution: Use the transformation

$$u = \sqrt{xy}, \qquad v = \sqrt{\frac{y}{x}}.$$

we have

$$x = \frac{u}{v}, \qquad y = uv.$$

Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{2u}{v}.$$

Therefore

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy = \int_1^2 \int_1^3 2(u + \frac{u^2}{v})\, dudv = \frac{52}{3}\ln 2 + 8.$$

14. (10 pts) Find the mass of a thin wire that lies along the curve

$$\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, \quad 0 \le t \le 2,$$

if the density is $\delta(x, y, z) = 3\sqrt{25 + x + 2y}$.

Solution:

$$\mathbf{v} = (1, 2, \sqrt{t})$$

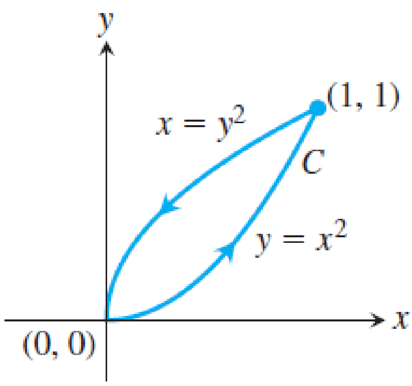
$$|\mathbf{v}| = \sqrt{5 + t}$$

$$\mathbf{M} = \int_0^2 3\sqrt{5}(5 + t)\, dt = 36\sqrt{5}$$

15. (10 pts) Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

$$\mathbf{F} = (xy + y^2)\mathbf{i} + (x - y)\mathbf{j}\,;$$

where C is shown in the figure below.



Solution: The counterclockwise circulation is

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\, dydx = -\frac{7}{60}.$$

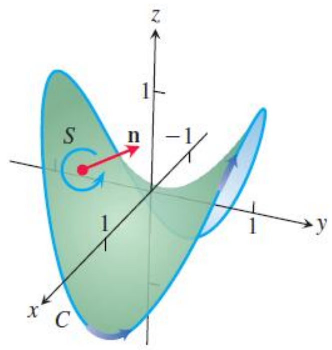
The outward flux is

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (y - 1)\, dydx = -\frac{11}{60}.$$

16. (10 pts) The surface S is formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the right circular cylinder of radius one around the z-axis. Let C be the boundary curve of S (see the figure below). Calculate

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$

where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$, and $\mathbf{n}$ is the unit normal vector of the surface S.



Solution:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos 2t)\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t, 2\sin 2t)$$

$$\mathbf{F} = (\sin t, -\cos t, \cos^2 t)$$

$$\int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + 2\sin 2t \cos^2 t\right) \, dt = -2\pi.$$

17. (15 pts) Consider the line integral

$$\int_{(1,1,1)}^{(1,2,3)} 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz \, .$$

(1) Show that the differential form in the integral is exact;

(2) Find a scalar function f such that $df = 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz$;

(3) Evaluate the integral.

Solution:

(1) Prove that it satisfies the component test or $\nabla \times \mathbf{F} = \mathbf{0}$.

(2) $x^3 + z^2 \ln y$.

(3) $9 \ln 2$.

18. (10 pts) Use the Divergence Theorem to find the outward flux of

$$\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$$

across the **boundary** of the solid sphere $D: \; x^2 + y^2 + z^2 \leq 4$.

Solution:

$$\nabla \cdot \mathbf{F} = 2x + 3$$

$$\iiint_D \nabla \cdot \mathbf{F} \, dv = 32\pi.$$