## Step-1

**(a)**

The square matrix $A$ has independent columns.

The pseudo-inverse of $A$ with independent columns is $A^+ = \left(A^T A\right)^{-1} A^T$

It is a left inverse of $A$ as follows:

$$\begin{aligned} A^+ A &= \left(A^T A\right)^{-1} A^T A \\ &= \left(A^T A\right)^{-1} \left(A^T A\right) \\ &= I \end{aligned}$$

Further,

$$\begin{aligned} A^+ &= \left(A^T A\right)^{-1} A^T \\ &= A^{-1} A^{-T} A^T \\ &= A^{-1} I \\ &= A^{-1} \end{aligned}$$

Thus, the left inverse $\left(A^T A\right)^{-1} A^T$ is $A^+$.

## Step-2

As the square matrix $A$ has independent columns, the columns of $A$ span the entire vector space $\mathbf{R}^n$.

But for a matrix, its row rank is equal to its column rank.

Therefore, the rows of $A$ too span the entire vector space $\mathbf{R}^n$.

This indicates that $x^+ = A^+ b$ is in the row space.

Now verify the following:

$$\begin{aligned} \left(A^T A\right) x^+ &= \left(A^T A\right)\left(A^+ b\right) \\ &= \left(A^T A\right)\left[\left(A^T A\right)^{-1} A^T\right] b \\ &= \left(A^T A\right)\left[A^{-1} \left(A^T\right)^{-1} A^T\right] b \end{aligned}$$

$$= A^{\mathrm{T}}\left(AA^{-1}\right)\left[\left(A^{\mathrm{T}}\right)^{-1}A^{\mathrm{T}}\right]b$$
$$= A^{\mathrm{T}}\cdot I\cdot I\cdot b$$
$$= A^{\mathrm{T}}b$$

Thus, $A^{\mathrm{T}}Ax^{+} = A^{\mathrm{T}}b$.

## Step-3

**(b)**

The square matrix $A$ has independent rows.

Therefore, the rows of $A$ span the entire vector space $\mathbf{R}^n$.

In this case, the right inverse of $A$ is given by,

$A^{+} = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$ This can be verified as follows:

$$\begin{aligned} AA^{+} &= AA^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1} \\ &= \left(AA^{\mathrm{T}}\right)\left(AA^{\mathrm{T}}\right)^{-1} \\ &= I \end{aligned}$$

Further,

$$\begin{aligned} A^{+} &= A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1} \\ &= A^{\mathrm{T}}\left(A^{-T}A^{-1}\right) \\ &= \left(A^{\mathrm{T}}A^{-T}\right)A^{-1} \\ &= IA^{-1} \\ &= A^{-1} \end{aligned}$$

## Step-4

Let $x^{+} = A^{+}b$, then $x^{+} = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}b$.

Obviously, $x^{+}$ is in the row space, because $A^{\mathrm{T}}$ multiplied by any vector has to be in the row space.

Now verify the following:

$$
\begin{aligned}
A^T A x^+ &= \left(A^T A\right)\left[A^T\left(AA^T\right)^{-1} b\right] \\
&= \left(A^T A\right)\left[A^T \cdot \left(A^T\right)^{-1} A^{-1}\right] b \\
&= \left(A^T A\right)\left[A^T \cdot \left(A^T\right)^{-1}\right] A^{-1} b
\end{aligned}
$$

$$
\begin{aligned}
&= \left(A^T A\right) \cdot I \cdot A^{-1} b \\
&= A^T \left(AA^{-1}\right) b \\
&= A^T \cdot I \cdot b \\
&= A^T b
\end{aligned}
$$

Thus, $A^{\mathrm{T}} A x^+ = A^{\mathrm{T}} b$