Suggested Solutions of Homework 1 MA327

Ex.1. α(t) = (sin 2πt, cos 2πt), t $\in$ [0, 1]. (You may also write down other forms of α.)

Ex.2. Let $l(t)$ be the distance from $\alpha(t)$to origin, so

$$l^2(t) = \alpha(t) \cdot \alpha(t).$$

Since $l(t)$ gets its minimum only if $l'(t) = 0$,

$$l'(t) = \frac{\alpha(t) \cdot \alpha'(t)}{l(t)}$$

$$\Longrightarrow \quad l'(t_0) = \frac{\alpha(t_0) \cdot \alpha'(t_0)}{l(t_0)} = 0.$$

Hence, $\alpha(t_0) \cdot \alpha'(t_0) = 0$ and the position vector $\alpha(t_0)$ is orthogonal to $\alpha(t_0)$.

Ex.3.
Necessity: Since $\alpha(t) \cdot \alpha(t) = |\alpha(t)|^2 = \text{Const}$, $\alpha(t) \cdot \alpha'(t) = 0$. Thus, $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.
Sufficiency: $|\alpha(t)|^2$ is the integral of $2\alpha(t)\cdot\alpha'(t)$, which is zero, so $|\alpha(t)|^2$ and $|\alpha(t)|$ is constant. Moreover, since $\alpha'(t) \neq 0$, $\alpha(t)$ is nonzero.

Ex.4. It is easy to verify that $\mathbf{n} = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$ is the unit vector which lays on the line which is the intersection of two planes $y = 0$ and $z = x$. Then $\alpha'(t) = (3, 6t, 6t^2)$ and

$$\frac{\alpha'(t) \cdot \mathbf{n}}{|\alpha'(t)|} = \frac{\sqrt{2}}{2},$$

which is constant. Thus, the tangent lines to $\alpha(t)$ make a constant angle with the line which is the intersection of two planes $y = 0$ and $z = x$.

Ex.5.

(a)

$$\alpha(t) = (t - \sin t, 1 - \cos t), t \in \mathbb{R}$$

$$\alpha'(t) = (1 - \cos t, \sin t) = 0 \Leftrightarrow t = 2\pi k, k \in \mathbb{Z}$$

Thus, (2πk, 0), k $\in \mathbb{Z}$ are the singular points of $\alpha(t)$.

(b) According to arc length formula:

$$s(t) = \int_{t_0}^{t} |\alpha'(\tau)| d\tau$$

where $|\alpha'(\tau)| = \sqrt{(1-\cos\tau)^2 + (\sin\tau)^2} = \sqrt{2 - 2\cos\tau}$.

In this question, a complete rotation of the disk means that $t_0 = 0$ and $t = 2\pi$, so the arc length of the cycloid corresponding to a complete rotation of the disk should be

$$s(2\pi) = 8.$$

Ex.6. (a) Calculate the derivative of $\alpha(t)$

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\sin t}), \quad t \in (0, \pi)$$

If $t \neq \pi/2$, then $\cos t \neq 0 \Longrightarrow \alpha'(t) \neq 0$. If $t = \pi/2$, then $\alpha'(t) = 0$. In conclusion, α is a differentiable parametrized curve, regular except at $t = \pi/2$.

(b) Let t be the parameter of α. When $t \neq \pi/2$, there exists the tangent of α. Let l_t be the the tangent of α at t, and s be the parameter of l_t:

$$\begin{aligned} l_t(s) &= \alpha(t) + s\alpha'(t) \\ &= (\sin t, \cos t + \log\tan\frac{t}{2}) + s(\cos t, -\sin t + \frac{1}{\sin t}) \\ &= (\sin t + s\cos t, \cos t + \log\tan\frac{t}{2} - s\cos t + \frac{s}{\sin t}) \end{aligned}$$

The intersection of l_t and y-axis is $(0, \cos t + \log\tan\frac{t}{2} - \cos t)$, denoted by $\mathbf{p}$. The distance d from $l_t(0)$ to $\mathbf{p}$ is

$$d = |l_t(0) - \mathbf{p}| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

As a result, the length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Ex.7.

(a) It is easy to verify that in polar coordinates $\alpha(t) = (ae^{bt}, t), \quad t \in \mathbb{R}$.

(b) Through simple calculation we know that

$$\begin{aligned} \alpha'(t) &= (abe^{bt}\cos t - ae^{bt}\sin t, abe^{bt}\sin t + ae^{bt}\cos t) \\ |\alpha'(t)| &= ae^{bt}(b^2+1)^{1/2} \end{aligned}$$

Because $a > 0, b < 0$, $\alpha'(t) \to 0 \quad as \quad t \to +\infty$ and that

$$\lim_{t\to+\infty} \int_{t_0}^{t} |\alpha'(\tau)| d\tau = -b^{-1}a\sqrt{b^2+1}e^{bt_0}$$

, which is finite.

Ex.8.

(a)

$$t(s) = \alpha'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}) \Longrightarrow |\alpha'(s)| = 1$$

$$\Longrightarrow \qquad \text{s is the arc length.}$$

(b) The curvature is a/c^2 and the torsion is $-b/c^2$. They can be calculated as follows:

$$\begin{aligned}
&\alpha''(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0)\\
\Longrightarrow\quad &k(s) = |\alpha''(s)| = \frac{a}{c^2}, \quad n = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)\\
\Longrightarrow\quad &b(s) = t(s)\wedge n(s) = (\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c})\\
\Longrightarrow\quad &b'(s) = (\frac{b}{c^2}\cos\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0) = -\frac{b}{c^2}n(s)\\
\Longrightarrow\quad &\tau(s) = -\frac{b}{c^2}
\end{aligned}$$

(c) p = (x, y, z) is in the osculation plane of α if and only if $(p-\alpha(s))\cdot b(s) = 0$. This plane should be

$$(\frac{b}{a}\sin\frac{s}{c})x - (\frac{b}{a}cos\frac{s}{c})y + z = \frac{b}{c}s$$

(d) $n(s)\cdot(0,0,1) = 0$

(e) $t(s)\cdot(0,0,1) = \frac{b}{c}$, which is constant.

Ex.9. Remind that $b' = \tau(s)n(s)$, we can get the formula of τ by taking the inner product of b' and n. Moreover, $t = \alpha'$, $n = \alpha''/k(s)$ then

$$\begin{aligned}
n' &= \frac{\alpha'''k(s) - \alpha''k'(s)}{k^2(s)}\\
b &= t\wedge n = \alpha'\wedge\alpha''/k(s)\\
b' = t\wedge n' &= \frac{\alpha'\wedge\alpha'''k(s) - \alpha'\wedge\alpha''k'(s)}{k^2(s)}
\end{aligned}$$

Finally

$$\begin{aligned}
\tau(s) &= b'\cdot n\\
&= (\alpha'\wedge\alpha'''/k(s))\cdot n\\
&= \frac{1}{k(s)}(\alpha'\wedge\alpha''')\cdot(b\wedge\alpha')\\
&= \frac{1}{k(s)}((\alpha'\cdot b)(\alpha'''\cdot\alpha) - (\alpha'''\cdot b)(\alpha'\cdot\alpha'))\\
&= -\frac{b\cdot\alpha'''}{k(s)}\\
&= -\frac{(\alpha'\wedge\alpha'')\cdot\alpha'''}{k^2(s)}
\end{aligned}$$

Ex.10.

(a) (i) Take arbitrary vector u, the norm of ρu satisfies

$$|\rho u| = \sqrt{\rho u\cdot\rho u} = \sqrt{u\cdot u} = |u|$$

.

(ii) Let θ be the angle of vectors u, v and ϕ be the angle of vectors $\rho u, \rho v$. Then

$$\cos\phi = \frac{\rho u\cdot\rho v}{|\rho u||\rho v|} = \frac{u\cdot v}{|u||v|} = \cos\theta.$$

Since θ and ϕ are in $[0, \pi]$, we have $\phi = \theta$.

(b) Take arbitrary vector u, v, w, the vector product of ρu and ρv satisfies $(\rho u \wedge \rho v) \cdot (\rho w) = \det(\rho u, \rho v, \rho w) = \det(\rho(u, v, w)) = \det(\rho) * \det(u, v, w)$. Remind that $\det(\rho) = 1$, so $(\rho u \wedge \rho v) \cdot (\rho w) = \det(u, v, w) = (u \wedge v) \cdot w = \rho(u \wedge v) \cdot \rho w$. Because w is arbitrary, $\rho u \wedge \rho v = \rho(u \wedge v)$.

The assertion is false if we drop the condition on the determinant.

(c) Define $\beta(t) = \rho\alpha(t) + v$. Let $\tilde{s}$, $\tilde{k}$ and $\tilde{\tau}$ be the arc length, curvature and torsion of β in respective and let s, k and τ be the arc length, curvature and torsion of α in respective . Then

$$\tilde{s}(t) = \int |\beta'(t)| = \int |\rho\alpha'(t)| = \int |\alpha'(t)| = s(t).$$

If α is parametrized by arc length, then so is β.

$$\tilde{k}(s) = |\beta''(s)| = |\rho\alpha''(s)| = |\alpha''(s)| = k(s).$$

$$\begin{aligned}\tilde{\tau}(s) &= -\frac{(\beta' \wedge \beta'') \cdot \beta'''}{\tilde{k}^2} \\ &= -\frac{(\rho\beta' \wedge \rho\beta'') \cdot \rho\beta'''}{\tilde{k}^2} \\ &= -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{k^2} \\ &= \tau(s)\end{aligned}$$

Consequently, the arc length, the curvature, and the torsion of a parametrized curve are invariant rigid motions.

Ex.11. (**One may also see a solution given in Lecture 3.**)

(a) Remind that

$$s(t) = \int_{t_0}^{t} |\alpha'(\tau)| d\tau,$$

so $\frac{ds}{dt} = |\alpha'(t)|$ and $\frac{dt}{ds} = \frac{1}{|\alpha'|}$. Second,

$$\begin{aligned}\frac{d^2t}{ds^2} &= \frac{d}{ds}(\frac{dt}{ds}) \\ &= \frac{dt}{ds} \cdot \frac{d}{dt}(\frac{dt}{ds}) \\ &= -\frac{|\alpha'|'}{|\alpha'|^3}\end{aligned}$$

Because $|\alpha'|' = \frac{\alpha'' \cdot \alpha'}{|\alpha'|}$,

$$\frac{d^2t}{ds^2} = -\frac{\alpha' \cdot \alpha''}{|\alpha'|^4}.$$

Notation: the subscripts of k, n, b represents the parameters of curve. For instance, k_β means the curvature of curve in arc length parameter.

(b) In above notation, we want to calculate k_α.

$$\alpha'(t) = \beta'(s(t)) = \beta'(s(t)) \cdot \frac{ds}{dt} \tag{1}$$

$$\Longrightarrow \quad \alpha''(t) = (\beta'(s(t)) \cdot \frac{ds}{dt})' \tag{2}$$

$$= k_\beta(s(t))(\frac{ds}{dt})^2 n_\beta(s(t)) + \frac{d^2s}{dt^2}\beta'(s(t)) \tag{3}$$

Combine (1) and (3), we have

$$\begin{aligned}
\alpha' \wedge \alpha'' &= k_\alpha(t)(\frac{ds}{dt})^3 b_\beta(s(t)) \\
\implies \qquad |\alpha' \wedge \alpha''| &= k_\alpha(t)(\frac{ds}{dt})^3 \\
\implies \qquad k_\alpha(t) &= \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}
\end{aligned}$$

(c) In above notation, we want to calculate τ_α. By **Ex 9**, we know that

$$\begin{aligned}
\tau_\alpha(t(s)) &= \tau_\beta(s) \\
&= -\frac{(\beta'(s) \wedge \beta''(s)) \cdot \beta'''(s)}{k_\beta^2(s)} \\
&= -\frac{((\alpha \circ t)'(s) \wedge (\alpha \circ t)''(s)) \cdot (\alpha \circ t)'''(s)}{(k_\alpha \circ t)^2(s)} \\
&= -\frac{(\alpha'(t(s))t'(s) \wedge \alpha''(t(s))(t'(s))^2) \cdot \alpha'''(t(s))(t'(s))^3}{\frac{|\alpha'(t(s)) \wedge \alpha''(t(s))|^2}{|\alpha'(t(s))|^6}} \\
&= -\frac{(\alpha'(t(s)) \wedge \alpha''(t(s))) \cdot \alpha'''(t(s))}{|\alpha'(t(s)) \wedge \alpha''(t(s))|^2}
\end{aligned}$$

Because t is surjective by t=t(s), we have

$$\tau_\alpha(t) = -\frac{(\alpha'(t) \wedge \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \wedge \alpha''(t)|^2}$$

(d) Recall the definition of signed curvature of a curve parametrized by arc length first. Let $\beta(s) = (x(s), y(s))$ be a plane curve parametrized by arc length s. Then the signed curvature of β,

$$k_\beta(s) := \ddot{\beta} \cdot n,$$

where $\dot{\beta}$ denotes the derivative of β w.r.t. s and n is the normal vector such that $\{t, n\}$ and $\{e_1, e_2\}$ has the same orientation. It just means that $n = (-\dot{y}, \dot{x})$. Therefore, $k_\beta(s) = -\ddot{x}\dot{y} + \ddot{y}\dot{x}$.

Now let $\alpha(t) = (x(t), y(t))$ be a curve which may not be parametrized by arc length and $\beta(s)$ be its reparametrization by arc length s. Then

$$\ddot{\beta} = \alpha'' \cdot (\frac{dt}{ds})^2 + \alpha' \cdot \frac{d^2t}{dt^2}.$$

Here α' denotes the derivative of α w.r.t. t. Now $n = \frac{dt}{ds}(-y', x')$. Thus, we have

$$k_\alpha = (\alpha'' \cdot (\frac{dt}{ds})^2 + \alpha' \cdot \frac{d^2t}{dt^2}) \cdot \frac{dt}{ds}(-y', x') = (\frac{dt}{ds})^3 \alpha'' \cdot (-y', x') = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}}.$$

Ex.12.

Necessity: Let t, n, b be the Fernet's coordinate of $\alpha(s)$. $\alpha(I)$ lies on a sphere means that there exists a point p, such that $(\alpha(s) - p)^2 = Const.$. Differentiate three times of that equation:

$$(\alpha(s) - p) \cdot t(s) = 0 \tag{4}$$
$$1 + ((\alpha(s) - p) \cdot \alpha''(s) = 0 \tag{5}$$
$$(\alpha(s) - p) \cdot \alpha'''(s) = 0 \tag{6}$$

In fact, $\alpha(s) - p$ can be expressed by the Fernet's coordinate, $\alpha(s) - p = \phi(s)t(s) + \zeta(s)n(s) + \lambda(s)b(s)$. $\phi(s) = 0$ by Eq.(4) and $\zeta(s) = -R$ by Eq.(5). As for $\lambda(s)$, remind that $\alpha''' = k'n - k^2 t - k\tau b$, so $\lambda(s) = R'T$.

$\Longrightarrow \qquad \alpha(s) - p = -Rn(s) + R'Tb(s)$. Hence $R^2 + (R')^2 T^2 = constant$.

Sufficiency: Define $\beta(s) = \alpha(s) + Rn(s) - R'Tb(s)$ and it suffices to prove $\beta(s)$ is a constant vector. Differentiate $\beta(s)$:

$$\beta'(s) = -(\tau R + (R'T)')b$$

Differentiate $R^2 + (R')^2 T^2 = constant$:

$$\begin{aligned} RR' + R'T(R'T)' &= 0 \\ \Longrightarrow \qquad \tau R + (R'T)' &= 0 \end{aligned}$$

Thus, $\beta'(s) = 0$ and $\beta(s)$ is constant. The proof has finished.