MA215 Probability Theory

Assignment 01

1. Provide a strict proof for the following set relations.

   (1) $B \backslash A = B \cap A^c$;

   (2) $(A \backslash B) \cap C = (A \cap C) \backslash (B \cap C)$;

   (3) $(\bigcup_{k=1}^{\infty} A_k)^c = \bigcap_{k=1}^{\infty} A_k^c$;

   (4) $(\bigcap_{k=1}^{\infty} A_k)^c = \bigcup_{k=1}^{\infty} A_k^c$;

   (5) $A \cup (\bigcap_{k=1}^{\infty} B_k) = \bigcap_{k=1}^{\infty} (A \cup B_k)$;

   (6) $A \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} (A \cap B_k)$.

   As the generalizations of (3) to (6) we have the following general De Morgan's Laws and Distributive laws: For any index set I, we have

   (7) $(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} (A_i)^c$

   (8) $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} (A_i)^c$

   (9) $A \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A \cup B_i)$;

   (10) $A \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} (A \cap B_i)$.

2. A sequence of sets $\{A_1, A_2, \ldots, A_n, \ldots\}$ is called increasing if

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots .$$

   Similarly, a sequence of sets $\{A_1, A_2, \ldots, A_n, \ldots\}$ is called decreasing if

$$A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots .$$

   Show that

   (i) If $\{A_n; n \geqslant 1\}$ is an increasing set sequence, then for any $n \geqslant 1$,

$$\bigcup_{k=1}^{n} A_k = A_n, \quad \text{and} \quad \lim_{n \to \infty} A_n = \bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n.$$

   (ii) If $\{A_n; n \geqslant 1\}$ is a decreasing set sequence, then for any $n \geqslant 1$,

$$\bigcap_{k=1}^{n} A_k = A_n, \quad \text{and} \quad \lim_{n \to \infty} A_n = \bigcap_{k=1}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n.$$

3. Show that if $A_1, A_2, \ldots, A_n$ are all countable sets, then so is the n-tuple Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n.$$

   In particular, if A is a countable set, then so is A^n.

4. Suppose that the three sets A, B and C have the relationship $A \subset B \subset C$ and that $\mathrm{Card}(A) = \mathrm{Card}(C)$, then

$$\mathrm{Card}(A) = \mathrm{Card}(B) = \mathrm{Card}(C),$$

where $\mathrm{Card}(A)$ denotes the cardinal number of the set A etc.

5. Show that the set $[0, 1]$ is not countable.

6. Show that the Cardinal number of the real number R is equal to the cardinal number of the open unit internal $(0, 1)$.

7. Suppose $\{A_n; n = 1, 2, \ldots\}$ is an increasing sequence of sets. Define $B_1 = A_1, B_2 = A_2 \backslash A_1$, and in general, $B_n = A_n \backslash A_{n-1} (n \geqslant 2)$. Show that

   (i) $\{B_n; n \geqslant 1\}$ are disjoint.

   (ii) For any $k \geqslant 1$, $\bigcup_{n=1}^{k} B_n = A_k$.

   (iii) $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

8. Let S be the set of all the sequences with elements 0 and 1 only. Is S countable or not? Prove your conclusion.