Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node j to node k , then that row has -1 in column j and + 1 in column k .

The 6 by 4 incidence matrix A for the second graph in the figure is,

$$A = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Step-2

Find the vector in null space. In order to find the null space, set the given matrix is in the form of $Ax = 0 \text{ where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

So, $Ax = 0$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$
$$-x_1 + x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$-x_3 + x_4 = 0$$

$$-x_1 + x_4 = 0$$
$$-x_3 + x_4 = 0$$

Solving above equations then,

$$\begin{aligned} x_1 &= x_2 \\ &= x_3 \\ &= x_4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix}$$
$$= x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in the null space of A

The dimension of left null space is,

$$\begin{aligned} m - n + 1 &= 6 - 4 + 1 \\ &= 3 \end{aligned}$$

Therefore, the vector $(1,1,1,1)$ is the null space of A and it has 3 independent vectors.

Step-3

To show that three independent vectors that satisfy the $A^T y = 0$

If the matrix A is $\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ then the transpose of the matrix is $\begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

So,

Step-4

$A^T y = 0$

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 + R_3$ to A^T

$$= \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$ to A^T

$$= \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_4 \to R_4 + R_3$ to A^T

$$=\begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, the independent columns are $1,2,4.$ that is; $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Step-5

So, the system of equations from the above matrix is,

$$\begin{aligned} -y_1 - y_2 - y_5 &= 0 \\ -y_2 - y_3 - y_4 - y_5 &= 0 \\ -y_4 - y_5 - y_6 &= 0 \end{aligned}$$

From the first equation is,

$$\begin{aligned} -y_1 - y_2 - y_5 &= 0 \\ y_1 &= -y_2 - y_5 \end{aligned}$$

Plug this value in second equation,

$$\begin{aligned} -y_2 - y_3 - y_4 - y_5 &= 0 \\ -y_2 - y_3 + y_6 &= 0 \\ y_6 &= -y_2 - y_3 \end{aligned}$$

Plug this value in third equation,

$$\begin{aligned} -y_4 - y_5 - y_6 &= 0 \\ y_3 &= -y_2 - y_4 - y_5 \end{aligned}$$

So,

Step-6

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} -y_2 - y_5 \\ y_2 \\ -y_2 - y_4 - y_5 \\ y_4 \\ y_5 \\ -y_2 - y_3 \end{bmatrix}$$

$$= y_2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, the three independent vectors that satisfy $A^T y = 0$ and the three vectors y are $\boxed{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$