Step-1

Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

We have to solve $Ax = b$ by least-squares.

Step-2

We know that the least-squares solution to a problem is $\hat{x} = \left(A^T A\right)^{-1} A^T b$

Now

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(0)+1(1) & 1(0)+0(1)+1(1) \\ 0(1)+1(0)+1(1) & 0(0)+1(1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & 0+0+1 \\ 0+0+1 & 0+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step-3

Now

$$\left(A^T A\right)^{-1} = \frac{1}{4-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left(\begin{array}{l} \text{Since if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{then } A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} b & -b \\ -c & a \end{bmatrix} \end{array} \right)$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Therefore, $\left(A^T A\right)^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Step-4

Now

$$\begin{aligned}
\left(A^T A\right)^{-1} A^T &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}(1)-\frac{1}{3}(0) & \frac{2}{3}(0)-\frac{1}{3}(1) & \frac{2}{3}(1)-\frac{1}{3}(1) \\ -\frac{1}{3}(1)+\frac{2}{3}(0) & -\frac{1}{3}(0)+\frac{2}{3}(1) & -\frac{1}{3}(1)+\frac{2}{3}(1) \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}
\end{aligned}$$

Step-5

Therefore,

$$\hat{x} = \left(A^T A\right)^{-1} A^T b$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}(1) - \frac{1}{3}(1) + \frac{1}{3}(0) \\ -\frac{1}{3}(1) + \frac{2}{3}(1) + \frac{1}{3}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} - \frac{1}{3} \\ -\frac{1}{3} + \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Hence the least-square solution to the given system is $\boxed{\hat{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}$

Step-6

Now we have to verify that the error $b - p$ is perpendicular to the columns of A.

$$p = A\hat{x}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Step-7

Therefore, the error

$$
\begin{aligned}
e &= b - p \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}
\end{aligned}
$$

Hence $e = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$

Step-8

Let a_1, a_2 are columns of A

Now

$$
\begin{aligned}
e^T a_1 &= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{2}{3}(1) + \frac{2}{3}(0) - \frac{2}{3}(1) \\
&= \frac{2}{3} + 0 - \frac{2}{3} \\
&= 0
\end{aligned}
$$

Step-9

And

$$
\begin{aligned}
e^T a_2 &= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{2}{3}(0) + \frac{2}{3}(1) - \frac{2}{3}(1) \\
&= 0 + \frac{2}{3} - \frac{2}{3} \\
&= 0
\end{aligned}
$$

Hence the error e is perpendicular to both columns of A.