SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Course Name: DG MA327 **Department:** Mathematics
Due date: 10th May **Exam Paper Setter:** HUANG Shaochuang

Question No.	1	2	3	4	5	6	7	8	9	10	11
Score	18	5	8	5	10	8	12	6	8	10	10

When you use some fact in the textbook or lecture notes, state it clearly first.

1.*(18 points)* (a)(6 points) Show that every regular parametrized differentiable curve can be re-parametrized by arc length.

Solution: Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curve, $t \in I$ be the parameter. The arc length function $s(t) = \int_{t_0}^{t} |\alpha'(\tau)| d\tau$. Then $\frac{ds}{dt} = |\alpha'(t)| > 0$. By inverse function theorem, $s(t)$ has a differentiable inverse function $t(s), s \in J := s(I)$. Now let $\beta(s) = \alpha \circ t(s)$, then $|\dot{\beta}| = |\alpha' \cdot \frac{dt}{ds}| = 1$. Therefore, β is the re-parametrization by arc length.

(b)(6 points) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized (by arc length s) curve with nowhere vanishing curvature. Show that α is a plane curve (i.e. $\alpha(I)$ is contained in a plane) if and only if its torsion τ is identically equal to 0.

Solution: ($\Rightarrow$) If α is a plane curve, then the osculating planes along $\alpha(s)$ do not change (i.e. the plane containing α). Thus, $\tau \equiv 0$.

($\Leftarrow$) If $\tau \equiv 0$, then $b(s) = b_0 = Const.$. We have $(\alpha(s) \cdot b_0)' = \alpha' \cdot b_0 = 0$. Then $\alpha(s) \cdot b_0 = Const. = \alpha(s_0) \cdot b_0$. Namely, $(\alpha(s) - \alpha(s_0)) \cdot b_0 = 0$. In another word, α is contained in the plane which passes through $\alpha(s_0)$ and with normal b_0.

(c)(6 points) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curve and let $[a, b] \subset I$. Show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt.$$

Solution: For any fixed constant vector v with $|v| = 1$, we have

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

Now take $v = \frac{\alpha(b)-\alpha(a)}{|\alpha(b)-\alpha(a)|}$, we obtain

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt.$$

2.*(5 points)* Is there a simple closed curve in the plane with length equal to 6 meters and bounding an area of 3 square meters. (Write down yes or no and then explain your answer.)

Solution: No. By isoperimetric inequality, we have $l^2 \geq 4\pi A$. Now $l^2 = 36 < 4\pi A = 12\pi$ which is impossible.

3. *(8 points)* Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curves with curvature nowhere vanishing. Suppose every osculating plane along α passes through a fixed point, show that α is a plane curve.

Solution: We may assume every osculating plane along α passes through the origin, then we have $\alpha(s) \cdot b(s) = 0$ for all s. Then we can write

$$\alpha(s) = A(s)t(s) + B(s)n(s),$$

for all s. We then take derivatives and apply Frenet formula, we have

$$t = A't + Akn + B'n + B(-kt - \tau b)$$

for all s. Now we assume there exists s_0 such that $\tau(s_0) \neq 0$. Then by continuity, $\tau(s) \neq 0$ near s_0. This implies $B(s) = 0$ near s_0. Then $B'(s) = 0$ and $A(s)k(s) = 0$ near s_0. Since its curvature is nowhere vanishing, we have $A(s) = 0$ near s_0. This implies $\alpha(s) = 0$ near s_0 which is impossible. Thus, we obtain $\tau(s) \equiv 0$ for all s. By Q1(b), α is a plane curve.

4. *(5 points)* Construct a regular parametrized differentiable curve $\alpha : I \to \mathbb{R}^2$ which is injective and there exists $t_0 \in I$ such that for all small open disk $B(\alpha(t_0), \varepsilon)$ in $\mathbb{R}^2$ with center $\alpha(t_0)$ and radius ε, $B(\alpha(t_0), \varepsilon) \cap \alpha(I)$ is not homeomorphic to an open interval in $\mathbb{R}$. (You may draw some example with clear explanation or write down α explicitly by functions and then explain your example clearly.)

Solution: One example is the following: Consider the map $\alpha : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\alpha(t) = (2\cos(2\arctan t - \frac{\pi}{2}), \sin 2(2\arctan t - \frac{\pi}{2})).$$

It likes a figure "8" lying down. It is an injective with never vanishing tangent vector. Since $\alpha(0) = (0, 0)$ and $\alpha(t) \to (0, 0)$ as $t \to \pm\infty$, all small open disk $B(\alpha(0), \varepsilon)$ in $\mathbb{R}^2$ with center $\alpha(0)$ and radius ε, $B(\alpha(0), \varepsilon) \cap \alpha(\mathbb{R})$ is not homeomorphic to an open interval in $\mathbb{R}$.

5. *(10 points)* (a)(2 points) Write down the definition of a regular surface clearly.

(b)(4 points) Write down or draw with explanation an example of some "two dimensional" object which is not a regular surface and explain why your example is not a regular surface without proof.

(c)(4 points) Write down an example of a regular surface and prove your example is a regular surface (you may use some facts in the textbook or lecture notes without proof).

Solution: This question have many solutions.

6. *(8 points)* Prove that the tangent planes of a regular surface given by the graph of $z = x \cdot f(\frac{y}{x})$, $x \neq 0$, where f is a differentiable function, all pass through the origin $(0, 0, 0)$.

Solution: See Lecture 8 Page 14.

7. *(12 points)* (a)(6 points) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curves with curvature nowhere vanishing which is parametrized by arc-length s. Consider the following parametrized surface

$$\mathbf{x}(s, v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v), \quad s \in I.$$

Here r is some fixed positive constant, n is the normal of α and b is the bi-normal of α. Find the unit normal vector of $\mathbf{x}$ whenever it is regular. (Write down your computation clearly.)

Solution: See Lecture 8 Page 15,16.

(b)(6 points) Find the area of the above surface. (It depends on r and the length l of α. Write down your computation clearly.)

Solution: $2\pi rl$. One may compute it by the definition of area in our lectures.

8. *(6 points)* Compute the first fundamental form of the following parametrization for the sphere:

$$\mathbf{x}(u,v) = (a\cos u\cos v, a\cos u\sin v, a\sin u),$$

where a is a positive constant. Consider two curves $u = v$ and $v = v_0$ with v_0 a fixed constant on the sphere, compute $\cos\beta$. Here β is the angle between these two curves where they intersect.

Solution: $E = a^2, F = 0, G = a^2\cos^2 u, \cos\beta = \frac{1}{\sqrt{1+\cos^2 v_0}}$.

9. *(8 points)* Consider the paraboloid i.e. the graph of $z = x^2 + ky^2$ with k a positive constant and $p = (0,0,0)$. Prove that the unit vector of the x axis and the y axis are eigenvectors of dN_p, with eigenvalue 2 and $2k$, respectively (assuming that N is pointing outwards from the region bounded by the paraboloid).

Solution: $N = \frac{(2x,2ky,-1)}{\sqrt{1+4x^2+4k^2y^2}}$. Then $N(t) = \frac{(2x(t),2ky(t),-1)}{\sqrt{1+4x^2(t)+4k^2y^2(t)}}$. Then $N'(0) = (2x'(0), 2ky'(0), 0)$. Then

$$dN_p(x'(0), y'(0), 0) = (2x'(0), 2ky'(0), 0).$$

Thus $dN_p(1,0,0) = 2\cdot(1,0,0)$ and $dN_p(0,1,0) = 2k\cdot(0,1,0)$.

10. *(10 points)* Prove that if all normal lines to a connected regular surface S meet a fixed point, then S is a piece of a sphere.

Solution: We first show it locally. Let $X(u,v)$ be a parametrization near p and $N(u,v)$ be the normal. Then by assumption, there exists a function $\lambda(u,v)$ and a constant point c_0 such that

$$X(u,v) + \lambda(u,v)N(u,v) = c_0.$$

We want to show λ is a constant. To see it, we take derivatives, then we have

$$X_u + \lambda_u N + \lambda N_u = 0$$

and

$$X_v + \lambda_v N + \lambda N_v = 0.$$

Then we take inner product with N to the above equations, we have $\lambda_u = \lambda_v = 0$. Thus $\lambda(u,v) \equiv \lambda_0$. Namely,

$$X(u,v) + \lambda_0 N(u,v) = c_0.$$

This implies $|X(u,v) - c_0|^2 = \lambda_0^2$. Therefore, $X(u,v)$ is a piece of a sphere. *(8 points)*

Let q be any other point in S, by connectedness, one can find a closed curve joining p and q in S. Since this curve is compact, one can find finite parametrizations covering it. Then q is also in the same sphere. *(2 points)*

11. *(10 points)* (a)(8 points) Show that a surface which is compact (i.e. bounded and closed in $\mathbb{R}^3$) has an elliptic point.

(b)(2 points) Prove that there are no compact minimal surfaces in $\mathbb{R}^3$.

Solution: See Homework 5.