Step-1

Consider the higher order equation $y'' + y = 0$.

This equation can be written as a first order system by introducing the velocity y' as another unknown is given by $\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ -y \end{bmatrix}$

The objective is to find the matrix A such that $\frac{du}{dt} = Au$ where $u = \begin{bmatrix} y \\ y' \end{bmatrix}$ and its eigenvalues and eigenvectors and compute the solution that starts from $y(0) = 2, y'(0) = 0$.

Step-2

Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then $Au = \begin{bmatrix} y' \\ -y \end{bmatrix}$, solve this;

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ -y \end{bmatrix}$$

This gives;

$$\begin{aligned} ay + cy' &= y' \\ by + dy' &= -y \end{aligned}$$

The possible solution is $a = 0, c = 1, b = -1, d = 0$.

Therefore, the matrix is $\boxed{A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$.

Step-3

Now find the eigenvalues of matrix A as $\det(A - \lambda I) = 0$;

$$\begin{aligned} \det\begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix} &= 0 \\ (-\lambda)(-\lambda) - (1)(-1) &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \end{aligned}$$

Hence eigenvalues are $\boxed{\pm i}$.

Step-4

Find the eigenvector for $\lambda_1 = -i$ as;

$$(A - \lambda_1 I)X = 0$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

From the above, the equation is $ix_1 + x_2 = 0$

Let $x_2 = k$ then,

$$\begin{aligned} x_1 &= -\frac{k}{i} \\ &= ik \end{aligned}$$

Therefore, the Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} i \\ 1 \end{bmatrix}$

The solution (the first eigenvector) is any nonzero multiple of x_1, thus eigenvector for $\lambda_1 = -i$ is $\boxed{v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}}$.

Step-5

Now, find the eigenvector for $\lambda_1 = i$.

$$(A - \lambda_1 I)X = 0$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Equation is $-ix_1 + x_2 = 0$

Let $x_2 = k$ then,

$$x_1 = \frac{k}{i}$$
$$= -ik$$

Therefore, the Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} -i \\ 1 \end{bmatrix}$

The solution (the second eigenvector) is any nonzero multiple of x_2, thus eigenvector for $\lambda_2 = i$ is $v_2 = \boxed{\begin{bmatrix} -i \\ 1 \end{bmatrix}}$.

The eigenvectors are $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Step-6

Now, the solution of the differential equation is given by,

$$\begin{aligned} u(t) &= c_1 e^{-\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \\ &= c_1 e^{-it} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{it} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} ic_1 e^{-it} \\ c_1 e^{-it} \end{bmatrix} + \begin{bmatrix} -ic_2 e^{it} \\ c_2 e^{it} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} ic_1 e^{-it} - ic_2 e^{it} \\ c_1 e^{-it} + c_2 e^{it} \end{bmatrix}$$
$$= \begin{bmatrix} ic_1 \left(\cos(t) - i\sin(t)\right) - ic_2 \left(\cos(t) + i\sin(t)\right) \\ c_1 \left(\cos(t) - i\sin(t)\right) + c_2 \left(\cos(t) + i\sin(t)\right) \end{bmatrix}$$

Step-7

Apply the initial condition $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ then,

$$u(0)=\begin{pmatrix} ic_1\left(\cos(0)-i\sin(0)\right)-ic_2\left(\cos(0)+i\sin(0)\right) \\ c_1\left(\cos(0)-i\sin(0)\right)+c_2\left(\cos(0)+i\sin(0)\right) \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}=\begin{pmatrix} ic_1\left(1-i(0)\right)-ic_2\left(1+i(0)\right) \\ c_1\left(1-i(0)\right)+c_2\left(1+i(0)\right) \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}=\begin{pmatrix} ic_1-ic_2 \\ c_1+c_2 \end{pmatrix}$$

Now, solve the equations, $c_1-c_2=-2i, c_1+c_2=0$

Add these two equations then $c_1=-i$ and $c_2=i$.

Step-8

Substitute the constants in the solution as,

$$\begin{aligned} u(t)&=\begin{bmatrix} i(-i)\left(\cos(t)-i\sin(t)\right)-i(i)\left(\cos(t)+i\sin(t)\right) \\ (-i)\left(\cos(t)-i\sin(t)\right)+(i)\left(\cos(t)+i\sin(t)\right) \end{bmatrix} \\ &=\begin{bmatrix} \cos(t)-i\sin(t)+\cos(t)+i\sin(t) \\ -i\cos(t)-\sin(t)+i\cos(t)-\sin(t) \end{bmatrix} \\ &=\begin{bmatrix} 2\cos(t) \\ -2\sin(t) \end{bmatrix} \end{aligned}$$

Hence, the required equation is $\boxed{y(t)=2\cos(t)}$.