Step-1

Objective is to prove $D_n = D_{n-1} - D_{n-2}$, and find the value of D_{1000}.

Step-2

Tridiagonal matrix:

The tridiagonal matrix is the matrix that has non-zero entries only on the main diagonal, first diagonal above the main diagonal and first diagonal below the main diagonal.

Step-3

(a)

Consider the following tridiagonal matrices,

$$A_1 = [1]$$
$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Let D_n denotes the determinant of A_n.

The objective is to show that $D_n = D_{n-1} - D_{n-2}$.

Consider;

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Determinant of A_4 is given by;

$$D_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \left(1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\right)$$

$$= 0 - 1 - (0 - 0)$$

$$= -1$$

Similarly,

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 0 - 1$$

$$= -1$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1 - 1$$

$$= 0$$

Therefore,

$$D_4 = D_3 - D_2$$

$$= -1 - 0$$

$$= -1$$

Thus,

$D_n = D_{n-1} - D_{n-2}$ â€¦â€¦ (1)

Step-4

(b)

The objective is to find D_{1000}.

Step-5

From the part (a)

$D_1 = 1$
$D_2 = 0$

Use $D_n = D_{n-1} - D_{n-2}$ and get;

$$\begin{aligned} D_3 &= D_2 - D_1 \\ &= -1 \\ D_4 &= D_3 - D_2 \\ &= -1 \end{aligned}$$

Step-6

Similarly

$$\begin{aligned} D_5 &= D_4 - D_3 \\ &= 0 \\ D_6 &= D_5 - D_4 \\ &= 1 \end{aligned}$$

$$\begin{aligned} D_7 &= D_6 - D_5 \\ &= 1 \\ D_8 &= D_7 - D_6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} D_9 &= D_8 - D_7 \\ &= -1 \\ D_{10} &= D_9 - D_8 \\ &= -1 \end{aligned}$$

It can be generalized that $D_{3n+2} = 0$ and

$D_{3n} = D_{3n+1} = -1$, when n is odd.

$D_{3n} = D_{3n+1} = 1$, when n is even.

Now,

$1000 = 3 \times 333 + 1$

Since 333 is odd. Hence, $\boxed{D_{1000} = -1}$