Step-1

The objective is to determine a vector x orthogonal to the row space matrix A and vector y orthogonal to the column space and vector z orthogonal to the null space.

Step-2

Consider the matrix $A=\begin{bmatrix}1&2&1\\2&4&3\\3&6&4\end{bmatrix}$.

Use the row method.

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

So,

$$\begin{aligned}A&=\begin{bmatrix}1&2&1\\2&4&3\\3&6&4\end{bmatrix}\\&=\begin{bmatrix}1&2&1\\0&0&1\\0&0&1\end{bmatrix}\end{aligned}$$

Now, replace row 3 with R_1-R_3.

Then,

$$\begin{aligned}A&=\begin{bmatrix}1&2&1\\0&0&1\\0&0&1\end{bmatrix}\\&=\begin{bmatrix}1&2&1\\0&0&1\\1&2&0\end{bmatrix}\end{aligned}$$

Let (a,b,c) is the vector of orthogonal of the row space.

So,

$$\begin{aligned}a+2b+c&=0\\c&=0\\a+2b&=0\end{aligned}$$

Then

$a = -2b$

Let $b = 1$

$$\begin{aligned} a &= -2b \\ &= -2 \end{aligned}$$

Thus, an orthogonal vector of orthogonal of the row space is $x = (-2,1,0)$.

Step-3

Take a transpose of the provided matrix.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

Replace row 2 with $R_2 - 2R_1$ and row 3 with $R_3 - R_1$.

$$\begin{aligned} A^T &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

The column space is the transpose of the row space. Let $c = k$,

$$\begin{aligned} a + 2b + 3c &= 0 \\ b + c &= 0 \\ c &= k \end{aligned}$$

Then,

$$\begin{aligned} b &= -k \\ a &= -k \end{aligned}$$

Thus, an orthogonal vector of the column space is $y = (-k,-k,k)$ let $k = 1$ so, the column space vector $y = (-1,-1,1)$.

Step-4

For null space vector, $Ax = 0 = \lambda x$ consider the row space matrix.

$$A=\begin{bmatrix}1&2&1\\0&0&1\\0&0&1\end{bmatrix}$$

Replace $R_3 \to R_3 - R_2$.

$$\begin{aligned}A&=\begin{bmatrix}1&2&1\\0&0&1\\0&0&1\end{bmatrix}\\&=\begin{bmatrix}1&2&1\\0&0&1\\0&0&0\end{bmatrix}\end{aligned}$$

Thus, a null space vector is:

$$\begin{aligned}a+2b+c&=0\\c&=0\end{aligned}$$

The orthogonal vector is $z=(1,2,0)$.

Hence, a vector $\boxed{x=(-2,1,0)}$ orthogonal to the row space matrix A and vector $\boxed{y=(-1,-1,1)}$ orthogonal to the column space and vector $\boxed{z=(1,2,0)}$ orthogonal to the null space.