Step-1

(a)

The objective is to provide examples of matrices A and B for which $A+B$ is not invertible although A and B are invertible.

Step-2

Assume that,

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } B=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A matrix is invertible when its determinant is not equal to zero.

$$\begin{aligned} \det A &= 1-0 \\ &= 1 \\ \det B &= (-1)(-1)-0 \\ &= 1 \end{aligned}$$

Therefore, the matrices A and B are invertible.

Step-3

Addition of matrices A and B is,

$$\begin{aligned} A+B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1-1 & 0+0 \\ 0+0 & 1-1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The determinant of $A+B$ is,

$$\begin{aligned} \det(A+B) &= 0-0 \\ &= 0 \end{aligned}$$

Hence $A+B$ is not invertible although A and B are invertible.

Step-4

(b)

The objective is to provide examples of matrices A and B for which $A+B$ is invertible although A and B are not invertible.

Step-5

Assume that,

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \text{ and } B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Addition of matrices A and B is,

$$\begin{aligned}A+B&=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}1+0 & 0+0\\ 0+0 & 0+1\end{bmatrix}\\ &=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\end{aligned}$$

The determinant of $A+B$ is,

$$\begin{aligned}\det(A+B)&=1-0\\ &=1\\ &\neq 0\end{aligned}$$

Therefore $A+B$ is invertible.

Step-6

The determinant of the matrix A is,

$$\begin{aligned}\det A&=(1)(0)-(0)(0)\\ &=0\\ \det B&=(0)(1)-(0)(0)\\ &=0\end{aligned}$$

Therefore, the matrices A and B are not invertible.

Hence, $A+B$ is invertible although A and B are not invertible.

Step-7

(c)

The objective is to provide examples of matrices A and B for which A and B are invertible and $A+B$ is also invertible.

Step-8

Assume that,

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \text{ and } B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

The determinant of matrix A is,

$$\begin{aligned}\det A &= 1-0\\ &= 1\\ &\neq 0\end{aligned}$$

Step-9

The determinant of matrix B is,

$$\begin{aligned}\det B &= 0-(1)-1\\ &= 1\\ &\neq 0\end{aligned}$$

Addition of matrices A and B is,

$$\begin{aligned}A+B &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\ &= \begin{bmatrix}1+0 & 0+1\\ 0-1 & 1+0\end{bmatrix}\\ &= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\end{aligned}$$

The determinant of matrix $A+B$ is,

$$\begin{aligned}\det(A+B) &= (1)(1)-(1)(-1)\\ &= 1+1\\ &= 2\\ &\neq 0\end{aligned}$$

Hence, the matrices A, B, and $A+B$ are invertible.

Step-10

Consider,

$$A^{-1}(A+B)B^{-1} = B^{-1}+A^{-1}$$

Sum of two invertible matrices is also invertible.

Therefore, $C = B^{-1} + A^{-1}$ is invertible.

The inverse of *C* is,

$$\begin{aligned}
\left(B^{-1}+A^{-1}\right)^{-1} &= \left(A^{-1}(A+B)B^{-1}\right)^{-1} \\
&= \left(B^{-1}\right)^{-1}(A+B)^{-1}\left(A^{-1}\right)^{-1} && \text{Since, } (ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \\
C^{-1} &= \boxed{B(A+B)^{-1}A} && \text{Since, } \left(A^{-1}\right)^{-1} = A.
\end{aligned}$$

Substitute, *A*, and *B* values in $C,$ obtained as,

$$\begin{aligned}
C &= B^{-1} + A^{-1} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

The determinant of *C* is,

$$\begin{aligned}
\det C &= 1+1 \\
&= 2 \\
&\neq 0
\end{aligned}$$

Hence, *C* is invertible.