CHAPTER

7

Systems of First Order Linear Equations

7.1

1. Introduce the variables $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -2u - 0.5\,u'.$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5\,x_2\,. \end{aligned}$$

2. First divide both sides of the equation by t^2, and write

$$u'' = -\frac{1}{t}\,u' - (1 - \frac{1}{4t^2})u\,.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{1}{t}\,u' - (1 - \frac{1}{4t^2})u\,.$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -(1 - \frac{1}{4t^2})x_1 - \frac{1}{t}\,x_2\,. \end{aligned}$$

4. Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + 0.25\,x_2 + 4\,x_1 = 2\cos 3t$$

with the initial conditions $x_1(0) = 1$ and $x_2(0) = -2$. The equivalent first order system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4\,x_1 - 0.25\,x_2 + 2\cos 3t \end{aligned}$$

with the above initial conditions.

6.(a) Solving the first equation for x_2, we have $x_2 = x_1' + 2x_1$.

(b) Substitution into the second equation results in $(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$. That is, $x_1'' + 4\,x_1' + 3\,x_1 = 0$.

(c) The general solution is $x_1(t) = c_1e^{-t} + c_2e^{-3t}$.

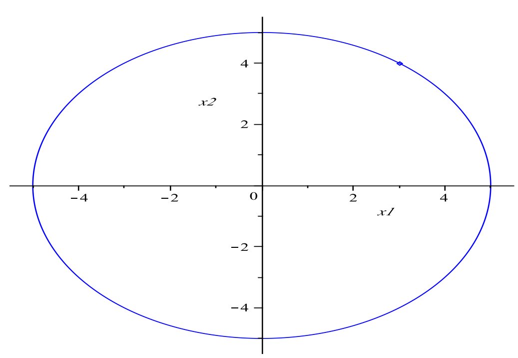
(d) With x_2 given in terms of x_1, it follows that $x_2(t) = c_1e^{-t} - c_2e^{-3t}$.

8.(a) Solving the first equation for x_2, we have $x_2 = x_1'/2$. Substitution into the second equation results in $x_1''/2 = -2\,x_1$. The resulting equation is $x_1'' + 4\,x_1 = 0$.

(b) The general solution is $x_1(t) = c_1\cos 2t + c_2\sin 2t$. With x_2 given in terms of x_1, it follows that $x_2(t) = -c_1\sin 2t + c_2\cos 2t$. Imposing the specified initial conditions, we obtain $c_1 = 3$ and $c_2 = 4$. Hence

$$x_1(t) = 3\cos 2t + 4\sin 2t \ \text{ and } \ x_2(t) = -3\sin 2t + 4\cos 2t\,.$$

(c)



10. Solving the first equation for V, we obtain $V = L\cdot I'$. Substitution into the second equation results in

$$L\cdot I'' = -\frac{I}{C} - \frac{L}{RC}I'\,.$$

Rearranging the terms, the single differential equation for I is

$$LRC \cdot I'' + L \cdot I' + R \cdot I = 0\,.$$

12. Let $x = c_1x_1(t) + c_2x_2(t)$ and $y = c_1y_1(t) + c_2y_2(t)$. Then

$$\begin{aligned} x' &= c_1x_1'(t) + c_2x_2'(t) \\ y' &= c_1y_1'(t) + c_2y_2'(t). \end{aligned}$$

Since $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$ are solutions for the original system,

$$\begin{aligned} x' &= c_1(p_{11}x_1(t) + p_{12}y_1(t)) + c_2(p_{11}x_2(t) + p_{12}y_2(t)) \\ y' &= c_1(p_{21}x_1(t) + p_{22}y_1(t)) + c_2(p_{21}x_2(t) + p_{22}y_2(t)). \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} x' &= p_{11}(c_1x_1(t) + c_2x_2(t)) + p_{12}(c_1y_1(t) + c_2y_2(t)) \\ y' &= p_{21}(c_1x_1(t) + c_2x_2(t)) + p_{22}(c_1y_1(t) + c_2y_2(t)), \end{aligned}$$

and so x and y solve the original system.

13. Based on the hypothesis,

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1(t) + p_{12}(t)y_1(t) + g_1(t) \\ x_2'(t) &= p_{11}(t)x_2(t) + p_{12}(t)y_2(t) + g_1(t)\,. \end{aligned}$$

Subtracting the two equations,

$$x_1'(t) - x_2'(t) = p_{11}(t)\left[x_1'(t) - x_2'(t)\right] + p_{12}(t)\left[y_1'(t) - y_2'(t)\right]\,.$$

Similarly,

$$y_1'(t) - y_2'(t) = p_{21}(t)\left[x_1'(t) - x_2'(t)\right] + p_{22}(t)\left[y_1'(t) - y_2'(t)\right]\,.$$

Hence the difference of the two solutions satisfies the homogeneous ODE.

14. For rectilinear motion in one dimension, Newton's second law can be stated as

$$\sum F = m\,x''.$$

The resisting force exerted by a linear spring is given by $F_s = k\,\delta$, in which δ is the displacement of the end of a spring from its equilibrium configuration. Hence, with $0 < x_1 < x_2$, the first two springs are in tension, and the last spring is in compression. The sum of the spring forces on m_1 is

$$F_s^1 = -k_1x_1 - k_2(x_2 - x_1)\,.$$

The total force on m_1 is

$$\sum F^1 = -k_1x_1 + k_2(x_2 - x_1) + F_1(t)\,.$$

Similarly, the total force on m_2 is

$$\sum F^2 = -k_2(x_2 - x_1) - k_3x_2 + F_2(t)\,.$$

15. One of the ways to transform the system is to assign the variables

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = x_1', \qquad y_4 = x_2'.$$

Before proceeding, note that

$$\begin{aligned} x_1'' &= \frac{1}{m_1}\left[-(k_1+k_2)x_1 + k_2 x_2 + F_1(t)\right] \\ x_2'' &= \frac{1}{m_2}\left[k_2 x_1 - (k_2+k_3)x_2 + F_2(t)\right] . \end{aligned}$$

Differentiating the new variables, we obtain the system of four first order equations

$$\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= \frac{1}{m_1}(-(k_1+k_2)y_1 + k_2 y_2 + F_1(t)) \\ y_4' &= \frac{1}{m_2}(k_2 y_1 - (k_2+k_3)y_2 + F_2(t)) . \end{aligned}$$

18. Let I_1, I_2, I_3,and I_4 be the current through the resistors, inductor, and capacitor, respectively. Assign V_1, V_2, V_3,and V_4 as the respective voltage drops. Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1 + V_3 + V_4 = 0, \quad V_1 + V_3 + V_2 = 0 \quad \text{and} \quad V_4 - V_2 = 0\,.$$

Applying Kirchhoff's first law to the upper-right node,

$$I_3 - (I_2 + I_4) = 0\,.$$

Likewise, in the remaining nodes,

$$I_1 - I_3 = 0 \quad \text{and} \quad I_2 + I_4 - I_1 = 0\,.$$

That is,

$$V_4 - V_2 = 0, \quad V_1 + V_3 + V_4 = 0 \quad \text{and} \quad I_2 + I_4 - I_3 = 0\,.$$

Using the current-voltage relations,

$$V_1 = R_1 I_1, \quad V_2 = R_2 I_2, \quad L\, I_3' = V_3, \quad C\, V_4' = I_4.$$

Combining these equations,

$$R_1 I_3 + L\, I_3' + V_4 = 0 \quad \text{and} \quad C\, V_4' = I_3 - \frac{V_4}{R_2}.$$

Now set $I_3 = I$ and $V_4 = V$, to obtain the system of equations

$$L\, I' = -R_1 I - V \quad \text{and} \quad C\, V' = I - \frac{V}{R_2}.$$

7.2

2.(a)

$$\mathbf{A}-2\,\mathbf{B}=\begin{pmatrix}1+i-2i & -1+2i-6\\ 3+2i-4 & 2-i+4i\end{pmatrix}=\begin{pmatrix}1-i & -7+2i\\ -1+2i & 2+3i\end{pmatrix}.$$

(b)

$$3\,\mathbf{A}+\,\mathbf{B}=\begin{pmatrix}3+3i+i & -3+6i+3\\ 9+6i+2 & 6-3i-2i\end{pmatrix}=\begin{pmatrix}3+4i & 6i\\ 11+6i & 6-5i\end{pmatrix}.$$

(c)

$$\begin{aligned}\mathbf{AB}\; &=\begin{pmatrix}(1+i)i+2(-1+2i) & 3(1+i)+(-1+2i)(-2i)\\ (3+2i)i+2(2-i) & 3(3+2i)+(2-i)(-2i)\end{pmatrix}\\ &=\begin{pmatrix}-3+5i & 7+5i\\ 2+i & 7+2i\end{pmatrix}.\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{BA}\; &=\begin{pmatrix}(1+i)i+3(3+2i) & (-1+2i)i+3(2-i)\\ 2(1+i)+(-2i)(3+2i) & 2(-1+2i)+(-2i)(2-i)\end{pmatrix}\\ &=\begin{pmatrix}8+7i & 4-4i\\ 6-4i & -4\end{pmatrix}.\end{aligned}$$

3.(c,d)

$$\begin{aligned}\mathbf{A}^T+\mathbf{B}^T\; &=\begin{pmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -3 & 1\end{pmatrix}+\begin{pmatrix}1 & 3 & -2\\ 2 & -1 & 1\\ 3 & -1 & 0\end{pmatrix}\\ &=\begin{pmatrix}-1 & 4 & 0\\ 3 & -1 & 0\\ 5 & -4 & 1\end{pmatrix}=(\mathbf{A}+\mathbf{B})^T.\end{aligned}$$

4.(b)

$$\overline{\mathbf{A}}=\begin{pmatrix}3+2i & 1-i\\ 2+i & -2-3i\end{pmatrix}.$$

(c) By definition,

$$\mathbf{A}^*=\overline{\mathbf{A}^T}=(\overline{\mathbf{A}})^T=\begin{pmatrix}3+2i & 2+i\\ 1-i & -2-3i\end{pmatrix}.$$

6. Let $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$. The given operations in (a)-(d) are performed elementwise. That is,

(a) $a_{ij}+b_{ij}=b_{ij}+a_{ij}$.

(b) $a_{ij}+(b_{ij}+c_{ij})=(a_{ij}+b_{ij})+c_{ij}$.

(c) $\alpha(a_{ij}+b_{ij})=\alpha\,a_{ij}+\alpha\,b_{ij}$.

(d) $(\alpha+\beta)\,a_{ij} = \alpha\,a_{ij} + \beta\,a_{ij}$.

In the following, let $\mathbf{A}= (a_{ij})$, $\mathbf{B}= (b_{ij})$ and $\mathbf{C}= (c_{ij})$.

(e) Calculating the generic element,

$$(\mathbf{BC})_{ij} = \sum_{k=1}^{n} b_{ik}\,c_{kj} \,.$$

Therefore

$$[\mathbf{A}(\mathbf{BC})]_{ij} = \sum_{r=1}^{n} a_{ir}(\sum_{k=1}^{n} b_{rk}\,c_{kj}) = \sum_{r=1}^{n}\sum_{k=1}^{n} a_{ir}\,b_{rk}\,c_{kj} = \sum_{k=1}^{n}(\sum_{r=1}^{n} a_{ir}\,b_{rk})\,c_{kj}.$$

The inner summation is recognized as

$$\sum_{r=1}^{n} a_{ir}\,b_{rk} = (\mathbf{AB})_{ik} \,,$$

which is the ik-th element of the matrix $\mathbf{AB}$. Thus $[\mathbf{A}(\mathbf{BC})]_{ij} = [(\mathbf{AB})\mathbf{C}]_{ij}$.

(f) Likewise,

$$[\mathbf{A}(\mathbf{B}+\mathbf{C})]_{ij} = \sum_{k=1}^{n} a_{ik}(\,b_{kj} +\, c_{kj}) = \sum_{k=1}^{n} a_{ik}\,b_{kj} + \sum_{k=1}^{n} a_{ik}\,c_{kj} = (\mathbf{AB})_{ij} + (\mathbf{AC})_{ij} \,.$$

7.(a) $\mathbf{x}^T\mathbf{y}= 2(-1+i) + 2(3i) + (1-i)(3-i) = 4i$.

(b) $\mathbf{y}^T\mathbf{y}= (-1+i)^2 + 2^2 + (3-i)^2 = 12 - 8i$.

(c) $(\mathbf{x},\mathbf{y}) = 2(-1-i) + 2(3i) + (1-i)(3+i) = 2+2i$.

(d) $(\mathbf{y},\mathbf{y}) = (-1+i)(-1-i) + 2^2 + (3-i)(3+i) = 16$.

9. First augment the given matrix by the identity matrix:

$$[\mathbf{A}\,|\,\mathbf{I}\,] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Divide the first row by 3 , to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Adding -6 times the first row to the second row results in

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.$$

Divide the second row by 4 , to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Finally, adding $1/3$ times the second row to the first row results in

$$\begin{pmatrix} 1 & 0 & 1/6 & 1/12 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}.$$

12. Elementary row operations yield

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}.$$

Finally, combining the first and third rows results in

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

13. Elementary row operations on the augmented matrix yield the row-reduced form of the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1/7 & 0 & 1/7 & 2/7 \\ 0 & 1 & 3/7 & 0 & 4/7 & 1/7 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{pmatrix}.$$

The left submatrix cannot be converted to the identity matrix. Hence the given matrix is singular.

14. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

15. Suppose that there exist matrices $\mathbf{B}$ and $\mathbf{C}$, such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$. Then $\mathbf{CAB} = \mathbf{IB} = \mathbf{B}$, also, $\mathbf{CAB} = \mathbf{CI} = \mathbf{C}$. This shows that $\mathbf{B} = \mathbf{C}$.

17. First note that

$$\mathbf{x}' = \begin{pmatrix}1\\0\end{pmatrix} e^t + 2\begin{pmatrix}1\\1\end{pmatrix}(e^t + t\,e^t) = \begin{pmatrix}3e^t + 2t\,e^t\\2e^t + 2t\,e^t\end{pmatrix}.$$

We also have

$$\begin{aligned}\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\mathbf{x} &= \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} e^t + \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}(t\,e^t)\\ &= \begin{pmatrix}2\\3\end{pmatrix} e^t + \begin{pmatrix}2\\2\end{pmatrix}(t\,e^t) = \begin{pmatrix}2e^t + 2t\,e^t\\3e^t + 2t\,e^t\end{pmatrix}.\end{aligned}$$

It follows that

$$\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\-1\end{pmatrix} e^t = \begin{pmatrix}3e^t + 2t\,e^t\\2e^t + 2t\,e^t\end{pmatrix}.$$

18. It is easy to see that

$$\mathbf{x}' = \begin{pmatrix}-6\\8\\4\end{pmatrix} e^{-t} + \begin{pmatrix}0\\4\\-4\end{pmatrix} e^{2t} = \begin{pmatrix}-6e^{-t}\\8e^{-t} + 4e^{2t}\\4e^{-t} - 4e^{-2t}\end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\begin{pmatrix}1 & 1 & 1\\2 & 1 & -1\\0 & -1 & 1\end{pmatrix}\mathbf{x} &= \begin{pmatrix}1 & 1 & 1\\2 & 1 & -1\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}6\\-8\\-4\end{pmatrix} e^{-t} + \begin{pmatrix}1 & 1 & 1\\2 & 1 & -1\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\2\\-2\end{pmatrix} e^{2t}\\ &= \begin{pmatrix}-6\\8\\4\end{pmatrix} e^{-t} + \begin{pmatrix}0\\4\\-4\end{pmatrix} e^{2t}.\end{aligned}$$

20. Differentiation, elementwise, results in

$$\mathbf{\Psi}' = \begin{pmatrix}e^t & -2e^{-2t} & 3e^{3t}\\-4e^t & 2e^{-2t} & 6e^{3t}\\-e^t & 2e^{-2t} & 3e^{3t}\end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\begin{pmatrix}1 & -1 & 4\\3 & 2 & -1\\2 & 1 & -1\end{pmatrix}\mathbf{\Psi} &= \begin{pmatrix}1 & -1 & 4\\3 & 2 & -1\\2 & 1 & -1\end{pmatrix}\begin{pmatrix}e^t & e^{-2t} & e^{3t}\\-4e^t & -e^{-2t} & 2e^{3t}\\-e^t & -e^{-2t} & e^{3t}\end{pmatrix}\\ &= \begin{pmatrix}e^t & -2e^{-2t} & 3e^{3t}\\-4e^t & 2e^{-2t} & 6e^{3t}\\-e^t & 2e^{-2t} & 3e^{3t}\end{pmatrix}.\end{aligned}$$

7.3

4. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 2 & 1 & 1 & | & 0 \\ 1 & -1 & 2 & | & 0 \end{pmatrix}.$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & -3 & 3 & | & 0 \end{pmatrix}.$$

Adding the negative of the second row to the third row results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$x = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

5. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 3 & 1 & 1 & | & 0 \\ -1 & 1 & 2 & | & 0 \end{pmatrix}.$$

Adding -3 times the first row to the second row and adding the first row to the last row yields

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}.$$

Now add the negative of the second row to the third row to obtain

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}.$$

We end up with an equivalent linear system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3\,x_3 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

Hence the unique solution of the given system of equations is $x_1 = x_2 = x_3 = 0$.

7. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $\det(\mathbf{X}) = 0$. Hence the vectors are linearly dependent. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & 5/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We obtain the system of equations

$$c_1 - c_3/2 = 0$$
$$c_2 + 5c_3/2 = 0\,.$$

Setting $c_3 = 2$, it follows that $c_1 = 1$ and $c_3 = -5$. Hence

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}\,.$$

8. The matrix containing the given vectors as columns is

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}.$$

We find that $\det(\mathbf{X}) = -70$. Hence the given vectors are linearly independent.

9. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}.$$

The four vectors are necessarily linearly dependent. Hence there are nonzero scalars such that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\begin{pmatrix} 1 & 3 & 2 & 4 & | & 0 \\ 2 & 1 & -1 & 3 & | & 0 \\ -2 & 0 & 1 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \end{pmatrix}.$$

We end up with an equivalent linear system

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 &= 0\,. \end{aligned}$$

Let $c_4 = -1$. Then $c_1 = 1$ and $c_2 = 1$. Therefore we find that

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}\,.$$

10. The matrix containing the given vectors as columns, $\mathbf{X}$, is of size $n \times m$. Since $n < m$, we can augment the matrix with $m - n$ rows of zeros. The resulting matrix, $\tilde{\mathbf{X}}$, is of size $m \times m$. Since $\tilde{\mathbf{X}}$ is a square matrix, with at least one row of zeros, it follows that $\det(\tilde{\mathbf{X}}) = 0$. Hence the column vectors of $\tilde{\mathbf{X}}$ are linearly dependent. That is, there is a nonzero vector, $\mathbf{c}$, such that $\tilde{\mathbf{X}}\mathbf{c} = \mathbf{0}_{m\times 1}$. If we write only the first n rows of the latter equation, we have $\mathbf{X}\mathbf{c} = \mathbf{0}_{n\times 1}$. Therefore the column vectors of $\mathbf{X}$ are linearly dependent.

11. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $3\,\mathbf{x}^{(1)}(t) - 6\,\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are linearly dependent.

15. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(3-\lambda)(-1-\lambda) + 8 = 0$, that is,

$$\lambda^2 - 2\lambda + 5 = 0\,.$$

The eigenvalues are $\lambda_1 = 1 - 2\,i$ and $\lambda_2 = 1 + 2\,i$. The components of the eigenvector $\mathbf{x}^{(1)}$ are solutions of the system

$$\begin{pmatrix} 2+2\,i & -2 \\ 4 & -2+2\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two equations reduce to $(1+i)x_1 = x_2$. Hence $\mathbf{x}^{(1)} = (1\,, 1+i)^T$. Now setting $\lambda = \lambda_2 = 1 + 2\,i$, we have

$$\begin{pmatrix} 2-2\,i & -2 \\ 4 & -2-2\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solution given by $\mathbf{x}^{(2)} = (1\,, 1-i)^T$.

16. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-2-\lambda)(-2-\lambda) - 1 = 0$, that is,

$$\lambda^2 + 4\lambda + 3 = 0\,.$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. For $\lambda_1 = -3$, the system of equations becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,,-1)^T$. Substituting $\lambda = \lambda_2 = -1$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = x_2$. Hence a solution vector is given by $\mathbf{x}^{(2)} = (1\,,1)^T$.

17. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, the determinant of the coefficient matrix must be zero. That is,

$$\lambda^2 - 4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Substituting the first eigenvalue, $\lambda = -2$, yields

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system is equivalent to the equation $\sqrt{3}\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,,-\sqrt{3}\,)^T$. Substitution of $\lambda = 2$ results in

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 = \sqrt{3}\,x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (\sqrt{3}\,,1)^T$.

19. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, with roots $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Setting $\lambda = \lambda_1 = 1$, we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,,0\,,-1)^T$. Setting $\lambda = \lambda_2 = 2\,$, the reduced system of equations is

$$\begin{aligned} x_1 + 2\,x_2 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (-2\,,1\,,0)^T$. Finally, setting $\lambda = \lambda_3 = 3\,$, the reduced system of equations is

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (0\,,1\,,-1)^T$.

20. For computational purposes, note that if λ is an eigenvalue of $\mathbf{B}$, then $c\,\lambda$ is an eigenvalue of the matrix $\mathbf{A}= c\,\mathbf{B}$. Eigenvectors are unaffected, since they are only determined up to a scalar multiple. So with

$$\mathbf{B} = \begin{pmatrix} 11 & -2 & 8 \\ -2 & 2 & 10 \\ 8 & 10 & 5 \end{pmatrix},$$

the associated characteristic equation is $\mu^3 - 18\mu^2 - 81\mu + 1458 = 0$, with roots $\mu_1 = -9\,$, $\mu_2 = 9$ and $\mu_3 = 18\,$. Hence the eigenvalues of the given matrix, $\mathbf{A}$, are $\lambda_1 = -1\,$, $\lambda_2 = 1$ and $\lambda_3 = 2\,$. Setting $\lambda = \lambda_1 = -1\,$, (which corresponds to using $\mu_1 = -9$ in the modified problem) the reduced system of equations is

$$\begin{aligned} 2\,x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,,2\,,-2)^T$. Setting $\lambda = \lambda_2 = 1\,$, the reduced system of equations is

$$\begin{aligned} x_1 + 2\,x_3 &= 0 \\ x_2 - 2\,x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (2\,,-2\,,-1)^T$. Finally, setting $\lambda = \lambda_2 = 1\,$, the reduced system of equations is

$$\begin{aligned} x_1 - x_3 &= 0 \\ 2\,x_2 - x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (2\,,1\,,2)^T$.

21.(b) By definition,

$$(\mathbf{A}\mathbf{x}\,,\mathbf{y}) = \sum_{i=0}^{n} (\mathbf{A}\mathbf{x})_i\,\overline{y_i} = \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\,x_j\;\overline{y_i}\,.$$

Let $b_{ij} = \overline{a_{ji}}$, so that $a_{ij} = \overline{b_{ji}}$. Now interchanging the order or summation,

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{j=0}^{n} x_j \sum_{i=0}^{n} a_{ij}\,\overline{y_i} = \sum_{j=0}^{n} x_j \sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i}\,.$$

Now note that

$$\sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i} = \overline{\sum_{i=0}^{n} b_{ji}\,y_i} = \overline{(\mathbf{A}^*\mathbf{y})}_j\,.$$

Therefore

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{j=0}^{n} x_j\,\overline{(\mathbf{A}^*\mathbf{y})}_j = (\mathbf{x}\,,\mathbf{A}^*\mathbf{y})\,.$$

(c) By definition of a Hermitian matrix, $\mathbf{A}$=$\mathbf{A}^*$.

22. Suppose that $\mathbf{Ax}=\mathbf{0}$, but that $\mathbf{x}\neq\mathbf{0}$. Let $\mathbf{A}=(a_{ij})$. Using elementary row operations, it is possible to transform the matrix into one that is not upper triangular. If it were upper triangular, backsubstitution would imply that $\mathbf{x}=\mathbf{0}$. Hence a linear combination of all the rows results in a row containing only zeros. That is, there are n scalars, β_i, one for each row and not all zero, such that for each for column j,

$$\sum_{i=1}^{n} \beta_i\,a_{ij} = 0\,.$$

Now consider $\mathbf{A}^* = (b_{ij})$. By definition, $b_{ij} = \overline{a_{ji}}$, or $a_{ij} = \overline{b_{ji}}$. It follows that for each j,

$$\sum_{i=1}^{n} \beta_i\,\overline{b_{ji}} = \sum_{k=1}^{n} \overline{b_{jk}}\,\beta_k = \sum_{k=1}^{n} b_{jk}\,\overline{\beta_k} = 0\,.$$

Let $\mathbf{y}=(\overline{\beta_1},\overline{\beta_2},\cdots,\overline{\beta_n})^T$. Hence we have a nonzero vector, $\mathbf{y}$, such that $\mathbf{A}^*\mathbf{y}=\mathbf{0}$.

24. By linearity,

$$\mathbf{A}(\mathbf{x}^{(0)} + \alpha\,\boldsymbol{\xi}) = \mathbf{Ax}^{(0)} + \alpha\,\mathbf{A}\boldsymbol{\xi} = \mathbf{b} + \mathbf{0} = \mathbf{b}\,.$$

25. Let $c_{ij} = \overline{a_{ji}}$. By the hypothesis, there is a nonzero vector, $\mathbf{y}$, such that

$$\sum_{j=1}^{n} c_{ij}\,y_j = \sum_{j=1}^{n} \overline{a_{ji}}\;y_j = 0\,,\ i = 1,2,\cdots,n\,.$$

Taking the conjugate of both sides, and interchanging the indices, we have

$$\sum_{i=1}^{n} a_{ij}\;\overline{y_i} = 0\,.$$

This implies that a linear combination of each row of $\mathbf{A}$ is equal to zero. Now consider the augmented matrix $[\mathbf{A}\,|\mathbf{b}]$. Replace the last row by

$$\sum_{i=1}^{n} \overline{y_i}\,[a_{i1}\,, a_{i2}\,, \cdots, a_{in}\,, b_i] = \left[0\,, 0\,, \cdots, 0\,, \sum_{i=1}^{n} \overline{y_i}\, b_i\right].$$

We find that if $(\mathbf{b}\,, \mathbf{y}) = 0$, then the last row of the augmented matrix contains only zeros. Hence there are $n-1$ remaining equations. We can now set $x_n = \alpha$, some parameter, and solve for the other variables in terms of α . Therefore the system of equations $\mathbf{Ax}$=$\mathbf{b}$ has a solution.

26. If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then there is a nonzero vector, $\mathbf{x}$, such that

$$\mathbf{Ax} = \lambda\,\mathbf{x} = \mathbf{0}\,.$$

That is, $\mathbf{Ax}= \mathbf{0}$ has a nonzero solution. This implies that the mapping defined by $\mathbf{A}$ is not 1-to-1, and hence not invertible. On the other hand, if $\mathbf{A}$ is singular, then $\det(\mathbf{A}) = 0$. Thus, $\mathbf{Ax}= \mathbf{0}$ has a nonzero solution. The latter equation can be written as $\mathbf{Ax}= 0\,\mathbf{x}$.

27.(a) Based on Problem 21, $(\mathbf{Ax}\,, \mathbf{x}) = (\mathbf{x}\,, \mathbf{Ax})\,.$

(b) Let $\mathbf{x}$ be an eigenvector corresponding to an eigenvalue λ . It then follows that $(\mathbf{Ax}\,, \mathbf{x}) = (\lambda\mathbf{x}\,, \mathbf{x})$ and $(\mathbf{x}\,, \mathbf{Ax}) = (\mathbf{x}\,, \lambda\mathbf{x})$. Based on the properties of the inner product, $(\lambda\mathbf{x}\,, \mathbf{x}) = \lambda(\mathbf{x}\,, \mathbf{x})$ and $(\mathbf{x}\,, \lambda\mathbf{x}) = \overline{\lambda}(\mathbf{x}\,, \mathbf{x})$. Then from part (a),

$$\lambda(\mathbf{x}\,, \mathbf{x}) = \overline{\lambda}(\mathbf{x}\,, \mathbf{x})\,.$$

(c) From part (b),

$$(\lambda - \overline{\lambda})(\mathbf{x}\,, \mathbf{x}) = 0\,.$$

Based on the definition of an eigenvector, $(\mathbf{x}\,, \mathbf{x}) = \|\mathbf{x}\|^2 > 0$. Hence we must have $\lambda - \overline{\lambda} = 0$, which implies that λ is real.

28. From Problem 21(c),

$$(\mathbf{Ax}^{(1)}\,, \mathbf{x}^{(2)}) = (\mathbf{x}^{(1)}\,, \mathbf{Ax}^{(2)})\,.$$

Hence

$$\lambda_1(\mathbf{x}^{(1)}\,, \mathbf{x}^{(2)}) = \overline{\lambda_2}(\mathbf{x}^{(1)}\,, \mathbf{x}^{(2)}) = \lambda_2(\mathbf{x}^{(1)}\,, \mathbf{x}^{(2)})\,,$$

since the eigenvalues are real. Therefore

$$(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}\,, \mathbf{x}^{(2)}) = 0\,.$$

Given that $\lambda_1 \neq \lambda_2$, we must have $(\mathbf{x}^{(1)}\,, \mathbf{x}^{(2)}) = 0\,.$

7.4

1.(a) Let $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$, and $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. Then

$$\mathbf{x}^{(1)\,\prime} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t,$$
$$A\,\mathbf{x}^{(1)} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t = \mathbf{x}^{(1)\,\prime}$$

Likewise

$$\mathbf{x}^{(2)\,\prime} = -\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} e^{-t},$$
$$A\,\mathbf{x}^{(2)} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} e^{-t} = \mathbf{x}^{(2)\,\prime}$$

Thus the given functions are solutions of the given system of differential equations.

(b) Let $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \begin{pmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{pmatrix}$. Then

$$A\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1e^t + c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{pmatrix} = \begin{pmatrix} c_1e^{-t} - c_2e^{-t} \\ c_1e^t - 3c_2e^{-t} \end{pmatrix} = \mathbf{x}'$$

Thus $\mathbf{x}$ satisfies the system of differential equations.

(c) To show that $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$ is a fundamental set of solutions of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, we must first show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions. This was done in part a). We must also show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent for all t; that is, that the only solution of $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$ is $c_1 = c_2 = 0$. This equation may written as the system of equations

$$c_1e^t + c_2e^{-t} = 0$$
$$c_1e^t + 3c_2e^{-t} = 0$$

Subtracting these equations yields $-2c_2e^{-t} = 0$, so $c_2 = 0$. The first equation then becomes $c_1e^t = 0$, so $c_1 = 0$ as well.

(d) Since $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we have the system of equations

$$c_1 + c_2 = 1$$
$$c_1 + 3c_2 = 2$$

This system of equations may be solved to find that $c_1 = 1/2$ and $c_2 = 1/2$, so

$$\mathbf{x}(t) = \frac{1}{2}\mathbf{x}^{(1)} + \frac{1}{2}\mathbf{x}^{(2)} = \begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{pmatrix}$$

(e)

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \det \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} = e^t(3e^{-t}) - e^t(e^{-t}) = 3 - 1 = 2$$

(f) We compute that $(p_{11}(t) + p_{22}(t))W = (2 + (-2))W = 0$, while $W = 2$ implies that $W' = 0$. Thus Abel's equation is satisfied.

3.(a) Let $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$, $\mathbf{x}^{(1)} = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}$, and $\mathbf{x}^{(2)} = \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$. Then

$$\begin{aligned} \mathbf{x}^{(1)\,\prime} &= \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix}, \\ A\,\mathbf{x}^{(1)} &= \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} = \begin{pmatrix} -5\sin t \\ -2\sin t + \cos t \end{pmatrix} = \mathbf{x}^{(1)\,\prime} \end{aligned}$$

Likewise

$$\begin{aligned} \mathbf{x}^{(2)\,\prime} &= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}, \\ A\,\mathbf{x}^{(2)} &= \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix} = \begin{pmatrix} 5\cos t \\ \sin t + 2\cos t \end{pmatrix} = \mathbf{x}^{(2)\,\prime} \end{aligned}$$

Thus the given functions are solutions of the given system of differential equations.

(b) Let $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \begin{pmatrix} 5c_1\cos t + 5c_2\sin t \\ (2c_2 - c_1)\cos t + (c_1 + 2c_2)\sin t \end{pmatrix}$. Then

$$\begin{aligned} A\mathbf{x} &= \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5c_1\cos t + 5c_2\sin t \\ (2c_2 - c_1)\cos t + (c_1 + 2c_2)\sin t \end{pmatrix} \\ &= \begin{pmatrix} 5c_2\cos t - 5c_1\sin t \\ (c_1 + 2c_2)\cos t + (-2c_1 + c_2)\sin t \end{pmatrix} = \mathbf{x}' \end{aligned}$$

Thus $\mathbf{x}$ satisfies the system of differential equations.

(c) To show that $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$ is a fundamental set of solutions of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$, we must first show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions. This was done in part a). We must also show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent for all t; that is, that the only solution of $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$ is $c_1 = c_2 = 0$. This equation may written as the system of equations

$$\begin{aligned} 5c_1\cos t + 5c_2\sin t &= 0 \\ (2c_1 - c_2)\cos t + (c_1 + 2c_2)\sin t &= 0 \end{aligned}$$

Consider the first equation. Define φ so that $\cos\varphi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$ and $\sin\varphi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}$ if either $c_1 \neq 0$ or $c_2 \neq 0$, and define $\varphi = 0$ if $c_1 = c_2 = 0$. Then assuming that $c_1 \neq 0$

or $c_2 \neq 0$, the first equation becomes

$$\begin{aligned}5c_1 \cos t + 5c_2 \sin t &= 5\sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin t\right)\\ &= 5\sqrt{c_1^2+c_2^2}\,(\cos\varphi\cos t + \sin\varphi \sin t) = 5\sqrt{c_1^2+c_2^2}\cos(\varphi - t) = 0\end{aligned}$$

Since this equation has no solution for all t when $c_1 \neq 0$ or $c_2 \neq 0$ the assumption was false and $c_1 = c_2 = 0$.

(d) Since $\mathbf{x}(0) = \begin{pmatrix}1\\2\end{pmatrix}$, we have the system of equations

$$\begin{aligned}5c_1 &= 1\\ 2c_1 - c_2 &= 2\end{aligned}$$

This system of equations may be solved to find that $c_1 = 1/5$ and $c_2 = -8/5$, so

$$\mathbf{x}(t) = \frac{1}{5}\mathbf{x}^{(1)} - \frac{8}{5}\mathbf{x}^{(2)} = \begin{pmatrix}\cos t - 8\sin t\\ 2\cos t - 3\sin t\end{pmatrix}$$

(e)

$$\begin{aligned}W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) &= \det\begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t + \sin t & 2\sin t - \cos t\end{pmatrix}\\ &= 5\cos t(2\sin t - \cos t) - 5\sin t(2\cos t + \sin t)\\ &= -5\cos^2 t - 5\sin^2 t = -5\end{aligned}$$

(f) We compute that $(p_{11}(t) + p_{22}(t))W = (2 + (-2))W = 0$, while $W = -5$ implies that $W' = 0$. Thus Abel's equation is satisfied.

5.(a) Let $A = \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}$, $\mathbf{x}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}t$, and $\mathbf{x}^{(2)} = \begin{pmatrix}1\\3\end{pmatrix}t^{-1}$. Then

$$\begin{aligned}t\mathbf{x}^{(1)\prime} &= \begin{pmatrix}1\\1\end{pmatrix}t,\\ A\,\mathbf{x}^{(1)} &= \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}t = \begin{pmatrix}1\\1\end{pmatrix}t = t\mathbf{x}^{(1)\prime}\end{aligned}$$

Likewise

$$\begin{aligned}t\mathbf{x}^{(2)\prime} &= t\begin{pmatrix}1\\3\end{pmatrix}(-t^{-2}) = \begin{pmatrix}-1\\-3\end{pmatrix}t^{-1},\\ A\,\mathbf{x}^{(2)} &= \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}t^{-1} = \begin{pmatrix}-1\\-3\end{pmatrix}t^{-1} = t\mathbf{x}^{(2)\prime}\end{aligned}$$

Thus the given functions are solutions of the given system of differential equations.

(b) Let $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \begin{pmatrix} c_1t + c_2t^{-1} \\ c_1t + 3c_2t^{-1} \end{pmatrix}$. Then

$$A\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1t + c_2t^{-1} \\ c_1t + 3c_2t^{-1} \end{pmatrix} = \begin{pmatrix} c_1t - c_2t^{-1} \\ c_1t - 3c_2t^{-1} \end{pmatrix} = t \begin{pmatrix} c_1 - c_2t^{-2} \\ c_1 - 3c_2t^{-2} \end{pmatrix} = t\mathbf{x}'$$

Thus $\mathbf{x}$ satisfies the system of differential equations.

(c) To show that $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$ is a fundamental set of solutions of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, we must first show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions. This was done in part a). We must also show that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent for all t; that is, that the only solution of $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$ is $c_1 = c_2 = 0$. This equation may written as the system of equations

$$\begin{aligned} c_1t + c_2t^{-1} &= 0 \\ c_1t + 3c_2t^{-1} &= 0 \end{aligned}$$

Subtracting these equations yields $-2c_2t^{-1} = 0$, so $c_2 = 0$. The first equation then becomes $c_1t = 0$, so $c_1 = 0$ as well.

(d) Since $\mathbf{x}(2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we have the system of equations

$$\begin{aligned} 2c_1 + \frac{1}{2}c_2 &= 1 \\ 2c_1 + \frac{3}{2}c_2 &= 2 \end{aligned}$$

This system of equations may be solved to find that $c_1 = 1/4$ and $c_2 = 1$, so

$$\mathbf{x}(t) = \frac{1}{4}\mathbf{x}^{(1)} + \mathbf{x}^{(2)} = \begin{pmatrix} \frac{1}{4}t + t^{-1} \\ \frac{1}{4}t + 3t^{-1} \end{pmatrix}$$

(e)

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \det \begin{pmatrix} t & t^{-1} \\ t & 3t^{-1} \end{pmatrix} = t(3t^{-1}) - t(t^{-1}) = 3 - 1 = 2$$

(f) We compute that $(p_{11}(t) + p_{22}(t))W = ((2/t) + (-2/t))W = 0$, while $W = 2$ implies that $W' = 0$. Thus Abel's equation is satisfied.

9. Equation (14) states that the Wronskian satisfies the first order linear ODE

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$

The general solution of this is given by Equation (15):

$$W(t) = C\, e^{\int (p_{11} + p_{22} + \cdots + p_{nn})dt},$$

in which C is an arbitrary constant. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices representing two sets of fundamental solutions. It follows that

$$\det(\mathbf{X}_1) = W_1(t) = C_1 e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt}$$
$$\det(\mathbf{X}_2) = W_2(t) = C_2 e^{\int (p_{11}+p_{22}+\cdots+p_{nn})dt}\ .$$

Hence $\det(\mathbf{X}_1)/\det(\mathbf{X}_2) = C_1/C_2$. Note that $C_2 \neq 0$.

10. First note that $p_{11} + p_{22} = -p(t)$. As shown in Problem 3,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = c\, e^{-\int p(t)dt}.$$

For second order linear ODE, the Wronskian (as defined in Chapter 3) satisfies the first order differential equation $W' + p(t)W = 0$. It follows that

$$W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] = c_1\, e^{-\int p(t)dt}.$$

Alternatively, based on the hypothesis,

$$\mathbf{y}^{(1)} = \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12}$$
$$\mathbf{y}^{(2)} = \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12}\ .$$

Direct calculation shows that

$$\begin{aligned} W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] &= \begin{vmatrix} \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12} & \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12} \\ \alpha_{11}\, x_{11}' + \alpha_{12}\, x_{12}' & \alpha_{21}\, x_{11}' + \alpha_{22}\, x_{12}' \end{vmatrix} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{12}' - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{11}' \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{22} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{21}\ . \end{aligned}$$

Here we used the fact that $\mathbf{x}_1' = \mathbf{x}_2$. Hence

$$W\left[\mathbf{y}^{(1)}, \mathbf{y}^{(2)}\right] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right]\ .$$

11. The particular solution satisfies the ODE $(\mathbf{x}^{(p)})' = \mathbf{P}(t)\mathbf{x}^{(p)} + \mathbf{g}(t)$. Now let $\mathbf{x}$ be any solution of the homogeneous equation, $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. We know that $\mathbf{x} = \mathbf{x}^{(c)}$, in which $\mathbf{x}^{(c)}$ is a linear combination of some fundamental solution. By linearity of the differential equation, it follows that $\mathbf{x} = \mathbf{x}^{(p)} + \mathbf{x}^{(c)}$ is a solution of the ODE. Based on the uniqueness theorem, all solutions must have this form.

13.(a) By definition,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = (t^2 - 2t)e^t.$$

(b) The Wronskian vanishes at $t_0 = 0$ and $t_0 = 2$. Hence the vectors are linearly independent on $\mathcal{D} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

(c) It follows from Theorem 7.4.3 that one or more of the coefficients of the ODE must be discontinuous at $t_0 = 0$ and $t_0 = 2$. If not, the Wronskian would not vanish.

(d) Let

$$\mathbf{x} = c_1\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Then

$$\mathbf{x}' = c_1\begin{pmatrix} 2t \\ 2 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\mathbf{x} &= c_1\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} c_1\left[p_{11}t^2 + 2p_{12}t\right] + c_2\left[p_{11} + p_{12}\right]e^t \\ c_1\left[p_{21}t^2 + 2p_{22}t\right] + c_2\left[p_{21} + p_{22}\right]e^t \end{pmatrix}.\end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned} p_{11}t^2 + 2p_{12}t &= 2t \\ p_{11} + p_{12} &= 1 \\ p_{21}t^2 + 2p_{22}t &= 2 \\ p_{21} + p_{22} &= 1\,. \end{aligned}$$

Solution of this system of equations results in

$$p_{11}(t) = 0, \quad p_{12}(t) = 1, \quad p_{21}(t) = \frac{2-2t}{t^2-2t}, \quad p_{22}(t) = \frac{t^2-2}{t^2-2t}\,.$$

Hence the vectors are solutions of the ODE

$$\mathbf{x}' = \frac{1}{t^2-2t}\begin{pmatrix} 0 & t^2-2t \\ 2-2t & t^2-2 \end{pmatrix}\mathbf{x}\,.$$

14. Suppose that the solutions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(m)}$ are linearly dependent at $t = t_0$. Then there are constants $c_1, c_2, \cdots, c_m$ (not all zero) such that

$$c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}\,.$$

Now let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$. Then clearly, $\mathbf{z}(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, with $\mathbf{z}(t_0) = 0$. Furthermore, $\mathbf{y}(t) \equiv \mathbf{0}$ is also a solution, with $\mathbf{y}(t_0) = 0$. By the uniqueness theorem, $\mathbf{z}(t) = \mathbf{y}(t) = \mathbf{0}$. Hence

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t) = \mathbf{0}$$

on the entire interval $\alpha < t < \beta$. Going in the other direction is trivial.

15.(a) Let $\mathbf{y}(t)$ be any solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. It follows that

$$\mathbf{z}(t) + \mathbf{y}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{y}(t)$$

is also a solution. Now let $t_0 \in (\alpha, \beta)$. Then the collection of vectors

$$\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \ldots, \mathbf{x}^{(n)}(t_0), \ \mathbf{y}(t_0)$$

constitutes $n+1$ vectors, each with n components. Based on the assertion in Problem 10, Section 7.3, these vectors are necessarily linearly dependent. That is, there are $n+1$ constants $b_1, b_2, \ldots, b_n, b_{n+1}$ (not all zero) such that

$$b_1\mathbf{x}^{(1)}(t_0) + b_2\mathbf{x}^{(2)}(t_0) + \cdots + b_n\mathbf{x}^{(n)}(t_0) + b_{n+1}\,\mathbf{y}(t_0) = \mathbf{0}\,.$$

From Problem 14, we have

$$b_1\mathbf{x}^{(1)}(t) + b_2\mathbf{x}^{(2)}(t) + \cdots + b_n\mathbf{x}^{(n)}(t) + b_{n+1}\,\mathbf{y}(t) = \mathbf{0}$$

for all $t \in (\alpha\,, \beta)$. Now $b_{n+1} \neq 0$, otherwise that would contradict the fact that the first n vectors are linearly independent. Hence

$$\mathbf{y}(t) = -\frac{1}{b_{n+1}}(b_1\mathbf{x}^{(1)}(t) + b_2\mathbf{x}^{(2)}(t) + \cdots + b_n\mathbf{x}^{(n)}(t)),$$

and the assertion is true.

(b) Consider $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\ \mathbf{x}^{(2)}(t) + \cdots + c_n\ \mathbf{x}^{(n)}(t)$, and suppose that we also have

$$\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + k_2\mathbf{x}^{(2)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)\,.$$

Based on the assumption,

$$(k_1 - c_1)\mathbf{x}^{(1)}(t) + (k_2 - c_2)\mathbf{x}^{(2)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}\,.$$

The collection of vectors

$$\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$$

is linearly independent on $\alpha < t < \beta$. It follows that $k_i - c_i = 0$, for $i = 1, 2, \cdots, n$.

7.5

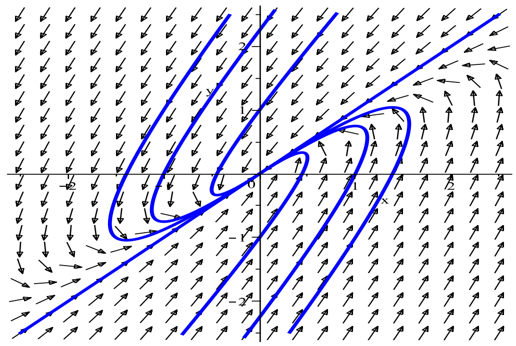
2.(a) Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 3r + 2 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = -2$. For $r = -1$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -2$ results in the single equation $3\xi_1 = 2\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}.$$

(b)



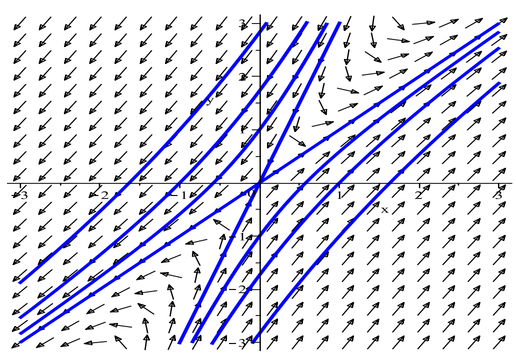
3.(a) Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -1$ results in the single equation $3\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

(b)



The system has an unstable eigendirection along $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Unless $c_1 = 0$, all solutions will diverge.

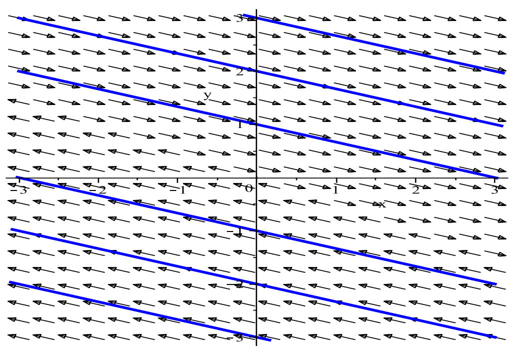
6.(a) Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & 6 \\ -1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = 0$. With $r = 1$, the system of equations reduces to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3, -1)^T$. For the case $r = 0$, the system is equivalent to the equation $\xi_1 + 2\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (2, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(b)



The entire line along the eigendirection $\boldsymbol{\xi}^{(2)} = (2, -1)^T$ consists of equilibrium points. All other solutions diverge. The direction field changes across the line $x_1 + 2x_2 = 0$. Eliminating the exponential terms in the solution, the trajectories are given by $x_1 + 3\,x_2 = -c_2$.

7. Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,, 1\,, 1)^T$. Setting $\lambda = 1$, the reduced system of equations is

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,-2\,,1)^T$. Finally, setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (1\,,0\,,-1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

8. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 6r^2 - 15r - 8 = 0$, with roots $r_1 = 8$, $r_2 = -1$ and $r_3 = -1$. Setting $r = r_1 = 8$, we have

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ 2\,\xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (2\,,1\,,2)^T$. Setting $r = -1$, the system of equations is reduced to the single equation

$$2\,\xi_1 + \xi_2 + 2\,\xi_3 = 0\,.$$

Two independent solutions are obtained as

$$\boldsymbol{\xi}^{(2)} = (1\,,-2\,,0)^T \text{ and } \boldsymbol{\xi}^{(3)} = (0\,,-2\,,1)^T.$$

Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t}.$$

10. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + 3\,c_2 &= -1\,. \end{aligned}$$

Hence $c_1 = 7/2$ and $c_2 = -3/2$, and the solution of the IVP is

$$\mathbf{x} = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

12. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 0 & -1 \\ 2 & -r & 0 \\ -1 & 2 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 4r^2 - r + 4 = 0$, with roots $r_1 = -1$, $r_2 = 1$ and $r_3 = 4$. Setting $r = r_1 = -1$, we have

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,-2\,,1)^T$. Setting $r = 1$, the system reduces to the equations

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2\,,-1)^T$. Finally, upon setting $r = 4$, the system is equivalent to the equations

$$\begin{aligned} 4\,\xi_1 + \xi_3 &= 0 \\ 8\,\xi_2 + \xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(3)} = (2\,,1\,,-8)^T$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}.$$

Invoking the initial conditions,

$$\begin{aligned} c_1 + c_2 + 2\,c_3 &= 7 \\ -2\,c_1 + 2\,c_2 + c_3 &= 5 \\ c_1 - c_2 - 8\,c_3 &= 5\,. \end{aligned}$$

It follows that $c_1 = 3$, $c_2 = 6$ and $c_3 = -1$. Hence the solution of the IVP is

$$\mathbf{x} = 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} + 6\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{t} - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} e^{4t}.$$

13. Set $\mathbf{x}= \boldsymbol{\xi}\, t^r$. Substitution into the system of differential equations results in

$$t \cdot rt^{r-1}\boldsymbol{\xi} \;= \mathbf{A}\,\boldsymbol{\xi}\, t^r,$$

which upon simplification yields is, $\mathbf{A}\,\boldsymbol{\xi} - r\boldsymbol{\xi} = \mathbf{0}$. Hence the vector $\boldsymbol{\xi}$ and constant r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$.

15. Setting $\mathbf{x}= \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

The Wronskian of this solution set is $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = 2t^6$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

16. As shown in Problem 13, solution of the ODE requires analysis of the equations

$$\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r = 0$. The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -2$. For $r = 0$, the system of equations reduces to $4\,\xi_1 = 3\,\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3\,,4)^T$. Setting $r = -2$ results in the single equation $2\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2)^T$. It follows that

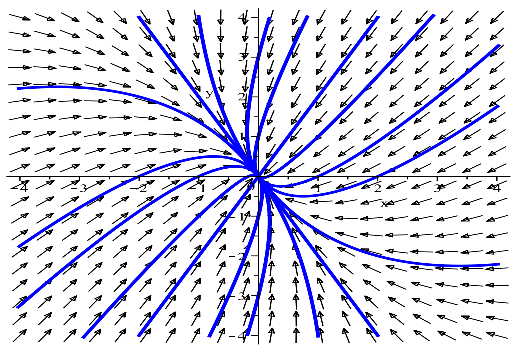
$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

The Wronskian of this solution set is $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = 2\,t^{-2}$. These solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

17.(a) The general solution is

$$x = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$



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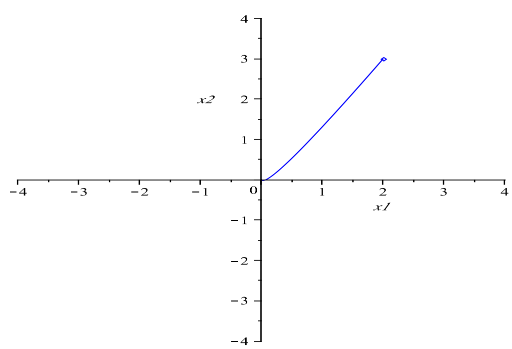

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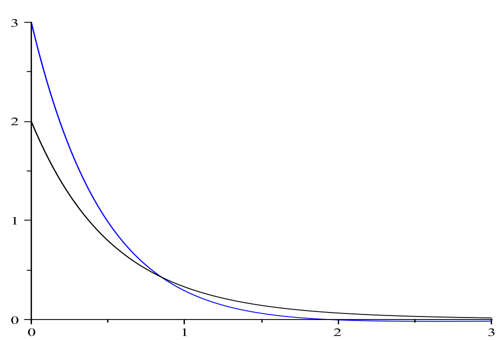


20.(a) We note that $(\mathbf{A} - r_i\mathbf{I})\boldsymbol{\xi}^{(i)} = \mathbf{0}$, for $i = 1, 2$.

(b) It follows that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{A}\,\boldsymbol{\xi}^{(1)} - r_2\boldsymbol{\xi}^{(1)} = r_1\boldsymbol{\xi}^{(1)} - r_2\boldsymbol{\xi}^{(1)}$.

(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then there exist constants c_1 and c_2, not both zero, such that $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$. Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)}) = \mathbf{0}$. On the other hand,

$$(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + \mathbf{0} = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}.$$

Since $r_1 \neq r_2$, we must have $c_1 = 0$, which leads to a contradiction.

(d) Note that $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(2)} = (r_2 - r_1)\boldsymbol{\xi}^{(2)}$.

(e) Let $n = 3$, with $r_1 \neq r_2 \neq r_3$. Suppose that $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ are indeed linearly dependent. Then there exist constants c_1, c_2 and c_3, not all zero, such that

$$c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)} = \mathbf{0}.$$

Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}) = \mathbf{0}$. On the other hand,

$$(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)}.$$

It follows that $c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)} = \mathbf{0}$. Based on the result of part (a), which is actually not dependent on the value of n, the vectors $\boldsymbol{\xi}^{(1)}$and $\boldsymbol{\xi}^{(3)}$ are linearly independent. Hence we must have $c_1(r_1 - r_2) = c_3(r_3 - r_2) = 0$, which leads to a contradiction.

21.(a) Let $x_1 = y$ and $x_2 = y'$. It follows that $x_1' = x_2$ and

$$x_2' = y'' = -\frac{1}{a}(c\,y + b\,y').$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{1}{a}(c\,x_1 + b\,x_2). \end{aligned}$$

(b) The coefficient matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.$$

Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have

$$\det(\mathbf{A} - r\mathbf{I}) = r^2 + \frac{b}{a}r + \frac{c}{a} = 0.$$

Multiplying both sides of the equation by a, we obtain $a\,r^2 + b\,r + c = 0$.

22.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -1/10 - r & 3/40 \\ 1/10 & -1/5 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = 0$. The characteristic equation is $80\,r^2 + 24\,r + 1 = 0$, with roots $r_1 = -1/4$ and $r_2 = -1/20$. With $r = -1/4$, the system of equations reduces to $2\,\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,, -2)^T$. Substitution of $r = -1/20$ results in the equation $2\,\xi_1 - 3\,\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (3\,, 2)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20}.$$

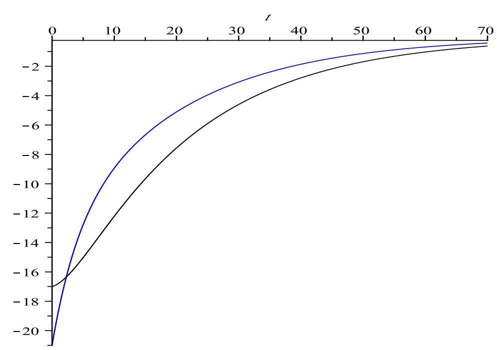
Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + 3\,c_2 &= -17 \\ -2\,c_1 + 2\,c_2 &= -21\,. \end{aligned}$$

Hence $c_1 = 29/8$ and $c_2 = -55/8$, and the solution of the IVP is

$$\mathbf{x} = \frac{29}{8} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4} - \frac{55}{8} \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20}.$$

(b)



(c) Both functions are monotone increasing. It is easy to show that $-0.5 \le x_1(t) < 0$ and $-0.5 \le x_2(t) < 0$ provided that $t > T \approx 74.39$.

24.(a) The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & -5/2 \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

Solution of the system requires analysis of the eigenvalue problem

$$\begin{pmatrix} -1/2 - r & -1/2 \\ 3/2 & -5/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2+3r+2=0$, with roots $r_1=-1$ and $r_2=-2$. With $r=-1$, the equations reduce to $\xi_1-\xi_2=0$. A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)}=(1,1)^T$. Setting $r=-2$, the system reduces to the equation $3\xi_1-\xi_2=0$. An eigenvector is $\boldsymbol{\xi}^{(2)}=(1,3)^T$. Hence the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}.$$

(b) The eigenvalues are distinct and both negative. We find that the equilibrium point $(0,0)$ is a stable node. Hence all solutions converge to $(0,0)$.

25.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -\frac{R_1}{L}-r & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is

$$r^2+(\frac{L+CR_1R_2}{LCR_2})r+\frac{R_1+R_2}{LCR_2}=0\,.$$

The eigenvectors are real and distinct, provided that the discriminant is positive. That is,

$$(\frac{L+CR_1R_2}{LCR_2})^2-4(\frac{R_1+R_2}{LCR_2})>0,$$

which simplifies to the condition

$$(\frac{1}{CR_2}-\frac{R_1}{L})^2-\frac{4}{LC}>0\,.$$

(b) The parameters in the ODE are all positive. Observe that the sum of the roots is

$$-\frac{L+CR_1R_2}{LCR_2}<0\,.$$

Also, the product of the roots is

$$\frac{R_1+R_2}{LCR_2}>0\,.$$

It follows that both roots are negative. Hence the equilibrium solution $I=0, V=0$ represents a stable node, which attracts all solutions.

(c) If the condition in part (a) is not satisfied, that is,

$$(\frac{1}{CR_2}-\frac{R_1}{L})^2-\frac{4}{LC}\le 0\,,$$

then the real part of the eigenvalues is

$$\mathrm{Re}(r_{1,2})=-\frac{L+CR_1R_2}{2\,LCR_2}\,.$$

As long as the parameters are all positive, then the solutions will still converge to the equilibrium point $(0\,,0)$.

7.6

1.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r + 5 = 0$. The roots of the characteristic equation are $r = -1 \pm 2i$. Substituting $r = -1 - 2i$, the two equations reduce to $\xi_1 + 2i\,\xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2i\,, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2i\,, 1)^T$. Hence one of the complex-valued solutions is given by

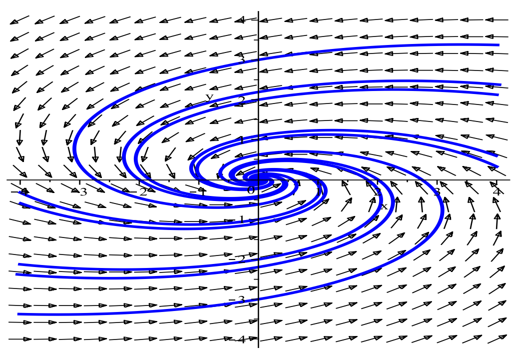
$$\mathbf{x}^{(1)} = \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-(1+2i)t} = \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-t}(\cos 2t - i \sin 2t) =$$

$$= e^{-t}\begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + i\, e^{-t}\begin{pmatrix} -2 \cos 2t \\ -\sin 2t \end{pmatrix}.$$

Based on the real and imaginary parts of this solution, the general solution is

$$\mathbf{x} = c_1\, e^{-t}\begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + c_2\, e^{-t}\begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}.$$

(b)



2.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,, 1)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos t + i \sin t) =$$

$$= \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.$$

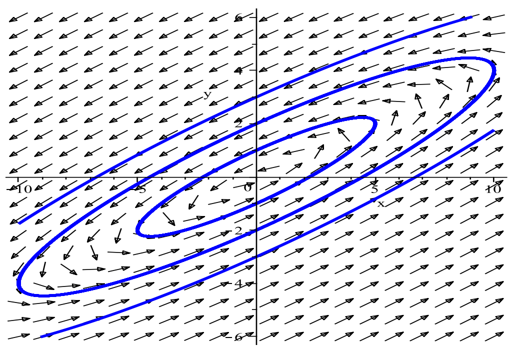
Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

(b)



3.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 - \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, 2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1, 2-i)^T$. Hence one of the complex-valued solutions is given by

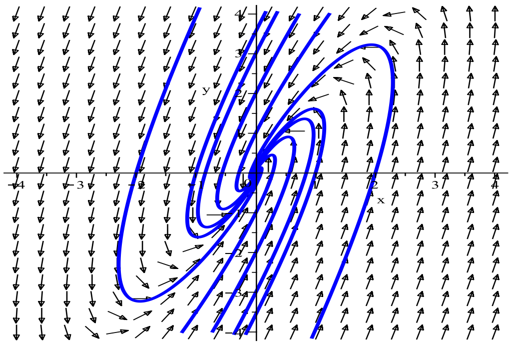
$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-(1+i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t}(\cos t - i\sin t) =$$

$$= e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

(b)



4.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 2 \\ -5 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 9 = 0$. The roots of the characteristic equation are $r = \pm 3\,i$. Setting $r = 3\,i$, the two equations reduce to $(1 - 3\,i)\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-2\,, 1 - 3i)^T$. Hence one of the complex-valued solutions is given by

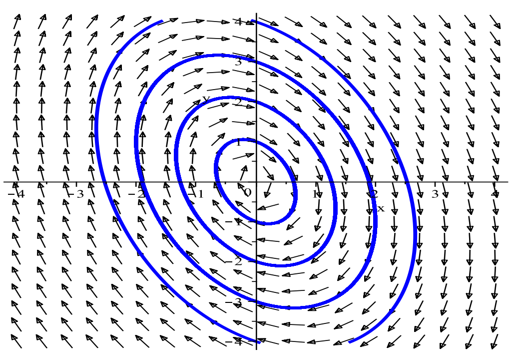
$$\mathbf{x}^{(1)} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} e^{3it} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} (\cos 3t + i \sin 3t) =$$

$$= \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + i \begin{pmatrix} -2 \sin 3t \\ -3 \cos 3t + \sin 3t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 2 \sin 3t \\ 3 \cos 3t - \sin 3t \end{pmatrix}.$$

(b)



6. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 + 4r^2 + 7r + 6 = 0$, with roots $r_1 = -2$, $r_2 = -1 - \sqrt{2}\, i$ and $r_3 = -1 + \sqrt{2}\, i$. Setting $r = -2$, the equations reduce to

$$\begin{aligned} -\xi_1 + 2\,\xi_3 &= 0 \\ \xi_1 + \xi_2 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2\,, -2\,, 1)^T$. With $r = -1 - \sqrt{2}\, i$, the system of equations is equivalent to

$$\begin{aligned} (2 - i\sqrt{2}\,)\xi_1 - 2\,\xi_3 &= 0 \\ \xi_1 + i\sqrt{2}\,\xi_2 &= 0\,. \end{aligned}$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (-i\sqrt{2}\,, 1\,, -1 - i\sqrt{2}\,)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-(1+i\sqrt{2})t} = \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-t}(\cos\sqrt{2}\,t - i\,\sin\sqrt{2}\,t) =$$

$$= e^{-t} \begin{pmatrix} -\sqrt{2}\,\sin\sqrt{2}\,t \\ \cos\sqrt{2}\,t \\ -\cos\sqrt{2}\,t - \sqrt{2}\,\sin\sqrt{2}\,t \end{pmatrix} + ie^{-t} \begin{pmatrix} -\sqrt{2}\,\cos\sqrt{2}\,t \\ -\sin\sqrt{2}\,t \\ -\sqrt{2}\,\cos\sqrt{2}\,t + \sin\sqrt{2}\,t \end{pmatrix}.$$

The other complex-valued solution is $\mathbf{x}^{(3)} = \overline{\boldsymbol{\xi}^{(2)}}\, e^{r_3 t}$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} +$$

$$+ c_2\, e^{-t} \begin{pmatrix} \sqrt{2}\,\sin\sqrt{2}\,t \\ -\cos\sqrt{2}\,t \\ \cos\sqrt{2}\,t + \sqrt{2}\,\sin\sqrt{2}\,t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sqrt{2}\,\cos\sqrt{2}\,t \\ \sin\sqrt{2}\,t \\ \sqrt{2}\,\cos\sqrt{2}\,t - \sin\sqrt{2}\,t \end{pmatrix}.$$

It is easy to see that all solutions converge to the equilibrium point $(0\,, 0\,, 0)$.

8. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 4\,r + 5 = 0$, with roots $r = -2 \pm i$. Substituting $r = -2 + i$, the equations are equivalent to $\xi_1 - (1 - i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1 - i\,, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} e^{-2t}(\cos\, t + i\,\sin\, t) =$$

$$= e^{-2t}\begin{pmatrix}\cos\, t + \sin\, t\\ \cos\, t\end{pmatrix} + ie^{-2t}\begin{pmatrix}-\cos\, t + \sin\, t\\ \sin\, t\end{pmatrix}.$$

Hence the general solution is

$$\mathbf{x} = c_1\, e^{-2t}\begin{pmatrix}\cos\, t + \sin\, t\\ \cos\, t\end{pmatrix} + c_2\, e^{-2t}\begin{pmatrix}-\cos\, t + \sin\, t\\ \sin\, t\end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 - \, c_2 &= 1\\ c_1 &= -2\,. \end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

$$\begin{aligned} \mathbf{x} &= -2\, e^{-2t}\begin{pmatrix}\cos\, t + \sin\, t\\ \cos\, t\end{pmatrix} - 3\, e^{-2t}\begin{pmatrix}-\cos\, t + \sin\, t\\ \sin\, t\end{pmatrix}\\ &= e^{-2t}\begin{pmatrix}\cos\, t - 5\, \sin\, t\\ -2\, \cos\, t - 3\, \sin\, t\end{pmatrix}. \end{aligned}$$

The solution converges to $(0, 0)$ as $t \to \infty$.

10. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix}-\frac{4}{5} - r & 2\\ -1 & \frac{6}{5} - r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

The characteristic equation is $25\, r^2 - 10\, r + 26 = 0$, with roots $r = 1/5 \pm i$. Setting $r = 1/5 + i$, the two equations reduce to $\xi_1 - (1 - i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1 - i\,, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix}1 - i\\ 1\end{pmatrix} e^{(\frac{1}{5}+i)t} = \begin{pmatrix}1 - i\\ 1\end{pmatrix} e^{t/5}(\cos\, t + i\, \sin\, t) =$$

$$= e^{t/5}\begin{pmatrix}\cos\, t + \sin\, t\\ \cos\, t\end{pmatrix} + ie^{t/5}\begin{pmatrix}-\cos\, t + \sin\, t\\ \sin\, t\end{pmatrix}.$$

Hence the general solution is

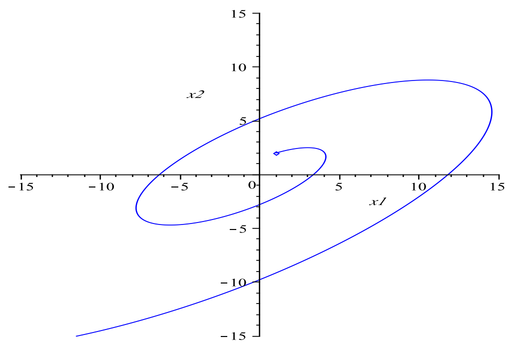
$$\mathbf{x} = c_1 e^{t/5}\begin{pmatrix}\cos\, t + \sin\, t\\ \cos\, t\end{pmatrix} + c_2 e^{t/5}\begin{pmatrix}-\cos\, t + \sin\, t\\ \sin\, t\end{pmatrix}.$$

(b) Let $\mathbf{x}(0) = (x_1^0\ , x_2^0)^T$. The solution of the initial value problem is

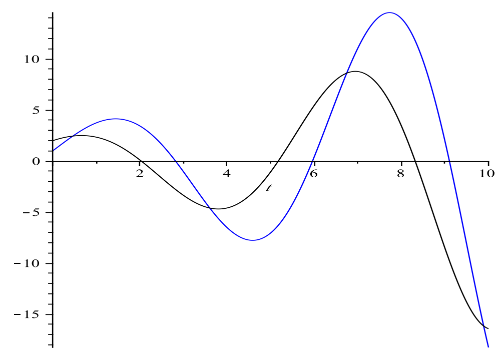
$$\begin{aligned} \mathbf{x} &= x_2^0\, e^{t/5}\begin{pmatrix}\cos\, t + \sin\, t\\ \cos\, t\end{pmatrix} + (x_2^0 - x_1^0)e^{t/5}\begin{pmatrix}-\cos\, t + \sin\, t\\ \sin\, t\end{pmatrix}\\ &= e^{t/5}\begin{pmatrix}x_1^0\, \cos\, t + (2\, x_2^0 - x_1^0)\, \sin\, t\\ x_2^0\, \cos\, t + (x_2^0 - x_1^0)\, \sin\, t\end{pmatrix}. \end{aligned}$$

With $\mathbf{x}(0) = (1\ , 2)^T$, the solution is

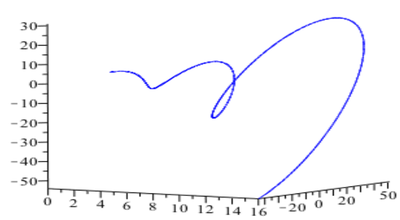
$$\mathbf{x} = e^{t/5}\begin{pmatrix}\cos\, t + 3\, \sin\, t\\ 2\, \cos\, t + \sin\, t\end{pmatrix}.$$



(c)



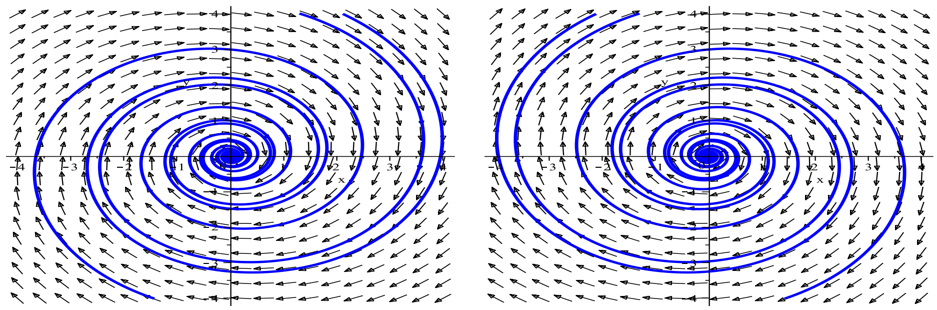
(d)



11.(a) The characteristic equation is $r^2 - 2\alpha r + 1 + \alpha^2 = 0$, with roots $r = \alpha \pm i$.

(b) When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0, 0)$ is a stable spiral and an unstable spiral, respectively. The equilibrium point is a center when $\alpha = 0$.

(c)



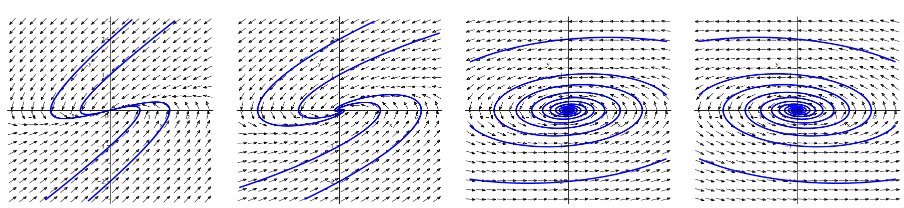
(a) $\alpha = -1/8$ (b) $\alpha = 1/8$

12.(a) The roots of the characteristic equation, $r^2 - \alpha\, r + 5 = 0$, are

$$r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}\,.$$

(b) Note that the roots are complex when $-\sqrt{20} < \alpha < \sqrt{20}$. For the case when $\alpha \in (-\sqrt{20}\,, 0)$, the equilibrium point $(0\,,0)$ is a stable spiral. On the other hand, when $\alpha \in (0\,,\sqrt{20}\,)$, the equilibrium point is an unstable spiral. For the case $\alpha = 0$, the roots are purely imaginary, so the equilibrium point is a center. When $\alpha^2 > 20$, the roots are real and distinct. The equilibrium point becomes a node, with its stability dependent on the sign of α. Finally, the case $\alpha^2 = 20$ marks the transition from spirals to nodes.

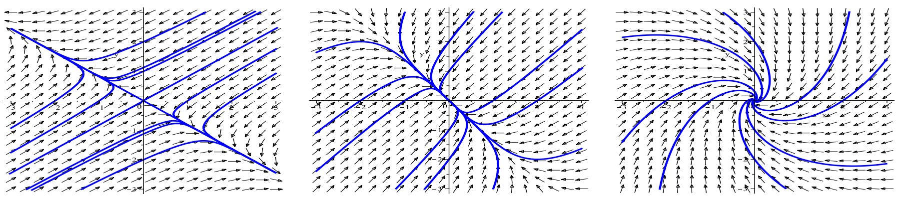
(c)



(a) $\alpha = -5$ (b) $\alpha = -3$ (c) $\alpha = -1/2$ (d) $\alpha = 1/2$

14. The characteristic equation of the coefficient matrix is $r^2 + 2r + 1 + \alpha = 0$, with roots given formally as $r_{1,2} = -1 \pm \sqrt{-\alpha}$. The roots are real provided that $\alpha \le 0$. First note that the sum of the roots is -2 and the product of the roots is $1 + \alpha$. For negative values of α, the roots are distinct, with one always negative.

When $\alpha < -1$, the roots have opposite signs. Hence the equilibrium point is a saddle. For the case $-1 < \alpha < 0$, the roots are both negative, and the equilibrium point is a stable node. $\alpha = -1$ represents a transition from saddle to node. When $\alpha = 0$, both roots are equal. For the case $\alpha > 0$, the roots are complex conjugates, with negative real part. Hence the equilibrium point is a stable spiral.

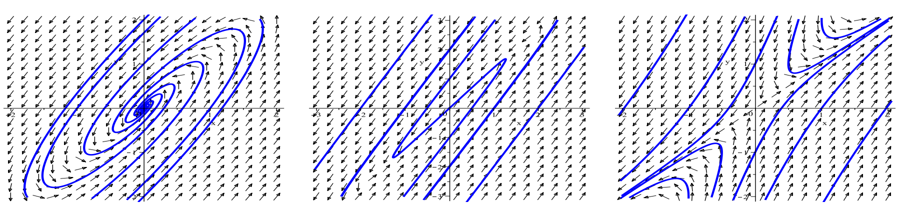


(a) $\alpha = -3/2$ (b) $\alpha = -1/2$ (c) $\alpha = 1/2$

15. The characteristic equation is $r^2 + 2\,r - (24 + 8\alpha) = 0$, with roots

$$r_{1,2} = -1 \pm \sqrt{25 + 8\alpha}\,.$$

The roots are complex when $\alpha < -25/8$. Since the real part is negative, the origin is a stable spiral. Otherwise the roots are real. When $-25/8 < \alpha < -3$, both roots are negative, and hence the equilibrium point is a stable node. For $\alpha > -3$, the roots are of opposite sign and the origin is a saddle.



(a) $\alpha = -4$ (b) $\alpha = -3.05$ (c) $\alpha = -2$

17. Based on the method in Problem 13 of Section 7.5, setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation for the system is $r^2 + 1 = 0$, with roots $r_{1,2} = \pm\, i$. With $r = i$, the equations reduce to the single equation $\xi_1 - (2+i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$. One complex-valued solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} t^i\,.$$

We can write $t^i = e^{i\ln t}$. Hence

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{i\,\ln t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} [\cos(\ln t) + i\,\sin(\ln t)] =$$

$$= \begin{pmatrix} 2\,\cos(\ln t) - \sin(\ln t) \\ \cos(\ln t) \end{pmatrix} + i \begin{pmatrix} \cos(\ln t) + 2\,\sin(\ln t) \\ \sin(\ln t) \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\,\cos(\ln t) - \sin(\ln t) \\ \cos(\ln t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(\ln t) + 2\,\sin(\ln t) \\ \sin(\ln t) \end{pmatrix}.$$

Other combinations are also possible.

19.(a) The characteristic equation of the system is

$$r^3 + \frac{2}{5}r^2 + \frac{81}{80}r - \frac{17}{160} = 0\,,$$

with eigenvalues $r_1 = 1/10$, and $r_{2,3} = -1/4 \pm i$. For $r = 1/10$, simple calculations reveal that a corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (0,0,1)^T$. Setting $r = -1/4 - i$, we obtain the system of equations

$$\begin{aligned} \xi_1 - i\,\xi_2 &= 0 \\ \xi_3 &= 0\,. \end{aligned}$$

A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (i\,,1\,,0)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10}.$$

Another solution, which is complex-valued, is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-(\frac{1}{4}+i)t} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-t/4}(\cos\,t - i\,\sin t) =$$

$$= e^{-t/4} \begin{pmatrix} \sin\,t \\ \cos\,t \\ 0 \end{pmatrix} + ie^{-t/4} \begin{pmatrix} \cos\,t \\ -\sin\,t \\ 0 \end{pmatrix}.$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

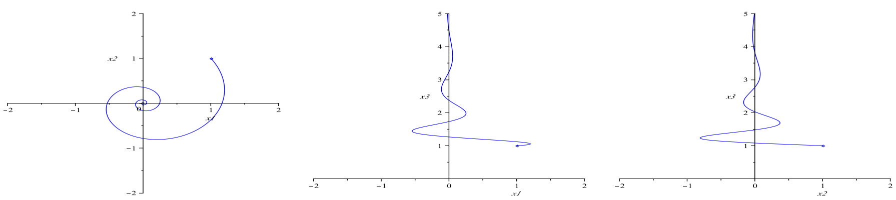
$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10} + c_2\,e^{-t/4} \begin{pmatrix} \sin\,t \\ \cos\,t \\ 0 \end{pmatrix} + c_3\,e^{-t/4} \begin{pmatrix} \cos\,t \\ -\sin\,t \\ 0 \end{pmatrix}.$$

(b) Let $\mathbf{x}(0) = (x_1^0\,, x_2^0\,, x_3^0)$. The solution can be written as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3^0\ e^{t/10} \end{pmatrix} + e^{-t/4} \begin{pmatrix} x_2^0 \text{ sin } t + x_1^0 \text{ cos } t \\ x_2^0 \text{ cos } t - x_1^0 \text{ sin } t \\ 0 \end{pmatrix}.$$

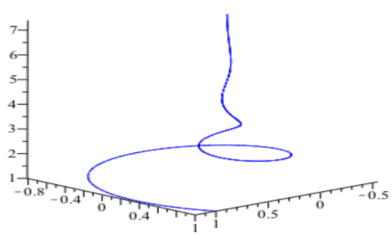
With $\mathbf{x}(0) = (1\,,1\,,1)$, the solution of the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ e^{t/10} \end{pmatrix} + e^{-t/4} \begin{pmatrix} \text{sin } t + \text{cos } t \\ \text{cos } t - \text{sin } t \\ 0 \end{pmatrix}.$$



(a) $x_1 - x_2$ (b) $x_1 - x_3$ (c) $x_2 - x_3$

(c)



20.(a) Based on Problems 16-17 of Section 7.1, the system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix},$$

since $R_1 = R_2 = 4$ ohms, $C = 1/2$ farads and $L = 8$ henrys.

(b) The eigenvalue problem is

$$\begin{pmatrix} -\frac{1}{2} - r & -\frac{1}{8} \\ 2 & -\frac{1}{2} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the system is $r^2+r+\frac{1}{2}=0$, with eigenvalues

$$r_{1,2}=-\frac{1}{2}\pm\frac{1}{2}i\,.$$

Setting $r=-1/2+i/2$, the algebraic equations reduce to $4i\xi_1+\xi_2=0$. It follows that $\boldsymbol{\xi}^{(1)}=(1\,,-4i)^T$. Hence one complex-valued solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)}=\begin{pmatrix} 1 \\ -4i \end{pmatrix}e^{(-1+i)t/2}=\begin{pmatrix} 1 \\ -4i \end{pmatrix}e^{-t/2}\left[\cos(t/2)+i\,\sin(t/2)\right]=$$

$$=e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\,\sin(t/2) \end{pmatrix}+ie^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\,\cos(t/2) \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}=c_1e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\,\sin(t/2) \end{pmatrix}+c_2e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\,\cos(t/2) \end{pmatrix}.$$

(c) Imposing the initial conditions, we arrive at the equations $c_1=2$ and $c_2=-3/4$, and

$$\begin{pmatrix} I \\ V \end{pmatrix}=e^{-t/2}\begin{pmatrix} 2\,\cos(t/2)-\frac{3}{4}\,\sin(t/2) \\ 8\,\sin(t/2)+3\,\cos(t/2) \end{pmatrix}.$$

(d) Since the eigenvalues have negative real parts, all solutions converge to the origin.

22.(a) Suppose that $c_1\mathbf{a}+c_2\mathbf{b}=\mathbf{0}$. Since $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the vector $\boldsymbol{\xi}^{(1)}$, respectively, $\mathbf{a}=(\boldsymbol{\xi}^{(1)}+\overline{\boldsymbol{\xi}^{(1)}})/2$ and $\mathbf{b}=(\boldsymbol{\xi}^{(1)}-\overline{\boldsymbol{\xi}^{(1)}})/2i$. Hence

$$c_1(\boldsymbol{\xi}^{(1)}+\overline{\boldsymbol{\xi}^{(1)}})-ic_2(\boldsymbol{\xi}^{(1)}-\overline{\boldsymbol{\xi}^{(1)}})=\mathbf{0}\,,$$

which leads to

$$(c_1-ic_2)\boldsymbol{\xi}^{(1)}+(c_1+ic_2)\overline{\boldsymbol{\xi}^{(1)}}=\mathbf{0}\,.$$

(b) Now since $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$ are linearly independent, we must have

$$\begin{aligned} c_1-ic_2&=0 \\ c_1+ic_2&=0\,. \end{aligned}$$

It follows that $c_1=c_2=0$.

(c) Recall that

$$\begin{aligned} \mathbf{u}(t)&=e^{\lambda t}(\mathbf{a}\,\cos\,\mu t-\mathbf{b}\,\sin\,\mu t) \\ \mathbf{v}(t)&=e^{\lambda t}(\mathbf{a}\,\cos\,\mu t+\mathbf{b}\,\sin\,\mu t)\,. \end{aligned}$$

Consider the equation $c_1\mathbf{u}(t_0)+c_2\mathbf{v}(t_0)=\mathbf{0}$, for some t_0. We can then write

$$c_1e^{\lambda t_0}(\mathbf{a}\,\cos\,\mu t_0-\mathbf{b}\,\sin\,\mu t_0)+c_2e^{\lambda t_0}(\mathbf{a}\,\cos\,\mu t_0+\mathbf{b}\,\sin\,\mu t_0)=\mathbf{0}\,.\ (*)$$

Rearranging the terms, and dividing by the exponential,

$$(c_1+c_2)\cos\,\mu t_0\,\mathbf{a}+(c_2-c_1)\sin\,\mu t_0\,\mathbf{b}=\mathbf{0}\,.$$

From part (b), since **a** and **b** are linearly independent, it follows that

$$(c_1 + c_2) \cos \mu t_0 = (c_2 - c_1) \sin \mu t_0 = 0 \,.$$

Without loss of generality, assume that the trigonometric factors are nonzero. Otherwise proceed again from Equation $(*)$, above. We then conclude that

$$c_1 + c_2 = 0 \;\text{ and }\; c_2 - c_1 = 0 \,,$$

which leads to $c_1 = c_2 = 0$. Thus $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ are linearly independent for some t_0, and hence the functions are linearly independent at every point.

23.(a) Let $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{k}{m}\, u \,.$$

In terms of the new variables, we obtain the system of two first order ODEs

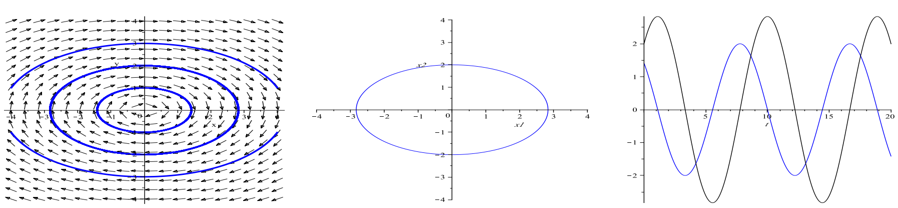
$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}\, x_1 \,. \end{aligned}$$

(b) The associated eigenvalue problem is

$$\begin{pmatrix} -r & 1 \\ -k/m & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + k/m = 0$, with roots $r_{1,2} = \pm\, i\sqrt{k/m}$.

(c) Since the eigenvalues are purely imaginary, the origin is a center. Hence the phase curves are ellipses, with a clockwise flow. For computational purposes, let $k = 1$ and $m = 2$.



(a) $k = 1$, $m = 2$ (b) $x_1 - x_2$ (c) x_1, x_2 vs t

(d) The general solution of the second order equation is

$$u(t) = c_1 \cos \sqrt{\frac{k}{m}}\, t + c_2 \sin \sqrt{\frac{k}{m}}\, t \,.$$

The general solution of the system of ODEs is given by

$$\mathbf{x} = c_1 \begin{pmatrix} \sqrt{\frac{m}{k}}\, \sin \sqrt{\frac{k}{m}}\, t \\ \cos \sqrt{\frac{k}{m}}\, t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{\frac{m}{k}}\, \cos \sqrt{\frac{k}{m}}\, t \\ - \sin \sqrt{\frac{k}{m}}\, t \end{pmatrix}.$$

It is evident that the natural frequency of the system is equal to $|r_1| = |r_2|$.

24.(a) Set $\mathbf{x} = (x_1\,, x_2\,)^T$. We can rewrite Equation (22) in the form

$$\begin{pmatrix} 2 & 0 \\ 0 & 9/4 \end{pmatrix} \begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -\frac{27}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Multiplying both sides of this equation by the inverse of the diagonal matrix, we obtain

$$\begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = \begin{pmatrix} -2 & 3/2 \\ 4/3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) Substituting $\mathbf{x} = \boldsymbol{\xi}\, \mathrm{e}^{rt}$,

$$r^2 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt} = \begin{pmatrix} -2 & 3/2 \\ 4/3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt},$$

which can be written as

$$(\mathbf{A} - r^2 \mathbf{I})\boldsymbol{\xi} = \mathbf{0}\ .$$

(c) The eigenvalues are $r_1^2 = -1$ and $r_2^2 = -4$, with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

(d) The linearly independent solutions are

$$\mathbf{x}^{(1)} = \tilde{C}_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{it} \text{ and } \mathbf{x}^{(2)} = \tilde{C}_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{2it}\,.$$

in which $\tilde{C}_1$ and $\tilde{C}_2$ are arbitrary complex coefficients. In scalar form,

$$x_1 = 3c_1 \cos\, t + 3c_2 \sin\, t + 3c_3 \cos\, 2t + 3c_4 \sin\, 2t$$
$$x_2 = 2c_1 \cos\, t + 2c_2 \sin\, t - 4c_3 \cos\, 2t - 4c_4 \sin\, 2t$$

(e) Differentiating the above expressions,

$$x_1' = -3c_1 \sin\, t + 3c_2 \cos\, t - 6c_3 \sin\, 2t + 6c_4 \cos\, 2t$$
$$x_2' = -2c_1 \sin\, t + 2c_2 \cos\, t + 8c_3 \sin\, 2t - 8c_4 \cos\, 2t$$

It is evident that $\mathbf{y} = (x_1, x_2, x_1', x_2')^T$ as in Equation (31).

7.7

Each of the problems 1 through 8, except 2 and 6, has been solved in one of the previous sections. Thus a fundamental matrix for the given systems can be readily written down. The fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(0) = \mathbf{I}$ can then be found as shown in the following problems.

1.(a) The eigenvalues and eigenvectors were found in Problem 1, Section 7.5.

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ 2\,e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2\,e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-t} & 2\,e^{2t} \\ 2\,e^{-t} & e^{2t} \end{pmatrix}.$$

(b) We now have

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \mathbf{\Psi}^{-1}(0) = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

So that

$$\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0) = \frac{1}{3} \begin{pmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3.(a) The general solution, found in Problem 2, Section 7.6, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} 5 \cos t & 5 \sin t \\ 2 \cos t + \sin t & -\cos t + 2 \sin t \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$5c_1 = 1$$
$$2c_1 - c_2 = 0\,,$$

resulting in $c_1 = 1/5\,,\ c_2 = 2/5\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$5c_1 = 0$$
$$2c_1 - c_2 = 1\,,$$

resulting in $c_1 = 0$, $c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -5\sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}.$$

5.(a) The general solution, found in Problem 10, Section 7.5, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 + c_2 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2$, $c_2 = 3/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} \\ -3e^{2t} + 3e^{4t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 + c_2 &= 1\,, \end{aligned}$$

resulting in $c_1 = 1/2$, $c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} e^{2t} - e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & e^{2t} - e^{4t} \\ -3e^{2t} + 3e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.$$

6.(a) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-t}\cos t & e^{-t}\sin t \\ 2e^{-t}\cos t + e^{-t}\sin t & -e^{-t}\cos t + 2e^{-t}\sin t \end{pmatrix}.$$

(b) The specific solution corresponding to the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$ is

$$\mathbf{x} = e^{-t}\begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix}.$$

For the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, the solution is

$$\mathbf{x} = e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = e^{-t}\begin{pmatrix} \cos t + 2\sin t & -\sin t \\ 5\sin t & \cos t - 2\sin t \end{pmatrix}.$$

7.(a) The general solution is given by

$$\mathbf{x} = c_1\begin{pmatrix} 4e^{-2t} \\ -5e^{-2t} \\ -7e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ -4e^{-t} \\ -2e^{-t} \end{pmatrix} + c_3\begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} 4e^{-2t} & 3e^{-t} & 0 \\ -5e^{-2t} & -4e^{-t} & e^{2t} \\ -7e^{-2t} & -2e^{-t} & -e^{2t} \end{pmatrix}.$$

(b) Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 1 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2\,,\ c_2 = 1\,,\ c_3 = 3/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -2e^{-2t} + 3e^{-t} \\ 5e^{-2t}/2 - 4e^{-t} + 3e^{2t}/2 \\ 7e^{-2t}/2 - 2e^{-t} - 3e^{2t}/2 \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 1 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/4\,,\ c_2 = 1/3\,,\ c_3 = 13/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ 5e^{-2t}/4 - 4e^{-t}/3 + 13e^{2t}/12 \\ 7e^{-2t}/4 - 2e^{-t}/3 - 13e^{2t}/12 \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(3)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 1\,, \end{aligned}$$

resulting in $c_1 = -1/4\,,\ c_2 = 1/3\,,\ c_3 = 1/12$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ 5e^{-2t}/4 - 4e^{-t}/3 + e^{2t}/12 \\ 7e^{-2t}/4 - 2e^{-t}/3 - e^{2t}/12 \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{12}\begin{pmatrix} -24e^{-2t} + 36e^{-t} & -12e^{-2t} + 12e^{-t} & -12e^{-2t} + 12e^{-t} \\ 30e^{-2t} - 48e^{-t} + 18e^{2t} & 15e^{-2t} - 16e^{-t} + 13e^{2t} & 15e^{-2t} - 16e^{-t} + e^{2t} \\ 42e^{-2t} - 24e^{-t} - 18e^{2t} & 21e^{-2t} - 8e^{-t} - 13e^{2t} & 21e^{-2t} - 8e^{-t} - e^{2t} \end{pmatrix}.$$

9. The solution of the initial value problem is given by

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}(0) = \begin{pmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ \frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} =$$

$$= e^{-t}\begin{pmatrix} 3\cos 2t - 2\sin 2t \\ \frac{3}{2}\sin 2t + \cos 2t \end{pmatrix}.$$

10. Let

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}(t_0) = \begin{pmatrix} x_1^{(1)}(t_0) & \cdots & x_1^{(n)}(t_0) \\ \vdots & & \vdots \\ x_n^{(1)}(t_0) & \cdots & x_n^{(n)}(t_0) \end{pmatrix}$$

is a scalar matrix, which is invertible, since the solutions are linearly independent. Let $\mathbf{\Psi}^{-1}(t_0) = (c_{ij})$. Then

$$\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

The j-th column of the product matrix is

$$\left[\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\right]^{(j)} = \sum_{k=1}^{n} c_{kj}\,\mathbf{x}^{(k)},$$

which is a solution vector, since it is a linear combination of solutions. Furthermore, the columns are all linearly independent, since the vectors $\mathbf{x}^{(k)}$ are. Hence the product is a fundamental matrix. Finally, setting $t = t_0$, $\mathbf{\Psi}(t_0)\mathbf{\Psi}^{-1}(t_0) = \mathbf{I}$. This is precisely the definition of $\mathbf{\Phi}(t)$.

11. The fundamental matrix $\mathbf{\Phi}(t)$ for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

is given by

$$\mathbf{\Phi}(t) = \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Direct multiplication results in

$$\begin{aligned} \mathbf{\Phi}(t)\mathbf{\Phi}(s) &= \frac{1}{16} \begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} \begin{pmatrix} 2e^{3s} + 2e^{-s} & e^{3s} - e^{-s} \\ 4e^{3s} - 4e^{-s} & 2e^{3s} + 2e^{-s} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 8(e^{3t+3s} + e^{-t-s}) & 4(e^{3t+3s} - e^{-t-s}) \\ 16(e^{3t+3s} - e^{-t-s}) & 8(e^{3t+3s} + e^{-t-s}) \end{pmatrix}. \end{aligned}$$

Hence

$$\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \frac{1}{4} \begin{pmatrix} 2e^{3(t+s)} + 2e^{-(t+s)} & e^{3(t+s)} - e^{-(t+s)} \\ 4e^{3(t+s)} - 4e^{-(t+s)} & 2e^{3(t+s)} + 2e^{-(t+s)} \end{pmatrix} = \mathbf{\Phi}(t+s).$$

12.(a) Let s be arbitrary, but fixed, and t variable. Similar to the argument in Problem 10, the columns of the matrix $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are linear combinations of fundamental solutions. Hence the columns of $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are also solution of the system of equations. Further, setting $t = 0$, $\mathbf{\Phi}(0)\mathbf{\Phi}(s) = \mathbf{I}\,\mathbf{\Phi}(s) = \mathbf{\Phi}(s)$. That is, $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ is a solution of the initial value problem $\mathbf{Z}' = \mathbf{A}\mathbf{Z}$, with $\mathbf{Z}(0) = \mathbf{\Phi}(s)$. Now consider the change of variable $\tau = t + s$. Let $\mathbf{W}(\tau) = \mathbf{Z}(\tau - s)$. The given initial value problem can be reformulated as

$$\frac{d}{d\tau}\mathbf{W} = \mathbf{A}\mathbf{W}\,, \text{ with } \mathbf{W}(s) = \mathbf{\Phi}(s)\,.$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$, with $\mathbf{\Phi}(0) = \mathbf{I}$, it follows that

$$\mathbf{W}(\tau) = \left[\mathbf{\Phi}(\tau)\mathbf{\Phi}^{-1}(s)\right] \mathbf{\Phi}(s) = \mathbf{\Phi}(\tau).$$

That is, $\mathbf{\Phi}(t+s) = \mathbf{\Phi}(\tau) = \mathbf{W}(\tau) = \mathbf{Z}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(s)$.

(b) Based on part (a), $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{\Phi}(t + (-t)) = \mathbf{\Phi}(0) = \mathbf{I}$. Hence $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$.

(c) It also follows that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t + (-s)) = \mathbf{\Phi}(t)\mathbf{\Phi}(-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

13. Let $\mathbf{A}$ be a diagonal matrix, with $\mathbf{A} = \left[a_1\mathbf{e}^{(1)}, a_2\mathbf{e}^{(2)}, \cdots, a_n\mathbf{e}^{(n)}\right]$. Note that for any positive integer k,

$$\mathbf{A}^k = \left[a_1^k\,\mathbf{e}^{(1)}, a_2^k\,\mathbf{e}^{(2)}, \cdots, a_n^k\,\mathbf{e}^{(n)}\right].$$

It follows, from basic matrix algebra, that

$$\mathbf{I} + \sum_{k=1}^{m} \mathbf{A}^k \frac{t^k}{k!} = \begin{pmatrix} \sum_{k=0}^{m} a_1^k \frac{t^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{m} a_2^k \frac{t^k}{k!} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{m} a_n^k \frac{t^k}{k!} \end{pmatrix}.$$

It can be shown that the partial sums on the left hand side converge for all t. Taking the limit as $m \to \infty$ on both sides of the equation, we obtain

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{a_1 t} & 0 & \cdots & 0 \\ 0 & e^{a_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{a_n t} \end{pmatrix}.$$

Alternatively, consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Since the ODEs are uncoupled, the vectors $\mathbf{x}^{(j)} = e^{a_j t}\,\mathbf{e}^{(j)}$, $j = 1, 2, \cdots n$, are a set of linearly independent solutions. Hence the matrix

$$\mathbf{x} = \left[e^{a_1 t}\,\mathbf{e}^{(1)}, e^{a_2 t}\,\mathbf{e}^{(2)}, \cdots, e^{a_n t}\,\mathbf{e}^{(n)} \right]$$

is a fundamental matrix. Finally, since $\mathbf{X}(0) = \mathbf{I}$, it follows that

$$\left[e^{a_1 t}\,\mathbf{e}^{(1)}, e^{a_2 t}\,\mathbf{e}^{(2)}, \cdots, e^{a_n t}\,\mathbf{e}^{(n)} \right] = \mathbf{\Phi}(t) = e^{\mathbf{A}t}.$$

14.(a) Let $x_1 = u$ and $x_2 = u'$; then $u'' = x_2'$. In terms of the new variables, we have

$$x_2' + \omega^2 x_1 = 0$$

with the initial conditions $x_1(0) = u_0$ and $x_2(0) = v_0$. The equivalent first order system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega^2 x_1 \end{aligned}$$

which can be expressed in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

(b) Setting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

it is easy to show that

$$\mathbf{A}^2 = -\omega^2\,\mathbf{I},\ \mathbf{A}^3 = -\omega^2\,\mathbf{A} \text{ and } \mathbf{A}^4 = \omega^4\,\mathbf{I}.$$

It follows inductively that

$$\mathbf{A}^{2k} = (-1)^k \omega^{2k}\,\mathbf{I}$$

and

$$\mathbf{A}^{2k+1} = (-1)^k \omega^{2k}\,\mathbf{A}.$$

Hence

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{k=0}^{\infty}\left[(-1)^k\,\frac{\omega^{2k}\,t^{2k}}{(2k)!}\mathbf{I} + (-1)^k\,\frac{\omega^{2k}\,t^{2k+1}}{(2k+1)!}\,\mathbf{A}\right] \\ &= \left[\sum_{k=0}^{\infty}(-1)^k\,\frac{\omega^{2k}\,t^{2k}}{(2k)!}\right]\mathbf{I} + \frac{1}{\omega}\,\left[\sum_{k=0}^{\infty}(-1)^k\,\frac{\omega^{2k+1}\,t^{2k+1}}{(2k+1)!}\right]\mathbf{A} \end{aligned}$$

and therefore

$$e^{\mathbf{A}t} = \cos\,\omega\,t\mathbf{I} + \frac{1}{\omega}\,\sin\,\omega\,t\,\mathbf{A}.$$

(c) From Equation (28),

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \left[\cos\,\omega\,t\mathbf{I} + \frac{1}{\omega}\,\sin\,\omega\,t\,\mathbf{A}\right]\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ &= \cos\,\omega t\,\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \frac{1}{\omega}\,\sin\,\omega\,t\,\begin{pmatrix} v_0 \\ -\omega^2\,u_0 \end{pmatrix}. \end{aligned}$$

15.(a) Assuming that $\mathbf{x}=\phi(t)$ is a solution, then $\phi' = \mathbf{A}\phi$, with $\phi(0) = \mathbf{x}^0$. Integrate both sides of the equation to obtain

$$\phi(t) - \phi(0) = \int_0^t \mathbf{A}\phi(s)ds\,.$$

Hence

$$\phi(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi(s)ds\,.$$

(b) Proceed with the iteration

$$\phi^{(i+1)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\phi^{(i)}(s)ds\,.$$

With $\phi^{(0)}(t) = \mathbf{x}^0$, and noting that $\mathbf{A}$ is a constant matrix,

$$\phi^{(1)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\mathbf{x}^0 ds = \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t\,.$$

That is, $\phi^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0$.

(c) We then have

$$\phi^{(2)}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}t)\mathbf{x}^0 ds = \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t + \mathbf{A}^2\mathbf{x}^0\frac{t^2}{2} = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2})\mathbf{x}^0.$$

Now suppose that

$$\phi^{(n)}(t) = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!})\mathbf{x}^0.$$

It follows that

$$\int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!})\mathbf{x}^0 ds =$$

$$= \mathbf{A}(\mathbf{I}t + \mathbf{A}\frac{t^2}{2} + \mathbf{A}^2\frac{t^3}{3!} + \cdots + \mathbf{A}^n\frac{t^{n+1}}{(n+1)!})\mathbf{x}^0$$
$$= (\mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \cdots + \mathbf{A}^{n+1}\frac{t^n}{n!})\mathbf{x}^0.$$

Therefore

$$\phi^{(n+1)}(t) = (\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n+1}\frac{t^{n+1}}{(n+1)!})\mathbf{x}^0.$$

By induction, the asserted form of $\phi^{(n)}(t)$ is valid for all $n \geq 0$.

(d) Define $\phi^{(\infty)}(t) = \lim_{n\to\infty}\phi^{(n)}(t)$. It can be shown that the limit does exist. In fact,

$$\phi^{(\infty)}(t) = e^{\mathbf{A}t}\mathbf{x}^0.$$

Term-by-term differentiation results in

$$\begin{aligned}\frac{d}{dt}\phi^{(\infty)}(t) &= \frac{d}{dt}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots)\mathbf{x}^0\\ &= (\mathbf{A} + \mathbf{A}^2 t + \cdots + \mathbf{A}^n\frac{t^{n-1}}{(n-1)!} + \cdots)\mathbf{x}^0\\ &= \mathbf{A}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n-1}\frac{t^{n-1}}{(n-1)!} + \cdots)\mathbf{x}^0.\end{aligned}$$

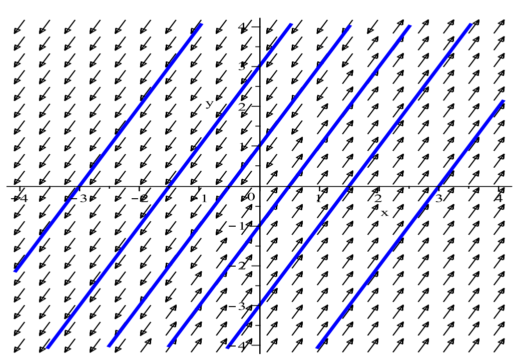
That is,

$$\frac{d}{dt}\phi^{(\infty)}(t) = \mathbf{A}\phi^{(\infty)}(t).$$

Furthermore, $\phi^{(\infty)}(0) = \mathbf{x}^0$. Based on uniqueness of solutions, $\phi(t) = \phi^{(\infty)}(t)$.

7.8

2.(a)



(b) All of the points on the line $x_2 = 2x_1$ are equilibrium points. Solutions starting at all other points become unbounded.

(c) Setting $\mathbf{x}= \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with the single root $r = 0$. Substituting $r = 0$ reduces the system of equations to $2\xi_1 - \xi_2 = 0$. Therefore the only eigenvector is $\boldsymbol{\xi} = (1\,, 2)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is a constant vector. In order to generate a second linearly independent solution, we must search for a generalized eigenvector. This leads to the system of equations

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system also reduces to a single equation, $2\eta_1 - \eta_2 = 1/2$. Setting $\eta_1 = k$, some arbitrary constant, we obtain $\eta_2 = 2k - 1/2$. A second solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} k \\ 2k - 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Note that the last term is a multiple of $\mathbf{x}^{(1)}$ and may be dropped. Hence

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right].$$

5. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r - 2 = 0$, with roots $r_1 = 2$ and $r_{2,3} = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $r = -1$, the system of equations is reduced to the single equation

$$\xi_1 + \xi_2 + \xi_3 = 0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,0\,,-1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -1$) is $\boldsymbol{\xi}^{(3)} = (0\,,1\,,-1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

6.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity two. Substituting $r = -3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

For a second linearly independent solution, we search for a generalized eigenvector. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} k+1/4 \\ k \end{pmatrix} e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

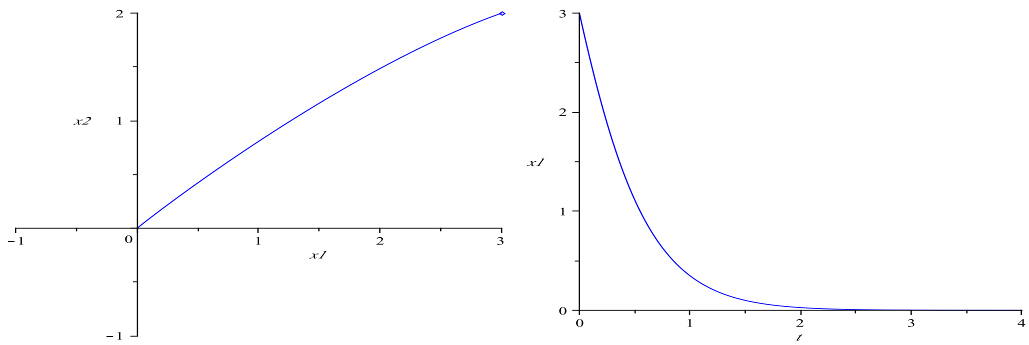
Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} \right].$$

Imposing the initial conditions, we require that $c_1 + c_2/4 = 3$, $c_1 = 2$, which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}.$$

(b)



7.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2}-r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} k \\ 2/3+k \end{pmatrix} e^{-t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

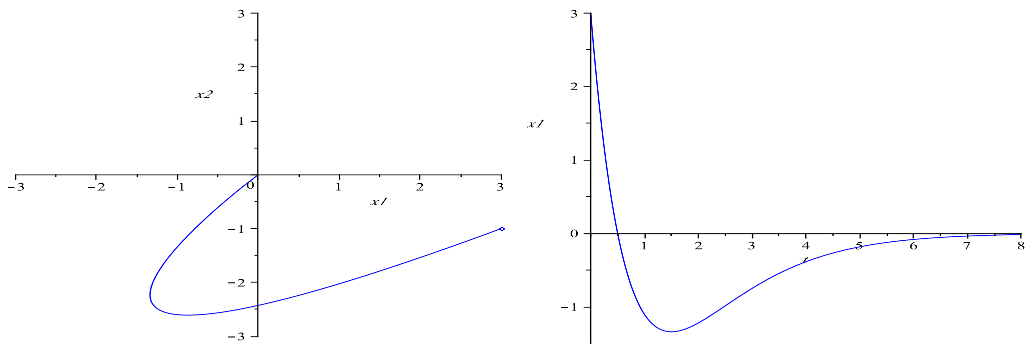
Dropping the last term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t}\right].$$

Imposing the initial conditions, we find that $c_1 = 3$, $c_1 + 2c_2/3 = -1$, so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} te^{-t}.$$

(b)



8.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 9 \\ -1 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with a single root $r = 0$. Setting $r = 0$, the two equations reduce to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (-3\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

which is a constant vector. A second linearly independent solution is obtained from the system

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $\eta_1 + 3\eta_2 = -1$. Let $\eta_2 = k$. We obtain $\eta_1 = -1 - 3k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1-3k \\ k \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

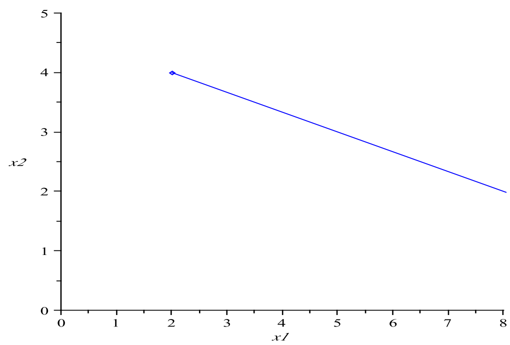
Dropping the last term, the general solution is

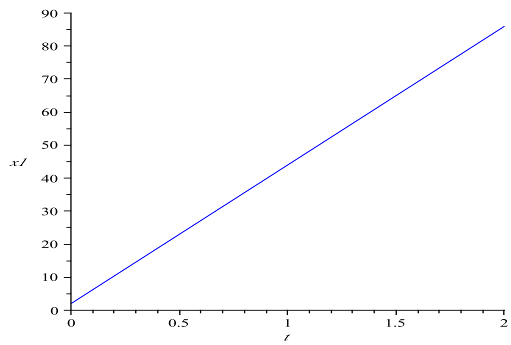
$$\mathbf{x} = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2\left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right].$$

Imposing the initial conditions, we require that $-3c_1 - c_2 = 2$, $c_1 = 4$, which results in $c_1 = 4$ and $c_2 = -14$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 14\begin{pmatrix} -3 \\ 1 \end{pmatrix} t.$$

(b)





11. Setting $\mathbf{x}=\boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 - 2r + 1 = 0$, with a single root of $r_{1,2} = 1$. With $r = 1$, the system reduces to a single equation $\xi_1 - 2\,\xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (2\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,.$$

In order to find a second linearly independent solution, we search for a generalized eigenvector whose components satisfy

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These equations reduce to $\eta_1 - 2\,\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. (Before proceeding, note that if we set $u = \ln t$, the original equation is transformed into a constant coefficient equation with independent variable u. Recall that a second solution is obtained by multiplication of the first solution by the factor u. This implies that we must multiply first solution by a factor of $\ln t$.) Hence a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,\ln\,t + \begin{pmatrix} 1+2k \\ k \end{pmatrix} t = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,\ln\,t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} t.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,\ln\,t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right].$$

14.(a) The eigenvalues of the matrix

$$\begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}$$

are

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{\sqrt{L^2 - 4R^2CL}}{2RCL}.$$

The discriminant vanishes when $L = 4R^2C$.

(b) The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & -1 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

The associated eigenvalue problem is

$$\begin{pmatrix} -r & \frac{1}{4} \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + r + 1/4 = 0$, with a single root of $r_{1,2} = -1/2$. Setting $r = -1/2$, the algebraic equations reduce to $2\xi_1 + \xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (1\,, -2)^T$. Hence one solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

A second solution is obtained from a generalized eigenvector whose components satisfy

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It follows that $\eta_1 = k$ and $\eta_2 = 4 - 2k$. A second linearly independent solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} k \\ 4-2k \end{pmatrix} e^{-t/2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2} + k\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

Dropping the last term, the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2}\right].$$

Imposing the initial conditions, we require that $c_1 = 1$, $-2c_1 + 4c_2 = 2$, which results in $c_1 = 1$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} te^{-t/2}.$$

16. (a) The adjoint of A is $A^* = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$. Since the characteristic polynomial of A^* is $\det(A^* - \lambda I) = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, the eigenvalues of A^* are $\lambda_{1,2} = 2$. The corresponding eigenvectors to $\lambda = 2$ are non-zero solutions of $(A^* - 2I)\mathbf{x} = \mathbf{0}$. Since $A^* - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, a corresponding eignevector is $\boldsymbol{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) The matrix A also has eigenvalues $\lambda_{1,2} = 2$, but a corresponding eigenvector to $\lambda = 2$ is $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since $(\boldsymbol{\xi}^{(1)}, \boldsymbol{\eta}^{(1)}) = 1 - 1 = 0$, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\eta}^{(1)}$ are orthogonal.

(c) Equation (16) is $\boldsymbol{\xi} + 2\,\boldsymbol{\eta} = A\,\boldsymbol{\eta}$, with $\boldsymbol{\xi} = \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ an eigenvector of A correpsonding to $\lambda = 2$. Equation (16) may be rewritten as $(A - 2I)\,\boldsymbol{\eta} = \boldsymbol{\xi}$. This system is solvable provided that $\boldsymbol{\xi}$ is orthogonal to any non-zero solution of the adjoint problem $(A - 2I)^* = \mathbf{0}$. But the non-zero solutions of $(A - 2I)^* = \mathbf{0}$ are the eigenvectors of A^* corrsponding to $\lambda = 2$, and we showed in (b) that $\boldsymbol{\xi}$ was orthogonal to all such eigenvectors. Thus Equation (16) has a solution.

18.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 5 - r & -3 & -2 \\ 8 & -5 - r & -4 \\ -4 & 3 & 3 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root of multiplicity three, $r = 1$. Setting $r = 1$, we have

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of algebraic equations reduces to a single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0\,.$$

An eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1\,,0\,,2)^T$. Since the last equation has two free variables, a second linearly independent eigenvector (associated with $r = 1$) is $\boldsymbol{\xi}^{(2)} = (0\,,2\,,-3)^T$. Therefore two solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

(b) It follows directly that $\mathbf{x}' = \boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t$. Hence the coefficient vectors must satisfy $\boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t = \mathbf{A}\boldsymbol{\xi} te^t + \mathbf{A}\boldsymbol{\eta} e^t$. Rearranging the terms, we have

$$\boldsymbol{\xi} e^t = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} te^t + (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} e^t.$$

Given an eigenvector $\boldsymbol{\xi}$, it follows that $(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ and $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.

(c) Clearly, $(\mathbf{A} - \mathbf{I})^2\boldsymbol{\eta} = (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$. Also,

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) We get that

$$\boldsymbol{\xi} = (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

This is an eigenvector:

$$\begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

(e) Given the three linearly independent solutions, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & 0 & -2t\,e^t \\ 0 & 2e^t & -4t\,e^t \\ 2e^t & -3e^t & 2t\,e^t + e^t \end{pmatrix}.$$

(f) We construct the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & 0 \\ 2 & 2 & 1 \end{pmatrix},$$

with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & -1/4 & 0 \\ -2 & 3/2 & 1 \end{pmatrix}.$$

The Jordan form of the matrix $\mathbf{A}$ is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

20.(a) Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix},\ \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix},\ \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\mathbf{J}^{n+1} = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda \cdot \lambda^n & 0 & 0 \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}.$$

Hence the result follows by mathematical induction.

(c) Using $e^{\mathbf{J}t} = \sum_{n=0}^{\infty} (\mathbf{J}t)^n/n!$, and noting, as previously seen in Problem 18, that $\sum_{n=0}^{\infty} \lambda^{n-1} t^n/n! = te^{\lambda t}$, it follows that

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) Setting $\lambda = 1$, and using the transformation matrix $\mathbf{T}$ in Problem 18,

$$\mathbf{T}e^{\mathbf{J}t} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & -2e^t & -2t\,e^t \\ 0 & -4e^t & -4t\,e^t \\ 2e^t & 2e^t & 2t\,e^t + e^t \end{pmatrix}.$$

Based on the form of $\mathbf{J}$, $e^{\mathbf{J}t}$ is the fundamental matrix associated with the solutions

$$\mathbf{y}^{(1)} = \boldsymbol{\xi}^{(1)} e^t, \mathbf{y}^{(2)} = -(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})e^t \text{ and } \mathbf{y}^{(3)} = -(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t.$$

Hence the resulting matrix is the fundamental matrix associated with the solution set

$$\left\{ \boldsymbol{\xi}^{(1)} e^t\,, -(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})e^t, -(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t \right\},$$

as opposed to the solution set in Problem 18, given by

$$\left\{ \boldsymbol{\xi}^{(1)} e^t\,, \boldsymbol{\xi}^{(2)} e^t, (2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)})te^t + \boldsymbol{\eta}e^t \right\}.$$

21.(a) Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{J}^{n+1} &= \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} & n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda \cdot \lambda^{n-2} \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}. \end{aligned}$$

The result follows by noting that

$$n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda \cdot \lambda^{n-2} = \left[n + \frac{n(n-1)}{2} \right] \lambda^{n-1} = \frac{n^2+n}{2}\lambda^{n-1}.$$

(c) We first observe that

$$\sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!} = e^{\lambda t}$$

$$\sum_{n=0}^{\infty} n\lambda^{n-1} \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} = t\, e^{\lambda t}$$

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} \lambda^{n-2} \frac{t^n}{n!} = \frac{t^2}{2} \sum_{n=2}^{\infty} \lambda^{n-2} \frac{t^{n-2}}{(n-2)!} = \frac{t^2}{2} e^{\lambda t}.$$

Therefore

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) Setting $\lambda = 2$, and using the transformation matrix $\mathbf{T}$,

$$\mathbf{T}e^{\mathbf{J}t} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2} e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 0 & e^{2t} & te^{2t} + 2e^{2t} \\ e^{2t} & te^{2t} + e^{2t} & \frac{t^2}{2} e^{2t} + te^{2t} \\ -e^{2t} & -te^{2t} & -\frac{t^2}{2} e^{2t} + 3e^{2t} \end{pmatrix}.$$

7.9

4. As shown in Problem 2, Section 7.8, the general solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - \frac{1}{2} \end{pmatrix}.$$

An associated fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \begin{pmatrix} 4t - 3 & -2t + 2 \\ 8t - 8 & -4t + 5 \end{pmatrix}.$$

We can now compute

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = -\frac{1}{t^3} \begin{pmatrix} 2t^2 + 4t - 1 \\ -2t - 4 \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\, dt = \begin{pmatrix} -\frac{1}{2} t^{-2} + 4t^{-1} - 2\ln t \\ -2t^{-2} - 2t^{-1} \end{pmatrix}.$$

Finally,

$$\mathbf{v}(t) = \mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\, dt,$$

where

$$v_1(t) = -\frac{1}{2}t^{-2} + 2t^{-1} - 2\ln t - 2, \qquad v_2(t) = 5t^{-1} - 4\ln t - 4\,.$$

Note that the vector $(2\,,4)^T$ is a multiple of one of the fundamental solutions. Hence we can write the general solution as

$$\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}t\\2t-\frac{1}{2}\end{pmatrix} - \frac{1}{t^2}\begin{pmatrix}1/2\\0\end{pmatrix} + \frac{1}{t}\begin{pmatrix}2\\5\end{pmatrix} - 2\ln t\begin{pmatrix}1\\2\end{pmatrix}.$$

5. The solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix} + c_2\begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix}.$$

Based on the simple form of the right hand side, we use the method of undetermined coefficients. Set $\mathbf{v}$=$\mathbf{a}\,e^t$. Substitution into the ODE yields

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}e^t = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}e^t + \begin{pmatrix}2\\-1\end{pmatrix}e^t.$$

In scalar form, after canceling the exponential, we have

$$\begin{aligned}a_1 &= a_1 + a_2 + 2\\ a_2 &= 4a_1 + a_2 - 1\,,\end{aligned}$$

with $a_1 = 1/4$ and $a_2 = -2$. Hence the particular solution is

$$\mathbf{v} = \begin{pmatrix}1/4\\-2\end{pmatrix}e^t,$$

so that the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix} + c_2\begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix} + \frac{1}{4}\begin{pmatrix}e^t\\-8e^t\end{pmatrix}.$$

7. Since the coefficient matrix is symmetric, the differential equations can be decoupled. The eigenvalues and eigenvectors are given by

$$r_1 = -4\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix}\sqrt{2}\\-1\end{pmatrix} \text{ and } r_2 = -1\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix}1\\\sqrt{2}\end{pmatrix}.$$

Using the normalized eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2} & 1\\-1 & \sqrt{2}\end{pmatrix} \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2} & -1\\1 & \sqrt{2}\end{pmatrix}.$$

Setting $\mathbf{x}$=$\mathbf{T}\mathbf{y}$, and $\mathbf{h}(t)$ =$\mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$\begin{aligned}y_1' &= -4y_1 + \frac{1}{\sqrt{3}}(1+\sqrt{2})e^{-t}\\ y_2' &= -y_2 + \frac{1}{\sqrt{3}}(1-\sqrt{2})\,e^{-t}\,.\end{aligned}$$

The solutions are easily obtained as

$$y_1(t) = k_1 e^{-4t} + \frac{1}{3\sqrt{3}}(1+\sqrt{2})e^{-t}, \qquad y_2(t) = k_2 e^{-t} + \frac{1}{\sqrt{3}}(1-\sqrt{2})te^{-t}.$$

Transforming back to the original variables, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}e^{-4t} + c_2\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix}2+\sqrt{2}+3\sqrt{3}\\3\sqrt{6}-\sqrt{2}-1\end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix}1-\sqrt{2}\\\sqrt{2}-2\end{pmatrix}te^{-t}.$$

Note that

$$\begin{pmatrix}2+\sqrt{2}+3\sqrt{3}\\3\sqrt{6}-\sqrt{2}-1\end{pmatrix} = \begin{pmatrix}2+\sqrt{2}\\-\sqrt{2}-1\end{pmatrix} + 3\sqrt{3}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}.$$

The second vector is an eigenvector, hence the solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}e^{-4t} + c_2\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix}2+\sqrt{2}\\-\sqrt{2}-1\end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix}1-\sqrt{2}\\\sqrt{2}-2\end{pmatrix}te^{-t}.$$

8. Based on the solution of Problem 2 of Section 7.6, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix}5\cos t & 5\sin t\\2\cos t+\sin t & -\cos t+2\sin t\end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{5}\begin{pmatrix}\cos t-2\sin t & 5\sin t\\2\cos t+\sin t & -5\cos t\end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix}\cos t\sin t\\-\cos^2 t\end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix}\frac{1}{2}\sin^2 t\\-\frac{1}{2}\cos t\sin t-\frac{1}{2}t\end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = \frac{5}{2}\cos t\sin t - \cos^2 t + \frac{5}{2}t + 1, \qquad v_2(t) = \cos t\sin t - \frac{1}{2}\cos^2 t + t + \frac{1}{2}.$$

Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}5\cos t\\2\cos t+\sin t\end{pmatrix} + c_2\begin{pmatrix}5\sin t\\-\cos t+2\sin t\end{pmatrix} - \\ - t\sin t\begin{pmatrix}5/2\\1\end{pmatrix} + t\cos t\begin{pmatrix}0\\1/2\end{pmatrix} - \cos t\begin{pmatrix}5/2\\1\end{pmatrix}.$$

9.(a) As shown in Problem 20 of Section 7.6, the solution of the homogeneous system is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\,\sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\,\cos(t/2) \end{pmatrix}.$$

Therefore the associated fundamental matrix is given by

$$\mathbf{\Psi}(t) = e^{-t/2}\begin{pmatrix} \cos(t/2) & \sin(t/2) \\ 4\,\sin(t/2) & -4\,\cos(t/2) \end{pmatrix}.$$

(b) The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{e^{t/2}}{4}\begin{pmatrix} 4\,\cos(t/2) & \sin(t/2) \\ 4\,\sin(t/2) & -\cos(t/2) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \frac{1}{2}\begin{pmatrix} \cos(t/2) \\ \sin(t/2) \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \sin(t/2) \\ -\cos(t/2) \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where $v_1(t) = 0$, $v_2(t) = 4\,e^{-t/2}$. Hence the general solution is

$$\mathbf{x} = c_1 e^{-t/2}\begin{pmatrix} \cos(t/2) \\ 4\,\sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} \sin(t/2) \\ -4\,\cos(t/2) \end{pmatrix} + 4\,e^{-t/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Imposing the initial conditions, we require that $c_1 = 0$, $-4c_2 + 4 = 0$, which results in $c_1 = 0$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\mathbf{x} = e^{-t/2}\begin{pmatrix} \sin(t/2) \\ 4 - 4\,\cos(t/2) \end{pmatrix}.$$

11. The general solution of the homogeneous problem is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2,$$

which can be verified by substitution into the system of ODEs. Since the vectors are linearly independent, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{pmatrix}.$$

The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{3}\begin{pmatrix} -t & 2t \\ 2t^{-2} & -t^{-2} \end{pmatrix}.$$

Dividing both equations by t, we obtain

$$\mathbf{g}(t) = \begin{pmatrix} -2 \\ t^3 - t^{-1} \end{pmatrix}.$$

Proceeding with the method of variation of parameters,

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} \frac{2}{3}t^4 + \frac{2}{3}t - \frac{2}{3} \\ -\frac{1}{3}t - \frac{4}{3}t^{-2} + \frac{1}{3}t^{-3} \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{2}{15}t^5 + \frac{1}{3}t^2 - \frac{2}{3}t \\ -\frac{1}{6}t^2 + \frac{4}{3}t^{-1} - \frac{1}{6}t^{-2} \end{pmatrix}.$$

Hence a particular solution is obtained as

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{5}t^4 + 3t - 1 \\ \frac{1}{10}t^4 + 2t - \frac{3}{2} \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2 + \frac{1}{10}\begin{pmatrix} -2 \\ 1 \end{pmatrix}t^4 + \begin{pmatrix} 3 \\ 2 \end{pmatrix}t - \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}.$$

12. Based on the hypotheses,

$$\phi'(t) = \mathbf{P}(t)\phi(t) + \mathbf{g}(t) \text{ and } \mathbf{v}'(t) = \mathbf{P}(t)\mathbf{v}(t) + \mathbf{g}(t)\ .$$

Subtracting the two equations results in

$$\phi'(t) - \mathbf{v}'(t) = \mathbf{P}(t)\phi(t) - \mathbf{P}(t)\mathbf{v}(t)\,,$$

that is,

$$[\phi(t) - \mathbf{v}(t)]' = \mathbf{P}(t)\,[\phi(t) - \mathbf{v}(t)]\,.$$

It follows that $\phi(t) - \mathbf{v}(t)$ is a solution of the homogeneous equation. According to Theorem 7.4.2,

$$\phi(t) - \mathbf{v}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t).$$

Hence

$$\phi(t) = \mathbf{u}(t) + \mathbf{v}(t),$$

in which $\mathbf{u}(t)$ is the general solution of the homogeneous problem.

13. (a) Let $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$ and

$$x_2' = y'' = 5y' - 6y + 2e^t = 5x_2 - 6x_1 + 2e^t$$

Therefore

$$\mathbf{x}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ 5x_2 - 6x_1 + 2e^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}$$

$$= A(t)\mathbf{x} + \mathbf{g}(t),\ \text{where } A(t) = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \text{ and } \mathbf{g}(t) = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}$$

(b) Let

$$\mathbf{x}^{(1)} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}.$$

Then

$$A(t)\mathbf{x}^{(1)} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} e^{2t} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} = \mathbf{x}^{(1)\,\prime}$$

and

$$A(t)\mathbf{x}^{(2)} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} e^{3t} = 3\begin{pmatrix} 1 \\ 3 \end{pmatrix} 3^{3t} = \mathbf{x}^{(2)\,\prime}$$

(c) The equation will be solved by $y = y_1(t)u_1(t) + y_2(t)u_2(t)$ when $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}; \text{that is, when } \begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}.$$

(d) The vector $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ solves the equation $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t)$ with $A(t) = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$ and $\mathbf{g}(t) = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}$ when $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$; that is, when

$$\begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}.$$

(e) The equivalence of the systems found in parts (c) and (d) follows directly from the observation that $\mathbf{\Psi}(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$.

14. (a) Let $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$ and

$$x_2' = y'' = \frac{t+2}{t}y' - \frac{t+2}{t^2}y + 2t = -\frac{t+2}{t^2}x_1 + \frac{t+2}{t}x_2 + 2t$$

Therefore

$$\begin{aligned} \mathbf{x}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' &= \begin{pmatrix} x_2 \\ -\frac{t+2}{t^2}x_1 + \frac{t+2}{t}x_2 + 2t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{t+2}{t^2} & \frac{t+2}{t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2t \end{pmatrix} \\ &= A(t)\mathbf{x} + \mathbf{g}(t), \text{ where } A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{t+2}{t^2} & \frac{t+2}{t} \end{pmatrix} \text{ and } \mathbf{g}(t) = \begin{pmatrix} 0 \\ 2t \end{pmatrix}. \end{aligned}$$

(b) Let

$$\mathbf{x}^{(1)} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} t \\ t+1 \end{pmatrix} e^t.$$

Then

$$A(t)\mathbf{x}^{(1)} = \begin{pmatrix} 0 & 1 \\ -\frac{t+2}{t^2} & \frac{t+2}{t} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}^{(1)\,\prime}$$

and

$$A(t)\mathbf{x}^{(2)} = \begin{pmatrix} 0 & 1 \\ -\frac{t+2}{t^2} & \frac{t+2}{t} \end{pmatrix} \begin{pmatrix} t \\ t+1 \end{pmatrix} e^t = \begin{pmatrix} t+1 \\ -\frac{t+2}{t} + \frac{(t+2)(t+1)}{t} \end{pmatrix} e^t = \begin{pmatrix} t+1 \\ t+2 \end{pmatrix} e^t = \mathbf{x}^{(2)\,\prime}$$

(c) The equation will be solved by $y = y_1(t)u_1(t) + y_2(t)u_2(t)$ when $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \end{pmatrix}; \text{that is, when } \begin{pmatrix} t & te^t \\ 1 & (t+1)e^t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \end{pmatrix}.$$

(d) The vector $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ solves the equation $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t)$ with $A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{t+2}{t^2} & \frac{t+2}{t} \end{pmatrix}$ and $\mathbf{g}(t) = \begin{pmatrix} 0 \\ 2t \end{pmatrix}$ when $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$; that is, when

$$\begin{pmatrix} t & te^t \\ 1 & (t+1)e^t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \end{pmatrix}.$$

(e) The equivalence of the systems found in parts (c) and (d) follows directly from the observation that $\mathbf{\Psi}(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$.

15. (a) Let $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$ and

$$x_2' = y'' = -p(t)y' - q(t)y + g(t) = -q(t)x_1 - p(t)x_2 + g(t)$$

Therefore

$$\begin{aligned} \mathbf{x}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' &= \begin{pmatrix} x_2 \\ -q(t)x_1 - p(t)x_2 + g(t) \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} = A(t)\mathbf{x} + \mathbf{g}(t), \end{aligned}$$

where $A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}$ and $\mathbf{g}(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$.

(b) Let

$$\mathbf{x}^{(1)} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}.$$

Then

$$A(t)\mathbf{x}^{(1)} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} y_1' \\ -q(t)y_1 - p(t)y_1' \end{pmatrix} = \begin{pmatrix} y_1' \\ y_1'' \end{pmatrix} = \mathbf{x}^{(1)\,\prime}$$

since $y_1'' + p(t)y_1' + q(t)y_1 = 0$, and

$$A(t)\mathbf{x}^{(2)} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2' \\ -q(t)y_2 - p(t)y_2' \end{pmatrix} = \begin{pmatrix} y_2' \\ y_2'' \end{pmatrix} = \mathbf{x}^{(2)\,\prime}$$

since $y_1'' + p(t)y_1' + q(t)y_1 = 0$.

(c) The equation will be solved by $y = y_1(t)u_1(t) + y_2(t)u_2(t)$ when $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

(d) The vector $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ solves the equation $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t)$ when $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$; that is, when

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

(e) The equivalence of the systems found in parts (c) and (d) follows directly from the observation that $\mathbf{\Psi}(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$.

16.(a) Setting $t_0 = 0$ in Equation (34),

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds\,.$$

It was shown in Problem 12(c) in Section 7.7 that $\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s)$. Therefore

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t-s)\mathbf{g}(s)ds\,.$$

(b) The principal fundamental matrix is identified as $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$. Hence

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)ds\,.$$

In Problem 22 of Section 3.6, the particular solution is given as

$$y(t) = \int_{t_0}^t K(t-s)g(s)ds\,,$$

in which the kernel $K(t)$ depends on the nature of the fundamental solutions.

17. Similarly to Eq.(43), here

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{G}(s) + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

where

$$\mathbf{G}(s) = \begin{pmatrix} 2/(s+1) \\ 3/s^2 \end{pmatrix} \quad \text{and} \quad s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}.$$

The transfer matrix is given by Eq.(46):

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}.$$

From these equations we obtain that

$$\mathbf{X}(s) = \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} + \frac{\alpha_1(s+2)}{(s+1)(s+3)} + \frac{\alpha_2}{(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} + \frac{\alpha_1}{(s+1)(s+3)} + \frac{\alpha_2(s+2)}{(s+1)(s+3)} \end{pmatrix}.$$

The inverse Laplace transform gives us that

$$\mathbf{x}(t) = \begin{pmatrix} \frac{4+\alpha_1+\alpha_2}{2}e^{-t} + \frac{-4+3\alpha_1-3\alpha_2}{6}e^{-3t} + t + te^{-t} - \frac{4}{3} \\ \frac{2+\alpha_1+\alpha_2}{2}e^{-t} + \frac{4-3\alpha_1+3\alpha_2}{6}e^{-3t} + 2t + te^{-t} - \frac{5}{3} \end{pmatrix},$$

so α_1 and α_2 should be chosen so that

$$\frac{4+\alpha_1+\alpha_2}{2} = c_2 + \frac{1}{2} \text{ and } \frac{-4+3\alpha_1-3\alpha_2}{6} = c_1.$$

This gives us $\alpha_1 = (-5+6c_1+6c_2)/6$ and $\alpha_2 = -c_1+c_2-13/6$.