Step-1

Let matrix A is defined as follows:

$$A=\begin{bmatrix}0 & -i & 0\\ i & 1 & i\\ 0 & -i & 0\end{bmatrix}$$

Step-2

Find the Eigen values and Eigen vectors of matrix A. Also find the property expected for the Eigen vectors. Is it true in the case of matrix A?

Firstly find the Eigen values and Eigen vectors of matrix A. Do the following calculations:

$$A-\lambda I=\begin{bmatrix}0-\lambda & -i & 0\\ i & 1-\lambda & i\\ 0 & -i & 0-\lambda\end{bmatrix}$$

$$\det\left(A-\lambda I\right)=0$$

$$\left(-\lambda\right)\left[-\left(1-\lambda\right)\lambda+i^2\right]+i\left(-i\lambda\right)=0$$

$$-\lambda^3+\lambda^2+2\lambda=0$$

After solving following values are obtained. Therefore, Eigen values are:

$$\boxed{\begin{matrix}\lambda_1=0\\ \lambda_2=-1\\ \lambda_3=2\end{matrix}}$$

Step-3

To calculate Eigen vectors do the following calculations:

$$\left(A-\lambda_1 I\right)x=0$$

$$\begin{bmatrix}0-\lambda & -i & 0\\ i & 1-\lambda & i\\ 0 & -i & 0-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -i & 0\\ i & 2 & i\\ 0 & -i & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

On solving values of x, y and z corresponding to $\lambda=-1$ is as follows:

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

Step-4

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 0-\lambda & -i & 0 \\ i & 1-\lambda & i \\ 0 & -i & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving the above matrix equation following values of x, y and z are obtained:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 2$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 0-\lambda & -i & 0 \\ i & 1-\lambda & i \\ 0 & -i & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -i & 0 \\ i & -1 & i \\ 0 & -i & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving the above matrix equation following values of x, y and z are obtained:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix}$$

Step-6

Therefore Eigen vectors are:

$$x_1 = \boxed{\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}}$$
$$x_2 = \boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$
$$x_3 = \boxed{\begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix}}$$

Step-7

Matrix A is Hermitian matrix as:

$$A^H = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}$$
$$= A$$

So, Eigen values of matrix A must be real values and Eigen vectors must be orthogonal to each other, if they come from different Eigen values.

Here, all Eigen vectors coming from different non-repeated Eigen values are orthogonal to each other.

Step-8

Therefore, the property is true in the case of matrix A.