MA215 Probability Theory

Assignment 02

1. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

   (a) List the sample space Ω.

   (b) List the elements that make up the following events:

   (1) $A =$ the sum of the two values is at least 5;

   (2) $B =$ the value for the first die is higher than the value of the second;

   (3) $C =$ the first value is 4.

   (c) List the elements of the following events:

   (1) $A \cap C$;

   (2) $B \cup C$;

   (3) $A \cap (B \cup C)$.

2. Let A and B be two arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

3. Suppose A and B are two events such that $A \subset B$. Show that

$$P(B \backslash A) = P(B) - P(A).$$

4. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions proved in the Lecture Notes.

$$\begin{aligned} P(A \cup B \cup C) = P(A) &+ P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C). \end{aligned}$$

5. Suppose that $\{A_n; n \geqslant 1\}$ is a sequence of events which may not be disjoint. Show that the following sub-additive property is true:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} P(A_n).$$

   Also, for any $k \geqslant 2$, we have

$$P\left(\bigcup_{n=1}^{k} A_n\right) \leqslant \sum_{n=1}^{k} P(A_n).$$

   In particular, for any two events A and B, we have $P(A \cup B) \leqslant P(A) + P(B)$.

6. Suppose $\{A_i; 1 \leqslant i \leqslant n\}$ are events.

(1) Show that the following inclusion-exclusion formula is true.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j\leqslant n} P(A_i \cap A_j) + \sum_{i<j<k\leqslant n} P(A_i \cap A_j \cap A_k)$$
$$- \cdots + (-1)^{n-1} P(A_1 \cap A_2 \ldots A_n).$$

(2) Write down this formula for cases of $n = 2, n = 3, n = 4$, and $n = 5$ clearly.

7. (i) If $\{A_n; n \geqslant 1\}$ is an increasing sequence of events, i.e. for all $n \geqslant 1, A_n \subset A_{n+1}$, show that $\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$.

(ii) If $\{A_n; n \geqslant 1\}$ is a decreasing sequence of events, i.e. for all $n \geqslant 1, A_n \supset A_{n+1}$, show that $\lim_{n\to\infty} P(A_n\Big) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$.