Step-1

Consider the matrix,

$$A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}.$$

The objective is to determine whether the matrix A is positive definite, positive semidefinite or indefinite or not by using any three tests.

Step-2

Use the result that a matrix M is said to be positive definite if it satisfies any of the following conditions:

1) For any nonzero real vector x, the matrix M satisfies $x^T Mx>0$.

2) All the eigenvalues of M are positive. That if λ is an eigenvalue of M then $\lambda>0$.

3) All the upper left submatrices M_k have positive determinants.

Step-3

Let $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be any vector.

Then $x^T=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}$.

Now compute $x^T Ax$.

$$\begin{aligned}
x^T Ax&=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\
&=\begin{bmatrix}x_1+x_2+x_3 & x_1+x_2+x_3 & x_1+x_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\
&=\left(x_1+x_2+x_3\right)x_1+\left(x_1+x_2+x_3\right)x_2+\left(x_1+x_2\right)x_3\\
&=x_1^2+x_1x_2+x_1x_3+x_1x_2+x_2^2+x_2x_3+x_1x_3+x_2x_3\\
&=x_1^2+2x_1x_2+2x_1x_3+x_2^2+2x_2x_3+x_1x_3
\end{aligned}$$

Note that if x_1,x_2,x_3 are all negative, then there is a possibility for $x^T Ax$ such that $x^T Ax<0$.

Therefore, by the 1st condition of the result is not satisfied.

Hence, the matrix A is **indefinite**.

Step-4

Now find the eigenvalues of the matrix A.

The characteristic equation of A is

$$\det(A-\lambda I)=0$$

$$\det\begin{bmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & -\lambda\end{bmatrix}=0$$

$$(1-\lambda)\big((1-\lambda)(-\lambda)-0\big)-1(-\lambda-1)+1\big(1-(1-\lambda)\big)=0$$

$$\lambda^3-2\lambda^2-2\lambda=0$$

$$\lambda\left(\lambda^2-2\lambda-2\right)=0$$

$$\lambda=0 \text{ or } \lambda^2-2\lambda-2=0$$

$$\lambda=0 \text{ or } \lambda=1\pm\sqrt{3}$$

Thus, the eigenvalues of A are $\lambda=0$ or $\lambda=1\pm\sqrt{3}$.

As the eigenvalue $\lambda=0\not\succ 0$, so by the 2nd condition of the result is not satisfied.

Hence, the matrix A is **indefinite**.

Step-5

Consider the submatrices of the matrix A.

The first upper submatrix of A is $A_1=[1]$.

Then the determinant of $A_1=[1]$ is

$\det A_1=\det[1]=1.$

The second upper submatrix of A is $A_2=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$.

Then the determinant of the matrix A_2 is

$$\det A_2 = \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= 1(1) - 1(1)$$
$$= 1 - 1$$
$$= 0$$

As the determinant of the second submatrix A_2 is 0, so the 3rd condition of the result is not satisfied.

Hence, the matrix A is **indefinite**.

Consider the matrix,

$$B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector.

Then $x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

Now compute $x^T Bx$.

$$x^T Ax = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 2x_1 + x_2 + 2x_3 & x_1 + x_2 + x_3 & 2x_1 + x_2 + 2x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= (2x_1 + x_2 + 2x_3)x_1 + (x_1 + x_2 + x_3)x_2 + (2x_1 + x_2 + 2x_3)x_3$$
$$= 2x_1^2 + 2x_1x_2 + 4x_1x_3 + x_2^2 + 2x_2x_3 + 2x_3^2$$

Note that if x_1, x_2, x_3 are all nonnegative, then there is no possibility for $x^T Bx$ such that $x^T Bx < 0$.

Therefore, by the 1st condition of the result is satisfied.

Hence, the matrix B is **positive definite**.

Step-6

Now find the eigenvalues of the matrix B.

The characteristic equation of *B* is

$$\det(B-\lambda I)=0$$

$$\det\begin{bmatrix}2-\lambda & 1 & 2\\ 1 & 1-\lambda & 1\\ 2 & 1 & 2-\lambda\end{bmatrix}=0$$

$$(2-\lambda)\left((1-\lambda)(2-\lambda)-1\right)-1\left((2-\lambda)(1-\lambda)-1\right)+2\left(1-2(1-\lambda)\right)=0$$

$$\begin{aligned}\lambda^3-5\lambda^2+2\lambda&=0\\ \lambda\left(\lambda^2-5\lambda+2\right)&=0\\ \lambda=0 \text{ or } \lambda^2-5\lambda+2&=0\\ \lambda=0 \text{ or } \lambda&=\frac{5\pm\sqrt{7}}{2}\end{aligned}$$

Thus, the eigenvalues of *B* are $\lambda=0 \text{ or } \lambda=\frac{5\pm\sqrt{7}}{2}$.

As the eigenvalue $\lambda=0 \not> 0$, so by the 2nd condition of the result is not satisfied.

Hence, the matrix *B* is **indefinite**.

Step-7

Consider the submatrices of the matrix *B*.

The first upper submatrix of *B* is $B_1=[2]$.

Then the determinant of B_1 is

$\det B_1=\det[2]=2.$

The second upper submatrix of *B* is $B_2=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$.

Then the determinant of the matrix B_2 is

$$\begin{aligned}\det B_2&=\det\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\\ &=2(1)-1(1)\\ &=2-1\\ &=1\end{aligned}$$

Step-8

The third upper submatrix of B is $B_3 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

Then the determinant of the matrix B_3 is

$$\begin{aligned}
\det B_3 &= \det \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \\
&= 2(2-1) - 1(2-2) + 2(1-2) \\
&= 2(1) - 1(0) + 2(-1) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

Step-9

As the determinant of the third submatrix B_3 is 0, so the 3rd condition of the result is not satisfied.

Hence, the matrix A is **indefinite**.