Step-1

Let $$A = \begin{bmatrix} 1 & 1 & - & - & 1 \\ 1 & 1 & - & - & 1 \\ - & - & - & - & - \\ - & - & - & - & - \\ 1 & 1 & - & - & 1 \end{bmatrix}_{n\times n}$$

Applying elementary row operations $R_i \to R_i - R_1, 2 \le i \le n$, this matrix becomes

$$\sim \begin{bmatrix} 1 & 1 & - & - & 1 \\ 0 & 0 & - & - & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & - & - & 0 \end{bmatrix}_{n\times n}$$

Step-2

Again applying the column operations $C_i \to C_i - C_1, 2 \le i \le n$, it will become $$\sim \begin{bmatrix} 1 & 0 & - & - & 0 \\ 0 & 0 & - & - & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & - & - & 0 \end{bmatrix}_{n\times n}$$

Step-3

We observe that the entries in the principal diagonal are 1â€™s or 0â€™s and the entries below the principal diagonal are 0â€™s.

So, this is the echelon form.

The rank of the matrix is nothing but the number of non zero rows or non zero columns in the echelon form.

Therefore, the rank A is 1.

Step-4

Further, the characteristic equation of A can be written as $$|A - \lambda \mathrm{I}| = \begin{vmatrix} 1-\lambda & 1 & - & - & 1 \\ 1 & 1-\lambda & - & - & 1 \\ - & - & - & - & - \\ - & - & - & - & - \\ 1 & 1 & - & - & 1-\lambda \end{vmatrix}_{n\times n} = 0$$

Applying elementary row operations on this, we can make it

$$R_1 \to \sum_{i=1}^{n} R_i \Rightarrow \begin{vmatrix} n-\lambda & n-\lambda & - & - & n-\lambda \\ 1 & 1-\lambda & - & - & 1 \\ - & - & - & - & - \\ - & - & - & - & - \\ 1 & 1 & - & - & n-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (n-\lambda)\begin{vmatrix} 1 & 1 & - & - & 1 \\ 1 & 1-\lambda & - & - & 1 \\ - & - & - & - & - \\ - & - & - & - & - \\ 1 & 1 & - & - & n-\lambda \end{vmatrix} = 0$$

Step-5

Applying $R_i \to R_i - R_1, 2 \le i \le n$, we get $(n-\lambda)\begin{vmatrix} 1 & 1 & - & - & 1 \\ 0 & -\lambda & - & - & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & - & - & -\lambda \end{vmatrix} = 0$

We observe that the entries below the principal diagonal are zero and so, the determinant is the product of diagonal entries.

i.e., $|A-\lambda I| = (n-\lambda)(-\lambda)^{n-1} = 0$

Thus, the eigen values of A are n, 0, 0,â€¦, 0

Step-6

$C = \begin{bmatrix} 1 & 0 & 1 & - & 1 \\ 0 & 1 & 0 & - & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 1 & - & - & 0 \end{bmatrix}_{n\times n}$ The entries of last row and last column depend on n is even or odd.

If n is even, then the last row entries are 0, 1, 0, 1,â€¦, 1.

If n is odd, then the last row and last column are identical to the first row and first column respectively.

To find the rank of this matrix, we apply the elementary row operations and reduce it to the echelon form.

Step-7

Applying $R_{2i-1} \to R_{2i-1} - R_1, R_{2i} \to R_{2i} - R_2, 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, we get

Step-8

$$C \sim \begin{bmatrix} 1 & 0 & 1 & - & 1 \\ 0 & 1 & 0 & - & 0 \\ 0 & 0 & 0 & - & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}_{n\times n}$$

The entries below the principal diagonal are zero and so, it is the echelon form.

The rank = number of non zero rows in the echelon form

= 2

Step-9

The characteristic equation of C is $$|C - \lambda \mathrm{I}| = \begin{vmatrix} 1-\lambda & 0 & 1 & - & - \\ 0 & 1-\lambda & 0 & - & - \\ 1 & 0 & - & - & - \\ - & 1 & 0 & - & 0 \\ 1 & 0 & 1 & - & 1-\lambda \end{vmatrix}$$

Applying the row operation $R_1 \to \sum_{i=1}^{n} R_i$, we get $$\begin{vmatrix} \left\lfloor \frac{n}{2} \right\rfloor - \lambda & \left\lfloor \frac{n}{2} \right\rfloor - \lambda & - & - & \left\lfloor \frac{n}{2} \right\rfloor - \lambda \\ 0 & 1-\lambda & 0 & - & - \\ 1 & 0 & - & - & - \\ - & 1 & 0 & - & 0 \\ 1 & 0 & 1 & - & 1-\lambda \end{vmatrix}$$

Step-10

It can be written as $$\left(\left\lfloor \frac{n}{2} \right\rfloor - \lambda \right) \begin{vmatrix} 1 & 1 & - & - & 1 \\ 0 & 1-\lambda & 0 & - & - \\ 1 & 0 & - & - & - \\ - & 1 & 0 & - & 0 \\ 1 & 0 & 1 & - & 1-\lambda \end{vmatrix} = 0$$

Applying the row operation $R_{2i-1} \to R_{2i-1} - R_1, 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, we get

$$\left(\left\lfloor \frac{n}{2} \right\rfloor - \lambda\right)\begin{vmatrix} 1 & 1 & - & - & 1 \\ 0 & -\lambda & 0 & - & - \\ 0 & -1 & - & - & - \\ - & 0 & 0 & - & 0 \\ 0 & -1 & 0 & - & 0-\lambda \end{vmatrix} = 0$$

Step-11

The other $\left\lfloor \frac{n}{2} \right\rfloor$ operations provide the echelon form and the entries of the principal diagonal are $\left(\left\lfloor \frac{n}{2} \right\rfloor + \lambda\right)(-\lambda)^{n-2}$

Thus, the determinant is $\left(\left\lfloor \frac{n}{2} \right\rfloor - \lambda\right)\left(\left\lfloor \frac{n}{2} \right\rfloor + \lambda\right)(-\lambda)^{n-2}$ =0

Therefore, the eigen values of *C* are 0, 0,â€¦, 0, $\frac{n}{2}, -\frac{n}{2}$