Step-1

Consider the following system:

$$\begin{aligned} u \quad\; + w &= 4 \\ u + v \quad\; &= 3 \\ u + v + w &= 6 \end{aligned}$$

Then the coefficient matrix is,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Factor the given matrix into $A = LU$ or $PA = LU$.

Suppose $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$

Then

$$A = LU$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Step-2

On comparing the coefficients, to get

$$\begin{aligned} u_{11} &= 1, \\ u_{12} &= 0, \\ u_{13} &= 1, \end{aligned}$$

$$l_{21}u_{11} = 1 \Rightarrow l_{21}(1) = 1 \Rightarrow l_{21} = 1,$$

$$l_{21}u_{12} + u_{22} = 1 \Rightarrow (1)(0) + u_{22} = 1 \Rightarrow u_{22} = 1,$$

$$l_{21}u_{13} + u_{23} = 0 \Rightarrow (1)(1) + u_{23} = 0 \Rightarrow u_{23} = -1,$$

$$l_{31}u_{11} = 1 \Rightarrow l_{31}(1) = 1 \Rightarrow l_{31} = 1,$$

$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow (1)(0) + l_{32}(1) = 1 \Rightarrow l_{32} = 1,$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \Rightarrow (1)(1) + (1)(-1) + u_{33} = 1 \Rightarrow u_{33} = 1.$

Hence, the matrices $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ such that $A = LU.$

Step-3

Consider the following system:

$$\begin{aligned} v + w &= 0 \\ u + \quad w &= 0 \\ u + v \quad &= 6 \end{aligned}$$

Then the coefficient matrix is,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

In the matrix A, the pivot is zero, 0.

So, to change the pivot entry, exchange the first and second rows so that the entry 1 in the second row moves into the pivot.

Step-4

For this row exchange, use the permutation matrix P which is the same as an identity matrix.

So, the PA matrix will be as follows:

$$\begin{aligned} PA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Here, the permutation matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Step-5

Factor the given matrix into $A = LU$ or $PA = LU$.

Suppose $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$

Then

$$PA = LU$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Step-6

On comparing the coefficients, to get

$$u_{11} = 1,$$
$$u_{12} = 0,$$
$$u_{13} = 1,$$

$$l_{21}u_{11} = 0 \Rightarrow l_{21} = 0,$$

$$l_{21}u_{12} + u_{22} = 1 \Rightarrow (0)(0) + u_{22} = 1 \Rightarrow u_{22} = 1,$$

$$l_{21}u_{13} + u_{23} = 1 \Rightarrow (0)(1) + u_{23} = 0 \Rightarrow u_{23} = 0,$$

$$l_{31}u_{11} = 0 \Rightarrow l_{31} = 1,$$

$$l_{31}u_{12} + l_{32}u_{22} = 0 \Rightarrow (1)(0) + l_{32}(1) = 0 \Rightarrow l_{32} = 0,$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 0 \Rightarrow (1)(1) + (0)(0) + u_{33} = 0 \Rightarrow u_{33} = -1.$$

Hence, the matrices $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ such that $PA = LU$.