Step-1

Consider the matrices,

$$A=\begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

The objective is to determine the conditions on the entries such that A and B are invertible.

Step-2

An $n \times n$ square matrix is invertible if and only if elimination yields the same number of pivots as rows.

Do elimination on A and B and see the conditions on their entries such that we get a pivot in every row.

Do elimination on matrix A:

Consider the matrix,

$$A=\begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$$

Observe that, if $f=0$, then the third row is all zeros and there can never be a third pivot.

So for matrix A is invertible it must be the case that $f \neq 0$. This implies that there is a pivot in the first column; to make the pivot occur at f, switch rows 1 and 3.

$$\begin{bmatrix} f & 0 & 0 \\ d & e & 0 \\ a & b & c \end{bmatrix}$$

$$R_2 \to R_2 - \frac{d}{f} R_1$$

$$R_3 \to R_3 - \frac{a}{f} R_1$$

$$\begin{bmatrix} f & 0 & 0 \\ 0 & e & 0 \\ 0 & b & c \end{bmatrix}$$

If $e=0$ the second row is all zero, means that there can never be a pivot in that row.

Thus, for matrix A is invertible it must be the case that $e \neq 0$. This implies that there is a pivot in the second column.

Step-3

Eliminate entry below e by subtracting $\frac{b}{e}$ times row 2 from row 3.

$$\begin{bmatrix} f & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & c \end{bmatrix}$$

We already know there are pivots in the first two rows; there will be a pivot in the third row only if $c \neq 0$. Thus, for matrix A is invertible it must be the case that $c \neq 0$.

Therefore, the conditions on entries of matrix A such that A is invertible are:

$$\boxed{c \neq 0,\ \ e \neq 0,\ \ f \neq 0}$$

Step-4

Do elimination on matrix B:

Consider the matrix,

$$B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

Observe that, if $e = 0$, then the third row is all zeros and there can never be a third pivot.

So, for matrix B is invertible it must be the case that $e \neq 0$.

Step-5

To have a pivot in the first column, then either a or c must be nonzero.

If a is nonzero, we can eliminate c by subtracting $\frac{c}{a}$ times row 1 from row 2.

$$\begin{bmatrix} a & b & 0 \\ 0 & d - \frac{c}{a}b & 0 \\ 0 & 0 & e \end{bmatrix}$$

Then, to have a pivot the second row, it must be the case that,

$$d-\frac{c}{a}b\neq 0$$

This implies,

$$ad-bc\neq 0$$

Step-6

On the other hand, if $c\neq 0$, we can switch row 1 and 2 to get,

$$\begin{bmatrix} c & d & 0 \\ a & b & 0 \\ 0 & 0 & e \end{bmatrix}$$

Eliminate a by subtracting $\frac{a}{c}$ times row 1 from row 2.

$$\begin{bmatrix} c & d & 0 \\ 0 & b-\frac{a}{c}d & 0 \\ 0 & 0 & e \end{bmatrix}$$

Then, to have a pivot the second row, it must be the case that,

$$b-\frac{a}{c}d\neq 0$$

This implies,

$$bc-ad\neq 0$$

Therefore, to matrix B invertible, either $a\neq 0$ and $ad-bc\neq 0$, or, $c\neq 0$ and $bc-ad\neq 0$.

Step-7

Note that, $ad-bc\neq 0$ is equivalent to $bc-ad\neq 0$, and this inequality required that either a or c is nonzero(if both were zero then the left hand side would be zero).

Hence the simplified conditions under which matrix B is invertible are:

$$\boxed{ad-bc\neq 0 \text{ and } e\neq 0}$$