5

Eigenvalues and Eigenvectors （特征值与特征向量）

5.1

EIGENVALUES AND EIGENVECTORS

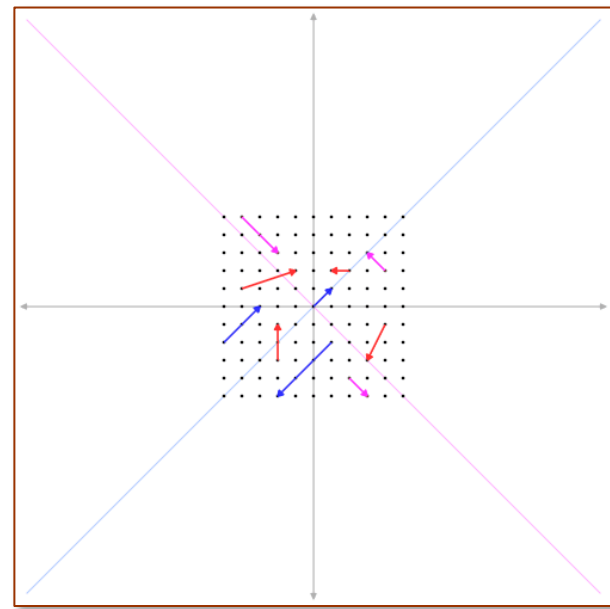Introduction

Definition

Calculations

Properties

Application
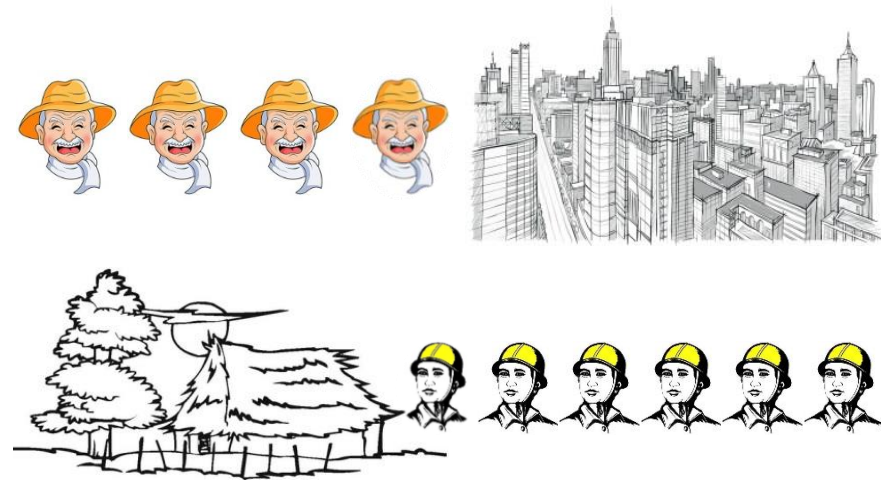
I. Introductory example - 人口流动问题

简化模型

❶ 假设：

从事农业工作的人员中每年有四分之一转为从事非农业工作，

从事非农业工作的人员中每年有六分之一转为从事农业工作.

❷ 人口总数不变.

预测多年之后劳动力从业情况的发展趋势.

分析 设最初农业人员和非农业人员的数量分别为 y_0, z_0，

第 1 年末数量为 y_1, z_1，

$$\begin{cases} y_1 = \dfrac{3}{4} y_0 + \dfrac{1}{6} z_0, \\[2mm] z_1 = \dfrac{1}{4} y_0 + \dfrac{5}{6} z_0. \end{cases} \longrightarrow \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\[2mm] \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

分析 设最初农业人员和非农业人员的数量分别为 y_0, z_0，第 1 年末数量为 y_1, z_1，

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

矩阵 A

分析 设最初农业人员和非农业人员的数量分别为 y_0, z_0，第 k 年末数量为 y_k, z_k，

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

矩阵 A

如何计算方阵 A 的幂 A^k ?

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix} \begin{bmatrix} y_{k-1} \\ z_{k-1} \end{bmatrix} = \ldots = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

最简单的方阵 Q： $\quad Q = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad Q^k = \begin{bmatrix} \lambda_1{}^k & \\ & \lambda_2{}^k \end{bmatrix}.$

有无可能？

设想

$$A = PQP^{-1}$$ ⟶ $$A^k = PQ^kP^{-1}$$

（其中Q为对角阵）

$$A^2 = AA = PQP^{-1}PQP^{-1} = PQ^2P^{-1}$$

$$A^3 = AAA = PQP^{-1}PQP^{-1}PQP^{-1} = PQ^3P^{-1}$$

$$AP = PQ$$

$$AP_1 = \lambda_1 P_1, \quad AP_2 = \lambda_2 P_2$$

$$A[P_1 \mid P_2] = [P_1 \mid P_2]\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$ ⟷ $$[AP_1 \quad AP_2] = [\lambda_1 P_1 \quad \lambda_2 P_2]$$

（其中P_1, P_2为2维列向量）

$$A_{2\times 2}\, x_{2\times 1} = \lambda x_{2\times 1}$$

II. Eigenvalues and Eigenvectors – Definition & Calculation

Definition 1 Let A be a *square matrix* of degree n.

If there exist a non-zero vector x and a scalar λ such that

$$A\,x = \lambda\,x \,,$$

then λ is called an **eigenvalue (特征值)** of A, and x is called an **eigenvector (特征向量)**, *corresponding to* the eigenvalue λ.

德语 *eigen* (有特征的, 自身的, 个体的)

思考

A 的特征值与特征向量有何直观含义?

$$R^n \rightarrow R^n$$
$$A : x \rightarrow Ax$$
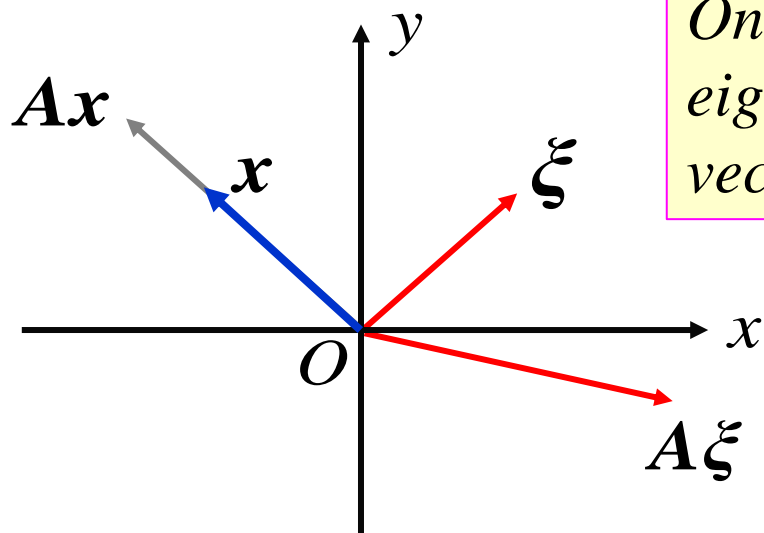
$$R^n \rightarrow R^n$$
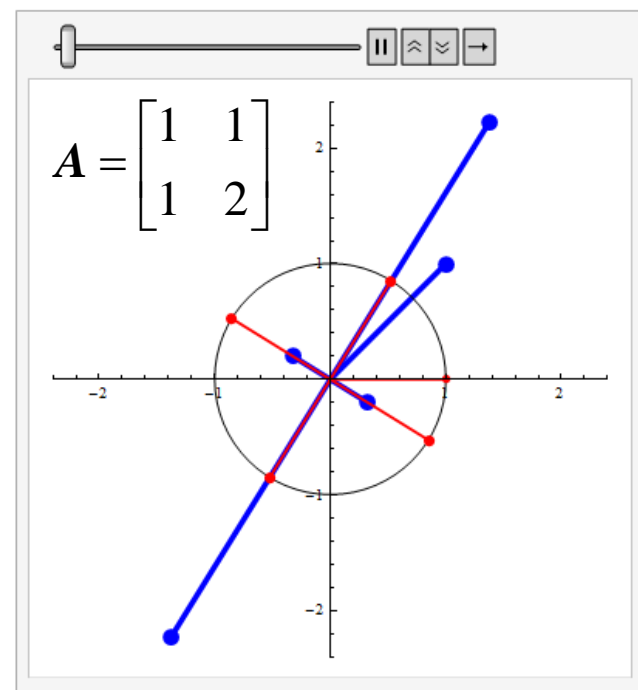$$\lambda : x \rightarrow \lambda x$$

David Hilbert
1862-1943

给定方阵$A \in \mathbf{R}^{n \times n}$, 一般说来, 对于$\xi \in R^n$, 向量 $A\xi$ 与 ξ <u>不在</u>同一方向上.

但也可能**存在**向量 x, **使得 Ax在x的方向上**. $\boxed{Ax \text{ is a multiple of } x.}$

$\boxed{\textit{Only certain special numbers are eigenvalues, and only certain special vectors are eigenvectors.}}$

A 的特征值与特征向量的**直观含义**:

特征向量: **与 A 左乘下的像"共线"**

特征 值: **"伸缩比例"**

For example, $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which shows that 4 is an eigenvalue of A, $(1,1)^{\text{T}}$ is an

eigenvector for 4 (an eigenvector corresponding to 4).

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -\dfrac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ \dfrac{2}{3} \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -\dfrac{2}{3} \end{bmatrix},$$

which tells us that -1 is also an eigenvalue of A, and

$(1, -2/3)^{\text{T}}$ is an eigenvector corresponding to -1.

$$A\boxed{x} = \boxed{\lambda}\,x$$

How to get?

!dea Equivalently, an eigenvalue λ and a corresponding eigenvector $\boldsymbol{x}$ satisfy:

$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\,\boldsymbol{x} = \boldsymbol{0}.$$

Since $\boldsymbol{x}$ is a *non-zero* vector, the matrix $\boldsymbol{A} - \lambda\,\boldsymbol{I}$ is not invertible, and thus the determinant $|\boldsymbol{A} - \lambda\,\boldsymbol{I}|$ equals 0.

注: 特征向量 $\boldsymbol{x}$ 是非零向量, 是齐次线性方程组

$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\,\boldsymbol{x} = \boldsymbol{0}$$

的非零解. λ 应满足

$$|\boldsymbol{A} - \lambda\,\boldsymbol{I}| = 0.$$

(**characteristic equation**: 特征方程)

We note that the determinant $|A - \lambda I|$ is a polynomial of degree n in λ, called the **characteristic polynomial** of A.

Definition 2   Let $A=[a_{ij}]_{n \times n}$ be a *square matrix* of degree n. Then

$$f(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

is called the **characteristic polynomial (特征多项式)** of **A.**

n 阶矩阵**A**的特征多项式是λ的 n 次多项式.

Lemma 1 *A scalar λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial.*
(λ 是 **A** 的特征值, 当且仅当 λ 是特征多项式的根.)

n 阶矩阵 A 的特征多项式是 λ 的 n 次多项式.

Gauss
1777-1855

代数基本定理(Fundamental theorem of algebra**)**

在复数范围内每个 n 次复系数方程恰有 n 个根.

注释：n 阶方阵 A 在复数范围内有 n 个特征值.

n 阶矩阵 A 的特征多项式在复数域上的 n 个根都是矩阵 A 的特征值, 其 k 重根叫做 k 重特征值.

当 $n \geq 5$ 时, 特征多项式没有一般的求根公式.

(Galois and Abel proved that there can be no algebraic formula for the roots of a fifth-degree polynomial).

Galois 1811-1832

即使是三阶矩阵, 一般也难以求根.

解决方法： "计算方法（数值分析）"

Abel 1802-1829

Example 1 Let $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$. Then

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & -5 \\ 2 & -3 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 2.$$

The roots are $\lambda_1 = -1$ and $\lambda_2 = 2$. Eigenvectors of A can be obtained as follows.

For $\lambda_1 = -1$, we solve the system of linear equations $(A - \lambda_1 I)x = (A + I)x = 0$, i.e.,

$$\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that $x_1 = k_1(1,1)^{\mathrm{T}}$ $(k_1 \in \mathbf{R}, k_1 \neq 0)$ are eigenvectors corresponding to the eigenvalue $\lambda_1 = -1$.

For $\lambda_2 = 2$, we solve the system of linear equations $(A - \lambda_2 I)x = (A - 2I)x = 0$, i.e.,

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that $x_2 = k_2(5,2)^{\mathrm{T}}$ $(k_2 \in \mathbf{R}, k_2 \neq 0)$ are eigenvectors corresponding to the eigenvalue $\lambda_2 = 2$.

This example illustrates a method for finding eigenvalues and eigenvectors of a given matrix, which is important in the area of matrix theory and many applications.

*A **Process** for finding eigenvalues and eigenvectors of a matrix **A** :*

1. ***Compute the determinant of*** $A - \lambda I$.

With λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.

2. ***Find the roots of this polynomial***.

The n roots are the eigenvalues of A.

3. ***For each eigenvalue solve the equation*** $(A - \lambda I) x = 0$.

Since the determinant is zero, there are solutions other than $x = 0$. Those are the eigenvectors.

特征值与特征向量的求解 步骤

第一步　计算 A 的特征多项式；

第二步　求出特征多项式的全部根，即得 A 的全部特征值；

第三步　将每一个特征值代入相应的线性方程组进行求解，即得该特征值的特征向量.

$$A \implies |A - \lambda I| = 0 \implies (A - \lambda_i I)x = 0$$

求特征值 λ_i　　　　　　　求特征向量

Remark: 另一种等价求法

$$A \implies |\lambda I - A| = 0 \implies (\lambda_i I - A)x = 0$$

求特征值 λ_i　　　　　　　求特征向量

Example 2 Everything is clear when A is a *diagonal matrix*:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ has } \lambda_1 = 3 \text{ with } x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\lambda_2 = 2 \text{ with } x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On each eigenvector A acts like: $Ax_1 = 3x_1$ and $Ax_2 = 2x_2$.

Other vectors like $x = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ are mixtures $x_1 + 5x_2$ of the two eigenvectors,

and A acts like: $A(x_1 + 5x_2) = 3x_1 + 10x_2 = \begin{bmatrix} 3 \\ 10 \end{bmatrix}.$

This is Ax for a typical vector x—not an eigenvector. But the action of A is determined by its eigenvectors and eigenvalues.

Remark: 1. The eigenvalues are on the main diagonal when A is *diagonal*.

(n 阶对角矩阵A的特征值是它的n个主对角元 $a_{11}, a_{22}, \cdots, a_{nn}$.)

This is true since the characteristic polynomial of A is

$$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

2. The eigenvalues are on the main diagonal when B is *triangular*.

(n 阶上(下)三角形矩阵B的特征值也是它的 n 个主对角元 $b_{11}, b_{22}, \cdots, b_{nn}$.)
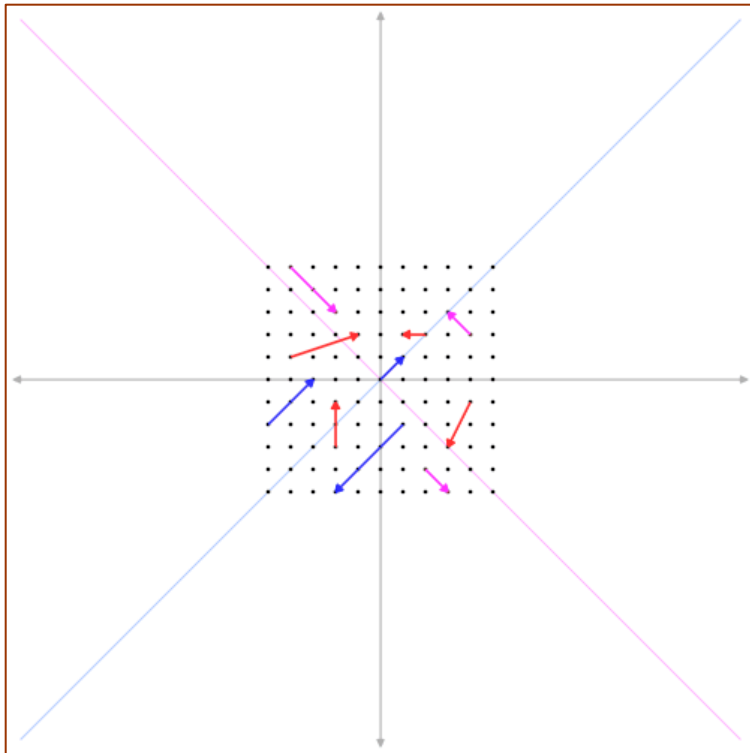
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Exercise

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Example 3 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The Eigenvectors

$$k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(k_1 \neq 0) \qquad (k_2 \neq 0)$$

corresponding respectively to

the Eigenvalues:

$$\lambda_1 = 1 \qquad \lambda_2 = 3$$

Example 4   If a square matrix A satisfies $A^2 = A$.
Show that the only possible eigenvalues of A are 0 or 1.

Proof Let λ be the eigenvalue of A , then $Ax = \lambda x$. And

$$A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda\lambda x = \lambda^2 x$$

So $(\lambda^2 - \lambda)x = 0$.

Since $x \neq 0$, we have $\lambda^2 - \lambda = 0$,   and $\lambda = 0$ or $\lambda = 1$.

For example,

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues: 0   1   1, 0

Remark: The eigenvalues of a *projection matrix* are 1 or 0.
(投影矩阵的特征值为0或1)

Remarks: Give a matrix A of degree n.

• A has exactly n eigenvalues, some of them might be *repeated*.

• Some eigenvalues of A may be *complex* numbers; some matrices may have no *real* eigenvalue, for instance, rotation matrices.

$$A_1 = \begin{bmatrix} 0 & -2 & -2 \\ 2 & -4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

has eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -2$ (二重特征值, ***a root of multiplicity 2***).
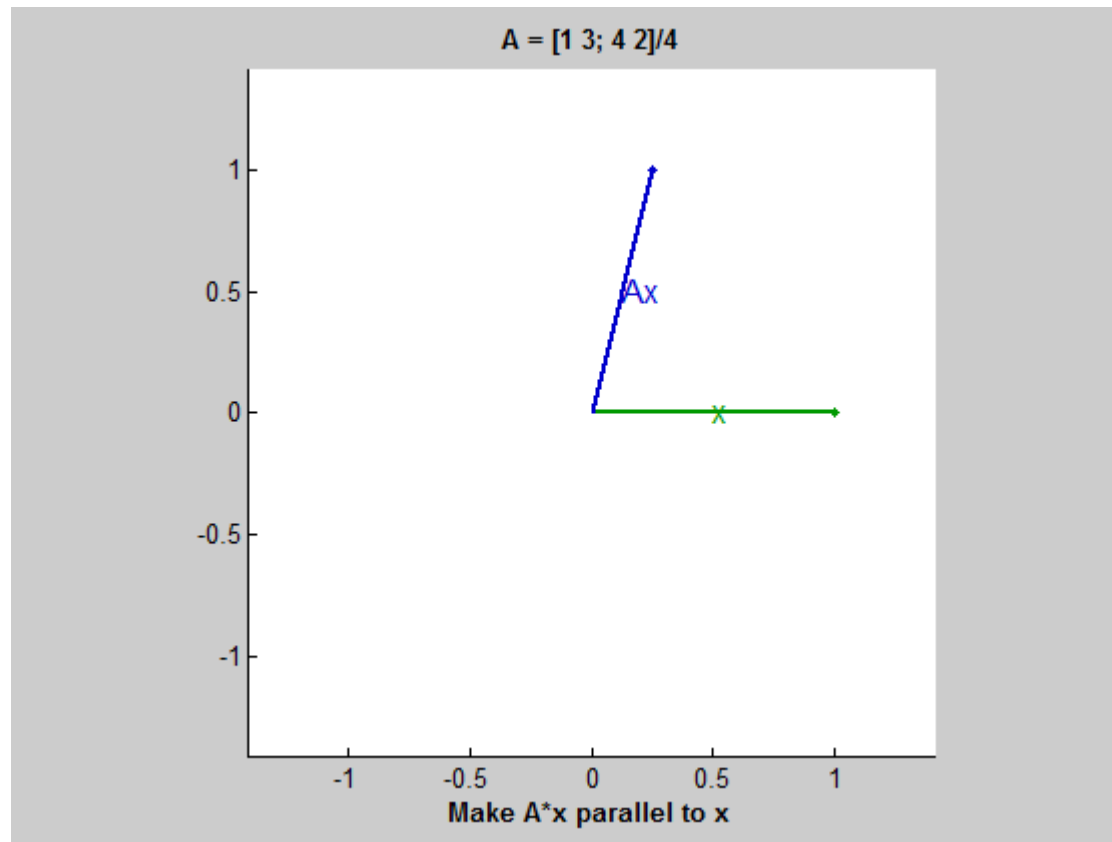
$$\left| A_1 - \lambda I \right| = -\lambda(\lambda + 2)^2$$

$$A_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has eigenvalues: $\lambda = \cos\theta \pm i\sin\theta$.

$$\left| A_2 - \lambda I \right| = \lambda^2 - (2\cos\theta)\lambda + 1$$

Matlab demo: eigshow

$$A = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \qquad \lambda_1 = -0.5, \\ \lambda_2 = 1.25.$$

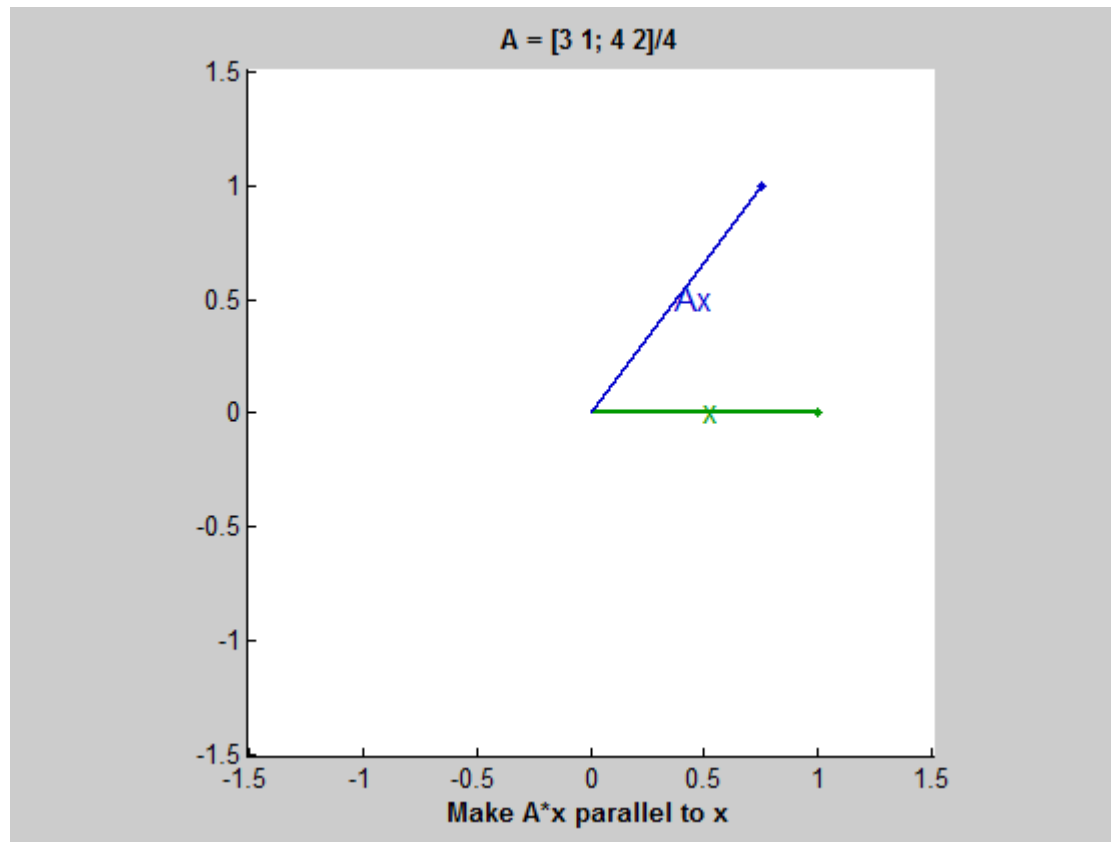A = [1 3; 4 2]/4

Make A*x parallel to x

https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c

Matlab demo: eigshow

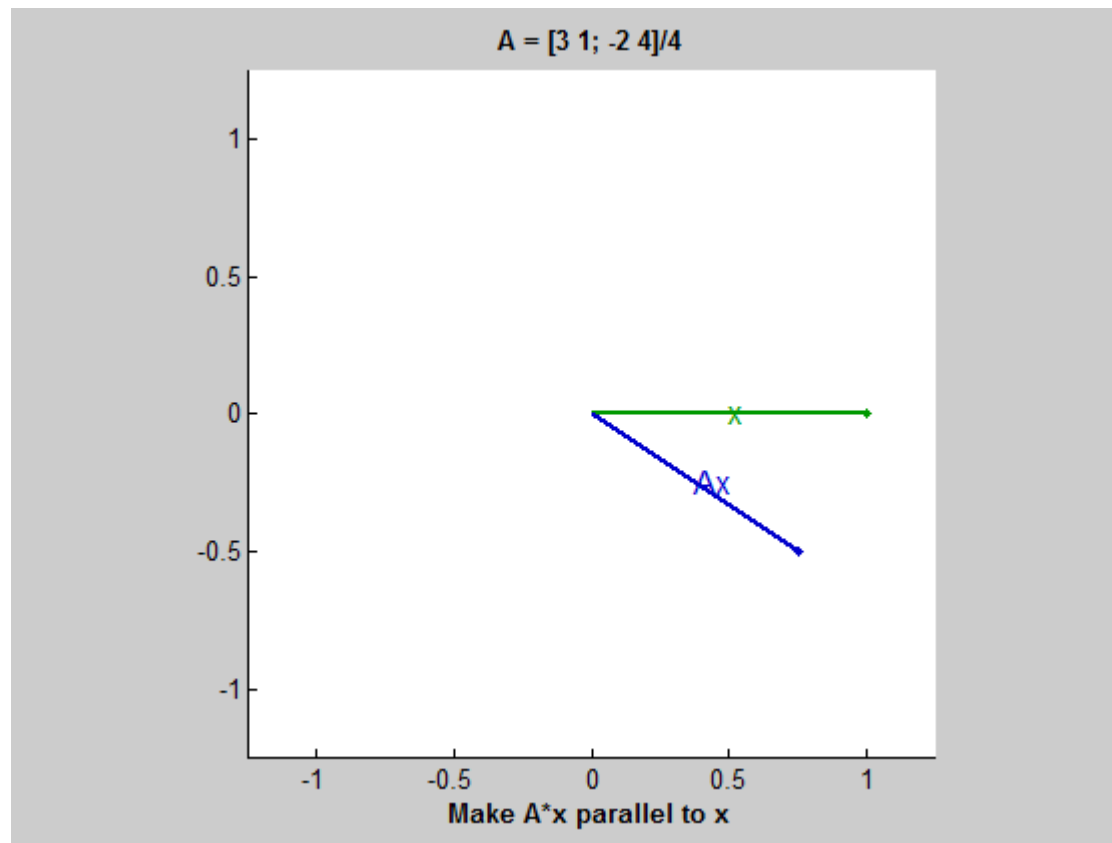$$A = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\lambda_1 = 1.14,$$
$$\lambda_2 = 0.11.$$

https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c

Matlab demo: eigshow

$$A = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

$$\lambda_1 = 0.8750 + 0.3307i,$$
$$\lambda_2 = 0.8750 - 0.3307i.$$

https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c

III. Eigenvalues and Eigenvectors – Properties

Theorem 1   If x_1, x_2 are two eigenvectors of A corresponding to the eigenvalue λ_0, then $k_1 x_1 + k_2 x_2$ is also an eigenvector for λ_0, where k_1, k_2 are any numbers that make $k_1 x_1 + k_2 x_2 \neq 0$.

定理1   若 x_1, x_2 是 A 属于 λ_0 的两个的特征向量, 则 $k_1 x_1 + k_2 x_2$ 也是 A 属于 λ_0 的特征向量 (其中 k_1, k_2 是任意常数, 但 $k_1 x_1 + k_2 x_2 \neq 0$).

Proof x_1, x_2 are solutions to the following homogeneous system of linear equations:

$$(A - \lambda_0 I)\, x = 0 \,,$$

So $k_1 x_1 + k_2 x_2$ is also a solution.

Therefore, nonzero vector $k_1 x_1 + k_2 x_2$ is also an eigenvector of A corresponding to the eigenvalue λ_0.

Theorem 2 Let $A = [a_{ij}]_{n \times n}$, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the n eigenvalues of A.

Then

$$(1) \prod_{i=1}^{n} \lambda_i = |A| \, (\text{i.e., } \det A),$$

$$(2) \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \text{trace}(A)(\text{i.e., } \text{tr}(A)).$$

The sum of all diagonal entries of A is called the **trace of A (A 的迹)**.

*Some properties about the traces of matrices:

$\text{tr}(A_{n \times n} + B_{n \times n}) = \text{tr}(A_{n \times n}) + \text{tr}(B_{n \times n})$, $\text{tr}(A_{m \times n} B_{n \times m}) = \text{tr}(B_{n \times m} A_{m \times n})$.

Proof

(1) Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the square matrix A.

Then the characteristic polynomial of A can be expressed as

$$| A - \lambda I | = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

Let $\lambda = 0$ on both sides, we can immediately get

$$| A | = \prod_{i=1}^{n} \lambda_i.$$

(make a clever choice of λ)

(2)
$$| A - \lambda I | = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix},$$

Among the expansion of the determinant, there is a term $(a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda),$ and the terms λ^n, λ^{n-1} in the characteristic polynomial *only* come from this term.

Therefore, $| A - \lambda I |$
$$= (-\lambda)^n + (a_{11} + a_{22} + \cdots + a_{nn})(-\lambda)^{n-1} + \cdots + |A|.$$

On the other hand
$$| A - \lambda I | = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$
$$= (-\lambda)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-\lambda)^{n-1}\cdots + \prod_{i=1}^{n}\lambda_i,$$

Finally, $\quad \sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n}a_{ii}.$ **(find the coefficient of $(-\lambda)^{n-1}$ and compare)**

Remark: Zero is an eigenvalue of A if and only if A is not invertible.

矩阵A可逆的充要条件是 A的任意一个特征值不等于零.

A为奇异(singular)矩阵的充要条件是A至少有一个特征值等于零.

Note: A certain eigenvector of A cannot be corresponding to different eigenvalues. （A的一个特征向量不能属于不同的特征值.)

If x were an eigenvector of A corresponding to different eigenvalues $\lambda_1, \lambda_2 (\lambda_1 \neq \lambda_2)$,

i.e., $A x = \lambda_1 x$ and $A x = \lambda_2 x$,

then $\lambda_1 x = \lambda_2 x$, that is, $(\lambda_1 - \lambda_2) x = 0$.

Since $\lambda_1 - \lambda_2 \neq 0$, then $x = 0$, which is impossible for an eigenvector.

Property 1 If $\lambda_1, \lambda_2,..., \lambda_n$ are the eigenvalues of $\boldsymbol{A}$, then $\lambda_1^k, \lambda_2^k,..., \lambda_n^k$ are the eigenvalues of $\boldsymbol{A}^k$, where k is a positive integer, and k may equal -1 if $\boldsymbol{A}$ is invertible.

Moreover, if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to λ, then $\boldsymbol{x}$ is also an eigenvector of $\boldsymbol{A}^k$ corresponding to λ^k.

Proof. Notice that, if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to λ_i, then

$$\boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda_i\boldsymbol{x}) = \lambda_i\boldsymbol{A}\boldsymbol{x} = \lambda_i(\lambda_i\boldsymbol{x}) = \lambda_i^2\boldsymbol{x},$$

The proposition then follows.

性质1 若λ是$\boldsymbol{A}$的特征值, $\boldsymbol{x}$ 是$\boldsymbol{A}$的属于λ 的特征向量. 则

(1) $k\lambda$ 是$k\boldsymbol{A}$的特征值（k为任意常数）；

(2) λ^m是$\boldsymbol{A}^m$的特征值;

(3) 若$\boldsymbol{A}$可逆，则λ^{-1}为$\boldsymbol{A}^{-1}$的一个特征值;

且 $\boldsymbol{x}$ 仍然是矩阵$k\boldsymbol{A}$, $\boldsymbol{A}^m$和$\boldsymbol{A}^{-1}$的分别对应于特征值 $k\lambda, \lambda^m$ 和 λ^{-1} 的特征向量.

Example 5 Suppose a 3×3 matrix $\boldsymbol{A}$ has eigenvalues $1, -1, 2$. And $\boldsymbol{B} = \boldsymbol{A}^3 - 5\boldsymbol{A}^2$. Find $|\boldsymbol{B}|$.

Solution The eigenvalues of $\boldsymbol{B}$ are: $1^3 - 5\cdot 1^2 = -4$,

$$(-1)^3 - 5\cdot(-1)^2 = -6, \qquad 2^3 - 5\cdot 2^2 = -12.$$

So $|\boldsymbol{B}| = (-4)(-6)(-12) = -288$.

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Property 2 The matrices $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ have same eigenvalues.

(**性质2** 矩阵 $\boldsymbol{A}$ 和 $\boldsymbol{A}^{\mathrm{T}}$ 的特征值相同.)

Proof. $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det(\boldsymbol{A} - \lambda\boldsymbol{I})^{\mathrm{T}}$

$$= \det(\boldsymbol{A}^{\mathrm{T}} - (\lambda\boldsymbol{I})^{\mathrm{T}}) = \det(\boldsymbol{A}^{\mathrm{T}} - \lambda\boldsymbol{I}).$$

Thus $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ have the same characteristic polynomials.

The proposition then follows.

回顾：人口流动问题

Possible? **Yes!**

$$A = PQP^{-1}$$

$$A^k = PQ^k P^{-1} \quad (Q: \text{diagonal})$$

$$AP = PQ$$

$$A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{7}{12}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\ \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & \\ & \dfrac{7}{12} \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = PQ^k P^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{5}(y_0 + z_0) + \dfrac{1}{5} \times \left(\dfrac{7}{12}\right)^k (3y_0 - 2z_0) \\[4mm] \dfrac{3}{5}(y_0 + z_0) + \dfrac{1}{5} \times \left(\dfrac{7}{12}\right)^k (2z_0 - 3y_0) \end{bmatrix} \rightarrow \begin{bmatrix} \dfrac{2}{5}(y_0 + z_0) \\[4mm] \dfrac{3}{5}(y_0 + z_0) \end{bmatrix}, \quad \text{当} k \rightarrow \infty \text{时}.$$

农业与非农业人口比例的趋势为 $2 : 3$.

? 思考

1 y_0 与 z_0 的比例对稳定的趋势有多大影响?

2 稳定的趋势与 A 的什么特征有联系?

3 如果人口总数是变化的, 如何建模?

$$A = \begin{bmatrix} \dfrac{3}{4} & \dfrac{1}{6} \\[2mm] \dfrac{1}{4} & \dfrac{5}{6} \end{bmatrix}.$$

$$= P \, Q \, P^{-1}.$$

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 7/12 \end{bmatrix}$$

对 $A_{n \times n}$,

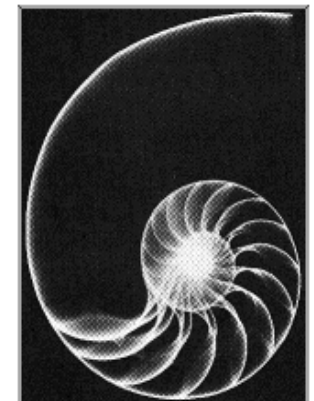- 不同的特征值对应的特征向量线性无关吗?

- 是否存在可逆矩阵 P 和对角矩阵 Q, 使得 $A=PQP^{-1}$?

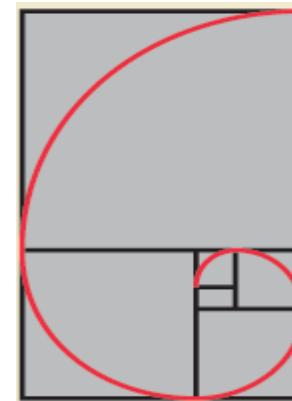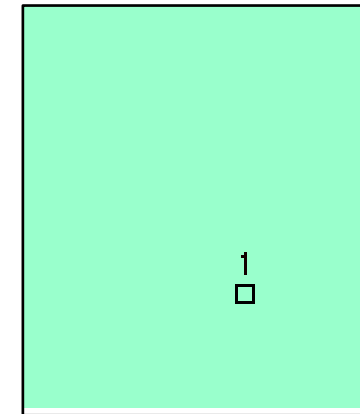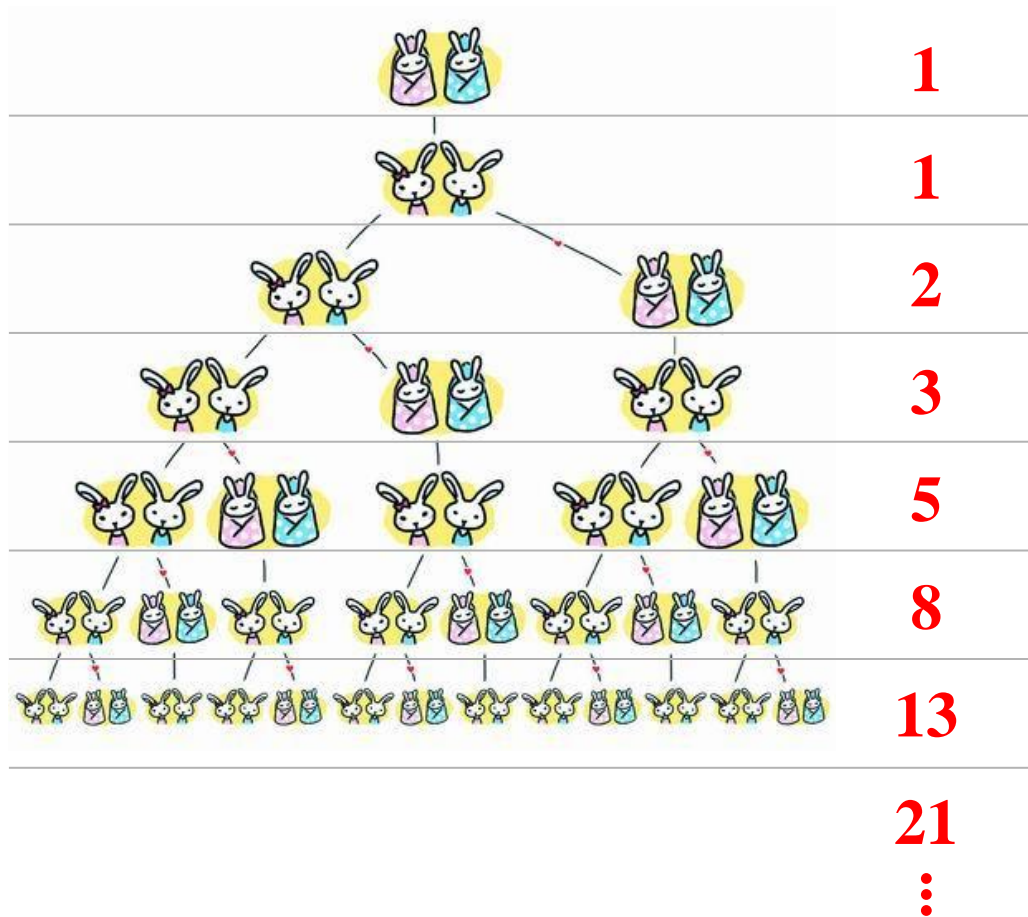Come back later in 5.2: *Diagonalization* (对角化) *of a Matrix*.

Another interesting application…

用矩阵方法求*Fibonacci数列*的通项公式

Fibonacci
1175-1250

1

1

2

3

5

8

13

21

⋮

*Fibonacci*数列的递推关系为

$$F_0=0, \ F_1=1, \ F_{n+2}=F_{n+1}+F_n, \ n=0,1,2,\ldots$$

首先构造一组恒等式

$$\begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_{n+1} = F_{n+1}. \end{cases} \quad \Longrightarrow \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

$$\begin{array}{cc} A & u_n \end{array}$$

$$\Longrightarrow \quad \boxed{u_n = A u_{n-1} = A^n u_0.}$$

A的特征值为

$$\lambda_1 = \frac{1-\sqrt{5}}{2}, \lambda_2 = \frac{1+\sqrt{5}}{2}.$$

进而求出数列通项

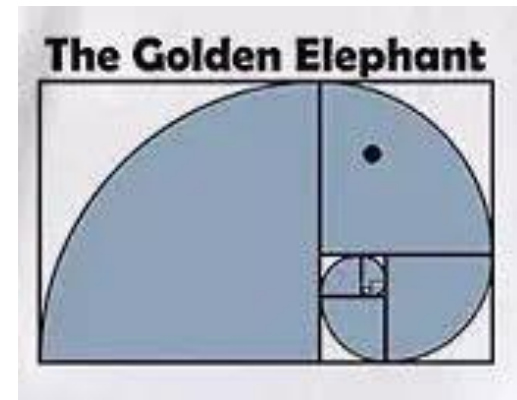$$\boxed{F_n = \frac{\sqrt{5}}{5}\left[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right].}$$

Key words:

Definition

Calculation

Properties

Examples

The Golden Elephant

Homework

See Blackboard

Homework