Suggested Solutions of Homework 6 MA327

Ex 1 Let $p \in \mathbf{x}(U), w \in T_p(S)$.

Then w is tangent to a curve $\mathbf{x}(\alpha(t))$ at $t = 0$, where $\alpha(t) = (u(t), v(t))$ in a curve in U; thus, w may be written $(t = 0)$ $w = \mathbf{x}_u u' + \mathbf{x}_v v'$.

By definition, $d\varphi_p(w)$ is tangent to the curve $\underline{\mathbf{x}}(\alpha(t))$ at $t = 0$.

$$d\varphi_p(w) = \underline{\mathbf{x}_u} u' + \underline{\mathbf{x}_v} v'.$$

So let v_1, v_2 be two tangent vector at $T_p(S)$, we have

$$\langle d\varphi_p(v_1), d\varphi_p(v_2) \rangle_{\varphi(p)} = \underline{E} u_1' u_2' + \underline{F}(u_1' v_2' + u_2' v_1') + \underline{G} v_1' v_2' = 1/\lambda^2 \langle v_1, v_2 \rangle_p.$$

Ex 2 (i)Since $\varphi : S \to \overline{S}$ is a diffeomorphism as an isometry map, $\varphi \circ \mathbf{x}$ is a parametrization. In detail: $\overline{\mathbf{x}}$ is differentiable since $\mathbf{x}$ and φ are differentiable; $\overline{\mathbf{x}}$ is a homeomorphism since $\mathbf{x}$ and φ are both homeomorphisms; $d\overline{\mathbf{x}}_q$ is one to one for $q \in \overline{S}$ because $d\overline{\mathbf{x}_q} = d\varphi_{\varphi^{-1}(q)} \circ d\mathbf{x}_{\varphi^{-1}(q)}$ is one-to-one.

(ii)Let $w \in T_p(S), \forall p \in S$. Since φ is isometry, $\langle d\varphi_p(w), d\varphi_p(w) \rangle = \langle w, w \rangle$ where $d\varphi_p(w) \in T_{\varphi(p)}(\overline{S})$.

Suppose w is the tangent vector of a curve $\mathbf{x}(\alpha(t))$ at $t = 0$, then $w = \mathbf{x}_u u' + \mathbf{x}_v v'$. And that

$$\overline{E}(u')^2 + 2\overline{F} u' v' + \overline{G}(v')^2 = E(u')^2 + 2F u' v' + (v')^2$$

Because u', v' are arbitrary,

$$E = \overline{E}, F = \overline{F}, G = \overline{G}$$

Ex 3. $\Rightarrow$:If $\varphi : S \to \overline{S}$ is an isometry, then for any parametrized curve $\alpha(t) = \mathbf{x}(u(t), v(t)), t \in (-\epsilon, \epsilon)$,, its arc length is

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} d\tau.$$

The arc length of the image curve $\varphi \circ \mathbf{x}(u(t), v(t))$ is

$$S(t) = \int_0^t \sqrt{\overline{E}(u')^2 + 2\overline{F}u'v' + \overline{G}(v')^2} d\tau.$$

Since φ is isometry, $E = \overline{E}, F = \overline{F}, G = \overline{G}$, their arc length is equal.

$\Leftarrow$: Since every parametrized curve and its image have the same arc length, we have

$$\int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} - \sqrt{\overline{E}(u')^2 + 2\overline{F}u'v' + \overline{G}(v')^2} d\tau = 0, \forall -\epsilon \leq a < b \leq \epsilon.$$

By the continuity of first fundamental form, we have

$$E(u')^2 + 2Fu'v' + G(v')^2 = \overline{E}(u')^2 + 2\overline{F}u'v' + \overline{G}(v')^2.$$

Since u', v' are arbitrary, $E = \overline{E}, F = \overline{F}, G = \overline{G}$. Thus φ is isometry.

Ex 4. Let the parametrization of plane be $\mathbf{x}$ = identity, the parametrization of sphere by stereographic projection be $\overline{\mathbf{x}}$. Then the first fundamental form of $\mathbf{x}$ is

$$E = G = 1, F = 0;$$

the first fundamental form of $\overline{\mathbf{x}}$ is

$$\overline{E} = \overline{G} = \frac{16}{(u^2 + v^2 + 4)^2}, \overline{F} = 0.$$

Apply the conclusion in Ex 1, the sphere is locally conformal to a plane.

Ex 5.

(a) Because φ is a diffeomorphism, its inverse φ^{-1} is a diffeomorphism. And that $\forall q \in S_2, w_1, w_2 \in T_q(S_2), \exists! p \in S_1, v_1, v_2 \in T_p(S_1)$ such that $q = \varphi(p), d\varphi_p(v_1) = w_1, d\varphi_p(v_2) = w_2$. So

$$\begin{aligned}
\langle d\varphi_q^{-1}(w_1), d\varphi_q^{-1}(w_2)\rangle &= \langle d\varphi_q^{-1}(d\varphi_p(v_1)), d\varphi_q^{-1}(d\varphi_p(v_2))\rangle \\
&= \langle v_1, v_2\rangle \\
&= \langle d\varphi_p(v_1), d\varphi_p(v_2)\rangle \\
&= \langle w_1, w_2\rangle.
\end{aligned}$$

Hence, φ^{-1} is an isometry.

(b)Let $p \in S_1$, $\varphi(p) = q \in S_2$, $w_1, w_2 \in T_p(S_1)$. Then

$$\begin{aligned}
\langle d(\psi \circ \varphi)_p(w_1), d(\psi \circ \varphi)_p(w_2)\rangle &= \langle d\psi_q(\varphi_p(w_1)), d\psi_q(\varphi_p(w_2))\rangle \\
&= \langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle \\
&= \langle w_1, w_2\rangle.
\end{aligned}$$

Ex 6. WLOG, assume the rotation axis is z-axis.

Let $\mathbf{x} = (x(u,v), y(u,v), z(u,v))$, the parametrization after rotation is $\overline{\mathbf{x}} = (\cos\phi x(u,v) - \sin\phi y(u,v), \sin\phi x(u,v) + \cos\phi y(u,v), z(u,v))$. By calculation, $\mathbf{x}, \overline{\mathbf{x}}$ have the same fundamental forms, so the rotation is isometry.

Ex 7

$\Rightarrow$:

$$\cos(d\varphi_p(v_1), d\varphi_p(v_2)) = \frac{\langle d\varphi_p(v_1), d\varphi_p(v_2)\rangle}{\sqrt{\langle d\varphi_p(v_1), d\varphi_p(v_1)\rangle}\sqrt{\langle d\varphi_p(v_2), d\varphi_p(v_2)\rangle}} = \frac{\lambda^2\langle v_1, v_2\rangle}{\lambda|v_1|\lambda|v_2|}$$

$$= \frac{\langle v_1, v_2\rangle}{|v_1||v_2|} = \cos(v_1, v_2)$$

$\Leftarrow$: Let e_1, e_2 be orthonormal basis in $T_p(S)$. Suppose

$$\begin{aligned}
\langle d\varphi_p(e_1), d\varphi_p(e_1)\rangle &= \lambda_1, \\
\langle d\varphi_p(e_1), d\varphi_p(e_2)\rangle &= \mu, \\
\langle d\varphi_p(e_2), d\varphi_p(e_2)\rangle &= \lambda_2,
\end{aligned}$$

Take $v_1 = e_1, v_2 = \cos\theta e_1 + \sin\theta e_2$. Then

$$\cos\theta = \frac{\lambda_1\cos\theta + \mu\sin\theta}{\sqrt{\lambda_1(\lambda_1\cos^2\theta + 2\mu\sin\theta\cos\theta + \lambda_2\sin^2\theta)}}.$$

Take $\theta = \pi/2$ then we know $\mu = 0$. And that

$$\lambda_1 = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.$$

Thus $\lambda_1 = \lambda_2$. i.e. φ is locally conformal.

Ex 8 Let identity map $id : \mathbb{R}^2 - Q \to \mathbb{R}^2 - Q$ be a parametrization of $\mathbb{R}^2 - Q$. Then the first fundamental form of id is $E = G = 1, F = 0$.

Seeing φ as $\varphi \circ id$. Then the first fundamental form of φ are

$$\begin{aligned} \overline{E} &= \langle (u_x, v_x), (u_x, v_x) \rangle = u_x^2 + v_x^2 = u_x^2 + u_y^2 \\ \overline{F} &= \langle (u_x, v_x), (u_y, v_y) \rangle = u_x u_y + v_x v_y = 0 \\ \overline{G} &= \langle ((u_y, v_y), (u_y, v_y) \rangle = u_y^2 + v_y^2 = u_x^2 + u_y^2 \end{aligned}$$

and since $u_x^2 + u_y^2 \neq 0$, put $\lambda = \frac{1}{\sqrt{u_x^2+u_y^2}}$ and we see that $E = \lambda^2 \overline{E}, F = \lambda^2 \overline{F}, G = \lambda^2 \overline{G}$, hence φ is local conformal.

Ex 9. When $F = 0$, solve Christoffel symbols and we have

$$\Gamma_{11}^1 = \frac{E_u}{2E} \quad \Gamma_{11}^2 = -\frac{E_v}{2G} \quad \Gamma_{12}^1 = \frac{E_v}{2E} \quad \Gamma_{12}^2 = \frac{G_u}{2G} \quad \Gamma_{22}^1 = -\frac{G_u}{2E} \quad \Gamma_{22}^2 = \frac{G_v}{2G}$$

$$\Rightarrow (\Gamma_{11}^2)_v = \frac{-GE_{vv} + G_v E_v}{2G^2} \quad (\Gamma_{12}^2)_u = \frac{G_{uu}G - (G_u)^2}{2G^2}$$

Substitute into Equation(5) at do Carmo's section 4.3 and

$$\begin{aligned} K &= -\frac{1}{E}\left(\frac{G_{uu}G - (G_u)^2 + GE_{vv} - G_v E_v}{2G^2} + \frac{-E_v^2}{4EG} + \frac{G_u^2}{4G^2} + \frac{E_v G_v}{4G^2} - \frac{E_u G_u}{4EG}\right) \\ &= -\frac{G_{uu} + E_{vv}}{2EG} + \frac{E_v^2 G + E_u G_u G + G_u^2 E + E_v G_v E}{4E^2 G^2} \end{aligned}$$

Note that

$$\begin{aligned} -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) &= -\frac{1}{2\sqrt{EG}}\left(\frac{E_{vv}\sqrt{EG} - \frac{E_v^2 G + E_v G_v E}{2\sqrt{EG}} + G_{uu}\sqrt{EG} - \frac{G_u^2 E + E_u G_u G}{2\sqrt{EG}}}{EG}\right) \\ &= -\frac{G_{uu} + E_{vv}}{2EG} + \frac{E_v^2 G + E_u G_u G + G_u^2 E + E_v G_v E}{4E^2 G^2} \end{aligned}$$

Hence, we prove the formula.

Ex 10.

$$\Delta \ln \lambda = \left(\frac{1}{\lambda}\lambda_u\right)_u + \left(\frac{1}{\lambda}\lambda_v\right)_v$$

By Ex 9,

$$\begin{aligned} K &= -\frac{1}{2\lambda}\left[\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right] \\ &= -\frac{1}{2\lambda}\Delta \ln \lambda \end{aligned}$$

When $E = G = \lambda = (u^2 + v^2 + c)^{-2}, F = 0$,

$$(\ln \lambda)_u = \frac{-4u}{u^2 + v^2 + c} \quad (\ln \lambda)_v = \frac{-4v}{u^2 + v^2 + c}$$

$$(\ln \lambda)_{uu} = \frac{4u^2 - 4v^2 - 4c}{(u^2 + v^2 + c)^2} \quad (\ln \lambda)_{vv} = \frac{4v^2 - 4u^2 - 4c}{(u^2 + v^2 + c)^2}$$

$$K = -\frac{1}{2\lambda}\Delta \ln \lambda = -\frac{1}{2}(u^2+v^2+c)^2 \cdot \frac{4u^2-4v^2-4c+4v^2-4u^2-4c}{(u^2+v^2+c)^2} = 4c$$

Ex 11. We have

$$E = 1 + 1/u^2, \quad F = 0, \quad G = u^2,$$
$$\overline{E} = 1, \quad \overline{F} = 0, \quad \overline{G} = u^2 + 1$$

Then we can calculate by exercise 9 that

$$K = \overline{K} = -\frac{1}{(u^2+1)^2}.$$

One can check $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ does not preserve the inner product, e.g. plugging the vector $\mathbf{x}_u$, one get different norms of this vector.

Ex 12. By Gauss Theorem, if they are isometric, then they have the same Gaussian curvature. But we know that the Gaussian curvature is 1 for the unit sphere and 0 for the plane. Thus, they are not isometric.