Step-1

(a)

Consider the statement given below:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D|$$

The objective is to prove that why the given statement is true. Somehow *B* doesn't enter.

Step-2

Consider *A* is *n* by *n* matrix and *I* is *m* by *m* identity matrix and *B* is *n* by *m* matrix.

First, show that $\begin{vmatrix} A & B \\ 0 & I \end{vmatrix} = |A|$ using mathematical induction.

Expand along the last row.

If $m=1$, then for matrix *A* that is *n* by *n*, expansion along the last row is given by,

$$\begin{vmatrix} & b_1 \\ A & \vdots \\ & b_n \\ 0\cdots 0 & 1 \end{vmatrix} = (-1)^{(n+1)+(n+1)}|A|$$
$$= |A|$$

Step-3

If $m \geq 1$, then for matrix *A* that is *n* by *n*, expansion along the last row is given by

$$\begin{vmatrix} A & B \\ 0 & I_{m+1} \end{vmatrix} = \begin{vmatrix} A & B_1 & B_2 \\ 0 & I_m & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= (-1)^{(n+m+1)+(n+m+1)}\begin{vmatrix} A & B_1 \\ 0 & I_m \end{vmatrix}$$
$$= |A| \qquad \text{â€¦â€¦ (1)}$$

Step-4

Thus, we have proved that, for all $m \geq 1$, $\begin{vmatrix} A & B \\ 0 & I \end{vmatrix} = |A|$.

Similarly, we can prove that $\begin{vmatrix} I & B \\ 0 & D \end{vmatrix} = |D|$

Consider *A* is *n* by *n* matrix and *I* is *m* by *m* identity matrix and *B* is *n* by *m* matrix.

Let $B = AX + YD$

Here, *X* is *n* by *m* and *Y* is *m* by *n*.

Step-5

So,

$$\begin{aligned} \begin{vmatrix} A & B \\ 0 & D \end{vmatrix} &= \begin{vmatrix} A & AX+YD \\ D & I \end{vmatrix} \\ &= \begin{vmatrix} A & Y \\ 0 & I_m \end{vmatrix} \begin{vmatrix} I_n & X \\ 0 & D \end{vmatrix} \\ &= |A||D| \qquad \text{(from equation (1))} \end{aligned}$$

Therefore, the first statement $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D|$ is $\boxed{\text{true}}$.

Step-6

(b)

The objective is to prove that the equality fails to hold when 0 is replaced by matrix C as shown below:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D|$$

Step-7

Consider the following matrix.

$$\begin{aligned} \begin{vmatrix} A & B \\ C & D \end{vmatrix} &= \left|\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right| \\ &= 1 \end{aligned}$$

Step-8

And

$$A=\begin{vmatrix}0&1\\0&0\end{vmatrix}=0$$

$$B=\begin{vmatrix}0&0\\1&0\end{vmatrix}=0$$

$$C=\begin{vmatrix}0&0\\1&0\end{vmatrix}=0$$

$$D=\begin{vmatrix}0&1\\0&0\end{vmatrix}=0$$

So, we have $|A||D|=0$.

We know that $1\neq 0$, therefore $\boxed{\begin{vmatrix}A&B\\C&D\end{vmatrix}\neq|A||D|}$, when C enters.

Step-9

(c)

The objective is to show by example that $\det(AD-CB)$ is also wrong.

Step-10

That is, to prove $\begin{vmatrix}A&B\\C&D\end{vmatrix}\neq|A||D|-|C||B|$, by the example.

Consider the following matrix.

$$\begin{vmatrix}A&B\\C&D\end{vmatrix}=\left|\begin{array}{cc|cc}0&1&0&0\\0&0&1&0\\\hline 0&0&0&1\\1&0&0&0\end{array}\right|$$
$$=1$$

And

$$A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$B = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$D = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

So, we have $|A||D| - |C||B| = 0$.

We know that $1 \neq 0$, therefore $\boxed{\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|}$.