

Step-1

Given vectors are $b = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The projection of b onto a is $\hat{x}a = \frac{a^T b}{a^T a} a$ (1)

$$\begin{aligned} a^T b &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ &= \cos \theta \end{aligned}$$

$$\begin{aligned} a^T a &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

So,

$$\begin{aligned} \hat{x} &= \frac{\cos \theta}{1} \\ &= \cos \theta \end{aligned} \quad (2)$$

Step-2

Use (2) in (1), to get $p_1 = \hat{x} a$

$$\begin{aligned} &= \cos \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \\ 0 \end{bmatrix} \end{aligned}$$

The projection matrix suitable is

$$\begin{aligned}
 P_1 &= \frac{aa^T}{a^T a} \\
 &= \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \\
 &= \frac{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}{1} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \hat{a} \in \hat{a} \in (3)
 \end{aligned}$$

Step-3

Given vectors are $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The projection of b on to a is $\hat{x}a = \frac{a^T b}{a^T a} a$

$$\begin{aligned}
 a^T b &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
 &= 1 - 1 \\
 &= 0
 \end{aligned}$$

And $a^T a = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 + 1 = 2$

So,

$$\begin{aligned}
 \hat{x} &= \frac{0}{2} \\
 &= 0
 \end{aligned}$$

Therefore $p_2 = \hat{x}a$

$$\begin{aligned}
 &= 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

Step-4

The matrix suitable to this projection is $P_1 = \frac{aa^T}{a^T a}$

$$\begin{aligned}
 &= \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \\
 &= \frac{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}{2} \\
 &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (4)
 \end{aligned}$$

Step-5

Use (3) and (4), and get $P_1 + P_2 = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

Step-6

$$\begin{aligned}
 P_1 P_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \\
 (P_1 + P_2)^2 &= \frac{1}{2} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \\
 P_1^2 + P_2^2 &= \frac{1}{2} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}
 \end{aligned}$$

From the above observations, it follows that $\boxed{(P_1 + P_2)^2 \neq P_1^2 + P_2^2}$ while $P_1 P_2 \neq 0$