Step-1

Consider the following orthogonal matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the unit Eigen vectors and unitary matrix. Determine the property of *P* that makes these Eigen vectors orthogonal.

Step-2

First step is to find the Eigen values of matrix *P*. To calculate the Eigen values do the following calculations:

$$P - \lambda I = \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & 0 & 0-\lambda \end{bmatrix}$$

$$\det(P - \lambda I) = 0$$
$$(-\lambda)(\lambda^2) + 1 = 0$$
$$(\lambda^3 - 1) = 0$$

After solving following values are obtained:

$$\lambda_1 = 1$$
$$\lambda_2 = e^{2\pi i/3}$$
$$\lambda_3 = e^{4\pi i/3}$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(P - \lambda I)x = 0$$

$$\begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & 0 & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving, values of *x*, *y* and *z* corresponding to $\lambda = 1$ are as follows:

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step-4

Similarly, Eigen vector corresponding to Eigen value $\lambda = e^{2\pi i/3}$ is as follows:

$$(P - \lambda I)x = 0$$
$$\begin{bmatrix} -e^{2\pi i/3} & 1 & 0 \\ 0 & -e^{2\pi i/3} & 1 \\ 1 & 0 & -e^{2\pi i/3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ e^{2\pi i/3} \\ e^{4\pi i/3} \end{bmatrix}$$

Step-5

Similarly, Eigen vector corresponding to Eigen value $\lambda = e^{4\pi i/3}$ will be:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ e^{4\pi i/3} \\ e^{2\pi i/3} \end{bmatrix}$$

Step-6

Therefore, Eigen vectors of matrix P are as follows:

$$\boxed{(1,1,1),\left(1,e^{2\pi i/3},e^{4\pi i/3}\right),\left(1,e^{4\pi i/3},e^{2\pi i/3}\right)}$$

Step-7

To put the Eigen vectors in unitary matrix make them orthonormal by dividing the length of the vector.

$$\begin{aligned}\|x\|^2 &= \left|(1)^2\right| + \left|\left(e^{2\pi i/3}\right)^2\right| + \left|\left(e^{4\pi i/3}\right)^2\right| \\ &= |1| + |1| + |1| \\ &= 3\end{aligned}$$

Let the length be L. So $L = \sqrt{3}$.

Step-8

Therefore, unitary matrix will be as follows:

$$U = \boxed{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}}$$

Step-9

The property of P that makes these Eigen vectors orthogonal is the orthogonality of matrix P and Eigen value absolute value is $|\lambda| = 1$