南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目： Probability Theory (MA215) **开课单位：** Department of Mathematics
考试时长： 2021/01/06: 10:30–12:30 **命题教师：** Prof. Z. Liu & Prof. J. Sun

题号	1	2	3	4	5
分值	30 分	20 分	30 分	10 分	10 分

本试卷共（5）大题，满分（100）分

（考试结束后请将试卷、答题卡、草稿纸一起交给监考老师）

1. Let (X, Y) be a random vector with joint probability density function

$$f(x,y) = \begin{cases} ce^{-2x-y}, & 0 < x < \infty,\ 0 < y < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

where c is a constant.

(i) Find the value of c. [4 marks]

(ii) Find the two marginal probability density functions f_X and f_Y and identify the distributions of X and Y, respectively, and determine with reasons whether X and Y are independent. [10 marks]

(iii) Compute the probabilities $\mathbb{P}\{X > 2, Y < 1\}$ and $\mathbb{P}\{X < Y\}$. [8 marks]

(iv) Find the probability density function $f_Z(z)$ of $Z := \frac{X}{Y}$, $z \in \mathbb{R}$. [8 marks]

[Total for Question 1: 30 marks]

1. 令随机向量 (X, Y) 的联合密度函数为

$$f(x,y) = \begin{cases} ce^{-2x-y}, & 0 < x < \infty,\ 0 < y < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

其中 c 为常数.

(i) 求出常数 c. [4 分]

(ii) 求出 X 与 Y 的边缘概率密度函数 f_X 和 f_Y, 并确定 X 和 Y 的分布. 试判断 X 和 Y 是否独立, 并说明理由. [10 分]

(iii) 计算 $\mathbb{P}(X > 2, Y < 1)$ 和 $\mathbb{P}(X < Y)$. [8 分]

(iv) 求出随机变量 $Z := \frac{X}{Y}$ 的概率密度函数 $f_Z(z)$, $z \in \mathbb{R}$. [8 分]

[题 1 总分: 30 分]

2. Suppose that X and Y are independent random variables with the same moment generating function $M_X(t) = M_Y(t) = \frac{4}{\sqrt{16-t}}$ $(t < 16)$ and $Z = \ln(X+Y)$.

   (i) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$. [6 marks]

   (ii) Find the moment generating function $M_{X+Y}(t)$ of $X+Y$, $t < 16$, and identify its distribution. [4 marks]

   (iii) Find a probability density function $f_Z(z)$ of Z, $z \in \mathbb{R}$. [5 marks]

   (iv) Find the moment generating function $M_Z(t)$ of Z, $t > 0$. [5 marks]

   Hint: You may need to represent the result in (iv) by the gamma function Γ (see Appendix).

[Total for Question 2: 20 marks]

2. 假设 X 和 Y 为独立的随机变量, 它们具有相同的矩生成函数 $M_X(t) = M_Y(t) = \frac{4}{\sqrt{16-t}}$ $(t < 16)$; $Z = \ln(X+Y)$.

   (i) 求 $\mathbb{E}(X)$ 和 $\mathrm{Var}(X)$. [6 分]

   (ii) 找出 $X+Y$ 的矩生成函数 $M_{X+Y}(t)$ 和分布, $t < 16$. [4 分]

   (iii) 求出 Z 的概率密度函数 $f_Z(z)$, $z \in \mathbb{R}$. [5 分]

   (iv) 求出 Z 的矩生成函数 $M_Z(t)$, $t > 0$. [5 分]

   提示: 你可能需要用伽马函数 Γ (表达式见附录) 表示出 (iv) 的结果.

[题 2 总分: 20 分]

3. Let $\{X_1, X_2, \cdots\}$ be a sequence of independent identically distributed $N(2,5)$-random variables. Denote $S_n = \sum_{i=1}^n X_i$, $\overline{X}_n = S_n/n$, and $Z_n := (\overline{X}_n - 2)/\sqrt{5/n}$ for $n \in \mathbb{N}_+$.

   (i) What distributions do S_{10}, S_{20}, and $\overline{X}_{20}$ obey? [3 marks]

   (ii) Estimate $\mathbb{P}\{1.5 < \overline{X}_{20} < 2.49\}$ and $\mathbb{P}\{S_{20} \geq 49\}$. [6 marks]

   (iii) Are S_{10} and S_{20} uncorrelated or correlated? Find the correlation coefficient $\sigma(S_{10}, S_{20})$ of S_{10} and S_{20} to support your conclusion. [5 marks]

(iv) Find the moment generating functions $M_{\overline{X}_n}(t)$ and $M_{Z_n}(t)$ of $\overline{X}_n$ and Z_n, respectively, $t \in \mathbb{R}$. [6 marks]

(v) Determine the random variables (a constant is also viewed as a random variable) whose moment generating functions are $\phi(t) := \lim_{n\to\infty} M_{\overline{X}_n}(t)$ and $\psi(t) := \lim_{n\to\infty} M_{Z_n}(t)$, respectively, $t \in \mathbb{R}$. [4 marks]

(vi) Prove $\mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} \geq 1 - \frac{5}{n\varepsilon^2}$ for any $n \in \mathbb{N}_+$, $\varepsilon > 0$ and $\lim_{n\to\infty} \mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} = 1$ for any $\varepsilon > 0$. [6 marks]

Hint: Use law of large numbers and central limit theorem. You may need to use the table

x	0.02	0.1	0.9	0.98	1	1.98
$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathrm{d}y$	0.5080	0.5398	0.8159	0.8365	0.8413	0.9761

[Total for Question 3: 30 marks]

3. 令 $\{X_1, X_2, \cdots\}$ 是独立同分布的 $N(2,5)$-随机变量序列. 对 $n \in \mathbb{N}_+$, 定义 $S_n = \sum_{i=1}^{n} X_i$, $\overline{X}_n = S_n/n$ 和 $Z_n := (\overline{X}_n - 2)/\sqrt{5/n}$.

(i) S_{10}, S_{20}, 与 $\overline{X}_{20}$ 分别服从何种分布? [3 分]

(ii) 估计 $\mathbb{P}\{1.5 < \overline{X}_{20} < 2.49\}$ 与 $\mathbb{P}\{S_{20} \geq 49\}$. [6 分]

(iii) S_{10} 与 S_{20} 是否相关? 求出 S_{10} 与 S_{20} 的相关系数 $\sigma(S_{10}, S_{20})$ 来证明你的结论. [5 分]

(iv) 分别求出 $\overline{X}_n$ 与 Z_n 的矩生成函数 $M_{\overline{X}_n}(t)$ 和 $M_{Z_n}(t)$, $t \in \mathbb{R}$. [6 分]

(v) 确定其矩生成函数分别为 $\phi(t) := \lim_{n\to\infty} M_{\overline{X}_n}(t)$ 和 $\psi(t) := \lim_{n\to\infty} M_{Z_n}(t)$ $t \in \mathbb{R}$ 的随机变量 (常数也被视为随机变量). [4 分]

(vi) 证明对任意的 $\varepsilon > 0$ 和 $n \in \mathbb{N}_+$, $\mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} \geq 1 - \frac{5}{n\varepsilon^2}$; 对任意的 $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} = 1$. [6 marks]

提示: 应用大数定律和中心极限定理. 可能用到的数据附下表:

x	0.02	0.1	0.9	0.98	1	1.98
$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathrm{d}y$	0.5080	0.5398	0.8159	0.8365	0.8413	0.9761

[题 3 总分: 30 分]

4. Suppose that X and Y are independent **discrete** random variables taking values of $\{x_i\}$ and $\{y_j\}$ and having probability mass functions f_X, f_Y and cumulative distribution functions F_X, F_Y, respectively. Find the following probability or distribution represented by f_X, f_Y, F_X, F_Y, $\{x_i\}$, and $\{y_j\}$.

(i) Compute the probability $\mathbb{P}\{X > Y\}$. [5 marks]

(ii) Find the distribution $F_{X+Y}(z) := \mathbb{P}\{X + Y \leq z\}$, $z \in \mathbb{R}$. [5 marks]

Hint: Compute probabilities by conditioning (or use the total probability formula).

[Total for Question 4: 10 marks]

4. 假定 X 与 Y 是取值为 $\{x_i\}$ 和 $\{y_j\}$ 的独立**离散**型随机变量, 他们分别具有概率质量函数 $f_X(x_i)$, $f_Y(y_j)$ 和累积分布函数 F_X, F_Y. 求出以下由 f_X, f_Y, F_X, F_Y, $\{x_i\}$ 和 $\{y_j\}$ 表示的概率或分布.

(i) 计算 $\mathbb{P}(X > Y)$. [5 分]

(ii) 求出 $F_{X+Y}(z) := \mathbb{P}(X + Y \leq z)$, $z \in \mathbb{R}$. [5 分]

提示: 通过条件概率计算 (或利用全概率公式).

[题 4 总分: 10 分]

5. A miner is trapped in a mine containing four doors. The first door leads to a tunnel which takes him to safety after 2h's travel. Another three doors lead to a tunnel which returns him to the mine after 3h's, 4h's, and 5h's travel, respectively. Assume that the miner is equally to choose one of the doors. Let X denote the time until the miner reaches safety. Calculate $\mathbb{E}(X)$, i.e., find the expected length of time until the miner reaches safety, under the following two situations.

(i) The door where the miner had tried but had not reached safety is not excluded (can not be marked), and it is still in the next choice. [5 marks]

(ii) The door where the miner had tried but had not reached safety is excluded (can be marked), and it is not in the next choice. [5 marks]

[Total for Question 5: 10 marks]

5. 一名矿工被困在一个有四扇门的矿井里. 第一扇门通向一条通道, 沿此通道在经过 2 小时之后可到达安全地带. 另外三扇门通向三条通道, 沿这些通道会分别在 3 小时、4 小时和 5 小时后重新返回矿井. 假设矿工等可能地选择其中一个通道. 令 X (小时) 表示矿工到达安全地带的时间. 在下面两种情形下计算 $\mathbb{E}(X)$, 即矿工到达安全地带的预期时间长度.

(i) 矿工在试过但没有到达安全地带的通道不排除 (不能做标记), 仍在下次选择之列. [5 分]

(ii) 矿工在试过但没有到达安全地带的通道 (做标记) 排除在外, 不在下次选择之列. [5 分]

[题 5 总分: 10 分]