Step-1

Consider the equations,

$$-u'' = 2,$$
$$u(0) = 0,$$
$$u(1) = 1$$

By using four intervals and three hat functions, with $h = \frac{1}{2}$, the matrix A(3 by 3) is given by,

$$A = 4\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let

$$f(x) = 2$$

Therefore, we get,

$$\begin{aligned} b &= hf(x) \\ &= \left(\frac{1}{4}\right)2 \\ &= \frac{1}{2} \end{aligned}$$

Step-2

By substituting A, and b into $Ay = b$, we get,

$$4\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y = \frac{1}{8}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y = \frac{1}{8}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The inverse matrix A is given by,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Step-3

On substitution, we get,

$$y=\frac{1}{32}\begin{bmatrix}3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$=\frac{1}{32}\begin{bmatrix}3+2+1\\ 2+4+2\\ 1+2+3\end{bmatrix}$$
$$=\frac{1}{32}\begin{bmatrix}6\\8\\6\end{bmatrix}$$
$$=\begin{bmatrix}\frac{3}{16}\\ \frac{4}{16}\\ \frac{3}{16}\end{bmatrix}$$

The linear finite element is given by,

$$U(x)=\frac{3}{16}V_1+\frac{4}{16}V_2+\frac{3}{16}V_3$$

Step-4

Thus, we get,

$u(x)=\frac{3}{16},\frac{4}{16},\frac{3}{16}$ at the notes $x=\frac{1}{4},\frac{1}{2},\frac{3}{4}$

Consider $x=\frac{1}{4}$.

By substituting $x=\frac{1}{4}$ into $u=x-x^2$, we get,

$$u=\frac{1}{4}-\frac{1}{16}$$
$$=\frac{3}{16}$$

Therefore, this agrees with the exact solution $u=x-x^2$.

Step-5

Thus, the linear finite element is $\boxed{U(x)=\frac{3}{16}V_1+\frac{4}{16}V_2+\frac{3}{16}V_3}$.