## Step-1

Let A be $A = \begin{bmatrix} 1 & 2 \\ 3 & 9 \\ 4 & 8 \end{bmatrix}$ and its rank is 2.

In order to find the null space basis, we need to set it as homogenous equation.

$Ax = 0$

Apply $R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 4R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to \frac{R_2}{3}$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 = 0$$

$$x_2 = 0$$

$$\Rightarrow x_1 = -2x_2$$

$$= 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $(1,0,0),(0,1,0)$ is in column space.

The basis for $C(A)$ is $\{(1,0,0)(0,1,0)\}$.

The basis for null space is $\{(0,0,1)\}$.

## Step-2

The row space contains all multiples of $(1,2)$. Therefore, the basis for row space is $\{(1,2)\}$.

The left null space basis is,

The transpose of the matrix is $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 8 \end{bmatrix}$

The homogeneous equations using this,

$$A^T x = 0$$
$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$

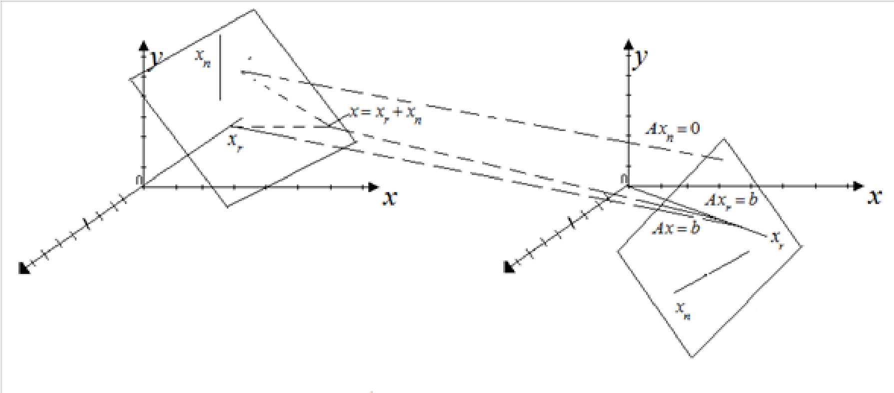
$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow x_1 + 3x_2 + 4x_3 = 0$$
$$3x_2 = 0$$
$$\Rightarrow x_2 = 0$$
$$x_1 = -4x_3$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 0 \\ x_3 \end{bmatrix}$$
$$= x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

The basis for null space is $\{(-4,0,1)\}$.

## Step-3

The four subspaces of this matrix is as shown below.




## Step-4

We know that $\dim(N(A)) + \dim(C(A)) = \text{number of columns}$

$\Rightarrow \dim(N(A)) + \dim(C(A)) = 2$

$\Rightarrow \dim(N(A)) + 2 = 2$

$\Rightarrow \dim(N(A)) = 0$

So, null space is $2(\text{zero vector only})$

Therefore, the null space of any vector $(x_1, x_2)$ in $\mathrm{R}^2$ and $x_n = 0$ for $n = 1,2$

Therefore, $\boxed{\text{rank } r = n \text{ means nullspae is zero vector and } x_n = 0}$.