Step-1

Suppose A and B are square matrices.

We have to show that $I-BA$ is invertible if $I-AB$ is invertible.

Step-2

Take $B(I-AB)=(I-BA)B$

Multiplying both sides with B^{-1}, we get

$$
\begin{aligned}
&B(I-AB)B^{-1}=(I-BA)BB^{-1}\\
&\Rightarrow B(I-AB)B^{-1}=(I-BA)I \qquad \left(\text{Since } BB^{-1}=I\right)\\
&\Rightarrow I-BA=B(I-AB)B^{-1}
\end{aligned}
$$

Now

$$
\begin{aligned}
(I-BA)^{-1}&=\left(B(I-AB)B^{-1}\right)^{-1}\\
&=\left(B^{-1}\right)^{-1}(I-AB)^{-1}B^{-1} \qquad \left(\text{Since } (AB)^{-1}=B^{-1}A^{-1}\right)\\
&=B(I-AB)^{-1}B^{-1}
\end{aligned}
$$

Therefore, $I-BA$ is invertible if B and $I-AB$ is invertible.

Step-3

Suppose $I-BA$ is not invertible.

Then $BA=I$

And $BAx=x$, for some nonzero x has no solution.

Multiplying left side of $BAx=x$ with A gives

$$
\begin{aligned}
&ABAx=Ax\\
&\Rightarrow ABy=y \qquad \left(\text{Since let } Ax=y\right)\\
&\Rightarrow (I-AB)y=0
\end{aligned}
$$

Therefore, $I-AB$ could not be invertible which is a contradiction to the hypothesis.

Hence $I-BA$ is invertible.