

Step-1

We have to mark all the sixth roots of 1 in the complex plane, we have to find the primitive root w_6 , and we have to find that which power of w_6 is equal to $\frac{1}{w_6}$. Also we have to find the value of $1 + w + w^2 + w^3 + w^4 + w^5$.

Step-2

The equation $z^6 = 1$ denotes sixth roots of unity in complex plane.

$$\begin{aligned} z^6 &= 1 \\ \Rightarrow z &= 1^{1/6} \\ \Rightarrow z &= (\cos 2k\pi + i \sin 2k\pi)^{1/6} \text{ for } k = 0, 1, 2, 3, 4, 5. \\ \Rightarrow z &= (e^{i2k\pi})^{1/6} \\ &= \left(e^{\frac{2i\pi}{6}} \right)^k \text{ for } k = 0, 1, 2, 3, 4, 5. \end{aligned}$$

Step-3

$$\Rightarrow z = 1, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{\frac{3i\pi}{3}}, e^{\frac{4i\pi}{3}}, e^{\frac{5i\pi}{3}} \hat{=}$$

$$\hat{=} \Rightarrow z = 1, w, w^2, w^3, w^4, w^5. \hat{=}| \hat{=}| (1)$$

$$\text{Here } w = e^{\frac{2i\pi}{6}}$$

Step-4

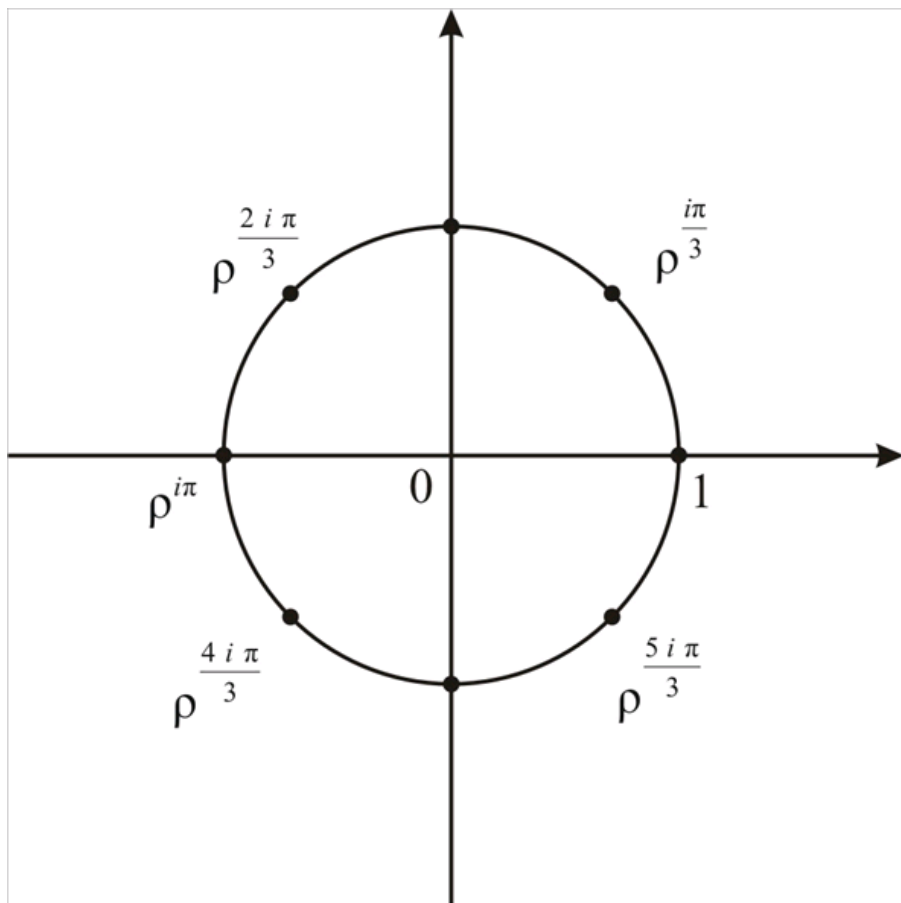
$$\hat{=} \hat{=} \hat{=} \hat{=} \hat{=} \hat{=}$$

Hence

$$z = 1, \frac{1+i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}, -1, \frac{-1-i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2} \hat{=}| \hat{=}| (2)$$

Step-5

The sixth roots of 1 in the complex plane are shown as follows:



Step-6

The primitive root w_6 is $\cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6}$

$$= \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$= \boxed{\frac{1 + i\sqrt{3}}{2}}$$

Step-7

Now

$$\begin{aligned}
 w^5 &= e^{\frac{5i\pi}{3}} \\
 &= \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \\
 &= \cos \left(2\pi - \frac{\pi}{3} \right) + i \sin \left(2\pi - \frac{\pi}{3} \right)
 \end{aligned}$$

Step-8

$$\begin{aligned}
 &= \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \\
 &= \cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \\
 &= e^{\frac{-i\pi}{3}} \\
 &= w^{-1} \\
 &= \frac{1}{w}
 \end{aligned}$$

Hence 5th power of w is equal to $\frac{1}{w}$

Step-9

From (1) and (2),

$$\begin{aligned}
 w &= \frac{1+i\sqrt{3}}{2} \\
 w^2 &= \frac{-1+i\sqrt{3}}{2} \\
 w^3 &= -1 \\
 w^4 &= \frac{-1-i\sqrt{3}}{2} \\
 w^5 &= \frac{1-i\sqrt{3}}{2}
 \end{aligned}$$

Step-10

$$\begin{aligned}
 &1 + w + w^2 + w^3 + w^4 + w^5 \\
 &= 1 + \frac{1+i\sqrt{3}}{2} + \frac{-1+i\sqrt{3}}{2} - 1 + \frac{-1-i\sqrt{3}}{2} + \frac{1-i\sqrt{3}}{2} \\
 &= 0
 \end{aligned}$$

Hence $\boxed{1 + w + w^2 + w^3 + w^4 + w^5 = 0}$