南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目：高等数学(下) A　　开课单位：数 学 系

考试时长：180 分钟　　命题教师：王融、吴纪桃 等

题 号	1	2	3	4	5	6	7	8	9	10
分 值	9 分	15 分	9 分	7 分	7 分	7 分	7 分	7 分	7 分	7 分
题 号	11	12	13							
分 值	7 分	7 分	4 分							

本试卷共 13 大题, 满分 100 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

注意：本试卷里的中文为直译（即完全按英文字面意思直接翻译），所有数学词汇的定义请参照教材（Thomas' Calculus，13th Edition）中的定义。如果其中有些数学词汇的定义不同于中文书籍（比方说同济大学的高等数学教材）里的定义，以教材（Thomas' Calculus，13th Edition）中的定义为准。

1. (9 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

 (1) If $a_n > 0, \forall n$, and $\lim\limits_{n\to\infty} na_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

 (2) The plane $x + y - 2z = 1$ is perpendicular to the plane $x + y + z = 1$.

 (3) If $f(x, y)$ has two local maxima, then f must have a local minimum.

一、　（9分）判断题：

 (1) 若$\forall n$, $a_n > 0$, 且 $\lim\limits_{n\to\infty} na_n = 0$, 那么级数$\sum\limits_{n=1}^{\infty} a_n$ 收敛.

 (2) 平面$x + y - 2z = 1$和平面$x + y + z = 1$垂直.

 (3) 如果函数$f(x, y)$有两个局部极大值点，那么f必有局部极小值点.

2. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

 (1) Which one of the following series diverges?

 (A) $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$.　　(B) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+\frac{1}{n}}}$.

 (C) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n \ln n}{\sqrt{n}}$.　　(D) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n(3 + (-1)^n \cdot 2)^n}{6^n}$.

(2) The iterated integral $\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} f(r\cos\theta, r\sin\theta)r\,drd\theta$ can be written as

(A) $\int_0^1\int_0^{\sqrt{y-y^2}} f(x,y)\,dxdy$. (B) $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy$.

(C) $\int_0^1\int_0^1 f(x,y)\,dydx$. (D) $\int_0^1\int_0^{\sqrt{x-x^2}} f(x,y)\,dydx$.

(3) For the function, $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0), \end{cases}$ which of the following statements is correct?

(A) f is not continuous at $(0,0)$.

(B) f is continuous at $(0,0)$, but its partial derivative f_x and f_y do not exist at $(0,0)$.

(C) Both partial derivatives f_x and f_y exist everywhere and are also continuous at $(0,0)$.

(D) f is not differentiable at $(0,0)$.

(4) For the critical points of the function $f(x,y) = 2x^4+y^4-2x^2-2y^2$, which one of the following statements is correct?

(A) $(0,0)$ is a local minima.

(B) $(0,1)$ is a local maxima.

(C) $(0,-1)$ is a saddle point.

(D) There are no local maxima among all the critical points.

(5) If the function $f(x,y)$ has the continuous first partial derivatives $\frac{\partial f}{\partial x} > 0$ and $\frac{\partial f}{\partial y} < 0$, $\forall (x,y)\in \mathbf{R}^2$, which one of the following statements is correct?

(A) $f(0,0) > f(1,1)$. (B) $f(0,0) < f(1,1)$.

(C) $f(0,1) > f(1,0)$. (D) $f(0,1) < f(1,0)$.

二、　（15分）单项选择题:

(1) 下列哪个级数发散?

(A) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$. (B) $\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$.

(C) $\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{\sqrt{n}}$. (D) $\sum_{n=1}^{\infty}\frac{(-1)^n(3+(-1)^n\cdot 2)^n}{6^n}$.

(2) 累次积分$\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} f(r\cos\theta, r\sin\theta)r\,drd\theta$ 可以写成

(A) $\int_0^1\int_0^{\sqrt{y-y^2}} f(x,y)\,dxdy$. (B) $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy$.

(C) $\int_0^1\int_0^1 f(x,y)\,dydx$. (D) $\int_0^1\int_0^{\sqrt{x-x^2}} f(x,y)\,dydx$.

(3) 对于函数$f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0), \end{cases}$ 以下哪个说法是正确的?

(A) f在$(0,0)$处不连续.

(B) f在$(0,0)$连续但其偏导数f_x和f_y在$(0,0)$处不存在.

(C) 两个偏导数f_x和f_y都处处存在且在$(0,0)$处连续.

(D) f在$(0,0)$处不可微.

(4) 关于函数$f(x,y) = 2x^4 + y^4 - 2x^2 - 2y^2$的临界点，以下哪种说法正确?

(A) $(0,0)$是局部极小值点.

(B) $(0,1)$是局部极大值点.

(C) $(0,-1)$是鞍点.

(D) 在所有的临界点中不存在局部极大值点.

(5) 设$f(x,y)$在$xy-$ 平面上有一阶连续偏导数，并且$\forall(x,y) \in \mathbf{R}^2$，都有$\frac{\partial f}{\partial x} > 0$，和$\frac{\partial f}{\partial y} < 0$. 则以下哪种说法正确?

(A) $f(0,0) > f(1,1)$.　　　　(B) $f(0,0) < f(1,1)$.

(C) $f(0,1) > f(1,0)$.　　　　(D) $f(0,1) < f(1,0)$.

3. (9 pts) Please fill in the blank for the questions below.

(1) Compute the limit: $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2} =$ ________________.

(2) The direction (unit vector) in which the function $f(x,y) = x^2 + xy + y^2 - y$ increases most rapidly at the point $(-1,2)$ is ________________.

(3) $\int_0^1 \int_y^1 \frac{\tan x}{x}\, dxdy =$ ________________.

三、 （9分）填空题:

(1) 计算极限: $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2} =$ ________________.

(2) 如果函数$f(x,y) = x^2 + xy + y^2 - y$在$(-1,2)$处的方向导数$D_{\mathbf{u}}f(-1,2)$沿单位向量$\mathbf{u}$ 达到最大值，那么$\mathbf{u} =$ ________________.

(3) $\int_0^1 \int_y^1 \frac{\tan x}{x}\, dxdy =$ ________________.

4. (7 pts)

(1) Find the interval of convergence of the series $\sum\limits_{n=2}^{\infty} \frac{(-1)^n (x-1)^{2n+1}}{\sqrt{n + 9012 \ln n}}$.

(2) For what values of x does the series converge absolutely, or conditionally?

四、 （7分）

(1) 求级数$\sum\limits_{n=2}^{\infty} \frac{(-1)^n (x-1)^{2n+1}}{\sqrt{n + 9012 \ln n}}$ 的收敛区间.

(2) x取哪些值时级数绝对收敛，取哪些值时条件收敛?

5. (7 pts) The region D is bounded by $z = \sqrt{x^2+y^2}$ and $z = \sqrt{1-x^2-y^2}$. Consider the following integral

$$\iiint\limits_D (x+z)\, dxdydz,$$

(1) Convert the above integral to an equivalent iterated integral in cylindrical coordinates;

(2) Convert the above integral to an equivalent iterated integral in spherical coordinates.

五、 （7分）区域 D由 $z = \sqrt{x^2+y^2}$ 与 $z = \sqrt{1-x^2-y^2}$ 所围成. 考虑积分

$$\iiint_D (x+z)\, dxdydz,$$

(1) 将上述积分化为柱坐标下对应的累次积分（要求写出累次积分上下限）.

(2) 将上述积分化为球坐标下对应的累次积分（要求写出累次积分上下限）.

6. (7 pts) Assume we can put a cuboid into the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Use the method of **Lagrange multipliers** to find the length, width and height of the cuboid such that it achieve the maximum volume.

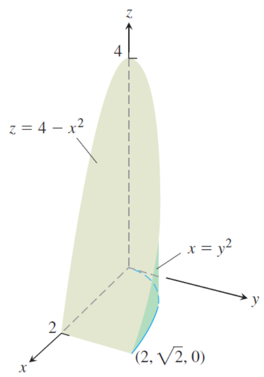
六、 （7分）在椭球$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$内嵌入有最大体积的长方体，请使用拉格朗日乘子法给出这个长方体的长宽高分别等于多少.

7. (7 pts) Find the equation of the osculating circle for the parabola $y = x^2$ at $x = 1$.

七、 （7分）写出抛物线$y = x^2$ 当$x = 1$ 处的曲率圆的方程.

8. (7 pts) A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the figure below). Its density function is $\delta(x, y, z) = xy$. Find the center of the mass for the solid.

八、 （7分）设D是由xy-平面、xz-平面、曲面$z = 4 - x^2$ 和$x = y^2$ 所围成的闭区域，其密度函数为$\delta(x, y, z) = xy$. 计算D 的质心.



9. (7 pts) Use the **substitution in double integral** (please find the transformation by yourself) to evaluate the integral

$$\iint_D e^{\frac{y-x}{y+x}}\, dxdy,$$

here D is the triangular region bounded by the lines $x = 0$, $y = 0$, and $x + y = 2$.

九、 （7分）用换元法来求二重积分

$$\iint\limits_{D} e^{\frac{y-x}{y+x}}\, dxdy,$$

其中D 是由x 轴、y 轴和直线$x+y=2$ 所围成的三角形闭区域.

10. (7 pts) Consider the line integral

$$\int_{(1,1,1)}^{(1,3,\pi)} e^x \ln y\, dx + \left(\frac{e^x}{y} + \sin z\right) dy + y\cos z\, dz.$$

(1) Show that the differential form in the integral is exact.

(2) Evaluate the integral.

十、 （7分）考虑曲线积分

$$\int_{(1,1,1)}^{(1,3,\pi)} e^x \ln y\, dx + \left(\frac{e^x}{y} + \sin z\right) dy + y\cos z\, dz.$$

(1) 证明积分中的微分形式是恰当的.

(2) 求积分的值.

11. (7 pts) Evaluate

$$\iint\limits_{S} \nabla \times (4x\mathbf{j}) \cdot \mathbf{n}\, d\sigma,$$

where S is the hemisphere $x^2+y^2+z^2=16$, $z \geq 0$. Use the normal vectors pointed away from the origin.

十一、 （7分）计算

$$\iint\limits_{S} \nabla \times (4x\mathbf{j}) \cdot \mathbf{n}\, d\sigma,$$

其中S 是半球面$x^2+y^2+z^2=16$, $z \geq 0$. 法向$\mathbf{n}$ 指向远离原点的方向.

12. (7 pts) Find the outward flux of $\mathbf{F} = (6x+y)\mathbf{i} - (x+z)\mathbf{j} + 4yz\mathbf{k}$ across the boundary of D, where D is the region in the first octant bounded by the cone $z = \sqrt{x^2+y^2}$, the cylinder $x^2+y^2=1$, and the coordinate planes.

十二、 （7分）设D是第一卦限中由锥面$z=\sqrt{x^2+y^2}$，柱面$x^2+y^2=1$以及坐标平面围成的区域，求$\mathbf{F} = (6x+y)\mathbf{i} - (x+z)\mathbf{j} + 4yz\mathbf{k}$向外穿过$D$的边界的通量.

13. (4 pts) The sequences $\{a_n\}$ and $\{b_n\}$ satisfy $0 < a_n < \frac{\pi}{2}$, $0 < b_n < \frac{\pi}{2}$, and $\cos a_n - a_n = \cos b_n$, $n = 1, 2, 3, \cdots$. The series $\sum\limits_{n=1}^{\infty} b_n$ converges. Show that $\lim\limits_{n\to\infty} a_n = 0$.

十三、 （4分）设数列$\{a_n\}$，$\{b_n\}$ 满足$0 < a_n < \frac{\pi}{2}$，$0 < b_n < \frac{\pi}{2}$，$\cos a_n - a_n = \cos b_n$，且级数$\sum\limits_{n=1}^{\infty} b_n$ 收敛. 证明：$\lim\limits_{n\to\infty} a_n = 0$.