1 Matrices and Gaussian Elimination

1.4

MATRIX OPERATIONS

（矩阵运算）

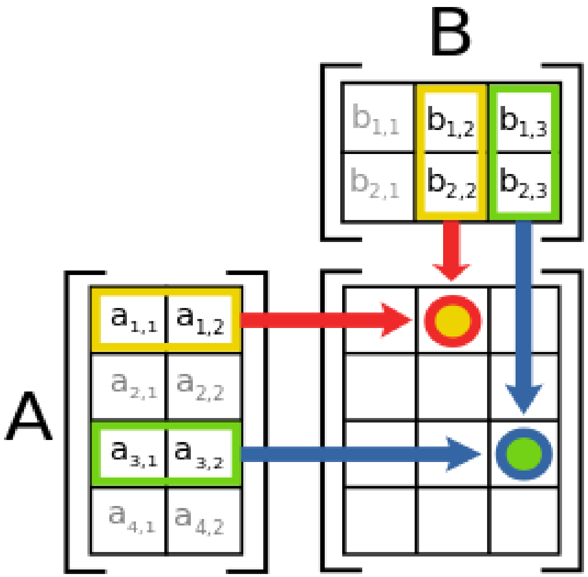
Addition

Scalar multiplication

Multiplication

Power

Elementary Matrices



MATRIX NOTATION

A matrix is an arrangement of mn elements with m rows and n columns, denoted by

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \text{Column } j & & \phantom{a_{1n}} \end{array} \\
\text{Row } i & \left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] = A
\end{array}
$$

$$\boxed{\boldsymbol{A} = \left[a_{ij} \right]_{m\times n}}$$

If $\boldsymbol{A}$ is an $m\times n$ matrix, that is, a matrix with m rows and n columns, then the scalar entry in the ith row and jth column of $\boldsymbol{A}$ is denoted by a_{ij} and is called the (i, j)-entry of $\boldsymbol{A}$.

THE ORDER IS IMPORTANT: *rows × columns*

- If two matrices have the same number of rows and the same number of columns, then they are called **matrices of the same size (同型矩阵)**.

For example, $\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 3 & 7 \end{bmatrix}$ and $\begin{bmatrix} 14 & 3 \\ -8 & 4 \\ 3i & 9 \end{bmatrix}$.

- If $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$ are matrices of the same size, and the corresponding entries are the same, i.e.,

$$a_{ij} = b_{ij} \ (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n),$$

then $\boldsymbol{A}$ and $\boldsymbol{B}$ are **equal (相等)**, denoted by $\boldsymbol{A} = \boldsymbol{B}$.

Attention! ***equal*** vs ***equivalent*** 等价 $\quad \boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow[\text{行等价}]{} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \boldsymbol{B}$

Some special matrices

(1) A matrix with the same number of rows and columns are called a **square matrix(方阵)**. An $n \times n$ matrix is also called a matrix of degree n / order n (n 阶方阵).

For instance, $\begin{bmatrix} 13 & 6 & 2i \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ is a complex matrix of order 3 (3 阶复方阵).

复方阵

(2) $1 \times n$ matrix $\boldsymbol{A} = [a_1, a_2, \ldots, a_n]$: **行矩阵** (或 **行向量, row vector**).

$m \times 1$ matrix $\boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$: **列矩阵**(或**列向量, column vector**).

(3) $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is called a **diagonal matrix**, (**对角矩阵**或**对角阵**)

(主) 对角线
(main) diagonal

denoted as $\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

A diagonal matrix with the same entry on the diagonal is called a **scalar matrix(数量矩阵)**. 主对角线上的元素全部相等

The following diagonal matrix is called an **identity matrix (单位矩阵)**, denoted as I_n or I .

主对角线上的元素全部是1

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(4) The matrix with each entry as zero is called a **zero matrix (零矩阵)**, denoted as **0** .

Zero matrices of different sizes are treated as different matrices.

(5)

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}, \qquad B=\begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & \\ \vdots & \vdots & \ddots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

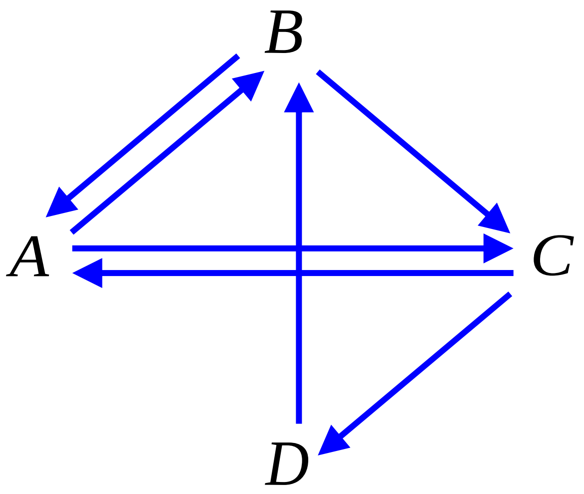
upper triangular matrix (上三角矩阵) **lower triangular matrix (下三角矩阵)**

对角线左下(右上)方的元素都为 0 的方阵称为

(upper/lower triangular matrix) .

引例(introductory example):

城市间的航班图



如果从 A 到 B 有航班, 则用带箭头的线连接 A 与 B.

航班图可用矩阵来表示:

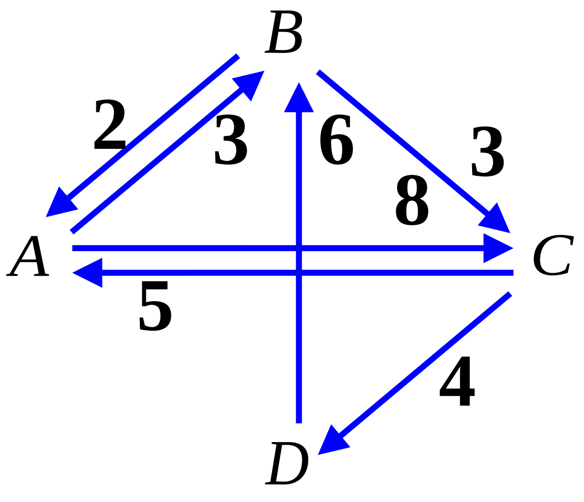
到达城市

出发城市

$$
\begin{array}{c|cccc}
 & A & B & C & D \\
A & & & & \\
B & & & & \\
C & & & & \\
D & & & &
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

引例 (introductory example):

城市间的航班图



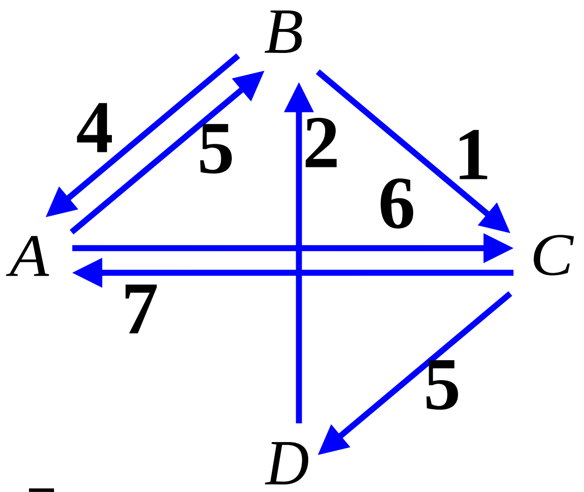
如果从 A 到 B 有航班, 则用带箭头的线连接 A 与 B.

航班量也可用矩阵来表示:

$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第一天　　第二天

城市间的航班图



如果从 ***A*** 到 ***B*** 有航班, 则用带箭头的线连接 ***A*** 与 ***B***.

航班量也可用矩阵来表示:

$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

第一天　　第二天　　第三天...

问题: 各城市2天内发送的航班量?

Matrix operations（矩阵的运算）

Matrix operations
(矩阵的运算)

I. Addition (矩阵的加法)

II. Scalar multiplication (数与矩阵相乘)

III. Multiplication (矩阵的乘法)

IV. Power (方阵的幂)

Elementary Matrices (初等矩阵)

I. 矩阵的加法(Addition)

each entry in $\boldsymbol{A}+\boldsymbol{B}$ is the sum of the corresponding entries in $\boldsymbol{A}$ and $\boldsymbol{B}$.

1. 定义 (Definition)

设有两个 $m\times n$ 矩阵 $\boldsymbol{A}=[a_{ij}]$ 和 $\boldsymbol{B}=[b_{ij}]$，那么矩阵 $\boldsymbol{A}$ 与 $\boldsymbol{B}$ 的和(sum)记作 $\boldsymbol{A}+\boldsymbol{B}$，规定为

$$\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

注　只有两个矩阵同型时，才能进行加法运算；

(The sum $\boldsymbol{A}+\boldsymbol{B}$ is defined only when $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same size.)

For example,

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & -9 & 0 \\ 3 & 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 9 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+1 & 3+8 & -5+9 \\ 1+6 & -9+5 & 0+4 \\ 3+3 & 6+2 & 8+1 \end{bmatrix} = \begin{bmatrix} 13 & 11 & 4 \\ 7 & -4 & 4 \\ 6 & 8 & 9 \end{bmatrix}.$$

2. 矩阵加法的运算规律

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be matrices of the same size, then

(1) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$;

(2) $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$;

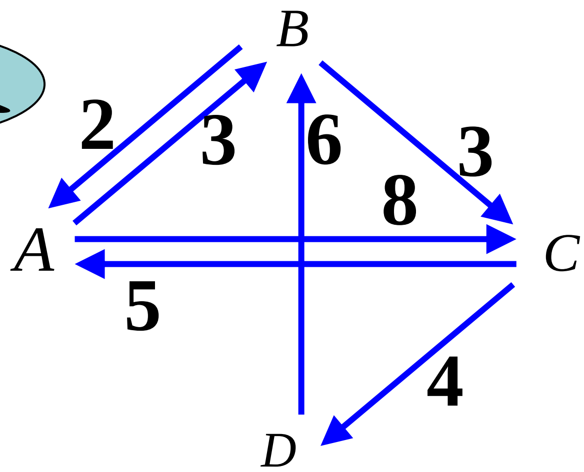
(3) $$-\boldsymbol{A}=\begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m1} & \cdots & -a_{mn} \end{bmatrix}=[-a_{ij}];$$

(4) $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{0}$, $\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B})$.

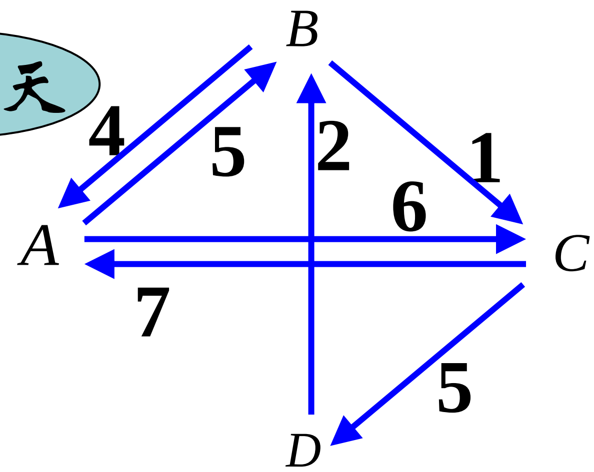
定义**矩阵的减法(subtraction)**

思考 城市间航班客流量

第一天



第二天



$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

问题1: 各城市2天内发送的航班量?

问题2: 收取的机场建设费(航空基金)有多少?

II. 数与矩阵相乘(Scalar multiplication)

1. 定义

数 λ 与矩阵 A 的乘积 (简称为数乘, scalar multiple)

记作 λA 或 $A\lambda$, 规定为

$$\lambda \boldsymbol{A} = \boldsymbol{A}\lambda = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}.$$

If λ is a scalar and $\boldsymbol{A}$ is a matrix, then the **scalar multiple** $\lambda\boldsymbol{A}$ is the matrix whose columns are λ times the corresponding columns in $\boldsymbol{A}$.

2. 数乘矩阵的运算规律

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be matrices of the same size ($m \times n$), and let λ and μ be scalars, then

(1) $(\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$;

(2) $(\lambda+\mu)\boldsymbol{A} = \lambda\boldsymbol{A} + \mu\boldsymbol{A}$;

(3) $\lambda(\boldsymbol{A}+\boldsymbol{B}) = \lambda\boldsymbol{A} + \lambda\boldsymbol{B}$.

矩阵的加法与数乘统称为矩阵的**线性运算**.

Example 1 Let $\boldsymbol{A}-3\boldsymbol{B}=4\boldsymbol{A}-\boldsymbol{C}$, where

$$\boldsymbol{A}=\begin{bmatrix}1 & -1\\ 0 & 2\\ 3 & 1\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}1 & -1\\ 0 & 1\\ -1 & 0\end{bmatrix}.$$

Find $\boldsymbol{C}$.

Solution From $\boldsymbol{A}-3\boldsymbol{B}=4\boldsymbol{A}-\boldsymbol{C}$, we have

$$\boldsymbol{C}=4\boldsymbol{A}-\boldsymbol{A}+3\boldsymbol{B}=3(\boldsymbol{A}+\boldsymbol{B}),$$

so

$$\boldsymbol{C}=\begin{bmatrix}3(1+1) & 3(-1-1)\\ 3(0+0) & 3(2+1)\\ 3(3-1) & 3(1+0)\end{bmatrix}=\begin{bmatrix}6 & -6\\ 0 & 9\\ 6 & 3\end{bmatrix}.$$

III. 矩阵乘法(Multiplication)

引例1 超市购物 =>Product of Matrices

同样的商品在不同的超市内的售价是不尽相同的. 这样, 在一次需要购买多种商品时, 就有到哪一家超市去买花费最少的问题.

这就要用到价格矩阵, 如

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \qquad \begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$$

可用来表示3家超市里4种商品的“价目表”

第1行的元依次表示超市1里4种商品的售价

III. 矩阵乘法(Multiplication)

引例1 超市购物

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{cccc} \text{商品1} & \text{商品2} & \text{商品3} & \text{商品4} \end{array} \\ \begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{超市1} \\ \text{超市2} \\ \text{超市3} \end{array}$$

购物者1对4种商品的需求分别为$a_{11}, a_{21}, a_{31}, a_{41}$，则在不同超市去购买所需花费总额为?

若有n名购物者，则可将他们的需求构成<u>**需求矩阵**</u>

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$$

那么这n名购物者的采购方案可以用一个数表来表示：

<u>总价矩阵</u> = ?

价格矩阵 $\times$ 需求矩阵 $=$ 总价矩阵

$$
\begin{array}{c}
 \\ \text{超市1} \\ \text{超市2} \\ \text{超市3}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{商品1} & \text{商品2} & \text{商品3} & \text{商品4} \end{array} \\
\begin{bmatrix}
\color{blue}{1.7} & \color{blue}{1.1} & \color{blue}{21} & \color{blue}{7} \\
1.5 & 1.4 & 26 & 9 \\
1.8 & 1.3 & 28 & 8
\end{bmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccc} \text{购物者1} & \text{购物者2} & \text{……} & \text{购物者}n \end{array} \\
\begin{bmatrix}
\color{blue}{a_{11}} & a_{12} & \cdots & a_{1n} \\
\color{blue}{a_{21}} & a_{22} & \cdots & a_{2n} \\
\color{blue}{a_{31}} & a_{32} & \cdots & a_{3n} \\
\color{blue}{a_{41}} & a_{42} & \cdots & a_{4n}
\end{bmatrix}
\end{array}
\begin{array}{c}
 \\ \text{需求1} \\ \text{需求2} \\ \text{需求3} \\ \text{需求4}
\end{array}
$$

$$
\begin{array}{c}
 \\ \text{超市1} \\ \text{超市2} \\ \text{超市3}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{购物者1} & \text{购物者2} & \text{……} & \text{购物者}n \end{array} \\
\begin{bmatrix}
\color{blue}{*} & * & \ldots & * \\
* & * & \ldots & * \\
* & * & \ldots & *
\end{bmatrix}
\end{array}
$$

矩阵乘法(Multiplication)

引例2 数学例子

设 x_1,x_2,x_3 和 y_1,y_2 是两组变量, 它们之间的关系为

$$y_1=a_{11}x_1+a_{12}x_2+a_{13}x_3$$
$$y_2=a_{21}x_1+a_{22}x_2+a_{23}x_3$$

$$\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=[a_{ik}]_{2\times 3}$$

又设 t_1,t_2 是另一组变量, 它们与 x_1,x_2,x_3 的关系为

$$x_1=b_{11}t_1+b_{12}t_2$$
$$x_2=b_{21}t_1+b_{22}t_2$$
$$x_3=b_{31}t_1+b_{32}t_2$$

$$\boldsymbol{B}=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}=[b_{kj}]_{3\times 2}$$

则

$$y_1=(\overbrace{\underline{a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}}}^{c_{11}})t_1+(a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32})t_2$$
$$y_2=(a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31})t_1+(a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32})t_2$$

矩阵$\boldsymbol{C}=[c_{ij}]_{2\times 2}$是矩阵$\boldsymbol{A}$与$\boldsymbol{B}$的一个运算, 定义为矩阵的乘积.

矩阵乘法(Multiplication)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

其中 $$\boxed{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj},}$$

*Row-column rule for computing **AB***

上式右边(i, j)元素 c_{ij} 等于左边的第一个矩阵的第 i 行与第二个矩阵的第 j 列对应元素乘积之和.

矩阵运算中具有的特殊规律，主要产生于矩阵的乘法运算.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

Each entry of ***AB*** is the product (乘积) of a ***row*** and a ***column***:

$(\boldsymbol{AB})_{ij}$=(row i of $\boldsymbol{A}$) times (column j of $\boldsymbol{B}$)

Each column of ***AB*** is the product of a ***matrix*** and a ***column***:

column j of $\boldsymbol{AB}$= $\boldsymbol{A}$ times (column j of $\boldsymbol{B}$)

Each row of ***AB*** is the product of a ***row*** and a ***matrix***:

row i of $\boldsymbol{AB}$=(row i of $\boldsymbol{A}$) times $\boldsymbol{B}$

矩阵乘法(Multiplication)

1. Definition

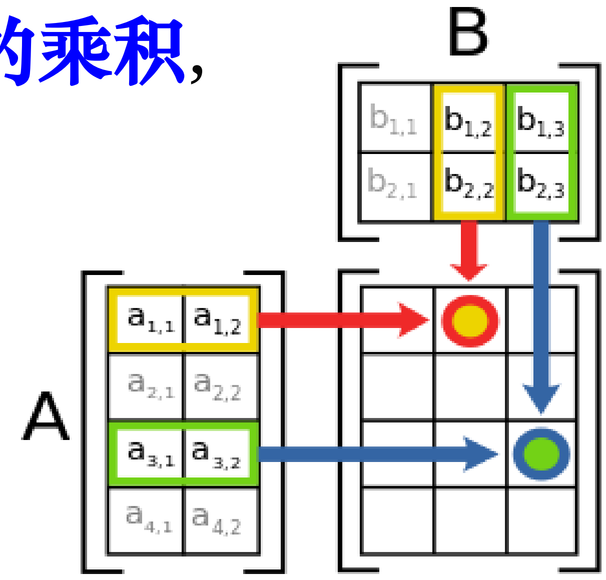
设 $\boldsymbol{A}=[a_{ik}]_{m\times p}$, $\boldsymbol{B}=[b_{kj}]_{p\times n}$ 为两个矩阵, 令

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ip}b_{pj}=\sum_{k=1}^{p}a_{ik}b_{kj},$$

$$i=1,2,\ldots m;\ j=1,2,\ldots,n,$$

称 $m\times n$ 矩阵 $\boldsymbol{C}=[c_{ij}]_{m\times n}$ 为 **$\boldsymbol{A}$ 与 $\boldsymbol{B}$ 的乘积**,

记为 $\boldsymbol{C}=\boldsymbol{AB}$.



注 只有当第一个矩阵的列数等于第二个矩阵的行数时, 两个矩阵才能相乘.

Example 2 Find $\boldsymbol{AB}$, where

$$\boldsymbol{A}=\begin{bmatrix}1 & 0 & -1 & 2\\ -1 & 1 & 3 & 0\\ 0 & 5 & -1 & 4\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}0 & 3 & 4\\ 1 & 2 & 1\\ 3 & 1 & -1\\ -1 & 2 & 1\end{bmatrix}.$$

Solution:

$$\boldsymbol{C}=\boldsymbol{AB}=\underset{3\times 4}{\begin{bmatrix}1 & 0 & -1 & 2\\ -1 & 1 & 3 & 0\\ 0 & 5 & -1 & 4\end{bmatrix}}\underset{4\times 3}{\begin{bmatrix}0 & 3 & 4\\ 1 & 2 & 1\\ 3 & 1 & -1\\ -1 & 2 & 1\end{bmatrix}}=\underset{3\times 3}{\begin{bmatrix}-5 & 6 & 7\\ 10 & 2 & -6\\ -2 & 17 & 10\end{bmatrix}}.$$

Match

Size of ***AB***

Exercises

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 6 & 8 \\ 6 & 0 & 1 \end{bmatrix} = ?$$

注 只有当第一个矩阵的列数等于第二个矩阵的行数时，两个矩阵才能相乘.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times 3 + 2\times 2 + 3\times 1 \end{bmatrix} = 10$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Example Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times 1$ and $1 \times n$ matrices, and

$$\boldsymbol{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \boldsymbol{B} = [b_1 \quad b_2 \quad \cdots \quad b_n].$$

Compute $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

Solution

$$\boldsymbol{AB} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{bmatrix}.$$

$$\boldsymbol{BA} = [a_1 b_1 + a_2 b_2 + \cdots a_n b_n].$$

A system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The i-th equation:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$(a_{i1}, a_{i2}, \ldots, a_{in}) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_i \quad (i = 1, 2, \ldots, m)$$

$$\text{Let } \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boxed{\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}}$$

System of Linear Equations

$$\begin{cases} \boxed{a_{11}x_1} + \boxed{a_{12}x_2} + \cdots + \boxed{a_{1n}x_n} = \boxed{b_1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\downarrow$$

$$\boxed{\boldsymbol{\alpha}_1 x_1} + \boxed{\boldsymbol{\alpha}_2 x_2} + \cdots + \boxed{\boldsymbol{\alpha}_n x_n} = \boxed{\boldsymbol{b}}$$

Vector Equation

Coefficient Matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n]$$

Augmented Matrix

$$(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n \quad \boldsymbol{b}]$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Solution (解向量)

Matrix Equation

$$\boxed{\boldsymbol{Ax} = \boldsymbol{b}.}$$

System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

1

Coefficient Matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n]$$

$$\boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2 + \cdots + \boldsymbol{\alpha}_n x_n = \boldsymbol{b}$$

Vector Equation

2

Augmented Matrix

$$(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n \quad \boldsymbol{b}]$$

Matrix Equation

$$\boxed{\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.}$$

3

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Solution (解向量)

2. Rules for Matrix Multiplication

(1) Let $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{B}=[b_{ij}]_{n\times p}$, $\boldsymbol{C}=[c_{ij}]_{p\times r}$, then

$$(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC}), \qquad k(\boldsymbol{AB}) = (k\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(k\boldsymbol{B}).$$

associative law of multiplication

(2) Let $\boldsymbol{A}=[a_{ij}]_{m\times p}$, $\boldsymbol{B}=[b_{ij}]_{p\times n}$, $\boldsymbol{C}=[c_{ij}]_{p\times n}$, $\boldsymbol{D}=[d_{ij}]_{n\times s}$, then

$$\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}, \quad (\boldsymbol{B}+\boldsymbol{C})\boldsymbol{D} = \boldsymbol{BD}+\boldsymbol{CD}.$$

left distributive law **right distributive law**

(3) Let $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{I}_m$, $\boldsymbol{I}_n$ are identity matrices of degree m and n respectively, then

$$\boldsymbol{A} = \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n; \qquad k\boldsymbol{A} = (k\boldsymbol{I}_m)\boldsymbol{A} = \boldsymbol{A}(k\boldsymbol{I}_n).$$

identity for matrix multiplication

证明: $(AB)C=A(BC)$.

设 $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{n\times p}$, $C=(c_{ij})_{p\times r}$, 则$(AB)C$与$A(BC)$都是$m\times r$矩阵.

只需证明: $\forall i=1,\cdots, m$, $\forall j=1,\cdots, r$, 有

$$[(AB)C]_{ij} = \sum_{k=1}^{p}(AB)_{ik}C_{kj} = \sum_{k=1}^{p}(\sum_{l=1}^{n}a_{il}b_{lk})c_{kj}$$

交换和号顺序

$$= \sum_{l=1}^{n}a_{il}(\sum_{k=1}^{p}b_{lk}c_{kj})$$

$$= \sum_{l=1}^{n}a_{il}(BC)_{lj} = [A(BC)]_{ij}$$

所以 $(AB)C=A(BC)$.

问题：矩阵乘法是否满足交换律(commutative law)，
即 $\boldsymbol{AB} = \boldsymbol{BA}$？

$$\begin{bmatrix} 1 & 6 & 8 \\ 6 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

不一定同时有意义

不一定同型

不相等

例外：可交换

The product of two matrices is not commutative:
$\boldsymbol{AB}$ is not necessarily equal to $\boldsymbol{BA}$.

For example, if $\boldsymbol{A}=\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then

$$\boldsymbol{AB}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{BA}=\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix},$$

therefore $\boldsymbol{AB} \neq \boldsymbol{BA}$.

矩阵运算中：
AB=0不能推出A=0或B=0
AB=AC不能推出B=C

Warnings:

1. In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$.
2. If a product $\boldsymbol{AB}$ is the zero matrix, you *cannot* conclude in general that either $\boldsymbol{A}=\mathbf{0}$ or $\boldsymbol{B}=\mathbf{0}$.
3. The cancellation laws do *not* hold for matrix multiplication. That is, if $\boldsymbol{AB}=\boldsymbol{AC}$, then it is *not* true in general that $\boldsymbol{B}=\boldsymbol{C}$.

投票： 两个上(下)三角阵$\boldsymbol{A}$与$\boldsymbol{B}$的乘积$\boldsymbol{AB}$是否仍是上(下)三角阵?

$$\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \boldsymbol{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$$

$$\boldsymbol{C}=\boldsymbol{AB}=(c_{ij})_{n\times n}$$

其主对角元$(\boldsymbol{AB})_{ii}=?$

证明： 两个上(下)三角阵$\boldsymbol{A}$与$\boldsymbol{B}$的乘积$\boldsymbol{AB}$仍是上(下)三角阵, 且其主对角元$(\boldsymbol{AB})_{ii}=a_{ii}b_{ii}$.

证　设 $\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \boldsymbol{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$　$\boldsymbol{C}=\boldsymbol{AB}=(c_{ij})_{n\times n}$

$i>j$时， $c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=\sum_{k=1}^{i-1}a_{ik}b_{kj}+\sum_{k=i}^{n}a_{ik}b_{kj}$

$\boxed{a_{ik}=0\ (k=1,2,\cdots,i-1)}$

$\boxed{b_{kj}=0\ (k=i,i+1,\cdots,n)}$

即$i>j$时， $c_{ij}=0$　　$\boldsymbol{C}$ 为上三角阵

而$c_{ii}=\sum_{k=1}^{n}a_{ik}b_{ki}=\sum_{k=1}^{i-1}a_{ik}b_{ki}+\sum_{k=i}^{n}a_{ik}b_{ki}$

$=0+a_{ii}b_{ii}=a_{ii}b_{ii}$.

$\boxed{a_{ik}=0\ (k=1,2,\cdots,i-1)}$

$\boxed{b_{ki}=0\ (k=i+1,\cdots,n)}$

IV. 方阵的幂(Power)

1. 定义

设 $\boldsymbol{A}$ 是 n 阶方阵, 定义 **$\boldsymbol{A}$ 的幂**为

$$\boldsymbol{A}^0 = \boldsymbol{I}, \quad \boldsymbol{A}^1 = \boldsymbol{A}, \quad \boldsymbol{A}^2 = \boldsymbol{A}^1\boldsymbol{A}^1, \ \ldots, \quad \boldsymbol{A}^{k+1} = \boldsymbol{A}^k\boldsymbol{A}^1.$$

注 只有方阵, 它的幂才有意义.

2. 矩阵的幂的运算规律

设 k, l 为非负整数, 则

(1) $\boldsymbol{A}^{k+l} = \boldsymbol{A}^k\boldsymbol{A}^l$;

(2) $(\boldsymbol{A}^k)^l = \boldsymbol{A}^{kl}$.

由矩阵乘法不满足交换律，一般地 $(\boldsymbol{AB})^k \neq \boldsymbol{A}^k\boldsymbol{B}^k$.

但也有例外，如设 $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$, 则有

$$\boldsymbol{AB} = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, \boldsymbol{BA} = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, \Rightarrow \boldsymbol{AB} = \boldsymbol{BA}.$$

注 当 $\boldsymbol{A}$ 与 $\boldsymbol{B}$ 可交换时, 有

$$(\boldsymbol{AB})^k = \boldsymbol{A}^k\boldsymbol{B}^k,$$

$$(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2,$$

$$(\boldsymbol{A}-\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{A}^2 - \boldsymbol{B}^2.$$

设

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

为一元多项式, 其中 $a_0, a_1, \ldots, a_n$ 为多项式的系数.

设 $\boldsymbol{A}$ 为方阵, 则称

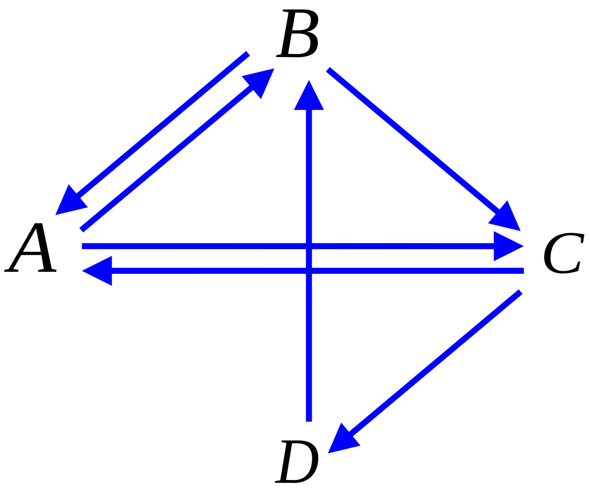
$$p(\boldsymbol{A}) = a_n \boldsymbol{A}^n + a_{n-1} \boldsymbol{A}^{n-1} + \cdots + a_1 \boldsymbol{A} + a_0 \boldsymbol{I}$$

为**方阵 $\boldsymbol{A}$ 的多项式**.

(3) 设 $f(x)$, $g(x)$, $h(x)$ 为一元多项式, $\boldsymbol{A}$ 是方阵.

如果 $f(x) = g(x)\,h(x)$, 则 $f(\boldsymbol{A}) = g(\boldsymbol{A})\,h(\boldsymbol{A})$.

城市间航班图



航班图可用矩阵来表示:

$$\boldsymbol{M}=\begin{bmatrix}0&1&1&0\\1&0&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix}.$$

练习

1: 计算 $\boldsymbol{M}^2$?

2: 思考 $\boldsymbol{M}^2$ 中元素的实际含义.

$$\boldsymbol{M}^2=\begin{bmatrix}2&0&1&1\\1&1&1&1\\0&2&1&0\\1&0&1&0\end{bmatrix}.$$

Example 3 Compute $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n$.

Solution Let

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

拆开之后好算，并且AB=BA

then $\boldsymbol{A}$ is a scalar matrix, and $\boldsymbol{AB} = \boldsymbol{BA}$, Therefore,

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n = (\boldsymbol{A}+\boldsymbol{B})^n = \sum_{k=0}^{n} \mathrm{C}_n^k \boldsymbol{A}^{n-k} \boldsymbol{B}^k$$

$$= \mathrm{C}_n^0 \boldsymbol{A}^n \boldsymbol{B}^0 + \mathrm{C}_n^1 \boldsymbol{A}^{n-1} \boldsymbol{B} + \mathrm{C}_n^2 \boldsymbol{A}^{n-2} \boldsymbol{B}^2 + \cdots \mathrm{C}_n^n \boldsymbol{B}^n.$$

都是0

Since $\boldsymbol{B}^2 = \mathbf{0}$, we have $\boldsymbol{B}^2 = \boldsymbol{B}^3 = \ldots = \boldsymbol{B}^n = \mathbf{0}$. And

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n = \boldsymbol{A}^n + n\boldsymbol{A}^{n-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} n3^{n-1} & 0 \\ 0 & n3^{n-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} 0 & n3^{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{bmatrix}.$$

V. Elementary Matrices

定义 由单位矩阵 I 经过一次初等变换得到的方阵称为**初等矩阵**. (An **elementary matrix** is one that is obtained by performing a single elementary operation on an identity matrix.)

三种初等变换对应着三种初等矩阵:

(1) 对调两行或两列——**初等对换矩阵**;

(2) 以数 $k \neq 0$ 乘某行或某列——**初等倍乘矩阵**;

(3) 以数 k 乘某行(列)加到另一行(列)上去——**初等倍加矩阵**.

(1)初等对换矩阵:

将单位矩阵的第 i, j 行(或列)对换

$$
\boldsymbol{P}_{ij} = \begin{array}{c@{}l}
\begin{bmatrix}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & & \cdots & & 1 & & & \\
& & & & 1 & & & & & & \\
& & & \vdots & & \ddots & & \vdots & & & \\
& & & & & & 1 & & & & \\
& & & 1 & & \cdots & & 0 & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{bmatrix}
&
\begin{array}{l}
\\ \\ \\ \leftarrow 第\, i\, 行 \\ \\ \\ \\ \leftarrow 第\, j\, 行 \\ \\ \\ \\
\end{array}
\end{array}
$$

第 i 列　　第 j 列

(2)初等倍乘矩阵:

将单位矩阵第 i 行(或列)乘 $k \neq 0$

$$\boldsymbol{D}_i(k)=\begin{bmatrix}1&&&&&&\\&\ddots&&&&&\\&&1&&&&\\&&&k&&&\\&&&&1&&\\&&&&&\ddots&\\&&&&&&1\end{bmatrix}\begin{matrix}\\ \\ \\ \leftarrow 第\,i\,行\\ \\ \\ \\ \end{matrix}$$

第 i 列

(3)初等倍加矩阵:

将单位矩阵第 i 行乘 k 加到第 j 行，

或将第 j 列乘 k 加到第 i 列

$$E_{ij}(k)=\begin{array}{cl}
\begin{bmatrix}
1 &&&&&& \\
& \ddots &&&&& \\
&& 1 &&&& \\
&& \vdots & \ddots &&& \\
&& k & \cdots & 1 && \\
&&&&& \ddots & \\
&&&&&& 1
\end{bmatrix} &
\begin{array}{l} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \\ \end{array} \\
\begin{array}{ccccccc} \quad & \quad & \text{第 } i \text{ 列} & \quad & \text{第 } j \text{ 列} & \quad & \quad \end{array} &
\end{array}$$

- **Example 4** Let

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Compute $\boldsymbol{K}_1\boldsymbol{A}$, $\boldsymbol{K}_2\boldsymbol{A}$, and $\boldsymbol{K}_3\boldsymbol{A}$, and describe how these products can be obtained by elementary row operations on $\boldsymbol{A}$.

Solution:

左乘！

Addition of -4 times row 1 of $\boldsymbol{A}$ to row 3 produces $\boldsymbol{K}_1\boldsymbol{A}$.

$$\boldsymbol{K}_1\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}.$$

k在第i行第j列，结果在第i行上加上k倍第j行

$$\boldsymbol{K}_2\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

An interchange of rows 1 and 2 of $\boldsymbol{A}$ produces $\boldsymbol{K}_2\boldsymbol{A}$.

第i行和第j行交换（第i列和第j列交换），结果第i行和第j行交换

$$\boldsymbol{K}_3\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Multiplication of row 3 of $\boldsymbol{A}$ by 5 produces $\boldsymbol{K}_3\boldsymbol{A}$.

第i行第i列为k，结果第i行乘k倍

- ***Left-multiplication*** (左乘, that is, multiplication on the left) by $\boldsymbol{K}_1$ in Example 4 has the same effect on any $3\times n$ matrix.

- Since $\boldsymbol{K}_1\boldsymbol{I}=\boldsymbol{K}_1$, we see that $\boldsymbol{K}_1$ *itself* is produced by this same row operation on the identity. (注：由单位矩阵 $\boldsymbol{I}$ 经过一次初等变换得到的方阵称为**初等矩阵**)

- Example 4 illustrates the following general fact about elementary matrices.
- If an elementary row operation is performed on an $m\times n$ matrix $\boldsymbol{A}$, the resulting matrix can be written as $\boldsymbol{KA}$, where the $m\times m$ matrix $\boldsymbol{K}$ is created by performing the *same row operation* on $\boldsymbol{I}_m$.

What if -- right-multiplication?

Addition of -4 times column 3 of ***A*** to column 1 produces $\boldsymbol{AK}_1$.

$$\boldsymbol{AK}_1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a-4c & b & c \\ d-4f & e & f \\ g-4i & h & i \end{bmatrix}.$$

k在第i行第j列，结果在第j列上加上k倍第i列

$$\boldsymbol{AK}_2 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}.$$

An interchange of columns 1 and 2 of ***A*** produces $\boldsymbol{AK}_2$.

第i行和第j行交换（第i列和第j列交换），结果第i列和第j列交换

$$\boldsymbol{AK}_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & b & 5c \\ d & e & 5f \\ g & h & 5i \end{bmatrix}.$$

Multiplication of column 3 of ***A*** by 5 produces $\boldsymbol{AK}_3$.

第i行第i列为k，结果第i列乘k倍

初等矩阵与矩阵的乘积

1) 用 m 阶初等矩阵 $\boldsymbol{P}_{ij}$ 左乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，得

$$\boldsymbol{P}_{ij}\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \\ \end{array}$$

相当于把矩阵 $\boldsymbol{A}$ 第 i 行与第 j 行对调 $(r_i \leftrightarrow r_j)$.

2) 用 m 阶初等矩阵 $\boldsymbol{D}_i(k)$ 左乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，得

$$\boldsymbol{D}_i(k)\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{第 } i \text{ 行}$$

相当于以数 k 乘矩阵 $\boldsymbol{A}$ 的第 i 行 (kr_i) ($k\neq 0$).

3) 用 $\boldsymbol{E}_{ij}(k)$ 左乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，得

$$\boldsymbol{E}_{ij}(k)\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1}+ka_{i1} & a_{j2}+ka_{i2} & \cdots & a_{jn}+ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \\ \end{array}$$

相当于把 $\boldsymbol{A}$ 的第 i 行乘 k 加到第 j 行 (r_j+kr_i).

4) 用 n 阶初等矩阵 $\boldsymbol{P}_{ij}$ 右乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，相当于把矩阵 $\boldsymbol{A}$ 第 i 列与第 j 列对调 ($c_i \leftrightarrow c_j$).

5) 用 n 阶初等矩阵 $\boldsymbol{D}_i(k)$ 右乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，相当于以数 k 乘矩阵 $\boldsymbol{A}$ 的第 i 列 (kc_i).

6) 用 n 阶初等矩阵 $\boldsymbol{E}_{ij}(k)$ 右乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，相当于将矩阵 $\boldsymbol{A}$ 的第 j 列乘数 k 加到第 i 列 (c_i+kc_j).

三种初等矩阵左乘矩阵A是对A作相应的初等行变换
三种初等矩阵右乘矩阵B是对B作相应的初等列变换

Note:
$\boldsymbol{K}_1=\boldsymbol{P}_{13}$, $\boldsymbol{K}_2=\boldsymbol{E}_{14}(c)$, $\boldsymbol{K}_3=\boldsymbol{D}_2(k)$

Example 5 Let

$$\boldsymbol{K}_1=\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix},\quad \boldsymbol{K}_2=\begin{bmatrix}1&&&\\0&1&&\\0&0&1&\\c&0&0&1\end{bmatrix},\quad \boldsymbol{K}_3=\begin{bmatrix}1&&&\\&k&&\\&&1&\\&&&1\end{bmatrix}.$$

Find $\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3$.

Solution We note that $\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3=\boldsymbol{P}_{13}\,\boldsymbol{E}_{14}(c)\,\boldsymbol{D}_2(k)$,

so

$$\boldsymbol{K}_2\boldsymbol{K}_3=\boldsymbol{E}_{14}(c)\,\boldsymbol{D}_2(k)=\begin{bmatrix}1&&&\\&1&&\\&&1&\\c&&&1\end{bmatrix}\cdot\begin{bmatrix}1&&&\\&k&&\\&&1&\\&&&1\end{bmatrix}=\begin{bmatrix}1&&&\\&k&&\\&&1&\\c&&&1\end{bmatrix}$$

Note:

$\boldsymbol{K}_1=\boldsymbol{P}_{13}$, $\boldsymbol{K}_2=\boldsymbol{E}_{14}(c)$, $\boldsymbol{K}_3=\boldsymbol{D}_2(k)$

$\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3= \boldsymbol{P}_{13}\boldsymbol{K}_2\boldsymbol{K}_3$, and

$$\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3 = \boldsymbol{P}_{13}\begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & k & 0 & 0 \\ 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 \end{bmatrix}$$

Remark We can also use right multiplication and the corresponding ***elementary column operations*** to do the calculation. It leads to the same result.

Example 6 (将初等矩阵概念用于消元：Elimination)

For 3 equations in 3 unknowns:

Suppose $\boldsymbol{E}$ subtracts twice the first equation from the second.

Suppose $\boldsymbol{F}$ is the matrix for the next step, *to add row* 1 *to row* 3.

$$\boldsymbol{E}=\begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}, \qquad \boldsymbol{F}=\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}.$$

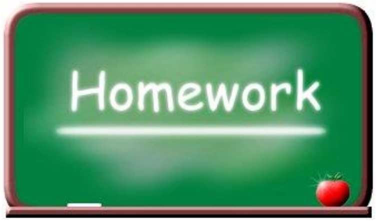
These two matrices do commute and the product does both steps at once:

$$\boldsymbol{EF}=\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 1 & & 1 \end{bmatrix}=\boldsymbol{FE}.$$

In either order, $\boldsymbol{EF}$ or $\boldsymbol{FE}$, this changes rows 2 and 3 using row 1.

What if : $\boldsymbol{E}$ is the same but $\boldsymbol{G}$ *add row* 2 *to row* 3**?** $\boldsymbol{EG}\neq\boldsymbol{GE}$.

Homework



- See Blackboard announcement
- ***Hardcover*** **textbook + Supplementary problems**
- Pay attention to the notation

In the textbook, the ***elementary matrix*** $\boldsymbol{E}_{ij}$ subtracts l times row j from row i.

$$\boldsymbol{E}_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -l & & 1 \end{bmatrix},$$

Deadline (DDL):

- Next tutorial class