Step-1

In Hilbert space, to find the length of the vector $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \ldots\right)$ and the length of the function $f(x) = e^x$ (over the interval $0 \le x \le 1$), and

Also to find the inner product of;

$$f(x) = e^x$$
$$g(x) = e^{-x}$$

Over this interval

Step-2

In Hilbert space, if $v = (v_1, v_2, v_3, \ldots)$ then

$$\|v\|^2 = v_1^2 + v_2^2 + v_3^2 + \ldots$$

Given, $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \ldots\right)$

$$\|v\|^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$
$$= \frac{a}{1-r} (\text{Sum of infinite G.P})$$
$$= \frac{1/2}{1 - \frac{1}{2}}$$
$$= \frac{1/2}{1/2}$$
$$= 1$$

Length of the vector $v = \|v\|$

$$= \sqrt{1}$$
$$= \boxed{1}$$

Step-3

Now,

$$\|f\|^2 = \int_0^1 [f(x)]^2 dx$$
$$= \int_0^1 (e^x)^2 dx$$
$$= \int_0^1 e^{2x} dx$$
$$= \left[\frac{e^{2x}}{2}\right]_0^1$$
$$= \frac{e^2}{2} - \frac{e^0}{2}$$
$$= \frac{e^2 - 1}{2}$$

Step-4

Therefore the length of the function $f(x)$;

$$= \|f(x)\|$$
$$= \boxed{\sqrt{\frac{e^2 - 1}{2}}}$$

Step-5

Let

$$f(x) = e^x,$$
$$g(x) = e^{-x}$$

Inner product of f and g;

$$= (f, g)$$
$$= \int_0^1 e^x e^{-x} dx$$
$$= \int_0^1 e^0 dx$$

$$= \int_0^1 1 dx$$

$$= [x]_0^1$$

$$= 1 - 0$$

$$= 1$$

Hence inner product of f and g is 1