

Step-1

Row space:

By the definition, nonzero rows of any matrix form the basis for the row spaces and matrix U have two pivot element. So,

$$\dim C(A^T) = \dim C(U^T) \\ = 2$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

And transpose of the nonzero rows form the basis for the row spaces.

Step-2

Column space:

In matrix U , only first two columns are pivot columns. So, these will form the basis for column space of U ,

$$\dim C(U) = 2, \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For the basis of column space of matrix A , one has to take the same pivot column (that is, first two) but the elements from A , that is,

$$\dim C(A) = 2, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Step-3

Null space:

For the null space, find the solution of the system $Ux = 0$:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Then there will be two equations

$$x_1 + 2x_2 + x_4 = 0$$

$$x_2 + x_3 = 0$$

Substitute $x_2 = -x_3$ in first equation and get, $x_1 = 2x_3 - x_4$, where x_3, x_4 are arbitrary real numbers. Thus, the solution of the system is:

$$\begin{aligned} x &= \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Hence, dimension of null space of matrix A and U will be 2 with the basis:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Step-4

Left null space:

The left null space of matrix U is the null space of U^T . That is, find the solution of the system $U^T x = 0$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Then it gives $x_1 = 0, x_2 = 0, x_3 \in R$. Thus, the solution of the system is:

$$x = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, $\dim N(U^T) = 1$, with basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Step-5

The left null space of matrix A is the null space of A^T . That is, find the solution of the system $A^T x = 0$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Then it gives $x_1 + x_3 = 0$, $x_2 = 0$, $x_3 \in \mathbb{R}$. Thus, the solution of the system is:

$$\begin{aligned} x &= \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Step-6

Hence, $\dim N(A^T) = 1$, with basis $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.