

题 号	1	2	3	4	5	6	7
分 值	15 分	25 分	15 分	15 分	10 分	12 分	8 分

本试卷共 ( 7 ) 大题, 满分 ( 100 ) 分. 请将所有答案写在答题本上.

This exam includes **7** questions and the score is 100 in total. **Write all your answers on the examination book.**

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let  $A$  be an  $m \times n$  matrix and suppose  $Ax = 0$  has a nonzero solution. Which of the following must be true? ( )

- (A) The row vectors of  $A$  are linearly dependent.
- (B) The column vectors of  $A$  are linearly independent.
- (C) The rank of  $A$  is  $< n$ .
- (D)  $m = n$  and  $\det(A) = 0$ .

设  $A$  为  $m \times n$  矩阵. 假设  $Ax = 0$  有非零解. 下列哪一项一定是正确的? ( )

- (A)  $A$  的行向量线性相关.
- (B)  $A$  的列向量线性无关.
- (C)  $A$  的秩  $< n$ .
- (D)  $m = n$  且  $\det(A) = 0$ .

(2) Let  $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$  and let  $\alpha_1, \alpha_2, \alpha_3$  be linearly independent column vectors in  $\mathbb{R}^3$ . Then the rank of the vector system  $A\alpha_1, A\alpha_2, A\alpha_3$  ( )

- (A) must be 1.
- (B) must be 2.
- (C) must be 3.
- (D) can be 1 or 2.

设  $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ ,  $\alpha_1, \alpha_2, \alpha_3$  为  $\mathbb{R}^3$  中线性无关的向量组. 则向量组  $A\alpha_1, A\alpha_2, A\alpha_3$  的秩 ( )

- (A) 一定是 1.
- (B) 一定是 2.
- (C) 一定是 3.
- (D) 可能是 1 也可能是 2.

- (3) Let  $A$  and  $P$  be square matrices of order 3 with  $P$  invertible. Suppose  $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ . If  $P = (\alpha_1, \alpha_2, \alpha_3)$  and  $Q = (\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$ , then  $Q^{-1}AQ =$  ( )

(A)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C)  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D)  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

设  $A$  和  $P$  为 3 阶方阵,  $P$  可逆. 假设  $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ . 若  $P = (\alpha_1, \alpha_2, \alpha_3)$ ,

$Q = (\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$ , 则  $Q^{-1}AQ =$  ( )

(A)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C)  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D)  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- (4) Let  $A$  and  $B$  be real symmetric matrices of order  $n$ . Suppose  $A$  and  $B$  are congruent. Then ( )

(A) The null spaces  $N(A)$  and  $N(B)$  have the same dimension

(B)  $A$  and  $B$  have the same eigenvalues

(C)  $A$  and  $B$  have the same column space

(D)  $A$  and  $B$  have the same determinant

设  $A$  与  $B$  均为  $n$  阶实对称矩阵. 假设  $A$  与  $B$  合同 (也称相合). 则 ( )

- (A) 零空间  $N(A)$  与  $N(B)$  有相同的维数
- (B)  $A$  与  $B$  有相同的特征值
- (C)  $A$  与  $B$  有相同的列空间
- (D)  $A$  与  $B$  有相同的行列式

(5) Let  $Q$  be a real orthogonal matrix of order 3. Which of the following is false? ( )

- (A) For every real symmetric matrix  $A$  of order 3,  $Q^{-1}AQ$  is symmetric.
- (B) For every column vector  $v \in \mathbb{R}^3$ , the vectors  $Qv$  and  $v$  have the same length.
- (C) There is a nonzero column vector  $v \in \mathbb{R}^3$  such that  $Qv = v$  or  $Qv = -v$ .
- (D) There is an invertible real matrix  $P$  of order 3 such that  $P^{-1}QP$  is diagonal.

设  $Q$  为 3 阶实正交矩阵. 下列哪一项论断是错误的? ( )

- (A) 对任何 3 阶实对称阵  $A$ ,  $Q^{-1}AQ$  仍为对称阵.
- (B) 对任何列向量  $v \in \mathbb{R}^3$ , 向量  $Qv$  和  $v$  的长度相同.
- (C) 存在非零列向量  $v \in \mathbb{R}^3$  使得  $Qv = v$  或  $Qv = -v$ .
- (D) 存在 3 阶可逆实矩阵  $P$  使得  $P^{-1}QP$  为对角阵.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let  $A, B, C$  and  $D$  be square matrices of order  $n$ . Suppose  $A$  is invertible. Find two square matrices  $X, Y$  such that  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$ . (We denote by  $I_n$  the identity matrix of order  $n$ .)

Answer:  $X =$  \_\_\_\_\_,  $Y =$  \_\_\_\_\_.

设  $A, B, C, D$  均为  $n$  阶方阵. 假设  $A$  可逆. 写出两个方阵  $X, Y$  使得  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} =$

$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$ . (我们用  $I_n$  表示  $n$  阶单位矩阵.)

答案:  $X =$  \_\_\_\_\_,  $Y =$  \_\_\_\_\_.

(2) Let  $A$  be a  $3 \times 3$  matrix with determinant  $|A| = 4$ . Then  $|2A^{-1}| =$  \_\_\_\_\_

设  $A$  为  $3 \times 3$  矩阵, 行列式  $|A| = 4$ . 则  $|2A^{-1}| =$  \_\_\_\_\_

(3) Let  $A$  be a  $3 \times 3$  matrix. Suppose that the sum of the diagonal entries of  $A$  is  $-5$ , and  $A^2 + 2A - 3I = 0$ , then the three eigenvalues of  $A$  are \_\_\_\_\_.

设  $A$  是 3 阶矩阵. 假设  $A$  的主对角线元素之和为  $-5$ , 且满足  $A^2 + 2A - 3I = 0$ . 则矩阵  $A$  的三个特征值是 \_\_\_\_\_.

(4) Let  $L \subseteq \mathbb{R}^3$  be the line through the vector  $\beta = (1, -2, 2)^T$  (and the origin). Then the projection of the vector  $\alpha = (1, 0, -1)^T$  onto the line  $L$  is \_\_\_\_\_

设  $L \subseteq \mathbb{R}^3$  为经过 (原点和) 向量  $\beta = (1, -2, 2)^T$  的直线. 则向量  $\alpha = (1, 0, -1)^T$  在直线  $L$  上的投影是 \_\_\_\_\_

(5) Suppose that the matrix  $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & 2 & 3 \end{bmatrix}$  is similar to the matrix  $B = \begin{bmatrix} 3 & & \\ & 4 & \\ & & -1 \end{bmatrix}$ .

Then  $b =$  \_\_\_\_\_.

假设矩阵  $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & 2 & 3 \end{bmatrix}$  和  $B = \begin{bmatrix} 3 & & \\ & 4 & \\ & & -1 \end{bmatrix}$  相似. 则  $b =$  \_\_\_\_\_.

3. (15 points) Let  $V = \mathbf{M}_2(\mathbb{R})$  be the space of real square matrices of order 2. Let  $T$  be the linear transformation

$$T : V \longrightarrow V; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & c \\ b & a \end{bmatrix}.$$

(a) Find the matrix  $A$  of  $T$  in the ordered basis  $v_1, v_2, v_3, v_4$ , where

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) Is  $T$  invertible? Why?

(c) Investigate whether the matrix  $A$  is diagonalizable.

(15 分) 设  $V = \mathbf{M}_2(\mathbb{R})$  为 2 阶实方阵构成的向量空间. 令  $T$  表示如下线性变换

$$T : V \longrightarrow V; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & c \\ b & a \end{bmatrix}.$$

(a) 求  $T$  在有序基  $v_1, v_2, v_3, v_4$  下的矩阵  $A$ , 其中

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(b)  $T$  是否是可逆的? 为什么?

(c) 判定矩阵  $A$  是否可对角化.

4. (15 points) Let  $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$ .

(a) Decide whether  $A$  is positive (or negative) definite, or positive (or negative) semidefinite.

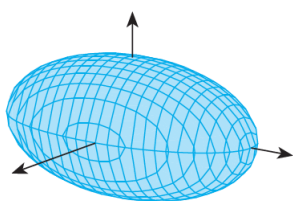
(b) Find an orthogonal matrix  $Q$  such that  $Q^{-1}AQ$  is a diagonal matrix.

(c) Let  $S$  be the surface in  $\mathbb{R}^3$  defined by the equation  $2x^2 - 4xy + y^2 - 4yz + 1 = 0$ .

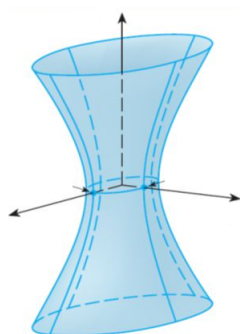
Which of the following graphs best illustrates the shape of the surface  $S$  (when the coordinate axes are suitably chosen)? (A), (B) or (C)?

(15 分) 设  $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$ .

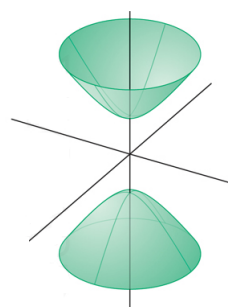
- (a) 判定  $A$  是否正定或负定、是否半正定或半负定.  
 (b) 找出一个正交矩阵  $Q$  使  $Q^{-1}AQ$  为对角阵.  
 (c) 设  $S$  为  $\mathbb{R}^3$  中由方程  $2x^2 - 4xy + y^2 - 4yz + 1 = 0$  定义的曲面.  
 (当坐标轴适当选取时) 以下那个图最适合描述曲面  $S$  的形状? (A), (B) 还是 (C)?



(A) An ellipsoid  
椭球面



(B) A hyperboloid  
of one sheet  
单叶双曲面



(C) A hyperboloid  
of two sheets  
双叶双曲面

5. (10 points) Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ .

- (a) Find all the singular values of  $A$ .  
 (b) Find the singular value decomposition of  $A$ . That is, find two orthogonal matrices  $U$  and  $V$  (of suitable size) such that  $A = U\Sigma V^T$ .

(10 分) 令  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ .

- (a) 求  $A$  的所有奇异值.  
 (b) 求  $A$  的奇异值分解. 即, 找出两个 (适当大小的) 正交矩阵  $U$  和  $V$  使得  $A = U\Sigma V^T$ .

6. (12 points) Let  $A$  be an  $m \times n$  complex matrix and set  $B = A^H A$  (where  $A^H = \overline{A}^T$  denotes the conjugate transpose of  $A$ ).

- (a) Prove that the eigenvalues of  $B$  in  $\mathbb{C}$  are all real numbers.
- (b) Suppose  $m < n$ . Show that 0 is an eigenvalue of  $B$ .
- (c) Suppose  $m = n > 1$ . Is it possible that  $-1$  is an eigenvalue of  $B$ ? If yes, write down explicitly a matrix  $A$  with this property and justify your answer. Otherwise explain why such a phenomenon is impossible.

(12 分) 设  $A$  为  $m \times n$  复矩阵,  $B = A^H A$  (其中  $A^H = \overline{A}^T$  表示  $A$  的共轭转置).

- (a) 证明  $B$  在  $\mathbb{C}$  中的特征值都是实数.
- (b) 假设  $m < n$ . 证明 0 是  $B$  的一个特征值.
- (c) 假设  $m = n > 1$ . 是否有可能  $-1$  是  $B$  的一个特征值? 若是, 请具体写出一个满足此条件的矩阵  $A$  并且解释你给的答案为何满足要求. 若否, 请解释为何此现象不可能出现.

7. (8 points) Let  $A$  be a real (symmetric) positive definite matrix of order  $n$  and let  $\alpha_1, \dots, \alpha_n$  be column nonzero vectors in  $\mathbb{R}^n$  such that for all distinct indices  $i, j \in \{1, 2, \dots, n\}$ ,  $\alpha_i^T A \alpha_j = 0$ .

Prove that the vectors  $\alpha_1, \dots, \alpha_n$  are linearly independent.

(8 分) 设  $A$  为  $n$  阶实 (对称) 正定矩阵. 设  $\alpha_1, \dots, \alpha_n$  为  $\mathbb{R}^n$  中的非零列向量. 假设对任意不同的指标  $i, j \in \{1, 2, \dots, n\}$  均有  $\alpha_i^T A \alpha_j = 0$ .

证明向量组  $\alpha_1, \dots, \alpha_n$  是线性无关的.