Step-1

Consider the following matrix:

$$A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$$

Find the limit of $A^k u_0$ when $k \to \infty$ for the following value:

$$u_0 = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step-2

Difference equation can be written as follows:

$$\begin{aligned} u_{k+1} &= A^k u_0 \\ &= S\Lambda^k S^{-1} \cdot u_0 \end{aligned}$$

Step-3

Firstly find the Eigen value and Eigen vectors of matrix A. To find the Eigen values calculate the following:

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} \frac{2}{5} - \lambda & \frac{3}{10} \\ \frac{3}{5} & \frac{7}{10} - \lambda \end{bmatrix} = 0$$

$$10\lambda^2 - 11\lambda + 1 = 0$$

$$10\lambda(\lambda - 1) - 1(\lambda - 1) = 0$$

After solving following values are obtained:

$$\lambda = 1$$

$$\lambda = \frac{1}{10}$$

Step-4

Letâ€™s take one Eigen value to calculate Eigen vectors:

$$\lambda = 1$$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} \frac{2}{5}-1 & \frac{3}{10} \\ \frac{3}{5} & \frac{7}{10}-1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-3}{5} & \frac{3}{10} \\ \frac{3}{5} & \frac{-3}{10} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-5

After solving Eigen value is as follows:

Step-6

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Step-7

Another Eigen value to calculate Eigen vector is:

$$\lambda = \frac{1}{10}$$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} \frac{2}{5}-\frac{1}{10} & \frac{3}{10} \\ \frac{3}{5} & \frac{7}{10}-\frac{1}{10} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{10} & \frac{3}{10} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-8

After solving Eigen value is as follows:

Step-9

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-10

Following is the Eigen vector matrix.

$$S = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix}$$

Step-11

Now, calculate the following:

$$\begin{aligned} u_{k+1} &= S\Lambda^k S^{-1} \cdot u_0 \\ &= \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

Step-12

Taking the limit $k \to \infty$ makes the element $(1/10)^k$ very small, so neglect it.

$$\begin{aligned} u_\infty &= \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} a+b \\ 2a+2b \end{bmatrix} \end{aligned}$$

Step-13

Therefore, the limit of $A^k u_0$ is as follows:

$$\boxed{\frac{1}{3}\begin{bmatrix} a+b \\ 2a+2b \end{bmatrix}}$$