Step-1

$$A_1 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Applying row operations on this, we can write $\xrightarrow[R_3 \leftrightarrow R_4/4]{R_2/2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is the reduced matrix.

So, the number of non zero rows in this matrix = 3 is the dimension of the row space

In other words, any three non zero rows of the given matrix span the row space of A_1.

Step-2

To find the row null space of A_1, we solve the homogeneous system $A_1 x = 0$ using the reduced matrix.

i.e., $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Writing the equations from below, we get

$$\begin{aligned} &x_4 = 0 \\ &x_2 + x_3 + x_4 = 0 \\ &x_1 + 2x_2 + 3x_4 = 0 \end{aligned}$$

Consequently,

$$\begin{aligned} &x_3 = -x_2 \\ &x_1 = -2x_2 \end{aligned}$$

Putting $x_2 = k$ a parameter, we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

Putting k = -1, the row null space of A_1 is spanned by $\left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}\right\}$

Step-3

Further, we transpose the given matrix and reduce it. $A_1^T = \begin{bmatrix}1&0&0&0\\2&2&0&0\\0&2&0&0\\3&2&0&4\end{bmatrix}$

$$\xrightarrow[R_4\to R_4-3R_1]{R_2\to R_2-2R_1}\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&2&0&0\\0&2&0&4\end{bmatrix}$$

$$\xrightarrow[R_4\to R_4-R_2]{R_3\to R_3-R_2}\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&4\end{bmatrix}$$

$$\xrightarrow[R_3\leftrightarrow R_4/4]{R_2/2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Step-4

This is the reduced matrix. So, the number of non zero rows = 3

So, the column space of A_1 is spanned by any three non zero columns of A_1

To get the column null space of A_1, we solve the homogeneous system $A_1^T x = 0$ using the reduced matrix.

i.e., $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Writing the equations, we get $x_1 = 0, x_2 = 0, x_4 = 0$

From this, we get $x_3 = k$ is any parameter.

The solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix}$

Putting $k = 1$, the column null space of A_1 is spanned by $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Step-5

$$A_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

Row reduction is $\begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

So, row space of A_2 is spanned by $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$

Step-6

To get the row null space of A_2, we solve $\begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 4x_2 = 0$$

Putting $x_2 = k$ a parameter, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Putting $k = 1$, the row null space of A_2 is spanned by $\left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$

Step-7

We now transpose A_2 and find the other two.

$$A_2^T = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \end{bmatrix}$$

Using row operations, we reduce it to $\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So, the rank of A_2^T is the number of non zero rows in the reduced matrix = 1

Therefore, the column space of A_2 is spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Step-8

To get the column null space, we solve the homogeneous system $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

Using $x_2 = k, x_3 = m,$ parameters, we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + m \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Therefore, the column null space of A_2 is spanned by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$