Step-1

We have to describe the set of attainable right-hand sides b (in the column space) for

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, by finding the constraints on b that turn the third equation into $0 = 0$(after elimination), also we have to find the rank, and a particular solution.

Step-2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 - 3b_2 \end{bmatrix}$$

Therefore the system is consistent if $b_3 - 2b_1 - 3b_2 = 0$

Step-3

Let the given system is $Ax = b$. Then if is converted to $Rx = C$ where R is reduced echelon form, therefore the rank of A is 2.

Since the system is consistent if $b_3 - 2b_1 - 3b_2 = 0$,

let $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

Step-4

Therefore $Ax = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ is converted to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u=1, v=1$$

$$\Rightarrow \begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore $(1,1)$ is a particular solution of given system.