Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

General solution is given as follows:

$$u(t) = c_1 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find the matrix A.

Step-2

General solution can be written as follows:

$$\begin{aligned} u(t) &= c_1 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= Se^{\Lambda t}c \end{aligned}$$

Step-3

Recall that $A = S\Lambda S^{-1}$. Therefore,

$$\begin{aligned} A &= S\Lambda S^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 6 \\ -3 & 8 \end{bmatrix} \end{aligned}$$

Step-4

Therefore,

$$\boxed{A = \begin{bmatrix} -1 & 6 \\ -3 & 8 \end{bmatrix}}$$