

Step-1

Consider the matrix A_1 as shown below:

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$$

In this matrix,

$$a = 5$$

$$b = 6$$

$$c = 7$$

Since $a > 0$ but $ac < b^2$, the matrix A_1 has one positive and one negative eigenvalue.

Step-2

Consider the matrix A_2 as shown below:

$$A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$$

In this matrix,

$$a = -1$$

$$b = -2$$

$$c = -5$$

Since $a < 0$ and $ac > b^2$, the matrix A_2 has negative eigenvalues.

Step-3

Consider the matrix A_3 as shown below:

$$A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$$

In this matrix,

$$a = 1$$

$$b = 10$$

$$c = 100$$

Since $a > 0$ and $ac = b^2$, the matrix A_3 has one positive and one zero eigenvalue.

Step-4

Consider the matrix A_4 as shown below:

$$A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$$

In this matrix,

$$a = 1$$

$$b = 10$$

$$c = 101$$

Since $a > 0$ and $ac > b^2$, the matrix A_4 has positive eigenvalues.

Step-5

We want a vector x , such that $x^T A_4 x < 0$. Let $x = (x_1, x_2)^T$.

Therefore, we get

$$\begin{aligned} x^T A_4 x &< 0 \\ (x_1, x_2) \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &< 0 \\ (x_1, x_2) \begin{bmatrix} 5x_1 + 6x_2 \\ 6x_1 + 7x_2 \end{bmatrix} &< 0 \\ 5x_1^2 + 6x_1x_2 + 6x_1x_2 + 7x_2^2 &< 0 \end{aligned}$$

This gives $5x_1^2 + 12x_1x_2 + 7x_2^2 < 0$. Note that the quantities $5x_1^2$ and $7x_2^2$ cannot be negative. Therefore, in order to satisfy this inequality, the necessary condition is that exactly one of the quantities x_1 and x_2 should be negative and the other should be positive.

Fix $x_1 = a$ and let $x_2 = x$. We will vary x so that $5a^2 + 12ax + 7x^2 < 0$.

Observe the following:

$$\begin{aligned} 5a^2 + 12ax + 7x^2 &< 0 \\ 7x^2 + 12ax + 5a^2 &< 0 \\ (\sqrt{7}x)^2 + 2(\sqrt{7})\left(\frac{6}{\sqrt{7}}\right)ax + \frac{36a^2}{7} - \frac{a^2}{7} &< 0 \end{aligned}$$

$$\begin{aligned} (\sqrt{7}x)^2 + 2(\sqrt{7})\left(\frac{6}{\sqrt{7}}\right)ax + \frac{36a^2}{7} &< \frac{a^2}{7} \\ \left(x\sqrt{7} + \frac{6a}{\sqrt{7}}\right)^2 &< \left(\frac{a}{\sqrt{7}}\right)^2 \end{aligned}$$

Therefore, whenever $x\sqrt{7} + \frac{6a}{\sqrt{7}} < \frac{a}{\sqrt{7}}$, we are through. That is, $x\sqrt{7} < -\frac{5a}{\sqrt{7}}$. This is same as $x < -\frac{5a}{7}$.

Step-6

Therefore, if $x_1 = a$, x_2 should be less than $-\frac{5a}{7}$. Let $x_1 = 7$ and $x_2 = -6$. Now observe the following:

$$\begin{aligned} x^T A_1 x &= (7, -6) \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} \\ &= (7, -6) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &= -7 \end{aligned}$$

Thus, we have obtained $\boxed{x^T A_1 x < 0}$.