

Step-1

Consider the matrix,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

The objective is to find $PA = LDU$ factorization.

Step-2

In the matrix A , the pivot is zero, 0.

So, to change the pivot entry, exchange the first and second rows so that the entry 1 in the second row moves into the pivot.

For this row exchange, use the permutation matrix P which is the same as an identity matrix.

So, the PA matrix will be as follows:

$$\begin{aligned} PA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \end{aligned}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here, the permutation matrix

Step-3

Find the lower and upper (L and U) triangular matrices for the matrix PA .

$$\text{Find the upper triangular matrix } U \text{ for the matrix } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Step 1: Subtract two times the first row from the third row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

Step 2: Subtract three times the second row from the third row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore, the upper triangular matrix U is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Step-4

Find the lower triangular matrix L for the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

To find the matrix L , do the same row operations reversely on the identity matrix I .

Step 1: Add two times the first row from the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Step 2: Add three times the second row from the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 3R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Therefore, the lower triangular matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$.

Step-5

Find the matrix D :

Use the fact that D is the diagonal matrix of pivots, and the pivots are the diagonal entries of the upper triangular matrix U .

Here, the pivots are $1, 1, -1$ because, $1, 1, -1$ are the diagonal entries of the upper triangular matrix U .

So, the matrix D is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Step-6

The LU factorization is the following:

$$PA = LU$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Step-7

To make the diagonal entries all 1 in the matrix U , one needs to split LU into LDU such that $DU' = U$ as follows:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Here, L and U are the lower and upper triangular matrices which contain 1 on diagonal

D is the diagonal matrices which consists pivots 1, 1,-1 on the diagonal.

Therefore, write the $PA = LDU$ factorizations as follows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step-8

Check:

Find LDU :

$$\begin{aligned}
 LDU &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\
 &= PA
 \end{aligned}$$

Therefore, by row column multiplication of the above two sides we get the same matrices.

Step-9

Consider the second matrix,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

In the matrix A , the pivot is 4.

So, to change the pivot entry, exchange the second and third rows, so that the entry 1 in the third row moves into the pivot.

So, PA matrix will be the following:

$$\begin{aligned}
 PA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}
 \end{aligned}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Here, the permutation matrix

Step-10

Find the lower and upper (L and U) triangular matrices for the matrix PA .

Find the upper triangular matrix U for the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$.

Step 1: Subtract one the first row from the second row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 - R_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

Step 2: Subtract two times the first row from the third row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the upper triangular matrix U is the following:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step-11

Find the lower triangular matrix L for the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$:

To find the matrix L , do the same row operations reversely on the identity matrix I .

Step-12

Step 1: Add one the first row to the second row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: Add two times the first row to the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Therefore, the lower triangular matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

Step-13

Find the matrix D :

Use the fact that D is the diagonal matrix of pivots, and the pivots are the diagonal entries of the upper triangular matrix U .

Here, the pivots are $1, -1, 0$ because, $1, -1, 0$ are the diagonal entries of the upper triangular matrix U .

So, the matrix D is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Step-14

The LU factorization is as follows:

$$PA = LU$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To make the diagonal entries all 1 in the matrix U , one needs to split LU into LDU such that $DU' = U$ as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, L and U are the lower and upper triangular matrices, which contain 1 on diagonal

D is the diagonal matrices which consists pivots $1, -1, 0$ on the diagonal.

Therefore, $PA = LDU$ factorization is as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step-15

Check:

Find LDU :

$$\begin{aligned} LDU &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \\ &= PA \end{aligned}$$

Therefore, by row column multiplication of the above two sides we get the same matrices.