Step-1

The objective is to determine the left inverse or right inverse.

Step-2

Consider the matrix $A=\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$, $M=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$, and $T=\begin{bmatrix}a&b\\0&a\end{bmatrix}$.

First, consider the matrix $A=\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$. This matrix is an echelon form. So, the rank of matrix A is the number of its non-zero rows

Therefore, the rank is $r(A)=2$.

Therefore, a right inverse of the matrix has a dimension (3×2). It does not have a left inverse.

Now, compute the right inverse matrix $[B]$.

$AB=I$

Where the term I is the identity matrix.

So,

$$
\begin{aligned}
AB&=I\\
\begin{bmatrix}1&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\\b_{31}&b_{32}\end{bmatrix}&=\begin{bmatrix}1&0\\0&1\end{bmatrix}\\
\begin{bmatrix}b_{11}+b_{21}&b_{12}+b_{22}\\b_{21}+b_{31}&b_{22}+b_{32}\end{bmatrix}&=\begin{bmatrix}1&0\\0&1\end{bmatrix}
\end{aligned}
$$

Then,

$$
\begin{aligned}
b_{11}+b_{21}&=1\\
b_{21}+b_{31}&=0\\
b_{12}+b_{22}&=0\\
b_{22}+b_{32}&=1
\end{aligned}
$$

There are four equation and six unknown variables.

Step-3

Now, evaluate the value of all six variables.

$$
\begin{aligned}
b_{11} &= 1 \\
b_{21} &= 0 \\
b_{31} &= 0 \\
b_{32} &= 1
\end{aligned}
$$

And

$$
\begin{aligned}
b_{22} &= 0 \\
b_{12} &= 0
\end{aligned}
$$

The matrix B is:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Verified the values,

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}
$$

Hence, **itâ€™s verified** and the right-inverse of A is $\boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}$.

Step-4

Now, consider the matrix M, $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ to reduce the matrix by use of echelon form.

Replace $R_2 \to R_2 - R_1$

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Now, replace $R_3 \to R_3 - R_1$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, the rank is $r(M) = 2$ that is equal to the number of the column that is $r = n$. It consists of left inverse matric, not a right inverse matrix.

Step-5

Now, compute the left inverse matrix by $CM = I$ and the matrix M is the transpose form of a matrix A.

$$M = A^T$$

Step-6

So,

$$CM = I$$
$CA^T = I$ here, $CA^T = AC^T$

Then,

$$AC^T = AB$$
$$C^T = B$$

Thus, the left inverse matrix is the transpose form of a matrix B.

$$C^T = B$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the left inverse matrix is $\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$.

Step-7

Now, verify the result.

$$\begin{aligned} CM &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Therefore, **itâ€™s verified.**

Step-8

Now consider the third matrix, $T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. If $a = 0$ then T has neither left inverse nor right inverse. If $a = b$ then,

$$\begin{aligned} T &= \begin{bmatrix} a & a \\ 0 & a \end{bmatrix} \\ &\underline{R_2 - R_1}\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{rank of } T &= 1 \\ &\neq m, n \end{aligned}$$

So that T has neither left inverse nor right inverse.

If $a \neq b$ then,

$$T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

The invertible matrix is:

$$T^{-1} = \frac{1}{a^2}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$$

Hence, the invertible matrix is $\boxed{T^{-1} = \frac{1}{a^2}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}}$.