[MA303] Partial Differential Equations
2022 Fall Semester Midterm

Name ______________________ **Student ID** ______________________

1. (10 points) For each equation below, find its order, linearity and homogeneity.

(a) $u_{tx} - e^u + u_{xxx} = 0$

(b) $u_t - (a(x)u_x)_x = 0$

(c) $u_x + 2022u_y = xu + \cos y$

(d) $u_{xx}u_{yy} - u_{xy}^2 = 3x + 4y$

2. (8 points) Classify each of the following PDE as hyperbolic, elliptic, or parabolic. If the type changes in the xy-plane, find the region for each type.

(a) $yu_{xx} + u_{yy} - u_x = 0$

(b) $u_{xx} - 2u_{xy}\cos x - e^y u_{yy} = 0$

3. (a) (8 points) Use method of characteristics to find the solution $u(x, y)$ of

$$\begin{cases} u_t - u_x = t, & x > 0, t > 0, \\ u(x, 0) = \phi(x), & x > 0. \end{cases}$$

(b) (4 points) Please draw several characteristic curves and explain why we do not need the boundary condition on the half line $\{x = 0, t > 0\}$.

4. Consider the following initial value problem for Burger's equation

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = \phi(x) = \begin{cases} 0, & x \le -1, \\ 1 + x, & -1 < x \le 0, \\ 1 - x, & 0 < x \le 1, \\ 0, & 1 < x. \end{cases} \end{cases}$$

(a) (4 points) Use the method of characteristics to find an implicit formula for u with **general** initial data $\phi(x)$ in details.

(b) (4 points) Derive the breakdown time t_s in details for the above **special** $\phi(x)$. Hint: Consider the point x_0 such that $\phi'(x_0) < 0$ and see what happens for u_x when t becomes large along the characteristic line issued from x_0 on the x-axis.

(c) (6 points) Solve the above problem (for the above **special** $\phi(x)$) before time t_s.

(d) (4 points) Show that for the above **special** $\phi(x)$,

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x, \quad \forall\, 0 < t < t_s.$$

And explain this physically.

5. Solve the following two eigenvalue problems in details:

(a) (6 points)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, l), \\ X'(0) = 0, \quad X'(l) = 0. \end{cases}$$

(b) (8 points)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, l), \\ X(0) = 0, \quad X'(l) + hX(l) = 0. \end{cases}$$

Here h is a constant that is strictly smaller than $-\frac{1}{l}$. You may write down your solution with the help of graph.

6. (16 points) Solve the following boundary-initial value problem using the method of separation of variables:

$$\begin{cases} u_t - u_{xx} = 0, & x \in (0, \pi), \quad t \in (0, +\infty), \\ u(x, 0) = 0, & x \in (0, \pi), \\ u_x(0, t) = At, \quad u_x(\pi, t) = At, & t \in (0, +\infty). \end{cases}$$

Here A is a constant.

7. Recall the fundamental solution of heat equation (heat kernel):

$$G(x,t;\xi) = \frac{1}{2a\sqrt{\pi t}} \exp(-\frac{(x-\xi)^2}{4a^2 t}), \quad t > 0.$$

(a) (2 points) Write down a solution of the following Cauchy problem for heat equation without proof:

$$\begin{cases} u_t - a^2 u_{xx} = 0, & x \in (-\infty, +\infty), \quad t > 0, \\ u(x,0) = \phi(x), & x \in (-\infty, +\infty). \end{cases}$$

Here $\phi(x)$ is a bounded and continuous function on $\mathbb{R}$.

(b) (4 points) Show that if ϕ is odd and u is bounded in (a), then u is also odd in x for all $t > 0$.

(c) (4 points) If $\phi(x)$ in (a) is nonnegative and positive somewhere, what can you conclude for the bounded solution u for $t > 0$? Use the conclusion to explain that the heat equation has infinite propagation speed.

8. (a) (4 points) State clearly the weak maximum principle for a bounded domain without proof.

(b) (8 points) Let $u(x,t)$ be an nonnegative solution of the following heat equation:

$$u_t - u_{xx} = u, \quad (x,t) \in D_T,$$

where $D_T := \{(x,t)|0 < x < l, 0 < t \le T\}$. Let Γ be its parabolic boundary. Assume there exists a positive constant M such that $u|_\Gamma \le M$. Show that for all $(x,t) \in \overline{D_T}$, we have

$$u(x,t) \le Me^t.$$