Step-1

Orthogonal Matrix: If matrix A is skew- symmetric then e^{At} is an orthogonal matrix.

Skew symmetric: If transpose of matrix A $\left(A^T\right)$ is equal to negative of matrix A, then matrix A is skew-symmetric.

In a conservative system following are observed:

$$\begin{aligned} A^T &= -A \\ \left(e^{At}\right)^T &= e^{-At} \\ \left\|e^{At}u(0)\right\| &= \left\|u(0)\right\| \end{aligned}$$

Step-2

Compute the Eigen values and Eigen vectors of the following matrix:

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

Without any computation comment on the orthogonality of e^{At} and $\left\|u(t)\right\|^2 = u_1^{\ 2} + u_2^{\ 2} + u_3^{\ 2}$ will be constant.

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 0-\lambda & -3 & 0 \\ 3 & 0-\lambda & -4 \\ 0 & 4 & 0-\lambda \end{bmatrix} \\ \det\left(A - \lambda I\right) &= 0 \\ \left(-\lambda\right)^3 + 25\left(-\lambda\right) &= 0 \\ \lambda\left(\lambda^2 + 25\right) &= 0 \end{aligned}$$

After solving following values are obtained:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 5i \\ \lambda_3 &= -5i \end{aligned}$$

Therefore, Eigen values are $\boxed{0, 5i, -5i}$.

Step-4

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 0-\lambda & -3 & 0 \\ 3 & 0-\lambda & -4 \\ 0 & 4 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving, values of x, y and z corresponding to $\lambda=0$ are as follows:

$$x_1=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$=\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda=5i$ is as follows:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 0-\lambda & -3 & 0 \\ 3 & 0-\lambda & -4 \\ 0 & 4 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0-5i & -3 & 0 \\ 3 & 0-5i & -4 \\ 0 & 4 & 0-5i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-6

On solving values of x, y and z are as follows:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -3/4 \\ 5i/4 \\ 1 \end{bmatrix}$$

Similarly, Eigen vectors corresponding to Eigen value $\lambda = -5i$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 0-\lambda & -3 & 0 \\ 3 & 0-\lambda & -4 \\ 0 & 4 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0+5i & -3 & 0 \\ 3 & 0+5i & -4 \\ 0 & 4 & 0+5i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -3/4 \\ -5i/4 \\ 1 \end{bmatrix}$$

Step-7

Therefore Eigen values are:

$$x_1 = \boxed{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}$$
$$x_2 = \boxed{\begin{bmatrix} -3/4 \\ 5i/4 \\ 1 \end{bmatrix}}$$
$$x_3 = \boxed{\begin{bmatrix} -3/4 \\ -5i/4 \\ 1 \end{bmatrix}}$$

Step-8

Next:

$$\begin{aligned}A^T &= \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} \\ &= -A\end{aligned}$$

Matrix A is skew-symmetric so, e^{At} will be an orthogonal matrix.

Step-9

Now,

$$\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$$

Differentiate the following:

$$\frac{d\|u(t)\|^2}{dt} = 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$$

Step-10

Do the following calculations for right hand side as given below:

$$\begin{aligned}\frac{du}{dt} &= Au \\ &= \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\end{aligned}$$

Calculate the following:

$$\begin{aligned}u_1' &= -3u_2 \\ u_2' &= 3u_1 - 4u_3 \\ u_3' &= 4u_2\end{aligned}$$

Now,

$$u_1u_1' = -3u_2u_1$$
$$u_2u_2' = 3u_1u_2 - 4u_3u_2$$
$$u_3u_3' = 4u_2u_2$$
$$u_1u_1' + u_2u_2' + u_3u_3' = 0$$

Substitute the above result, to get derivate of $\|u(t)\|^2$ equal to zero. This shows that $\|u(t)\|^2$ is constant.

Recall that $u(t) = e^{At}u(0)$, then

$$\begin{aligned} \|u(t)\|^2 &= \|e^{At}u(0)\|^2 \\ &= \|u(0)\|^2 \\ &= \text{constant} \end{aligned}$$

Step-11

Therefore, $\boxed{\|u(t)\|^2 = \|u(0)\|^2}$