2021/2022 Fall Final A: MA215 – Solutions

1. (i) By the property $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$, we have

$$1 = \int_0^\infty \int_0^\infty ce^{-2x-y}dxdy = \frac{c}{2} \quad [\textbf{2 marks}] \quad \Rightarrow \mathbf{c = 2}. \quad [\textbf{2 marks}]$$

(ii) If $x < 0$, $f_X(x) = 0$. [**1 mark**]
If $x > 0$, $f_X(x) = \int_0^\infty f(x,y)dy = \int_0^\infty 2e^{-2x-y}dy = 2e^{-2x}$. [**1 mark**]
Hence, $f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ and $X \sim Exp(2)$. [**2 marks**]
Similarly, if $y < 0$, $f_Y(y) = 0$. [**1 mark**]
If $y > 0$, $f_Y(y) = \int_0^\infty f(x,y)dx = \int_0^\infty 2e^{-2x-y}dx = e^{-y}$. [**1 mark**]
Hence, $f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & y \le 0 \end{cases}$ and $Y \sim Exp(1)$. [**2 marks**]
Since $f_X(x)f_Y(y) = f(x,y)$, X are Y independent. [**2 marks**]

(iii) By the independence of X and Y established in (ii), we have

$$\begin{aligned} \mathbb{P}(X > 2, Y < 1) &= \mathbb{P}(X > 2)\mathbb{P}(Y < 1) \quad [\textbf{1 mark}] \\ &= \int_2^\infty 2e^{-2x}dx \int_0^1 e^{-y}dy \quad [\textbf{1 mark}] \\ &= e^{-4}(1 - e^{-1}). \quad [\textbf{2 marks}] \end{aligned}$$

Define $D := \{(x,y) \in \mathbb{R} : \ 0 < x < y, y > 0\}$. Then

$$\begin{aligned} \mathbb{P}(X < Y) &= \int\int_D f(x,y)dxdy \quad [\textbf{1 mark}] \\ &= \int_0^\infty \int_0^y 2e^{-2x-y}dxdy \quad [\textbf{1 mark}] \\ &= \int_0^\infty e^{-y}(-e^{-2y} + 1)dy = \frac{2}{3}. \quad [\textbf{2 marks}] \end{aligned}$$

(vi) If $z \le 0$, $F_Z(z) = \mathbb{P}(X/Y \le 0) = 0$. [**2 marks**]

If $z > 0$, define $B := \{(x, y) \in \mathbb{R} : \ x > 0, y > 0, x/y \le z\}$. Then

$$\begin{aligned} F_Z(z) =& \mathbb{P}(X/Y \le z) = \int\int_B 2e^{-2x-y} dxdy \quad [\textbf{1 mark}] \\ =& \int_0^\infty \int_0^{zy} 2e^{-2x-y} dxdy \quad [\textbf{1 mark}] \\ =& \int_0^\infty e^{-y}(-e^{-2yz}+1)dy = 1 - \frac{1}{2z+1} \ [\textbf{2 marks}] \\ \Rightarrow f_Z(z) =& F_Z'(z) = \frac{2}{(2z+1)^2}. \quad [\textbf{2 marks}] \end{aligned}$$

Hence $f_Z(z) = \begin{cases} \frac{2}{(2z+1)^2}, & z > 0; \\ 0, & otherwise. \end{cases}$

2. (i) $E(X) = M_X'(0) = 2(16-t)^{-3/2}|_{t=0} = \frac{1}{32}$, **[2 marks]**
$E(X^2) = M_X''(0) = 3(16-t)^{-5/2}|_{t=0} = \frac{3}{1024}$, **[2 marks]**
and $Var(X) = E(X^2) - (E(X))^2 = \frac{1}{512}$. **[2 marks]**

(ii) Since X and Y are independent, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \frac{16}{16-t}$, $t < 16$. **[2 marks]**
So $X + Y \sim Exp(16)$. **[2 marks]**

(iii) Note that $X + Y \sim Exp(16)$, then for $z \in \mathbb{R}$,

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z < z) = \mathbb{P}(\ln(X+Y) < z) = \mathbb{P}(X+Y < e^z) \quad [\textbf{1 mark}] \\ &= \int_{-\infty}^{e^z} f_{X+Y}(x)dx = \int_0^{e^z} 16e^{-16x}dx \quad [\textbf{2 marks}] \\ \Rightarrow \quad f_Z(z) &= F_Z'(z) \quad [\textbf{1 mark}] \\ &= 16e^{-16e^z} \cdot (e^z)' = 16\exp(z - 16e^z). \quad [\textbf{1 mark}] \end{aligned}$$

(iv) By the definition of m.g.f., we have

$$\begin{aligned} M_Z(t) &= \mathbb{E}e^{Zt} = \mathbb{E}e^{t\ln(X+Y)} = \mathbb{E}(X+Y)^t \quad [\textbf{1 mark}] \\ &= \int_{-\infty}^\infty x^t f_{X+Y}(x)dx = \int_0^\infty x^t 16e^{-16x}dx \quad [\textbf{1 mark}] \\ &= 16^{-t}\int_0^\infty y^{(t+1)-1}e^{-y}dy \quad (\text{let } 16x = y) \quad [\textbf{1 mark}] \\ &= 16^{-t}\Gamma(t+1) (= t16^{-t}\Gamma(t) \quad \text{as } \Gamma(t+1) = t\Gamma(t)). \quad [\textbf{2 marks}] \end{aligned}$$

3. (i) Since $X_i \sim N(2,5)$, we have $M_{X_i} = \exp(2t + \frac{5t^2}{2})$ for all i. The independence of $\{X_1, X_2, \ldots\}$ implies that

$$M_{S_n}(t) = \Pi_{i=1}^n M_{S_n}(t) = \Pi_{i=1}^n \exp(2t + \frac{5t^2}{2}) = \exp(2nt + \frac{5nt^2}{2})$$

$$\Rightarrow \quad S_n \sim N(2n, 5n) \quad \text{for all } n \in \mathbb{N}_+.$$

So $S_{10} \sim N(20, 50)$, $S_{20} \sim N(40, 100)$. **[2 marks]**
Similarly,

$$M_{\overline{X}_n}(t) = M_{S_n}(t/n) = \exp(2t + \frac{5t^2}{2n})$$

$$\Rightarrow \quad \overline{X}_n \sim N(2, 5/n) \quad \text{for all } n \in \mathbb{N}_+.$$

So $\overline{X}_{20} \sim N(2, 1/4)$. **[1 mark]**

(ii) Method I: By (i) we know $\overline{X}_{20} \sim N(2, 1/4)$. So

$$\begin{aligned}\mathbb{P}(1.5 < \overline{X}_{20} < 2.49) &= \mathbb{P}\left(\frac{1.5-2}{1/2} < \frac{\overline{X}_{20}-2}{1/2} < \frac{2.49-2}{1/2}\right) \quad \textbf{[1 mark]} \\ &= \Phi(0.98) - \Phi(-1) = \Phi(0.98) + \Phi(1) - 1 \quad \textbf{[1 mark]} \\ &\approx 0.6778. \quad \textbf{[1 mark]}\end{aligned}$$

Method II: By the central limit theorem, we have

$$\begin{aligned}&\mathbb{P}(1.5 < \overline{X}_{20} < 2.49) \\ &= \mathbb{P}\left(\frac{1.5 \times 20 - 20 \times 2}{\sqrt{20 \times 5}} < \frac{\sum_{i=1}^{20} X_i - 20 \times 2}{\sqrt{20 \times 5}} < \frac{2.49 \times 20 - 20 \times 2}{\sqrt{20 \times 5}}\right) \quad \textbf{[1 mark]} \\ &= \mathbb{P}\left(\frac{-10}{10} < \frac{\sum_{i=1}^{20} X_i - 40}{10} < \frac{9.8}{10}\right) \approx \Phi(0.98) - \Phi(-1) \quad \textbf{[1 mark]} \\ &= \Phi(0.98) + \Phi(1) - 1 \approx 0.6778. \quad \textbf{[1 mark]}\end{aligned}$$

By the central limit theorem, we have

$$\begin{aligned}\mathbb{P}(S_{20} \geq 49) &= \mathbb{P}\left(\frac{S_{20} - 20 \times 2}{\sqrt{20 \times 5}} \geq \frac{49 - 20 \times 2}{\sqrt{20 \times 5}}\right) \quad \textbf{[1 mark]} \\ &= 1 - \mathbb{P}\left(\frac{S_{20} - 40}{10} < \frac{9}{10}\right) \quad \textbf{[1 mark]} \\ &\approx 1 - \Phi(0.9) \approx 0.1841. \quad \textbf{[1 mark]}\end{aligned}$$

(iii) By the (i) we know $Var(S_{20}) = 100$, $Var(S_{10}) = 50$, then

$$\begin{aligned}
\sigma(S_{10}, S_{20}) &= \frac{Cov(S_{10}, S_{20})}{\sqrt{Var(S_{10})}\sqrt{Var(S_{20})}} \quad [\textbf{1 mark}] \\
&= \frac{Cov\left(S_{10}, S_{10} + \sum_{i=11}^{20} X_i\right)}{50\sqrt{2}} \quad [\textbf{1 mark}] \\
&= \frac{Var(S_{10})}{50\sqrt{2}} = \frac{\sqrt{2}}{2} > 0. \quad [\textbf{1 mark}]
\end{aligned}$$

So S_{10} and S_{20} are (positively) correlated. **[2 marks]**

(iv) By the define of moment generating function, we have

$$M_{\overline{X}_n}(t) = M_{S_n}(t/n) = \exp(2t + \frac{5t^2}{2n}), \quad [\textbf{2 marks}]$$

and

$$\begin{aligned}
M_{Z_n}(t) = \mathbb{E}e^{Z_n t} &= \mathbb{E}\exp\left(\frac{t\overline{X}_n}{\sqrt{5/n}} - \frac{2t}{\sqrt{5/n}}\right) \quad [\textbf{1 mark}] \\
&= \exp\left(-\frac{2t}{\sqrt{5/n}}\right)\mathbb{E}\exp\left(\frac{t\overline{X}_n}{\sqrt{5/n}}\right) \quad [\textbf{1 mark}] \\
&= \exp\left(-\frac{2t}{\sqrt{5/n}}\right)M_{\overline{X}_n}\left(\frac{t}{\sqrt{5/n}}\right) \quad [\textbf{1 mark}] \\
&= \exp(t^2/2). \quad [\textbf{1 mark}]
\end{aligned}$$

(v) By (iv), we have $M_{\overline{X}_n}(t) = \exp(2t + \frac{5t^2}{2n}), M_{Z_n}(t) = \exp(t^2/2)$. Then for any $t \in \mathbb{R}$,

$$\begin{aligned}
\phi(t) &= \lim_{n\to\infty} M_{\overline{X}_n}(t) = \lim_{n\to\infty} \exp(2t + \frac{5t^2}{2n}) = \exp(2t), \quad [\textbf{1 mark}] \\
\psi(t) &= \lim_{n\to\infty} M_{Z_n}(t) = \lim_{n\to\infty} \exp(t^2/2) = \exp(t^2/2). \quad [\textbf{1 mark}]
\end{aligned}$$

So, the m.g.f. $\phi(t)$ corresponds to a constant r.v. with a constant value 2, **[1 mark]** the m.g.f. $\psi(t)$ corresponds to a normal distributed r.v. with mean 0 and variance 1. **[1 mark]**

(vi) Note that $\mathbb{E}\overline{X}_n = E\frac{1}{n}\sum_{i=1}^n X_i = 2$, and $Var\overline{X}_n = Var\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{VarX_i}{n} = \frac{5}{n}$, **[1 mark]** then by Chebychev inequality,

$$\mathbb{P}(|\overline{X}_n - 2| \geq \varepsilon) \leq \frac{Var\overline{X}_n}{\varepsilon^2} = \frac{5}{n\varepsilon^2} \quad [\textbf{2 mark}]$$

$$\Rightarrow \quad \mathbb{P}(|\overline{X}_n - 2| < \varepsilon) = 1 - \mathbb{P}(|\overline{X}_n - 2| \geq \varepsilon) \geq 1 - \frac{5}{n\varepsilon^2}. \quad [\textbf{1 mark}]$$

So $\lim_{n\to\infty} \mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} \geq \lim_{n\to\infty}(1 - \frac{5}{n\varepsilon^2}) = 1.$ **[1 mark]**
As $\mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} \leq 1$, we have $\lim_{n\to\infty} \mathbb{P}\{|\overline{X}_n - 2| < \varepsilon\} = 1$.
[1 mark]

4. (i) By the total probability formula and the independence of X and Y, we have

$$\begin{aligned}
\mathbb{P}(X > Y) &= 1 - \mathbb{P}(X \leq Y) && [\textbf{1 mark}] \\
&= 1 - \sum_j \mathbb{P}(X \leq Y | Y = y_j)\mathbb{P}(Y = y_j) && [\textbf{2 marks}] \\
&= 1 - \sum_j \mathbb{P}(X \leq y_j)\mathbb{P}(Y = y_j) && [\textbf{1 mark}] \\
&= 1 - \sum_j F_X(y_j) f_Y(y_j). && [\textbf{1 mark}]
\end{aligned}$$

$$\begin{aligned}
\text{Or} \quad \mathbb{P}(X > Y) &= \sum_j \mathbb{P}(X > Y | Y = y_j)\mathbb{P}(Y = y_j) && [\textbf{2 marks}] \\
&= \sum_j \mathbb{P}(X > y_j)\mathbb{P}(Y = y_j) && [\textbf{1 mark}] \\
&= \sum_j (1 - \mathbb{P}(X \leq y_j))\mathbb{P}(Y = y_j) && [\textbf{1 mark}] \\
&= \sum_j (1 - F_X(y_j)) f_Y(y_j). && [\textbf{1 mark}]
\end{aligned}$$

(ii) By the total probability formula and the independence of X and

Y, we have

$$
\begin{aligned}
F_{X+Y}(Z) &= \mathbb{P}(X+Y \leq z) \\
&= \sum_j \mathbb{P}(X+Y \leq z | Y = y_j)\mathbb{P}(Y = y_j) \qquad [\textbf{2 marks}] \\
&= \sum_j \mathbb{P}(X \leq z - Y | Y = y_j)\mathbb{P}(Y = y_j) \qquad [\textbf{1 mark}] \\
&= \sum_j \mathbb{P}(X \leq z - y_j)\mathbb{P}(Y = y_j) \qquad [\textbf{1 mark}] \\
&= \sum_j F_X(z - y_j) f_Y(y_j). \qquad [\textbf{1 mark}]
\end{aligned}
$$

5. (i) By the total expectation formula, we have

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|Y]] \quad [\textbf{1 mark}] \\
&= \mathbb{E}[X|Y=1]\mathbb{P}(Y=1) + \mathbb{E}[X|Y=2]\mathbb{P}(Y=2) \\
&\quad + \mathbb{E}[X|Y=3]\mathbb{P}(Y=3) + \mathbb{E}[X|Y=4]\mathbb{P}(Y=4) \quad [\textbf{1 mark}] \\
&= 1/4\,(2 + 3 + \mathbb{E}[X] + 4 + \mathbb{E}[X] + 5 + \mathbb{E}[X]) \quad [\textbf{1 mark}]
\end{aligned}
$$

$$\Rightarrow \mathbb{E}[X] = 14. \quad [\textbf{2 mark}]$$

(ii) Possible values of X are: 2,5,6,7,9,10,11,14, and we have

$$
\begin{aligned}
&\mathbb{P}[X=2] = \frac{1}{4}; \quad [\textbf{1 mark}] \\
&\mathbb{P}[X=5] = \mathbb{P}[X=6] = P[X=7] = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}; \quad [\textbf{1 mark}] \\
&\mathbb{P}[X=9] = \mathbb{P}[X=10] = \mathbb{P}[X=11] = C_2^1 \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{12}; \quad [\textbf{1 mark}] \\
&\mathbb{P}[X=14] = C_3^1 \times \frac{1}{4} \times C_2^1 \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{4}. \quad [\textbf{1 mark}] \\
&\Rightarrow \quad \mathbb{E}[X] = \frac{1}{4} \times 2 + \frac{1}{12} \times (5+6+7) + \frac{1}{12} \times (9+10+11) + \frac{1}{4} \times 14 \\
&\qquad\qquad = 8. \quad [\textbf{1 mark}]
\end{aligned}
$$