

Step-1

The objective is to decide the stability of $u' = Au$ for the given matrices.

We know that the differential equation $\frac{du}{dt} = Au$ is

stable, when all $\operatorname{Re} \lambda_i < 0$

neutrally stable, when all $\operatorname{Re} \lambda_i \leq 0$ and $\operatorname{Re} \lambda_i = 0$

unstable and e^{At} is unbounded if any Eigen values has $\operatorname{Re} \lambda_i > 0$.

Step-2

(a)

Consider $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, we have to find Eigen values of matrix A .

To find the Eigen values of matrix A , we have to solve for $\det(A - \lambda I) = 0$.

$$\begin{aligned}\det(A - \lambda I) &= 0 \\ \begin{vmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(5-\lambda) - 12 &= 0 \\ \lambda^2 - 7\lambda - 2 &= 0\end{aligned}$$

Step-3

Solve the equation $\lambda^2 - 7\lambda - 2 = 0$, using the following formula.

$$\begin{aligned}\lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \lambda &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{7 \pm \sqrt{49+8}}{2} \\ &= \frac{7 \pm \sqrt{57}}{2}\end{aligned}$$

Solving the above equation, we get λ_1 and λ_2 as

$$\lambda_1 = \frac{7 + \sqrt{57}}{2}$$

$$\lambda_2 = \frac{7 - \sqrt{57}}{2}$$

Step-4

Hence, the Eigen values of matrix A are $\lambda_1 = \frac{7 + \sqrt{57}}{2}$ and $\lambda_2 = \frac{7 - \sqrt{57}}{2}$.

Since the Eigen value $\lambda_1 = \frac{7 + \sqrt{57}}{2}$ is greater than 0, that is $\text{Re } \lambda_1 > 0$, so the differential equation $\frac{du}{dt} = Au$ is unstable.

Step-5

(b)

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, we have to find Eigen values of matrix A .

To find the Eigen values of matrix A , we have to solve for $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 1 - \lambda & 2 \\ 3 & -1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(-1 - \lambda) - 6 = 0$$

$$-1 - \lambda + \lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 = 7$$

Step-6

Taking square root of both side of equation $\lambda^2 = 7$, we get.

$$\lambda = \pm\sqrt{7}$$

Hence, the Eigen values of matrix A are $\lambda_1 = \sqrt{7}$ and $\lambda_2 = -\sqrt{7}$.

Since the Eigen value $\lambda_1 = \sqrt{7}$ is greater than 0, that is $\text{Re } \lambda_1 > 0$, so the differential equation $\frac{du}{dt} = Au$ is unstable.

Step-7

(c)

Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, we have to find Eigen values of matrix A .

To find the Eigen values of matrix A , we have to solve for $\det(A - \lambda I) = 0$.

$$\begin{aligned}\det(A - \lambda I) &= 0 \\ \begin{vmatrix} 1-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(-2-\lambda) - 1 &= 0 \\ \lambda^2 + \lambda - 3 &= 0\end{aligned}$$

Solve the equation $\lambda^2 + \lambda - 3 = 0$, using the following formula.

$$\begin{aligned}\lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \lambda &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-3)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{13}}{2}\end{aligned}$$

Solving the above equation, we get λ_1 and λ_2 as

$$\begin{aligned}\lambda_1 &= \frac{-1 + \sqrt{13}}{2} \\ \lambda_2 &= \frac{-1 - \sqrt{13}}{2}\end{aligned}$$

Step-8

Hence, the Eigen values of matrix A are $\lambda_1 = \frac{-1 + \sqrt{13}}{2}$ and $\lambda_2 = \frac{-1 - \sqrt{13}}{2}$.

Since the Eigen value $\lambda_1 = \frac{-1 + \sqrt{13}}{2}$ is greater than 0, that is $\text{Re } \lambda_1 > 0$, so the differential equation $\frac{du}{dt} = Au$ is unstable.

Step-9

(d)

Consider $A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, we have to find Eigen values of matrix A .

To find the Eigen values of matrix A , we have to solve for $\det(A - \lambda I) = 0$.

$$\begin{bmatrix} -1-\lambda & -1 \\ -1 & -1-\lambda \end{bmatrix} = 0$$
$$(-1-\lambda)(-1-\lambda) - 1 = 0$$
$$(-1)(1+\lambda)(-1)(1+\lambda) - 1 = 0$$
$$(1+\lambda)^2 - 1 = 0$$

Step-10

Solve the equation $(1+\lambda)^2 - 1 = 0$, to find the value of λ .

$$(1+\lambda)^2 - 1 = 0$$
$$(1+\lambda)^2 = 1$$
$$1+\lambda = \pm\sqrt{1}$$

Solving the above equation, we get λ_1 and λ_2 as

$$\lambda_1 = 0$$
$$\lambda_2 = -2$$

Step-11

Hence, the Eigen values of matrix A are $\lambda_1 = 0$ and $\lambda_2 = -2$.

Since the Eigen values $\lambda_1 = 0$ and $\lambda_2 = -2$, that is $\operatorname{Re} \lambda_1 \leq 0$ and $\operatorname{Re} \lambda_1 = 0$, so the differential equation $\frac{du}{dt} = Au$ is neutrally stable.