Step-1

Let **V** and **W** be subspaces of some vector space. Let us assume that the zero vector is the only common vector to these spaces.

In such case **V** + **W** is called the direct sum of **V** and **W** and is denoted by $\mathbf{V} \oplus \mathbf{W}$.

Step-2

Let **V** be spanned by $(1,1,1)$ and $(1,0,1)$. We need to find **W** such that $\mathbf{V} \cap \mathbf{W} = \{0\}$.

Since **V** is spanned by two vectors, **V** is a plane in $\mathbf{R}^3$. Therefore, **W** should be a line, perpendicular to **V**.

If (a,b,c) is a vector along **W**, then we get the following:

$$\begin{aligned} (a,b,c)(1,1,1) &= 0 \\ a+b+c &= 0 \\ (a,b,c)(1,0,1) &= 0 \\ a+c &= 0 \end{aligned}$$

Step-3

Thus, we get $a=-c$ and therefore, $b=0$. Thus, the vector $(1,0,-1)$ is perpendicular to the plane, spanned by $(1,1,1)$ and $(1,0,1)$.

Thus, **W** is a line, which contains the vector $(1,0,-1)$.

It is clear that $\mathbf{V} \cap \mathbf{W} = \{0\}$.

Therefore, $\mathbf{R}^3 = \mathbf{V} \oplus \mathbf{W}$.

Step-4

Let if possible, $x = v_1 + w_1$ and $x = v_2 + w_2$, where $v_1, v_2 \in \mathbf{V}$ and $w_1, w_2 \in \mathbf{W}$. Then we get

$$\begin{aligned} 0 &= x - x \\ &= (v_1 + w_1) - (v_2 + w_2) \\ &= v_1 - v_2 + w_1 - w_2 \end{aligned}$$

This is possible, only if $v_1 - v_2 = 0$ and $w_1 - w_2 = 0$. This shows that $v_1 = v_2$ and $w_1 = w_2$. Therefore, when $x = v + w$, where $v \in \mathbf{V}$ and $w \in \mathbf{W}$, then this the unique way to express x as a sum of a vector from **V** and a vector from **W**.