MA215 Probability Theory

Assignment 12

1. The covariance between X and Y, denoted by $\mathrm{Cov}\,(X, Y)$, is defined by

$$\mathrm{Cov}\,(X, Y) \triangleq E[(X - E(X))(Y - E(Y))].$$

Show that

$$\mathrm{Cov}\,(X, Y) = E(XY) - E(X)E(Y).$$

2. Let X be a discrete random variable with p.m.f.

$$P\{X = 0\} = P\{X = 1\} = P\{X = -1\} = \frac{1}{3}.$$

Define

$$Y = \begin{cases} 0, & \text{if} \quad X \neq 0, \\ 1, & \text{if} \quad X = 0. \end{cases}$$

(i) Show that $\mathrm{Cov}\,(X, Y) = 0$.

(ii) Find the joint p.m.f. of X and Y, and show that X and Y are not independent.

3. Show that the following conclusions are true:

(i) $\mathrm{Cov}\,(X, Y) = \mathrm{Cov}\,(Y, X)$;

(ii) $\mathrm{Cov}\,(X, X) = \mathrm{Var}\,(X)$;

(iii) $\mathrm{Cov}\,(aX, Y) = a\mathrm{Cov}\,(X, Y)$, where a is a constant;

(iv) $\mathrm{Cov}\,\left(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{Cov}\,(X_i, Y_j)$;

(v) If X is a random variable and C is a constant, then $\mathrm{Cov}\,(X, C) = 0$.

(vi) Show that the following statements are true:

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}\,(X_i) + \sum_{1 \leqslant i \neq j \leqslant n} \mathrm{Cov}\,(X_i, X_j),$$

or, equivalently,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}\,(X_i) + 2 \sum_{1 \leqslant i < j \leqslant n} \mathrm{Cov}\,(X_i, X_j).$$

Further show that if $X_1, \ldots, X_n$ are pairwise independent (i.e., X_i and X_j are independent for $1 \leqslant i \neq j \leqslant n$), then we have

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}\,(X_i).$$

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common expectation μ and common variance σ^2. Let $\bar{X}$ and S^2 be defined as follows.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2.$$

Find $E[\bar{X}]$, $\mathrm{Var}\,(\bar{X})$, and $E\left[\frac{S^2}{n-1}\right]$.

5. Let I_A and I_B be the indicator variables for the events A and B. That is,

$$I_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases} \qquad I_B(\omega) = \begin{cases} 1, & \omega \in B, \\ 0, & \omega \notin B. \end{cases}$$

Show that

(i) $E[I_A] = P(A)$, $E[I_B] = P(B)$, $E[I_A I_B] = P(AB)$.

(ii) $\mathrm{Cov}\,(I_A, I_B\Big) = P(AB) - P(A)P(B)$.

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common variance σ^2. Show that for any fixed i $(1 \leqslant i \leqslant n)$,

$$\mathrm{Cov}\,(X_i - \bar{X}, \bar{X}) = 0,$$

where $\bar{X}$ is the sample mean (i.e. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$).