Applied stochastic processes
Chapter 3

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Chapter 3. Markov chain: Basics

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3.1. Definitions and examples

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3.1. Definitions and examples

Example (Gambler's ruin)

In a game, you win \$1 with prob. $p = .4$ or lose \$1 w/ prob. $1 - p = .6$. Suppose you will stop if your fortune reaches $\$N$ (or \$0 when you have to).

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Namely, given X_n, the state X_{n+1} is indep. of the past $X_0,\ X_1,\ \cdots,\ X_{n-1}$.

Definition

$\{X_n\}$ is a *Markov chain* if

$$\mathbb{P}(X_{n+1} = j | X_n = i,\ X_{n-1} = i_{n-1},\ \cdots,\ X_0 = i_0) = p(i,j),$$

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Remark

$$\mathbb{P}(X_{n+1} = j | X_n = i) = p(i, j).$$

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$$\sum_j p(i,j) = 1. \qquad \text{(After } X_n = i,\ X_{n+1} \text{ will be at a state } j\text{)}.$$

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Find the transition matrix.

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$$P = \begin{bmatrix} 0 & 0 & .1 & .2 & .4 & .3 \\ 0 & 0 & .1 & .2 & .4 & .3 \\ & & & & & \\ & & & & & \\ & & & & & \\ 0 & 0 & .1 & .2 & .4 & .3 \end{bmatrix}$$

If $X_n = 2$, then $X_{n+1} = (2 - D_{n+1})^+ = 2,\ 1,\ 0,\ 0.$

$$P = \begin{bmatrix} 0 & 0 & .1 & .2 & .4 & .3 \\ 0 & 0 & .1 & .2 & .4 & .3 \\ .3 & .4 & .3 & 0 & 0 & 0 \\ .1 & .2 & .4 & .3 & 0 & 0 \\ 0 & .1 & .2 & .4 & .3 & 0 \\ 0 & 0 & .1 & .2 & .4 & .3 \end{bmatrix}$$

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In the population, each individual gives birth (indep. of others) to k children with prob. p_k, $k = 0, 1, 2, \cdots$.

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Q: What is the extinction probability?

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Q: What is the probability that the population fixates in the all A state? i.e. $X_n=N$?

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So,

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3.2 C-K equation

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$$p(i,j) = \mathbb{P}(X_{n+1} = j | X_n = i).$$

How about m-step transition probabilities? $m > 1$.

$$p^{(m)}(i,j) = \mathbb{P}(X_{n+m} = j | X_n = i).$$

Example (7)

Let $\{X_n\}$ be a MC (social mobility) with states $\{1,2,3\}$ (poor, middle, rich) and transition matrix

$$P = \begin{bmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{bmatrix}$$

Find $p^{(2)}(2,1)$.

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Solution: Do it on board! Recall $\mathbb{P}(AB) = \mathbb{P}(A|B)\mathbb{P}(B)$. So

$$
\begin{aligned}
p^{(2)}(2,1) &= \mathbb{P}(X_2 = 1 | X_0 = 2) \\
&= \sum_{j=1}^{3} \mathbb{P}(X_2 = 1,\ X_1 = j | X_0 = 2) \\
&= \sum_{j=1}^{3} \mathbb{P}(X_2 = 1 | X_1 = j,\ X_0 = 2)\mathbb{P}(X_1 = j | X_0 = 2) \\
&= \sum_{j=1}^{3} \mathbb{P}(X_2 = 1 | X_1 = j)\mathbb{P}(X_1 = j | X_0 = 2) \\
&= \sum_{j=1}^{3} p(2,j)p(j,1) \\
&= .3 \times .7 + .5 \times .3 + .2 \times .2 = .4
\end{aligned}
$$

$\square$

Remark

If the transition matrix is P, then the 2-step transition matrix is P^2, namely,

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Theorem (C-K equation)

The m-step transition matrix is P^m, namely, $P^{(m)} = P^m$.

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3.3. Classification of states

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Notation:

$$\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x) \qquad \text{conditional prob. given } X_0 = x.$$

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$$\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x) \qquad \text{conditional prob. given } X_0 = x.$$

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Note the book use notation: $f_{i,j}$ or f_i.

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$$T_y^1 = T_y, \quad T_y^k = \inf\{n > T_y^{k-1} : \ X_n = y\}.$$

Then,

$$\mathbb{P}_y(T_y^k < \infty) = \rho_{yy}^k.$$

Definition

i) y is *transient* if $\rho_{yy} < 1$.

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Proof: Do it on board!

i)

$$\begin{aligned}\mathbb{P}_y(X_n = y,\ i.o.) &= \mathbb{P}_y(T_y^k < \infty,\ \forall k) \\ &\leq \mathbb{P}_y(T_y^n < \infty) \qquad \forall n.\end{aligned}$$

Thus,

$$\mathbb{P}_y(X_n = y,\ i.o.) \leq \rho_{yy}^n \to 0.$$

ii)

$$\begin{aligned}1 - \mathbb{P}_y(X_n = y,\ i.o.) &= \mathbb{P}_y(\exists k,\ \ T_y^k = \infty) \\ &\leq \sum_k \mathbb{P}_y(T_y^k = \infty) \\ &\leq \sum_k (1 - \rho_{yy}^k) = 0.\end{aligned}$$

$\square$

Example (Gambler's ruin $N = 4$)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 & 0 \\ 0 & .6 & 0 & .4 & 0 \\ 0 & 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Classify all 5 states.

Solution: Do it on board!

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As 0 and 4 are absorbing state, they are clearly recurrent. e.g.

$$\rho_{00} = \mathbb{P}_0(T_0 < \infty) = \mathbb{P}_0(1 < \infty) = 1.$$

Claim: 1, 2, 3 are transient. e.g.

$$\mathbb{P}_1(T_1 = \infty) \geq p(1, 0) = .6$$

so,

$$\rho_{11} = \mathbb{P}_1(T_1 < \infty) \leq 1 - .6 = .4 < 1.$$

The MC gets stuck in either 0 or 4 eventually. □

Example (Social mobility)

$$P = \begin{bmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{bmatrix}$$

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Proof: Do it later!

Now, we will give some general results about state classification.

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Definition

We say that x *communicates with* y, write $x \to y$, if

$$\rho_{xy} \equiv \mathbb{P}_x(T_y < \infty) > 0.$$

Lemma

If $x \to y$ *and* $y \to z$, *then* $x \to z$.

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Proof: Do it on board!

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If $x \to y$ and $y \to z$, then $x \to z$.

Proof: Do it on board!
Since $x \to y$, $\exists$ m s.t. $p^m(x, y) > 0$. Similarly, $\exists$ n, $p^n(y, z) > 0$.
Then,

$$p^{m+n}(x, z) \geq p^m(x, y)p^n(y, z) > 0.$$

Hence, $x \to z$. $\square$

Theorem (14)

If $\rho_{xy} > 0$ and $\rho_{yx} < 1$, then x is transient.

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Proof: Do it on board!

Let k be the smallest s.t. $p^k(x,y) > 0$. Then,

$$0 < p^k(x,y)(1-\rho_{yx}) \leq \mathbb{P}_x(T_x = \infty) = 1 - \rho_{xx}.$$

So, $\rho_{xx} < 1$ and hence, x is transient. $\square$

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Corollary (1.5)

If x recurrent and $\rho_{xy} > 0$, then $\rho_{yx} = 1$.

Example

A seven-state chain

$$P = \begin{bmatrix} .7 & 0 & 0 & 0 & .3 & 0 & 0 \\ .1 & .2 & .3 & .4 & 0 & 0 & 0 \\ 0 & 0 & .5 & .3 & .2 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 & 0 \\ .6 & 0 & 0 & 0 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & .8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

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Classify all states.

Solution: Do it on board!

To identify recurrent and transient states, we draw a graph of 1-step communication.

$$
\begin{tikzcd}
1 \arrow[dd, shift left] & 2 \arrow[l] \arrow[r] \arrow[dd] & 4 \arrow[dd] & \\
 & & & 7 \arrow[ul] \\
5 \arrow[uu, shift left] & 3 \arrow[l] \arrow[uur] & 6 \arrow[ur] &
\end{tikzcd}
$$

$\rho_{21} > 0$ but $\rho_{12} = 0 < 1$, so 2 is transient (by Theorem 14). Similarly, 3 is transient.

We will prove all other states are recurrent later. $\square$

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A set A is *closed* if it is impossible to get out.

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Example (Conti.) $\{1,5\}$, $\{4,6,7\}$, $\{1,5,4,6,7\}$, $\{1,2,3,4,5,6,7\}$ are all closed.

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A set B is *irreducible* if $\forall\ i, j \in B,\ i \to j$.

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Example (Conti.) $\{1, 5\}$, $\{4, 6, 7\}$ are irreducible. $\{1, 5, 4, 6, 7\}$, $\{1, 2, 3, 4, 5, 6, 7\}$ are not.

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If C is a finite closed irreducible set, then all states in C are recurrent.

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Proof: Later.

Example (conti.) Seven-state, $\{1, 5\}$ is closed, irreducible. So, 1 and 5 are recurrent.

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Theorem

If the state space S is finite, then S can be written as a disjoint union

$$S = T \cup R_1 \cup \cdots \cup R_k,$$

where T is a set of transient states and R_i, $1 \leq i \leq k$, are closed irreducible sets of recurrent states.

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where T is a set of transient states and R_i, $1 \leq i \leq k$, are closed irreducible sets of recurrent states.

Proof: Do it on board!

Proof: Let T be the set of all transient states. Then, $S \setminus T$ contains recurrent states. Let $x \in S \setminus T$ and

$$R_1 = \{y \in S \setminus T : \quad x \to y\}.$$

$\forall y \in R_1$, since $\rho_{xy} > 0$ and x recurrent, $\rho_{yx} = 1 > 0$. Hence, $y \to x$.
For any $z \in R_1$, then $y \to x \to z$. Thus, R_1 is irreducible. It is clear that R_1 is closed.
Replacing $S \setminus T$ by $S \setminus (T \cup R_1)$, we can construct R_2.
Eventually, we obtain the desired decomposition. $\square$

Recall

$$T_y^k = \inf\{n > T_y^{k-1} : \; X_n = y\} \text{ and } \rho_{xy} = \mathbb{P}_x(T_y < \infty).$$

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Then

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Let

$$N(y) = \# \text{ of visits to } y \text{ at } n \geq 1.$$

Lemma (23)

$$\mathbb{E}_x N(y) = \frac{\rho_{xy}}{1 - \rho_{yy}}.$$

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Lemma (23)

$$\mathbb{E}_x N(y) = \frac{\rho_{xy}}{1-\rho_{yy}}.$$

Proof: Do it on board! Note that for $\mathbb{Z}_+$-valued r.v. X,

$$\mathbb{E}X = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k).$$

Thus,

$$
\begin{aligned}
\mathbb{E}_x N(y) &= \sum_{k=1}^{\infty} \mathbb{P}_x(N(y) \geq k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}_x(T_y^k < \infty) \\
&= \sum_{k=1}^{\infty} \rho_{xy} \rho_{yy}^{k-1} \\
&= \frac{\rho_{xy}}{1 - \rho_{yy}}.
\end{aligned}
$$

$\square$

Lemma

$$\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} p^n(x,y).$$

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Proof: Do it on board!

As

$$N(y) = \sum_{n=1}^{\infty} 1_{X_n=y},$$

we have

$$\mathbb{E}_x N(y) = \mathbb{E}_x \sum_{n=1}^{\infty} 1_{X_n=y} = \sum_{n=1}^{\infty} \mathbb{P}_x(X_n = y) = \sum_{n=1}^{\infty} p^n(x,y).$$

$\square$

Theorem

y is recurrent iff

$$\sum_{n=1}^{\infty} p^n(y,y) = \mathbb{E}_y N(y) = \infty.$$

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Proof: Do it on board! "$\Rightarrow$" Since $X_n = y$ i.o., $N(y) = \infty$. Thus, $\mathbb{E}_y N(y) = \infty$.
"$\Leftarrow$" By Lemma 20 ,

$$\frac{\rho_{yy}}{1-\rho_{yy}} = \mathbb{E}_y N(y) = \infty.$$

$\rho_{yy} = 1$. $\square$

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Proof: Do it on board!

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Proof: Do it on board! By Cor. 1.4, $\rho_{yx} = 1 > 0$. Let j, ℓ be s.t.

$$p^j(y,x) > 0 \text{ and } p^\ell(x,y) > 0.$$

Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} p^n(y,y) &\geq \sum_{k=0}^{\infty} p^{j+k+\ell}(y,y) \\
&\geq \sum_{k=0}^{\infty} p^j(y,x)p^k(x,x)p^\ell(x,y) \\
&= p^j(y,x)p^\ell(x,y)\mathbb{E}_x N(x) = \infty.
\end{aligned}$$

So, y is recurrent. □

Lemma

In a finite closed set there has to be at lease one recurrent state.

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Lemma

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Proof: Do it on board!

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Lemma

In a finite closed set there has to be at lease one recurrent state.

Proof: Do it on board! Suppose all states in the closed set C are transient. Then, $\mathbb{E}_x N(y) < \infty$, $\forall\ x, y \in C$. Thus

$$\begin{aligned} \infty &> \sum_{y \in C} \mathbb{E}_x N(y) = \sum_{y \in C} \sum_{n=1}^{\infty} p^n(x,y) \\ &= \sum_{n=1}^{\infty} \sum_{y \in C} p^n(x,y) = \sum_{n=1}^{\infty} 1 = \infty. \end{aligned}$$

$\square$

Recall

Theorem (18)

If C is a finite closed irreducible set, then all states in C are recurrent.

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Proof of Theorem 18: Do it on board!

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Theorem (18)

If C is a finite closed irreducible set, then all states in C are recurrent.

Proof of Theorem 18: Do it on board! Let $x \in C$ be recurrent. Let $y \in C$. Since C is irreducible, $x \to y$. Thus, y is recurrent. Thus, all states in C are recurrent. $\square$

HW: Ch4, 5, 8, 13, 14.