Suggested Solutions of Homework 7 MA327

Ex 1. Recall one of compatibility equations

$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2.$$

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0, \quad \Gamma_{12}^2 = -\tan u, \quad \Gamma_{22}^1 = \cos u \sin u.$$

$$LHS = 0, \quad RHS = \cos^3 u \sin u + \tan u.$$

Hence the compatibility equation is not satisfied, this surface does not exist.

Ex 2.
(a)$\mathbf{x}(u, v) = (u, v, 0)$

$$E = G = 1 \quad F = 0$$

$$\Rightarrow \Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = \Gamma_{22}^2 = 0$$

By Gauss formula, there exists $K = 0$.
(b)$\mathbf{x}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$

$$E = 1 \quad F = 0 \quad G = \rho^2$$

$$\Rightarrow \Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0 \quad \Gamma_{12}^2 = 1/\rho \quad \Gamma_{22}^1 = -\rho.$$

By Gauss formula, there exists $K = -(-1/\rho^2 + 0 + 0 + 1/\rho^2 - 0 - 0) = 0$.

Ex 3.
(a) $K > 0$
(b) $K = 0$
(c) $K < 0$
Since these surfaces have different Gauss curvature, they are not pairwise locally isometric.

Ex 4. Let $\alpha : I \to C$ be the parametrization of C by arc length t.
First, because α is geodesic:

$$\alpha'' = \underline{\frac{D\alpha'(t)}{dt}} + kN = kN,$$

$$\Rightarrow n//N.$$

where n is the normal vector of curve C, N is the normal vector of surface.
On the other hand, α is a line of curvature, so

$$(N \circ \alpha)'(t) = \lambda(t)\alpha'(t),$$

$$\Rightarrow \alpha' \wedge n' = 0$$

Then we find that

$$b'(t) = \alpha''(t) \wedge n(t) + \alpha'(t) \wedge n'(t) = 0.$$

i.e. the torsion of C vanishes everywhere and C is a plane curve.

Ex 5. Suppose $\alpha(t) = (u(t), v(t)), t \in \mathbb{R}$ is a geodesics of a plane. Geodesic equations

$$\begin{cases} u'' + \Gamma_{11}^1 (u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 (v')^2 = 0 \\ v'' + \Gamma_{11}^2 (u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 (v')^2 = 0. \end{cases}$$

In cartesian coordinates, the Christoffel symbols are all zero. Then $\alpha(t) = (a_1 t + a_2, b_1 t + b_2)$ with $t \in \mathbb{R}$ is the straight line.

Ex 6. We can write

$$\frac{d}{dt} v(t) = \frac{Dv(t)}{dt} + aN, \quad \frac{d}{dt} w(t) = \frac{Dv(t)}{dt} + bN$$

for some scalar functions a, b. Then

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \langle \frac{d}{dt} v(t), w(t)\rangle + \langle v(t), \frac{d}{dt} w(t)\rangle$$

$$= \langle \frac{Dv(t)}{dt} + aN, w(t)\rangle + \langle v(t), \frac{Dv(t)}{dt} + bN\rangle$$

$$= \langle \frac{Dv(t)}{dt}, w(t)\rangle + \langle v(t), \frac{Dv(t)}{dt}\rangle.$$

Ex 7.

Let $\alpha(s)$ be a parametrization of C by arc length. Note that $k^2 = k_g^2 + k_n^2$. Therefore, $k = 0$ is equivalent to k_g and k_n are both 0. This is exactly the conclusion in Question 1 by recalling the definitions of asymptotic curves and geodesics.

Ex 8.

Let

$$\mathbf{x}(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$$

be a parametrization of the torus. Then the maximum parallel can be parametrized by $\mathbf{x}(0, v)$, the minimum parallel can be parametrized by $\mathbf{x}(\pi, v)$, and the upper parallel can be parametrized by $\mathbf{x}(\frac{\pi}{2}, v)$. To check these curves whether are or not geodesics, asymptotic curves and lines of curvature, we use the differential equations of geodesics, asymptotic curves and lines of curvature, respectively. First of all, we compute the first fundamental form, second fundamental form, and Christoffel symbols of $\mathbf{x}(u, v)$:

$$E = r^2, F = 0, G = (a + r\cos u)^2, e = r, f = 0, g = \cos u(a + r\cos u),$$

and

$$\Gamma_{22}^2 = 0, \Gamma_{22}^1 = \frac{\sin u(a + r\cos u)}{r}, \Gamma_{12}^2 = \frac{-r\sin u}{a + r\cos u}, \Gamma_{12}^1 = 0, \Gamma_{11}^2 = 0, \Gamma_{11}^1 = 0.$$

Then the differential equations of geodesics becomes

$$\begin{cases} v'' - \frac{2r\sin u}{a+r\cos u} u'v' = 0 \\ u'' + \frac{\sin u(a+r\cos u)}{r} (v')^2 = 0. \end{cases}$$

Therefore, the maximum, minimum parallels are geodesics and the upper parallel is not a geodesic.

The differential equation of asymptotic curves becomes

$$r(u')^2 + \cos u(a + r\cos u)(v')^2 = 0.$$

Therefore, the upper parallel is an asymptotic curve and the other two are not asymptotic curves.

Similarly, when you write down the differential equation of lines of curvature for the torus, you can find that the maximum, minimum and upper parallels are lines of curvature.

Ex 9.

First, one can compute that the normal vector N of the torus along the upper parallel is $(0, 0, -1)$. (one may also choose the other direction)

Second, we need to re-parametrize the upper parallel by arc length: $\alpha(s) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}, r)$.

Finally, we compute the geodesic curvature of $\alpha(s)$ using the formula: $k_g = \langle \alpha'', N \wedge \alpha' \rangle$. One can obtain $k_g = -\frac{1}{a}$ along the upper parallel. One may get the positive sign if one choose $N = (0, 0, 1)$ or change the orientation of the upper parallel.

Ex 10.

Let $\alpha(s)$ be a parametrization of C by arc length with $\alpha(0) = p$. Then the plane curve obtained by projecting C onto T_pS along the normal to the surface at p is

$$\beta(s) = \alpha(s) + \langle p - \alpha(s), N(p) \rangle N(p).$$

Note that $\beta(s)$ may not be parametrized by arc length. We then have $\beta(0) = p, \beta'(0) = \alpha'(0)$ and $\beta''(0) = \frac{D\alpha'}{ds}(0)$. Now we compute the signed curvature of the plane curve β at p:

$$\kappa_\beta(p) = \langle \beta''(0), n_\beta(0) \rangle = \langle \beta''(0), N(p) \wedge \beta'(0) \rangle = \langle \frac{D\alpha'}{ds}(0), N(p) \wedge \alpha'(0) \rangle = (k_g)_\alpha(p).$$

Remark: The first equality holds when $s = 0$ and note that the original definition of signed curvature is for curves parametrized by arc length.

Ex 11.

Using the fact that T, N, V are orthonormal and take derivatives to $\langle T, T \rangle = 1, \langle T, N \rangle = 0$, etc., we can obtain

$$\frac{dT}{ds} = 0 + aV + bN,$$
$$\frac{dV}{ds} = -aT + 0 + cN,$$
$$\frac{dN}{ds} = -bT - cV + 0,$$

where $a = a(s), b = b(s), c = c(s), s \in I$.

Next we will find the coefficients a, b, c. First, from the third equation, by taking inner product with V, we obtain $c = -\langle dN/ds, V \rangle$. Now $c \equiv 0$ is equivalent to $\langle N', N \wedge \alpha' \rangle \equiv 0$. If $\alpha(I)$ is a line of curvature, then N' is collinear with α'. Thus, $\langle N', N \wedge \alpha' \rangle \equiv 0$. On the other hand, if $\langle N', N \wedge \alpha' \rangle \equiv 0$, we want to show N' is collinear with α' which implies that $\alpha(I)$ is a line of curvature. Combining the fact that $\langle N', N \rangle = 0$ with $\langle N', N \wedge \alpha' \rangle = 0$, we can conclude that N' is collinear with α'.

Second, $b = \langle \frac{dT}{ds}, N \rangle = \langle \alpha'', N \rangle$. By the definition of normal curvature, we conclude b is the normal curvature.

Finally, $a = \langle \frac{dT}{ds}, V \rangle = \langle \alpha'', N \wedge \alpha' \rangle$. By the definition of geodesic curvature, we conclude a is the geodesic curvature.

Ex 12. First, we see that every compact surface in $\mathbb{R}^3$ has a point where the Gaussian curvature is positive. (We proved it in previous homework.) Second, by Gauss-Bonnet Theorem (and the classification theorem for compact surfaces in $\mathbb{R}^3$), we have $\int_S K d\sigma \leq 0$. Therefore, there must be some point with negative Gaussian curvature. By continuity of Gaussian curvature, there also exists some point with zero Gaussian curvature.