

Step-1

Consider the matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Thus, matrix D is given by,

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We know that,

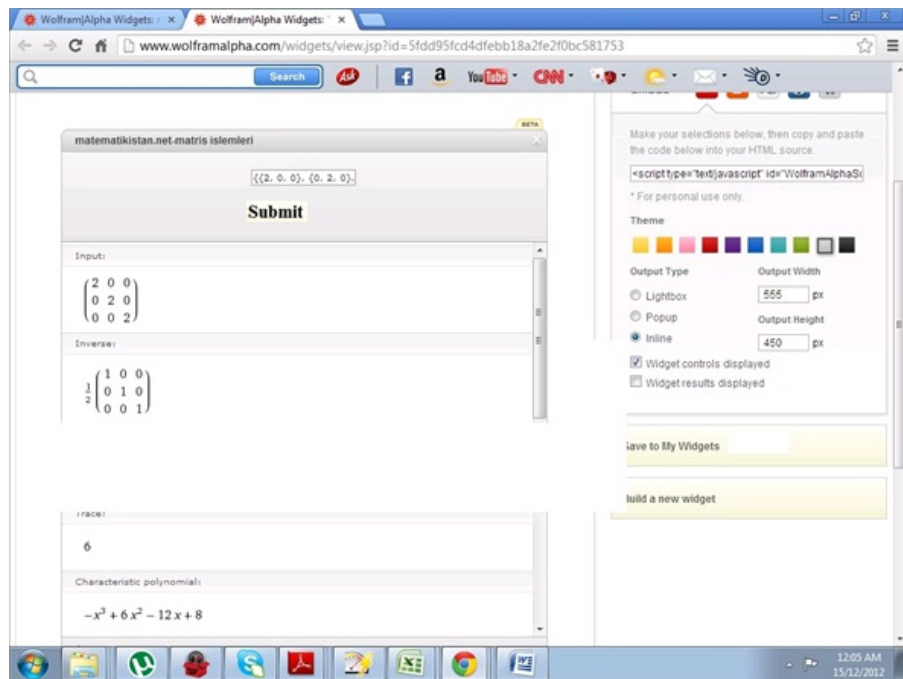
$$A = D + L + U$$

Therefore, we get,

$$\begin{aligned} -L - U &= D - A \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Step-2

By using matrix calculator (the screenshot is given below), the inverse of D is given by,



Therefore,

$$D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Step-3

By multiplying D^{-1} and $\hat{A} \in L \hat{A} \in U$, we get,

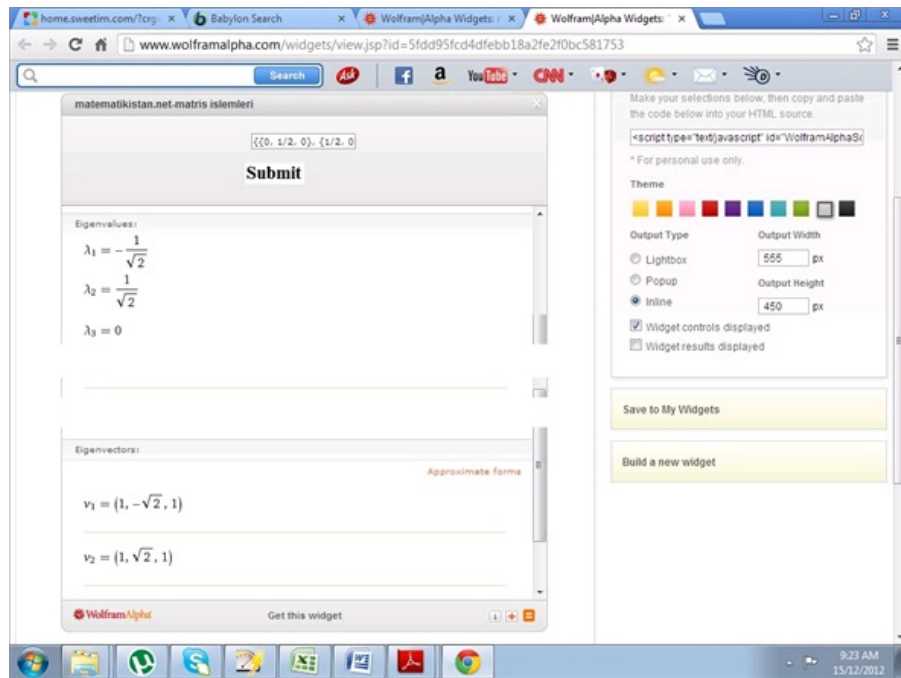
$$D^{-1}(-L-U) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Step-4

By using matrix calculator (the screenshot is given below), the eigenvalues of $D^{-1}(-L-U)_D$ are given by,

Step-5



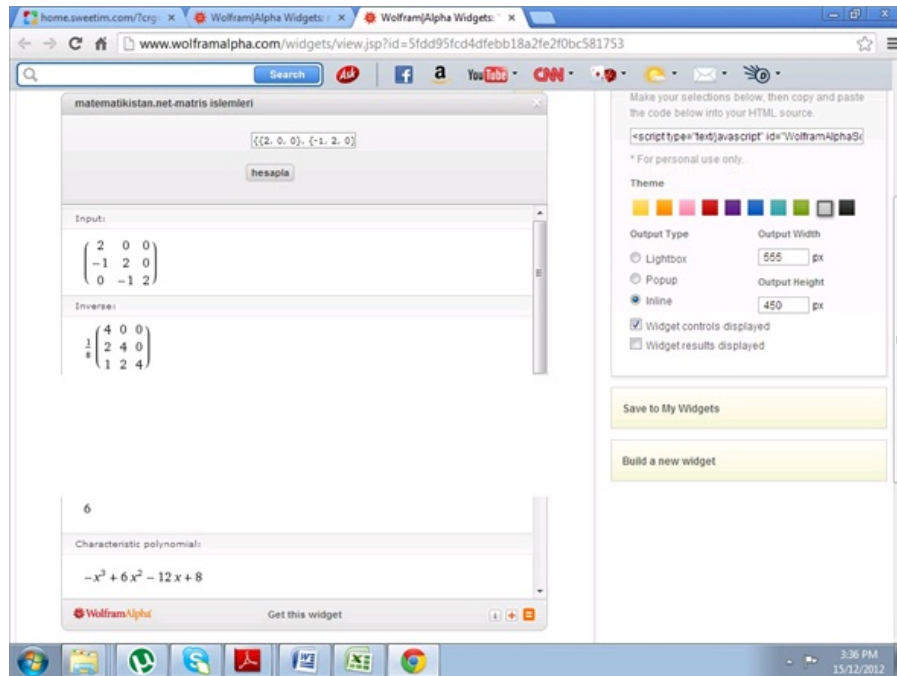
Similarly,

$$D+L = A-U$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

By using matrix calculator (the screenshot is given below), the inverse of $D+L$ is given by,



Step-6

Thus,

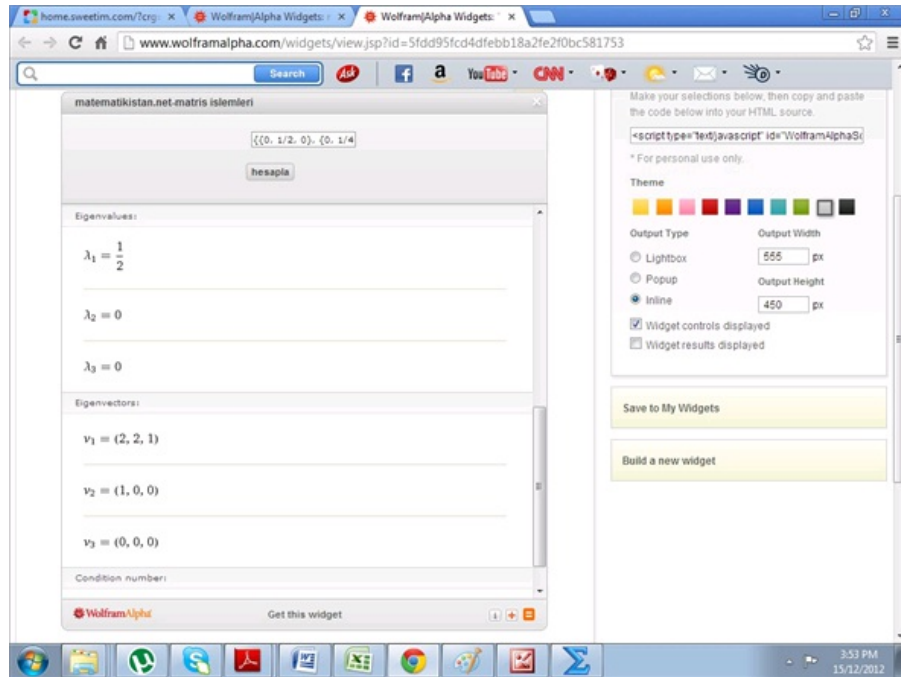
$$(D+L)^{-1} = \frac{1}{8} \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

By multiplying $(D+L)^{-1}$ and $A-U$, we get,

$$\begin{aligned}
 (D+L)^{-1}(-U) &= \frac{1}{8} \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= \frac{1}{8} \begin{bmatrix} 0 & 4 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}
 \end{aligned}$$

Step-7

By using matrix calculator (the screenshot is given below), the eigenvalues of $(D+U)^{-1}(-U)$ are given by,



The maximum eigenvalue is given by,

$$\mu_{\max} = \frac{1}{\sqrt{2}}$$

By substituting $\mu_{\max} = \frac{1}{\sqrt{2}}$ into $\omega_{opt} = \frac{2(1 - \sqrt{1 - \mu_{\max}^2})}{\mu_{\max}^2}$, we get,

$$\begin{aligned}\omega_{opt} &= \frac{2(1 - \sqrt{1 - \mu_{\max}^2})}{\mu_{\max}^2} \\ &= \frac{2\left(1 - \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}\right)}{\left(\frac{1}{\sqrt{2}}\right)^2}\end{aligned}$$

On simplification, we get,

$$\begin{aligned}\omega_{opt} &= 4\left(1 - \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= 4\left(1 - \frac{\sqrt{2}}{2}\right) \\ &= 2(2 - \sqrt{2}) \\ &= 4 - 2\sqrt{2}\end{aligned}$$

Step-8

The λ_{\max} is given by,

$$\begin{aligned}\lambda_{\max} &= \omega_{opt} - 1 \\ &= 4 - 2\sqrt{2} - 1 \\ &= 3 - 2\sqrt{2} \\ &\approx 0.2\end{aligned}$$

Step-9

Thus, $\boxed{\omega_{opt} = 4 - 2\sqrt{2}}$ and $\boxed{\lambda_{\max} = 3 - 2\sqrt{2} \approx 0.2}$.