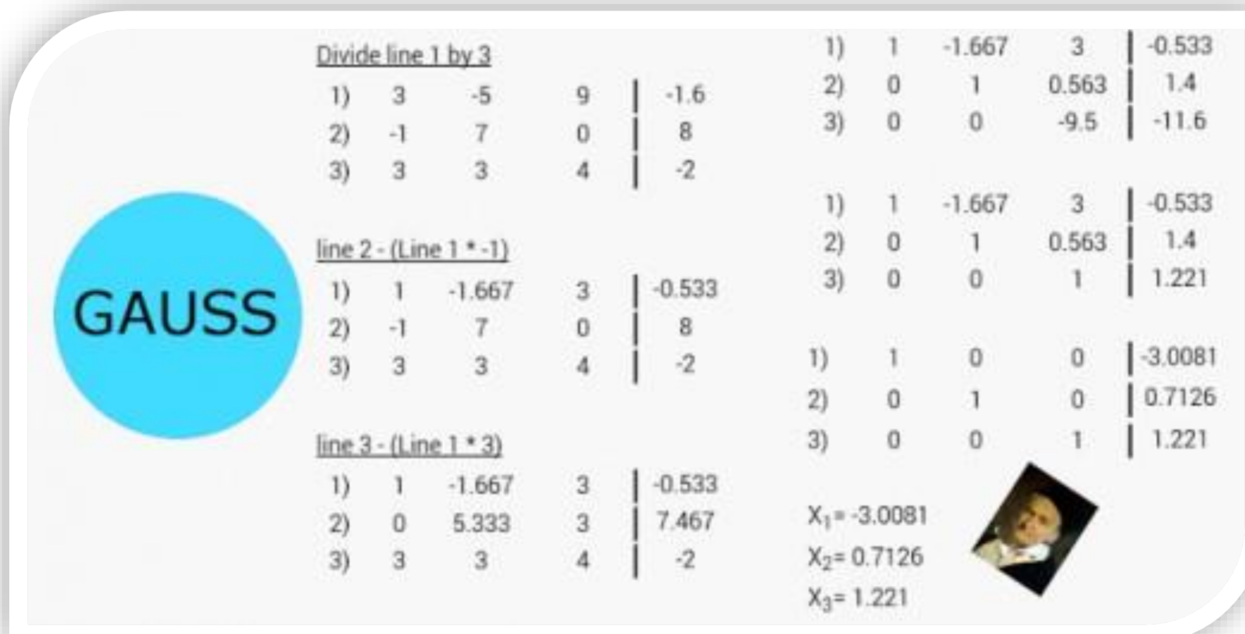
1 Matrices and Gaussian Elimination

1.2

GEOMETRY OF LINEAR EQUATIONS （线性方程组的几何解释）



- A system of linear equations has
  1. *exactly one solution*, or → **1&3: consistent(相容)**
  2. *no solution*, or
  3. *infinitely many solutions*.

Why?

2&3: Singular(奇异) cases: *none* or *too many* solutions

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

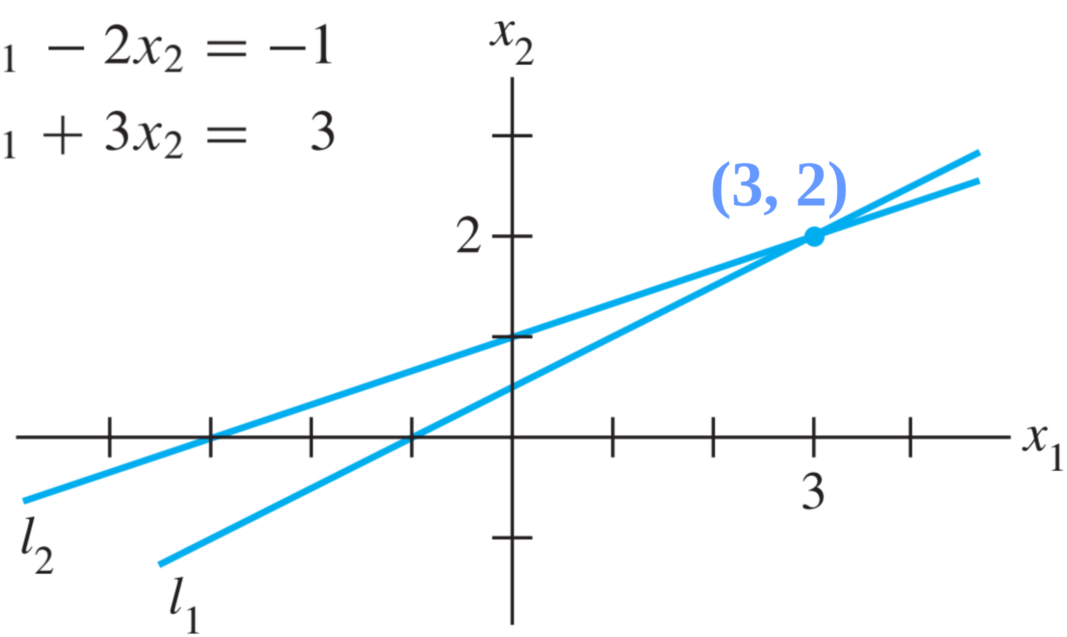


FIGURE 1 Exactly one solution.

(a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

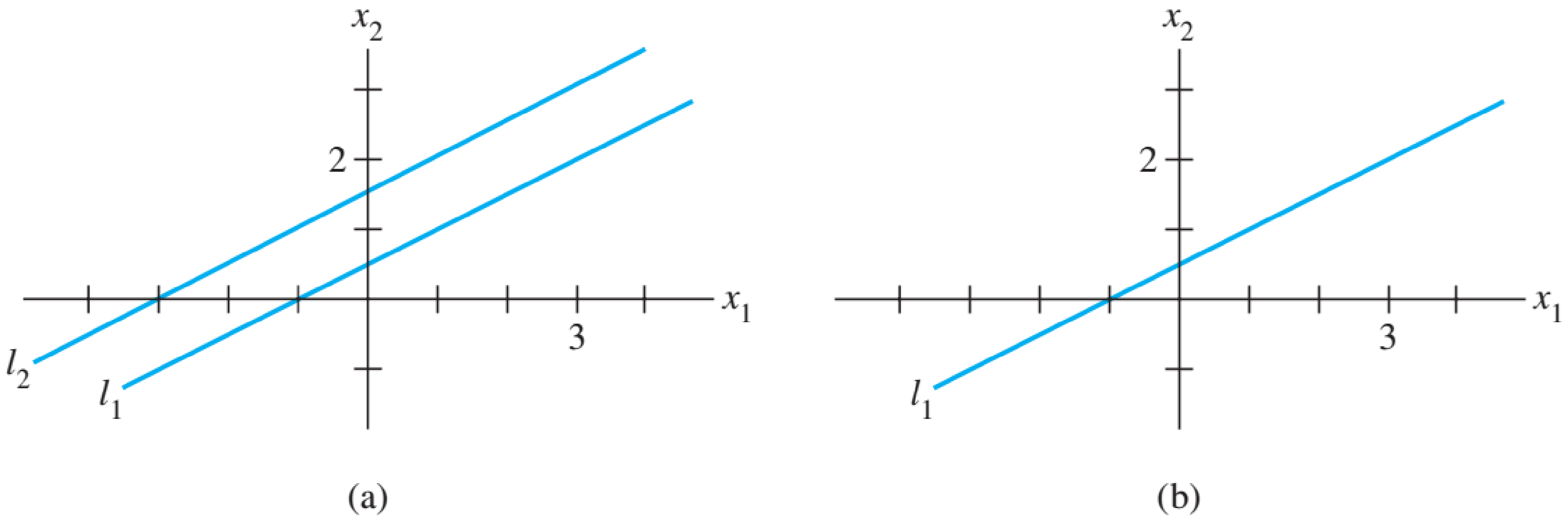
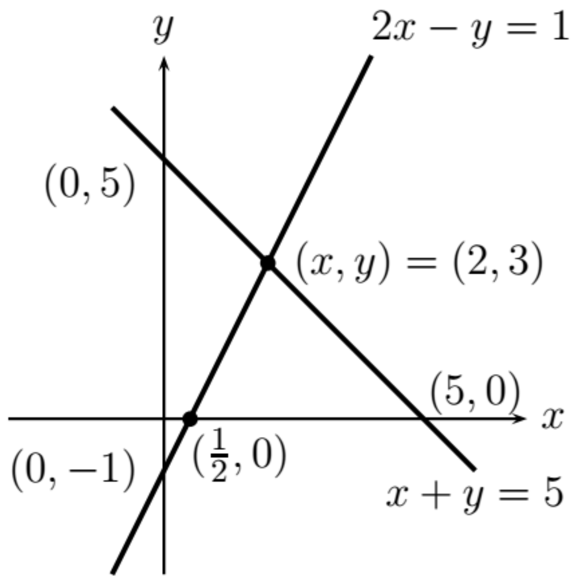


FIGURE 2 (a) No solution. (b) Infinitely many solutions.

$n = 2$:
Row Picture

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

We can look at that system *by rows* and *by columns*.



(a) Lines meet at $x = 2$, $y = 3$

Row picture
(two lines)

理解Column Picture之前，回顾：*What is a vector* (向量)?

n 维向量

定义1 n 个有序的数 $a_1, a_2, \ldots, a_n$ 所组成的数组称为 n **维向量**, 记为

$$\boldsymbol{\alpha} = (a_1, a_2, \ldots, a_n),$$

其中 a_i 称为向量 $\boldsymbol{\alpha}$ 的第 i 个分量(或坐标).

分量全为零的向量称为**零向量**，记为 $\mathbf{0} = (0, 0, \ldots, 0)$.

分量全为实数的向量称为**实向量(real vector)**,

分量为复数的向量称为**复向量(complex vector)**.

全体 n 维实向量的集合记为 $\mathbf{R}^n$.

例如

$(1, 2, 3, \ldots, n) \longrightarrow$ n 维实向量

$(\underbrace{1+2i}_{第1个分量}, \underbrace{2+3i}_{第2个分量}, \ldots, \underbrace{n+(n+1)i}_{第n个分量}) \longrightarrow$ n 维复向量

$(1, 2i, 3+4i) \longrightarrow$ 3维复向量

n 维向量写成一行，称为**行向量(row vector)**，如

$$\boldsymbol{\alpha} = (a_1, a_2, \ldots, a_n);$$

n 维向量写成一列，称为**列向量(column vector)**, 如

$$\boldsymbol{\beta} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = (a_1, a_2, \ldots, a_n)^{\mathrm{T}}.$$

1. 在做运算时，行向量和列向量总被看作是两个不同的向量.

2. 当未说明是行向量还是列向量时, 都当作列向量.

定义2 设 $\boldsymbol{\alpha}=(a_1, a_2,\cdots, a_n)\in P^n$, $\boldsymbol{\beta}=(b_1, b_2,\cdots, b_n)\in P^n$, $\lambda\in P$,

P为数域

(1) $\boldsymbol{\alpha}=\boldsymbol{\beta}$ 当且仅当 $a_i=b_i$, $i=1,2,\cdots, n$

(2) **向量加法**（$\boldsymbol{\alpha}$ 与 $\boldsymbol{\beta}$ 之和 ）： *addition of vectors*

$$\boldsymbol{\alpha}+\boldsymbol{\beta}=(a_1+b_1, a_2+b_2,\cdots, a_n+b_n)$$

(3) **向量数乘**（数量乘法，数 λ 与 $\boldsymbol{\alpha}$ 之乘积）：

$$\lambda\boldsymbol{\alpha}=(\lambda a_1,\lambda a_2, \cdots, \lambda a_n)$$

multiplication by a scalar

向量的加法与数量乘法统称为向量的线性运算

$\lambda=-1$时， $-\boldsymbol{\alpha}=(-a_1,-a_2,\cdots,-a_n)$ **负向量**

$$\boldsymbol{\beta}-\boldsymbol{\alpha}=\boldsymbol{\beta}+(-\boldsymbol{\alpha})$$

线性方程组的向量表示

$$\begin{cases} \boxed{a_{11}x_1} + \boxed{a_{12}x_2} + \cdots + \boxed{a_{1n}x_n} = \boxed{b_1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\boxed{\boldsymbol{\alpha}_1 x_1} + \boxed{\boldsymbol{\alpha}_2 x_2} + \cdots + \boxed{\boldsymbol{\alpha}_n x_n} = \boxed{\boldsymbol{b}}$$

系数矩阵

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n]$$

增广矩阵

$$(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n \quad \boldsymbol{b}]$$

线性方程组与增广矩阵的列向量组之间一一对应.

线性组合(linear combination)

— one of the central ideas of linear algebra

定义3 给定向量组 $\boldsymbol{\beta}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$, 若存在一组数 $k_1, k_2, \ldots, k_m$, 使得

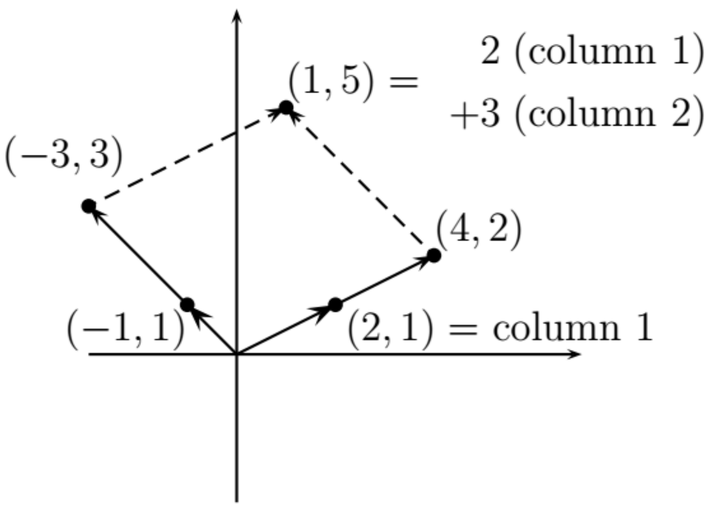
$$\boldsymbol{\beta} = k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_m\boldsymbol{\alpha}_m = \sum_{i=1}^{m} k_i\boldsymbol{\alpha}_i,$$

则称 $\boldsymbol{\beta}$ 为向量组 $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$ 的**线性组合**, 或称向量 $\boldsymbol{\beta}$ 可由向量组 $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$ **线性表示（线性表出）**.

It uses *both* of the basic operations; vectors are *multiplied by numbers and then added*. The result is called a ***linear combination***.

$n = 2$: *Column Picture*

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$



(b) Columns combine with 2 and 3

Column form

$$\boxed{x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix}} = \begin{bmatrix}1\\5\end{bmatrix}$$

vector equation

Linear combination (线性组合)

Column picture
(combine columns)

The problem is ***to find the <u>combination of the column vectors</u> on the left side that produces the vector on the right side***.

$n = 3$

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v \qquad = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

u=1, v=1, w=2

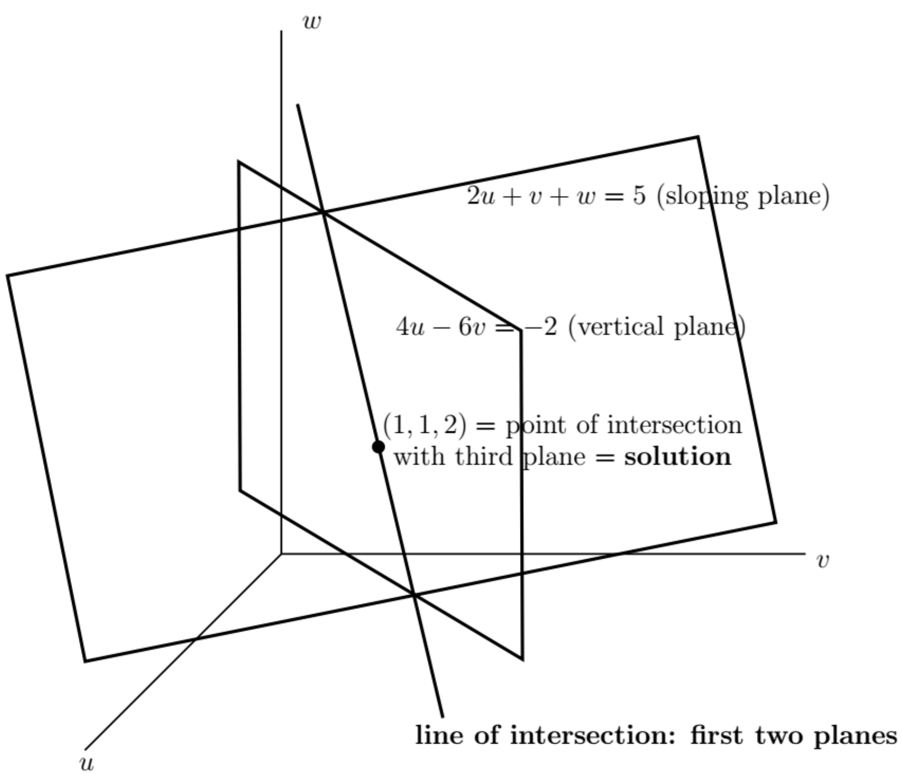


Figure 1.3: The row picture: three intersecting planes from three linear equations.

Linear algebra can operate with any number of equations. The first equation produces an (n-1)-dimensional plane in n dimensions, … ***Assuming all goes well***, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all n equations. ***It is the solution!***

Geometry of Linear Equations

Column form
$$\overset{1}{u}\begin{bmatrix}2\\4\\-2\end{bmatrix}+\overset{1}{v}\begin{bmatrix}1\\-6\\7\end{bmatrix}+\overset{2}{w}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=\boldsymbol{b}.$$

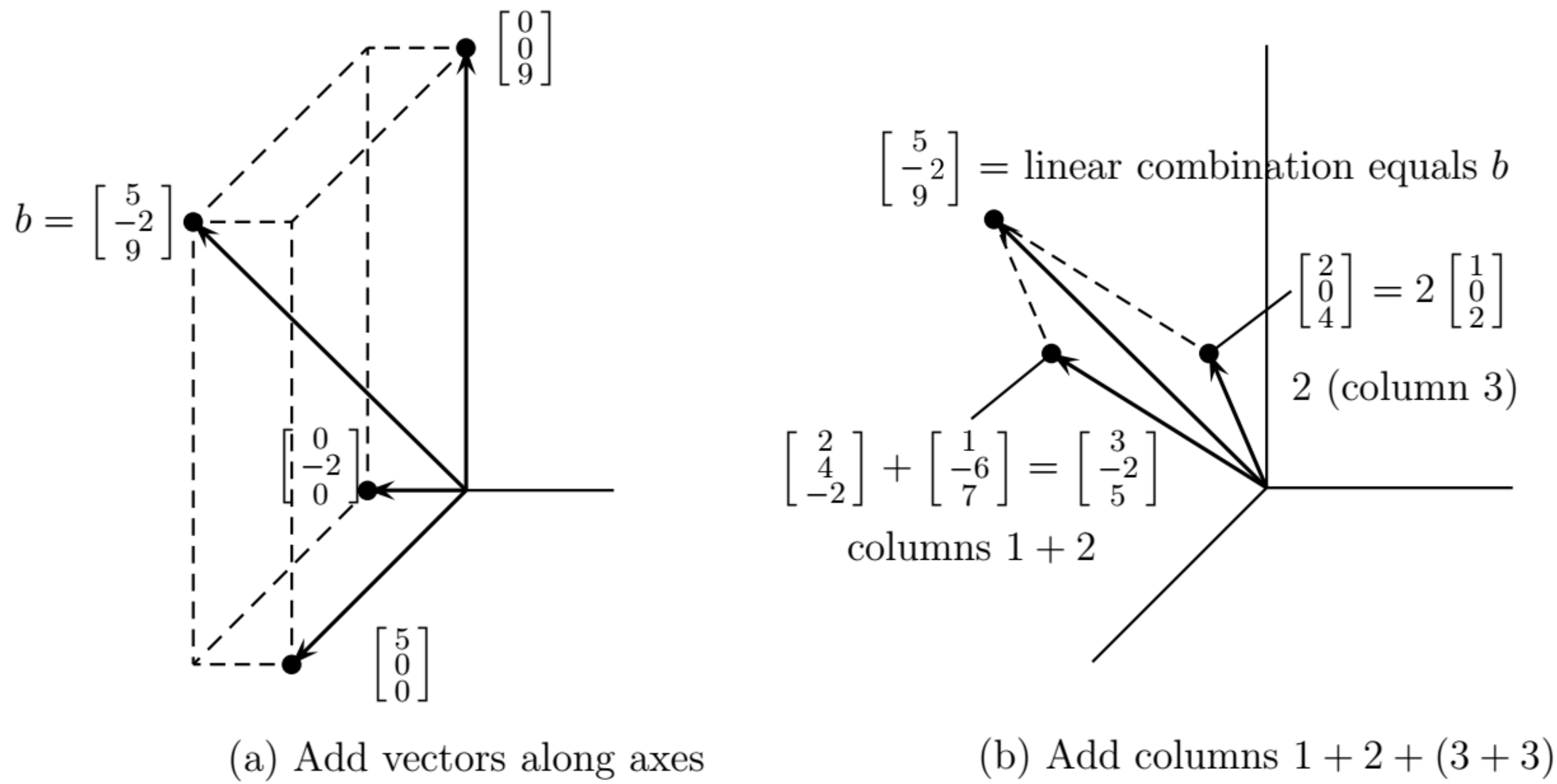


Figure 1.4: The column picture: linear combination of columns equals b.

Linear combination
(线性组合)

$$1\begin{bmatrix}2\\4\\-2\end{bmatrix}+1\begin{bmatrix}1\\-6\\7\end{bmatrix}+2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}.$$

n dimensions

- With n equations in n unknowns, there are n planes in the row picture.
- There are n vectors in the column picture, plus a vector $\boldsymbol{b}$ on the right side. The equations ask for a ***linear combination of the*** n ***columns that equals*** b.

  (*For certain equations that will be impossible.*)

- ***Row picture***: Intersection of planes ("平面"的交点)
- ***Column picture***: Combination of columns (列的组合)

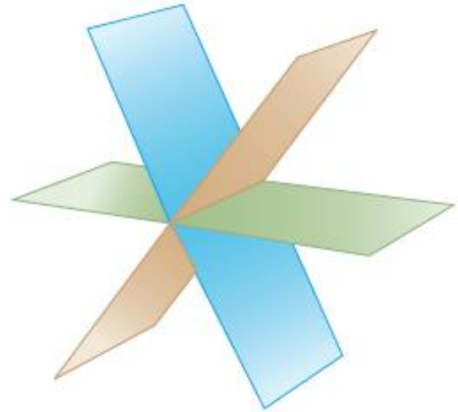
$n = 3$:
Row Picture

The solutions (x,y,z) of a single linear equation

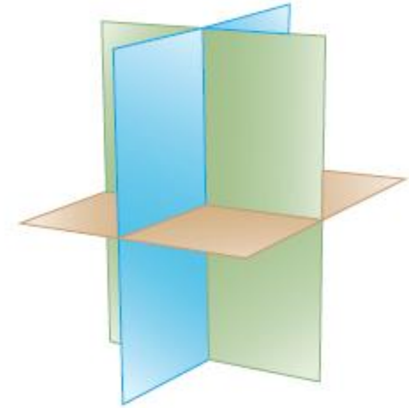
$$ax + by + cz = d$$

form a plane in $\mathbf{R}^3$ when a, b and c are not all zero.

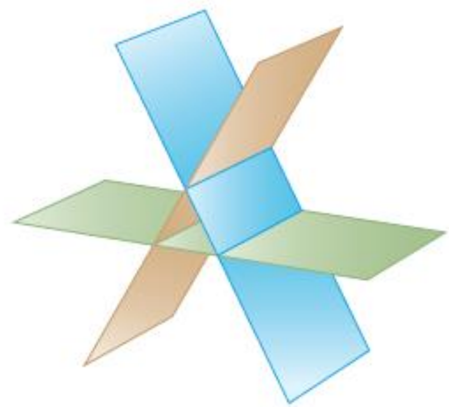
Construct sets of three linear equations whose graphs
(a) intersect in a single line,
(b) intersect in a single point, and
(c) have no points in common.



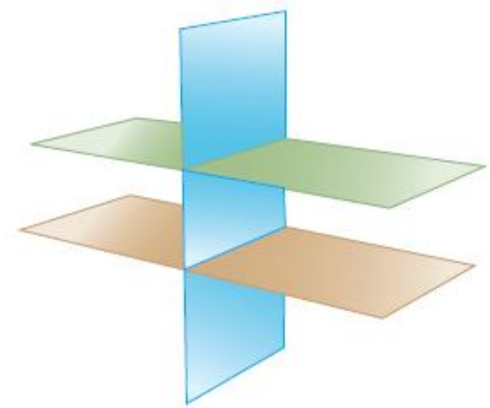
Three planes intersecting in a line
(a)



Three planes intersecting in a point
(b)



Three planes with no intersection
(c)



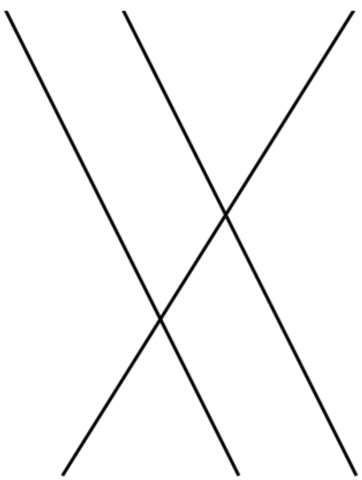
Three planes with no intersection
(c')

Singular Cases 1 ($n = 3$): ***Row picture*** -- no solution

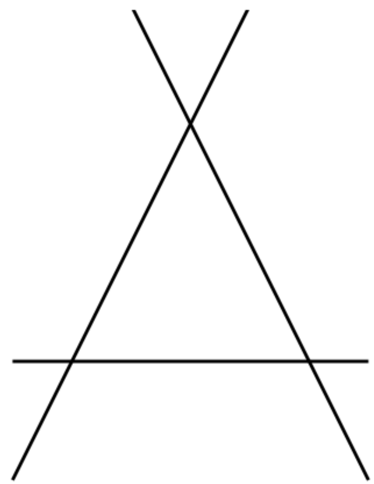
$$\begin{cases} 2u + v + w = 5 \\ 4u + 2v + 2w = 11 \\ -2u + 7v + 2w = 9 \end{cases}$$

$$\begin{cases} u + v + w = 2 \\ 2u \quad + 3w = 5 \\ 3u + v + 4w = 6 \end{cases}$$

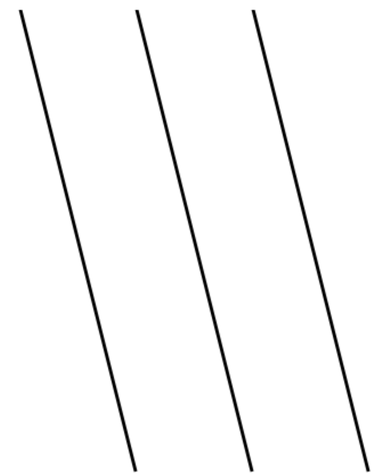
$$\begin{cases} 2u + v + w = 5 \\ 4u + 2v + 2w = 11 \\ 6u + 3v + 3w = 20 \end{cases}$$



two parallel planes



no intersection



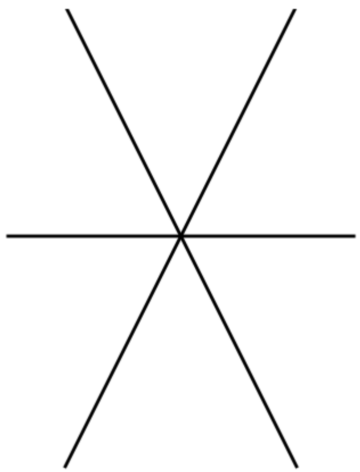
all planes parallel

no solution (end view)

Singular Cases 2 (*n* = 3): ***Row picture*** -- infinitely many solutions

$$\begin{cases} u+v+w=2 \\ 2u \quad +3w=5 \\ 3u+v+4w=7 \end{cases}$$

$$\begin{cases} 2u+v+w=2 \\ 4u+2v+2w=4 \\ 6u+3v+3w=6 \end{cases} \qquad \begin{cases} 2u+v+w=0 \\ 4u+2v+2w=0 \\ 6u+3v+3w=0 \end{cases}$$



line of intersection

move over to the same plane

an infinity of solutions (end view)

https://www.geogebra.org/m/hqwufeg4

Singular Cases ($n = 3$): ***the column picture***

Why?

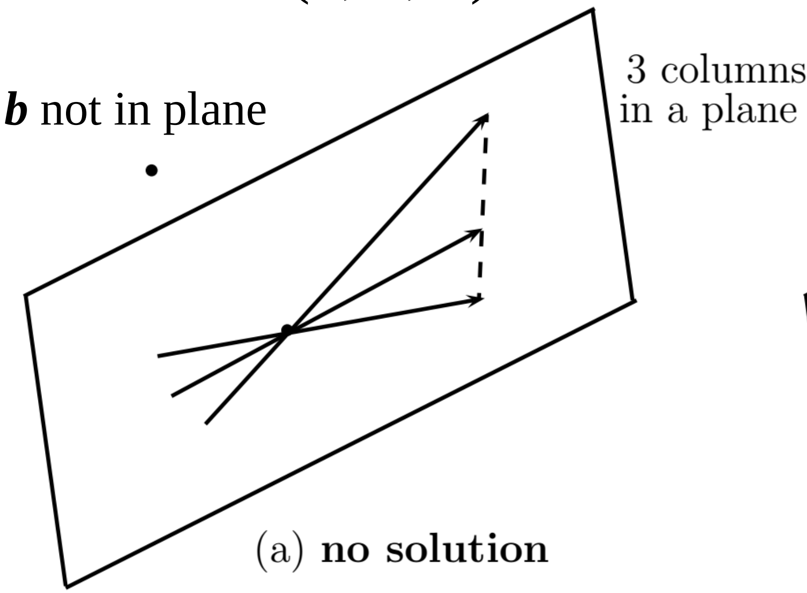
$$u\begin{bmatrix}1\\2\\3\end{bmatrix}+v\begin{bmatrix}1\\0\\1\end{bmatrix}+w\begin{bmatrix}1\\3\\4\end{bmatrix}=\boldsymbol{b}$$

$\boxed{\textbf{Three columns in the same plane;}}$

Solvable only for *b* in that plane

https://www.geogebra.org/m/cgtdgg3x

$\boldsymbol{b} = (2, 5, 6)^T$ $\boldsymbol{b} = (2, 5, 7)^T$



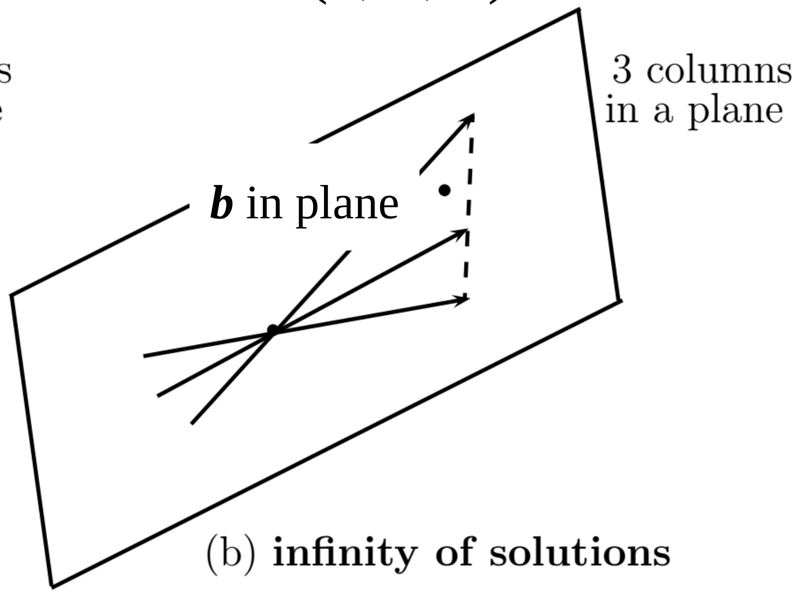


Figure 1.6: Singular cases: *b* outside or inside the plane with all three columns.

Three columns lie in the same plane?

Find a combination of the columns that adds to zero.

$$k^* \quad 3\begin{bmatrix}1\\2\\3\end{bmatrix} - 1\begin{bmatrix}1\\0\\1\end{bmatrix} - 2\begin{bmatrix}1\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Only two columns are *independent*.

The vector $\boldsymbol{b} = (2, 5, 7)^T$ is in that plane of the columns:

$$+ \quad 1\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\end{bmatrix} + 1\begin{bmatrix}1\\3\\4\end{bmatrix} = \begin{bmatrix}2\\5\\7\end{bmatrix},$$

so $(1, 0, 1)^T$ is a solution.

We can add a multiple of the combination $(3, -1, -2)^T$ *that gives* $\boldsymbol{b} = (0, 0, 0)^T$.

So there is a whole line of solutions—as we know from the row picture.

The truth is that we *knew* the columns would combine to give zero, because the rows did.

That is a fact of mathematics, not of computation—and it remains true in dimension *n*.

***If the* n *planes have no point in common, or infinitely many points, then the* n *columns lie in the same plane*.**

If the row picture breaks down, so does the column picture.

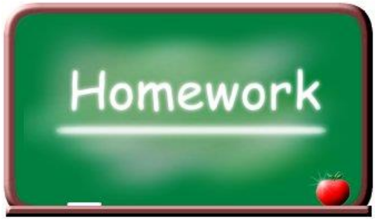
With a decent notation (*matrix notation*) and a decent algorithm (*elimination*), this becomes clear.

And it will be even more interesting after we learn the concept of ***the rank of a matrix* (矩阵的秩)**.

This is one of the most important theorems in linear algebra.

"***row rank* = *column rank*.**"

Homework



- See Blackboard announcement
- ***Hardcover*** **textbook + Supplementary problems**

Deadline (DDL):

- Next tutorial class