MA208 Midterm Exam (April 4, 2022)

Note: All answers must be justified!

1. (14 points) Let X be an exponential random variable with parameter λ. Given $X = x$, Y is a uniform random variable on $(0, x)$. Calculate $E(Y)$ and $V(Y)$.

2. (12 poins) Let (X, Y) be a 2-dimensional random vector uniformly distributed over the region $\{(x, y) : \ x, y \geq 0, \ x + y \leq 1\}$. Find the pdf of the random variable $Z = XY$.

3. (16 points) Suppose that X is a random variable with pdf $f(x) = \frac{1}{2}e^{-|x|}$.
i) Find the moment generating function (mgf) of X.
ii) Using the mgf to calculate $E(X^n)$.

4. (20 points) Let X_n, $n = 0, 1, 2, \cdots$ be a Markov chain (MC) with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} .4 & .6 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 \\ 0 & .2 & 0 & 0 & .8 \\ 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & .6 & .4 \end{bmatrix}.$$

i) Decompose $S = T \cup R_1 \cup \cdots \cup R_k$ into transient and recurrent classes.
ii) Find its stationary distribution(s).

(Problems on other side)

5. (16=5+3+5+3 points) Let X_n, $n = 0, 1, 2, \cdots$ be a Markov chain (MC) with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0 & .4 & .6 \\ .6 & 0 & .4 \\ .3 & .7 & 0 \end{bmatrix}.$$

i) Does the detailed balanced condition hold?
ii) Find its stationary distribution(s).
iii) Find its 2-step transition matrix.
iv) Find the long term proportion for the process to spend at state 3.

6. (12=2+10 points) Suppose that you will play a sequence of games. For each game, you will toss a die first. If you observe x, ($x = 1, 2, 3, 4, 5, 6$), then you will shoot x free throws. You will pay \$2 for each game, and get \$1 for each free throw made. Let X_n be money you have after n games. Suppose that your free throw rate is p and the free throws are independent of each other.
i) Is (X_n) a Markov chain?
ii) If the answer to i) is Yes, write out the state space and the transition probabilities.

7. (10 points) Suppose that you bring \$5 to a casino to play a sequence of independent games until you reach \$10 or \$0. For each game, you will win \$1 or lose \$1 with probability p and $1 - p$, respectively, where $p < .5$. What is the probability that you will walk away from the casino as a winner?