Step-1

(a) We have to find an orthonormal basis for the column space of A, where

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$, where

$$a_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}, a_2 = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

Step-2

$$\|a_1\| = \sqrt{1^2 + 3^2 + 4^2 + 5^2 + 7^2}$$
$$= \sqrt{100}$$
$$= 10$$

Step-3

$$q_1 = \frac{a_1}{\|a_1\|}$$
$$= \frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$
$$= \begin{bmatrix} 1/10 \\ 3/10 \\ 4/10 \\ 5/10 \\ 7/10 \end{bmatrix}$$

Step-4

$q_2 = \frac{\beta}{\|\beta\|}$ where

$\beta = a_2 - \left(q_1^T a_2\right) q_1$

Now

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{4}{10} & \frac{5}{10} & \frac{7}{10} \end{bmatrix} \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

$$= \frac{-6+18+32+0+56}{10}$$

$$= 10$$

Step-5

Therefore $\beta = a_2 - \left(q_1^T a_2\right) q_1$

$$\beta = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

Step-6

Therefore

$$\|\beta\| = \sqrt{49+9+16+25+1}$$

$$= 10$$

Hence

$$q_2 = \frac{1}{10}\begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7/10 \\ 3/10 \\ 4/10 \\ -5/10 \\ 1/10 \end{bmatrix}$$

Step-7

Hence an orthonormal basis for the column space of A is

$$= \{q_1, q_2\}$$

$$= \boxed{\left\{ \begin{bmatrix} 1/10 \\ 3/10 \\ 4/10 \\ 5/10 \\ 7/10 \end{bmatrix}, \begin{bmatrix} -7/10 \\ 3/10 \\ 4/10 \\ -5/10 \\ 1/10 \end{bmatrix} \right\}}$$

Step-8

(b) We have to write A as $QR,$ where Q has orthonormal columns and R is upper triangular.

$$q_1^T a_1 = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{4}{10} & \frac{5}{10} & \frac{7}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1+9+16+25+49}{10}$$

$$= 10$$

Step-9

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{4}{10} & \frac{5}{10} & \frac{7}{10} \end{bmatrix} \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

$$= \frac{-6+18+32+0+56}{10}$$

$$= 10$$

Step-10

$$q_1^T a_2 = \begin{bmatrix} \frac{-7}{10} & \frac{3}{10} & \frac{4}{10} & \frac{-5}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

$$= \frac{42+18+32+0+8}{10}$$

$$= 10$$

Step-11

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1/10 & -7/10 \\ 3/10 & 3/10 \\ 4/10 & 4/10 \\ 5/10 & -5/10 \\ 7/10 & 1/10 \end{bmatrix} \begin{bmatrix} 10 & 10 \\ 0 & 10 \end{bmatrix} = QR}$$

Step-12

(c) We have to find the least squares solution to $Ax = b$, if $b = [-3, 7, 1, 0, 4]$.

By the method of least squares,

$R\hat{x} = Q^T b$ where $b = [-3, 7, 1, 0, 4]$

By part (b), $Q=\begin{bmatrix}1/10 & -7/10\\ 3/10 & 3/10\\ 4/10 & 4/10\\ 5/10 & -5/10\\ 7/10 & 1/10\end{bmatrix},\ R=\begin{bmatrix}10 & 10\\ 0 & 10\end{bmatrix}$

And $\hat{x}=\begin{bmatrix}C\\ D\end{bmatrix}$

$R\,\hat{x}=Q^T b$

$$\Rightarrow\begin{bmatrix}10 & 10\\ 0 & 10\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}=\begin{bmatrix}1/10 & 3/10 & 4/10 & 5/10 & 7/10\\ -7/10 & 3/10 & 4/10 & -5/10 & 1/10\end{bmatrix}\begin{bmatrix}-3\\ 7\\ 1\\ 0\\ 4\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}10C+10D\\ 10D\end{bmatrix}=\begin{bmatrix}\dfrac{-3+21+4+0+28}{10}\\ \dfrac{21+21+4+0+4}{10}\end{bmatrix}$$

$$=\begin{bmatrix}5\\ 5\end{bmatrix}$$

Step-13

$\Rightarrow 10C+10D=5$
and $10D=5$

$\Rightarrow 2C+2D=1$ and $D=\dfrac{1}{2}$

$\Rightarrow D=\dfrac{1}{2}$

Step-14

$2C=1-2\left(\dfrac{1}{2}\right)$

$=0$

$\Rightarrow C=0,$

$D=\dfrac{1}{2}$

Hence $\hat{x} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$