

Step-1

Consider the systems:

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

And

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} d & e & g \\ 0 & f & h \\ 0 & 0 & i \end{pmatrix}$$

Consider that when zero appears in a pivot position, $A = LU$ is not possible (non-zero pivots d, f, i in U).

Objective is to show directly why these are both impossible.

Step-2

Consider the $A = LU$ system:

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

Take the matrix multiplication in right hand side.

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} d+0 & e+0 \\ ld+0 & le+f \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} d & e \\ dl & le+f \end{pmatrix}$$

Step-3

Compare the similar elements of the equal matrices.

$$d = 0, e = 1$$

$$dl = 2$$

$$le + f = 3$$

Product of d and l is 2.

Hence, d is not equal to zero.

But d is zero on compare $d = 0$ which is not allowed so $A = LU$ is not allowed.

Step-4

Consider the $A = LU$ system:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} d & e & g \\ 0 & f & h \\ 0 & 0 & i \end{pmatrix}$$

Take the matrix multiplication on right hand side.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} d+0+0 & e+0+0 & g+0+0 \\ ld+0+0 & le+f+0 & lg+h+0 \\ md+0+0 & me+nf+0 & mg+nh+i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} d & e & g \\ ld & le+f & lg+h \\ md & me+nf & mg+nh+i \end{pmatrix}$$

Step-5

Compare the similar elements of the equal matrices.

$$d = 1, e = 1, g = 0$$

$$ld = 1 \text{ So } l = 1$$

$$le + f = 1$$

$$1 + f = 1$$

$$f = 0$$

$$lg + h = 2$$

$$0 + h = 2$$

$$h = 2$$

$$md = 1 \text{ So } m = 1$$

$$me + nf = 2$$

$$1 + 0 = 2$$

But it is not possible.

Which is not allowed so $A = LU$ is not allowed.

This implies there is no pivot in row 2.

Hence, $A = LU$ is not possible for the both.