

Series Solutions of Second Order Linear Equations

5.1

1. Apply the ratio test:

$$\lim_{n \rightarrow \infty} \frac{|(x-3)^{n+1}|}{|(x-3)^n|} = \lim_{n \rightarrow \infty} |x-3| = |x-3|.$$

Hence the series converges absolutely for $|x-3| < 1$. The radius of convergence is $\rho = 1$. The series diverges for $x = 2$ and $x = 4$, since the n -th term does not approach zero.

3. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n! x^{2n+2}|}{|(n+1)! x^{2n}|} = \lim_{n \rightarrow \infty} \frac{x^2}{n+1} = 0.$$

The series converges absolutely for all values of x . Thus the radius of convergence is $\rho = \infty$.

4. Apply the ratio test:

$$\lim_{n \rightarrow \infty} \frac{|2^{n+1} x^{n+1}|}{|2^n x^n|} = \lim_{n \rightarrow \infty} 2|x| = 2|x|.$$

Hence the series converges absolutely for $2|x| < 1$, or $|x| < 1/2$. The radius of convergence is $\rho = 1/2$. The series diverges for $x = \pm 1/2$, since the n -th term does not approach zero.

5. Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n(x - x_0)^{n+1}|}{|(n+1)(x - x_0)^n|} = \lim_{n \rightarrow \infty} \frac{n}{n+1} |x - x_0| = |x - x_0|.$$

Hence the series converges absolutely for $|x - x_0| < 1$. The radius of convergence is $\rho = 1$. At $x = x_0 + 1$, we obtain the harmonic series, which is divergent. At the other endpoint, $x = x_0 - 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is conditionally convergent.

6. Apply the ratio test :

$$\lim_{n \rightarrow \infty} \frac{|3^n(n+1)^2(x+2)^{n+1}|}{|3^{n+1}n^2(x+2)^n|} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{3n^2} |x+2| = \frac{1}{3} |x+2|.$$

Hence the series converges absolutely for $\frac{1}{3} |x+2| < 1$, or $|x+2| < 3$. The radius of convergence is $\rho = 3$. At $x = -5$ and $x = +1$, the series diverges, since the n -th term does not approach zero.

8. We have $f(x) = e^x$, with $f^{(n)}(x) = e^x$, for $n = 1, 2, \dots$. Therefore $f^{(n)}(0) = 1$. Hence the Taylor expansion about $x_0 = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|n!x^{n+1}|}{|(n+1)!x^n|} = \lim_{n \rightarrow \infty} \frac{1}{n+1} |x| = 0.$$

The radius of convergence is $\rho = \infty$.

9. We have $f(x) = x$, with $f'(x) = 1$ and $f^{(n)}(x) = 0$, for $n = 2, \dots$. Clearly, $f(1) = 1$ and $f'(1) = 1$, with all other derivatives equal to zero. Hence the Taylor expansion about $x_0 = 1$ is

$$x = 1 + (x - 1).$$

Since the series has only a finite number of terms, it converges absolutely for all x .

12. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \dots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \rightarrow \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

13. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \dots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1}n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = - \sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n \rightarrow \infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

14.(a,b,c) Applying the ratio test,

$$\lim_{n \rightarrow \infty} \frac{|(n+1)x^{n+1}|}{|n x^n|} = \lim_{n \rightarrow \infty} \frac{n+1}{n} |x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} n^2(n-1) x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \dots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^2(n+1) x^n.$$

17. Shifting the index in the second series, that is, setting $n = k+1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Hence

$$\begin{aligned} \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^k. \end{aligned}$$

18. Shifting the index by 2, that is, setting $m = n - 2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.\end{aligned}$$

21. Clearly,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Shifting the index in the first series, that is, setting $k = n - 1$,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.$$

Shifting the index in the second series, that is, setting $k = n + 1$,

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Combining the series, and starting the summation at $n = 1$,

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}] x^n.$$

22. Shifting the index in the first series, that is, setting $k = n - 2$,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k$$

Combining the series and starting the summation at $n = 0$,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n.$$

5.2

1.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n] x^n = 0.$$

Equating all the coefficients to zero,

$$(n+2)(n+1)a_{n+2} - a_n = 0, \quad n = 0, 1, 2, \dots$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, \dots$$

The subscripts differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \dots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \dots = \frac{a_1}{(2k+1)!}.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

The linearly independent solutions are

$$y_1 = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots = \cosh x$$

$$y_2 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sinh x.$$

(c) The Wronskian at 0 is 1.

2.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1}] x^n = 0.$$

Setting the coefficients equal to zero, we have

$$(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} = (n+1)[(n+2)a_{n+2} + 3a_{n+1}] = 0$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{3}{n+2}a_{n+1}, \quad \text{for } n = 0, 1, 2, \dots$$

If we set $a_0 = 1$ and $a_1 = 0$, we find that all other coefficients are 0 so $y_1 = 1$ is a solution to the differential equation. If we set $a_0 = 0$ and $a_1 = 1$, we find that $a_2 = -3/2$, $a_3 = 3/2$, $a_4 = -9/8 \dots$. More precisely, we could note that if $a_0 = 0$ and $a_1 = 1$, then $a_n = (-3)^{n-1}/n!$ for $n = 1, 2, 3, \dots$, so

$$y_2 = x - \frac{3}{2!}x^2 + \frac{3^2}{3!}x^3 - \frac{3^3}{4!}x^4 + \dots$$

is also a solution to the differential equation.

(c) The Wronskian at 0 is 1.

5.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + k^2x^2 \sum_{n=0}^{\infty} a_nx^n = 0.$$

Rewriting the second summation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} k^2a_{n-2}x^n = 0,$$

that is,

$$2a_2 + 3 \cdot 2a_3x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + k^2a_{n-2}]x^n = 0.$$

Setting the coefficients equal to zero, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2a_{n-2} = 0, \quad \text{for } n = 2, 3, 4, \dots$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2a_{n-2}}{(n+2)(n+1)}, \quad n = 2, 3, 4, \dots$$

The indices differ by four, so a_4, a_8, a_{12}, \dots are defined by

$$a_4 = -\frac{k^2a_0}{4 \cdot 3}, \quad a_8 = -\frac{k^2a_4}{8 \cdot 7}, \quad a_{12} = -\frac{k^2a_8}{12 \cdot 11}, \quad \dots$$

Similarly, a_5, a_9, a_{13}, \dots are defined by

$$a_5 = -\frac{k^2a_1}{5 \cdot 4}, \quad a_9 = -\frac{k^2a_5}{9 \cdot 8}, \quad a_{13} = -\frac{k^2a_9}{13 \cdot 12}, \quad \dots$$

The remaining coefficients are zero. Therefore the general solution is

$$y = a_0 \left[1 - \frac{k^2}{4 \cdot 3}x^4 + \frac{k^4}{8 \cdot 7 \cdot 4 \cdot 3}x^8 - \frac{k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}x^{12} + \dots \right] + a_1 \left[x - \frac{k^2}{5 \cdot 4}x^5 + \frac{k^4}{9 \cdot 8 \cdot 5 \cdot 4}x^9 - \frac{k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 4 \cdot 4}x^{13} + \dots \right].$$

Note that for the even coefficients,

$$a_{4m} = -\frac{k^2 a_{4m-4}}{(4m-1)4m}, \quad m = 1, 2, 3, \dots$$

and for the odd coefficients,

$$a_{4m+1} = -\frac{k^2 a_{4m-3}}{4m(4m+1)}, \quad m = 1, 2, 3, \dots$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (k^2 x^4)^{m+1}}{3 \cdot 4 \cdot 7 \cdot 8 \dots (4m+3)(4m+4)}$$

$$y_2(x) = x \left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (k^2 x^4)^{m+1}}{4 \cdot 5 \cdot 8 \cdot 9 \dots (4m+4)(4m+5)} \right].$$

(c) The Wronskian at 0 is 1.

7.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + n a_n + 2a_n] x^n = 0.$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \dots$. Note that the indices differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \dots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \dots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \dots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \dots (2k)}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \dots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \dots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \dots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \dots (2n)}.$$

(c) The Wronskian at 0 is 1.

9.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$(3 - x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

It follows that

$$6a_2 - a_0 + (-4a_1 + 18a_3)x + \sum_{n=2}^{\infty} [3(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3na_n - a_n] x^n = 0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2)a_{n+2} = (n+1)a_n, \quad n = 0, 1, 2, \dots$$

The indices differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = \frac{(2k-1)a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1)a_{2k-4}}{3^2(2k-2)(2k)} = \dots = \frac{3 \cdot 5 \dots (2k-1) a_0}{3^k \cdot 2 \cdot 4 \dots (2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \dots = \frac{2 \cdot 4 \cdot 6 \dots (2k) a_1}{3^k \cdot 3 \cdot 5 \dots (2k+1)}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \dots = 1 + \sum_{n=1}^{\infty} \frac{3 \cdot 5 \dots (2n-1) x^{2n}}{3^n \cdot 2 \cdot 4 \dots (2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \dots = x + \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \dots (2n) x^{2n+1}}{3^n \cdot 3 \cdot 5 \dots (2n+1)}.$$

(c) The Wronskian at 0 is 1.

10.(a,b,d) Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} + n a_n + 3a_n] x^n = 0.$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0$$

for $n = 0, 1, 2, \dots$. The indices differ by two, so for $k = 1, 2, \dots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \dots \\ &= \frac{(-1)^k 3 \cdot 5 \dots (2k+1)}{2^k (2k)!} a_0. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \dots \\ &= \frac{(-1)^k 4 \cdot 6 \dots (2k)(2k+2)}{2^k (2k+1)!} a_1. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \dots (2n+1)}{2^n (2n)!} x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \dots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \dots (2n+2)}{2^n (2n+1)!} x^{2n+1}.$$

(c) The Wronskian at 0 is 1.

12.(a) From Problem 3, we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}.$$

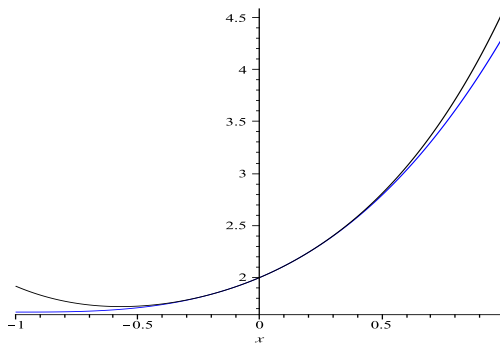
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \dots$$

The four- and five-term polynomial approximations are

$$p_4 = 2 + x + x^2 + x^3/3, \text{ and } p_5 = 2 + x + x^2 + x^3/3 + x^4/4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.7$.

13.(a) From Problem 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \dots (2n-1)} \quad \text{and} \quad y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \dots (2n)}.$$

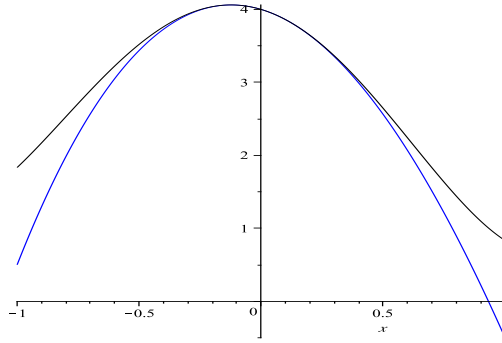
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \dots$$

The four- and five-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3, \text{ and } p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

(b)



(c) The four-term approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.5$.

16. The given recurrence relation is

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

and the proposed solution is given in Eq.(10) as

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1, \text{ for } k = 1, 2, 3, \dots$$

To prove the solution by induction, we first note that for $k = 0$,

$$\frac{(-1)^0}{(2(0)+1)!} a_1 = a_1$$

so Eq.(10) is satisfied for $k = 0$. Next we assume that Eq.(10) is true for $k = m$; that is, we assume that

$$a_{2m+1} = \frac{(-1)^m}{(2m+1)!} a_1$$

and compute $a_{2(m+1)+1} = a_{2m+3}$:

$$\begin{aligned} a_{2m+3} &= \frac{-1}{(2m+3)(2m+2)} a_{2m+1} && \text{(from the recurrence relation)} \\ &= \frac{-1}{(2m+3)(2m+2)} \cdot \frac{(-1)^m}{(2m+1)!} a_1 && \text{(from the inductive hypothesis)} \\ &= \frac{(-1)^{m+1}}{(2m+3)!} a_1 = \frac{(-1)^{m+1}}{(2(m+1)+1)!} a_1 \end{aligned}$$

and Eq.(10) holds for $k = m + 1$. Therefore Eq.(10) is true for $k = 1, 2, 3, \dots$

17. Two linearly independent solutions of Airy's equation (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}.$$

Applying the ratio test to the terms of $y_1(x)$,

$$\lim_{n \rightarrow \infty} \frac{|2 \cdot 3 \cdots (3n-1)(3n) x^{3n+3}|}{|2 \cdot 3 \cdots (3n+2)(3n+3) x^{3n}|} = \lim_{n \rightarrow \infty} \frac{1}{(3n+1)(3n+2)(3n+3)} |x|^3 = 0.$$

Similarly, applying the ratio test to the terms of $y_2(x)$,

$$\lim_{n \rightarrow \infty} \frac{|3 \cdot 4 \cdots (3n)(3n+1) x^{3n+4}|}{|3 \cdot 4 \cdots (3n+3)(3n+4) x^{3n+1}|} = \lim_{n \rightarrow \infty} \frac{1}{(3n+2)(3n+3)(3n+4)} |x|^3 = 0.$$

Hence both series converge absolutely for all x .

18. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n a_n x^n.$$

We then obtain

$$2a_2 + \lambda a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - 2n a_n + \lambda a_n] x^n = 0.$$

Setting the coefficients equal to zero, it follows that

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)} a_n$$

for $n = 0, 1, 2, \dots$. Note that the indices differ by two, so for $k = 1, 2, \dots$

$$\begin{aligned} a_{2k} &= \frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k} = \frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} \\ &= (-1)^k \frac{\lambda \cdots (\lambda-4k+8)(\lambda-4k+4)}{(2k)!} a_0. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= \frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)} = \frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} \\ &= (-1)^k \frac{(\lambda-2) \dots (\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!} a_1. \end{aligned}$$

Hence the linearly independent solutions of the Hermite equation (about $x_0 = 0$) are

$$\begin{aligned} y_1(x) &= 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6!}x^6 + \dots \\ y_2(x) &= x - \frac{\lambda-2}{3!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7!}x^7 + \dots \end{aligned}$$

(b) Based on the recurrence relation

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)} a_n,$$

the series solution will terminate as long as λ is a nonnegative even integer. If $\lambda = 2m$, then one or the other of the solutions in part (b) will contain at most $m/2 + 1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda = 0, 2, 4, 6, 8, 10$:

$\lambda = 0$	$y_1(x) = 1$
$\lambda = 2$	$y_2(x) = x$
$\lambda = 4$	$y_1(x) = 1 - 2x^2$
$\lambda = 6$	$y_2(x) = x - 2x^3/3$
$\lambda = 8$	$y_1(x) = 1 - 4x^2 + 4x^4/3$
$\lambda = 10$	$y_2(x) = x - 4x^3/3 + 4x^5/15$

(c) Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (-1)^k \frac{2n \dots (2n-4k+8)(2n-4k+4)}{(2k)!}$$

and

$$a_{2k+1} = (-1)^k \frac{(2n-2) \dots (2n-4k+6)(2n-4k+2)}{(2k+1)!}.$$

for $k = 1, 2, \dots, [n/2]$. It follows that the coefficient of x^n , in y_1 and y_2 , is

$$a_n = \begin{cases} (-1)^k \frac{4^k k!}{(2k)!} & \text{for } n = 2k \\ (-1)^k \frac{4^k k!}{(2k+1)!} & \text{for } n = 2k+1 \end{cases}$$

Then by definition,

$$H_n(x) = \begin{cases} (-1)^k 2^n \frac{(2k)!}{4^k k!} y_1(x) = (-1)^k \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (-1)^k 2^n \frac{(2k+1)!}{4^k k!} y_2(x) = (-1)^k \frac{2^{2k+1}}{k!} y_2(x) & \text{for } n = 2k+1 \end{cases}$$

Therefore the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

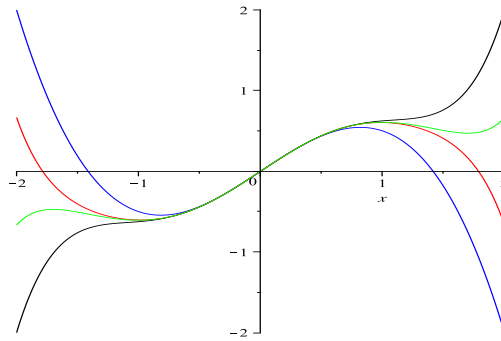
$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

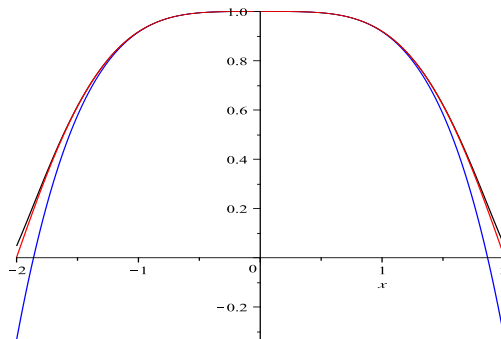
20. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \frac{x^9}{2 \cdot 4 \cdot 6 \cdot 8} - \dots$$



22. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \dots$$



23. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitution into the ODE results in

$$(1-x) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + n a_n - 2 a_n] x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + (n-2)a_n = 0$$

for $n = 1, 2, \dots$. Writing out the individual equations,

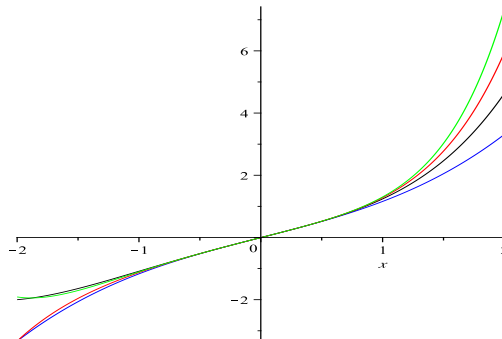
$$\begin{aligned} 3 \cdot 2 a_3 - 2 \cdot 1 a_2 - a_1 &= 0 \\ 4 \cdot 3 a_4 - 3 \cdot 2 a_3 &= 0 \\ 5 \cdot 4 a_5 - 4 \cdot 3 a_4 + a_3 &= 0 \\ 6 \cdot 5 a_6 - 5 \cdot 4 a_5 + 2 a_4 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$\begin{aligned} 3 \cdot 2 a_3 - 1 &= 0 \\ 4 \cdot 3 a_4 - 3 \cdot 2 a_3 &= 0 \\ 5 \cdot 4 a_5 - 4 \cdot 3 a_4 + a_3 &= 0 \\ 6 \cdot 5 a_6 - 5 \cdot 4 a_5 + 2 a_4 &= 0 \\ &\vdots \end{aligned}$$

That is, $a_3 = 1/6$, $a_4 = 1/12$, $a_5 = 1/24$, $a_6 = 1/45, \dots$. Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \dots$$



5.3

3. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2 y' - (\sin x)y.$$

Differentiating twice,

$$y''' = -x^2 y'' - (2x + \sin x)y' - (\cos x)y$$

$$y^{(4)} = -x^2 y''' - (4x + \sin x)y'' - (2 + 2\cos x)y' + (\sin x)y.$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the first equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{(4)}(0) = -4a_1$.

4. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x . Hence the series solutions converge everywhere.

10. The Taylor series expansion of e^x , about $x_0 = 0$, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \right] + x \sum_{n=0}^{\infty} a_nx^n = 0.$$

First note that

$$x \sum_{n=0}^{\infty} a_nx^n = \sum_{n=1}^{\infty} a_{n-1}x^n = a_0x + a_1x^2 + a_2x^3 + \dots + a_{n-1}x^n + \dots$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2 \frac{1}{n!} + 6a_3 \frac{1}{(n-1)!} + 12a_4 \frac{1}{(n-2)!} + \dots$$

$$\dots + (n+1)n a_{n+1} + (n+2)(n+1)a_{n+2}.$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \dots$$

$$\dots + a_0x + a_1x^2 + a_2x^3 + \dots = 0.$$

Setting the coefficients equal to zero, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$, $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0$, \dots . Hence the general solution is

$$y(x) = a_0 + a_1x - a_0 \frac{x^3}{6} + (a_0 - a_1) \frac{x^4}{12} + (2a_1 - a_0) \frac{x^5}{40} + \left(\frac{4}{3}a_0 - 2a_1\right) \frac{x^6}{120} + \dots$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \dots$$

$$y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \dots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$.

11. The Taylor series expansion of $\cos x$, about $x_0 = 0$, is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \right] + \sum_{n=1}^{\infty} na_nx^n - 2 \sum_{n=0}^{\infty} a_nx^n = 0.$$

The coefficient of x^n in the product of the two series is

$$c_n = 2a_2b_n + 6a_3b_{n-1} + 12a_4b_{n-2} + \dots + (n+1)na_{n+1}b_1 + (n+2)(n+1)a_{n+2}b_0,$$

in which $\cos x = b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_nx^n + \sum_{n=1}^{\infty} (n-2)a_nx^n = 0.$$

Expanding the product of the series, it follows that

$$\begin{aligned} 2a_2 - 2a_0 + 6a_3x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \dots \\ \dots - a_1x + a_3x^3 + 2a_4x^4 + \dots = 0. \end{aligned}$$

Setting the coefficients equal to zero, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, \dots . Hence the general solution is

$$y(x) = a_0 + a_1x + a_0x^2 + a_1\frac{x^3}{6} + a_0\frac{x^4}{12} + a_1\frac{x^5}{60} + a_0\frac{x^6}{120} + a_1\frac{x^7}{560} + \dots$$

We find that two linearly independent solutions ($W(y_1, y_2)(0) = 1$) are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \dots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \dots$$

The nearest zero of $P(x) = \cos x$ is at $x = \pm\pi/2$. Hence $\rho_{\min} = \pi/2$.

12. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting y_2 into the ODE results in

$$2P(x) + 2xQ(x) + x^2R(x) = 0.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting y_1 into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that y_1 and y_2 were the only two solutions. But note that an arbitrary value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

13. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n]x^n = 0.$$

Setting the coefficients equal to zero, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \dots$. It is easy to see that $a_n = a_0/(n!)$. Therefore the general solution is

$$y(x) = a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right] = a_0 e^x.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

14. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - x \sum_{n=0}^{\infty} a_nx^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - a_{n-1}]x^n = 0.$$

Setting the coefficient equal to zero, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \dots$. Note that the indices differ by two, so for $k = 1, 2, \dots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \dots = \frac{a_0}{2 \cdot 4 \dots (2k)}$$

and

$$a_{2k+1} = 0.$$

Hence the general solution is

$$y(x) = a_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \dots + \frac{x^{2n}}{2^n n!} + \dots \right] = a_0 e^{x^2/2}.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

15. Let $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$. Substituting into the ODE,

$$(1-x) \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_nx^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - n a_n - a_n] x^n = 0.$$

Setting the coefficients equal to zero, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \dots$. Hence the general solution is

$$y(x) = a_0 [1 + x + x^2 + x^3 + \dots + x^n + \dots] = a_0 \frac{1}{1-x}.$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

18. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1 + \sum_{m=1}^n (-1)^m \frac{2^m n(n-1) \dots (n-m+1)(2n+1)(2n+3) \dots (2n+2m-1)}{(2m)!} x^{2m}.$$

$\alpha = 0$	1
$\alpha = 2$	$1 - 3x^2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x + \sum_{m=1}^n (-1)^m \frac{2^m n(n-1) \dots (n-m+1)(2n+3)(2n+5) \dots (2n+2m+1)}{(2m+1)!} x^{2m+1}.$$

$\alpha = 1$	x
$\alpha = 3$	$x - \frac{5}{3}x^3$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

19.(a) Based on Problem 18,

$\alpha = 0$	1	$y_1(1) = 1$
$\alpha = 2$	$1 - 3x^2$	$y_1(1) = -2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$	$y_1(1) = \frac{8}{3}$

Normalizing the polynomials, we obtain

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= -\frac{1}{2} + \frac{3}{2}x^2 \\ P_4(x) &= \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4 \end{aligned}$$

$\alpha = 1$	x	$y_2(1) = 1$
$\alpha = 3$	$x - \frac{5}{3}x^3$	$y_2(1) = -\frac{2}{3}$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$	$y_2(1) = \frac{8}{15}$

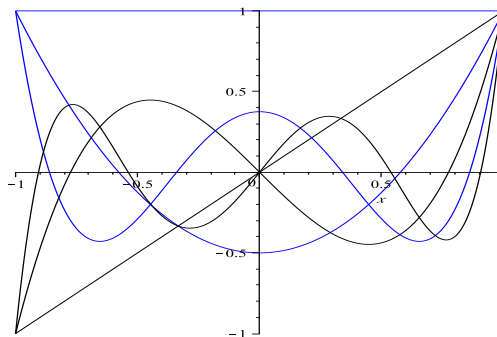
Similarly,

$$P_1(x) = x$$

$$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$

$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

(b)



(c) $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The zeros of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The zeros of $P_3(x)$ are $x = 0, \pm\sqrt{3}/5$. The roots of $P_4(x)$ are given by $x^2 = (15 + 2\sqrt{30})/35, (15 - 2\sqrt{30})/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = (35 + 2\sqrt{70})/63, (35 - 2\sqrt{70})/63$.

21. We first compute the n^{th} derivatives of $(x^2 - 1)^n$ for $n = 0, 1, 2, 3$:

$$\frac{d^0}{dx^0}(x^2 - 1)^0 = 1$$

$$\frac{d}{dx}(x^2 - 1) = 2x$$

$$\frac{d^2}{dx^2}(x^2 - 1)^2 = -4 + 12x^2$$

$$\frac{d^3}{dx^3}(x^2 - 1)^3 = 120x^3 - 72x$$

Thus we have that

$$\begin{aligned}\frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2 - 1)^0 &= 1 = P_0(x) \\ \frac{1}{2^1 1!} \frac{d}{dx} (x^2 - 1) &= \frac{2x}{2} = x = P_1(x) \\ \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2 - 1)^2 &= \frac{-4 + 12x^2}{8} = -\frac{1}{2} + \frac{3}{2}x^2 = P_2(x) \\ \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2 - 1)^3 &= \frac{-72x + 120x^3}{48} = -\frac{3}{2}x + \frac{5}{2}x^3 = P_3(x)\end{aligned}$$

23. Since the $n + 1$ polynomials P_0, P_1, \dots, P_n are linearly independent, and the degree of P_k is k , any polynomial f of degree n can be expressed as a linear combination

$$f(x) = \sum_{k=0}^n a_k P_k(x).$$

Multiplying both sides by P_m and integrating,

$$\int_{-1}^1 f(x) P_m(x) dx = \sum_{k=0}^n a_k \int_{-1}^1 P_k(x) P_m(x) dx.$$

Based on Problem 22,

$$\int_{-1}^1 P_k(x) P_m(x) dx = \frac{2}{2m+1} \delta_{km}.$$

Hence

$$\int_{-1}^1 f(x) P_m(x) dx = \frac{2}{2m+1} a_m.$$

5.4

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) + 4r + 2 = r^2 + 3r + 2$. The roots are $r = -2, -1$. Hence the general solution, for $x \neq 0$, is $y = c_1 x^{-2} + c_2 x^{-1}$.

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) - 3r + 4 = r^2 - 4r + 4$. The root is $r = 2$, with multiplicity two. Hence the general solution, for $x \neq 0$, is $y = (c_1 + c_2 \ln|x|) x^2$.

4. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) - r + 1 = r^2 - 2r + 1$. The root is $r = 1$, with multiplicity two. Hence the general solution, for $x \neq 0$, is $y = (c_1 + c_2 \ln|x|) x$.

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + 5r - 1$. The roots are $r = -(5 \pm \sqrt{29})/2$. Hence the general solution, for $x \neq 0$, is $y = c_1 |x|^{-(5+\sqrt{29})/2} + c_2 |x|^{-(5-\sqrt{29})/2}$.

6. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 - 3r + 3$. The roots are complex, with $r = (3 \pm i\sqrt{3})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{3/2} \cos\left(\frac{\sqrt{3}}{2} \ln |x|\right) + c_2 |x|^{3/2} \sin\left(\frac{\sqrt{3}}{2} \ln |x|\right).$$

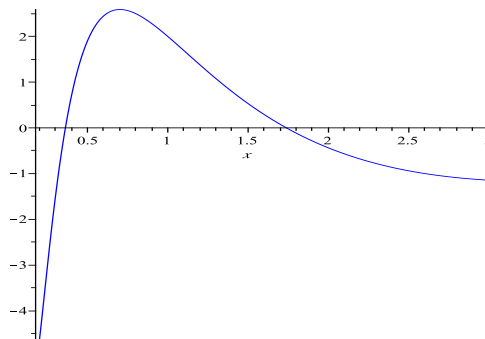
8. Substitution of $y = (x - 2)^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + 4r + 8$. The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is $y = c_1 (x - 2)^{-2} \cos(2 \ln |x - 2|) + c_2 (x - 2)^{-2} \sin(2 \ln |x - 2|)$.

10. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = 4r^2 + 4r + 17$. The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is $y = c_1 x^{-1/2} \cos(2 \ln x) + c_2 x^{-1/2} \sin(2 \ln x)$. Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2, \quad -\frac{1}{2}c_1 + 2c_2 = -3.$$

Hence the solution of the initial value problem is

$$y(x) = 2x^{-1/2} \cos(2 \ln x) - x^{-1/2} \sin(2 \ln x).$$



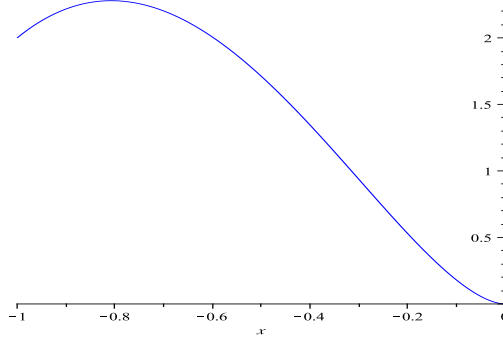
As $x \rightarrow 0^+$, the solution decreases without bound.

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 - 4r + 4$. The root is $r = 2$, with multiplicity two. Hence the general solution, for $x < 0$, is $y = (c_1 + c_2 \ln |x|) x^2$. Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2, \quad -2c_1 - c_2 = 3.$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7 \ln |x|) x^2.$$



We find that $y(x) \rightarrow 0$ as $x \rightarrow 0^-$.

13. We see that $P(x) = 0$ when $x = 0$ and 1 . Since the three coefficients have no factors in common, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{2x}{x^2(1-x)^2} = 2.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{4}{x^2(1-x)^2} = 4.$$

The singular point $x = 0$ is regular. Considering $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{2x}{x^2(1-x)^2}.$$

The latter limit does not exist. Hence $x = 1$ is an irregular singular point.

14. $P(x) = 0$ when $x = 0$ and 1 . Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{x-2}{x^2(1-x)}.$$

The limit does not exist, and so $x = 0$ is an irregular singular point. Considering $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{x-2}{x^2(1-x)} = 1.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{-3x}{x^2(1-x)} = 0.$$

Hence $x = 1$ is a regular singular point.

15. $P(x) = 0$ when $x = 0$ and ± 1 . Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{2}{x^3(1-x^2)}.$$

The limit does not exist, and so $x = 0$ is an irregular singular point. Near $x = -1$,

$$\lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} (x+1) \frac{2}{x^3(1-x^2)} = -1.$$

$$\lim_{x \rightarrow -1} (x+1)^2 q(x) = \lim_{x \rightarrow -1} (x+1)^2 \frac{2}{x^3(1-x^2)} = 0.$$

Hence $x = -1$ is a regular singular point. At $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{2}{x^3(1-x^2)} = -1.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{2}{x^3(1-x^2)} = 0.$$

Hence $x = 1$ is a regular singular point.

17. The only singular point is at $x = 0$. We find that

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{x}{x^2} = 1.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{x^2 - \nu^2}{x^2} = -\nu^2.$$

Hence $x = 0$ is a regular singular point.

18. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x) = \frac{3}{(x+2)^2} \quad \text{and} \quad q(x) = \frac{-2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1 . For $x = -2$,

$$\lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} (x+2) \frac{3}{(x+2)^2}.$$

The limit does not exist. Hence $x = -2$ is an irregular singular point. For $x = 1$,

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} (x-1) \frac{3}{(x+2)^2} = 0.$$

$$\lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} (x-1)^2 \frac{-2}{(x+2)(x-1)} = 0.$$

Hence $x = 1$ is a regular singular point.

19. $P(x) = 0$ when $x = 0$ and 3 . Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{x+1}{x(3-x)} = \frac{1}{3}.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{-2}{x(3-x)} = 0.$$

Hence $x = 0$ is a regular singular point. For $x = 3$,

$$\lim_{x \rightarrow 3} (x-3)p(x) = \lim_{x \rightarrow 3} (x-3) \frac{x+1}{x(3-x)} = -\frac{4}{3}.$$

$$\lim_{x \rightarrow 3} (x-3)^2 q(x) = \lim_{x \rightarrow 3} (x-3)^2 \frac{-2}{x(3-x)} = 0.$$

Hence $x = 3$ is a regular singular point.

21. Note that $p(x) = \ln|x|$ and $q(x) = 3x$. Evidently, $p(x)$ is not analytic at $x_0 = 0$. Furthermore, the function $x p(x) = x \ln|x|$ does not have a Taylor series about $x_0 = 0$. Hence $x = 0$ is an irregular singular point.

23. The singular points are located at $x = \pm n\pi$, $n = 0, 1, \dots$. Dividing the ODE by $x \sin x$, we find that $x p(x) = 3 \csc x$ and $x^2 q(x) = x^2 \csc x$. Evidently, $x p(x)$ is not even defined at $x = 0$. Hence $x = 0$ is an irregular singular point. On the other hand, the Taylor series of $x \csc x$, about $x = 0$, is

$$x \csc x = 1 + x^2/6 + 7x^4/360 + \dots$$

Noting that $\csc(x \mp n\pi) = (-1)^n \csc x$,

$$\begin{aligned} (x \mp n\pi)p(x) &= 3(-1)^n(x \mp n\pi) \csc(x \mp n\pi)/x \\ &= 3(-1)^n(x \mp n\pi) \csc(x \mp n\pi) \left[\frac{1}{(x \mp n\pi) \pm n\pi} \right]. \end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi)^2 \csc x = (-1)^n(x \mp n\pi)^2 \csc(x \mp n\pi),$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are regular.

25. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = (1 \pm i\sqrt{4\beta - 1})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2} \cos\left(\frac{1}{2}\sqrt{4\beta - 1} \ln|x|\right) + c_2 |x|^{1/2} \sin\left(\frac{1}{2}\sqrt{4\beta - 1} \ln|x|\right).$$

Since the trigonometric factors are bounded, $y(x) \rightarrow 0$ as $x \rightarrow 0$. If $\beta = 1/4$, the roots are equal, and

$$y = c_1 |x|^{1/2} + c_2 |x|^{1/2} \ln|x|.$$

Since $\lim_{x \rightarrow 0} \sqrt{|x|} \ln|x| = 0$, $y(x) \rightarrow 0$ as $x \rightarrow 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = (1 \pm \sqrt{1 - 4\beta})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2 + \sqrt{1 - 4\beta}/2} + c_2 |x|^{1/2 - \sqrt{1 - 4\beta}/2}.$$

Evidently, solutions approach zero as long as $1/2 - \sqrt{1 - 4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4.$$

Hence all solutions approach zero for $\beta > 0$.

26. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1, 2$. Hence the general solution, for $x \neq 0$, is $y = c_1 x^{-1} + c_2 x^2$. Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 1, \quad -c_1 + 2c_2 = \gamma$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3}x^{-1} + \frac{1+\gamma}{3}x^2.$$

The solution is bounded, as $x \rightarrow 0$, if $\gamma = 2$.

29. To show that $W[x^\lambda \cos(\mu \ln x), x^\lambda \sin(\mu \ln x)](x) = \mu x^{2\lambda-1}$, we proceed by direct computation. Let $y_1 = x^\lambda \cos(\mu \ln x)$ and $y_2 = x^\lambda \sin(\mu \ln x)$. Then

$$y_1' = \lambda x^{\lambda-1} \cos(\mu \ln x) + x^\lambda \left(-\sin(\mu \ln x) \cdot \frac{\mu}{x} \right) = x^{\lambda-1} (\lambda \cos(\mu \ln x) - \mu \sin(\mu \ln x))$$

$$y_2' = \lambda x^{\lambda-1} \sin(\mu \ln x) + x^\lambda \left(\cos(\mu \ln x) \cdot \frac{\mu}{x} \right) = x^{\lambda-1} (\lambda \sin(\mu \ln x) + \mu \cos(\mu \ln x))$$

and the Wronskian will be

$$\begin{aligned} W[y_1, y_2](x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x^\lambda \cos(\mu \ln x) & x^\lambda \sin(\mu \ln x) \\ x^{\lambda-1} (\lambda \cos(\mu \ln x) - \mu \sin(\mu \ln x)) & x^{\lambda-1} (\lambda \sin(\mu \ln x) + \mu \cos(\mu \ln x)) \end{vmatrix} \\ &= x^\lambda \cos(\mu \ln x) x^{\lambda-1} (\lambda \sin(\mu \ln x) + \mu \cos(\mu \ln x)) \\ &\quad - x^\lambda \sin(\mu \ln x) x^{\lambda-1} (\lambda \cos(\mu \ln x) - \mu \sin(\mu \ln x)) \\ &= x^{2\lambda-1} (\lambda \cos(\mu \ln x) \sin(\mu \ln x) + \mu \cos^2(\mu \ln x) \\ &\quad - \lambda \sin \mu \ln x \cos(\mu \ln x) + \mu \sin^2(\mu \ln x)) \\ &= \mu x^{2\lambda-1} (\cos^2(\mu \ln x) + \sin^2(\mu \ln x)) = \mu x^{2\lambda-1} \end{aligned}$$

31. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{3}{2x} = \frac{3}{2},$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2},$$

so $x = 0$ is a regular singular point. Let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$. Substitution into the ODE results in

$$2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 3x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - (1+x) \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$2 \sum_{n=2}^{\infty} n(n-1)a_n x^n + 3 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty} [2n(n-1)a_n + 3na_n - a_n - a_{n-1}]x^n = 0.$$

Equating the coefficients to zero, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}, \quad n = 2, 3, \dots$$

We conclude that all the a_n are equal to zero. Hence $y(x) = 0$ is the only solution that can be obtained.

33. Based on Problem 32, the change of variable, $x = 1/\xi$, transforms the ODE into the form

$$\xi^4 \frac{d^2 y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim_{\xi \rightarrow 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$ ($x = \infty$) is an irregular singular point.

35. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \left(1 - \frac{1}{\xi^2}\right) \frac{d^2 y}{d\xi^2} + \left[2\xi^3 \left(1 - \frac{1}{\xi^2}\right) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0,$$

that is,

$$(\xi^4 - \xi^2) \frac{d^2 y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \quad \text{and} \quad q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)}.$$

It follows that

$$\begin{aligned} \lim_{\xi \rightarrow 0} \xi p(\xi) &= \lim_{\xi \rightarrow 0} \xi \frac{2\xi}{\xi^2 - 1} = 0, \\ \lim_{\xi \rightarrow 0} \xi^2 q(\xi) &= \lim_{\xi \rightarrow 0} \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)} = -\alpha(\alpha+1). \end{aligned}$$

Hence $\xi = 0$ ($x = \infty$) is a regular singular point.

36. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2 y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \lambda y = 0,$$

that is,

$$\xi^4 \frac{d^2 y}{d\xi^2} + 2(\xi^3 + \xi) \frac{dy}{d\xi} + \lambda y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2 + 1)}{\xi^3} \quad \text{and} \quad q(\xi) = \frac{\lambda}{\xi^4}.$$

It immediately follows that the limit $\lim_{\xi \rightarrow 0} \xi p(\xi)$ does not exist. Hence $\xi = 0$ ($x = \infty$) is an irregular singular point.

37. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2 y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi \rightarrow 0} \xi p(\xi) = \lim_{\xi \rightarrow 0} \xi \frac{2}{\xi} = 2,$$

but

$$\lim_{\xi \rightarrow 0} \xi^2 q(\xi) = \lim_{\xi \rightarrow 0} \xi^2 \frac{(-1)}{\xi^5}.$$

The latter limit does not exist. Hence $\xi = 0$ ($x = \infty$) is an irregular singular point.

5.5

1.(a) $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \frac{1}{2x} = \frac{1}{2}.$$

$$\lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \frac{1}{2} = 0.$$

Hence $x = 0$ is a regular singular point.

(b) Let

$$y = x^r (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots) = \sum_{n=0}^{\infty} a_n x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}.$$

Substitution into the ODE results in

$$2 \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

That is,

$$2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0.$$

It follows that

$$\begin{aligned} a_0 [2r(r-1) + r] x^r + a_1 [2(r+1)r + r + 1] x^{r+1} \\ + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-2}] x^{r+n} = 0. \end{aligned}$$

Assuming that $a_0 \neq 0$, we obtain the indicial equation $2r^2 - r = 0$, with roots $r_1 = 1/2$ and $r_2 = 0$. It immediately follows that $a_1 = 0$. Setting the remaining coefficients equal to zero, we have

$$a_n = \frac{-a_{n-2}}{(r+n)[2(r+n)-1]}, \quad n = 2, 3, \dots$$

(c) For $r = 1/2$, the recurrence relation becomes

$$a_n = \frac{-a_{n-2}}{n(1+2n)}, \quad n = 2, 3, \dots$$

Since $a_1 = 0$, the odd coefficients are zero. Furthermore, for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(1+4k)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)} = \frac{(-1)^k a_0}{2^k k! 5 \cdot 9 \cdot 13 \dots (4k+1)}.$$

(d) For $r = 0$, the recurrence relation becomes

$$a_n = \frac{-a_{n-2}}{n(2n-1)}, \quad n = 2, 3, \dots$$

Since $a_1 = 0$, the odd coefficients are zero, and for $k = 1, 2, \dots$,

$$a_{2k} = \frac{-a_{2k-2}}{2k(4k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)} = \frac{(-1)^k a_0}{2^k k! 3 \cdot 7 \cdot 11 \dots (4k-1)}.$$

The two linearly independent solutions are

$$\begin{aligned} y_1(x) &= \sqrt{x} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! 5 \cdot 9 \cdot 13 \dots (4k+1)} \right] \\ y_2(x) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! 3 \cdot 7 \cdot 11 \dots (4k-1)}. \end{aligned}$$

3.(a) Note that $x p(x) = 0$ and $x^2 q(x) = x$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

and after multiplying both sides of the equation by x ,

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0.$$

It follows that

$$a_0 [r(r-1)] x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + a_{n-1}] x^{r+n} = 0.$$

Setting the coefficients equal to zero, the indicial equation is $r(r-1) = 0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)(r+n-1)}, \quad n = 1, 2, \dots$$

(c) For $r = 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)}, \quad n = 1, 2, \dots$$

Hence for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \dots = \frac{(-1)^n a_0}{n!(n+1)!}.$$

Therefore one solution is

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+1)!}.$$

5.(a) Note that $x p(x) = 1$ and $x^2 q(x) = x - 2$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} - 2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$\begin{aligned} a_0 [r(r-1) + r - 2] x^r \\ + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - 2a_n + a_{n-1}] x^{r+n} = 0. \end{aligned}$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 - 2 = 0$, with roots $r = \pm \sqrt{2}$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)^2 - 2}, \quad n = 1, 2, \dots$$

Note that $(r+n)^2 - 2 = (r+n+\sqrt{2})(r+n-\sqrt{2})$.

(c) For $r = \sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n(n+2\sqrt{2})}, \quad n = 1, 2, \dots$$

It follows that

$$a_n = \frac{(-1)^n a_0}{n!(1+2\sqrt{2})(2+2\sqrt{2})\dots(n+2\sqrt{2})}, \quad n = 1, 2, \dots$$

(d) For $r = -\sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n(n-2\sqrt{2})}, \quad n = 1, 2, \dots,$$

and therefore

$$a_n = \frac{(-1)^n a_0}{n!(1-2\sqrt{2})(2-2\sqrt{2})\dots(n-2\sqrt{2})}, \quad n = 1, 2, \dots$$

The two linearly independent solutions are

$$y_1(x) = x^{\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(1+2\sqrt{2})(2+2\sqrt{2})\dots(n+2\sqrt{2})} \right]$$

$$y_2(x) = x^{-\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!(1-2\sqrt{2})(2-2\sqrt{2})\dots(n-2\sqrt{2})} \right].$$

6.(a) Here $x p(x) = 1 - x$ and $x^2 q(x) = -x$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} - \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

After multiplying both sides by x ,

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} - \sum_{n=0}^{\infty} (r+n)a_n x^{r+n+1} - \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

After adjusting the indices in the last two series, we obtain

$$a_0 [r(r-1) + r] x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - (r+n)a_{n-1}] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 = 0$, with roots $r_1 = r_2 = 0$. Setting the remaining coefficients equal to zero, the recurrence relation is

$$a_n = \frac{a_{n-1}}{r+n}, \quad n = 1, 2, \dots$$

(c) With $r = 0$,

$$a_n = \frac{a_{n-1}}{n}, \quad n = 1, 2, \dots$$

Hence one solution is

$$y_1(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = e^x.$$

8.(a) Dividing through by the leading coefficient, the ODE can be written as

$$y'' - \frac{x}{1-x^2} y' + \frac{\alpha^2}{1-x^2} y = 0.$$

For $x = 1$,

$$p_0 = \lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} \frac{x}{x+1} = \frac{1}{2}.$$

$$q_0 = \lim_{x \rightarrow 1} (x-1)^2 q(x) = \lim_{x \rightarrow 1} \frac{\alpha^2(1-x)}{x+1} = 0.$$

For $x = -1$,

$$p_0 = \lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} \frac{x}{x-1} = \frac{1}{2}.$$

$$q_0 = \lim_{x \rightarrow -1} (x+1)^2 q(x) = \lim_{x \rightarrow -1} \frac{\alpha^2(x+1)}{(1-x)} = 0.$$

Hence $x = -1$ and $x = 1$ are regular singular points. As shown in Example 1, the indicial equation is given by

$$r(r-1) + p_0 r + q_0 = 0.$$

In this case, both sets of roots are $r_1 = 1/2$ and $r_2 = 0$.

(b) Let $t = x - 1$, and $u(t) = y(t+1)$. Under this change of variable, the differential equation becomes

$$(t^2 + 2t)u'' + (t+1)u' - \alpha^2 u = 0.$$

Based on part (a), $t = 0$ is a regular singular point. Set $u = \sum_{n=0}^{\infty} a_n t^{r+n}$. Substitution into the ODE results in

$$\begin{aligned} & \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n t^{r+n} + 2 \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n t^{r+n-1} \\ & + \sum_{n=0}^{\infty} (r+n)a_n t^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n t^{r+n-1} - \alpha^2 \sum_{n=0}^{\infty} a_n t^{r+n} = 0. \end{aligned}$$

Upon inspection, we can also write

$$\sum_{n=0}^{\infty} (r+n)^2 a_n t^{r+n} + 2 \sum_{n=0}^{\infty} (r+n)(r+n-\frac{1}{2}) a_n t^{r+n-1} - \alpha^2 \sum_{n=0}^{\infty} a_n t^{r+n} = 0.$$

After adjusting the indices in the second series, it follows that

$$a_0 \left[2r(r-\frac{1}{2}) \right] t^{r-1} + \sum_{n=0}^{\infty} \left[(r+n)^2 a_n + 2(r+n+1)(r+n+\frac{1}{2}) a_{n+1} - \alpha^2 a_n \right] t^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, the indicial equation is $2r^2 - r = 0$, with roots $r = 0, 1/2$. The recurrence relation is

$$(r+n)^2 a_n + 2(r+n+1)(r+n+\frac{1}{2}) a_{n+1} - \alpha^2 a_n = 0, \quad n = 0, 1, 2, \dots$$

With $r_1 = 1/2$, we find that for $n \geq 1$,

$$a_n = \frac{4\alpha^2 - (2n-1)^2}{4n(2n+1)} a_{n-1} = (-1)^n \frac{[1-4\alpha^2][9-4\alpha^2] \dots [(2n-1)^2-4\alpha^2]}{2^n(2n+1)!} a_0.$$

With $r_2 = 0$, we find that for $n \geq 1$,

$$a_n = \frac{\alpha^2 - (n-1)^2}{n(2n-1)} a_{n-1} = (-1)^n \frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2] \dots [(n-1)^2-\alpha^2]}{n! \cdot 3 \cdot 5 \dots (2n-1)} a_0.$$

The two linearly independent solutions of the Chebyshev equation are

$$y_1(x) = |x-1|^{1/2} \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{(1-4\alpha^2)(9-4\alpha^2) \dots ((2n-1)^2-4\alpha^2)}{2^n(2n+1)!} (x-1)^n \right)$$

$$y_2(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha(-\alpha)(1-\alpha^2)(4-\alpha^2) \dots ((n-1)^2-\alpha^2)}{n! \cdot 3 \cdot 5 \dots (2n-1)} (x-1)^n.$$

9.(a) Here $xp(x) = 1-x$ and $x^2q(x) = \lambda x$, which are both analytic at $x = 0$. In fact,

$$p_0 = \lim_{x \rightarrow 0} xp(x) = 1 \text{ and } q_0 = \lim_{x \rightarrow 0} x^2q(x) = 0.$$

(b) The indicial equation is $r(r-1) + r = 0$, with roots $r_{1,2} = 0$.

(c) Set

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

Substitution into the ODE results in

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} na_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} na_nx^n + \lambda \sum_{n=0}^{\infty} a_nx^n = 0.$$

It follows that

$$a_1 + \lambda a_0 + \sum_{n=1}^{\infty} [(n+1)^2 a_{n+1} - (n-\lambda)a_n] x^n = 0.$$

Setting the coefficients equal to zero, we find that $a_1 = -\lambda a_0$, and

$$a_n = \frac{(n-1-\lambda)}{n^2} a_{n-1}, \quad n = 2, 3, \dots$$

That is, for $n \geq 2$,

$$a_n = \frac{(n-1-\lambda)}{n^2} a_{n-1} = \dots = \frac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{(n!)^2} a_0.$$

Therefore one solution of the Laguerre equation is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{(n!)^2} x^n.$$

Note that if $\lambda = m$, a positive integer, then $a_n = 0$ for $n \geq m+1$. In that case, the solution is a polynomial

$$y_1(x) = 1 + \sum_{n=1}^m \frac{(-\lambda)(1-\lambda)\dots(n-1-\lambda)}{(n!)^2} x^n.$$

5.6

2.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $xp(x) = -2 - x$ and $x^2q(x) = 2 + x^2$. It follows that

$$p_0 = \lim_{x \rightarrow 0} (-2 - x) = -2$$

$$q_0 = \lim_{x \rightarrow 0} (2 + x^2) = 2$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) - 2r + 2 = 0$, that is, $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$.

3. The coefficients $P(x)$, $Q(x)$, and $R(x)$ are analytic for all $x \in \mathbb{R}$. Hence there are no singular points.

4.(a) $P(x) = 0$ for $x = 0$ and $x = -2$. We note that $p(x) = x^{-1}(x+2)^{-1}/2$, and $q(x) = -(x+2)^{-1}/2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \rightarrow 0} \frac{1}{2(x+2)} = \frac{1}{4}$$

$$q_0 = \lim_{x \rightarrow 0} \frac{-x^2}{2(x+2)} = 0$$

and therefore $x = 0$ is a regular singular point.

For the singularity at $x = -2$,

$$p_0 = \lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{1}{2x} = -\frac{1}{4}$$

$$q_0 = \lim_{x \rightarrow -2} (x+2)^2 q(x) = \lim_{x \rightarrow -2} \frac{-(x+2)}{2} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) For $x = 0$: the indicial equation is given by $r(r-1) + r/4 = 0$, that is, $r^2 - 3r/4 = 0$, with roots $r_1 = 3/4$ and $r_2 = 0$.

For $x = -2$: the indicial equation is given by $r(r-1) - r/4 = 0$, that is, $r^2 - 5r/4 = 0$, with roots $r_1 = 5/4$ and $r_2 = 0$.

5.(a) $P(x) = 0$ only for $x = 0$. Furthermore, $x p(x) = 1/2 + \sin x / 2x$ and $x^2 q(x) = 1$. It follows that

$$p_0 = \lim_{x \rightarrow 0} x p(x) = 1$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x) = 1$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by

$$r(r-1) + r + 1 = 0,$$

that is, $r^2 + 1 = 0$, with complex conjugate roots $r = \pm i$.

7.(a) $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x(x-2)^{-2}(x+2)^{-1}$, and $q(x) = 3(x-2)^{-1}(x+2)^{-1}$. For the singularity at $x = 2$,

$$\lim_{x \rightarrow 2} (x-2)p(x) = \lim_{x \rightarrow 2} \frac{2x}{x^2 - 4},$$

which is undefined. Therefore $x = 2$ is an irregular singular point. For the singularity at $x = -2$,

$$p_0 = \lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{2x}{(x-2)^2} = -\frac{1}{4}$$

$$q_0 = \lim_{x \rightarrow -2} (x+2)^2 q(x) = \lim_{x \rightarrow -2} \frac{3(x+2)}{x-2} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) - r/4 = 0$, that is, $r^2 - 5r/4 = 0$, with roots $r_1 = 5/4$ and $r_2 = 0$.

8.(a) $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x/(4 - x^2)$, and $q(x) = 3/(4 - x^2)$. For the singularity at $x = 2$,

$$p_0 = \lim_{x \rightarrow 2} (x-2)p(x) = \lim_{x \rightarrow 2} \frac{-2x}{x+2} = -1$$

$$q_0 = \lim_{x \rightarrow 2} (x-2)^2 q(x) = \lim_{x \rightarrow 2} \frac{3(2-x)}{x+2} = 0$$

and therefore $x = 2$ is a regular singular point.

For the singularity at $x = -2$,

$$p_0 = \lim_{x \rightarrow -2} (x+2)p(x) = \lim_{x \rightarrow -2} \frac{2x}{2-x} = -1$$

$$q_0 = \lim_{x \rightarrow -2} (x+2)^2 q(x) = \lim_{x \rightarrow -2} \frac{3(x+2)}{2-x} = 0$$

and therefore $x = -2$ is a regular singular point.

(b) For $x = 2$: the indicial equation is given by $r(r-1) - r = 0$, that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

For $x = -2$: the indicial equation is given by $r(r-1) - r = 0$, that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

10.(a) Here $x p(x) = 2x$ and $x^2 q(x) = 6xe^x$. Both of these functions are analytic at $x = 0$, therefore $x = 0$ is a regular singular point. Note that $p_0 = q_0 = 0$.

(b) The indicial equation is given by $r(r-1) = 0$, that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+1} x^{n+1} + 2 \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} + 6e^x \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

After adjusting the indices in the first two series, and expanding the exponential function,

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n + 2 \sum_{n=1}^{\infty} n a_{n-1} x^n + 6a_0 x + (6a_0 + 6a_1)x^2$$

$$+ (6a_2 + 6a_1 + 3a_0)x^3 + (6a_3 + 6a_2 + 3a_1 + a_0)x^4 + \dots = 0.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 2a_1 + 2a_0 + 6a_0 &= 0 \\ 6a_2 + 4a_1 + 6a_0 + 6a_1 &= 0 \\ 12a_3 + 6a_2 + 6a_2 + 6a_1 + 3a_0 &= 0 \\ 20a_4 + 8a_3 + 6a_3 + 6a_2 + 3a_1 + a_0 &= 0 \\ &\vdots \end{aligned}$$

Setting $a_0 = 1$, solution of the system results in $a_1 = -4$, $a_2 = 17/3$, $a_3 = -47/12$, $a_4 = 191/120, \dots$. Therefore one solution is

$$y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \dots$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a y_1(x) \ln x + 1 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

Substituting into the ODE, we obtain

$$a L[y_1(x)] \cdot \ln x + 2a y_1'(x) + 2a y_1(x) - a \frac{y_1(x)}{x} + L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 0.$$

Since $L[y_1(x)] = 0$, it follows that

$$L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = -2a y_1'(x) - 2a y_1(x) + a \frac{y_1(x)}{x}.$$

More specifically,

$$\begin{aligned} \sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + 2 \sum_{n=1}^{\infty} n c_n x^n + 6 + (6+6c_1)x \\ + (6c_2 + 6c_1 + 3)x^2 + \dots = -a + 10ax - \frac{61}{3}ax^2 + \frac{193}{12}ax^3 + \dots \end{aligned}$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 6 &= -a \\ 2c_2 + 8c_1 + 6 &= 10a \\ 6c_3 + 10c_2 + 6c_1 + 3 &= -\frac{61}{3}a \\ 12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 &= \frac{193}{12}a \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $a = -6$. In order to solve the remaining equations, set $c_1 = 0$. Then $c_2 = -33$, $c_3 = 449/6$, $c_4 = -1595/24, \dots$. Therefore a second solution is

$$y_2(x) = -6 y_1(x) \ln x + \left[1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \dots \right].$$

11.(a) After multiplying both sides of the ODE by x , we find that $x p(x) = 0$ and $x^2 q(x) = x$. Both of these functions are analytic at $x = 0$, hence $x = 0$ is a regular singular point.

(b) Furthermore, $p_0 = q_0 = 0$. So the indicial equation is $r(r - 1) = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c) In order to find the solution corresponding to $r_1 = 1$, set $y = x \sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

That is,

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}] x^n = 0.$$

Setting the coefficients equal to zero, we find that for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)}.$$

It follows that

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{(n!)^2(n+1)}.$$

Hence one solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \cdots$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a y_1(x) \ln x + 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

Substituting into the ODE, we obtain

$$a L[y_1(x)] \cdot \ln x + 2a y_1'(x) - a \frac{y_1(x)}{x} + L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = 0.$$

Since $L[y_1(x)] = 0$, it follows that

$$L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] = -2a y_1'(x) + a \frac{y_1(x)}{x}.$$

Now

$$\begin{aligned} L \left[1 + \sum_{n=1}^{\infty} c_n x^n \right] &= 1 + (2c_2 + c_1)x + (6c_3 + c_2)x^2 + (12c_4 + c_3)x^3 \\ &\quad + (20c_5 + c_4)x^4 + (30c_6 + c_5)x^5 + \cdots \end{aligned}$$

Substituting for $y_1(x)$, the right hand side of the ODE is

$$-a + \frac{3}{2}ax - \frac{5}{12}ax^2 + \frac{7}{144}ax^3 - \frac{1}{320}ax^4 + \cdots$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 1 &= -a \\ 2c_2 + c_1 &= \frac{3}{2}a \\ 6c_3 + c_2 &= -\frac{5}{12}a \\ 12c_4 + c_3 &= \frac{7}{144}a \\ &\vdots \end{aligned}$$

Evidently, $a = -1$. In order to solve the second equation, set $c_1 = 0$. We then find that $c_2 = -3/4$, $c_3 = 7/36$, $c_4 = -35/1728$, \dots . Therefore a second solution is

$$y_2(x) = -y_1(x) \ln x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \dots \right].$$

14.(a) After dividing by the leading coefficient, we find that

$$p_0 = \lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} \frac{\gamma - (1 + \alpha + \beta)x}{1 - x} = \gamma.$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} \frac{-\alpha\beta x}{1 - x} = 0.$$

Hence $x = 0$ is a regular singular point. The indicial equation is $r(r - 1) + \gamma r = 0$, with roots $r_1 = 1 - \gamma$ and $r_2 = 0$.

(b) For $x = 1$,

$$p_0 = \lim_{x \rightarrow 1} (x - 1)p(x) = \lim_{x \rightarrow 1} \frac{-\gamma + (1 + \alpha + \beta)x}{x} = 1 - \gamma + \alpha + \beta.$$

$$q_0 = \lim_{x \rightarrow 1} (x - 1)^2 q(x) = \lim_{x \rightarrow 1} \frac{\alpha\beta(x - 1)}{x} = 0.$$

Hence $x = 1$ is a regular singular point. The indicial equation is

$$r^2 - (\gamma - \alpha - \beta)r = 0,$$

with roots $r_1 = \gamma - \alpha - \beta$ and $r_2 = 0$.

(c) Given that $r_1 - r_2$ is not a positive integer, we can set $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution into the ODE results in

$$x(1 - x) \sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} + [\gamma - (1 + \alpha + \beta)x] \sum_{n=1}^{\infty} n a_n x^{n-1} - \alpha\beta \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=1}^{\infty} n(n + 1)a_{n+1}x^n - \sum_{n=2}^{\infty} n(n - 1)a_n x^n + \gamma \sum_{n=0}^{\infty} (n + 1)a_{n+1}x^n$$

$$-(1 + \alpha + \beta) \sum_{n=1}^{\infty} n a_n x^n - \alpha \beta \sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we obtain

$$\gamma a_1 - \alpha \beta a_0 + [(2 + 2\gamma)a_2 - (1 + \alpha + \beta + \alpha\beta)a_1]x + \sum_{n=2}^{\infty} A_n x^n = 0,$$

in which

$$A_n = (n+1)(n+\gamma)a_{n+1} - [n(n-1) + (1 + \alpha + \beta)n + \alpha\beta]a_n.$$

Note that $n(n-1) + (1 + \alpha + \beta)n + \alpha\beta = (n + \alpha)(n + \beta)$. Setting the coefficients equal to zero, we have $\gamma a_1 - \alpha \beta a_0 = 0$, and

$$a_{n+1} = \frac{(n + \alpha)(n + \beta)}{(n + 1)(n + \gamma)} a_n$$

for $n \geq 1$. Hence one solution is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 3!}x^3 + \dots$$

Since the nearest other singularity is at $x = 1$, the radius of convergence of $y_1(x)$ will be at least $\rho = 1$.

(d) Given that $r_1 - r_2$ is not a positive integer, we can set $y = x^{1-\gamma} \sum_{n=0}^{\infty} b_n x^n$. Then substitution into the ODE results in

$$\begin{aligned} x(1-x) \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n-\gamma-1} \\ + [\gamma - (1 + \alpha + \beta)x] \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n-\gamma} - \alpha\beta \sum_{n=0}^{\infty} a_n x^{n+1-\gamma} = 0. \end{aligned}$$

That is,

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma} \\ + \gamma \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n-\gamma} - (1 + \alpha + \beta) \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n+1-\gamma} \\ - \alpha\beta \sum_{n=0}^{\infty} a_n x^{n+1-\gamma} = 0. \end{aligned}$$

After adjusting the indices,

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=1}^{\infty} (n-\gamma)(n-1-\gamma)a_{n-1} x^{n-\gamma} \\ + \gamma \sum_{n=0}^{\infty} (n+1-\gamma)a_n x^{n-\gamma} - (1 + \alpha + \beta) \sum_{n=1}^{\infty} (n-\gamma)a_{n-1} x^{n-\gamma} - \alpha\beta \sum_{n=1}^{\infty} a_{n-1} x^{n-\gamma} = 0. \end{aligned}$$

Combining the series, we obtain

$$\sum_{n=1}^{\infty} B_n x^{n-\gamma} = 0,$$

in which

$$B_n = n(n+1-\gamma)b_n - [(n-\gamma)(n-\gamma+\alpha+\beta) + \alpha\beta] b_{n-1}.$$

Note that $(n-\gamma)(n-\gamma+\alpha+\beta) + \alpha\beta = (n+\alpha-\gamma)(n+\beta-\gamma)$. Setting $B_n = 0$, it follows that for $n \geq 1$,

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)} b_{n-1}.$$

Therefore a second solution is

$$y_2(x) = x^{1-\gamma} \left[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma)1!} x + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma)2!} x^2 + \dots \right].$$

(e) Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{1}{\xi} \left(1 - \frac{1}{\xi} \right) \frac{d^2 y}{d\xi^2} + \left\{ 2\xi^3 \frac{1}{\xi} \left(1 - \frac{1}{\xi} \right) - \xi^2 \left[\gamma - (1+\alpha+\beta) \frac{1}{\xi} \right] \right\} \frac{dy}{d\xi} - \alpha\beta y = 0.$$

That is,

$$(\xi^3 - \xi^2) \frac{d^2 y}{d\xi^2} + [2\xi^2 - \gamma\xi^2 + (-1 + \alpha + \beta)\xi] \frac{dy}{d\xi} - \alpha\beta y = 0.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{(2-\gamma)\xi + (-1 + \alpha + \beta)}{\xi^2 - \xi} \quad \text{and} \quad q(\xi) = \frac{-\alpha\beta}{\xi^3 - \xi^2}.$$

It follows that

$$p_0 = \lim_{\xi \rightarrow 0} \xi p(\xi) = \lim_{\xi \rightarrow 0} \frac{(2-\gamma)\xi + (-1 + \alpha + \beta)}{\xi - 1} = 1 - \alpha - \beta,$$

$$q_0 = \lim_{\xi \rightarrow 0} \xi^2 q(\xi) = \lim_{\xi \rightarrow 0} \frac{-\alpha\beta}{\xi - 1} = \alpha\beta.$$

Hence $\xi = 0$ ($x = \infty$) is a regular singular point. The indicial equation is

$$r(r-1) + (1 - \alpha - \beta)r + \alpha\beta = 0,$$

or $r^2 - (\alpha + \beta)r + \alpha\beta = 0$. Evidently, the roots are $r = \alpha$ and $r = \beta$.

5.7

3. Here $x p(x) = 1$ and $x^2 q(x) = 2x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} + 2 \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

After adjusting the indices in the last series, we obtain

$$a_0 [r(r-1) + r] x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n + 2a_{n-1}] x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the indicial equation is $r^2 = 0$, with double root $r = 0$. Setting the remaining coefficients equal to zero, we have for $n \geq 1$,

$$a_n(r) = -\frac{2}{(n+r)^2} a_{n-1}(r).$$

It follows that

$$a_n(r) = \frac{(-1)^n 2^n}{[(n+r)(n+r-1)\dots(1+r)]^2} a_0, \quad n \geq 1.$$

Since $r = 0$, one solution is given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. First note that

$$\frac{a'_n(r)}{a_n(r)} = -2 \left[\frac{1}{n+r} + \frac{1}{n+r-1} + \dots + \frac{1}{1+r} \right].$$

Setting $r = 0$,

$$a'_n(0) = -2 H_n a_n(0) = -2 H_n \frac{(-1)^n 2^n}{(n!)^2}.$$

Therefore,

$$y_2(x) = y_1(x) \ln x - 2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n H_n}{(n!)^2} x^n.$$

5. Let $y(x) = v(x)/\sqrt{x}$. Then $y' = x^{-1/2} v' - x^{-3/2} v/2$ and $y'' = x^{-1/2} v'' - x^{-3/2} v' + 3x^{-5/2} v/4$. Substitution into the ODE results in

$$\left[x^{3/2} v'' - x^{1/2} v' + 3x^{-1/2} v/4 \right] + \left[x^{1/2} v' - x^{-1/2} v/2 \right] + (x^2 - \frac{1}{4}) x^{-1/2} v = 0.$$

Simplifying, we find that

$$v'' + v = 0,$$

with general solution $v(x) = c_1 \cos x + c_2 \sin x$. Hence

$$y(x) = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x.$$

7. The absolute value of the ratio of consecutive terms is

$$\left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| = \frac{|x|^{2m+2} 2^{2m} (m+1)! m!}{|x|^{2m} 2^{2m+2} (m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}.$$

Applying the ratio test,

$$\lim_{m \rightarrow \infty} \left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| = \lim_{m \rightarrow \infty} \frac{|x|^2}{4(m+2)(m+1)} = 0.$$

Hence the series for $J_1(x)$ converges absolutely for all values of x . Furthermore, since the series for $J_0(x)$ also converges absolutely for all x , term-by-term differentiation results in

$$\begin{aligned} J_0'(x) &= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m! (m-1)!} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{2^{2m+1} (m+1)! m!} = \\ &= -\frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m+1)! m!}. \end{aligned}$$

Therefore, $J_0'(x) = -J_1(x)$.

8.(a) Note that $x p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are both analytic at $x = 0$. Thus $x = 0$ is a regular singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the indicial equation is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$\begin{aligned} a_0 [r(r-1) + r - \nu^2] x^r + a_1 [(r+1)r + (r+1) - \nu^2] x^{r+1} \\ + \sum_{n=2}^{\infty} [(r+n)(r+n-1) a_n + (r+n) a_n - \nu^2 a_n + a_{n-2}] x^{r+n} = 0. \end{aligned}$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2} a_{n-2},$$

for $n \geq 2$. It follows that $a_3 = a_5 = \dots = a_{2m+1} = \dots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)} a_{n-2}.$$

So for $m = 1, 2, \dots$,

$$a_{2m} = \frac{-1}{2m(2m+2\nu)} a_{2m-2} = \frac{(-1)^m}{2^{2m} m!(1+\nu)(2+\nu)\dots(m-1+\nu)(m+\nu)} a_0.$$

Hence one solution is

$$y_1(x) = x^\nu \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\dots(m-1+\nu)(m+\nu)} \left(\frac{x}{2}\right)^{2m} \right].$$

(c) Assuming that $r_1 - r_2 = 2\nu$ is not an integer, simply setting $r = -\nu$ in the above results in a second linearly independent solution

$$y_2(x) = x^{-\nu} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\dots(m-1-\nu)(m-\nu)} \left(\frac{x}{2}\right)^{2m} \right].$$

(d) The absolute value of the ratio of consecutive terms in $y_1(x)$ is

$$\begin{aligned} \left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| &= \frac{|x|^{2m+2} 2^{2m} m!(1+\nu)\dots(m+\nu)}{|x|^{2m} 2^{2m+2} (m+1)!(1+\nu)\dots(m+1+\nu)} \\ &= \frac{|x|^2}{4(m+1)(m+1+\nu)}. \end{aligned}$$

Applying the ratio test,

$$\lim_{m \rightarrow \infty} \left| \frac{a_{2m+2} x^{2m+2}}{a_{2m} x^{2m}} \right| = \lim_{m \rightarrow \infty} \frac{|x|^2}{4(m+1)(m+1+\nu)} = 0.$$

Hence the series for $y_1(x)$ converges absolutely for all values of x . The same can be shown for $y_2(x)$. Note also, that if ν is a positive integer, then the coefficients in the series for $y_2(x)$ are undefined.

9.(a) It suffices to calculate $L[J_0(x) \ln x]$. Indeed,

$$[J_0(x) \ln x]' = J_0'(x) \ln x + \frac{J_0(x)}{x}$$

and

$$[J_0(x) \ln x]'' = J_0''(x) \ln x + 2 \frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}.$$

Hence

$$\begin{aligned} L[J_0(x) \ln x] &= x^2 J_0''(x) \ln x + 2x J_0'(x) - J_0(x) \\ &\quad + x J_0'(x) \ln x + J_0(x) + x^2 J_0(x) \ln x. \end{aligned}$$

Since $x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$,

$$L[J_0(x) \ln x] = 2x J_0'(x).$$

(b) Given that $L[y_2(x)] = 0$, after adjusting the indices in part (a), we have

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty} (n^2 b_n + b_{n-2}) x^n = -2x J_0'(x).$$

Using the series representation of $J_0'(x)$ in Problem 8,

$$b_1x + 2^2b_2x^2 + \sum_{n=3}^{\infty} (n^2b_n + b_{n-2})x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n(2n)x^{2n}}{2^{2n}(n!)^2}.$$

(c) Equating the coefficients on both sides of the equation, we find that

$$b_1 = b_3 = \dots = b_{2m+1} = \dots = 0.$$

Also, with $n = 1$, $2^2b_2 = 1/(1!)^2$, that is, $b_2 = 1/[2^2(1!)^2]$. Furthermore, for $m \geq 2$,

$$(2m)^2b_{2m} + b_{2m-2} = -2 \frac{(-1)^m(2m)}{2^{2m}(m!)^2}.$$

More explicitly,

$$\begin{aligned} b_4 &= -\frac{1}{2^2 4^2} \left(1 + \frac{1}{2}\right) \\ b_6 &= \frac{1}{2^2 4^2 6^2} \left(1 + \frac{1}{2} + \frac{1}{3}\right) \\ &\vdots \end{aligned}$$

It can be shown, in general, that

$$b_{2m} = (-1)^{m+1} \frac{H_m}{2^{2m}(m!)^2}.$$

10. Bessel's equation of order one is

$$x^2 y'' + x y' + (x^2 - 1)y = 0.$$

Based on Problem 9, the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$. Substitution into the ODE results in

$$\begin{aligned} \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0. \end{aligned}$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0[r(r-1) + r - 1]x^r + a_1[(r+1)r + (r+1) - 1]$$

$$+ \sum_{n=2}^{\infty} [(r+n)(r+n-1)a_n + (r+n)a_n - a_n + a_{n-2}]x^{r+n} = 0.$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n(r) = \frac{-1}{(r+n)^2 - 1} a_{n-2}(r) = \frac{-1}{(n+r+1)(n+r-1)} a_{n-2}(r),$$

for $n \geq 2$. It follows that $a_3 = a_5 = \dots = a_{2m+1} = \dots = 0$. Solving the recurrence relation,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2 \dots (r+3)^2(r+1)} a_0.$$

With $r = r_1 = 1$,

$$a_{2m}(1) = \frac{(-1)^m}{2^{2m}(m+1)!m!} a_0.$$

For a second linearly independent solution, we follow the discussion in Section 5.6. Since $r_1 - r_2 = N = 2$, we find that

$$a_2(r) = -\frac{1}{(r+3)(r+1)},$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a = \lim_{r \rightarrow -1} (r+1) a_2(r) = -\frac{1}{2}.$$

Further,

$$(r+1) a_{2m}(r) = \frac{(-1)^m}{(2m+r+1) [(2m+r-1) \dots (3+r)]^2}.$$

Let $A_n(r) = (r+1) a_n(r)$. It follows that

$$\frac{A'_{2m}(r)}{A_{2m}(r)} = -\frac{1}{2m+r+1} - 2 \left[\frac{1}{2m+r-1} + \dots + \frac{1}{3+r} \right].$$

Setting $r = r_2 = -1$, we calculate

$$\begin{aligned} c_{2m}(-1) &= -\frac{1}{2}(H_m + H_{m-1})A_{2m}(-1) \\ &= -\frac{1}{2}(H_m + H_{m-1}) \frac{(-1)^m}{2^m [(2m-2) \dots 2]^2} = -\frac{1}{2}(H_m + H_{m-1}) \frac{(-1)^m}{2^{2m-1} m!(m-1)!}. \end{aligned}$$

Note that $a_{2m+1}(r) = 0$ implies that $A_{2m+1}(r) = 0$, so

$$c_{2m+1}(-1) = \left[\frac{d}{dr} A_{2m+1}(r) \right]_{r=r_2} = 0.$$

Therefore,

$$\begin{aligned} y_2(x) &= -\frac{1}{2} \left[x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!m!} \left(\frac{x}{2}\right)^{2m} \right] \ln x \\ &\quad + \frac{1}{x} \left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{m!(m-1)!} \left(\frac{x}{2}\right)^{2m} \right]. \end{aligned}$$

Based on the definition of $J_1(x)$,

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x} \left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{m!(m-1)!} \left(\frac{x}{2}\right)^{2m} \right].$$