Step-1

The matrix that connects r, θ to x, y where polar coordinates satisfy $x = r \cos \theta$ and $y = r \sin \theta$. Polar are $\int dr d\theta$ includes J:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \cos \theta$$
, $\frac{\partial x}{\partial \theta} = -r \sin \theta$, $\frac{\partial y}{\partial r} = \sin \theta$, $\frac{\partial y}{\partial \theta} = r \cos \theta$

$$J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

Step-2

And inverting the matrix we get

$$J^{-1} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{vmatrix}$$

Step-3

So that

$$\frac{\partial r}{\partial x} = \cos \theta, \frac{\partial r}{\partial y} = \sin \theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-\sin \theta}{r}, \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$$

Step-4

Therefore,

$$\frac{\partial x}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial x}{\partial \theta}.\frac{\partial \theta}{\partial x}$$

$$= (\cos \theta)(\cos \theta) + (-r\sin \theta) \left(\frac{-\sin \theta}{r}\right)$$
$$= \cos^2 \theta + \sin^2 \theta$$

=1

$$=\frac{\partial x}{\partial x}$$

Step-5

Thus, the product $JJ^{-1} = J$ gives the chain rule

$$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$
$$= 1$$