

Step-1

We have to find that which step is not justified in the following:

A has right inverse B

$$\Rightarrow AB = I$$

$$\Rightarrow A^T AB = A^T \text{ or } B = (A^T A)^{-1} A^T$$

But that satisfies $BA = I$; it is a left-inverse.

Step-2

We do not have $B = (A^T A)^{-1} A^T$

Because the inverse of $(A^T A)$ can exist or not

For example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-3

$$\begin{aligned} \Rightarrow AB &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Step-4

And

$$\begin{aligned} (A^T A) &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Step-5

Determinant of $(A^T A)$

$$= 1(2-1) - 1(1-0)$$

$$= 1 - 1$$

$$= 0$$

Therefore $(A^T A)$ has no inverse.

Hence $B = (A^T A)^{-1} A^T$ is not justified