Step-1

We have

$$V_4 = \det\begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}$$

Is designed to the determinant is zero at $x = a, x = b,$ and $x = c$. So, this cannot contain a higher power of x than x^3.

Step-2

By subtracting last row from each of 1, 2, 3 row we get

$$V_4 = \det\begin{bmatrix} 0 & a-x & a^2-x^2 & a^3-x^3 \\ 0 & b-x & b^2-x^2 & b^3-x^3 \\ 0 & c-x & c^2-x^2 & c^3-x^3 \\ 0 & x & x^2 & x^3 \end{bmatrix}$$

$$= \begin{bmatrix} a-x & a^2-x^2 & a^3-x^3 \\ b-x & b^2-x^2 & b^3-x^3 \\ c-x & c^2-x^2 & c^3-x^3 \end{bmatrix}$$ Expanding by 1st column

Step-3

$$V_4 = (a-x)(b-x)(c-x)\begin{bmatrix} 1 & a+x & a^2+ax+x^2 \\ 1 & b+x & b^2+bx+x^2 \\ 1 & c+x & c^2+cx+x^2 \end{bmatrix}$$

$\left(\text{taking } (a\text{-}x),(b-x)\forall(c-x)\text{common from 1st, II and III rows respectively}\right)$

$$= -(x-a)(x-b)(x-c)\begin{bmatrix} 1 & a+x & a^2+ax+x^2 \\ 0 & b-a & b^2-a^2+x(b-a) \\ 0 & c-b & c^2-b^2+x(c-b) \end{bmatrix}$$

Step-4

(Subtracting 2nd row from the thired row and 1st row from the 2nd row)

$$V_4 = -(x-a)(x-b)(x-c)\begin{vmatrix} 1 & a+x & a^2+ax+x^2 \\ 0 & b-a & (b-a)(b+a+x) \\ 0 & c-b & (c-b)(c+b+x) \end{vmatrix}$$

$$= -(x-a)(x-b)(x-c)(b-a)(c-b)\begin{vmatrix} 1 & a+x & a^2+ax+x^2 \\ 0 & 1 & b+a+x \\ 0 & 1 & c+b+x \end{vmatrix}$$

Step-5

$$V_4 = -(x-a)(x-b)(x-c)(b-a)(c-b)\left[(c+b+x)-(a+b+x)\right]$$

$$= (x-a)(x-b)(x-c)(a-b)(b-c)(c-a)$$

$$= (x-a)(x-b)(x-c)V_3$$

Step-6

Where $V_3 =$ cofactor of x^3 in V_4

$$= \det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

$$= (a-b)(b-c)(c-a)$$

Step-7

Can be obtained as below.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-b & c^2-b^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

Thus $\boxed{V_4 = (x-a)(x-b)(x-c)V_3}$