高数上 2022期中. Calculus I 2022 midterm. 细质答案. detailed solution.

1.(1) A: $x>0$ $y=x\sqrt{\sin x+2}$ $x<0$ $y=-x\sqrt{\sin x+2}$ $\lim\limits_{x\to0^-}y=-\lim\limits_{x\to0^+}y$ no

B: $y=\underset{\sim}{|x|}+\sqrt{\sin x+2}$ $|x|$ exist. no

C: $x>0$ $y=x\sin x$ $x<0$ $y=-x\sin x$ $\lim\limits_{x\to0^-}y=\lim\limits_{x\to0^+}y=0$ yes.

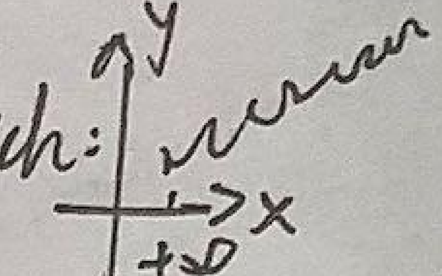
D: $|x|$ exist. no

Answer: C

(2) oblique asymptote $\Rightarrow$ $y=\dfrac{f(x)}{g(x)}$ $\#\underset{\uparrow\text{the largest}}{\text{power}}$ of $f(x)$ $-$ $\#\underset{\uparrow\text{the largest.}}{\text{power}}$ of $g(x)=1$ $\Leftrightarrow$ $\#A-\#B=1$

A: $\#A-\#B=\frac{1}{2}$ no B $\#A-\#B=1$ yes C: $\#A-\#B>1$ for $\sin x$ exist.

D $\#A-\#B>1$ no.

sketch: 

for B: $y=\dfrac{x^4+1}{x^3+\sin x}=\dfrac{x^3+\frac{1}{x}}{x^2+\frac{\sin x}{x}}$ $\lim\limits_{x\to+\infty}y$ $\lim\limits_{x\to+\infty}\dfrac{x^3}{x^2}=\lim\limits_{x\to+\infty}(x)$

for C. D. $\sin x$ in numerator will make $\lim\limits_{x\to+\infty}y$ to shake in $x\to+\infty$. no oblique asymptote.

Answer: B

(3) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\frac{\cos x}{\sin^2x}+\cos x\right)dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cot x\csc x\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos x\,dx=-[\csc x]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}+[\sin x]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$=-1-(-2)+1-\frac{1}{2}=\frac{3}{2}$ Answer: A

(4) $$f(x)=\begin{cases}\dfrac{1-\cos x}{x} & x>0\\ x\sin\dfrac{1}{x-1} & x\le0\end{cases}$$

$\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}\dfrac{1-\cos x}{x}=\lim\limits_{x\to0^+}\dfrac{2\sin^2\frac{x}{2}}{x}=1\times0=0$

$\lim\limits_{x\to0^-}f(x)=\lim\limits_{x\to0^-}x\sin\dfrac{1}{x-1}=0$ they are equal.

thus A false. so does B.

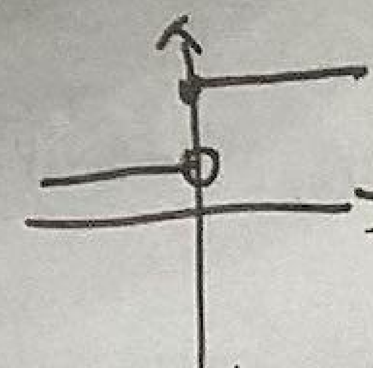
$$f'(x)=\begin{cases}\dfrac{x\sin x-1+\cos x}{x^2} & x>0\\ \sin\dfrac{1}{x-1}+x\cos\dfrac{1}{x-1}\cdot\left(-\dfrac{1}{(x-1)^2}\right) & x\le0\end{cases}$$

$\lim\limits_{x\to0^+}f'(x)=\lim\limits_{x\to0^+}\dfrac{x\sin x-1+1}{x^2}=\lim\limits_{x\to0}\dfrac{\sin x}{x}=1$

$\lim\limits_{x\to0^-}f'(x)=\sin 1$ not equal.

thus C ture D false.

Answer: C.

(5) jump discontinuity :  like this.

A: $\lim_{x\to0} f(x^2) = \lim_{x\to0^+} f(x^2)$ for ~~$x^2\ge0$~~ $\lim_{x\to0} x^2>0$ thus $\lim_{x\to0} f(x^2)$ must exist.

and for B. C. D. they don't solve $\lim_{x\to0^-} \cancel{f(x)}\, g(x) \ne \lim_{x\to0^+} g(x)$ for $g(x)=$ { function in ~~this~~ choices }

Answer : A

2 (1) $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$ Attention! it equals to $\int_0^1 \sin\pi x\,dx$ $\frac{1}{n}$ to $\frac{n}{n}$

no $\int_0^\pi \sin x\,dx$ for ~~this~~ , the add number in limit is not from $\frac{1}{n\pi}$ to $\frac{n\pi}{n\pi}$ but

so $\int_0^1 \sin\pi x\,dx = -\frac{1}{\pi}\cos\pi x\Big|_0^1 = \frac{2}{\pi}$

Answer: $\frac{2}{\pi}$

(2) $l: y=9x+b$ $k=9$ $y=x^3-3x$ $y'=3x^2-3=9$ $x=\pm2$ $y|_{x=\pm2}=2$ or -2

$l: y|_{x=2}=18+b=2$ $b=-16$ $y|_{x=-2}=-18+b=-2$ $b=16$ $\therefore b=\pm16$

Answer ~~b=~~ ±16

(3) $f(x)=\sqrt{x\sqrt{x+\sqrt{x}}}$ $f'(x)=\frac{1}{2\sqrt{x\sqrt{x+\sqrt{x}}}}\left(\sqrt{x+\sqrt{x}}+\frac{x}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right)$

$f'(1)=\frac{1}{2\sqrt[4]{2}}\left(\sqrt{2}+\frac{1}{2\sqrt{2}}\left(\frac{3}{2}\right)\right)=11\cdot2^{-\frac{15}{4}}$

Answer $11\cdot2^{-\frac{15}{4}}$

(4) $\lim_{x\to0}\frac{x\tan x}{1-\cos x}$ $\lim_{x\to0}\frac{x\tan x}{2\sin^2\frac{x}{2}}=\lim_{x\to0}\frac{2\tan x}{x}\left(\frac{(\frac{x}{2})^2}{\sin^2\frac{x}{2}}\right)=\lim_{x\to0}\frac{2\sin x}{x\cos x}(1)=2$

Answer : 2

(5) $f(x)=\tan x$ $f'(x)=\sec^2x$ $f''(x)=2\sec x(\sec x\tan x)=2\sec^2x\tan x$

$f'''(x)=4\sec^2x\tan^2x+\cancel{2\sec x}$ $f''''(x)=8\sec^2x\tan^3x+\cancel{0}+4\sec^2x(2\tan x\cdot\sec^2x)$
$+8\sec^3x\tan x$

$f^{(4)}(0)=0.$

Answer: 0.

3. prove by contridiction.

assume there exist two real roots. x_1 and x_2 for $f(x_1)=f(x_2)=0$ ($x_1 \neq x_2$). $f(x)=x^5+2x-100$

$f'(x)=5x^4+2>0$. thus $f(x)$ increase on $x\in(-\infty,+\infty)$.

but for $f(x_1)=f(x_2)=0$ there must exist $x_0\in(x_1,x_2)$ $f'(x_0)=0$. but $f'(x)>0$

contridiction. thus only has one real root.

4. $\int_0^1 (1+x)^2(1-x)^5\,dx$ set $u=1-x$ $du=-dx$ $\Rightarrow -\int_1^0 (2-u)^2 u^5\,du = \cancel{\int_0^1 (2u^5 - u^7)\,du}$

$\cancel{=\left(\frac{2}{6}u^6-\frac{1}{8}u^8\right)\Big|_0^1 = \frac{5}{24}}$ $=\int_0^1 (u^7-4u^6+4u^5)\,du = \left(\frac{1}{8}u^8-\frac{4}{7}u^7+\frac{2}{3}u^6\right)\Big|_0^1$

$=\frac{1}{8}-\frac{4}{7}+\frac{2}{3}=\frac{37}{168}$: Answer.

5. $L(x)=f(0)+f'(0)(x-0)$ $f'(x)=\frac{2}{(1-x)^2}+\frac{1}{2\sqrt{1+x}}$ $f'(0)=2+\frac{1}{2}=\frac{5}{2}$ $f(0)=2+1=3$

$=3+\frac{5}{2}x$ $\Rightarrow$ Answer: $L(x)=3+\frac{5}{2}x$

6. $\lim_{x\to1^+} f(x)=\lim_{x\to1^+}\frac{2x^2-x+b}{x-1}=a$ $\Rightarrow$ $2x^2-x+b=0$ $x=1\Rightarrow b=-1$

$\frac{2x^2-x-1}{x-1}=\frac{(x-1)(2x+1)}{x-1}=2x+1$ for $x=1$ $a=3$.

Answer: $a=3$ $b=-1$

7 (a) $f(x)=\frac{x^3}{2(x-1)^2}$ $x\neq1$ $f'(x)=\frac{3x^2\cdot2(x-1)^2-2x^3\cdot2(x-1)}{4(x-1)^4}=\frac{3x^2(x-1)-2x^3}{2(x-1)^3}=\frac{x^3-3x^2}{2(x-1)^3}=\frac{x^2(x-3)}{2(x-1)^3}$

$f''(x)=\frac{(3x^2-6x)2(x-1)^3-(x^3-3x^2)6(x-1)^2}{4(x-1)^6}=\frac{(3x^2-6x)(x-1)-(x^3-3x^2)3}{2(x-1)^4}=\frac{3x}{(x-1)^4}$

$(0,0)$ is inflection point. $f'(x)=0$ $x=0$ and $x=3$

$\lim_{x\to-\infty} f(x)=-\infty$ $\lim_{x\to+\infty} f(x)=+\infty$. $f(x)$ decrease on $x\in(-\infty,1)$ and $(1,3)$.
increase on $x\in(3,+\infty)$

thus $f(x)$ has local minima at $x=3$ $f(x)_{min}=\frac{27}{8}$ and $(0,0)$ is inflection point.

Answer: ↗

(b) set l: $y=kx+b$ as asymptotes. obviously $\lim_{x\to1^+} f(x)=\lim_{x\to1^-} f(x)=+\infty$. thus $x=1$ is a verticle asymptotes.

$\because \lim_{x\to\pm\infty} f(x)=\pm\infty$ $\therefore$ so no horizontal asymptotes.

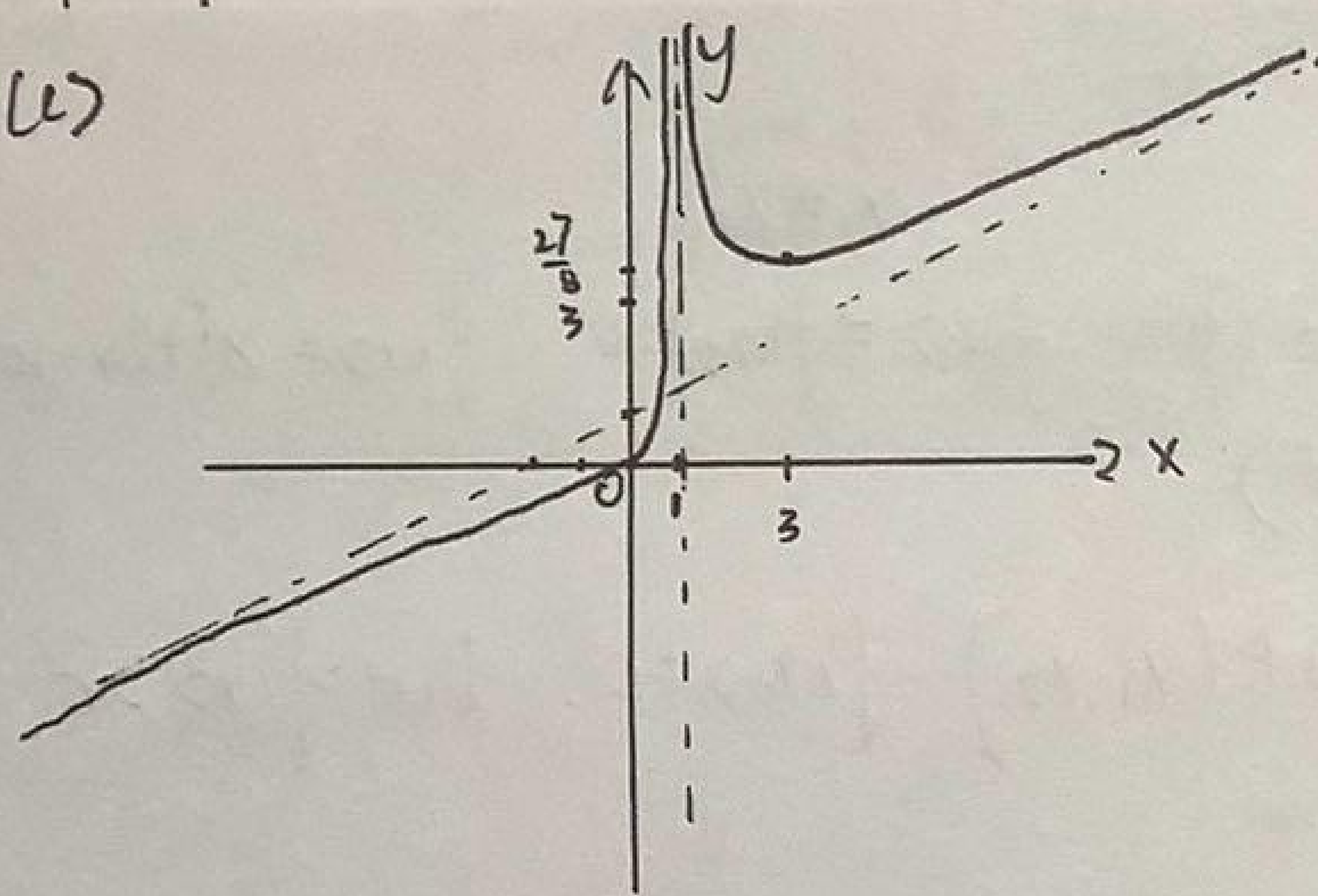
$\lim_{x\to\pm\infty} f'(x)=\frac{1}{2}$ $\Rightarrow k=\frac{1}{2}$ $\lim_{x\to\pm\infty}\left(f(x)-\frac{1}{2}x\right)=\lim_{x\to\pm\infty}\left(\frac{x^3}{2x^2-4x+2}-\frac{x^3-2x^2+x}{2x^2-4x+2}\right)=\lim_{x\to\pm\infty}\left(\frac{2x^2-x}{2x^2-4x+2}\right)$

$\therefore y=\frac{1}{2}x+1$ is an oblique asymptotes. $x=1$ verticle. no horizontal $=1 \Rightarrow b=1$

Answer ↗

(5) jump discontinuity : ... thus $\lim_{x \to 0} f(x)$ ... cho

(6)



sketch showed as left.

8. $x=1 \Rightarrow 2y^3-y^2+3y-4=0 \quad y=1 \Rightarrow (1,1)$ to find tangent line.

derivate: $6y^2y'-2yy'+3y+3xy'-4x=0 \quad y'=\frac{4x-3y}{6y^2-2y+3x} \quad y'|_{x=1\ y=1}=\frac{1}{7}$

$l: y=\frac{1}{7}(x-1)+1=\frac{1}{7}x-\frac{1}{7}+1=\frac{1}{7}x+\frac{6}{7}$

9. set $u=x^2-t^2 \quad du=-2t\,dt \quad F(x)=\int_0^x xtf(x^2-t^2)dt=-\frac{1}{2}\int_{x^2}^{0} xf(u)\,du$

$F(x)=\frac{1}{2}\int_0^{x^2} xf(u)\,du \quad F'(x)=\frac{1}{2}\frac{d}{dx}\int_0^{x^2} xf(u)\,du=\frac{1}{2}(2x^2f(x^2))=x^2f(x^2)$

$\therefore F'(x)=x^2f(x^2)$