MA215 Probability Theory

Assignment 13

1. (a) Suppose X is an continuous r.v. with p.d.f. $f_X(x)$. For any real value $-\infty < t < +\infty$, define a real-valued function, denoted by $M_X(t)$, as $M_X(t) = E(e^{tX})$. Further assume that $M_X(t)$ is well-defined for any $-\infty < t < +\infty$.

   (i) Write down the integration form of $M_X(t)$.

   (ii) If X is non-negative, show that $M_X(t)$ is a nondecreasing function of t.

   (iii) If X is non-negative, show that

   $$\text{if } t < 0 \text{ then } 0 \leqslant M_X(t) \leqslant 1 \text{ and } M_X(0) = 1.$$

   (iv) If $Y = aX + b$ where a and b are two constants. Show that

   $$M_Y(t) = e^{bt} M_X(at).$$

   (v) Suppose X and Y are two independent continuous r.v.s. Show that

   $$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

   (b) Suppose X is a discrete r.v. with p.m.f. $p_k = P(X = x_k)$, $k \geqslant 1$. For any real value $-\infty < t < +\infty$, define a real-valued function, denoted by $M_X(t)$, as $M_X(t) = E(e^{tX})$. Further assume that $M_X(t)$ is well-defined for any $t \in \mathbb{R}$.

   (i) Write down the series form of $M_X(t)$.

   (ii) If X is non-negative, show that $M_X(t)$ is a nondecreasing function of t.

   (iii) If X is non-negative, show that

   $$\text{if } t < 0 \text{ then } 0 \leqslant M_X(t) \leqslant 1 \text{ and } M_X(0) = 1.$$

   (iv) If $Y = aX + b$ where a and b are two constants. Show that

   $$M_Y(t) = e^{bt} M_X(at).$$

   (v) Suppose X and Y are two independent discrete r.v.s. Show that

   $$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

2. Find the m.g.f. of

   (i) the discrete random variable X with $P(X = 4) = 1$;

   (ii) the Bernoulli random variable with parameter p $(0 < p < 1)$, and then applying the properties of m.g.f. to find the m.g.f. of the Binomial random variable with parameter p $(0 < p < 1)$ and n where n is a positive integer;

   (iii) the Poisson random variable with parameter $\lambda > 0$;

(iv) the Geometric random variable with parameter p $(0 < p < 1)$, and then applying the properties of m.g.f. to find the m.g.f. of the Negative Binomial random variable with parameter p and r where r is a positive integer.

(v) the continuous random variable Y with probability density function

$$f_Y(y) = \begin{cases} 2y, & 0 \leqslant y \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

(vi) the random variable $X \sim U[a, b]$ $(-\infty < a < b < +\infty)$.

(vii) the exponential random variable with parameter $\lambda > 0$, and then applying the properties of m.g.f. to find the m.g.f. of the Gamma random variable with parameter $\lambda > 0$ and m where m is a positive integer;

(viii) the general Gamma random variable with parameter $\lambda > 0$ and α, where $\alpha > 0$ may NOT be a positive integer;

(ix) the standard normal random variable $Z \sim N(0, 1)$; Define $X = \mu + \sigma Z$ for real numbers μ, σ with $\sigma > 0$, use the properties of m.g.f. $M_Z(t)$ to find the m.g.f. $M_X(t)$ of X.

3. Suppose that the m.g.f. of a r.v. X is given by $M_X(t) = e^{3(e^t - 1)}$. What is the probability $P(X = 0)$? Also, find $E(X)$ and $\operatorname{Var}(X)$. (Hint: You do not need to do any detailed calculations. Just find what the distribution of the r.v. X is and then use the known results to answer this question.)