Step-1

The nullspace of a matrix has dimension

$n - r$

Where, $r =$ rank of the matrix and $n =$ number of column of matrix

The column space of any matrix is its rank.

Step-2

The objective is to find the dimension and construct a basis for the four subspaces associated with each of the matrices:

$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$ and

$U = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Step-3

Now, consider;

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$$\underline{R_2 - 2R_1} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Therefore

$Ax = 0$

implies

$Ux = 0$

This implies;

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 + 4x_3 = 0$$

$$x_2 = -4x_3$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -4x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The null space of U = Null space of A

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_1, x_2, x_4 \in R \right\}$$

Here $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent vectors.

Therefore the basis for the column space of U;

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So dimension of null space U = dimension of null space $A = 3$

And U is row reduced echelon form of A.

Column 1 is pivot column.

Therefore the column space of $A = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} / x \in R \right\}$

Therefore the basis for column space of A;

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}}$$

And its dimension is 1

The column space of $U = \left\{ x\begin{bmatrix}1\\2\end{bmatrix} / x \in R \right\}$

Therefore the basis for column space of A;

$$\boxed{\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}}$$

$\boxed{\text{And its dimension is 1}}$

Step-4

Now,

$$A^T = \begin{bmatrix}0 & 0\\1 & 2\\4 & 8\\0 & 0\end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix}0 & 0\\1 & 2\\4 & 8\\0 & 0\end{bmatrix}$$

$$\xrightarrow{R_{12}, R_3 - 4R_2} \begin{bmatrix}1 & 2\\0 & 0\\0 & 0\\0 & 0\end{bmatrix}$$

Null space of A^T;

$$A^T = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} : \begin{bmatrix}1 & 2\\0 & 0\\0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix}2x_2\\x_2\end{bmatrix} : x_2 \in R \right\}$$

$$= \left\{ x_2\begin{bmatrix}2\\1\end{bmatrix} : x_2 \in R \right\}$$

Therefore the basis for row space of A is $\begin{bmatrix}2\\1\end{bmatrix}$, and its dimension is 1

Now,

$$U^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix}$$

The columns space of U^T;

$$= \left\{ x \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} / x \in R \right\}$$

Therefore the basis for column space of U^T;

$$= \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

Therefore the row space of U;

$$= \left\{ x \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} : x \in R \right\}$$

Therefore the basis for the row space of U is;

$$= \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

Dimension of row space of $U = 1$