Step-1

Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

To solve the differential equation

$$\frac{du}{dt} = Au$$

$$u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

and find the two pure exponential solutions.

Step-2

The given differential equation is $\frac{du}{dt} = Au$ and $u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

Let pure exponential solution be $u(t) = e^{\lambda t}.u(0)$

The number λ is an Eigen value of A if and only if $|A - \lambda I| = 0$

This implies;

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

This implies;

$$(1-\lambda)(4-\lambda)-(-1)(2) = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda-2)(\lambda-3) = 0$$

This implies;

$\lambda = 2$ and
$\lambda = 3$

Step-3

If $\lambda_1 = 2$ then

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y + z = 0$$

$$\frac{y}{-1} = \frac{z}{1}$$

Eigen vector for $\lambda_1 = 2$ is $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step-4

If $\lambda_2 = 3$ and $(A - \lambda I)x = 0$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2y + z = 0$$

$$\frac{y}{-1} = \frac{z}{2}$$

Eigen vector for $\lambda_2 = 3$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Step-5

In the differential equation, this produces the special solution $u = e^{\lambda t}.x$.

They are the pure exponential solution to $\dfrac{du}{dt} = Au$.

Notice e^{3t} and e^{2t}:

$$u(t) = e^{\lambda_1 t} x_1$$

$$= e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u(t) = e^{\lambda_2 t}.x_2$$

$$= e^{3t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The complete solution $u(t) = C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$

Step-6

The initial condition; $u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

This implies;

$$C_1 x_1 + C_2 x_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

The constants are;

$C_1 = -6$ and
$C_2 = 6$.

Hence the solution to the original equation is;

$$u(t) = -6.e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 6.e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Writing the two components separately,

Step-7

$v(0) = 0$
$w(0) = 6$

Therefore the solution is;

$$\boxed{\begin{aligned} v(t) &= 6e^{2t} - 6e^{3t} \\ w(t) &= -6e^{2t} + 12e^{3t} \end{aligned}}$$