MA327 Homework 3

Due on 1st April

1. Prove that the rotations of a surface of revolution S about its axis are diffeomorphisms of S.

2. Parametrized surfaces are often useful to describe sets Σ which are regular surfaces except for a finite number of points and a finite number of lines. For instance, let C be the trace of a regular parametrized curve $\alpha : (a, b) \to \mathbb{R}^3$ which does not pass through the origin $O = (0, 0, 0)$. Let Σ be the set generated by the displacement of a straight line l passing through a moving point $p \in C$ and the fixed point O (a cone with vertex O, see Figure-1 in the file "HW3-Figures"). (a) Find a parametrized surface $\mathbf{x}$ whose trace is Σ. (b) Find the points where $\mathbf{x}$ is not regular. (c) What should be removed from Σ so that the remaining set is a regular surface?

3. Show that the definition of differentiability of a function $f : V \subset S \to \mathbb{R}$ given in Lecture 5 is equivalent to the following: f is differentiable at $p \in V$ if it it the restriction to V of a differentiable function defined in an open set of $\mathbb{R}^3$ containing p.

4. Let C be a regular curve and let $\alpha : I \subset \mathbb{R} \to C, \beta : J \subset \mathbb{R} \to C$ be two parametrizations of C in a neighborhood of $p \in \alpha(I) \cap \beta(J) =: W$. Let

$$h := \alpha^{-1} \circ \beta : \beta^{-1}(W) \to \alpha^{-1}(W)$$

be the change of parameters. Prove that: (a) h is a diffeomorphism. (b) The absolute value of the arc length of C in W does not depend on which parametrization is chosen to define it, that is,

$$|\int_{t_0}^{t} |\alpha'(t)|dt| = |\int_{\tau_0}^{\tau} |\beta'(\tau)|d\tau|, \quad t = h(\tau), t \in I, \tau \in J.$$

5. Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

6. Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z axis.

7. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with everywhere nonzero curvature. Consider the tangent surface of α:

$$\mathbf{x}(t, v) := \alpha(t) + v\alpha'(t), \quad t \in I, v \neq 0.$$

Show that the tangent planes along the curve $\mathbf{x}(Const., v)$ are all equal.

8. Let $f : S \to \mathbb{R}$ given by $f(p) = |p - p_0|^2$, where $p \in S$ and p_0 is a fixed point in $\mathbb{R}^3$. Show that $df_p(w) = 2w \cdot (p - p_0)$, for $w \in T_p(S)$.

9. Prove that if $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and $S \subset \mathbb{R}^3$ is a regular surface invariant under L, i.e. $L(S) \subset S$, then the restriction $L|S$ is a differential map and

$$dL_p(w) = L(w)$$

for $p \in S, w \in T_p(S)$.

10. Show that the normals to a parametrized surface given by

$$\mathbf{x}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)), \quad f(u) \neq 0, g' \neq 0,$$

all pass through the z axis.

11. A critical point of a differentiable function $f : S \to \mathbb{R}$ defined on a regular surface S is a point $p \in S$ such that $df_p = 0$. (a) Let $f : S \to \mathbb{R}$ be given by $f(p) = |p - p_0|, p \in S, p_0 \notin S$. Show that $p \in S$ is a critical point if and only if the line joining p to p_0 is normal to S at p. (b) Let $h : S \to \mathbb{R}$ be given by $h(p) = p \cdot v$ where $v \in \mathbb{R}^3$ is a unit vector. Show that $p \in S$ is a critical point if and only if v is a normal vector of S at p.

12. Let w be a tangent vector to a regular surface S at a point $p \in S$ and let $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(\bar{u},\bar{v})$ be two parametrizations at p. Suppose that the expressions of w in the bases associated to $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(\bar{u},\bar{v})$ are

$$w = a_1\mathbf{x}_u + a_2\mathbf{x}_v$$

and

$$w = b_1\bar{\mathbf{x}}_{\bar{u}} + b_2\bar{\mathbf{x}}_{\bar{v}}.$$

Show that the coordinates of w are related by

$$b_1 = a_1\frac{\partial\bar{u}}{\partial u} + a_2\frac{\partial\bar{u}}{\partial v},$$

$$b_2 = a_1\frac{\partial\bar{v}}{\partial u} + a_2\frac{\partial\bar{v}}{\partial v},$$

where $\bar{u} = \bar{u}(u,v)$ and $\bar{v} = \bar{v}(u,v)$ are the expressions of the change of coordinates.

13. (Chain Rule) Show that if $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are differentiable maps and $p \in S_1$, then

$$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p.$$

14. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

(a) $\mathbf{x}(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$; ellipsoid.

(b) $\mathbf{x}(u,v) = (au\cos v, bu\sin v, u^2)$; elliptic paraboloid.

(c) $\mathbf{x}(u,v) = (au\cosh v, bu\sinh v, u^2)$; hyperbolic paraboloid.

(d) $\mathbf{x}(u,v) = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u)$; hyperboloid of two sheets.

15. Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (recall it from Question 9 in Homework 2).

16. Show that

$$\mathbf{x}(u,v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha),$$

for $0 < u < \infty, 0 < v < 2\pi$ and $\alpha = Const.$ which is very small and positive, is a parametrization of the cone with 2α as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve

$$\mathbf{x}(c \cdot \exp(v \sin\alpha \cot\beta), v), \quad c = Const., \beta = Const.,$$

intersects the generators of the cone ($v = Const.$) under the constant angle β.