Step-1

Consider,

$$c = (1,0,1,0)$$

The objective is to compute $y = F_4 c$ by the three steps of the Fast Fourier Transform.

The three steps of the Fast Fourier Transform are,

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_0 \\ c_2 \\ c_1 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} F_2 c' \\ F_2 c'' \end{bmatrix} \rightarrow y$$

â€¦â€¦ (1)

Step-2

Since $c = (1,0,1,0)$, from (1),

$$c_0 = 1, c_1 = 0, c_2 = 1, c_3 = 0$$

So, the first step is,

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ â€¦â€¦ (2)

Therefore, the first step of Fast Fourier Transform is $\boxed{c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$.

Step-3

The second step is,

$$c' = \begin{matrix} c_{even} = \\ c_{odd} = \end{matrix} \begin{bmatrix} c_0 \\ c_2 \\ c_1 \\ c_3 \end{bmatrix}$$

$$\begin{matrix} c_{even} = \\ c_{odd} = \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ â€¦â€¦ (3)

Therefore, the second step of Fast Fourier Transform is $\boxed{c' = \begin{bmatrix} c_{even} \\ c_{odd} \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$.

Step-4

The Fourier matrix,

$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & w \end{bmatrix}$, where $w = e^{\frac{i2\pi}{2}} = -1$.

Hence, $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,

The third step is,

$$\begin{aligned} y' &= F_2 c' \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 1 \cdot (-1) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1+1 \\ 1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Step-5

Now, calculate $y'' = F_2 c''$ as follows:

$$
\begin{aligned}
y'' &= F_2 c'' \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 0 + 1\cdot 0 \\ 1\cdot 0 + (-1)\cdot 0 \end{bmatrix}
\end{aligned}
$$

Step-6

$$
= \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Therefore $\begin{matrix} y' = F_2 c' = \\ y'' = F_2 c'' = \end{matrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ â€¦â€¦ (4)

Therefore, the third step of Fast Fourier Transform is $\boxed{\begin{matrix} y' = \\ y'' = \end{matrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$.

Step-7

Substitute $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$ and $c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ in $y = F_4 c$,

$y = F_4 c$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^2 \cdot i \\ 1 & i^2 & \left(i^2\right)^2 & \left(i^2\right)^3 \\ 1 & i^2 \cdot i & \left(i^2\right)^3 & \left(i^2\right)^4 \cdot i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Step-8

Substitute $i^2 = -1$ in above matrix,

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -1 \cdot i \\ 1 & -1 & (-1)^2 & (-1)^3 \\ 1 & -1 \cdot i & (-1)^3 & (-1)^4 \cdot i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + i \cdot 0 + (-1) \cdot 1 + (-i) \cdot 0 \\ 1 \cdot 1 + (-1) \cdot 0 + 1 \cdot 1 + (-1) \cdot 0 \\ 1 \cdot 1 + (-i) \cdot 0 + (-1) \cdot 1 + i \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1+0 \\ 1+0-1+0 \\ 1+0+1+0 \\ 1+0-1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Â Â Â Â

So,

$$y = F_4 c = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$ â€¦â€¦ (5)

Step-9

Substitute equation (2), (3), (4) and (5) values in equation (1),

So,

$$c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_{even} = \\ c_{odd} = \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} y' = \\ y'' = \end{matrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow y = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Hence, the required three steps of the Fast Fourier Transform is,

$$\boxed{c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_{even} = \\ c_{odd} = \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} y' = \\ y'' = \end{matrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow y = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}}.$$