Consider the system of equations:

$$x+3y+5z=1$$

$$2x+6y+9z=5$$
and
$$-x-3y+3z=5$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Step-2

Now consider the system,

$$x + 3y + 5z = 1$$

$$2x + 6y + 9z = 5$$

$$-x-3y+3z=5$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix},$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

Step-3

Given that;

$$Ax = b$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} R_2 - 2R_1, \\ R_3 + R_1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$\frac{1}{3}R_{2}, \frac{1}{6}R_{3} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\underbrace{R_{3} - R_{2}, R_{1} - 3R_{2}}_{2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Step-5

This is in the form Rx = b'

R is in the reduced row echelon form.

First, three columns are pivot columns, thus x, z are pivot variables and y is the variable.

$$x + 3y + 3z = 1$$
$$z = 1$$

Step-6

Now

$$x = 1 - 3y - 3z$$
$$= -3y - 2$$

Thus the complete solution of the system is;

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y - 2 \\ y \\ 1 \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Step-7

Now consider the system:

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\frac{R_2 - 2R_1}{0} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{R_3 - R_2}{0} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This is in the row echelon form, first and third are pivot columns x, z are pivot variables y, t are free variables.

Step-9

$$x+3y+z+2t=1$$

$$2z+4t=1$$

$$R_3-R_2\begin{bmatrix} 1 & 3 & 1 & 2\\ 0 & 0 & 2 & 4\\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x\\ y\\ z\\ t \end{bmatrix} = \begin{bmatrix} 1\\ 1\\ 0 \end{bmatrix}$$

$$2z = 1 - 4t$$

$$z = \frac{1 - 4t}{2}$$

$$= \frac{1}{2} - 2t$$

Step-10

Therefore,

$$x+3y + \frac{1-4t}{2} + 2t = 1$$
$$2x+6y = 1$$
$$x = -3y + \frac{1}{2}$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -3y + \frac{1}{2} \\ y \\ -2t + \frac{1}{2} \\ t \end{bmatrix}$$

Hence the complete solution of the system is;

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$