Step-1

Let W be the space of all polynomials $p(x)$ of degree ≤ 3.

The objective is to find the basis for W.

Step-2

Since W consists the polynomials of degree ≤ 3, so all the polynomials in W are the linear combination of the vectors $1, x, x^2, x^3$.

And the vectors $1, x, x^2, x^3$ are linearly independent.

Hence, the basis for W is $\boxed{\{1, x, x^2, x^3\}}$.

Step-3

Now find the basis for the subspace of space of all polynomials of degree ≤ 3 with $p(1) = 0$.

Let S be the subspace of W such that $p(1) = 0$.

Let $p(x) = ax^3 + bx^2 + cx + d \in S$.

Then

$$\begin{aligned} p(1) &= 0 \\ a(1)^3 + b(1)^2 + c(1) + d &= 0 \\ a + b + c + d &= 0 \end{aligned}$$

Step-4

Here, b, c, and d are free variables.

Therefore, $a = -b - c - d$.

Thus, polynomial $p(x) = ax^3 + bx^2 + cx + d$ can be written as,

$$\begin{aligned} p(x) &= ax^3 + bx^2 + cx + d \\ &= (-b - c - d)x^3 + bx^2 + cx + d \\ &= (-1 + x^3)d + (-1 + x^2)b + (-1 + x)c \\ &= (x^3 - 1)d + (x^2 - 1)b + (x - 1)c \end{aligned}$$

Thus, $p(x) \in S$ can be written as a linear combination of the vectors $(x^3-1),(x^2-1),(x-1)$ and note that the vectors $(x^3-1),(x^2-1),(x-1)$ are linearly independent.

Hence, the basis for the subspace with $p(1)=0$ is $\boxed{\{(x^3-1),(x^2-1),(x-1)\}}$.