Step-1

Therefore, we get

$$
\begin{aligned}
A &= uv^{\mathrm{T}} + wz^{\mathrm{T}} \\
&= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} (b_1, b_2, \ldots, b_n) + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} (d_1, d_2, \ldots, d_n) \\
&= \begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 & a_nb_2 & \ldots & a_nb_n \end{bmatrix} + \begin{bmatrix} c_1d_1 & c_1d_2 & \ldots & c_1d_n \\ c_2d_1 & c_2d_2 & \ldots & c_2d_n \\ \vdots & \vdots & \vdots & \vdots \\ c_nd_1 & c_nd_2 & \ldots & c_nd_n \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1 + c_1d_1 & a_1b_2 + c_2d_2 & \ldots & a_1b_n + c_1d_n \\ a_2b_1 + c_2d_1 & a_2b_2 + c_2d_2 & \ldots & a_2b_n + c_2d_n \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 + c_nd_1 & a_nb_2 + c_nd_2 & \ldots & a_nb_n + c_nd_n \end{bmatrix}
\end{aligned}
$$

Step-2

(a) Since both uv^{T} and wz^{T} are rank one matrices, there is only one independent column in $\begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 & a_nb_2 & \ldots & a_nb_n \end{bmatrix}$ and $\begin{bmatrix} c_1d_1 & c_1d_2 & \ldots & c_1d_n \\ c_2d_1 & c_2d_2 & \ldots & c_2d_n \\ \vdots & \vdots & \vdots & \vdots \\ c_nd_1 & c_nd_2 & \ldots & c_nd_n \end{bmatrix}$.

Therefore, the column space of A can be spanned by the vectors u $\boxed{\text{and}}$ w.

Step-3

(b) As before, both uv^{T} and wz^{T} are rank one matrices. Therefore, there is only one independent row in $\begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ a_2b_1 & a_2b_2 & \ldots & a_2b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 & a_nb_2 & \ldots & a_nb_n \end{bmatrix}$ and $\begin{bmatrix} c_1d_1 & c_1d_2 & \ldots & c_1d_n \\ c_2d_1 & c_2d_2 & \ldots & c_2d_n \\ \vdots & \vdots & \vdots & \vdots \\ c_nd_1 & c_nd_2 & \ldots & c_nd_n \end{bmatrix}$.

Therefore, the row space of A can be spanned by the vectors v $\boxed{\text{and}}$ z.

Step-4

(c) We have $A=\begin{bmatrix} a_1b_1+c_1d_1 & a_1b_2+c_2d_2 & \dots & a_1b_n+c_1d_n \\ a_2b_1+c_2d_1 & a_2b_2+c_2d_2 & \dots & a_2b_n+c_2d_n \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1+c_nd_1 & a_nb_2+c_nd_2 & \dots & a_nb_n+c_nd_n \end{bmatrix}$. Then rank of A is less than 2 if and only if there is at the most only one independent row or one independent column in the matrix of A.

This is possible only when either, u and w are dependent or v and z are dependent.

Step-5

(d) Let us consider the following:

$$\begin{aligned} u &= (1,0,0) \\ v &= (0,0,1) \\ z &= (1,0,0) \\ w &= (0,0,1) \end{aligned}$$

Thus, we get

$$\begin{aligned} A &= uv^{\mathrm{T}} + wz^{\mathrm{T}} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [0,0,1] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [1,0,0] \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

It is clear that rank(A) = 2.

Thus, we have $\boxed{A=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$ and rank of A is 2.