Step-1

Similar matrices have the same eigenvalues. Consider the following 2 by 2 matrix:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Its eigenvalues can be obtained by solving $\det(A-\lambda I)=0$.

Consider

$$\begin{aligned} 0 &= \det(A-\lambda I) \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda)-bc \\ &= \lambda^2-(a+d)\lambda+(ad-bc) \end{aligned}$$

Step-2

Since, any of *a*, *b*, *c*, and *d* can be either 0 or 1, there are only 3 choices (0 or 1 or 2) possible for *a* + *d*.

Suppose $a+d=0$. Then $a=0$ and $d=0$. Now *bc* can be equal to 1 or 0. Thus, the following two equations are possible:

$$\lambda^2=0$$
$$\lambda^2-1=0$$

Suppose $a+d=1$. Then one of *a* and *d* is 1 and the other is 0. Now *bc* can be equal to 1 or 0. Thus, the following two equations are possible:

$$\lambda^2-\lambda=0$$
$$\lambda^2-\lambda-1=0$$

Suppose $a+d=2$. Then both *a* and *d* are equal to 1. Now *bc* can be equal to 1 or 0. Thus, the following two equations are possible:

$$\lambda^2-2\lambda=0$$
$$\lambda^2-2\lambda+1=0$$

Step-3

Let us obtain the roots of the 6 quadratic equations.

For $\lambda^2=0$, the roots are 0 and 0. For $\lambda^2-1=0$, the roots are 1 and â€“1. For $\lambda^2-\lambda=0$, the roots are 0 and 1. For $\lambda^2-\lambda-1=0$, the roots are $\frac{1\pm\sqrt{5}}{2}$. For $\lambda^2-2\lambda=0$, the roots are 0 and 2. Finally, for $\lambda^2-2\lambda+1=0$, the roots are 1 and 1.

Each of these roots corresponds to some eigenvalue of a 2 by 2 matrix. Thus, 6 different eigenvalues are possible. These are: â€“1, 0, 1, 2, $\frac{1 \pm \sqrt{5}}{2}$.

We have obtained 6 distinct quadratic equations and each of these have a unique set of 2 eigenvalues.

Thus, it is clear that there are $\boxed{6}$ families of similar matrices.

Step-4

Let us enlist all possible matrices in each of the 6 families:

Let the characteristic equation be $\lambda^2 = 0$. Then both *a* and *d* should be zero and either one or both of *b* and *c* should be zero. Thus, we get the following three matrices in this family:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - 1 = 0$. Then both *a* and *d* should be zero and both *b* and *c* should be 1. Thus, we get the following (one and only one) matrix in this family:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - \lambda = 0$. Then exactly one of *a* and *d* should be equal to one and at least one of *b* and *c* should be equal to zero. Thus, we get the following six matrices in this family:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - \lambda - 1 = 0$. Then exactly one of *a* and *d* should be equal to one and both *b* and *c* should be equal to one. Thus, we get the following two matrices in this family:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Step-5

Let the characteristic equation be $\lambda^2 - 2\lambda = 0$. Then both *a* and *d* should be equal to one and both *b* and *c* should be equal to one. Thus, we get the following one matrix in this family:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - 2\lambda + 1 = 0$. Then both *a* and *d* should be equal to one and at least one of *b* and *c* should be equal to zero. Thus, we get the following three matrices in this family:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-6

Thus, we have categorised all the 16 matrices in the 6 families.