Step-1

(a)

All sequences like $(1,0,1,0,...)$ that include infinitely many zeros is not a subspace.

Since,

$(1,0,1,0,...)+(1,0,1,0,...)=(2,0,2,0,...)$

That include infinitely many zeros does not belong to the set.

It is not closed under vector addition.

Hence , $\boxed{\text{not a subspace}}$

Step-2

(b)

Consider the set of all sequences $(x_1,x_2,...)$ with $x_j=0$ from some point onward.

Let,

$x=(x_1,x_2,\ldots)$ With $x_j=0$ for some point onward

$y=(y_1,y_2,\ldots)$ With $y_j=0$ for some point onward

Then,

$x+y=(x_1+y_1,x_2+y_2,\ldots)$ With x_l

Where $l=\max\{i,j\},x_l=0$ for some point l onward.

Let c be any scalar.

Then, $cx=(cx_1,cx_2,\cdots\cdots)$ with $cx_j=0$ for some point onward

Hence the given set is $\boxed{\text{a subspace}}$.

Step-3

(c)

Consider the set of all decreasing sequences: $x_{j+1} \le x_j$ for each j.

This set is not subspace of R^{∞}.

Since,

Let $x = (3,2,1,0,-1,\cdots)$, is a decreasing sequence. And let $c = -2$

Then,

$cx = (-6,-4,-2,0,2,\cdots)$ Is an increasing sequence

Therefore, it is not closed under scalar multiplication.

Hence, it is $\boxed{\text{not a subspace.}}$

Step-4

(d)

Given set is the set of all convergent sequences: then x_j have a limit as $j \to \infty$.

Let $S = \{x_j / x_j \text{ is a convergent sequence}\}$

Let x_j and y_j be the elements of S.

Then x_j and y_j are convergent sequences.

Let,

$$\lim_{j\to\infty} x_j = L \text{ and}$$
$$\lim_{j\to\infty} y_j = M .$$

Now,

$$\begin{aligned}\lim_{j\to\infty}(x_j + y_j) &= \lim_{j\to\infty} x_j + \lim_{j\to\infty} y_j \\ &= L + M\end{aligned}$$

Then, $x_j + y_j$ is also convergent

Therefore, $x_j + y_j \in S$

Step-5

Let c be any scalar and let $x_j \in S$

Then, x_j is a convergent sequence

Let,

$$\lim_{j\to\infty} x_j = L$$

Now,

$$\begin{aligned}\lim_{j\to\infty}\left(cx_j\right) &= c\left(\lim_{j\to\infty} x_j\right)\\ &= cL\end{aligned}$$

Then, cx_j is convergent

Therefore, $cx_j \in S$

Hence S is $\boxed{\text{a subspace}}$.

Step-6

(e)

Given set is the set of all arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.

This is a subspace.

Let,

$$x = x_1, x_2, \cdots, x_{n1}, \cdots$$

Then,

$$\begin{aligned}x_2 - x_1 &= x_3 - x_2\\ &\vdots\\ &= d\end{aligned}$$

And,

$$y = y_1, y_2, \cdots, y_{n1}, \cdots$$

Step-7

Then,

$$\begin{aligned}y_2 - y_1 &= y_3 - y_2\\ &\vdots\\ &= t\end{aligned}$$

Now,

$$\begin{aligned}x + y &= x_1 + y_1, x_2 + y_2, \cdots\\ x_2 + y_2 - x_1 - y_1 &= d - t\\ x_3 + y_3 - x_2 - y_2 &= d - t\end{aligned}$$

Therefore, vector addition is closed.

Step-8

And, cx is $cx_1, cx_2, cx_3, \cdots$

Now,

$$\begin{aligned}cx_2 - cx_1 &= cx_3 - cx_2\\ &\vdots\\ &= cd\end{aligned}$$

The scalar multiplication is closed.

Therefore, this is $\boxed{\text{a vector space}}$.

Step-9

(f)

Given set is the set of all geometric progressions $(x_1, kx_1, k^2x_1, \cdots)$ allowing all *k* and x_1.

It is not a subspace.

Since,

Let

$(x_1, kx_1, k^2x_1, \cdots), (y_1, ty_1, t^2y_1, t^3y_1, \cdots)$ Be the geometric progressions.

Addition these two

$(x_1 + y_1,\ kx_1 + ty_1 + k^2x_1 + t^2y_1, \ldots)$

Therefore, addition is not closed.

Hence it is $\boxed{\text{not a vector space}}$.