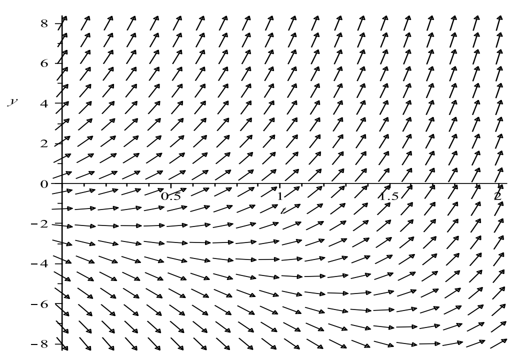
CHAPTER

2

First-Order Differential Equations

2.1

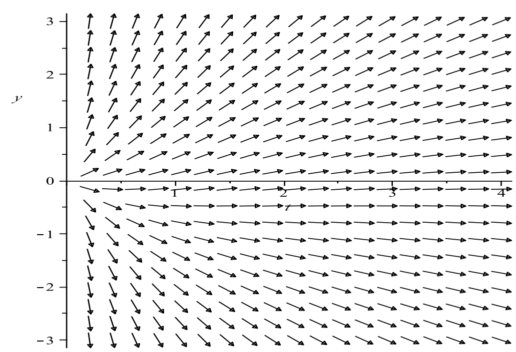
5.(a)



(b) If $y(0) > -3$, solutions eventually have positive slopes, and hence increase without bound. If $y(0) \leq -3$, solutions have negative slopes and decrease without bound.

(c) The integrating factor is $\mu(t) = e^{-\int 2dt} = e^{-2t}$. The differential equation can be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^{t} + c\,e^{2t}$. It follows that all solutions will increase exponentially if $c > 0$ and will decrease exponentially if $c \le 0$. Letting $c = 0$ and then $t = 0$, we see that the boundary of these behaviors is at $y(0) = -3$.

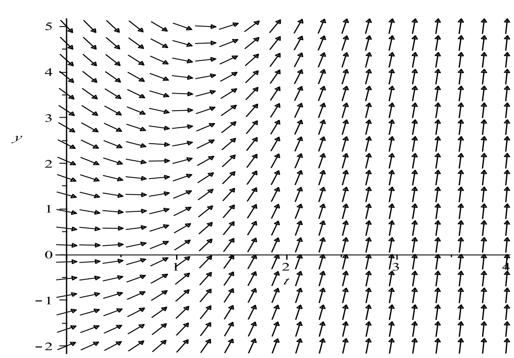
6.(a)



(b) For $y > 0$, the slopes are all positive, and hence the corresponding solutions increase without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c) First divide both sides of the equation by t $(t > 0)$. From the resulting standard form, the integrating factor is $\mu(t) = e^{-\int (1/t)\,dt} = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions diverge, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches zero as $t \to \infty$.

8.(a)



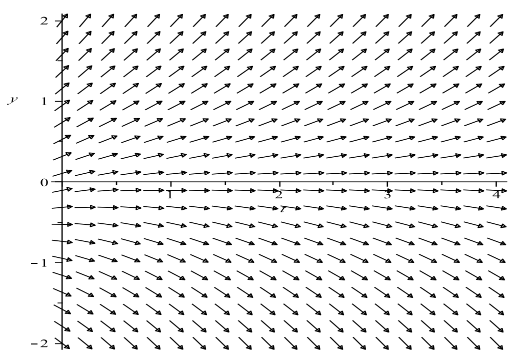
(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $(e^{t/2}\,y/2)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $3t^2 - 12t + 24$.

10. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $(e^{2t}\,y)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2 e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

11. The integrating factor is $\mu(t) = e^{\int (2/t)\,dt} = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = \cos t$. Integrating both sides of the equation results in the general solution $y(t) = \sin t/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to zero gives $c = 0$. Hence the specific solution is $y(t) = \sin t/t^2$.

14.(a)

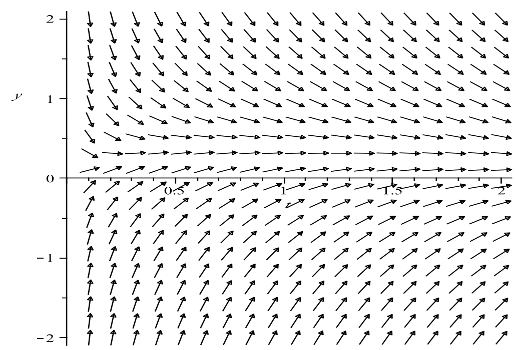


Solutions appear to grow infinitely large in absolute value, with signs depending on the initial value $y(0) = a_0$. The direction field appears horizontal for $a_0 \approx -1/8$.

(b) Dividing both sides of the given equation by 3, the integrating factor is $\mu(t) = e^{-2t/3}$. Multiplying both sides of the original differential equation by $\mu(t)$ and integrating results in $y(t) = (2\,e^{2t/3} - 2\,e^{-\pi t/2} + a(4+3\pi)\,e^{2t/3})/(4+3\pi)$. The qualitative behavior of the solution is determined by the terms containing $e^{2t/3}$: $2\,e^{2t/3} + a(4+3\pi)\,e^{2t/3}$. The nature of the solutions will change when $2 + a(4+3\pi) = 0$. Thus the critical initial value is $a_0 = -2/(4+3\pi)$.

(c) In addition to the behavior described in part (a), when $y(0) = -2/(4+3\pi)$, the solution is $y(t) = (-2\,e^{-\pi t/2})/(4+3\pi)$, and that specific solution will converge to $y = 0$.

15.(a)



As $t \to 0$, solutions increase without bound if $y(1) = a > 0.4$, and solutions decrease without bound if $y(1) = a < 0.4$.

(b) The integrating factor is $\mu(t) = e^{\int (t+1)/t\, dt} = t\, e^t$. The general solution of the differential equation is $y(t) = t\, e^{-t} + c\, e^{-t}/t$. Since $y(1) = a$, we have that $1 + c = ae$. That is, $c = ae - 1$. Hence the solution can also be expressed as $y(t) = t\, e^{-t} + (ae-1)\, e^{-t}/t$. For small values of t, the second term is dominant. Setting $ae - 1 = 0$, the critical value of the parameter is $a_0 = 1/e$.

(c) When $a = 1/e$, the solution is $y(t) = t\, e^{-t}$, which approaches 0 as $t \to 0$.

17. The integrating factor is $\mu(t) = e^{\int (1/2)\, dt} = e^{t/2}$. Therefore the general solution is $y(t) = (4\cos t + 8\sin t)/5 + c\, e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = (4\cos t + 8\sin t - 9\, e^{-t/2})/5$. Differentiating, it follows that $y'(t) = (-4\sin t + 8\cos t + 4.5\, e^{-t/2})/5$ and $y''(t) = (-4\cos t - 8\sin t - 2.25\, e^{-t/2})/5$. Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point in a local maximum. The coordinates of the point are $(1.3643\,, 0.82008)$.

18. The integrating factor is $\mu(t) = e^{\int (2/3)\, dt} = e^{2t/3}$, and the differential equation can be written as $(e^{2t/3}\, y)' = e^{2t/3} - t\, e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 + c\, e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21-6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = (3/2)\ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective value at the stationary point is $y(t_1) = 3/2 + (9/4)\ln 3 - (9/8)\ln(21 - 8y_0)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (21 - 9\, e^{4/3})/8 \approx -1.643$.

19.(a) The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $(e^{t/4}\, y)' = 3\, e^{t/4} + 2\, e^{t/4}\cos 2t$. After integration, we get that the general solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t)/65 + ce^{-t/4}$. Invoking the initial condition, $y(0) = 0$, the specific solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t - 788\, e^{-t/4})/65$. As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an average value of 12, with an amplitude of $8/\sqrt{65}$.

(b) Solving $y(t) = 12$, we obtain the desired value $t \approx 10.0658$.

21. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\, y)' = 3t\, e^{-3t/2} + 2\, e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\, e^t + c\, e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\, e^t + (y_0 + 16/3)\, e^{3t/2}$. Now as $t \to \infty$, the term containing $e^{3t/2}$ will dominate the solution. Its sign will determine the divergence properties. Hence the critical value of the initial condition is $y_0 = -16/3$. The corresponding solution, $y(t) = -2t - 4/3 - 4\, e^t$, will also decrease without bound.

Note on Problems 24-27:

Let $g(t)$ be given, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to 0$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a constant, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\, e^{-at}$. Hence $y(t) = c\, e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

24. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\, y)' = 3e^t$. The general solution is $y(t) = 3 + c\, e^{-t}$.

26. Here $g(t) = 2t - 5$. Consider the linear equation $y' + y = 2 + 2t - 5$. The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\, y)' = (2t - 3)e^t$. The general solution is $y(t) = 2t - 5 + c\, e^{-t}$.

27. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$ .The integrating factor is $\mu(t) = e^t$, and the equation can be written as $(e^t\, y)' = (4 - 2t - t^2)e^t$. The general solution is $y(t) = 4 - t^2 + c\, e^{-t}$.

28.(a) Differentiating y and using the fundamental theorem of calculus we obtain that $y' = Ae^{-\int p(t)dt} \cdot (-p(t))$, and then $y' + p(t)y = 0$.

(b) Differentiating y we obtain that

$$y' = A'(t)e^{-\int p(t)dt} + A(t)e^{-\int p(t)dt} \cdot (-p(t)).$$

If this satisfies the differential equation then

$$y' + p(t)y = A'(t)e^{-\int p(t)dt} = g(t)$$

and the required condition follows.

(c) Let us denote $\mu(t) = e^{\int p(t)dt}$. Then clearly $A(t) = \int \mu(t)g(t)dt$, and after substitution $y = \int \mu(t)g(t)dt \cdot (1/\mu(t))$, which is just Eq. (33).

30. We assume a solution of the form $y = A(t)e^{-\int(1/t)\,dt} = A(t)e^{-\ln t} = A(t)t^{-1}$, where $A(t)$ satisfies $A'(t) = 3t\cos 2t$. This implies that

$$A(t) = \frac{3\cos 2t}{4} + \frac{3t\sin 2t}{2} + c$$

and the solution is

$$y = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}.$$

2.2

Problems 1 through 16 follow the pattern of the examples worked in this section. The first eight problems, however, do not have an initial condition, so the integration constant c cannot be found.

2. The differential equation may be written as $y^{-2}dy = -\sin x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$. That is, $(c - \cos x)y = 1$, in which c is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(c - \cos x)$.

3. Write the differential equation as $\cos^{-2} 2y\,dy = \cos^2 x\,dx$, which also can be written as $\sec^2 2y\,dy = \cos^2 x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan 2y = \sin x\cos x + x + c$.

5. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

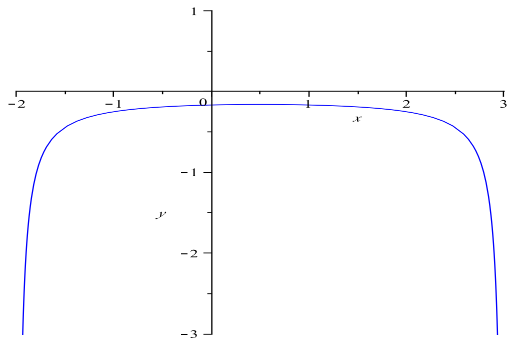
6. Write the differential equation as $(1 + y^2)dy = x^2\,dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3 + c$.

7. Write the differential equation as $y^{-1}\,dy = x^{-1}\,dx$. Integrating both sides of the equation, we obtain the relation $\ln|y| = \ln|x| + c$. Solving for y explicitly gives $y(x) = kx$. Note that k may be positive or negative due to the absolute values in the integrated equation.

8. Write the differential equation as $y\,dy = -x\,dx$. Integrating both sides of the equation, we obtain the relation $(1/2)y^2 = -(1/2)x^2 + c$. The explicit form of the solution is $y(x) = \pm\sqrt{x^2 + c}$. The initial condition would then be used to determine whether the positive of negative solution is to be used for a specific initial value problem.

9.(a) The differential equation is separable, with $y^{-2}dy = (1 - 2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y = 1/(x^2 - x - 6)$.

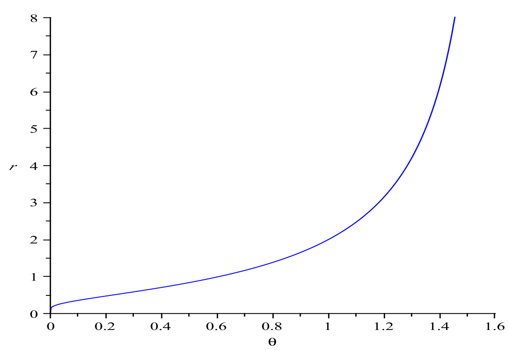
(b)



(c) Note that $x^2 - x - 6 = (x+2)(x-3)$. Hence the solution becomes singular at $x = -2$ and $x = 3$, so the interval of existence is $(-2, 3)$.

12.(a) Write the differential equation as $r^{-2}dr = \theta^{-1}\,d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = \ln\,\theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The explicit form of the solution is $r = 2/(1 - 2\,\ln\,\theta)$.

(b)



(c) Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $\ln\,\theta = 1/2$, that is, $\theta = \sqrt{e}$.

18. The differential equation can be written as $(3y^2 - 4)dy = 3x^2dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x \approx -1.276$, 1.598. Hence the solution is valid for $-1.276 < x < 1.598$.

22.(a) Write the differential equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$. Integrating both sides of the equation, we obtain $\ln|y| - \ln|y-4| = 4t - 4\ln|1+t| + c$. Taking the exponential of both sides $|y/(y-4)| = c\,e^{4t}/(1+t)^4$. It follows that as $t \to \infty$, $|y/(y-4)| = |1 + 4/(y-4)| \to \infty$. That is, $y(t) \to 4$.

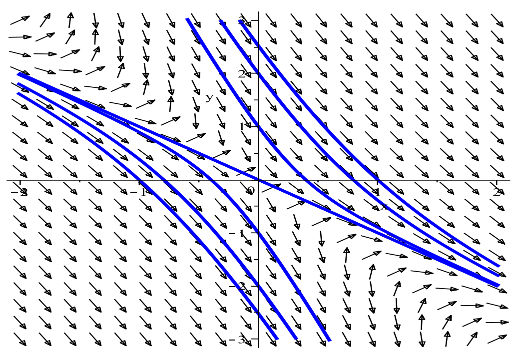
(b) Setting $y(0)=2$, we obtain that $c=1$. Based on the initial condition, the solution may be expressed as $y/(y-4)=-e^{4t}/(1+t)^4$. Note that $y/(y-4)<0$, for all $t\geq 0$. Hence $y<4$ for all $t\geq 0$. Referring back to the differential equation, it follows that y' is always positive. This means that the solution is monotone increasing. We find that the root of the equation $e^{4t}/(1+t)^4=399$ is near $t=2.844$.

(c) Note the $y(t)=4$ is an equilibrium solution. Examining the local direction field we see that if $y(0)>0$, then the corresponding solutions converge to $y=4$. Referring back to part (a), we have $y/(y-4)=[y_0/(y_0-4)]\,e^{4t}/(1+t)^4$, for $y_0\neq 4$. Setting $t=2$, we obtain $y_0/(y_0-4)=(3/e^2)^4y(2)/(y(2)-4)$. Now since the function $f(y)=y/(y-4)$ is monotone for $y<4$ and $y>4$, we need only solve the equations $y_0/(y_0-4)=-399(3/e^2)^4$ and $y_0/(y_0-4)=401(3/e^2)^4$. The respective solutions are $y_0=3.6622$ and $y_0=4.4042$.

29.(a) Observe that $-(4x+3y)/(2x+y)=-2-(y/x)\left[2+(y/x)\right]^{-1}$. Hence the differential equation is homogeneous.

(b) The substitution $y=x\,v$ results in $v+x\,v'=-2-v/(2+v)$. The transformed equation is $v'=-(v^2+5v+4)/(2+v)x$. This equation is separable, with general solution $(v+4)^2\,|v+1|=c/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2\,|x+y|=c$.

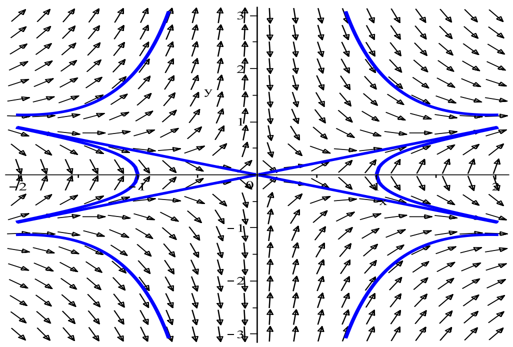
(c) The integral curves are symmetric with respect to the origin.



30.(a) The differential equation can be expressed as $y'=(1/2)(y/x)^{-1}-(3/2)(y/x)$. Hence the equation is homogeneous. The substitution $y=xv$ results in $xv'=(1-5v^2)/2v$. Separating variables, we have $2vdv/(1-5v^2)=dx/x$.

(b) Integrating both sides of the transformed equation yields $-(\ln|1-5v^2|)/5=\ln|x|+c$, that is, $1-5v^2=c/|x|^5$. In terms of the original dependent variable, the general solution is $5y^2=x^2-c/|x|^3$.

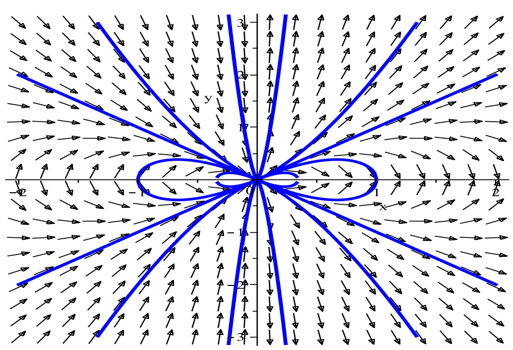
(c) The integral curves are symmetric with respect to the origin.



31.(a) The differential equation can be expressed as $y' = (3/2)(y/x) - (1/2)(y/x)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2 - 1)/2v$, that is, $2vdv/(v^2 - 1) = dx/x$.

(b) Integrating both sides of the transformed equation yields $\ln\left|v^2 - 1\right| = \ln|x| + c$, that is, $v^2 - 1 = c\,|x|$. In terms of the original dependent variable, the general solution is $y^2 = c\,x^2\,|x| + x^2$.

(c) The integral curves are symmetric with respect to the origin.



2.3

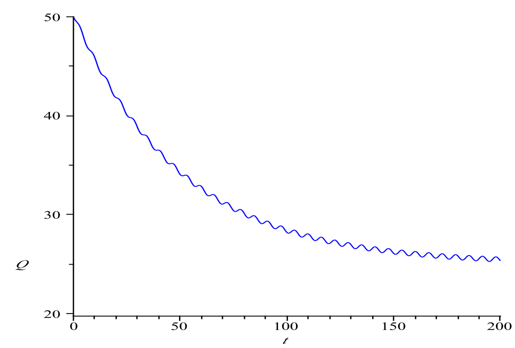
1. Let $Q(t)$ be the amount of dye in the tank at time t. Clearly, $Q(0) = 200$ g. The differential equation governing the amount of dye is $Q'(t) = -2Q(t)/200$. The solution of this separable equation is $Q(t) = Q(0)e^{-t/100} = 200e^{-t/100}$. We need the time T such that $Q(T) = 2$ g. This means we have to solve $2 = 200e^{-T/100}$ and we obtain that $T = -100\ln(1/100) = 100\ln 100 \approx 460.5$ min.

3.(a) Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,(1/4)(1 + (1/2)\sin\,t) = 1/2 + (1/4)\sin\,t$ oz/min. It leaves the tank at a rate of $2\,Q/100$ oz/min. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{Q}{50}\,.$$

The initial amount of salt is $Q_0 = 50$ oz. The governing differential equation is linear, with integrating factor $\mu(t) = e^{t/50}$. Write the equation as $(e^{t/50}Q)' = e^{t/50}(1/2 + (1/4)\sin t)$. The specific solution is $Q(t) = 25 + (12.5\sin t - 625\cos t + 63150\,e^{-t/50})/2501$ oz.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of approximate amplitude 1/4 about a level of 25 oz.

4.(a) Using the Principle of Conservation of Energy, the speed v of a particle falling from a height h is given by

$$\frac{1}{2}mv^2 = mgh\,.$$

(b) The outflow rate is (outflow cross-section area)×(outflow velocity): $\alpha\, a\sqrt{2gh}\,.$ At any instant, the volume of water in the tank is $V(h) = \int_0^h A(u)du$. The time rate of change of the volume is given by $dV/dt = (dV/dh)(dh/dt) = A(h)dh/dt$. Since the volume is decreasing, $dV/dt = -\alpha\, a\sqrt{2gh}\,.$

(c) With $A(h) = \pi$, $a = 0.01\,\pi$, $\alpha = 0.6$, the differential equation for the water level h is $\pi(dh/dt) = -0.006\,\pi\sqrt{2gh}$, with solution $h(t) = 0.000018gt^2 - 0.006\sqrt{2gh(0)}\,t + h(0)$. Setting $h(0) = 3$ and $g = 9.8$, $h(t) = 0.0001764\,t^2 - 0.046\,t + 3$, resulting in $h(t) = 0$ for $t \approx 130.4$ s.

5.(a) The equation governing the value of the investment is $dS/dt = r\,S$. The value of the investment, at any time, is given by $S(t) = S_0e^{rt}$. Setting $S(T) = 2S_0$, the required time is $T = \ln(2)/r$.

(b) For the case $r = .07$, $T \approx 9.9$ yr.

(c) Referring to part (a), $r = \ln(2)/T$. Setting $T = 8$, the required interest rate is to be approximately $r = 8.66\%$.

8.(a) Using Eq.(15) we have $dS/dt - 0.005S = -(800 + 10t)$, $S(0) = 150,000$. Using an integrating factor and integration by parts we obtain that $S(t) = 560,000 - 410,000e^{0.005t} + 2000t$. Setting $S(t) = 0$ and solving numerically for t yields $t = 146.54$ months.

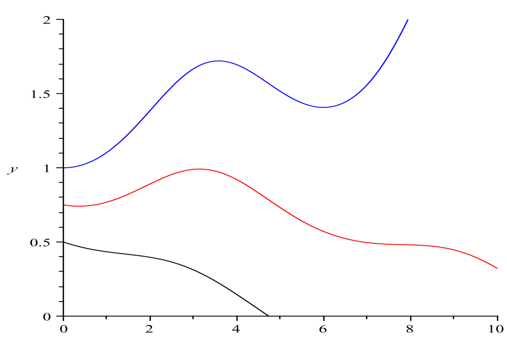
(b) The solution we obtained in part (a) with a general initial condition $S(0) = S_0$ is $S(t) = 560,000 - 560,000e^{0.005t} + S_0e^{0.005t} + 2000t$. Solving the equation $S(240) = 0$ yields $S_0 = 246,758$.

9.(a) Let $Q' = -r\,Q$. The general solution is $Q(t) = Q_0e^{-rt}$. Based on the definition of half-life, consider the equation $Q_0/2 = Q_0e^{-5730\,r}$. It follows that $-5730\,r = \ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ per year.

(b) The amount of carbon-14 is given by $Q(t) = Q_0\, e^{-1.2097\times10^{-4}t}$.

(c) Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times10^{-4}T}$. Solving for the decay time, the apparent age of the remains is approximately $T = 13,305$ years.

11.(a) The differential equation $dy/dt = r(t)\,y - k$ is linear, with integrating factor $\mu(t) = e^{-\int r(t)dt}$. Write the equation as $(\mu\, y)' = -k\,\mu(t)$. Integration of both sides yields the general solution $y = \left[-k\int \mu(\tau)d\tau + y_0\,\mu(0)\right]/\mu(t)$. In this problem, the integrating factor is $\mu(t) = e^{(\cos\, t - t)/5}$.



(b) The population becomes extinct, if $y(t^*) = 0$, for some $t = t^*$. Referring to part (a), we find that $y(t^*) = 0$ when

$$\int_0^{t^*} e^{(\cos\,\tau - \tau)/5} d\tau = 5\,e^{1/5} y_c.$$

It can be shown that the integral on the left hand side increases monotonically, from zero to a limiting value of approximately 5.0893. Hence extinction can happen only if $5\,e^{1/5}y_0 < 5.0893$. Solving $5e^{1/5}y_c = 5.0893$ yields $y_c = 0.8333$.

(c) Repeating the argument in part (b), it follows that $y(t^*) = 0$ when

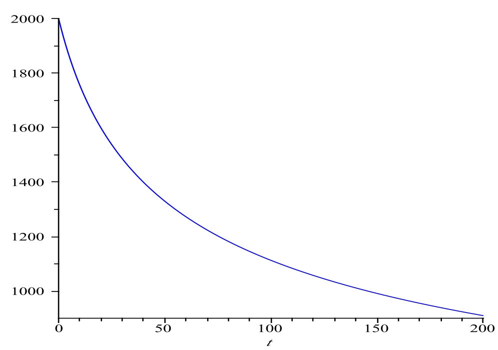
$$\int_0^{t^*} e^{(\cos\,\tau - \tau)/5} d\tau = \frac{1}{k}\, e^{1/5} y_c.$$

Hence extinction can happen only if $e^{1/5} y_0/k < 5.0893$, so $y_c = 4.1667\, k$.

(d) Evidently, y_c is a linear function of the parameter k .

13.(a) The solution of the governing equation satisfies $u^3 = u_0^3/(\,3\,\alpha\, u_0^3\, t + 1\,)$. With the given data, it follows that $u(t) = 2000/\sqrt[3]{6\,t/125 + 1}$.

(b)

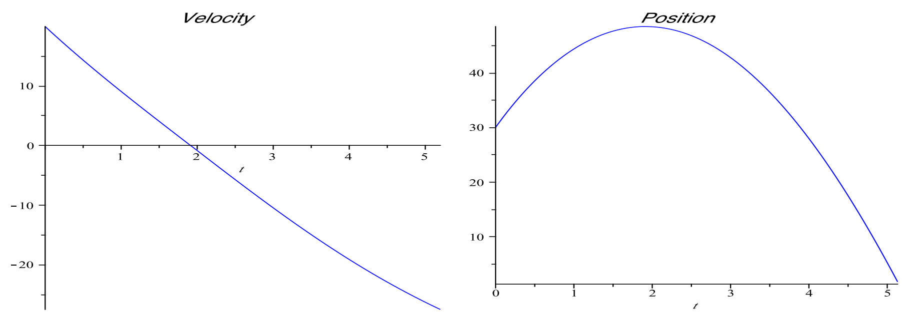


(c) Numerical evaluation results in $u(t) = 600$ for $t \approx 750.77$ s.

18.(a) The differential equation for the upward motion is $mdv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is separable, with $m/(\mu\, v^2 + mg)\, dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\, \tan(0.425 - 0.222\, t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916$ s . Integrating $v(t)$, the position is given by $x(t) = 198.75\, \ln\left[\cos(0.222\, t - 0.425)\right] + 48.57$. Therefore the maximum height is $x(t_1) = 48.56$ m.

(b) The differential equation for the downward motion is $m\, dv/dt = +\mu v^2 - mg$. This equation is also separable, with $m/(mg - \mu\, v^2)\, dv = -dt$. For convenience, set $t = 0$ at the top of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $\ln((44.13 - v)/(44.13 + v)) = t/2.25$. Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, we obtain $x(t) = 99.29\ln(e^{t/2.25}/(1 + e^{t/2.25})^2) + 186.2$. To estimate the duration of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276$ s. Hence the total time that the ball spends in the air is $t_1 + t_2 = 5.192$ s.

(c)



19.(a) Measure the positive direction of motion upward. The equation of motion is given by $mdv/dt = -k\,v - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)\ln[(mg + k\,v_0)/mg]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[(\frac{m}{k})^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g(\frac{m}{k})^2 \ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b) Recall that for $\delta \ll 1$, $\ln(1+\delta) = \delta - \delta^2/2 + \delta^3/3 - \delta^4/4 + \ldots$.

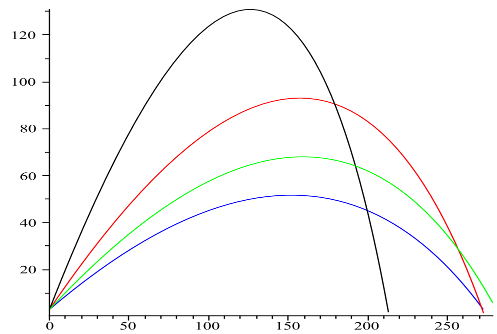
(c) The dimensions of the quantities involved are $[k] = MT^{-1}$, $[v_0] = LT^{-1}$, $[m] = M$ and $[g] = LT^{-2}$. This implies that kv_0/mg is dimensionless.

23.(a) Both equations are linear and separable. Initial conditions: $v(0) = u\cos A$ and $w(0) = u\sin A$. We obtain the solutions $v(t) = (u\cos A)e^{-rt}$ and $w(t) = -g/r + (u\sin A + g/r)e^{-rt}$.

(b) Integrating the solutions in part (a), and invoking the initial conditions, the coordinates are $x(t) = u\cos A(1 - e^{-rt})/r$ and

$$y(t) = -\frac{gt}{r} + \frac{g + ur\,\sin\,A + hr^2}{r^2} - (\frac{u}{r}\sin\,A + \frac{g}{r^2})e^{-rt}.$$

(c)



(d) Let T be the time that it takes the ball to go 350 ft horizontally. Then from above, $e^{-T/5} = (u \cos A - 70)/u \cos A$. At the same time, the height of the ball is given by

$$y(T) = -160T + 803 + 5u \sin A - \frac{(800 + 5u \sin A)(u \cos A - 70)}{u \cos A}.$$

Hence A and u must satisfy the equality

$$800 \ln \left[\frac{u \cos A - 70}{u \cos A} \right] + 803 + 5u \sin A - \frac{(800 + 5u \sin A)(u \cos A - 70)}{u \cos A} = 10$$

for the ball to touch the top of the wall. To find the optimal values for u and A, consider u as a function of A and use implicit differentiation in the above equation to find that

$$\frac{du}{dA} = -\frac{u(u^2 \cos A - 70u - 11200 \sin A)}{11200 \cos A}.$$

Solving this equation simultaneously with the above equation yields optimal values for u and A: $u \approx 145.3\,\text{ft/s}$, $A \approx 0.644\,\text{rad}$.

24.(a) Solving equation (i), $y'(x) = \left[(k^2 - y)/y\right]^{1/2}$. The positive answer is chosen, since y is an increasing function of x.

(b) Let $y = k^2 \sin^2 t$. Then $dy = 2k^2 \sin t \cos t dt$. Substituting into the equation in part (a), we find that

$$\frac{2k^2 \sin t \cos t dt}{dx} = \frac{\cos t}{\sin t}.$$

Hence $2k^2 \sin^2 t dt = dx$.

(c) Setting $\theta = 2t$, we further obtain $k^2 \sin^2(\theta/2)\, d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the origin, we obtain the solutions $x(\theta) = k^2(\theta - \sin\theta)/2$ and (from part (b)) $y(\theta) = k^2(1 - \cos\theta)/2$.

(d) Note that $y/x = (1 - \cos\theta)/(\theta - \sin\theta)$. Setting $x = 1$, $y = 2$, the solution of the equation $(1 - \cos\theta)/(\theta - \sin\theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.

2.4

2. The function $\tan t$ is discontinuous at odd multiples of $\pi/2$. Since $\pi/2 < \pi < 3\pi/2$, the initial value problem has a unique solution on the interval $(\pi/2, 3\pi/2)$.

4. The function $\ln t$ is defined and continuous on the interval $(0\,,\infty)$. At $t=1$, $\ln t = 0$, so the normal form of the differential equation has a singularity there. Also, $\cot t$ is not defined at integer multiples of π, so the initial value problem will have a solution on the interval $(1, \pi)$.

6. The function $f(t\,,y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1-t^2+y^2)} - 2\frac{y\,\ln|ty|}{(1-t^2+y^2)^2}$$

has the same points of discontinuity.

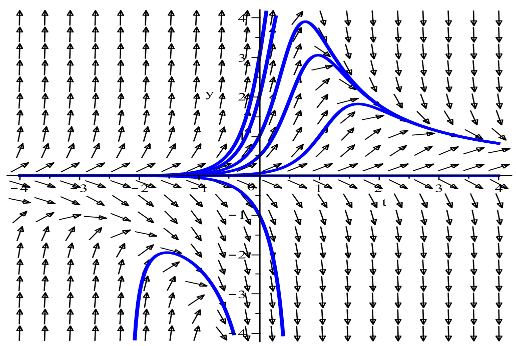
7. $f(t\,,y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

10. The equation is separable, with $dy/y^2 = 2t\,dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1-y_0t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \le 0$, solutions are defined for all t.

11. The equation is separable, with $dy/y^3 = -\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0^2t+1}$. Solutions exist as long as $2y_0^2t+1 > 0$, that is, $2y_0^2t > -1$. If $y_0 \ne 0$, solutions exist for $t > -1/2y_0^2$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t.

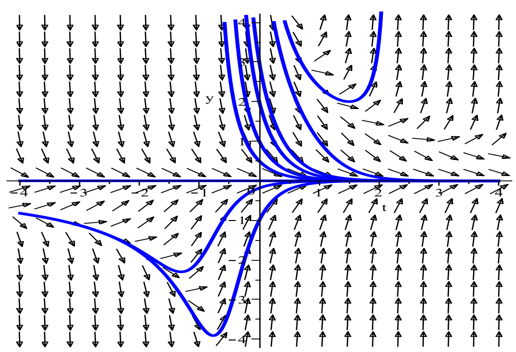
12. The function $f(t\,,y)$ is discontinuous along the straight lines $t = -1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is separable, with $y\,dy = t^2\,dt/(1+t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[(2/3)\ln\left|1+t^3\right| + y_0^2\right]^{1/2}$. Solutions exist as long as $(2/3)\ln\left|1+t^3\right| + y_0^2 \ge 0$, that is, $y_0^2 \ge -(2/3)\ln\left|1+t^3\right|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one), solutions exist as long as $\left|1+t^3\right| \ge e^{-3y_0^2/2}$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \ge e^{-3y_0^2/2} - 1$. It follows that the solutions are valid for $(e^{-3y_0^2/2} - 1)^{1/3} < t < \infty$.

14.



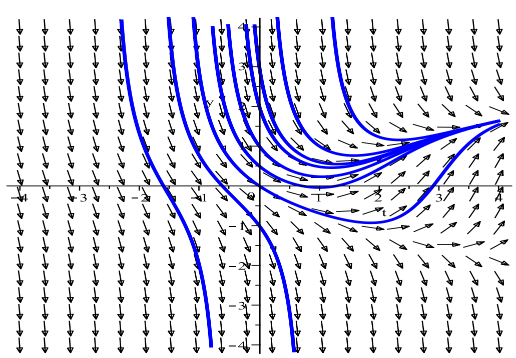
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes eventually become negative, and hence solutions tend to $-\infty$. For $y_0 > 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes eventually become negative, and solutions tend to zero. Furthermore, $y_0 = 0$ is an equilibrium solution. Note that slopes are zero along the curves $y = 0$ and $ty = 3$.

15.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to zero. For $y_0 \leq 9$, the solutions tend to 0; for $y_0 > 9$, the solutions tend to ∞. Also, $y_0 = 0$ is an equilibrium solution.

16.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the right of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions above the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions below the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

17.(a) No. There is no value of $t_0 \geq 0$ for which $(2/3)(t - t_0)^{2/3}$ satisfies the condition $y(1) = 1$.

(b) Yes. Let $t_0 = 1/2$ in Eq.(19).

(c) For $t_0 > 0$, $|y(2)| \leq (4/3)^{3/2} \approx 1.54$.

20. The assumption is $\phi'(t) + p(t)\phi(t) = 0$. But then $c\phi'(t) + p(t)c\phi(t) = 0$ as well.

22.(a) Recalling Eq.(33) in Section 2.1,

$$y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\, ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = 1/\mu(t)$ and $y_2(t) = (1/\mu(t)) \int_{t_0}^{t} \mu(s)g(s)\, ds$.

(b) By definition, $1/\mu(t) = e^{-\int p(t)dt}$. Hence $y_1' = -p(t)/\mu(t) = -p(t)y_1$. That is, $y_1' + p(t)y_1 = 0$.

(c) $y_2' = (-p(t)/\mu(t)) \int_0^t \mu(s)g(s)\, ds + \mu(t)g(t)/\mu(t) = -p(t)y_2 + g(t)$. This implies that $y_2' + p(t)y_2 = g(t)$.

25. Since $n = 3$, set $v = y^{-2}$. It follows that $v' = -2y^{-3}y'$ and $y' = -(y^3/2)v'$. Substitution into the differential equation yields $-(y^3/2)v' - \varepsilon y = -\sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(ve^{2\varepsilon t})' = 2\sigma e^{2\varepsilon t}$. The solution is given by $v(t) = \sigma/\varepsilon + ce^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2} = \pm(\sigma/\varepsilon + ce^{-2\varepsilon t})^{-1/2}$.

27. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore $y(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

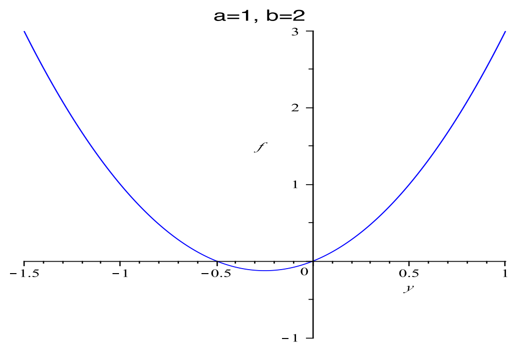
$$y(t) = \begin{cases} e^{-2t}, & 0 \leq t \leq 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

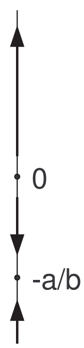
Note the discontinuity of the derivative

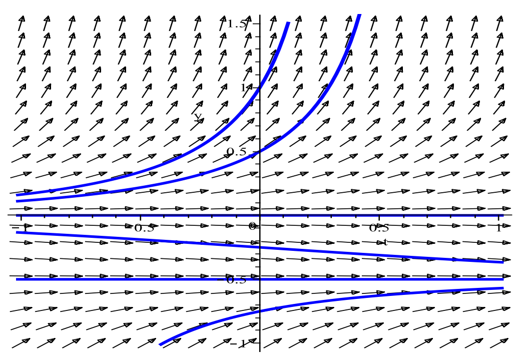
$$y'(t) = \begin{cases} -2e^{-2t}, & 0 < t < 1 \\ -e^{-1-t}, & t > 1 \end{cases}.$$

2.5

1.

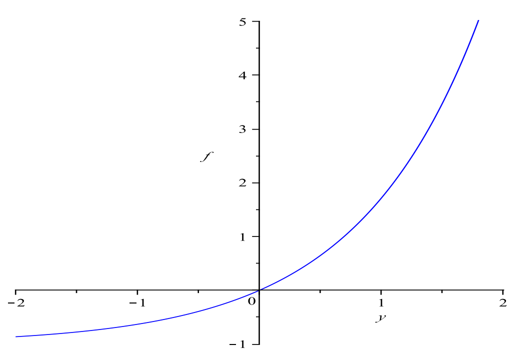


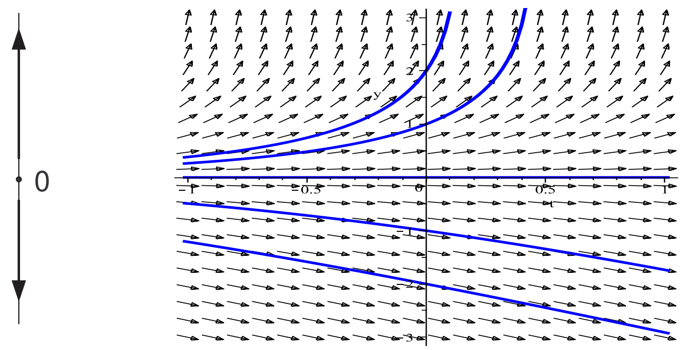




The equilibrium points are $y^* = -a/b$ and $y^* = 0$, and $y' > 0$ when $y > 0$ or $y < -a/b$, and $y' < 0$ when $-a/b < y < 0$, therefore the equilibrium solution $y = -a/b$ is asymptotically stable and the equilibrium solution $y = 0$ is unstable.

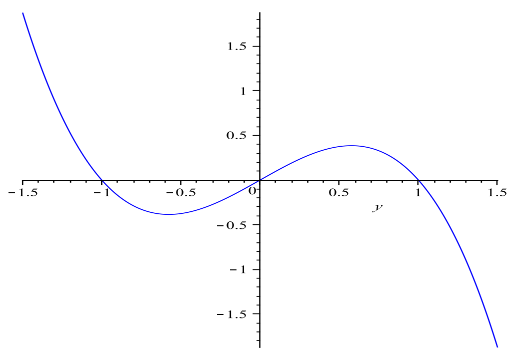
3.

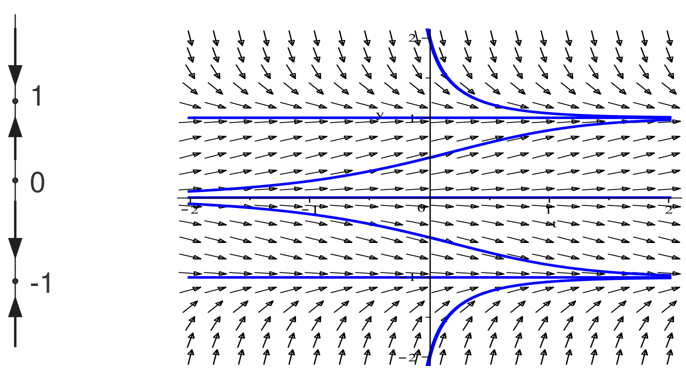




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y > 0$, $y' < 0$ when $y < 0$, hence the equilibrium solution $y = 0$ is unstable.

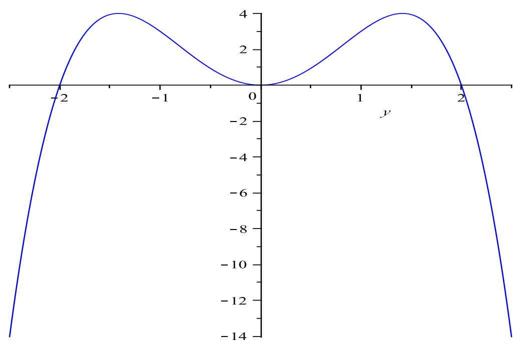
7.

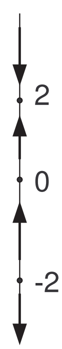


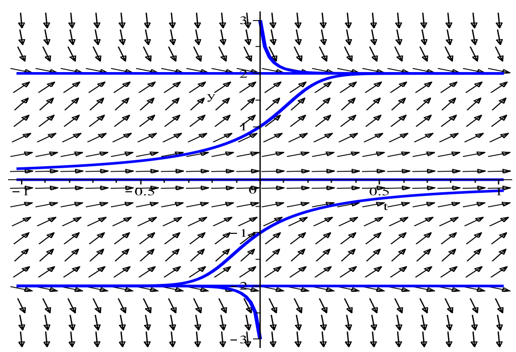


The equilibrium points are $y^* = 0, \pm 1$, and $y' > 0$ for $y < -1$ or $0 < y < 1$ and $y' < 0$ for $-1 < y < 0$ or $y > 1$. The equilibrium solution $y = 0$ is unstable, and the remaining two are asymptotically stable.

8.

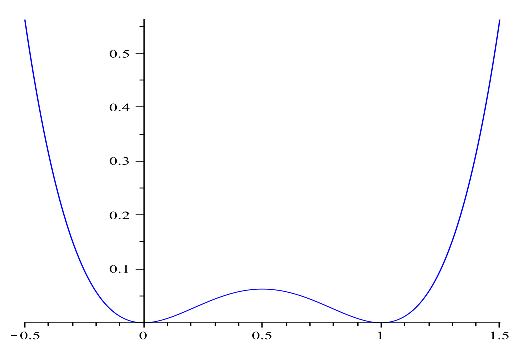


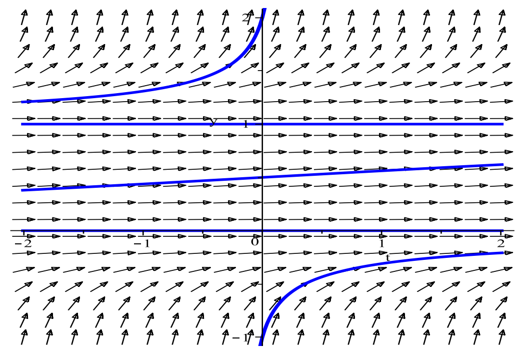




The equilibrium points are $y^* = 0, \pm 2$, and $y' < 0$ when $y < -2$ or $y > 2$, and $y' > 0$ for $-2 < y < 0$ or $0 < y < 2$. The equilibrium solutions $y = -2$ and $y = 2$ are unstable and asymptotically stable, respectively. The equilibrium solution $y = 0$ is semistable.

9.





The equilibrium points are $y^* = 0, 1$. $y' > 0$ for all y except $y = 0$ and $y = 1$. Both equilibrium solutions are semistable.

10. Eq.(10) is $y/(1-(y/K)) = Ce^{rt}$. Clearing the denominator gives $y = Ce^{rt}(1-(y/K)) = Ce^{rt} - Ce^{rt}(y/K)$. Thus $y + Ce^{rt}(y/K) = Ce^{rt}$, or $(1+(C/K)e^{rt})y = Ce^{rt}$. This last equation may easily be solved for y to give $y = Ce^{rt}/(1+(C/K)e^{rt})$. Applying the initial condition $y(0) = y_0$ gives $y_0 = C/(1+(C/K))$, which may be solved for C to give $C = y_0/(1-(y_0/K)) = Ky_0/(K-y_0)$. Using this last value of C in the solution for y gives $y(x) = ((Ky_0e^{rt}/(K-y_0))/(1+(y_0e^{rt}/(K-y_0))$, which may be simplified to yield Eq.(11).

11. To solve Eq.(12) for t, multiply each side of the equation by $(y_0/K) + [1-(y_0/K)]e^{-rt}$ to obtain $y((y_0/K) + [1-(y_0/K)]e^{-rt}) = y_0$, or $y_0/y = (y_0/K) + [1-(y_0/K)]e^{-rt}$. Multiplying each side of this equation by K gives $(y_0K)/y = y_0 + (K-y_0)e^{-rt}$, which may be solved for e^{-rt} to find that

$$e^{-rt} = \frac{(y_0K)/y - y_0}{K - y_0} = \frac{(y_0/y) - (y_0/K)}{1-(y_0/K)} = \frac{(y_0/K)[1-(y/K)]}{(y/K)[1-(y_0/K)]}$$

as in the text. Taking logarithms and dividing by $-r$ gives

$$t = -\frac{1}{r}\ln\frac{(y_0/K)[1-(y/K)]}{(y/K)[1-(y_0/K)]}$$

as given in Eq.(13).

12. To locate the time at which the solution given in Eq.(15) reaches its vertical asymptote, determine when the denominator of the solution is zero. Solving $y_0 + (T-y_0)e^{rt} = 0$ for e^{rt} gives $e^{rt} = y_0/(y_0-T)$, where the fact that $y_0 > T$ ensures that $y_0/(y_0-T) > 0$. Thus the equation $e^{rt} = y_0/(y_0-T)$ has a solution, which is $t = (1/r)\ln(y_0/(y_0-T))$ as given in Eq.(16).

13. To find the inflection points of a solution y to the differential equation $y' = f(y)$, first compute $y'' = f'(y)(dy/dt) = f'(y)f(y)$. Thus possible inflection points occur at solutions to $f'(y) = 0$ or $f(y) = 0$. The function $f(y) = -r(1-(y/T))(1-(y/K))y = 0$ at $y = 0$, $y = T$, and $y = K$, but Figure 2.5.7 shows that y'' does not change sign at any of these points, so they are not inflection points. To consider the points at which $f'(y) = 0$, note that $f(y) = (-r/(KT))(y-T)(y-K)y$, so $f'(y) = (-r/(KT))((y-K)y + (y-T)y + (y-T)(y-K)) = (-r/KT)(3y^2 - 2(K+T) +$

KT). Setting $f'(y) = 0$ and using the quadratic formula gives that $f'(y) = 0$ when $y = (K + T \pm \sqrt{K^2 - KT + T^2})/3$ as given in Eq.(18).

21.(a) The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $y = 0$ is unstable and the equilibrium solution $y = 1$ is asymptotically stable.

(b) The differential equation is separable, with $[y(1-y)]^{-1}\, dy = \alpha\, dt$. Integrating both sides and invoking the initial condition, the solution is

$$y(t) = \frac{y_0\, e^{\alpha t}}{1 - y_0 + y_0\, e^{\alpha t}} = \frac{y_0}{y_0 + (1 - y_0)e^{-\alpha t}}.$$

It is evident that (independent of y_0) $\lim_{t \to -\infty} y(t) = 0$ and $\lim_{t \to \infty} y(t) = 1$.

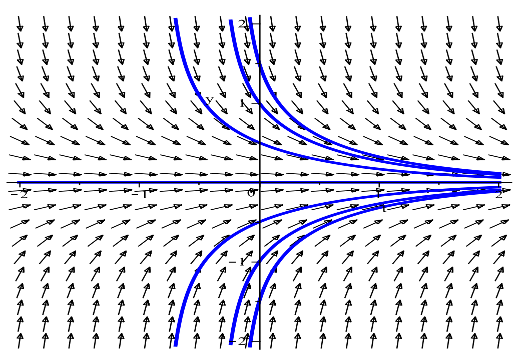
22.(a) $y(t) = y_0\, e^{-\beta t}$.

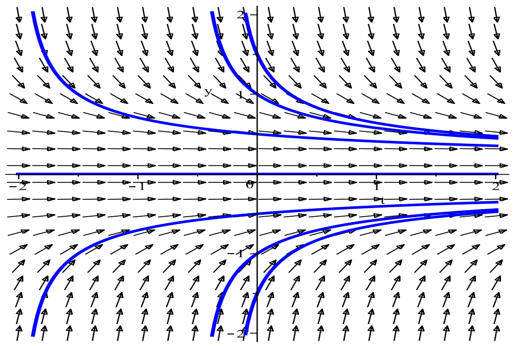
(b) From part (a), $dx/dt = -\alpha x y_0 e^{-\beta t}$. Separating variables, $dx/x = -\alpha y_0 e^{-\beta t} dt$. Integrating both sides, the solution is $x(t) = x_0\, e^{-\alpha\, y_0 (1 - e^{-\beta t})/\beta}$.

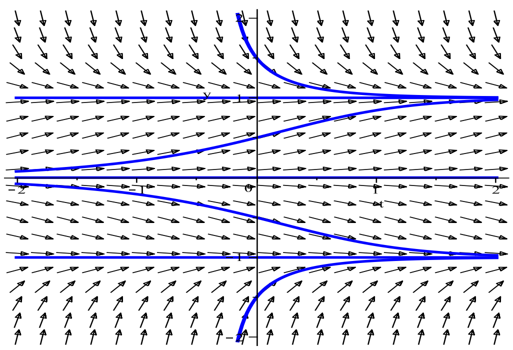
(c) As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_0\, e^{-\alpha\, y_0/\beta}$. Over a long period of time, the proportion of carriers vanishes. Therefore the proportion of the population that escapes the epidemic is the proportion of susceptibles left at that time, $x_0\, e^{-\alpha\, y_0/\beta}$.

25.(a) For $a < 0$, the only critical point is at $y = 0$, which is asymptotically stable. For $a = 0$, the only critical point is at $y = 0$, which is asymptotically stable. For $a > 0$, the three critical points are at $y = 0, \pm\sqrt{a}$. The critical point at $y = 0$ is unstable, whereas the other two are asymptotically stable.

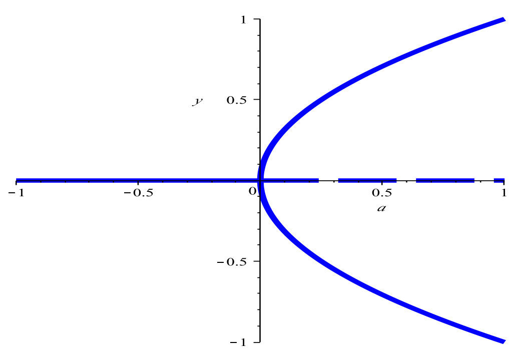
(b) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.





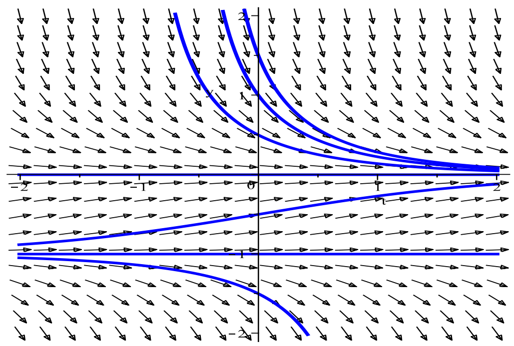


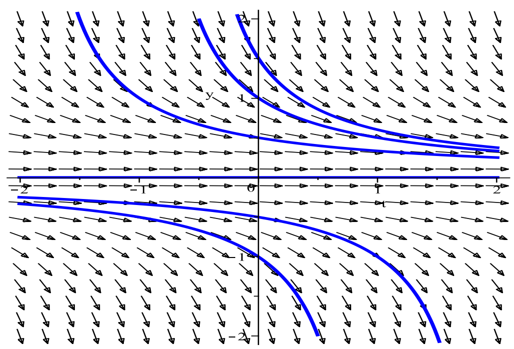
(c)

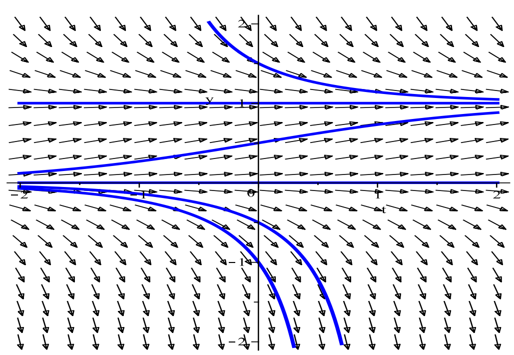


26.(a) $f(y) = y(a-y)$; $\quad f'(y) = a - 2y$. For $a < 0$, the critical points are at $y = a$ and $y = 0$. Observe that $f'(a) > 0$ and $f'(0) < 0$. Hence $y = a$ is unstable and $y = 0$ asymptotically stable. For $a = 0$, the only critical point is at $y = 0$, which is semistable since $f(y) = -y^2$ is concave down. For $a > 0$, the critical points are at $y = 0$ and $y = a$. Observe that $f'(0) > 0$ and $f'(a) < 0$. Hence $y = 0$ is unstable and $y = a$ asymptotically stable.

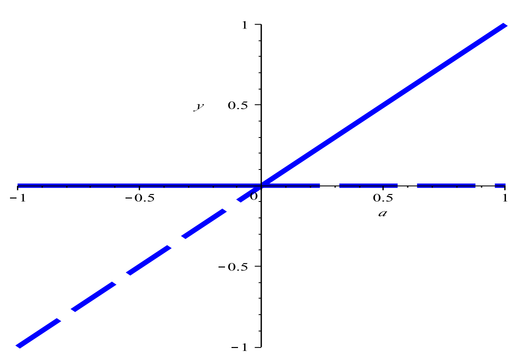
(b) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.







(c)



2.6

1. $M(x,y) = 2x+3$ and $N(x,y) = 2y-2$. Since $M_y = N_x = 0$, the equation is exact. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) = x^2+3x+h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = y^2-2y$. Hence $\psi(x,y) = x^2+3x+y^2-2y$, and the solution is defined implicitly as $x^2+3x+y^2-2y = c$.

2. $M(x,y) = 2x+4y$ and $N(x,y) = 2x-2y$. Note that $M_y \neq N_x$, and hence the differential equation is not exact.

5. Write the equation as $(ax-by)dx+(bx-cy)dy = 0$. Now $M(x,y) = ax-by$ and $N(x,y) = bx-cy$. Since $M_y \neq N_x$, the differential equation is not exact.

7. $M(x,y) = y/x+6x$ and $N(x,y) = \ln x - 2$. Since $M_y = N_x = 1/x$, the given equation is exact. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = y\ln x - 2y + h(x)$. Differentiating with respect to x, $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution is defined implicitly as $3x^2 + y\ln x - 2y = c$.

9. $M(x,y) = 2x-y$ and $N(x,y) = 2y-x$. Since $M_y = N_x = -1$, the equation is exact. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) = x^2-xy+h(y)$. Now $\psi_y = -x+h'(y)$. Equating ψ_y with N results in $h'(y) = 2y$, and hence $h(y) = y^2$. Thus $\psi(x,y) = x^2-xy+y^2$, and the solution is given implicitly as $x^2-xy+y^2 = c$. Invoking the initial condition $y(1) = 3$, the specific solution is $x^2-xy+y^2 = 7$. The explicit form of the solution is $y(x) = (x+\sqrt{28-3x^2})/2$. Hence the solution is valid as long as $3x^2 \leq 28$.

12. $M(x,y) = y\,e^{2xy}+x$ and $N(x,y) = bx\,e^{2xy}$. Note that $M_y = e^{2xy}+2xy\,e^{2xy}$, and $N_x = b\,e^{2xy}+2bxy\,e^{2xy}$. The given equation is exact, as long as $b = 1$. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = e^{2xy}/2+h(x)$. Now differentiating with respect to x, $\psi_x = y\,e^{2xy}+h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = x$, and hence $h(x) = x^2/2$. We conclude that $\psi(x,y) = e^{2xy}/2+x^2/2$. Hence the solution is given implicitly as $e^{2xy}+x^2 = c$.

13. Note that ψ is of the form $\psi(x,y) = f(x)+g(y)$, since each of the integrands is a function of a single variable. It follows that $\psi_x = f'(x)$ and $\psi_y = g'(y)$. That is, $\psi_x = M(x,y_0)$ and $\psi_y = N(x_0,y)$. Furthermore,

$$\frac{\partial^2\psi}{\partial x\partial y}(x_0,y_0) = \frac{\partial M}{\partial y}(x_0,y_0) \text{ and } \frac{\partial^2\psi}{\partial y\partial x}(x_0,y_0) = \frac{\partial N}{\partial x}(x_0,y_0),$$

based on the hypothesis and the fact that the point (x_0,y_0) is arbitrary, $\psi_{xy} = \psi_{yx}$ and $M_y(x,y) = N_x(x,y)$.

14. Observe that $(M(x))_y = (N(y))_x = 0$.

21. The equation is not exact, since $N_x - M_y = 2y-1$. However, $(N_x - M_y)/M =$

$(2y-1)/y$ is a function of y alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is separable, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = e^{2y - \ln y} = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\,e^{2y} - 1/y)dy = 0$. This equation is exact, and it is easy to see that $\psi(x,y) = x\,e^{2y} - \ln|y|$. Therefore the solution of the given equation is defined implicitly by $x\,e^{2y} - \ln|y| = c$.

22. Multiplying both sides of the ODE by $\mu = [xy(2x+y)]^{-1}$, the given equation is equivalent to $\left[(3x+y)/(2x^2+xy)\right]dx + \left[(x+y)/(2xy+y^2)\right]dy = 0$. Rewrite the differential equation as

$$\left[\frac{2}{x} + \frac{2}{2x+y}\right]dx + \left[\frac{1}{y} + \frac{1}{2x+y}\right]dy = 0\,.$$

It is easy to see that $M_y = N_x$. Integrating M with respect to x, while keeping y constant, results in $\psi(x,y) = 2\ln|x| + \ln|2x+y| + h(y)$. Now taking the partial derivative with respect to y, $\psi_y = (2x+y)^{-1} + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 1/y$, and hence $h(y) = \ln|y|$. Therefore $\psi(x,y) = 2\ln|x| + \ln|2x+y| + \ln|y|$, and the solution of the given equation is defined implicitly by $2x^3y + x^2y^2 = c$.

2.7

2. The Euler formula is given by $y_{n+1} = y_n + h(2y_n - 1) = (1+2h)y_n - h$.

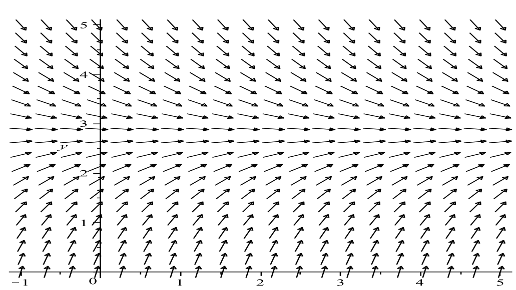
(a) 1.1, 1.22, 1.364, 1.5368

(b) 1.105, 1.23205, 1.38578, 1.57179

(c) 1.10775, 1.23873, 1.39793, 1.59144

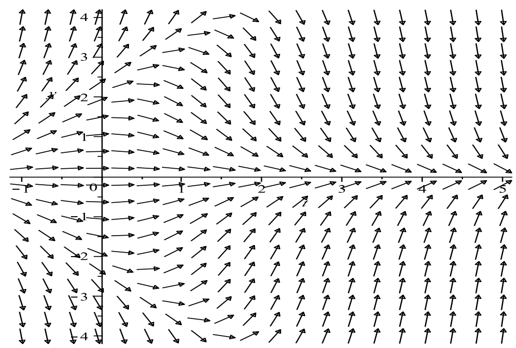
(d) The differential equation is linear with solution $y(t) = (1 + e^{2t})/2$. The values are 1.1107, 1.24591, 1.41106, 1.61277.

5.



All solutions seem to converge to $y = 25/9$.

7.



Solutions with initial conditions $|y(0)| > 2.5$ seem to diverge. On the other hand, solutions with initial conditions $|y(0)| < 2.5$ seem to converge to zero. Also, $y = 0$ is an equilibrium solution.

9. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

(a) 2.30800, 2.49006, 2.60023, 2.66773, 2.70939, 2.73521

(b) 2.30167, 2.48263, 2.59352, 2.66227, 2.70519, 2.73209

(c) 2.29864, 2.47903, 2.59024, 2.65958, 2.70310, 2.73053

(d) 2.29686, 2.47691, 2.58830, 2.65798, 2.70185, 2.72959

10. The Euler formula is $y_{n+1} = (1 + 3h)y_n - ht_n y_n^2$. The initial value is $(t_0, y_0) = (0, 0.5)$.

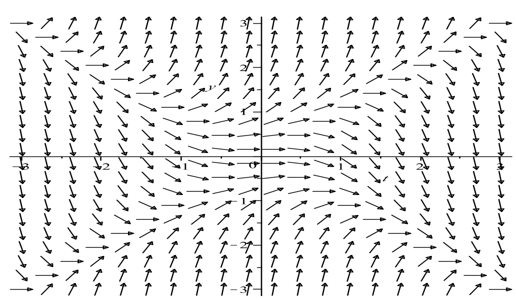
(a) 1.70308, 3.06605, 2.44030, 1.77204, 1.37348, 1.11925

(b) 1.79548, 3.06051, 2.43292, 1.77807, 1.37795, 1.12191

(c) 1.84579, 3.05769, 2.42905, 1.78074, 1.38017, 1.12328

(d) 1.87734, 3.05607, 2.42672, 1.78224, 1.38150, 1.12411

14.(a)



(b) The iteration formula is $y_{n+1} = y_n + h\, y_n^2 - h\, t_n^2$. The critical value α_0 appears to be between 0.67 and 0.68. For $y_0 > \alpha_0$, the iterations diverge.

15.(a) The ODE is linear, with general solution $y(t) = t + ce^t$. Invoking the specified initial condition, $y(t_0) = y_0$, we have $y_0 = t_0 + ce^{t_0}$. Hence $c = (y_0 - t_0)e^{-t_0}$. Thus the solution is given by $\phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b) The Euler formula is $y_{n+1} = (1 + h)y_n + h - h\, t_n$. Now set $k = n + 1$.

(c) We have $y_1 = (1 + h)y_0 + h - ht_0 = (1 + h)y_0 + (t_1 - t_0) - ht_0$. Rearranging the terms, $y_1 = (1 + h)(y_0 - t_0) + t_1$. Now suppose that $y_k = (1 + h)^k(y_0 - t_0) + t_k$, for some $k \geq 1$. Then $y_{k+1} = (1 + h)y_k + h - ht_k$. Substituting for y_k, we find that

$$y_{k+1} = (1 + h)^{k+1}(y_0 - t_0) + (1 + h)t_k + h - ht_k = (1 + h)^{k+1}(y_0 - t_0) + t_k + h.$$

Noting that $t_{k+1} = t_k + h$, the result is verified.

(d) Substituting $h = (t - t_0)/n$, with $t_n = t$, $y_n = (1 + (t - t_0)/n)^n(y_0 - t_0) + t$. Taking the limit of both sides, and using the fact that $\lim_{n\to\infty}(1 + a/n)^n = e^a$, pointwise convergence is proved.

16. The exact solution is $y(t) = e^t$. The Euler formula is $y_{n+1} = (1 + h)y_n$. It is easy to see that $y_n = (1 + h)^n y_0 = (1 + h)^n$. Given $t > 0$, set $h = t/n$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1 + t/n)^n = e^t$.

2.8

2. Let $z = y - 3$ and $\tau = t + 1$. It follows that $dz/d\tau = (dz/dt)(dt/d\tau) = dz/dt$. Furthermore, $dz/dt = dy/dt = 1 - y^3$. Hence $dz/d\tau = 1 - (z + 3)^3$. The new initial condition is $z(0) = 0$.

3.(a) The approximating functions are defined recursively by

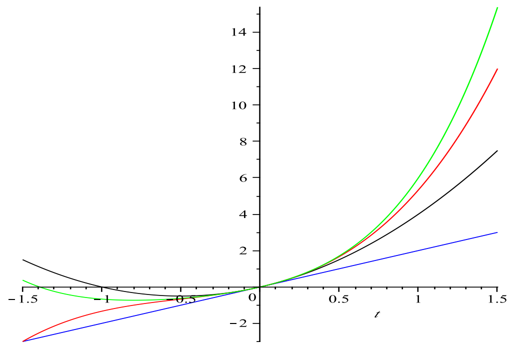
$$\phi_{n+1}(t) = \int_0^t 2\,[\phi_n(s) + 1]\, ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = 4t^3/3 + 2t^2 + 2t$, $\phi_4(t) = 2t^4/3 + 4t^3/3 + 2t^2 + 2t$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} 2^k t^k/k!$ and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_n(t) = 2t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have

$$\phi_{n+1}(t) = \int_0^t 2\,[\phi_n(s) + 1]\, ds = \int_0^t 2\left[\sum_{k=1}^{n} \frac{2^k}{k\,!} s^k + 1\right] ds = \sum_{k=1}^{n+1} \frac{2^k}{k\,!} t^k,$$

and we have verified our conjecture.

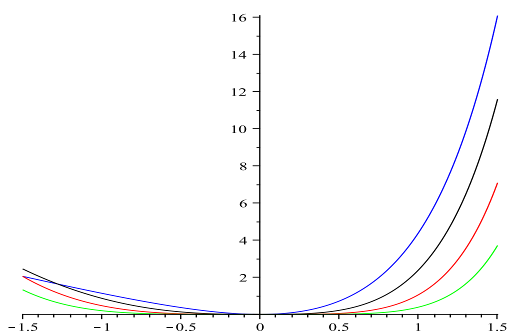
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k / k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} \frac{2^k}{k\,!} t^k = e^{2t} - 1\,.$$

(d)



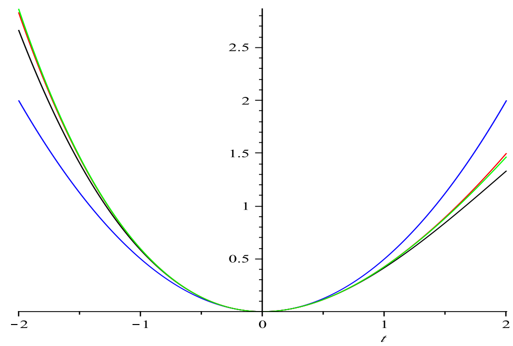
From the plot it appears that ϕ_4 is a good estimate for $|t| < 1/2$.

4.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[-\phi_n(s)/2 + s\right] ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = t^2/2$. Continuing, $\phi_2(t) = t^2/2 - t^3/12$, $\phi_3(t) = t^2/2 - t^3/12 + t^4/96$, $\phi_4(t) = t^2/2 - t^3/12 + t^4/96 - t^5/960$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} 4(-1/2)^{k+1} t^{k+1}/(k+1)!$ and use mathematical induction to verify this form for $\phi_n(t)$.

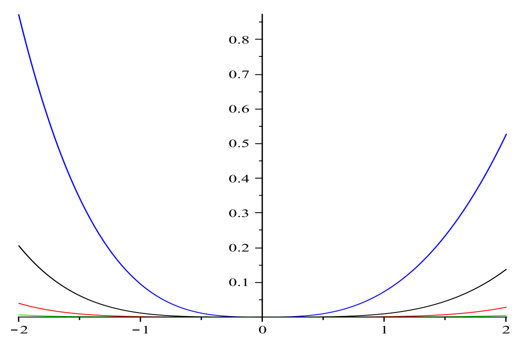
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k / k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} 4 \frac{(-1/2)^{k+1}}{k+1\,!} t^{k+1} = 4e^{-t/2} + 2t - 4\,.$$

(d)



From the plot it appears that ϕ_4 is a good estimate for $|t| < 2$.

6.(a) The approximating functions are defined recursively by

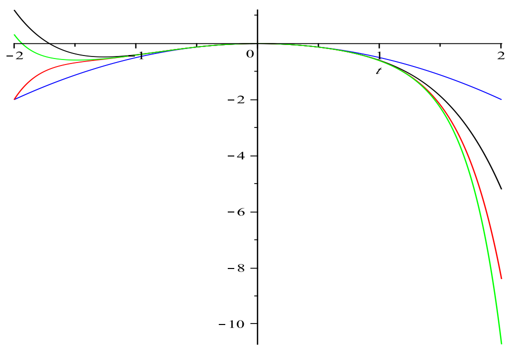
$$\phi_{n+1}(t) = \int_0^t \left[s^2 \phi_n(s) - s \right] ds\,.$$

Set $\phi_0(t) = 0$. The iterates are given by $\phi_1(t) = -t^2/2$, $\phi_2(t) = -t^2/2 - t^5/10$, $\phi_3(t) = -t^2/2 - t^5/10 - t^8/80$, $\phi_4(t) = -t^2/2 - t^5/10 - t^8/80 - t^{11}/880$,.... Upon inspection, it becomes apparent that

$$\phi_n(t) = -t^2 \left[\frac{1}{2} + \frac{t^3}{2 \cdot 5} + \frac{t^6}{2 \cdot 5 \cdot 8} + \ldots + \frac{(t^3)^{n-1}}{2 \cdot 5 \cdot 8 \ldots [2 + 3(n-1)]} \right] =$$

$$= -t^2 \sum_{k=1}^{n} \frac{(t^3)^{k-1}}{2 \cdot 5 \cdot 8 \ldots [2 + 3(k-1)]}.$$

(b)



(c) Using the identity $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \ldots + [\phi_n(t) - \phi_{n-1}(t)]$, consider the series $\phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$. Fix any t value now. We use the Ratio Test to prove the convergence of this series:

$$\left|\frac{\phi_{k+1}(t) - \phi_k(t)}{\phi_k(t) - \phi_{k-1}(t)}\right| = \left|\frac{\frac{(-t^2)(t^3)^k}{2\cdot 5\cdots(2+3k)}}{\frac{(-t^2)(t^3)^{k-1}}{2\cdot 5\cdots(2+3(k-1))}}\right| = \frac{|t|^3}{2+3k}.$$

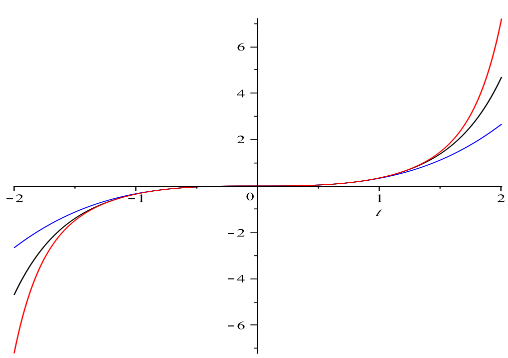
The limit of this quantity is 0 for any fixed t as $k \to \infty$, and we obtain that $\phi_n(t)$ is convergent for any t.

7.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2 + \phi_n^2(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^7/63$, $\phi_3(t) = t^3/3 + t^7/63 + 2t^{11}/2079 + t^{15}/59535\,.$

(b)



The iterates appear to be converging.

10.(a) The approximating functions are defined recursively by

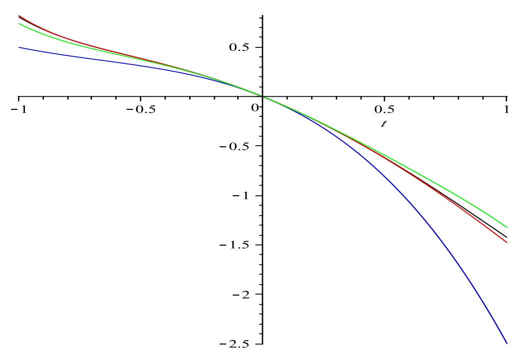
$$\phi_{n+1}(t) = \int_0^t \left[\frac{3s^2 + 4s + 2}{2(\phi_n(s) - 1)}\right] ds\,.$$

Note that $1/(2y-2) = -(1/2)\sum_{k=0}^{6} y^k + O(y^7)$. For computational purposes, use the geometric series sum to replace the above iteration formula by

$$\phi_{n+1}(t) = -\frac{1}{2}\int_0^t \left[(3s^2 + 4s + 2)\sum_{k=0}^{6} \phi_n^k(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first four approximations are given by $\phi_1(t) = -t - t^2 - t^3/2$, $\phi_2(t) = -t - t^2/2 + t^3/6 + t^4/4 - t^5/5 - t^6/24 + \ldots$, $\phi_3(t) = -t - t^2/2 + t^4/12 - 3t^5/20 + 4t^6/45 + \ldots$, $\phi_4(t) = -t - t^2/2 + t^4/8 - 7t^5/60 + t^6/15 + \ldots$

(b)



The approximations appear to be converging to the exact solution, which can be found by separating the variables: $\phi(t) = 1 - \sqrt{1 + 2t + 2t^2 + t^3}$.

12.(a) $\phi_n(0) = 0$, for every $n \geq 1$. Let $a \in (0\,,1]$. Then $\phi_n(a) = 2na\,e^{-na^2} = 2na/e^{na^2}$. Using l'Hospital's rule, $\lim_{z\to\infty} 2az/e^{az^2} = \lim_{z\to\infty} 1/ze^{az^2} = 0$. Hence $\lim_{n\to\infty}\phi_n(a) = 0$.

(b) $\int_0^1 2nx\,e^{-nx^2}dx = -e^{-nx^2}\big|_0^1 = 1 - e^{-n}$. Therefore,

$$\lim_{n\to\infty}\int_0^1 \phi_n(x)dx \neq \int_0^1 \lim_{n\to\infty}\phi_n(x)dx\,.$$

13.(a) Recall that Eq.(9) states that $\phi(t) = \int_0^t 2s[1 + \phi(s)]\,ds$. Since $\phi(t) = \sum_{k=1}^{\infty} t^{2k}/k!$,

$$2s[1 + \phi(s)] = 2s\sum_{k=0}^{\infty}\frac{s^{2k}}{k!} = 2\sum_{k=0}^{\infty}\frac{s^{2k+1}}{k!}$$

Integrating term-by-term,

$$\begin{aligned}\int_0^t 2s[1+\phi(s)]\,ds &= \int_0^t 2\sum_{k=0}^{\infty}\frac{s^{2k+1}}{k!}\,ds = 2\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t s^{2k+1}\,ds\\ &= 2\sum_{k=0}^{\infty}\frac{1}{k!}\frac{t^{2k+2}}{2k+2} = \sum_{k=0}^{\infty}\frac{t^{2(k+1)}}{(k+1)!} = \sum_{k=1}^{\infty}\frac{t^{2k}}{k!} = \phi(t)\end{aligned}$$

and $\phi(t)$ is a solution of Eq.(9).

(b) Recall that the initial value problem in Eq.(8) is $y' = 2t(1+y), y(0) = 0$. Letting $y = \phi(t) = \sum_{k=1}^{\infty} t^{2k}/k!$,

$$\begin{aligned} y' = \phi'(t) &= \sum_{k=1}^{\infty}\frac{2kt^{2k-1}}{k!} = 2\sum_{k=1}^{\infty}\frac{t^{2k-1}}{(k-1)!} = 2t\sum_{k=1}^{\infty}\frac{t^{2k-2}}{(k-1)!}\\ &= 2t\sum_{k=0}^{\infty}\frac{t^{2k}}{k!} = 2t\left[1+\sum_{k=1}^{\infty}\frac{t^{2k}}{k!}\right] = 2t[1+\phi(t)]\end{aligned}$$

and $y(0) = \phi(0) = \sum_{k=1}^{\infty} 0 = 0$, so $y = \phi(t)$ satisfies the initial value problem $y' = 2t(1+y), y(0) = 0$.

(c) Since $e^t = \displaystyle\sum_{k=0}^{\infty}\frac{t^k}{k!}$,

$$\phi(t) = \sum_{k=1}^{\infty}\frac{t^{2k}}{k!} = -1 + \sum_{k=0}^{\infty}\frac{t^{2k}}{k!} = -1 + \sum_{k=0}^{\infty}\frac{(t^2)^k}{k!} = -1 + e^{t^2}$$

(d) Separating the variables gives $(1/(1+y))\,dy = 2t\,dt$, and integration yields $\ln|1+y| = t^2 + c$. Applying the initial condition, $\ln 1 = 0 + c$, so $c = 0$ and $\ln|1+y| = t^2$. Solving for y first gives $|1+y| = e^{t^2}$, so $1+y = \pm e^{t^2}$. In order for $y(0) = 0$ to be true, we choose $1+y = e^{t^2}$, and thus $y = -1 + e^{t^2}$.

(e) Consider the first-order linear equation $y' - 2ty = 0$. The integrating factor will be $\mu(t) = e^{\int -2t\,dt} = e^{-t^2}$. Since multiplying the differential equation by $\mu(t)$ yields

$$\frac{d}{dt}\left(e^{-t^2}y\right) = 2te^{-t^2}$$

we have $e^{-t^2}y = \int 2te^{-t^2}\,dt = -e^{-t^2} + c$. The initial condition $y(0) = 0$ may now be applied to show that $c = 1$, and $y = e^{t^2}(-e^{-t^2} + 1) = -1 + e^{t^2}$.

14. Let t be fixed, such that $(t, y_1), (t, y_2) \in D$. Without loss of generality, assume that $y_1 < y_2$. Since f is differentiable with respect to y, the mean value theorem asserts that there exists $\xi \in (y_1, y_2)$ such that $f(t, y_1) - f(t, y_2) = f_y(t, \xi)(y_1 - y_2)$. This means that $|f(t, y_1) - f(t, y_2)| = |f_y(t, \xi)|\,|y_1 - y_2|$. Since, by assumption, $\partial f/\partial y$ is continuous in D, f_y attains a maximum K on any closed and bounded subset of D. Hence $|f(t, y_1) - f(t, y_2)| \le K\,|y_1 - y_2|$.

15. For a sufficiently small interval of t, $\phi_{n-1}(t)$, $\phi_n(t) \in D$. Since f satisfies a Lipschitz condition, $|f(t, \phi_n(t)) - f(t, \phi_{n-1}(t))| \leq K\,|\phi_n(t) - \phi_{n-1}(t)|$. Here $K = \max|f_y|$.

16.(a) $\phi_1(t) = \int_0^t f(s\,,0)ds$. Hence $|\phi_1(t)| \leq \int_0^{|t|} |f(s\,,0)|\,ds \leq \int_0^{|t|} M ds = M\,|t|$, in which M is the maximum value of $|f(t\,,y)|$ on D.

(b) By definition, $\phi_2(t) - \phi_1(t) = \int_0^t [f(s\,,\phi_1(s)) - f(s\,,0)]\,ds$. Taking the absolute value of both sides, $|\phi_2(t) - \phi_1(t)| \leq \int_0^{|t|} |[f(s\,,\phi_1(s)) - f(s\,,0)]|\,ds$. Based on the results in Problems 14 and 15,

$$|\phi_2(t) - \phi_1(t)| \leq \int_0^{|t|} K\,|\phi_1(s) - 0|\,ds \leq KM \int_0^{|t|} |s|\,ds\,.$$

Evaluating the last integral, we obtain that $|\phi_2(t) - \phi_1(t)| \leq MK\,|t|^2/2$.

(c) Suppose that

$$|\phi_i(t) - \phi_{i-1}(t)| \leq \frac{MK^{i-1}\,|t|^i}{i!}$$

for some $i \geq 1$. By definition,

$$\phi_{i+1}(t) - \phi_i(t) = \int_0^t [f(s, \phi_i(s)) - f(s, \phi_{i-1}(s))]\,ds\,.$$

It follows that

$$|\phi_{i+1}(t) - \phi_i(t)| \leq \int_0^{|t|} |f(s, \phi_i(s)) - f(s, \phi_{i-1}(s))|\,ds$$

$$\leq \int_0^{|t|} K\,|\phi_i(s) - \phi_{i-1}(s)|\,ds \leq \int_0^{|t|} K\frac{MK^{\,i-1}\,|s|^i}{i!}ds =$$

$$= \frac{MK^{\,i}\,|t|^{i+1}}{(i+1)!} \leq \frac{MK^{\,i}h^{i+1}}{(i+1)!}\,.$$

Hence, by mathematical induction, the assertion is true.

17.(a) Use the triangle inequality, $|a + b| \leq |a| + |b|$.

(b) For $|t| \leq h$, $|\phi_1(t)| \leq Mh$, and $|\phi_n(t) - \phi_{n-1}(t)| \leq MK^{\,n-1}h^n/(n\,!)$. Hence

$$|\phi_n(t)| \leq M\sum_{i=1}^{n} \frac{K^{\,i-1}h^i}{i\,!} = \frac{M}{K}\sum_{i=1}^{n} \frac{(Kh)^i}{i\,!}\,.$$

(c) The sequence of partial sums in (b) converges to $M(e^{Kh} - 1)/K$. By the comparison test, the sums in (a) also converge. Since individual terms of a convergent series must tend to zero, $|\phi_n(t) - \phi_{n-1}(t)| \to 0$, and it follows that the sequence $|\phi_n(t)|$ is convergent.

18.(a) Let $\phi(t) = \int_0^t f(s\,,\phi(s))ds$ and $\psi(t) = \int_0^t f(s\,,\psi(s))ds$. Then by linearity of the integral, $\phi(t) - \psi(t) = \int_0^t [f(s\,,\phi(s)) - f(s\,,\psi(s))]\,ds$.

(b) It follows that $|\phi(t) - \psi(t)| \leq \int_0^t |f(s\,,\phi(s)) - f(s\,,\psi(s))|\, ds$.

(c) We know that f satisfies a Lipschitz condition, $|f(t\,,y_1) - f(t\,,y_2)| \leq K\; |y_1 - y_2|$, based on $|\partial f/\partial y| \leq K$ in D. Therefore,

$$|\phi(t) - \psi(t)| \leq \int_0^t |f(s\,,\phi(s)) - f(s\,,\psi(s))|\, ds \leq \int_0^t K\,|\phi(s) - \psi(s)|\, ds.$$

2.9

1. Writing the equation for each $n \geq 0$, $y_1 = -0.9\,y_0$, $y_2 = -0.9\,y_1 = (-0.9^2)y_0$, $y_3 = -0.9\,y_2 = (-0.9)^3 y_0$ and so on, it is apparent that $y_n = (-0.9)^n\,y_0$. The terms constitute an alternating series, which converge to zero, regardless of y_0.

2. Write the equation for each $n \geq 0$, $y_1 = \sqrt{3}\,y_0$, $y_2 = \sqrt{4/2}\,y_1$, $y_3 = \sqrt{5/3}\,y_2$, ... Upon substitution, we find that $y_2 = \sqrt{(4 \cdot 3)/2}\,y_1$, $y_3 = \sqrt{(5 \cdot 4 \cdot 3)/(3 \cdot 2)}\,y_0$, ... It can be proved by mathematical induction, that

$$y_n = \frac{1}{\sqrt{2}}\sqrt{\frac{(n+2)!}{n!}}\;y_0\; = \frac{1}{\sqrt{2}}\sqrt{(n+1)(n+2)}\;y_0\,.$$

This sequence is divergent, except for $y_0 = 0$.

3. Writing the equation for each $n \geq 0$, $y_1 = -y_0$, $y_2 = y_1$, $y_3 = -y_2$, $y_4 = y_3$, and so on. It can be shown that

$$y_n = \begin{cases} y_0, & \text{for } n = 4k \text{ or } n = 4k-1 \\ -y_0, & \text{for } n = 4k-2 \text{ or } n = 4k-3 \end{cases}$$

The sequence is convergent only for $y_0 = 0$.

5. Let y_n be the balance at the end of the nth month. Then $y_{n+1} = (1 + r/12)y_n + 25$. We have $y_n = \rho^n[y_0 - 25/(1-\rho)] + 25/(1-\rho)$, in which $\rho = (1 + r/12)$. Here r is the annual interest rate, given as 8%. Thus $y_{36} = (1.0066)^{36}\,[1000 + 12 \cdot 25/r] - 12 \cdot 25/r = \$2,283.63$.

6. Let y_n be the balance due at the end of the nth month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\,y_n - P$. Here r is the annual interest rate and P is the monthly payment. The solution, in terms of the amount borrowed, is given by $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$, in which $\rho = (1 + r/12)$ and $y_0 = 8,000$. To figure out the monthly payment P, we require that $y_{36} = 0$. That is, $\rho^{36}[y_0 + P/(1-\rho)] = P/(1-\rho)$. After the specified amounts are substituted, we find that $P = \$258.14$.

7. Let y_n be the balance due at the end of the nth month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\,y_n - P$, in which $r = .09$ and P is the monthly payment. The initial value of the mortgage is $y_0 = \$100,000$. Then the balance

due at the end of the n-th month is $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$, where $\rho = (1 + r/12)$. In terms of the specified values, $y_n = (1.0075)^n[10^5 - 12P/r] + 12P/r$. Setting $n = 30 \cdot 12 = 360$, and $y_{360} = 0$, we find that $P = \$804.62$. For the monthly payment corresponding to a 20 year mortgage, set $n = 240$ and $y_{240} = 0$ to find that $P = \$899.73$. The total amount paid during the term of the loan is $360 \times 804.62 = \$289,663.20$ for the 30-year loan and is $240 \times 899.73 = \$215,935.20$ for the 20-year loan.

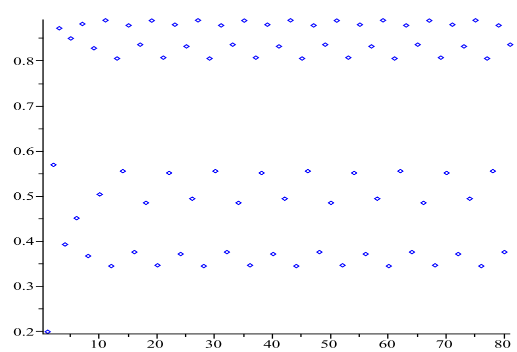
8. Let y_n be the balance due at the end of the nth month, with y_0 the initial value of the mortgage. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = 0.1$ and $P = \$1000$ is the maximum monthly payment. Given that the life of the mortgage is 20 years, we require that $y_{240} = 0$. The balance due at the end of the n-th month is $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$. In terms of the specified values for the parameters, the solution of $(1.00833)^{240}[y_0 - 12 \cdot 1000/0.1] = -12 \cdot 1000/0.1$ is $y_0 = \$103,624.62$.

15.(a) $\delta_2 = (\rho_2 - \rho_1)/(\rho_3 - \rho_2) = (3.449 - 3)/(3.544 - 3.449) = 4.7263$.

(b) diff$= (|\delta - \delta_2|/\delta) \cdot 100 = (|4.6692 - 4.7363|/4.6692) \cdot 100 \approx 1.22\%$.

(c) Assuming $(\rho_3 - \rho_2)/(\rho_4 - \rho_3) = \delta$, $\rho_4 \approx 3.5643$

(d) A period 16 solution appears near $\rho \approx 3.565$.



(e) Note that $(\rho_{n+1} - \rho_n) = \delta_n^{-1}(\rho_n - \rho_{n-1})$. With the assumption that $\delta_n = \delta$, we have $(\rho_{n+1} - \rho_n) = \delta^{-1}(\rho_n - \rho_{n-1})$, which is of the form $y_{n+1} = \alpha\, y_n$, $n \geq 3$. It follows that $(\rho_k - \rho_{k-1}) = \delta^{3-k}(\rho_3 - \rho_2)$ for $k \geq 4$. Then

$$\begin{aligned}
\rho_n &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + (\rho_4 - \rho_3) + \ldots + (\rho_n - \rho_{n-1}) \\
&= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[1 + \delta^{-1} + \delta^{-2} + \ldots + \delta^{3-n}\right] \\
&= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[\frac{1 - \delta^{4-n}}{1 - \delta^{-1}}\right].
\end{aligned}$$

Hence $\lim_{n\to\infty} \rho_n = \rho_2 + (\rho_3 - \rho_2)\left[\frac{\delta}{\delta - 1}\right]$. Substitution of the appropriate values yields

$$\lim_{n\to\infty} \rho_n = 3.5699$$

PROBLEMS

1. The equation is *linear*. It can be written in the form $y' + 2y/x = x^2$, and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ yields $x^2y' + 2yx = (yx^2)' = x^4$. Integration with respect to x and division by x^2 gives that $y = x^3/5 + c/x^2$.

5. The equation is *exact*. Algebraic manipulations give the symmetric form of the equation, $(2xy + y^2 + 1)dx + (x^2 + 2xy)dy = 0$. We can check that $M_y = 2x + 2y = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x,y) = x^2y + xy^2 + x + g(y)$, then $\psi_y = x^2 + 2xy + g'(y) = x^2 + 2xy$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^2y + xy^2 + x = c$.

6. The equation is *linear*. It can be written in the form $y' + (1 + (1/x))y = 1/x$ and the integrating factor is $\mu(x) = e^{\int 1+(1/x)\,dx} = e^{x+\ln x} = xe^x$. Multiplication by $\mu(x)$ yields $xe^xy' + (xe^x + e^x)y = (xe^xy)' = e^x$. Integration with respect to x and division by xe^x shows that the general solution of the equation is $y = 1/x + c/(xe^x)$. The initial condition implies that $0 = 1 + c/e$, which means that $c = -e$ and the solution is $y = 1/x - e/(xe^x) = x^{-1}(1 - e^{1-x})$.

7. The equation is *linear*. It can be written in the form $y' + 2y/x = \sin x/x^2$ and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ gives $x^2y' + 2xy = (x^2y)' = \sin x$, and after integration with respect to x and division by x^2 we obtain the general solution $y = (c - \cos x)/x^2$. The initial condition implies that $c = 4 + \cos 2$ and the solution becomes $y = (4 + \cos 2 - \cos x)/x^2$.

10. The equation is *exact*. It is easy to check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = x^3/3 + xy + g(y)$, then $\psi_y = x + g'(y) = x + e^y$, which means that $g'(y) = e^y$, so we obtain that $g(y) = e^y$. Therefore the solution is defined implicitly as $x^3/3 + xy + e^y = c$.

11. The equation is *exact*. We can check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = x^2/2 + xy + g(y)$, then $\psi_y = x + g'(y) = x + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the general solution is defined implicitly as $x^2/2 + xy + y^2 = c$. The initial condition gives us $c = 17$, so the solution is $x^2 + 2xy + 2y^2 = 34$.

12. The equation is *separable*. Separation of variables leads us to the equation

$$\frac{dy}{y} = \frac{1 - e^x}{1 + e^x}dx.$$

Note that $1 + e^x - 2e^x = 1 - e^x$. We obtain that

$$\ln|y| = \int \frac{1-e^x}{1+e^x}dx = \int 1 - \frac{2e^x}{1+e^x}dx = x - 2\ln(1+e^x) + \tilde{c}.$$

This means that $y = ce^x(1+e^x)^{-2}$, which also can be written as $y = c/\cosh^2(x/2)$ after some algebraic manipulations.

13. The equation is *exact*. The symmetric form is $(-e^{-x}\cos y + e^{2y}\cos x)dx + (-e^{-x}\sin y + 2e^{2y}\sin x)dy = 0$. We can check that $M_y = e^{-x}\sin y + 2e^{2y}\cos x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = e^{-x}\cos y + e^{2y}\sin x + g(y)$, then $\psi_y = -e^{-x}\sin y + 2e^{2y}\sin x + g'(y) = -e^{-x}\sin y + 2e^{2y}\sin x$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $e^{-x}\cos y + e^{2y}\sin x = c$.

14. The equation is *linear*. The integrating factor is $\mu(x) = e^{-\int 3\,dx} = e^{-3x}$, which turns the equation into $e^{-3x}y' - 3e^{-3x}y = (e^{-3x}y)' = e^{-x}$. We integrate with respect to x to obtain $e^{-3x}y = -e^{-x} + c$, and the solution is $y = ce^{3x} - e^{2x}$ after multiplication by e^{3x}.

15. The equation is *linear*. The integrating factor is $\mu(x) = e^{\int 2\,dx} = e^{2x}$, which gives us $e^{2x}y' + 2e^{2x}y = (e^{2x}y)' = e^{-x^2}$. The antiderivative of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 0 to x. We obtain that the left hand side turns into

$$\int_0^x (e^{2s}y(s))'ds = e^{2x}y(x) - e^0y(0) = e^{2x}y - 3.$$

The right hand side gives us $\int_0^x e^{-s^2}\,ds$. So we found that

$$y = e^{-2x}\int_0^x e^{-s^2}\,ds + 3e^{-2x}.$$

16. The equation is *exact*. Algebraic manipulations give us the symmetric form $(y^3 + 2y - 3x^2)dx + (2x + 3xy^2)dy = 0$. We can check that $M_y = 3y^2 + 2 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = xy^3 + 2xy - x^3 + g(y)$, then $\psi_y = 3xy^2 + 2x + g'(y) = 2x + 3xy^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is $xy^3 + 2xy - x^3 = c$.

17. The equation is *separable*, because $y' = e^{x+y} = e^xe^y$. Separation of variables yields the equation $e^{-y}dy = e^xdx$, which turns into $-e^{-y} = e^x + c$ after integration and we obtain the implicitly defined solution $e^x + e^{-y} = c$.

19. The equation is *linear*. Division by t gives $y' + (1 + (1/t))y = e^{2t}/t$, so the integrating factor is $\mu(t) = e^{\int(1+(1/t))dt} = e^{t+\ln t} = te^t$. The equation turns into $te^ty' + (te^t + e^t)y = (te^ty)' = e^{3t}$. Integration therefore leads to $te^ty = e^{3t}/3 + c$ and the solution is $y = e^{2t}/(3t) + ce^{-t}/t$.

22. The equation is *homogeneous.* (See Section 2.2, Problem 25) We can see that

$$y' = \frac{x+y}{x-y} = \frac{1+(y/x)}{1-(y/x)}.$$

We substitute $u = y/x$, which means also that $y = ux$ and then $y' = u'x + u = (1+u)/(1-u)$, which implies that

$$u'x = \frac{1+u}{1-u} - u = \frac{1+u^2}{1-u},$$

a separable equation. Separating the variables yields

$$\frac{1-u}{1+u^2} du = \frac{dx}{x},$$

and then integration gives $\arctan u - \ln(1+u^2)/2 = \ln|x| + c$. Substituting $u = y/x$ back into this expression and using that

$$-\ln(1+(y/x)^2)/2 - \ln|x| = -\ln(|x|\sqrt{1+(y/x)^2}) = -\ln(\sqrt{x^2+y^2})$$

we obtain that the solution is $\arctan(y/x) - \ln(\sqrt{x^2+y^2}) = c$.

23. The equation is *homogeneous.* (See Section 2.2, Problem 25) Algebraic manipulations show that it can be written in the form

$$y' = \frac{3y^2+2xy}{2xy+x^2} = \frac{3(y/x)^2+2(y/x)}{2(y/x)+1}.$$

Substituting $u = y/x$ gives that $y = ux$ and then

$$y' = u'x + u = \frac{3u^2+2u}{2u+1},$$

which implies that

$$u'x = \frac{3u^2+2u}{2u+1} - u = \frac{u^2+u}{2u+1},$$

a separable equation. We obtain that $(2u+1)du/(u^2+u) = dx/x$, which in turn means that $\ln(u^2+u) = \ln|x| + \tilde{c}$. Therefore, $u^2 + u = cx$ and then substituting $u = y/x$ gives us the solution $(y^2/x^3) + (y/x^2) = c$.

25. Let y_1 be a solution, i.e. $y_1' = q_1 + q_2 y_1 + q_3 y_1^2$. Now let $y = y_1 + (1/v)$ also be a solution. Differentiating this expression with respect to t and using that y is also a solution we obtain $y' = y_1' - (1/v^2)v' = q_1 + q_2 y + q_3 y^2 = q_1 + q_2(y_1 + (1/v)) + q_3(y_1 + (1/v))^2$. Now using that y_1 was also a solution we get that $-(1/v^2)v' = q_2(1/v) + 2q_3(y_1/v) + q_3(1/v^2)$, which, after some simple algebraic manipulations turns into $v' = -(q_2 + 2q_3 y_1)v - q_3$.

27.(a) The equation is $y' = (1-y)(x+by) = x + (b-x)y - by^2$. We set $y = 1 + (1/v)$ and differentiate: $y' = -v^{-2}v' = x + (b-x)(1+(1/v)) - b(1+(1/v))^2$, which, after simplification, turns into $v' = (b+x)v + b$.

(b) When $x = at$, the equation is $v' - (b + at)v = b$, so the integrating factor is $\mu(t) = e^{-bt-at^2/2}$. This turns the equation into $(v\mu(t))' = b\mu(t)$, so $v\mu(t) = \int b\mu(t)dt$, and then $v = (b\int \mu(t)dt)/\mu(t)$.

28. Substitute $v = y'$, then $v' = y''$. The equation turns into $t^2v' + 2tv = (t^2v)' = 1$, which yields $t^2v = t + c_1$, so $y' = v = (1/t) + (c_1/t^2)$. Integrating this expression gives us the solution $y = \ln t - (c_1/t) + c_2$.

29. Set $v = y'$, then $v' = y''$. The equation with this substitution is $tv' + v = (tv)' = 1$, which gives $tv = t + c_1$, so $y' = v = 1 + (c_1/t)$. Integrating this expression yields the solution $y = t + c_1 \ln t + c_2$.

30. Set $v = y'$, so $v' = y''$. The equation is $v' + tv^2 = 0$, which is a separable equation. Separating the variables we obtain $dv/v^2 = -tdt$, so $-1/v = -t^2/2 + c$, and then $y' = v = 2/(t^2 + c_1)$. Now depending on the value of c_1, we have the following possibilities: when $c_1 = 0$, then $y = -2/t + c_2$, when $0 < c_1 = k^2$, then $y = (2/k)\arctan(t/k) + c_2$, and when $0 > c_1 = -k^2$ then

$$y = (1/k)\ln|(t-k)/(t+k)| + c_2.$$

We also divided by $v = y'$ when we separated the variables, and $v = 0$ (which is $y = c$) is also a solution.

31. Substitute $v = y'$ and $v' = y''$. The equation is $2t^2v' + v^3 = 2tv$. This is a *Bernoulli* equation (See Section 2.4, Problem 19), so the substitution $z = v^{-2}$ yields $z' = -2v^{-3}v'$, and the equation turns into $2t^2v'v^3 + 1 = 2t/v^2$, i.e. into $-2t^2z'/2 + 1 = 2tz$, which in turn simplifies to $t^2z' + 2tz = (t^2z)' = 1$. Integration yields $t^2z = t + c$, which means that $z = (1/t) + (c/t^2)$. Now $y' = v = \pm\sqrt{1/z} = \pm t/\sqrt{t + c_1}$ and another integration gives

$$y = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2.$$

The substitution also loses the solution $v = 0$, i.e. $y = c$.

33. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + y = 0$, where the differentiation is with respect to y. This is a separable equation which simplifies to $vdv = -ydy$. We obtain that $v^2/2 = -y^2/2 + c$, so $y' = v(y) = \pm\sqrt{c - y^2}$. We separate the variables again to get $dy/\sqrt{c - y^2} = \pm dt$, so $\arcsin(y/\sqrt{c}) = t + d$, which means that $y = \sqrt{c}\sin(\pm t + d) = c_1 \sin(t + c_2)$.

34. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $yv'v - v^3 = 0$, where the differentiation is with respect to y. This separable equation gives us $dv/v^2 = dy/y$, which means that $-1/v = \ln|y| + c$, and then $y' = v = 1/(c - \ln|y|)$. We separate variables again to obtain $(c - \ln|y|)dy = dt$, and then integration yields the implicitly defined solution $cy - (y\ln|y| - y) = t + d$. Also, $y = c$ is a solution which we lost when we divided by $v = 0$.

37. Set $v = y'$, then $v' = y''$. The equation with this substitution turns into the equation $(1 + t^2)v' + 2tv = ((1 + t^2)v)' = -3t^{-2}$. Integrating this we get that

$(1+t^2)v = 3t^{-1} + c$, and $c = -5$ from the initial conditions. This means that $y' = v = 3/(t(1+t^2)) - 5/(1+t^2)$. The partial fraction decomposition of the first expression shows that $y' = 3/t - 3t/(1+t^2) - 5/(1+t^2)$ and then another integration here gives us that $y = 3\ln t - (3/2)\ln(1+t^2) - 5\arctan t + d$. The initial conditions identify $d = 2 + (3/2)\ln 2 + 5\pi/4$, and we obtained the solution.