Step-1

Consider the non-orthogonal vectors,

$$a=\begin{bmatrix}1\\1\\0\end{bmatrix}, b=\begin{bmatrix}1\\0\\1\end{bmatrix}, c=\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

The objective is to find the orthonormal vectors $q_1, q_2,$ and q_3 from the vectors *a, b,* and *c*.

Step-2

The orthonormal vectors must be computed by using Gram-Schmidt process.

To find the vector q_1, convert the first vector *a* into unit vector.

$$\begin{aligned}\|a\| &= \sqrt{1^2+1^2+0^2}\\ &= \sqrt{1+1+0}\\ &= \sqrt{2}\end{aligned}$$

Thus, $q_1=\frac{a}{\|a\|}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}.$

Step-3

Remaining vectors are found by using the following formulas:

$$B=b-\left(q_1^T b\right)q_1.$$

Now find $q_1^T b.$

$$\begin{aligned}q_1^T b &= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}(1)+\frac{1}{\sqrt{2}}(0)+0(1)\\ &= \frac{1}{\sqrt{2}}+0+0\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Step-4

Substitute the known values in the formula $B = b - \left(q_1^T b\right)q_1$.

$$B = b - \left(q_1^T b\right)q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Step-5

Further simplification is as follows:

$$= \begin{bmatrix} 1 - \frac{1}{2} \\ 0 - \frac{1}{2} \\ 1 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

The vector can be found by using $q_2 = \frac{B}{\|B\|}$.

$$\|B\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 1^2}$$
$$= \sqrt{\frac{1}{4} + \frac{1}{4} + 1}$$
$$= \sqrt{\frac{6}{4}}$$
$$= \frac{1}{2}\sqrt{6}$$

Step-6

Substitute the known values in $q_2 = \frac{B}{\|B\|}$.

$$q_2 = \frac{B}{\|B\|}$$
$$= \frac{1}{\frac{1}{2}\sqrt{6}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Therefore, $q_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

Step-7

Subtract the components along the vectors q_1 and q_2 from the vector c to find the vector q_3.

Use the formula $C = c - \left(q_1^T c\right) q_1 - \left(q_2^T c\right) q_2$.

$$
\begin{aligned}
q_1^T c &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1) + 0(1) \\
&= 0 + \frac{1}{\sqrt{2}} + 0 \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
q_2^T c &= \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{6}}(0) + \left(-\frac{1}{\sqrt{6}}\right)(1) + \left(\frac{2}{\sqrt{6}}\right)(1) \\
&= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} \\
&= \frac{1}{\sqrt{6}}
\end{aligned}
$$

Step-8

Substitute the known values in the formula $C = c - (q_1^T c) q_1 - (q_2^T c) q_2$.

$$
\begin{aligned}
C &= c - (q_1^T c) q_1 - (q_2^T c) q_2 \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \frac{1}{\sqrt{6}} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \\ \frac{2}{6} \end{bmatrix} \\
&= \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}
\end{aligned}
$$

Step-9

The vector q_3 can be found by using $q_3 = \frac{C}{\|C\|}$.

$$\begin{aligned}\|C\| &= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \\ &= \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} \\ &= \sqrt{\frac{12}{9}} \\ &= \frac{2}{3}\sqrt{3}\end{aligned}$$

Substitute the known values in $q_3 = \frac{C}{\|C\|}$.

$$\begin{aligned}q_3 &= \frac{C}{\|C\|} \\ &= \frac{1}{\frac{2}{3}\sqrt{3}}\begin{bmatrix}-2/3 \\ 2/3 \\ 2/3\end{bmatrix} \\ &= \begin{bmatrix}-\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}}\end{bmatrix}\end{aligned}$$

Hence, the required orthonormal vectors are $\boxed{q_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0\end{bmatrix}, q_2 = \begin{bmatrix}\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}}\end{bmatrix}, q_3 = \begin{bmatrix}-\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}}\end{bmatrix}}$.