Step-1

The given Hilbert matrix is $A=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{bmatrix}$

Suppose $A=\left[A\middle| \; I\right]$

$$=\left[\begin{array}{ccc|ccc}1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1\end{array}\right]$$

Step-2

Applying the row operations on both parts of the matrix, to reduce the square matrix on the left to identity matrix, parallelly, the matrix on the right changes to a new matrix which is the inverse of A.

$$\left.\begin{array}{l}R_2\to R_2-\frac{1}{2}R_1\\ R_3\to R_3-\frac{1}{3}R_1\end{array}\right\}\to\left[\begin{array}{ccc|ccc}1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0\\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0\\ 0 & \frac{1}{12} & \frac{4}{45} & \frac{-1}{3} & 0 & 1\end{array}\right]$$

Step-3

Continuation to the above

$$R_3\to R_3-R_2\Rightarrow\left[\begin{array}{ccc|ccc}1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0\\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0\\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1\end{array}\right]$$

$$\left.\begin{array}{l} R_2 \to R_2 - 15R_3 \\ R_1 \to R_1 - 60R_3 \end{array}\right\} \to \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & \frac{1}{12} & 0 & -3 & 16 & -15 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

Step-4

Continuation to the above

$$R_1 \to R_1 - 6R_2 \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & \frac{1}{12} & 0 & -3 & 16 & -15 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array}\right]$$

$$R_2(12), R_3(180) \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right] \approx \left[I \quad A^{-1}\right]$$

Thus, the given matrix is reduced as $A = \left[I | A^{-1}\right]$ where $A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$

Hence the inverse of the given Hilbert matrix is $\boxed{A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}}$.