

## Step-1

Given that  $V$  is spanned by  $(1,1,0,1)$  and  $(0,0,1,0)$

(a) We have to find a basis for the orthogonal complement  $V^\perp$

Let  $a_1 = (1,1,0,1), a_2 = (0,0,1,0)$  and write  $a_1, a_2$  are rows of  $A$

Therefore 
$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

## Step-2

The row space of  $A$  is equal to  $V$

This implies  $V^\perp = \text{null space of } A$

By definition of null space  $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + x_4 = 0 \text{ and } x_3 = 0$$

## Step-3

Put

$$x_2 = a, x_4 = b$$

$$\Rightarrow x_1 = -a - b$$

Therefore  $V^\perp$  (null space of  $A$ ) is given by the vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a-b \\ a \\ 0 \\ b \end{bmatrix}$$

$$= a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

## Step-4

Hence a basis for the orthogonal complement  $\mathbf{V}^\perp$  is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

## Step-5

(b) We have to find the projection matrix  $P$  onto  $V$ .

The projection matrix  $P = A^T (AA^T)^{-1} A$

Given  $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

## Step-6

And

$$\begin{aligned}
AA^T &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \\
(AA^T)^{-1} &= \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}
\end{aligned}$$

## Step-7

Therefore

$$\begin{aligned}
A^T (AA^T)^{-1} A &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Therefore the projection matrix  $P$  onto  $\mathbf{V}$

$$= \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$$

## Step-8

(c) We have to find the vectors in  $\mathbf{V}$  closest to the vector  $b = (0, 1, 0, -1)$  in  $\mathbf{V}^\perp$

The closest vector to  $b$  in  $\mathbf{V}$  = the projection of  $b$  onto  $\mathbf{V}$

$$= Pb$$

$$= \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$