Step-1

The derivative of $a+bx+cx^2$ is $b+2cx+0x^2$

a) $D\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 2c \\ 0 \end{bmatrix}$

The 3 by 3 matrix $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ such that $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$

So, the required matrix is $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Step-2

c) $D^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$

Step-3

This means D stands for the derivative and the third derivative of a second degree polynomial is zero.

The first derivative $= b+2cx$

The second derivative = $2c$

The derivative = 0

The third derivative of $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

Step-4

c) The characteristic equation of $D = |D - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda(\lambda^2) - 1(0) + 0 = 0$$
$$\Rightarrow \lambda^3 = 0$$

$\Rightarrow \lambda = 0, 0, 0$ are the eigen values of the derivative matrix.

Step-5

To find eigen vector corresponding to $\lambda = 0$, we solve the homogeneous system $(D - 0I)x = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 = 0, x_3 = 0$

We further, observe that x_1 is satisfied by any real number $x_1 = k$ and not necessarily zero.

So, the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

Putting $k = 1$, we get the eigen vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ corresponding to $\lambda = 0$.

Observe that the eigen values are repeated 3 times and the corresponding eigen vector is only one.