Step-1

Consider the solution for differential equation

$$\frac{du}{dt} = Au$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u$$

goes around in a circle $u = (\cos t, \sin t)$.

Matrix $I + A$ is given by

$$I + A = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Step-2

To find the eigenvalues of matrix $I + A$, solve the equation $\det((I + A) - \lambda I) = 0$, for λ.

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$
$$(\lambda - 1)^2 = -1$$
$$\lambda - 1 = \pm i$$

Therefore, the eigenvalues of matrix $I + A$ are $\lambda_1 = \boxed{1-i}$ and $\lambda_2 = \boxed{1+i}$.

Step-3

To calculate eigenvector x_1 do the following calculation:

$$\det((I + A) - \lambda_1 I) x_1 = 0$$
$$\begin{bmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z corresponding to $\lambda_1 = 1 - i$ is as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Step-4

To calculate eigenvector x_2 do the following calculation:

$$\det\left(\left(I+A\right)-\lambda_2 I\right)x_2 = 0$$
$$\begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of *y* and *z* corresponding to $\lambda_2 = 1+i$ is as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Step-5

Therefore, the eigenvectors for the matrix $I+A$ are $x_1 = \boxed{\begin{bmatrix} 1 \\ i \end{bmatrix}}$ and $x_2 = \boxed{\begin{bmatrix} 1 \\ -i \end{bmatrix}}$.

Step-6

Matrix $(I-A)^{-1}$ is given by

$$(I-A)^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Step-7

To find the eigenvalues of matrix $(I-A)^{-1}$, solve the equation $\det\left((I-A)^{-1} - \lambda I\right) = 0$, for λ.

$$\det\begin{bmatrix} 1/2-\lambda & -1/2 \\ 1/2 & 1/2-\lambda \end{bmatrix}=0$$
$$\left(\frac{1}{2}-\lambda\right)^2+\frac{1}{4}=0$$

Therefore, the eigenvalues of matrix $(I-A)^{-1}$ are $\lambda_1=\boxed{\frac{1}{2}-\frac{1}{2}i}$ and $\lambda_2=\boxed{\frac{1}{2}+\frac{1}{2}i}$.

To calculate eigenvector x_1 do the following calculation:

$$\det\left((I-A)^{-1}-\lambda_1 I\right)x_1=0$$
$$\begin{bmatrix} 1/2-(1/2-1/2\,i) & -1/2 \\ 1/2 & 1/2-(1/2-1/2\,i) \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of *y* and *z* corresponding to $\lambda_1=(1/2-1/2\,i)$ is as follows:

$$x_1=\begin{bmatrix} y \\ z \end{bmatrix}$$
$$=\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Step-8

To calculate eigenvector x_2 do the following calculation:

$$\det\left((I-A)^{-1}-\lambda_2 I\right)x_2=0$$
$$\begin{bmatrix} 1/2-(1/2+1/2\,i) & -1/2 \\ 1/2 & 1/2-(1/2+1/2\,i) \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of *y* and *z* corresponding to $\lambda_2=(1/2+1/2\,i)$ is as follows:

$$x_2=\begin{bmatrix} y \\ z \end{bmatrix}$$
$$=\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Step-9

Therefore, the eigenvectors for the matrix $(I-A)^{-1}$ are $\boxed{x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$.

Step-10

Consider matrix $B = \left(I - \frac{1}{2}A\right)^{-1}\left(I + \frac{1}{2}A\right)$ is given by

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4/5 & -2/5 \\ 2/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \end{aligned}$$

Step-11

To find the eigenvalues of matrix B, solve the equation $\det(B - \lambda I) = 0$, for λ.

$$\det\begin{bmatrix} 3/5-\lambda & -4/5 \\ 4/5 & 3/5-\lambda \end{bmatrix} = 0$$

$$\left(\frac{3}{5} - \lambda\right)^2 + \frac{16}{25} = 0$$

Therefore, the eigenvalues of matrix B are $\lambda_1 = \boxed{\frac{3}{5} - \frac{4}{5}i}$ and $\lambda_2 = \boxed{\frac{3}{5} + \frac{4}{5}i}$.

Step-12

To calculate eigenvector x_1 do the following calculation:

$$\det(B - \lambda_1 I)x_1 = 0$$

$$\begin{bmatrix} 3/5-(3/5-4/5i) & -4/5 \\ 4/5 & 3/5-(3/5-4/5i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z corresponding to $\lambda_1 = (3/5 - 4/5i)$ is as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Step-13

To calculate eigenvector x_2 do the following calculation:

$$\det\left(B - \lambda_2 I\right)x_2 = 0$$
$$\begin{bmatrix} 3/5-(3/5+4/5i) & -4/5 \\ 4/5 & 3/5-(3/5+4/5i) \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z corresponding to $\lambda_2 = (3/5+4/5i)$ is as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Step-14

Therefore, the eigenvectors for the matrix $(I-A)^{-1}$ are $\boxed{x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$.

Step-15

Consider (x_1, x_2) lies on unit circle then $\sqrt{(x_1)^2 + (x_2)^2} = 1$, that is

$(x_1)^2 + (x_2)^2 = 1$.

Take the forward difference (**F**):

$u_{n+1} = (I+A)u_n$

So, solving the following equation, we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

If (y_1, y_2) lies on unit circle then $y_1^2 + y_2^2 = 1$.

Solve the following equation:

$$\begin{aligned} y_1^2 + y_2^2 &= (x_1 - x_2)^2 + (x_1 + x_2)^2 \\ &= 2\left[(x_1)^2 + (x_2)^2\right] \\ &= 2(1) \\ &\neq 1 \end{aligned}$$

Step-16

Therefore, for the differential equation $u_{n+1} = (I + A)u_n$, solution does not stay on a circle.

Step-17

Consider the forward difference (**B**):

$$u_{n+1} = (I - A)^{-1} u_n$$

So, solving the following equation, we have

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2\, x_1 - 1/2\, x_2 \\ 1/2\, x_1 + 1/2\, x_2 \end{bmatrix}$$

If (z_1, z_2) lies on unit circle then $z_1^2 + z_2^2 = 1$. For this solve the following equation:

$$\begin{aligned} z_1^2 + z_2^2 &= \left(\frac{1}{2}x_1 - \frac{1}{2}x_2\right)^2 + \left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)^2 \\ &= \frac{1}{4}\left[(x_1)^2 + (x_2)^2\right] \\ &= \frac{1}{4}(1) \\ &\neq 1 \end{aligned}$$

Therefore, for the differential equation $u_{n+1} = (I - A)^{-1} u_n$, solution does not stay on a circle.

Step-18

Consider the forward difference (**C**):

$$u_{n+1} = \left(I - \frac{1}{2}A\right)^{-1}\left(I + \frac{1}{2}A\right)u_n$$

So, solving the following equation, we have

$$\begin{aligned}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 3/5\,x_1 - 4/5\,x_2 \\ 4/5\,x_1 + 3/5\,x_2 \end{bmatrix}\end{aligned}$$

If $(w_1,\ w_2)$ lies on unit circle then $w_1^{\,2} + w_2^{\,2} = 1$. For this solve the following equation:

$$\begin{aligned} w_1^{\,2} + w_2^{\,2} &= \left(\frac{3}{5}x_1 - \frac{4}{5}x_2\right)^2 + \left(\frac{4}{5}x_1 + \frac{3}{5}x_2\right)^2 \\ &= \frac{25}{25}\left[(x_1)^2 + (x_2)^2\right] \\ &= 1(1) \\ &= 1 \end{aligned}$$

Therefore, for the differential equation $u_{n+1} = \left(I - \frac{1}{2}A\right)^{-1}\left(I + \frac{1}{2}A\right)u_n$, solution does stay on a circle.