Suggested Solutions of Homework 2 MA327

Ex 1. Define $U = \{(a,b) \in \mathbb{R}^2 \big| \|a\| < 1\}$, $V^{1,0} = \{(x,y,z) \big| x > 0\}$, $V^{1,1} = \{(x,y,z) \big| x < 0\}$, $V^{2,0} = \{(x,y,z) \big| y > 0\}$, $V^{2,1} = \{(x,y,z) \big| y < 0\}$.

$$
\begin{aligned}
x_1 &: U \longrightarrow V^{1,0} : (a,b) \mapsto (\sqrt{1-a^2}, a, b) \\
x_2 &: U \longrightarrow V^{1,1} : (a,b) \mapsto (-\sqrt{1-a^2}, a, b) \\
x_3 &: U \longrightarrow V^{2,0} : (a,b) \mapsto (a, \sqrt{1-a^2}, b) \\
x_4 &: U \longrightarrow V^{2,1} : (a,b) \mapsto (a, -\sqrt{1-a^2}, b)
\end{aligned}
$$

Now show each x_i is a parametrization. WLOG, assume p $\in V^{1,0}$, it is clear that x_1 is differentiable and it is a homeomorphism. Next x_1 satisfies regularity condition since

$$
\begin{vmatrix} \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1
$$

so x_1 is a parametrization. Finally the cylinder is a regular surface because $\{(x_i, U)\}_{i=1}^4$ cover the cylinder.

You may find other parametrizations for the cylinder.

Ex 2. The set S = $\{(x,y,z) \in \mathbb{R}^3 \big| z = 0, x^2 + y^2 \le 1\}$ is not a regular surface, since the point (1, 0, 0) has no neighborhood homeomorphic to open sets in $\mathbb{R}^2$. You can verify that the fundamental group $\pi_1(V \cap S - (1,0,0)) = 0$ and $\pi_1(U - x^{-1}((1,0,0))) = \mathbb{Z}$. Here V is an open set in $\mathbb{R}^3$ containing $(1,0,0)$ and U is an open set in $\mathbb{R}^2$ containing $x^{-1}(1,0,0)$.

The set $S' = \{(x,y,z) \in \mathbb{R}^3 \big| z = 0, x^2 + y^2 < 1\}$ is a regular surface. Take $x : D \longrightarrow S'$: (x, y) $\mapsto$ (x, y, 0). It is easy to check x is a parametrization of S'.

Ex 3. For any neighborhood $V \subset \{(x,y,z) \in \mathbb{R}^3 \big| x^2 + y^2 - z^2 = 0\}$ of (0,0,0) in S, V is not any graph of z = f(x,y), y = g(x,z), x = h(y,z). Hence S is not a regular surface.

Ex 4. $df : \mathbb{R}^3 \longrightarrow \mathbb{R} : (x,y,z) \mapsto (0,0,2z)$. df is not surjective if and only if z = 0. Then 0 = f(x,y,0) is a critical value. $f^{-1}(0) = \{(x,y,z) \big| z = 0\} \cong \mathbb{R}^2$, which is a regular surface.

Ex 5. Yes. Condition 1: it is easy to verify x is differentiable.

Condition 3:

$$
d\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ v & u \end{bmatrix}
$$

Since u > v, rank d$\mathbf{x}$ = 2, so it satisfies the regular condition.

Condition 2: Try to define $x^{-1} : \mathbf{x}(U) \longrightarrow U$: suppose (x,x,z) $\in$ $\mathbf{x}$(U), then $x^2 - 4z > 0$ such that there are two distinct solutions of (u, v). As a result we can define $x^{-1} : (x,x,z) \mapsto (\frac{x+\sqrt{x^2-4z}}{2}, \frac{x-\sqrt{x^2-4z}}{2})$. What is more, x^{-1} is continuous, we have done.

Ex 6. (a) $f_x = f_y = f_z = 2(x+y+z-1)$. So the critical points of f are $\{(x,y,z) \in \mathbb{R}^3 \big| x+y+z-1=0\}$, and the set of critical values is 0.

(b) If c > 0, by prop 2.2.2, $f^{-1}(c)$ is a regular surface; if c = 0, $f^{-1}(c)$ is a plane, which is also regular. Thus all c which is not smaller than 0 can make $f^{-1}(c)$ a regular surface.

(c) Critical points = $\{(x,y,z) \big| \ either\ x = y = 0\ or\ z = 0\}$; set of critical values = $\{0\}$. Finally, $(0,0,0) \in f^{-1}(0)$ is not any graph of z = f(x,y), y = g(x,z), x = h(y,z), i.e. $f^{-1}(0)$ is not a regular surface. Thus for $c \neq 0$, $f^{-1}(c)$ is a regular surface.

Ex 7. $\mathrm{d}x_q = (\frac{\partial \mathbf{x}}{\partial u}\ \frac{\partial \mathbf{x}}{\partial v})$ is one-to-one $\quad\Leftrightarrow\quad$ rank $\mathrm{d}x_q = 2 \quad\Leftrightarrow\quad \frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0$

Ex 8. Consider function $f = (x/a)^2 + (y/b)^2 + (z/c)^2 - 1$, 0 is a regular value, so ellipsoid $= f^{-1}(0)$ is a regular surface.
Condition 1: Trivially, $\mathbf{x}$ is differentiable;
Condition 3:

$$d\mathbf{x} = \begin{bmatrix} a\cos u\cos v & -a\sin u\sin v \\ b\cos u\sin v & b\sin u\cos v \\ -c\sin u & 0 \end{bmatrix}$$

$$\left|x_u \wedge x_v\right|^2 = \sin^2 u(b^2c^2\sin^2 u\cos^2 v + a^2c^2\sin^2 u\sin^2 v + a^2b^2\cos^2 u) = 0$$

$$\Leftrightarrow \quad \sin u = 0$$

$$\Leftrightarrow \quad u = k\pi, k \in \mathbb{Z}$$

so dx is one-to-one.
Condition 2: It is easy to see x is one-to-one, referring prop 2.2.4, x^{-1} is continuous.

The curves $u = Const.$ on the ellipsoid is ellipse which is parallel to the xy-plane.

Ex 9. (a) Given a point (u, v) $\in \mathbb{R}^2$, then the line going through N and (u,v,0) is given by $l(t) = (u, v, -2)t + (0, 0, 2), t \in \mathbb{R}$. After solving for the intersection of the line and the sphere we get

$$t_1 = 0, t_2 = \frac{4}{u^2 + v^2 + 4}.$$

Since $l(0) = N$, the intersection point must be $l(t_2)$

$$\pi^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2}{u^2+v^2+4}).$$

For (a), one may also see an argument in Lecture 6.

(b) It is clear that π^{-1} is differentiable and is a homeomorphism. Then you can check

$$d\pi^{-1} = \frac{4}{(u^2+v^2+4)^2}\begin{bmatrix} -u^2+v^2+4 & -2uv \\ -2uv & u^2-v^2+4 \\ 4u & 4v \end{bmatrix}$$

hence $d\pi^{-1}$ is one-to-one for all (u,v) $\in \mathbb{R}^2$. Therefore π^{-1} is a parametrization of S^2. To cover S^2, we do a stereographic projection on $-N$, then S^2 could be covered.

Ex 10. Since $A^{-1} = A$, it suffices to prove A is differentiable, which means to prove A is differentiable at every point $p = (x,y,z)$. WLOG, assume $z > 0$, and we select the following parametrization:

$$x_1 : \mathbb{D} \longrightarrow U^{3,0} = \{(x,y,z)\big| z > 0\} : (x,y) \mapsto (x, y, \sqrt{1-x^2-y^2})$$
$$x_2 : \mathbb{D} \longrightarrow U^{3,1} = \{(x,y,z)\big| z < 0\} : (x,y) \mapsto (x, y, -\sqrt{1-x^2-y^2})$$

Trivially, $p \in U^{3,0}$ and $-p \in U^{3,1}$. Besides, $x_2^{-1} \circ A \circ x_1(x,y) = (-x,-y), (x,y) \in \mathbb{D}$, which is differentiable. Hence we have proved A and A^{-1} is differentiable and that A is a diffeomorphism.

Ex 11. Denote $P = \{(x,y,z) \big| z = x^2 + y^2\}$, $\mathbb{R}^2$ = plane. Define a map $\phi : P \longrightarrow \mathbb{R}^2 : (x,y,z) \mapsto (x,y)$. Trivially, ϕ is bijective and its inverse is: $\phi^{-1} : \mathbb{R}^2 \longrightarrow P : (x,y) \mapsto (x,y,x^2+y^2)$. Then $x_1 = \phi^{-1}$ is a parametrization of P, and $x_2 = id_{\mathbb{R}^2}$ is a parametrization of $\mathbb{R}^2$. Hence

$$x_2^{-1} \circ \phi \circ x_1 = id_{\mathbb{R}^2}$$
$$x_1^{-1} \circ \phi^{-1} \circ x_2 = id_{\mathbb{R}^2}.$$

They are definitely differentiable, so P is diffeomorphic to $\mathbb{R}^2$.

Ex 12. $A : Ellipsoid \to S^2 : (x,y,z) \mapsto (x/a, y/b, z/c)$

Ex 13. Define $p_0 = (x_0, y_0, z_0)$;

$$U \xrightarrow{x} S \xrightarrow{d} \mathbb{R} : (u,v) \mapsto (x(u,v), y(u,v), z(u,v)) \mapsto \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

$d \circ x$ is differentiable at every (u,v), so d is differentiable.

Ex 14. Given $A : M \longrightarrow N$. Select a point $p \in M$, there are $x : U \longrightarrow M$, $y : V \longrightarrow N$ such that $y^{-1} \circ A \circ x$ is differentiable at $x^{-1}(p)$. Then given any other maps: $x' : U' \longrightarrow M$, $y' : V' \longrightarrow N$ with $p \in x'(U')$ and $A(p) \in y'(V')$, then

$$y'^{-1} \circ A \circ x' = y'^{-1} \circ (y \circ y^{-1}) \circ A \circ (x^{-1} \circ x) \circ x' = (y'^{-1} \circ y) \circ (y^{-1} \circ A \circ x) \circ (x^{-1} \circ x')$$

in a small neighborhood which contain $x'^{-1}(p)$. Since change of coordinates is differentiable, $y'^{-1} \circ A \circ x'$ is differentiable at $x'^{-1}(p)$. Since p is arbitrary, the differentiability does not depend on parametrization.

Ex 15. Given a point p=(x,y,z) $\in S^2 - N - S$, the equation for l is

$$l(t) = (x,y,0)t + (0,0,z),$$

Solve the intersection we can define a map:

$$F : S^2 - N - S \longrightarrow H : (x,y,z) \mapsto (x\frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}, y\frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}, z).$$

Then it is going to prove F is differentiable at every point. WLOG, assume $x > 0$. Now parameterize a neighborhood of p and $F(p)$:

$$x : U \longrightarrow S^2 : (y,z) \mapsto (\sqrt{1-y^2-z^2}, y, z)$$
$$y : V \longrightarrow H : (y,z) \mapsto (\sqrt{1+z^2-y^2}, y, z)$$

Then $y^{-1} \circ F \circ x$ is differentiable at $x^{-1}(p)$. Thus F is differentiable at every p, i.e. F is differentiable.