

Step-1

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Observe that this matrix has a sub matrix of order 3×3 and also the determinant is zero.

So, the rank of the matrix A is less than 3.

And also, see that any sub matrix of order 2×2 has the determinant zero.

So, the rank of the matrix A is less than 2, that means rank of A is 1.

Step-2

Now, the characteristic equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

Use elementary row operations on this to reduce into the echelon form as follows.

$$R_1 \rightarrow R_1 + R_2 + R_3 + R_4$$

$$\begin{vmatrix} 4-\lambda & 4-\lambda & 4-\lambda & 4-\lambda \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

Step-3

Continue the above steps.

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$(4-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 0 \\ 0 & 0 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[1(-\lambda)(-\lambda)(-\lambda)] = 0$$

$$\lambda^3(\lambda-4) = 0$$

So, $\lambda = 0, 0, 0, 4$ are the Eigen values of the given matrix.

Step-4

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Now, take the matrix

Here, the elementary operations on the matrix C does not change the rank of C .

So, apply row operations on C to reduce it to echelon form.

Then we confirm the rank of C .

$$R_1 \leftrightarrow R_4, R_2 \leftrightarrow R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

For,

$$R_3 \rightarrow R_3 - R_1, R_4 \rightarrow R_4 - R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-5

Clearly, the entries in the principal diagonal are either zero or one.

And also all the entries below the main diagonal are 0's.

So, the matrix is in the echelon form.

Hence, the rank of C is equal to the number of non-zero rows in the matrix C .

Here, the number of non-zero rows is 2, and then the rank is 2.

Step-6

Further, to get the Eigen values of C , take the characteristic equation of C .

That is

$$|C - \lambda I| = 0$$

$$\left| \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 & 0 & 1 \\ 1 & 0-\lambda & 1 & 0 \\ 0 & 1 & 0-\lambda & 1 \\ 1 & 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix} = 0$$

Apply the row operations on this, then

$$R_1 \rightarrow R_1 + R_2 + R_3 + R_4 \Rightarrow$$

$$\begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & 0-\lambda & 1 & 0 \\ 0 & 1 & 0-\lambda & 1 \\ 1 & 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix} = 0$$

Step-7

Apply the following row-operations.

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$(2-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda-1 & 0 & -1 \\ 0 & 1 & -\lambda & 1 \\ 0 & -1 & 0 & -\lambda-1 \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} -\lambda-1 & 0 & -1 \\ 1 & -\lambda & 1 \\ -1 & 0 & -\lambda-1 \end{vmatrix} = 0$$

$$(2-\lambda)(-\lambda)\{(-\lambda-1)^2-1\} = 0$$

$$(2-\lambda)(-\lambda)\{(\lambda^2+2\lambda+1)-1\} = 0$$

$$(2-\lambda)(-\lambda)(\lambda^2+2\lambda) = 0$$

$$(2-\lambda)(-\lambda)(\lambda)(\lambda+2) = 0$$

$$-\lambda^2(2-\lambda)(\lambda+2) = 0$$

Therefore, the eigen values of the matrix C are $0, 0, -2$, and 2 .

Step-8

Now, find the Eigen vectors as follows.

Let $v_1 = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ be the required Eigen vector.

At $\lambda = 2$, the system $(C - \lambda I)v_1 = \mathbf{0}$ becomes as $(C - (2)I)v_1 = \mathbf{0}$.

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Here, the coefficient matrix is $\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$.

Apply the row operations to reduce into echelon form.

$$R_1 \leftrightarrow R_2$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1, R_4 \rightarrow R_4 - R_1$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

$$R_3 \rightarrow 3R_3 + R_2, R_4 \rightarrow 3R_4 + 2R_2$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & -4 \end{pmatrix}$$

$$R_4 \rightarrow R_4 + R_3$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now the system becomes as,

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The system of equations is,

$$x - 2y + z = 0 \quad \dots\dots (1)$$

$$-3y + 2z + t = 0 \quad \dots\dots (2)$$

$$-4z + 4t = 0 \quad \dots\dots (3)$$

From equation (3), $z = t$.

Substitute $z = t$ in the equation (2),

$$-3y + 2t + t = 0$$

$$-3y + 3t = 0$$

$$y = t$$

Then, substitute $z = t, y = t$ in the equation (1).

$$x - 2y + z = 0$$

$$x - 2t + t = 0$$

$$x - t = 0$$

$$x = t$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

So, the variable matrix is

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence, the Eigen vector for the Eigen value $\lambda = 2$ is

Step-9

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Similarly, the remaining Eigen vectors are for $\lambda = -2$.

$$v_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \text{ for } \lambda = 0 \text{ and } v_4 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ for } \lambda = 0.$$