SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 10 Solutions

1. Suppose a player plays the following gambling games which is known as the wheel of fortune. The player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, $i = 1, 2, 3$, then the player wins i units; on the other hand, if the number bet by the player does not appear on any of the dies, then the player loses 1 unit. Is this game fair to the player?

   Solution: Denote X(units) be the number of the player to play the wheel of fortune after get, then the range of X is $\{-1, 1, 2, 3\}$.

$$\mathrm{P}(X=-1) = \binom{3}{0}(\frac{1}{6})^0 \cdot (1-\frac{1}{6})^3 = \frac{125}{216}.$$

$$\mathrm{P}(X=1) = \binom{3}{1}(\frac{1}{6})^1 \cdot (1-\frac{1}{6})^2 = \frac{75}{216}.$$

$$\mathrm{P}(X=2) = \binom{3}{2}(\frac{1}{6})^2 \cdot (1-\frac{1}{6})^1 = \frac{15}{216}.$$

$$\mathrm{P}(X=3) = \binom{3}{3}(\frac{1}{6})^3 \cdot (1-\frac{1}{6})^0 = \frac{1}{216}.$$

$$\begin{aligned}\mathrm{E(X)} &= -1 \times \mathrm{P}(X=-1) + 1 \times \mathrm{P}(X=1) + 2 \times \mathrm{P}(X=2) + 3 \times \mathrm{P}(X=3) \\ &= -1 \times \frac{125}{216} + 1 \times \frac{125}{216} + 2 \times \frac{125}{216} + 3 \times \frac{125}{216} \\ &= -\frac{17}{216} < 0.\end{aligned}$$

   Hence, this game isn't fair to the player.

2. Suppose the random variable X takes non-negative integer values only. Show that

$$\mathrm{E(X)} = \sum_{\mathrm{n=0}}^{\infty} \mathrm{P(X > n)} = \sum_{\mathrm{n=1}}^{\infty} \mathrm{P(X \geq n)}.$$

   Proof:

$$\mathrm{E(X)} = \sum_{\mathrm{m=0}}^{\infty} \mathrm{mP(X = m)} = \sum_{\mathrm{m=0}}^{\infty}\sum_{\mathrm{n=0}}^{\mathrm{m-1}} \mathrm{P(X = m)} = \sum_{\mathrm{n=0}}^{\infty}\sum_{\mathrm{m=n+1}}^{\infty} \mathrm{P(X = m)} = \sum_{\mathrm{n=0}}^{\infty} \mathrm{P(X > n)}.$$

   Observed that the random variable X takes non-negative integer values only, so

$$\{X > n\} = \{X \geq n+1\}.$$

   yields

$$\sum_{n=0}^{\infty} \mathrm{P}(X > n) = \sum_{n=0}^{\infty} \mathrm{P}(X \geq n+1) = \sum_{n=1}^{\infty} \mathrm{P}(X \geq n).$$

   Method 2: Assume $\mathrm{E(X)} < \infty$, then we know absolutely convergent series by rearranging the order, without changing its value. Hence,

$$\mathrm{E(X)} = 1 \times \mathrm{P}(X=1) + 2 \times \mathrm{P}(X=2) + 3 \times \mathrm{P}(X=3) \ldots + n \times \mathrm{P}(X=n) + \ldots$$

$$
\begin{aligned}
&= 1\times \mathrm{P}(X=1) + 1\times \mathrm{P}(X=2) + 1\times \mathrm{P}(X=3) + \ldots + 1\times \mathrm{P}(X=n) + \ldots \\
&\qquad\qquad\qquad + 1\times \mathrm{P}(X=2) + 1\times \mathrm{P}(X=3) + \ldots + 1\times \mathrm{P}(X=n) + \ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad + 1\times \mathrm{P}(X=3) + \ldots + 1\times \mathrm{P}(X=n) + \ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
&= 1\times \mathrm{P}(X\ge 1) \\
&\qquad\qquad\qquad + 1\times \mathrm{P}(X\ge 2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + 1\times \mathrm{P}(X\ge 3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
&= \sum_{n=1}^{\infty} \mathrm{P}(X\ge n) = \sum_{n=0}^{\infty} \mathrm{P}(X\ge n+1) = \sum_{n=0}^{\infty} \mathrm{P}(X>n).
\end{aligned}
$$

3. (a) Suppose the random variable X obeys the uniformly distribution over interval $[a,b]$. Find $E(X)$.
   (b) Suppose the random variable X obeys the general Γ distribution with parameters λ and α where $\lambda > 0$ and $\alpha > 0$. Write down the p.d.f of this general Γ random variable and the analytic form of the Γ function $\Gamma(\alpha)$ for $\alpha > 0$ and hence find the E(X) of this general Γ random variable.
   (c) Suppose $Y = X^2$ where X is normally distributed with parameters μ and σ^2. Obtain the p.d.f of Y and then find E(Y).

Solution:

(a) Recall:

$$
f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b, \\ 0 & \textit{otherwise}. \end{cases}
$$

$$
\begin{aligned}
\mathrm{E(X)} &= \int_{-\infty}^{+\infty} x f(x)\mathrm{d}x \\
&= \int_{-\infty}^{a} x f(x)\mathrm{d}x + \int_{a}^{b} x f(x)\mathrm{d}x + \int_{b}^{+\infty} x f(x)\mathrm{d}x \\
&= \int_{-\infty}^{a} x\cdot 0\mathrm{d}x + \int_{a}^{b} x\cdot \frac{1}{b-a}\mathrm{d}x + \int_{b}^{+\infty} x\cdot 0\mathrm{d}x \\
&= \frac{1}{b-a}\int_{a}^{b} x\mathrm{d}x \\
&= \frac{1}{b-a}\cdot \frac{x^2}{2}\bigg|_a^b = \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} = \frac{a+b}{2}.
\end{aligned}
$$

(b) Since $X \sim \Gamma(\lambda,\alpha), \lambda > 0, \alpha > 0$
Then th p.d.f is:

$$
f(x) = \begin{cases} \dfrac{\lambda(\lambda x)^{\alpha-1}\,\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)} & x \ge 0, \\ 0 & \textit{otherwise}, \end{cases}
$$

where $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$.
Hence

$$
\begin{aligned}
\mathrm{E(X)} &= \int_{-\infty}^{+\infty} x f(x)\mathrm{d}x \\
&= \int_{0}^{+\infty} x\frac{\lambda(\lambda x)^{\alpha-1}\,\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)}\mathrm{d}x
\end{aligned}
$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (\lambda x)^{\alpha+1-1} \mathrm{e}^{-\lambda x} \, \mathrm{d}x$$
$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha.$$

(c) Let $F_X(x)$ and $F_Y(y)$ be the c.d.fs of the random variable X and Y,respectively. Let $f_X(x)$ and $f_Y(y)$ be the p.d.fs of the random variable X and Y,respectively. Then

$$F_Y(y) = \mathrm{P}\{Y \le y\} = \mathrm{P}\{X^2 \le y\}.$$

Now,

1° if $y < 0$, then the event $\{w \in \Omega : X^2(\omega) \le y\} = \{X^2 \le y\} = \emptyset$ and hence

$$F_Y(y) = \mathrm{P}\{Y \le y\} = \mathrm{P}\{X^2 \le y\} = \mathrm{P}(\emptyset) = 0.$$

So, $f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} 0 = 0$.

2° if $y = 0$, then the event $\{X^2 \le y\} = \{X^2 \le 0\} = \{X^2 = 0\} = \{X = 0\}$ and hence

$$F_Y(y) = \mathrm{P}\{Y \le y\} = \mathrm{P}\{X^2 \le y\} = \mathrm{P}(X = 0) = 0.$$

Since the probability density function in a point modify the value does not affect the distribution function, for $y = 0$, let $f_Y(y) = f_Y(0) = 0$.

3° if $y > 0$, then

$$\begin{aligned} F_Y(y) &= \mathrm{P}(X^2 \le y) = \mathrm{P}\{|X| \le \sqrt{y}\} = \mathrm{P}\{-\sqrt{y} \le X \le \sqrt{y}\} \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Notice that $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, it follows that

$$\begin{aligned} f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_X(\sqrt{y}) - \frac{\mathrm{d}}{\mathrm{d}y} F_X(-\sqrt{y}) \\ &= \frac{\mathrm{d}F_X(\sqrt{y})}{\mathrm{d}y} - \frac{\mathrm{d}F_X(-\sqrt{y})}{\mathrm{d}y} \\ &= \frac{\mathrm{d}F_X(u)}{\mathrm{d}u}\Big|_{u=\sqrt{y}} \cdot \frac{\mathrm{d}(\sqrt{y})}{\mathrm{d}y} - \frac{\mathrm{d}F_X(u)}{\mathrm{d}u}\Big|_{u=-\sqrt{y}} \cdot \frac{\mathrm{d}(-\sqrt{y})}{\mathrm{d}y} \\ &= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot (-1) \cdot \frac{1}{2\sqrt{y}} \\ &= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} = \frac{1}{2\sqrt{y}} \left[\frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(-\sqrt{y}-\mu)^2}{2\sigma^2}} \right] \\ &= \frac{1}{2\sqrt{2\pi y}\sigma} \left[\mathrm{e}^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} + \mathrm{e}^{-\frac{(\sqrt{y}+\mu)^2}{2\sigma^2}} \right]. \end{aligned}$$

So, for $y > 0$, we have $f_Y(y) = \frac{1}{2\sqrt{2\pi y}\sigma} [\mathrm{e}^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} + \mathrm{e}^{-\frac{(\sqrt{y}+\mu)^2}{2\sigma^2}}]$.

In a word, we get the p.d.f of random variable Y is

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{2\pi y}\sigma} [\mathrm{e}^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} + \mathrm{e}^{-\frac{(\sqrt{y}+\mu)^2}{2\sigma^2}}] & y > 0, \\ 0 & y \le 0. \end{cases}$$

$$\begin{aligned} \mathrm{E(Y)} &= \int_{-\infty}^{+\infty} y f_Y(y) \mathrm{d}y = \int_{-\infty}^{0} y f_Y(y) \mathrm{d}y + = \int_{0}^{+\infty} y f_Y(y) \mathrm{d}y \\ &= \int_0^{+\infty} \frac{y}{2\sqrt{2\pi y}\sigma} [\mathrm{e}^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} + \mathrm{e}^{-\frac{(\sqrt{y}+\mu)^2}{2\sigma^2}}] \mathrm{d}y \end{aligned}$$

$$\overset{t=\sqrt{y}}{=\!=\!=} \int_0^{+\infty} \frac{t^2}{2\sqrt{2\pi}t\sigma}[\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}} + \mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}]2t\mathrm{d}t = \int_0^{+\infty} [\frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}} + \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}]\mathrm{d}t$$

$$= \left\{\boxed{\int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t} + \boxed{\int_{\mu}^{+\infty} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t}\right\} + \left\{\boxed{\int_0^{-\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}\,\mathrm{d}t} + \boxed{\int_{-\mu}^{+\infty} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}\,\mathrm{d}t}\right\}$$

$$= \left\{\boxed{\int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t} + \boxed{\int_0^{-\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}\,\mathrm{d}t}\right\} + \boxed{\int_{\mu}^{+\infty} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t} + \boxed{\int_{-\mu}^{+\infty} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}\,\mathrm{d}t}$$

$$\triangleq I + II + III.$$

Where if $\mu < 0$, then we should understand $\int_0^{\mu} f(t)\mathrm{d}t = -\int_{\mu}^{0} f(t)\mathrm{d}t$, other similar.

$$\begin{aligned}
I &= \int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t + \int_0^{-\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}\,\mathrm{d}t \\
&= \int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t + \int_0^{\mu} \frac{(-s)^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(-s+\mu)^2}{2\sigma^2}}(-1)\cdot\mathrm{d}s \\
&= \int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t - \int_0^{\mu} \frac{s^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(s-\mu)^2}{2\sigma^2}}\,\mathrm{d}s \\
&= \int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t - \int_0^{\mu} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
II &= \int_{\mu}^{+\infty} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t \;\overset{s=\frac{t-\mu}{\sigma}}{\underset{t=\mu+\sigma s}{=\!=\!=\!=}}\; \int_0^{+\infty} \frac{(\mu+\sigma s)^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{s^2}{2}}\,\sigma\cdot\mathrm{d}s \\
&= \int_0^{+\infty} \frac{(\mu^2+2\mu\sigma s+\sigma^2 s^2)}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s \\
&= \mu^2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + 2\mu\sigma \int_0^{+\infty} \frac{s}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + \sigma^2 \int_0^{+\infty} \frac{s^2}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s.
\end{aligned}$$

$$\begin{aligned}
III &= \int_{-\mu}^{+\infty} \frac{t^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(t+\mu)^2}{2\sigma^2}}\,\mathrm{d}t \;\overset{s=\frac{t+\mu}{\sigma}}{\underset{t=-\mu+\sigma s}{=\!=\!=\!=}}\; \int_0^{+\infty} \frac{(-\mu+\sigma s)^2}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{s^2}{2}}\,\sigma\cdot\mathrm{d}s \\
&= \int_0^{+\infty} \frac{(\mu^2-2\mu\sigma s+\sigma^2 s^2)}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s \\
&= \mu^2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s - 2\mu\sigma \int_0^{+\infty} \frac{s}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + \sigma^2 \int_0^{+\infty} \frac{s^2}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{E(Y)} &= I + II + III \\
&= 0 + \mu^2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + 2\mu\sigma \int_0^{+\infty} \frac{s}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + \sigma^2 \int_0^{+\infty} \frac{s^2}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s \\
&\quad + \mu^2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s - 2\mu\sigma \int_0^{+\infty} \frac{s}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + \sigma^2 \int_0^{+\infty} \frac{s^2}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s \\
&= 2\mu^2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s + 2\sigma^2 \int_0^{+\infty} \frac{s^2}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}\,\mathrm{d}s \\
&\overset{t=\frac{s^2}{2}}{=\!=\!=} 2\mu^2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-t}\,\frac{1}{2}\cdot\frac{2}{\sqrt{2t}}\mathrm{d}t + 2\sigma^2 \int_0^{+\infty} \frac{2t}{\sqrt{2\pi}}\,\mathrm{e}^{-t}\,\frac{1}{2}\cdot\frac{2}{\sqrt{2t}}\mathrm{d}t \\
&= \frac{\mu^2}{\sqrt{\pi}} \int_0^{+\infty} t^{\frac{1}{2}-1}\,\mathrm{e}^{-t}\,\mathrm{d}t + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{+\infty} t^{\frac{3}{2}-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \\
&= \frac{\mu^2}{\sqrt{\pi}}\cdot\Gamma(\frac{1}{2}) + \frac{2\sigma^2}{\sqrt{\pi}}\cdot\Gamma(\frac{3}{2}) = \frac{\mu^2}{\sqrt{\pi}}\times\sqrt{\pi} + \frac{2\sigma^2}{\sqrt{\pi}}\times\frac{1}{2}\cdot\sqrt{\pi} = \mu^2+\sigma^2.
\end{aligned}$$

4. (a) Suppose that the two discrete random variables X and Y have joint p.m.f given by

$X \backslash Y$	$Y=1$	$Y=2$	$Y=3$	$Y=4$
$X=1$	$2/32$	$3/32$	$4/32$	$5/32$
$X=2$	$3/32$	$4/32$	$5/32$	$6/32$

Obtain E(X) and E(Y).

(b) Suppose that the two continuous random variables X and Y have joint p.d.f

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find E(X) and E(Y).

Solution:

(a) Notice that

$$\{X=1\} = \{X=1, Y=1\} \cup \{X=1, Y=2\} \cup \{X=1, Y=3\} \cup \{X=1, Y=4\}.$$

then, the marginal probability mass function of X is:

$$\begin{aligned} \mathrm{P}(X=1) &= \mathrm{P}\left(\{X=1, Y=1\} \cup \{X=1, Y=2\} \cup \{X=1, Y=3\} \cup \{X=1, Y=4\}\right) \\ &= \mathrm{P}\{X=1, Y=1\} + \mathrm{P}\{X=1, Y=2\} + \mathrm{P}\{X=1, Y=3\} + \mathrm{P}\{X=1, Y=4\} \\ &= \frac{2}{32} + \frac{3}{32} + \frac{4}{32} + \frac{5}{32} \\ &= \frac{14}{32} = \frac{7}{16}. \end{aligned}$$

Method 1:

$$\begin{aligned} \mathrm{P}(X=2) &= \mathrm{P}\left(\{X=2, Y=1\} \cup \{X=2, Y=2\} \cup \{X=2, Y=3\} \cup \{X=2, Y=4\}\right) \\ &= \mathrm{P}\{X=2, Y=1\} + \mathrm{P}\{X=2, Y=2\} + \mathrm{P}\{X=2, Y=3\} + \mathrm{P}\{X=2, Y=4\} \\ &= \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{6}{32} \\ &= \frac{18}{32} = \frac{9}{16}. \end{aligned}$$

Method 2: Observed that $\{X=1\} \cup \{X=2\} = \Omega$, yield $\{X=1\} = \{X=2\}^c$. Then

$$\mathrm{P}(X=2) = 1 - \mathrm{P}(X=1) = 1 - \frac{7}{16} = \frac{9}{16}.$$

Then,

$$\begin{aligned} \mathrm{E(X)} &= 1 \times \mathrm{P}(X=1) + 2 \times \mathrm{P}(X=2) \\ &= 1 \times \frac{7}{16} + 2 \times \frac{9}{16} \\ &= \frac{25}{16}. \end{aligned}$$

Notice that

$$\{Y=1\} = \{X=1, Y=1\} \cup \{X=2, Y=1\}.$$

then,

$$\mathrm{P}(Y=1) = \mathrm{P}\left(\{X=1, Y=1\} \cup \{X=2, Y=1\}\right)$$

$$
\begin{aligned}
&= \mathrm{P}(X=1, Y=1) + \mathrm{P}(X=2, Y=1) \\
&= \frac{2}{32} + \frac{3}{32} \\
&= \frac{5}{32}.
\end{aligned}
$$

Similarly,

$$\{Y=2\} = \{X=1, Y=2\} \cup \{X=2, Y=2\}.$$

then,

$$
\begin{aligned}
\mathrm{P}(Y=2) &= \mathrm{P}\left(\{X=1, Y=2\} \cup \{X=2, Y=2\}\right) \\
&= \mathrm{P}(X=1, Y=2) + \mathrm{P}(X=2, Y=2) \\
&= \frac{3}{32} + \frac{4}{32} \\
&= \frac{7}{32}.
\end{aligned}
$$

$$\{Y=3\} = \{X=1, Y=3\} \cup \{X=2, Y=3\}.$$

then,

$$
\begin{aligned}
\mathrm{P}(Y=3) &= \mathrm{P}\left(\{X=1, Y=3\} \cup \{X=2, Y=3\}\right) \\
&= \mathrm{P}(X=1, Y=3) + \mathrm{P}(X=2, Y=3) \\
&= \frac{4}{32} + \frac{5}{32} \\
&= \frac{9}{32}.
\end{aligned}
$$

$$\{Y=4\} = \{X=1, Y=4\} \cup \{X=2, Y=4\}.$$

then,

$$
\begin{aligned}
\mathrm{P}(Y=4) &= \mathrm{P}\left(\{X=1, Y=4\} \cup \{X=2, Y=4\}\right) \\
&= \mathrm{P}(X=1, Y=4) + \mathrm{P}(X=2, Y=4) \\
&= \frac{5}{32} + \frac{6}{32} \\
&= \frac{11}{32}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{E(Y)} &= 1 \times \mathrm{P}(Y=1) + 2 \times \mathrm{P}(Y=2) + 3 \times \mathrm{P}(Y=3) + 4 \times \mathrm{P}(Y=4) \\
&= 1 \times \frac{5}{32} + 2 \times \frac{7}{32} + 3 \times \frac{9}{32} + 4 \times \frac{11}{32} \\
&= \frac{90}{32} = \frac{45}{16}.
\end{aligned}
$$

(b) For $x < 0$ or $x > 1$, we have $f(x, y) = 0$, thus

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}y = \int_{-\infty}^{+\infty} 0\mathrm{d}y = 0.$$

For $0 \leq x \leq 1$,

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}y \\
&= \int_{-\infty}^{0} f(x, y)\mathrm{d}y + \int_{0}^{1} f(x, y)\mathrm{d}y + \int_{1}^{+\infty} f(x, y)\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} 0\mathrm{d}y + \int_{0}^{1} (x+y)\mathrm{d}y + \int_{1}^{+\infty} 0\mathrm{d}y \\
&= \int_{0}^{1} (x+y)\mathrm{d}y = (xy + \frac{y^2}{2})\Big|_{y=0}^{y=1} = x + \frac{1}{2}.
\end{aligned}
$$

Similarly, for $y < 0$ or $y > 1$, we have $f(x, y) = 0$, thus

$$
f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}x = \int_{-\infty}^{+\infty} 0\mathrm{d}x = 0.
$$

For $0 \le y \le 1$,

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}x \\
&= \int_{-\infty}^{0} f(x, y)\mathrm{d}x + \int_{0}^{1} f(x, y)\mathrm{d}x + \int_{1}^{+\infty} f(x, y)\mathrm{d}x \\
&= \int_{-\infty}^{0} 0\mathrm{d}x + \int_{0}^{1} (x+y)\mathrm{d}x + \int_{1}^{+\infty} 0\mathrm{d}x \\
&= \int_{0}^{1} (x+y)\mathrm{d}x = (\frac{x^2}{2} + yx)\Big|_{x=0}^{x=1} = y + \frac{1}{2}.
\end{aligned}
$$

In sum up, we obtain the marginal p.d.f of X:

$$
f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

The marginal p.d.f of Y:

$$
f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
\mathrm{E(X)} &= \int_{-\infty}^{+\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} x f_X(x)\mathrm{d}x + \int_{0}^{1} x f_X(x)\mathrm{d}x + \int_{1}^{+\infty} x f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} x \cdot 0\mathrm{d}x + \int_{0}^{1} x \cdot (x + \frac{1}{2})\mathrm{d}x + \int_{1}^{+\infty} x \cdot 0\mathrm{d}x \\
&= \int_{0}^{1} x(x + \frac{1}{2})\mathrm{d}x = \int_{0}^{1} x^2 + \frac{1}{2}x\mathrm{d}x \\
&= (\frac{x^3}{3} + \frac{1}{2} \cdot \frac{x^2}{2})\Big|_{0}^{1} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E(Y)} &= \int_{-\infty}^{+\infty} y f_Y(y)\mathrm{d}y \\
&= \int_{-\infty}^{0} y f(y)\mathrm{d}y + \int_{0}^{1} y f_Y(y)\mathrm{d}y + \int_{1}^{+\infty} y f_Y(y)\mathrm{d}y \\
&= \int_{-\infty}^{0} y \cdot 0\mathrm{d}y + \int_{0}^{1} y \cdot (y + \frac{1}{2})\mathrm{d}y + \int_{1}^{+\infty} y \cdot 0\mathrm{d}y \\
&= \int_{0}^{1} y(y + \frac{1}{2})\mathrm{d}y = \int_{0}^{1} y^2 + \frac{1}{2}y\mathrm{d}y \\
&= (\frac{y^3}{3} + \frac{1}{2} \cdot \frac{y^2}{2})\Big|_{0}^{1} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.
\end{aligned}
$$