Step-1

Consider the matrix,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

The objective is to find the singular value decomposition (SVD) and the pseudoinverse $V\Sigma^{+}U^{T}$.

The SVD of an $m \times n$ matrix A is $A = U\Sigma V^{T}$.

Here, the matrix U consists the unit eigenvectors of the matrix AA^{T} as columns, the matrix V consists the unit eigenvectors of the matrix $A^{T}A$ and the matrix Σ consists the square roots of the nonzero eigenvalues of the matrices AA^{T} and $A^{T}A$ on its main diagonal.

Step-2

The transpose of the matrix A is,

$$A^{T} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the product AA^{T}.

$$\begin{aligned} AA^{T} &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \left[1(1) + 1(1) + 1(1) + 1(1)\right] \\ &= \left[1+1+1+1\right] \\ &= \left[4\right] \end{aligned}$$

Step-3

Now find the product $A^{T}A$.

$$A^T A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1) & 1(1) & 1(1) & 1(1) \\ 1(1) & 1(1) & 1(1) & 1(1) \\ 1(1) & 1(1) & 1(1) & 1(1) \\ 1(1) & 1(1) & 1(1) & 1(1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step-4

Now find the eigenvalues and eigenvectors of the matrix AA^T.

The characteristic equation of the matrix AA^T is,

$$\begin{aligned} \det\left(AA^T - \lambda I\right) &= 0 \\ |4-\lambda| &= 0 \\ (4-\lambda) &= 0 \\ \lambda &= 4 \end{aligned}$$

Therefore, the eigenvalue of the matrix AA^T is $\lambda = 4$.

Step-5

Find the eigenvectors of the matrix AA^T corresponding to the eigenvalue $\lambda = 4$.

The eigenvector **x** to the matrix A is the solution space of the system $(A-\lambda I)\mathbf{x} = \mathbf{0}$.

For $\lambda = 4$, the system $(A-\lambda I)\mathbf{x} = \mathbf{0}$ becomes,

$$\begin{aligned} \left(AA^T - \lambda I\right)[x] &= 0 \\ [4-4][x] &= 0 \\ [0][x] &= 0 \end{aligned}$$

Here, x is a free variable.

So choose $x = t$, where t is any parameter.

Thus, the vector **x** can be written as,

$$\begin{aligned}\mathbf{x} &= [t] \\ &= t[1]\end{aligned}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 4$ is $[1]$.

Now find the eigenvalues and eigenvectors of the matrix $A^T A$.

The characteristic equation of the matrix $A^T A$ is,

$$\det\left(AA^T - \lambda I\right) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^4 - 4\lambda^3 = 0$$

$$\begin{aligned}\lambda^3(\lambda - 4) &= 0 \\ \lambda^3 = 0 \text{ and } \lambda - 4 &= 0 \\ \lambda = 0,0,0 \text{ and } \lambda &= 4\end{aligned}$$

Thus, the eigenvalues of $A^T A$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ with multiplicity 3.

Step-6

Find the eigenvectors of the matrix $A^T A$ corresponding to the eigenvalue $\lambda = 4$.

The eigenvector **x** to the matrix A is the solution space of the system $\left(A^T A - \lambda I\right)\mathbf{x} = \mathbf{0}$.

For $\lambda = 4$, the system $\left(A^T A - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(A^T A - 4I\right)\mathbf{x} = 0$$

$$\begin{bmatrix} 1-4 & 1 & 1 & 1 \\ 1 & 1-4 & 1 & 1 \\ 1 & 1 & 1-4 & 1 \\ 1 & 1 & 1 & 1-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-7

The reduced row echelon form of the matrix $\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, the system $\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From this, the following equations are obtained:

$x_1 - x_4 = 0, x_2 - x_4 = 0$ and $x_3 - x_4 = 0$.

Step-8

Here, x_4 is a free variable.

So choose $x_4 = t$, where t is any parameter.

Then $x_1 = t, x_2 = t, x_3 = t$.

Thus, the vector **x** can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 0$, the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(A^T A - 0I\right)\mathbf{x} = 0$$

$$\begin{bmatrix} 1-0 & 1 & 1 & 1 \\ 1 & 1-0 & 1 & 1 \\ 1 & 1 & 1-0 & 1 \\ 1 & 1 & 1 & 1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-9

The reduced row echelon form of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, the system $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

From this, the following equations are obtained:

$x_1 + x_2 + x_3 + x_4 = 0.$

Step-10

Here, x_2, x_3, x_4 are free variables.

So choose $x_2 = s, x_3 = t, x_4 = u$, where s, *t*, *u* are any parameters.

Then $x_1 = -s - t - u.$

Thus, the vector **x** can be written as,

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -s-t-u \\ s \\ t \\ u \end{bmatrix} \\ &= s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Hence, the eigenvectors corresponding to the eigenvalue $\lambda = 0$ is $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

Step-11

Find the unit vectors corresponding to the eigenvectors of the matrix $A^T A.$

The unit eigenvectors are,

$$\mathbf{v}_1 = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

Similarly, the remaining unit eigenvectors are obtained as,

$$\mathbf{v}_2 = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}-1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}.$$

The matrix U can be written as,

$U = [1]$.

The matrix V can be written as,

$$V = \begin{bmatrix}1/2 & -1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2}\\1/2 & 1/\sqrt{2} & 0 & 0\\1/2 & 0 & 1/\sqrt{2} & 0\\1/2 & 0 & 0 & 1/\sqrt{2}\end{bmatrix}.$$

Then $V^T = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$

And the matrix Σ can be written as,

$$\begin{aligned}\Sigma &= \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

Step-12

The SVD of the matrix A is,

$$\begin{aligned}A &= U\Sigma V^T \\ &= [1]\begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}\end{aligned}$$

Hence, the singular value decomposition of the matrix A is

$$\boxed{A = [1]\begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}}.$$

Step-13

Now find the pseudoinverse of the matrix A.

The pseudoinverse of the matrix A is $A^+ = V\Sigma^+U^T$.

The transpose of the matrix U is $U^T = [1]$.

The matrix Σ^+ can be written as,

$$\Sigma^{+}=\begin{bmatrix}1/2\\0\\0\\0\end{bmatrix}.$$

Thus, the pseudoinverse of the matrix A is,

$$\begin{aligned}A^{+}&=V\Sigma^{+}U^{T}\\&=\begin{bmatrix}1/2&-1/\sqrt{2}&-1/\sqrt{2}&-1/\sqrt{2}\\1/2&1/\sqrt{2}&0&0\\1/2&0&1/\sqrt{2}&0\\1/2&0&0&1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/2\\0\\0\\0\end{bmatrix}[1]\\&=\begin{bmatrix}1/4\\1/4\\1/4\\1/4\end{bmatrix}\end{aligned}$$

Hence, the pseudoinverse of the matrix A is $\boxed{A^{+}=\begin{bmatrix}1/4\\1/4\\1/4\\1/4\end{bmatrix}}$.

Step-14

Consider the matrix,

$$B=\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}.$$

The objective is to find the singular value decomposition (SVD) and the pseudoinverse of the matrix B.

The SVD of an $m\times n$ matrix B is $B=U_1\Sigma_1V_1^{T}$.

Here, the matrix U_1 consists the unit eigenvectors of the matrix BB^{T} as columns, the matrix V_1 consists the unit eigenvectors of the matrix $B^{T}B$ and the matrix Σ_1 consists the square roots of the nonzero eigenvalues of the matrices BB^{T} and $B^{T}B$. on its main diagonal.

Step-15

The transpose of the matrix B is,

$$B^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find the product BB^T.

$$\begin{aligned} BB^T &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0(0)+1(1)+0(0) & 0(1)+1(0)+0(0) \\ 1(0)+0(1)+0(0) & 1(1)+0(0)+0(0) \end{bmatrix} \\ &= \begin{bmatrix} 0+1+0 & 0+0+0 \\ 0+0+0 & 1+0+0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Step-16

Now find the product $B^T B$.

$$\begin{aligned} B^T B &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0(0)+1(1) & 0(1)+1(0) & 0(0)+1(0) \\ 1(0)+0(1) & 1(1)+0(0) & 1(0)+0(0) \\ 0(0)+0(1) & 0(1)+0(0) & 0(0)+0(0) \end{bmatrix} \\ &= \begin{bmatrix} 0+1 & 0+0 & 0+0 \\ 0+0 & 1+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Step-17

Now find the eigenvalues and eigenvectors of the matrix BB^T.

The characteristic equation of the matrix BB^T is,

$$\det\left(BB^T - \lambda I\right) = 0$$
$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)^2 = 0$$
$$1-\lambda = 0$$
$$\lambda = 1$$

Therefore, the eigenvalue of the matrix BB^T is $\lambda = 1$.

Step-18

Find the eigenvectors of the matrix BB^T corresponding to the eigenvalue $\lambda = 1$.

The eigenvector $\mathbf{x}$ to the matrix A is the solution space of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

For $\lambda = 1$, the system $\left(BB^T - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(BB^T - 1I\right)\mathbf{x} = 0$$
$$\begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Step-19

Here, x_1 and x_2 are free variables.

So choose $x_1 = s$ and $x_2 = t$, where *s, t* are any parameters.

Thus, the vector $\mathbf{x}$ can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} s \\ t \end{bmatrix}$$
$$= s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, the eigenvectors corresponding to the eigenvalue $\lambda = 1$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Therefore, the unit eigenvectors of the matrix BB^T are,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, the matrix $U_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Step-20

Now find the eigenvalues and eigenvectors of the matrix $B^T B$.

The characteristic equation of the matrix $B^T B$ is,

$$\begin{aligned}
\det\left(B^T B - \lambda I\right) &= 0 \\
\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(1-\lambda)(-\lambda) &= 0 \\
(1-\lambda)^2 = 0 \text{ or } \lambda &= 0 \\
\lambda = 1 \text{ or } \lambda &= 0
\end{aligned}$$

Therefore, the eigenvalue of the matrix $B^T B$ is $\lambda_1 = 0$ and $\lambda_2 = 1$.

Step-21

Find the eigenvectors of the matrix $B^T B$ corresponding to the eigenvalues

$\lambda_1 = 0$ and $\lambda_2 = 1$.

The eigenvector $\mathbf{x}$ to the matrix A is the solution space of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

For $\lambda = 0$, the system $\left(B^T B - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(BB^T - 0I\right)\mathbf{x} = 0$$

$$\begin{bmatrix} 1-0 & 0 & 0 \\ 0 & 1-0 & 0 \\ 0 & 0 & 0-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Step-22

From this the following equations are obtained:

$x_1 = 0$, $x_2 = 0$

Here, x_3 is a free variable.

So choose $x_3 = t$, where t is any parameter.

Thus, the vector **x** can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 0$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Step-23

For $\lambda = 1$, the system $\left(B^T B - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(BB^T - 1I\right)\mathbf{x} = 0$$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & 0-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Step-24

From this the following equations are obtained:

$x_3 = 0$

Here, x_1 and x_2 are free variables.

So choose $x_1 = s$ and $x_2 = t$, where *s, t* are any parameters.

Thus, the vector **x** can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Step-25

Hence, the eigenvectors corresponding to the eigenvalue $\lambda = 1$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Therefore, the unit eigenvectors of the matrix $B^T B$ are,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the matrix $V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

The transpose of the matrix V_1 is ,

$$V_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step-26

The singular values of the matrix *B* are the square roots of the nonzero eigenvalues.

Thus,

$$\begin{aligned} \sigma_1 &= \sqrt{1} \\ &= 1 \\ \sigma_2 &= \sqrt{1} \\ &= 1 \end{aligned}$$

Now write the matrix Σ_1.

$$\begin{aligned} \Sigma_1 &= \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Substitute the known matrices in $B = U_1 \Sigma_1 V_1^T$.

$$\begin{aligned} B &= U_1 \Sigma_1 V_1^T \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Hence, the SVD of the matrix B is $\boxed{B=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}$.

Step-27

Now find the pseudoinverse of the matrix B.

The pseudoinverse of the matrix B is $B^{+}=V_1\Sigma_1{}^{+}U_1{}^{T}$.

The transpose of the matrix U_1 is $U_1{}^{T}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

The matrix Σ^{+} can be written as,

$$\Sigma^{+}=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}.$$

Thus, the pseudoinverse of the matrix B is,

$$\begin{aligned}B^{+}&=V_1\Sigma_1{}^{+}U_1{}^{T}.\\&=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\\&=\begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}\end{aligned}$$

Hence, the pseudoinverse of the matrix B is $\boxed{B^{+}=\begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}}$.

Step-28

Consider the matrix,

$$C=\begin{bmatrix}1&1\\0&0\end{bmatrix}.$$

The objective is to find the singular value decomposition (SVD) and the pseudoinverse of the matrix C.

The SVD of an $m \times n$ matrix C is $C = U_2 \Sigma_2 V_2^T$.

Here, the matrix U_2 consists the unit eigenvectors of the matrix CC^T as columns, the matrix V_2 consists the unit eigenvectors of the matrix $C^T C$ and the matrix Σ_2 consists the square roots of the nonzero eigenvalues of the matrices CC^T and $C^T C$ on its main diagonal.

The transpose of the matrix C is,

$$C^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Find the product CC^T.

$$\begin{aligned} CC^T &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+1(1) & 1(0)+1(0) \\ 0(1)+0(1) & 0(0)+0(0) \end{bmatrix} \\ &= \begin{bmatrix} 1+1 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Step-29

Now find the product $C^T C$.

$$\begin{aligned} C^T C &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+0(0) & 1(1)+0(0) \\ 1(1)+0(0) & 1(1)+0(0) \end{bmatrix} \\ &= \begin{bmatrix} 1+0 & 1+0 \\ 1+0 & 1+0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Step-30

Now find the eigenvalues and eigenvectors of the matrix CC^T.

The characteristic equation of the matrix CC^T is,

$$\det\left(CC^T - \lambda I\right) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-\lambda) = 0$$

$$2-\lambda = 0 \quad \text{or} \quad \lambda = 0$$

$$\lambda = 2 \quad \text{or} \quad \lambda = 0$$

Therefore, the eigenvalue of the matrix CC^T is $\lambda_1 = 0$ and $\lambda_2 = 2$.

Step-31

Find the eigenvectors of the matrix CC^T corresponding to the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$.

The eigenvector **x** to the matrix A is the solution space of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

For $\lambda = 0$, the system $\left(CC^T - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(CC^T - 0I\right)\mathbf{x} = 0$$

$$\begin{bmatrix} 2-0 & 0 \\ 0 & 0-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Step-32

From this, the obtained equation is $2x_1 = 0$.

Here, x_2 is a free variable and $x_1 = 0$.

So choose $x_2 = t$, where t is any parameter.

Thus, the vector **x** can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 0$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Step-33

For $\lambda = 2$, the system $\left(CC^T - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(CC^T - 1I\right)\mathbf{x} = 0$$
$$\begin{bmatrix} 2-2 & 0 \\ 0 & 0-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

From this, the obtained equation is $-2x_2 = 0$.

Here, x_1 is a free variable and $x_2 = 0$.

So choose $x_1 = t$, where *t* is any parameter.

Thus, the vector **x** can be written as,

Step-34

$$\begin{aligned}\mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} t \\ 0 \end{bmatrix} \\ &= t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 2$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Therefore, the unit eigenvectors of the matrix CC^T are,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, the matrix $U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Step-35

Now find the eigenvalues and eigenvectors of the matrix $C^T C$.

The characteristic equation of the matrix $C^T C$ is,

$$\det\left(C^T C - \lambda I\right) = 0$$
$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 2\lambda = 0$$
$$\lambda(\lambda - 2) = 0$$
$$\lambda = 0 \quad \text{or } \lambda - 2 = 0$$
$$\lambda = 0 \quad \text{or } \lambda = 2$$

Therefore, the eigenvalue of the matrix $C^T C$ is $\lambda_1 = 0$ and $\lambda_2 = 2$.

Step-36

Find the eigenvectors of the matrix $C^T C$ corresponding to the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$.

The eigenvector **x** to the matrix A is the solution space of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

For $\lambda = 0$, the system $\left(C^T C - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\left(C^T C - 0I\right)\mathbf{x} = 0$$
$$\begin{bmatrix} 1-0 & 1 \\ 1 & 1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \left(\begin{array}{l} \text{Reduced row echelon form of } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ \text{Therefore, the systems are equivalent.} \end{array}\right)$$

Step-37

From this, the obtained equation is $x_1 + x_2 = 0$.

Here, x_2 is a free variable.

So choose $x_2 = t$, where t is any parameter.

Then $x_1 = -t.$

Thus, the vector **x** can be written as,

$$\begin{aligned}\mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -t \\ t \end{bmatrix} \\ &= t\begin{bmatrix} -1 \\ 1 \end{bmatrix}\end{aligned}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 0$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Step-38

For $\lambda = 2$, the system $\left(C^T C - \lambda I\right)\mathbf{x} = \mathbf{0}$ becomes,

$$\begin{aligned}\left(C^T C - 0I\right)\mathbf{x} &= 0 \\ \begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \qquad \left(\begin{array}{l}\text{Reduced row echelon form of } \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \\ \text{Therefore, the systems are equivalent.}\end{array}\right)\end{aligned}$$

From this, the obtained equation is $x_1 - x_2 = 0.$

Step-39

Here, x_2 is a free variable.

So choose $x_2 = t$, where t is any parameter.

Then $x_1 = t.$

Thus, the vector **x** can be written as,

$$\begin{aligned}\mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} t \\ t \end{bmatrix} \\ &= t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned}$$

Hence, the eigenvector corresponding to the eigenvalue $\lambda = 2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-40

The unit eigenvectors of the matrix $C^T C$ are,

$$\begin{aligned}\mathbf{v}_1 &= \frac{1}{\sqrt{(-1)^2 + (1)^2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_2 &= \frac{1}{\sqrt{(1)^2 + (1)^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\end{aligned}$$

Step-41

The matrix V_2 can be written as,

$$V_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Thus, the transpose of the matrix V_2 is,

$$V_2^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

The singular values of the matrix C are the square roots of the nonzero eigenvalues.

Thus, $\sigma = \sqrt{2}$.

Now write the matrix Σ_2.

$$\begin{aligned}\Sigma_2 &= \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Substitute the known matrices in $C = U_2 \Sigma_2 V_2^T$.

$$\begin{aligned}C &= U_2 \Sigma_2 V_2^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\end{aligned}$$

Hence, the SVD of the matrix C is $\boxed{C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$.

Step-42

Now find the pseudoinverse of the matrix C.

The pseudoinverse of the matrix C is $C^+ = V_2 \Sigma_2^+ U_2^T$.

The transpose of the matrix U_2 is $U_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The matrix Σ^+ can be written as $\Sigma^+ = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}$

Thus, the pseudoinverse of the matrix A is,

$$
\begin{aligned}
C^{+} &= V_2 \Sigma_2^{\ +} U_2^{\ T}. \\
&= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}
\end{aligned}
$$

Hence, the pseudoinverse of the matrix C is $\boxed{C^{+} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}}$.