Solution for Assignment 08

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PROBLEM 1. Suppose that the continuous random variable X has p.d.f:

$$f_X(x) = kx(1-x), 0 < x < 1; \;\; f_X(x) = 0, otherwise.$$

Evaluate the constant k, and then find the non-zero range of Y and the p.d.f $f_Y(y)$ of Y when

(a) $Y = -3X + 3$;

(b) $Y = \frac{1}{X}$.

SOLUTION. $k=6$

(a) Let $g(x) = -3x + 3$, g is a strictly monotone function and $g((0,1)) = (0,3)$.So

$$\begin{aligned} f_Y(y) =& f_X[g^{-1}(y)] * |\frac{d}{dy} g^{-1}(y)| \\ =& f_X(-\frac{y-3}{3}) * \frac{1}{3} \\ =& k * \frac{-y^2 + 3y}{27} \end{aligned}$$

for any $y \in (0,3)$. And $F_Y(y) = 0$ for otherwise.

或者 $F_Y(y) = 1 - F_X(-\frac{y-3}{3})$

$f_Y(y) = \frac{dF_Y(y)}{dy}$

$= \frac{d}{dy}(1 - F_X(-\frac{y-3}{3})) = \frac{1}{3} f_X(-\frac{y-3}{3})$

(b) Let $g(x) = \frac{1}{X}$, g is a strictly monotone function and $g((0,1)) = (1,\infty)$.So

$$\begin{aligned} f_Y(y) =& f_X[g^{-1}(y)] * |\frac{d}{dy}g^{-1}(y)| \\ =& f_X(\frac{1}{y}) * \frac{1}{y^2} \\ =& k * (\frac{1}{y^3} - \frac{1}{y^4}) \end{aligned}$$

for any $y \in (1,\infty)$. And $F_Y(y) = 0$ for otherwise.

PROBLEM 2. Suppose that the random variable X has c.d.f.

$$\begin{aligned} F_X(x) =& 0, & x < 0, \\ F_X(x) =& \frac{1 - \cos x}{2}, & 0 \le x \le \pi, \\ F_X(x) =& 1, & x > \pi. \end{aligned}$$

and that $Y = \sqrt{X}$. What is the non-zero range of Y ? Find the c.d.f.$F_Y(y)$ of Y, and hence find the p.d.f of Y .

SOLUTION. For $x < 0$, $F(x) = 0$;

For $0 \le x < \sqrt{\pi}$,

$$\begin{aligned} F(x) =& P(Y \le x) \\ =& P(\sqrt{X} \le x) \\ =& P(X \le x^2) \\ =& F_X(x^2) \\ =& \frac{1 - \cos x^2}{2}; \end{aligned}$$

For $x \ge \sqrt{\pi}, F(x) = 1$.

Derivate the equation for $0 \le x < \sqrt{\pi}$, we get

$$p_Y(x) = \frac{d}{dx}\frac{1 - \cos x^2}{2} = x \sin x^2.$$

PROBLEM 3. Suppose that the two random variables X and Y have joint probability c.d.f. $F(x,y)$. Show that $F(x,y)$ possesses the following properties:

(a) For any fixed x, $F(x,y)$ is a non-decreasing function of y and, similarly, for any fixed y, $F(x,y)$ is a non-decreasing function of x.

(b) $F(x,y) \to 1$ when both $x \to +\infty$ and $y \to +\infty$.

(c) $F(x,y) \to 0$ when either $x \to -\infty$ or $y \to -\infty$.

(d) If $x_1 < x_2$ and $y_1 < y_2$, then

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2,y_2) - F(x_2,y_1) - F(x_1,y_2) + F(x_1,y_1).$$

SOLUTION.

(a) For any fixed x, and $y_1 < y_2$

$$F(x,y_2) = P(X \le x, Y \le y_2) = P(X \le x, Y \le y_1) + P(X \le x, y_1 < Y \le y_2) \ge F(x,y_1).$$

So $F(x,y)$ is a non-decreasing function of y, and by same argument, we can get another side.

(b) By both $x \to +\infty$ and $y \to +\infty$, it equals to prove the limit equation by choosing any sequences $\{(x_n,y_n)\}_{n\ge 1}$,s.t. x_n and y_n both increasing to ∞, and

$$\lim_{n\to\infty} F(x_n,y_n) = 1$$

To show the equation, we first fix a x_m, and by the monotone property, we have

$$F(x_n,y_n) \ge F(x_m,y_n) = P(X \le x_m, Y \le y_n) \to P(X \le x_m).$$

So $\lim_{n\to\infty} F(x_n,y_n) \ge P(X \le x_m)$ for any $m \ge 1$. Let $m \to \infty$, we get

$$\lim_{n\to\infty} F(x_n,y_n) \ge \lim_{n\to\infty} P(X \le x_m) = 1.$$

The other side is completely similar.

(c) Without lose of generality, we suppose $x \to -\infty$.And $\{x_n\}_{n\geq 1}$ is asequence decreasing to $-\infty$,then

$$\lim_{n\to\infty} F(x_n, y) = \lim_{n\to\infty} P(X \leq x_n, Y \leq y) = P(X \leq \lim_{n\to\infty} x_n, Y \leq y) = 0$$

(d) If $x_1 < x_2$ and $y_1 < y_2$, $A = x_1 < X \leq x_2, y_1 < Y \leq y_2$,$B_1 = X \leq x_2, Y \leq y_2$,$B_2 = X \leq x_1, Y \leq y_2$,$B_3 = X \leq x_2, Y \leq y_1$,$B_4 = B_2 \cap B_3 = X \leq x_1, Y \leq y_1$,apply principle of inclusion-exclusion with $B_2 \cup B_3$, then

$$\begin{aligned}
P(x_1 < X \leq x_2, y_1 < Y \leq y_2) =& P(A) \\
=& P(B_1) - P(B_2 \cup B_3) \\
=& P(B_1) - P(B_2) - P(B_3) + P(B_4) \\
=& F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1).
\end{aligned}$$

PROBLEM 4. Suppose that the two discrete random variables X and Y have joint p.m.f. given by

	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$
$X = 1$	2/32	3/32	4/32	5/32
$X = 2$	3/32	4/32	5/32	6/32

Obtain the marginal p.m.f. of X.

SOLUTION.

$$p_X(1) = \sum_{k=1}^{4} P(X = 1, Y = k) = 14/32 = \frac{7}{16};$$

$$p_X(2) = 1 - p_X(1) = \frac{9}{16};$$

and $p_X(x) = 0$ for otherwise.