

Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Step-2

Find the Eigen values and Eigen vectors and write the solution in the form of $Se^{\Lambda t}S^{-1}$. Then find e^{At} from $Se^{\Lambda t}S^{-1}$.

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A . To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(-\lambda) = 0$$

$$\lambda^2 - \lambda = 0$$

After solving following values are obtained:

$$\lambda_1 = 1$$

$$\lambda_2 = 0$$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = 1$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix} \\ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0 \\ \begin{bmatrix} 1-0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix} \\ = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore Eigen values are:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-6

Recall that: $A = SAS^{-1}$. Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Step-7

Recall that $e^{At} = Se^{\Lambda t}S^{-1}$. Therefore,

$$\begin{aligned}
 e^{At} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} e^t & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

Step-8

Therefore,

$$e^{At} = \boxed{\begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}}.$$