Step-1

A survey gives, every month half of the people are become sick, and one fourth of the sick people are become dead.

The objective is to find the steady state for corresponding Markov process,

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$$

Step-2

Stability status: Stability depends on the eigenvalues. The difference equation $u_{k+1} = Au_k$ will have following stability status corresponding to the eigenvalues:

Stable: If all of the eigenvalues are, $|\lambda_i| < 1$.

Neutrally stable: If some $|\lambda_i| = 1$ and other $|\lambda_i| < 1$.

Unstable: If at least one eigenvalue is, $|\lambda_i| > 1$.

Step-3

The steady state for the following equation of matrices is,

$u_{k+1} = Au_k$.

Here, matrices u_{k+1}, A, u_k are defined as follows:

$$A = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix}$$

$$u_k = \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$$

Step-4

Matrix A is a triangular matrix, so main diagonal gives the eigenvalues. Thus, eigenvalues are as follows:

$$\lambda_1 = 1$$

$$\lambda_2 = \frac{3}{4}$$

$$\lambda_3 = \frac{1}{2}$$

From the above values see that $\lambda_1 = 1$ and all other Eigen values are $|\lambda_j| < 1$.

This shows that difference equation is neutrally stable,

Therefore, $u_{k+1} = Au_k$ is neutrally stable

Step-5

For $\lambda = 1$:

Consider the matrix equation, $(A - \lambda I)X = 0$.

$$\begin{bmatrix} 1-1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4}-1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2}-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding equations are,

$$\begin{aligned} \frac{1}{4}x_2 &= 0, \\ -\frac{1}{4}x_2 + \frac{1}{2}x_3 &= 0, \\ -\frac{1}{2}x_3 &= 0. \end{aligned}$$

Solving,

$$\begin{aligned} x_2 &= 0, \\ x_3 &= 0. \end{aligned}$$

Put, $x_1 = k_1$.

Step-6

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = 1$ is, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Step-7

For $\lambda = \frac{3}{4}$:

Consider the matrix equation, $(A - \lambda I)X = 0.$

$$\begin{bmatrix} 1-\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{3}{4}-\frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2}-\frac{3}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding equations are,

$$\begin{aligned} \frac{1}{4}x_1 + \frac{1}{4}x_2 &= 0, \\ \frac{1}{2}x_3 &= 0, \\ -\frac{1}{4}x_3 &= 0. \end{aligned}$$

Solving,

$x_3 = 0.$

Put, $x_2 = k_2$,

$x_1 = -k_2.$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = \frac{3}{4}$ is, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$

Step-8

For $\lambda = \frac{1}{2}$:

Consider the matrix equation, $(A - \lambda I)X = 0.$

$$\begin{bmatrix} 1-\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{3}{4}-\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2}-\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding equations are,

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 = 0,$$
$$\frac{1}{4}x_2 + \frac{1}{2}x_3 = 0.$$

Put, $x_3 = k_3,$

$$x_2 = -2k_3,$$
$$x_1 = k_3.$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = \frac{1}{2}$ is, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

Modal matrix is,

$$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Step-9

The inverse of the matrix S is,

$$S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A can be expressed as,

$$\begin{aligned} A &= S\Lambda S^{-1} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Step-10

To find A^k,

$$\begin{aligned} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} &= A^k \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 & 0 \\ 0 & \left(\frac{3}{4}\right)^k & 0 \\ 0 & 0 & \left(\frac{1}{2}\right)^k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \end{aligned}$$

The steady state is the eigenvector of matrix A corresponding to $\lambda = 1$.

That is,

$$\begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Or

$$Au_\infty = u_\infty$$

Since, u_∞ unchanged for eigenvector of matrix A corresponding to $\lambda = 1$.

Hence, steady state is,

$$\boxed{\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix} = \left(x_0 + y_0 + z_0\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}.$$