Step-1

The matrix corresponding to the quadratic $ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_1x_3 + 2fx_2x_3$ is

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}.$$

This is the conversion of quadratic form into matrix.

Step-2

(a)

Given that $f_1 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

The symmetric matrix corresponding to it is;

$$A_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Step-3

Now the matrix A_2 corresponding to $f_2 = x_1^2 + 2x_2^2 + 11x_3^2 - 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ is;

$$A_2 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 11 \end{pmatrix}.$$

Therefore,

$$\boxed{\begin{array}{l} A_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \text{and} \\ A_2 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 11 \end{pmatrix} \end{array}}.$$

Step-4

(b).

Given that $f_1 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

$= (x_1 - x_2 - x_3)^2$

Therefore, $\boxed{f_1 \text{is a perfect square}}$

For

$$\begin{aligned} x_1 &= 2 \\ x_2 &= x_3 \\ &= 1, \end{aligned}$$
$$f_1(x_1, x_2, x_3) = 0$$

Therefore, $\boxed{f_1 \text{is not positive definite}}$

Therefore,

$$\begin{aligned} f_1 &= (x_1 - x_2 - x_3)^2 \\ &= 0 \end{aligned}$$

$\boxed{\text{when } x_1 - x_2 - x_3 = 0}$.

Step-5

(c).

Now,

$$\begin{aligned} f_2 &= x_1^2 + 2x_2^2 + 11x_3^2 - 2x_1x_2 - 2x_1x_3 - 4x_2x_3 \\ &= x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 - 2x_1x_3 + x_2^2 + 10x_3^2 - 6x_2x_3 \\ &= (x_1 - x_2 - x_3)^2 + x_2^2 + 9x_3^2 - 6x_2x_3 + x_3^2 \\ &= (x_1 - x_2 - x_3)^2 + (x_2 - 3x_3)^2 + x_3^2. \end{aligned}$$

Now, write L as;

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

So that,

$$A=\begin{pmatrix}1&0&0\\-1&1&0\\-1&-3&1\end{pmatrix}\begin{pmatrix}1&-1&-1\\0&1&-3\\0&0&1\end{pmatrix}$$
$$=LL^T.$$

Therefore, $\boxed{f_2=\left(x_1-x_2-x_3\right)^2+\left(x_1-3x_3\right)^2+x_3^2;L=\begin{pmatrix}1&0&0\\-1&1&0\\-1&-3&1\end{pmatrix}}$.