1 Matrices and Gaussian Elimination

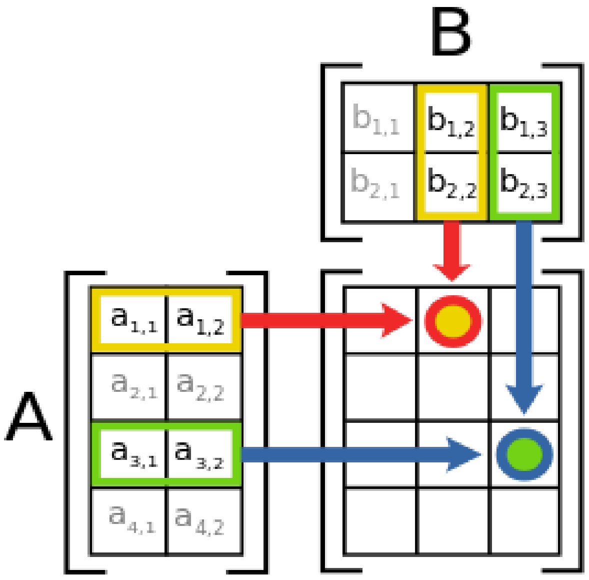
1.7

TRIANGULAR FACTORS AND ROW EXCHANGES

（矩阵的三角分解和换行）

LU Factorization

Row Exchanges



*** Textbook: Section 1.5 + Section 1.6 (part)**

I. Triangular Factors (矩阵的LU分解)

Example 1 将矩阵

$$A=\begin{bmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{bmatrix}$$

分解成为主对角元为1的下三角矩阵L和上三角矩阵U的乘积，即 $A=LU$（称为矩阵的LU分解 or *Triangular factorization* $A = LU$）.

解 利用倍加初等变换(Replacement)把 A 变为上三角矩阵:

$$E_{12}(-\frac{1}{2})A=\begin{bmatrix}2&1&0&0\\0&\frac{3}{2}&1&0\\0&1&2&1\\0&0&1&2\end{bmatrix},\qquad E_{23}(-\frac{2}{3})E_{12}(-\frac{1}{2})A=\begin{bmatrix}2&1&0&0\\0&\frac{3}{2}&1&0\\0&0&\frac{4}{3}&1\\0&0&1&2\end{bmatrix},$$

$$E_{34}(-\frac{3}{4})E_{23}(-\frac{2}{3})E_{12}(-\frac{1}{2})A=\boxed{\begin{bmatrix}2&1&0&0\\0&\frac{3}{2}&1&0\\0&0&\frac{4}{3}&1\\0&0&0&\frac{5}{4}\end{bmatrix}}=U$$

$$A=E_{12}^{-1}(-\frac{1}{2})E_{23}^{-1}(-\frac{2}{3})E_{34}^{-1}(-\frac{3}{4})U=E_{12}(\frac{1}{2})E_{23}(\frac{2}{3})E_{34}(\frac{3}{4})U=LU$$

其中 $L=E_{12}(\frac{1}{2})E_{23}(\frac{2}{3})E_{34}(\frac{3}{4}).$

$$L=\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\frac{2}{3}&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{3}{4}&1\end{bmatrix}=\boxed{\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\0&\frac{2}{3}&1&0\\0&0&\frac{3}{4}&1\end{bmatrix}}$$

Remark: In Example 1, $\boldsymbol{A}$ ($n \times n$ matrix) is written in the form $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is an $n \times n$ lower triangular matrix with 1's on the diagonal and $\boldsymbol{U}$ is an $n \times n$ upper triangular matrix.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

Divide out of U a diagonal pivot matrix D

$$= \underset{\boldsymbol{L}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}} \underset{\boldsymbol{D}}{\begin{bmatrix} \frac{2}{1} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}} \underset{\boldsymbol{V}}{\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$\boldsymbol{A}$ is symmetric:

$\boldsymbol{A}=\boldsymbol{LDL}^{\mathrm{T}}$.

习惯上仍记为$\boldsymbol{U}$

(此时为单位上三角矩阵)

The triangular factorization can be written $\boldsymbol{A} = \boldsymbol{LDU}$, *where* $\boldsymbol{L}$ *and* $\boldsymbol{U}$ *have* 1's *on the diagonal and* $\boldsymbol{D}$ *is the diagonal matrix of pivots.*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Remarks: 1. *The LDU factorization is* ***uniquely*** *determined by* $\boldsymbol{A}$ *if* $\boldsymbol{A}$ *is invertible.*
(Proof: P53, Problem Set1.6, #17)

2. Some matrices *cannot* be factored into $\boldsymbol{A} = \boldsymbol{LU}$ ***or*** $\boldsymbol{LDU}$.

For instance, $\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$.

练习 将矩阵

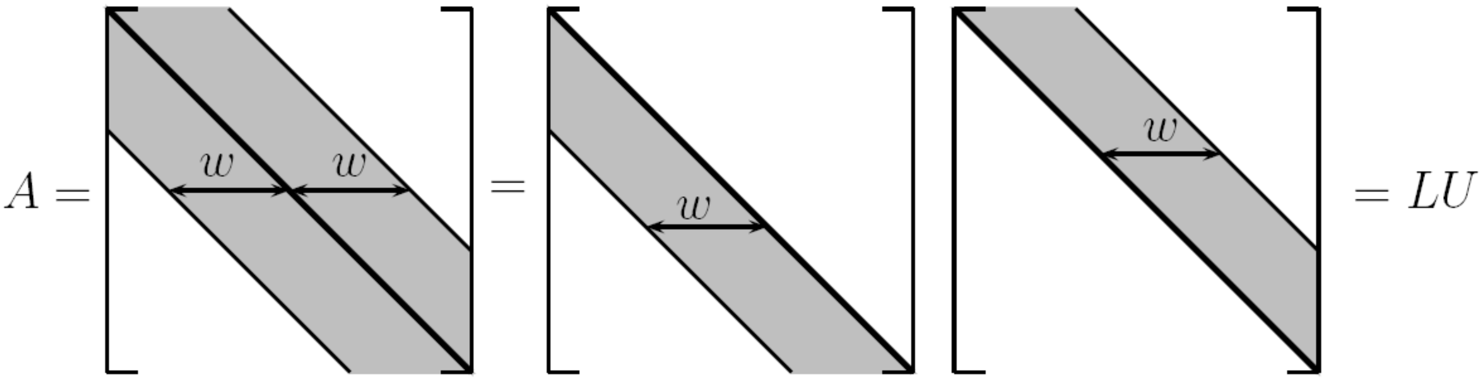
$$\boldsymbol{A}=\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 2\end{bmatrix}$$

分解成为主对角元为1的下三角矩阵$\boldsymbol{L}$ (invertible, unit lower triangular matrix)和上三角矩阵$\boldsymbol{U}$ (upper triangular matrix)的乘积, 即 $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{U}$.

解

$$A=\begin{bmatrix}1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1\\ & & -1 & 2\end{bmatrix}=\begin{bmatrix}1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1\\ & & & 1\end{bmatrix}.$$

Remark: band matrix (带状矩阵) (P61, Figure 1.8)



A band matrix ***A*** and its factors ***L*** and ***U***.

A band matrix ***A*** has $a_{ij} = 0$ except in the band $|i - j| < w$.

w : "half bandwidth"

$w = 1$: a diagonal matrix,

$w = 2$: a tridiagonal matrix (三对角矩阵), $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$w = n$: a full matrix.

Example 2 Solve $\boldsymbol{Ax} = \boldsymbol{b}$

$$\begin{aligned} x_1 - x_2 \quad\quad\quad\quad\quad &= 1 \\ -x_1 + 2x_2 - x_3 \quad\quad &= 1 \\ - x_2 + 2x_3 - x_4 &= 1 \\ - x_3 + 2x_4 &= 1 \end{aligned}$$

This is the previous matrix $\boldsymbol{A}$ with a right-hand side $\boldsymbol{b} = (1,1,1,1)^T$.

$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow$
$\boldsymbol{Ux} = \boldsymbol{c}$ & $\boldsymbol{Lc} = \boldsymbol{b}$

$\boldsymbol{Ax} = \boldsymbol{b}$ splits into $\boldsymbol{Lc} = \boldsymbol{b}$ and $\boldsymbol{Ux} = \boldsymbol{c}$

$Lc = b$

solved *forward*

$$\begin{aligned} c_1 \quad\quad\quad\quad &= 1 \\ -c_1 + c_2 \quad\quad &= 1 \\ - c_2 + c_3 \quad &= 1 \\ - c_3 + c_4 &= 1 \end{aligned} \quad \text{gives} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

$Ux = c$

solved *backward*

$$\begin{aligned} x_1 - x_2 \quad\quad\quad &= 1 \\ x_2 - x_3 \quad\quad &= 2 \\ x_3 - x_4 &= 3 \\ x_4 &= 4 \end{aligned} \quad \text{gives} \quad \boldsymbol{x} = \begin{bmatrix} 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}.$$

One Linear System = Two Triangular Systems

Splitting of $\boldsymbol{Ax = b}$

First $\boldsymbol{Lc = b}$ *and then* $\boldsymbol{Ux = c}$.

1. ***Factor*** (from $\boldsymbol{A}$ find its factors $\boldsymbol{L}$ and $\boldsymbol{U}$).
2. ***Solve*** (from $\boldsymbol{L}$ and $\boldsymbol{U}$ and $\boldsymbol{b}$ find the solution $\boldsymbol{x}$).

$$\boldsymbol{A} = \begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * \\ & \blacksquare & * & * \\ & & \blacksquare & * \\ & & & \blacksquare \end{bmatrix}$$

$$n \times n \qquad\qquad n \times n \qquad\qquad n \times n$$

What if – $\boldsymbol{A}$ is an $m \times n$ matrix ?

LU factorization

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

Notes: Assume that $\boldsymbol{A}$ is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.

Then $\boldsymbol{A}$ can be written in the form $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is an $m \times m$ lower triangular matrix with 1's on the diagonal and $\boldsymbol{U}$ is an $m \times n$ echelon form of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix},$$ which needs a row exchange,

cannot be factored into $\boldsymbol{A} = \boldsymbol{LU}$.

II. Row Exchanges and Permutation Matrices

$$\boldsymbol{A}=\begin{bmatrix}0 & 2\\ 3 & 4\end{bmatrix},$$ cannot be factored into $\boldsymbol{A}=\boldsymbol{LU}$.

Remedy: *Exchange the two rows*

$$\boldsymbol{P}_{12}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \qquad \boldsymbol{P}_{12}\boldsymbol{A}=\begin{bmatrix}3 & 4\\ 0 & 2\end{bmatrix}.$$

Permutation matrix (置换矩阵)

A permutation matrix has the same rows as the identity matrix but in some order.

There is a single “1” in every row and every column.

How many permutation matrices do we have for $n = 2$? $n = 3$? …

$n = 2$

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{P}_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$n = 3$

$$\boldsymbol{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad \boldsymbol{P}_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \qquad \boldsymbol{P}_{32}\boldsymbol{P}_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix},$$

$$\boldsymbol{P}_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \qquad \boldsymbol{P}_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \qquad \boldsymbol{P}_{21}\boldsymbol{P}_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There are $n! = n(n-1)\ldots(1)$ permutations of size n.

A zero in the pivot location raises two possibilities:
The trouble may be easy to fix, or it may be serious.

$$\boldsymbol{A} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix},$$

$$\boldsymbol{P}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{P}_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\boxed{\boldsymbol{P}_{23}\boldsymbol{P}_{13}}\boldsymbol{A} = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

$$\begin{aligned} d = 0 &\implies \text{no first pivot} \\ a = 0 &\implies \text{no second pivot} \\ c = 0 &\implies \text{no third pivot.} \end{aligned}$$

$$\boldsymbol{P} = \boldsymbol{P}_{23}\boldsymbol{P}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

With the rows in the right order $\boldsymbol{PA}$, any nonsingular matrix is ready for elimination.

$PA = LU$

Example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \underbrace{\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}}_{P}, PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$PA = \underbrace{\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}}_{L} \underbrace{\boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}}}_{U}$$

Elimination in a Nutshell: $PA = LU$

In the *nonsingular* case, there is a permutation matrix $\boldsymbol{P}$ that reorders the rows of $\boldsymbol{A}$ to avoid zeros in the pivot positions. Then $\boldsymbol{Ax} = \boldsymbol{b}$ has a *unique solution*.

With the rows reordered in advance, $\boldsymbol{PA}$ can be factored into $\boldsymbol{LU}$.

In the *singular* case, no $\boldsymbol{P}$ can produce a full set of pivots: elimination fails.

Remark:

In practice, we also consider a row exchange when the original pivot is *near* zero — even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error. (***partial pivoting***)

Partial pivoting (部分主元法) : Numerical note (P62)

Elimination with small pivot

$$\begin{cases} 0.0001u + v = 1 \\ u + v = 2 \end{cases} \longrightarrow \begin{cases} 0.0001u + v = 1 \\ -9999v = -9998 \end{cases}$$

Correct result $\begin{cases} 0.0001u + v = 1 \\ v = 0.9999 \end{cases} \longrightarrow u = 1$

Wrong result $\begin{cases} 0.0001u + v = 1 \\ v = 1 \end{cases} \longrightarrow u = 0$

The small pivot 0.0001 brought instability, and the remedy is clear – *exchange rows*.

Partial pivoting (部分主元法)

Exchange rows

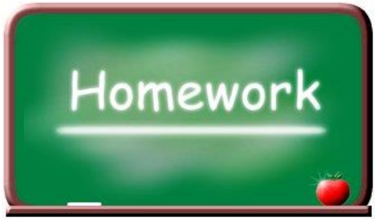
$$\begin{cases} 0.0001u + v = 1 \\ \quad\quad\quad u + v = 2 \end{cases} \xrightarrow{r_1 \longleftrightarrow r_2} \begin{cases} \quad\quad\quad u + v = 2 \\ 0.0001u + v = 1 \end{cases}$$

$$\begin{cases} u + v = 2 \\ 0.9999v = 0.9998 \end{cases} \longrightarrow v = 1, \quad u = 1.$$

Correct result

A small pivot forces a practical change in elimination. Normally we compare each pivot with all possible pivots in the same column. Exchanging rows to obtain the largest possible pivot (having the largest absolute value) is called **partial pivoting**.

Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

Deadline (DDL):

- Next tutorial class