南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目：线性代数精讲　　开课单位：数学系
考试时长：120 分钟　　命题教师：线性代数精讲教师团队

题 号	1	2	3	4	5	6	7
分 值	30 分	10 分	10 分	18 分	20 分	6 分	6 分

本试卷共（7）大题，满分（100）分.（考试结束后请将试卷、答题本、草稿纸一起交给监考老师）

This test includes 7 questions. Write *all your answers* on the examination book.

Please put away all books, calculators, cell phones and other devices. Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are over $\mathbb{F}$ and with finite dimensions, where $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$.

1. *(30 points, 6 points each)* Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) Let V be a finite-dimensional vector space over $\mathbb{F}$, U_1, U_2, W be subspaces of V. If $V=U_1\oplus W$, $V=U_2\oplus W$, then $U_1=U_2$.

   (b) Let $T\in\mathcal{L}(V)$ and v_1, v_2 be eigenvectors of T, then v_1+v_2 is an eigenvector of T.

   (c) Every finite-dimensional vector space has a basis.

   (d) Every linear functional is either surjective or the zero map.

   (e) Suppose v_1, v_2 are two vectors in V and U_1, U_2 are subspaces of V such that $v_1+U_1=v_2+U_2$. Then $U_1=U_2$.

2. (10 points) Supppose U_1, U_2, U_3 are subspaces of a finite-dimensional vector space V. Show that $U_1+U_2+U_3$ is the smallest subspace of V containing U_1, U_2, U_3.

3. (10 points) Show that -1, $\sin x$, $\cos^2 x$ is linearly independent in the space of real-valued functions, $\mathbb{R}^{\mathbb{R}}$.

4. (18 points) Let

$$S=\{A\in\mathbb{R}^{2\times 2}\mid A=A^T\}\text{ and }E=\left\{\begin{bmatrix}0&-1\\-1&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}.$$

   (1) Show that S is a subspace of $\mathbb{R}^{2\times 2}$.

   (2) Show that E is a basis of S.

(3) Recall that $\mathcal{P}_2(\mathbb{R})$ is the vector space of polynomials with real coefficients of degree less or equal to 2. Let $T: \mathcal{P}_2(\mathbb{R}) \to S$ be the linear map

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - 2a_1 - a_2 & a_0 + a_1 - a_2 \\ a_0 + a_1 - a_2 & a_0 + a_2 \end{bmatrix}.$$

$a_0v_1 + a_1v_2 + a$

Find the matrix representation of T with respect to the ordered basis

$$F = \{1,\ 1 - x,\ 1 + x^2\}$$

of $\mathcal{P}_2(\mathbb{R})$ and the ordered basis E of S.

5. (20 points) Define $T \in \mathcal{L}(\mathbb{R}^3)$ by

$$T(x, y, z) = (8x,\ 3x + 5y,\ y + 2z).$$

(a) Find all the eigenvalues of T.

(b) Find a basis of $\mathbb{R}^3$ with respect to which T has a diagonal matrix.

(c) Show that T is invertible.

(d) Let T^{-1} be the inverse of T, find $T^{-1}(1, 2, 3)$.

$(\lambda I - AB)x = 0$

6. (6 points) Suppose T is a linear map from $\mathbb{R}^{n,n}$ to $\mathbb{R}$ with

$$T(AB) = T(BA),\ \ A, B \in \mathbb{R}^{n,n}.$$

Show that there exists a real number λ such that $T(A) = \lambda\, tr(A)$ for all $A \in \mathbb{R}^{n,n}$. Where $\mathbb{R}^{n,n}$ denotes the vector space consisting of all $n \times n$ real matrices, and $tr(A)$ denotes the trace of matrix A.

7. (6 points) Suppose T is a linear map from V to W. Where V and W are finite-dimensional vector spaces over the same field $\mathbb{R}$. Let $w_1, w_2, \cdots, w_m$ be a basis of range T. Show that there exist linear functionals $g_1, g_2, \cdots, g_m$ defined on V such that

$$Tv = g_1(v)w_1 + g_2(v)w_2 + \cdots + g_m(v)w_m$$

$T(A) \cdot T(B)$

for all $v \in V$.

$T(AB)x = T(BA)x$

$T(\lambda x)$