Step-1

(i) Suppose a vector x is in $\mathbf{R}^n$

Then $x = x_r + x_n$ â€¦â€¦ (1)

We note that x_r is in the row space of A and x_n is in the null space of A

So, $Ax_n = 0$

Considering the product with A on either sides of (1), we get

$$\begin{aligned} Ax &= A(x_r + x_n) \\ &= Ax_r + Ax_n \\ &= Ax_r + 0 \\ &= Ax_r \end{aligned}$$

Therefore, $Ax = Ax_r$

Step-2

Further, the null space of A has the image 0 vector present in the intersection of the column space of A and left null space of A.

So, with respect to A, we see that Ax_r is assigned to the column space of A and Ax_n is assigned to the zero vector.

So, Ax_r is in the column space of A.

(ii) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

By the figure 3.4, we have $Ax = b$

$$\Rightarrow b = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Suppose $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then $Ax = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = Ax_r$

Step-3

Using row operations, we get $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Rewriting the system, we get $x_1 + x_2 = 1$ â€¦â€¦ (2)

So, $x_2 = 1 - x_1$ and the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ shows that $x_2 = x_1$

So, with these conditions (2) provides $x_2 = x_1 = \frac{1}{2}$

Thus, $x_r = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$