Step-1

We have to prove that if V and W are 3-dimensional subspaces of R⁵, then V and W must have a non zero vector in common.

Step-2

Let V and W are three dimensional subspace of R5

Let $\beta_1 = \{v_1, v_2, v_3\}$ is a basis for **V**

 $\beta_2 = \{w_1, w_2, w_3\}$ is a basis for **W**

Now $S = \{v_1, v_2, v_3, w_1, w_2, w_3\}$ is a subset of \mathbb{R}^5 .

Sine the dimension of R^5 is 5. The maximally linearly independent set have only 5 vectors.

Therefore the above set S is linearly dependent

 $\exists \text{ not all zero scalars } a_1, a_2, a_3, a_4, a_5, a_6 \text{ such that } a_1v_1 + a_2v_2 + a_3v_3 + a_4w_1 + a_5w_2 + a_6w_6 = 0$

$$\Rightarrow a_1 v_1 + a_2 v_2 + a_3 v_3 = -a_4 w_1 - a_5 w_2 - a_6 w_6$$

Step-3

Let $\alpha = a_1 v_1 + a_2 v_2 + a_3 v_3$

 $\therefore \alpha \in V \text{ (Since } \{v_1, v_2, v_3\} \text{ is a basis for } V \text{)}$

Then $\alpha = -a_4 w_1 - a_5 w_2 - a_6 w_6 \in W$ (since $\{w_1, w_2, w_3\}$ is a basis for W)

If possible $\alpha = 0$

$$\Rightarrow a_1v_1+a_2v_2+a_3v_3=0$$

$$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0$$

And hence, $-a_4w_1 - a_5w_2 - a_6w_6 = 0$

$$\Rightarrow a_4 = a_5 = a_6 = 0$$

Therefore $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$

This is a contradiction to $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ is linearly dependent.

Hence $\alpha \neq 0$

Therefore α is a non zero vector belongs to both W and V.