Step-1

Given the matrix $A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

In the given matrix, independent columns are 2 and 3. Its rank is 2(non-zero rows).

The basis of column space is $\{(3,0,1),(3,0,0)\}$ and its dimension is less than or equal to 2.

The basis of row space is $\{(0 \quad 3 \quad 3 \quad 3),(0 \quad 1 \quad 0 \quad 1)\}$ and its dimension is less than or equal to 2.

Step-2

In order to find the null space, set it in $AX = 0$. So,

$$\begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies;

$$\begin{aligned} 3x_2 + 3x_3 + 3x_4 &= 0 \\ x_2 + x_4 &= 0 \\ x_2 &= -x_4 \\ x_3 &= 0 \end{aligned}$$

So,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} x_1 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, the basis for null space is $\boxed{\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}}$ and its dimension is $\boxed{2}$.

Step-3

In order to find the left null space basis, the transpose of the matrix,

$$A^T=\begin{bmatrix}0&0&0\\3&0&1\\3&0&0\\3&0&1\end{bmatrix}$$

Letâ€™s write it in $A^TX=0$

$$\begin{bmatrix}0&0&0\\3&0&1\\3&0&0\\3&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This implies;

$$\begin{aligned}3x_1+3x_3&=0\\3x_1&=0\\x_3&=0\\x_1&=0\end{aligned}$$

So,

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\x_2\\0\\x_4\end{bmatrix}$$

$$=x_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Therefore, the basis for null space is $\boxed{\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}}$ and its dimension is $\boxed{2}$.

Step-4

Given the matrix $B=\begin{bmatrix}1&1\\4&4\\5&5\end{bmatrix}$

In this matrix, first column is independent.

Step-5

Column space basis is $C(B)=\begin{bmatrix}1\\4\\5\end{bmatrix}$ and its dimension is 1.

Null space basis is $\left(C\left(B^T\right)\right)=\begin{bmatrix}1\\1\end{bmatrix}$ and its dimension is 1.

Step-6

In order to find the null space, set it in $BX=0$. So,

$$\begin{bmatrix}1&1\\4&4\\5&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This implies;

$$\begin{aligned}x_1+x_2&=0\\4x_1+4x_2&=0\\5x_1+5x_2&=0\\x_1&=-x_2\end{aligned}$$

So,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$
$$= x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, basis for null space is $\boxed{B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ and its dimension is $\boxed{1}$.

Step-7

In order to find the left null space basis, the transpose of the matrix,

$$B^T = \begin{bmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{bmatrix}$$

Letâ€™s write it in $B^T X = 0$

$$\begin{bmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_1 + 4x_2 + 5x_3 = 0$$
$$x_1 + 4x_2 + 5x_3 = 0$$
$$x_1 = -4x_2 - 5x_3$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the basis for is $\boxed{N\left(B^T\right) = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}}$ and its dimension is $\boxed{2}$.