Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node j to node k , then that row has -1 in column j and + 1 in column k.

The incidence matrix A is $A=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$

We need to compute 3 by 3 matrix, and show that it is symmetric but singular, need to find the vectors in null space and need to show the 2 by 2 matrix is not singular(after removing the last column of A)

A symmetric matrix is a matrix that equals to its own transpose,

Step-2

The matrix A is $A=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$

In order to compute 3 by 3 matrix, we need find the transpose of the matrix A.

If $A=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$ then $A^T=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & -1\end{bmatrix}$

So,

$$\begin{aligned}A^T A &= \begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}\end{aligned}$$

The above matrix $A^T A$ is equals to its own transpose.

i.e., $\left(A^T A\right)^T = A^T A$

Therefore, $A^T A$ is a symmetric matrix.

Step-3

In order to show that $A^T A$ is singular, we need to show the determinant of $A^T A$ is zero.

$$\begin{aligned}\det\left(A^T A\right) &= 2(4-1)+1(-2-1)-1(1+2)\\ &= 6-3-3\\ &= 0\end{aligned}$$

Therefore, $A^T A$ is singular.

Therefore, the matrix $A^T A$ is symmetric but singular.

Step-4

In order to find the vectors are in its null space, we need to set the matrix is in

$$\left(A^T A\right)X = 0$$
$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to 2R_2 + R_1$ and $R_3 \to 2R_3 + R_1$ to A^T

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$ to A^T

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\Rightarrow 2x_1 - x_2 - x_3 &= 0\\ 3x_2 - 3x_3 &= 0\end{aligned}$$

From the second equation,

$$\Rightarrow 3x_2 - 3x_3 = 0$$
$$\Rightarrow x_2 = x_3$$

Plug this value in first equation,

$$\Rightarrow 2x_1 - x_2 - x_3 = 0$$
$$\Rightarrow 2x_1 - 2x_3 = 0$$
$$\Rightarrow x_1 = x_3$$

So, the vectors are,

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, $\boxed{N\left(A^T A\right) = \left\{ C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / C \in R \right\}}$.

Step-5

So, the matrix A is $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ and the matrix A^T is $A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

Step-6

Let $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ $\left(\text{ removing last column of } A\right)$

Let $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ $\quad$ $\left(\text{ removing last row of } A^T\right)$

So the new $A^T A$ matrix is,

$$CB = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \left(\text{Since } CB = A^T A\right)$$
$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

We need to show it is not singular. So,

$$\begin{aligned} \det CB &= 2 \cdot 2 - (-1)(-1) \\ &= 4 - 1 \\ &= 3 \\ &\neq 0 \end{aligned}$$

Therefore, $\boxed{CB \text{ is not singular}}$.