Step-1

We know that Singular Value Decomposition for any *m* by *n* matrix *A* can be factor into

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{pmatrix}\text{orthogonal} \\ U \text{ is } m\times m\end{pmatrix}\begin{pmatrix} m\times n \text{ matrix } \Sigma \\ \sigma_1\cdots\sigma_r \text{ on diagonal}\end{pmatrix}\begin{pmatrix}\text{orthogonal} \\ V \text{ is } n\times n\end{pmatrix}\end{aligned}$$

Here eigenvectors of AA^T are in *U*, eigenvectors of A^TA are in V.

The *r* singular-values on the diagonal of Σ are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Step-2

Suppose *A* is a invertible with $\sigma_1 > \sigma_2 > 0$.

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix}\begin{bmatrix} v_1 & v_2\end{bmatrix}^T$$

To make matrix *A* as singular matrix A_0, we have to do the smallest change in *A* is to set its smallest singular value σ_2 to 0.

Consider the factorization.

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix}\begin{bmatrix} v_1 & v_2\end{bmatrix}^T \\ &= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T \end{aligned}$$

So, in this factorization $\sigma_2 \to 0$, gives the closest rank-1 matrix as $u_1\sigma_1 v_1^T$.

Therefore, the singular matrix $\boxed{A_0 = u_1\sigma_1 v_1^T}$.