Step-1

Gibonacci number: Let G_{k+2} be the Gibonacci number which is defined by the average of two previous number G_{k+1}, G_k.

$$G_{k+2} = \frac{G_{k+1} + G_k}{2}$$

Step-2

Let following be the difference equation of matrices:

$$u_{k+1} = Au_k$$

Here, matrices u_{k+1}, A, u_k are defined as follows:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$
$$u_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$$
$$u_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

Step-3

(a) Find the Eigen values and Eigen vectors of matrix *A*:

To calculate the Eigen values do the following calculations;

$$\det(A - \lambda I) = 0$$
$$\left(\frac{1}{2} - \lambda\right)(-\lambda) - \frac{1}{2} = 0$$
$$2\lambda^2 - \lambda - 1 = 0$$

After solving following values are obtained:

$$\lambda_1 = 1$$
$$\lambda_2 = \frac{-1}{2}$$

Therefore, Eigen values are $\boxed{1, -1/2}$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of *y* and *z* corresponding to $\lambda=1$ is as follows:

$$x_1=\begin{bmatrix} y \\ z \end{bmatrix}$$
$$=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda=-\frac{1}{2}$ is as follows:

$$x_2=\begin{bmatrix} y \\ z \end{bmatrix}$$
$$=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore Eigen values are:

$$x_1=\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$
$$x_2=\boxed{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

Step-6

(b) Find the limit of the following matrices when $n\to\infty$.

$$A^n=S\Lambda^n S^{-1}$$

Matrix A can be written as follows:

$$A = S\Lambda S^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

Step-7

Power matrix:

$$A^n = S\Lambda^n S^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & \left(\frac{-1}{2}\right)^n \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

Take the limit $n \to \infty$. Value of $\left(\frac{-1}{2}\right)^n$ becomes very small, so neglect it and do the above calculations.

$$A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$
$$= \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

Step-8

Therefore, at $n \to \infty$ value of A^n is:

$$\boxed{A^n = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}}$$

Step-9

(c) Consider the following values:

$$G_0 = 0$$
$$G_1 = 1$$

To show that Gibonacci numbers approaches to $\frac{2}{3}$.

Step-10

Difference equation can be written as follows:

$$G_{k+1} = A^k G_0$$

Substitute the values to get the value of G_{k+1}

$$\begin{aligned} G_{k+1} &= \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} \end{aligned}$$

Step-11

Therefore, above result shows that Gibonacci numbers approaches to $\boxed{\frac{2}{3}}$.