Step-1

Given circulant matrix $C=\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$

If the eigen vector of C is $x=\left(1,w_j,w_j^{\,2},w_j^{\,3}\right)$ then its corresponding eigen value is

$e_j=c_0+c_1w_j+c_2w_j^{\,2}+c_3w_j^{\,3}$ for $j=0,1$ and $n=4$

If $x=(1,1,1,1)$ then its corresponding eigen value is

$$\begin{aligned} e_0 &= c_0+c_1w_0+c_2w_0^{\,2}+c_3w_0^{\,3} \\ &= c_0+c_1+c_2+c_3 \end{aligned}$$

If $x=\left(1,i,i^2,i^3\right)$ then its corresponding eigen value is

$$\begin{aligned} e_1 &= c_0+c_1w_1+c_2w_1^{\,2}+c_3w_1^{\,3} \\ &= c_0+ic_1+i^2c_2+i^3c_3 \\ &= c_0+ic_1-c_2-ic_3 \\ &= c_0-c_2+i\left(c_1-c_3\right) \end{aligned}$$

Therefore, the required eigen values of C are $e_0=c_0+c_1+c_2+c_3$ and $e_1=c_0-c_2+i\left(c_1-c_3\right)$