Step-1

Let us solve the Left Hand Side:

$$\begin{aligned}\int\left(-y_1V_1'' - y_2V_2'' - \ldots - y_nV_n''\right)V_j dx &= \int\left(-y_1V_1''V_j - y_2V_2''V_j - \ldots - y_nV_n''V_j\right)dx \\ &= \int -y_1V_1''V_j dx + \int -y_2V_2''V_j dx + \ldots + \int -y_nV_n''V_j dx \\ &= -\int y_1V_1''V_j dx - \int y_2V_2''V_j dx - \ldots - \int y_nV_n''V_j dx\end{aligned}$$

Step-2

Consider the following:

$$\begin{aligned}\int_0^1 -V_i''V_j dx &= -V_j\int_0^1 V_i'' dx - \int_0^1\left(\int -V_i'' dx\right)\left(\frac{d}{dx}V_j\right)dx \\ &= \left[-V_jV_i'\right]_0^1 + \int_0^1 V_i'V_j' dx \\ &= \int_0^1 V_i'V_j' dx - \left[V_jV_i'\right]_0^1\end{aligned}$$

$$\begin{aligned}\int_0^1 -V_i''V_j dx &= \int_0^1 V_i'V_j' dx - 0 \\ &= \int_0^1 V_i'V_j' dx \\ &= A_{ij}\end{aligned}$$

Step-3

This gives the following:

$$\begin{aligned}\int\left(-y_1V_1'' - y_2V_2'' - \ldots - y_nV_n''\right)V_j dx &= \int\left(A_{ij}y\right)V_j dx \\ &= \int f(x)V_j dx\end{aligned}$$

Thus, the equation $\int\left(-y_1V_1'' - y_2V_2'' - \ldots - y_nV_n''\right)V_j dx = \int f(x)V_j(x)dx$ is true.