Step-1

Properties of the determinant

1. The determinant of the identity matrix is 1

2. The determinant changes sign when two rows are exchanged

3. The determinant depends linearly on the first row.

4. If two rows of A are equal, then $\det A = 0$

5. Subtracting a multiple of one row from another row leaves the same determinant.

6. If A has a row of zeros, then $\det A = 0$

7. If A is triangular, then $\det A$ is the product $a_{11}a_{22}a_{33}\ldots a_{nn}$ of the diagonal entries.

8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$.

9. The determinant of AB is product of $\det A$ times $\det B$

10. The transpose of A has the same determinant as A itself; $\det A^T = \det A$

Step-2

a) We have

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 0 & b \\ 0 & a & 0 \\ c & 0 & 0 \end{vmatrix} \text{ interchaning first and second rows}$$

$$= -\begin{vmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} \text{ interchaning first and third rows}$$

Step-3

Therefore

$$|A|=\begin{vmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix}$$

$$=\boxed{abc}$$

Step-4

b) Given $B=\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix}$

Now

$$\det B=\begin{vmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{vmatrix}$$

$$=-\begin{vmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{vmatrix} \text{interchanging first and second rows}$$

Step-5

On solving

$$=\begin{vmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \text{interchanging fourth and third rows}$$

$$=-\begin{vmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \text{interchanging first and third rows}$$

$\det B = -(d)(a)(b)(c)$

$= \boxed{-abcd}$

Step-6

iii) We have

$$c = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}$$

$$\det c = \begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a \\ a & b & b \\ 0 & 0 & c-b \end{vmatrix} \text{Adding } -1 \text{ time second row to the thired row}$$

Step-7

$$= \begin{vmatrix} a & a & a \\ a & b-a & b-a \\ 0 & 0 & c-b \end{vmatrix} \text{Adding } -1 \text{ time the first row to the second row}$$

$= U$

$\det C = \det U$

$= a(b-a)(c-b)$

$= \boxed{a(a-b)(b-c)}$