Step-1

Consider the following 3 by 3 Vandermonde determinant:

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

The objective is to use the row operations to verify that the 3 by 3 Vandermonde determinant.

Use row operations and properties of the determinant,

$$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\xrightarrow{\substack{R_2\to R_2-R_1\\ R_3\to R_3-R_1}}\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}$$

By linearity of the determinant, we can pull out a factor of $(b-a)$ from row 2 and a factor of $(c-a)$ from row 3.

$$\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 1 & c+a\end{vmatrix}$$

Step-2

Use the cofactor formula or alternatively use row transformations.

From the cofactor formula, the determinant of the matrix is,

$$\begin{aligned}\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}&=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 1 & c+a\end{vmatrix}\\ &=(b-a)(c-a)(1)\begin{vmatrix}1 & b+a\\ 1 & c+a\end{vmatrix}\\ &=(b-a)(c-a)\left(c+a-(b+a)\right)\\ &=(b-a)(c-a)(c-b).\end{aligned}$$