# Step-1

Consider the following Hadamard matrix $H$:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

It is known that $H$ has orthogonal rows and the box is hypercube. Objective is to find $\det H$ which is same as the volume of hypercube in $R^4$.

# Step-2

For the determinant of $H$, use the cofactor expansion along with the first row and get,

$$\begin{aligned} \det H &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \\ &= 1\begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \end{aligned}$$

Now again use the cofactor expansion along with the first row for all these four determinants:

$$\begin{aligned} \det H &= 1\begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \left(1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix}\right) - \left(1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\right) \\ &\quad + \left(1\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}\right) - \left(1\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}\right) \end{aligned}$$

And then

$$\begin{aligned} \det H &= (0+2+2)-(0-2-2)+(2+2+0)-(-2-2-0) \\ &= 4+4+4+4 \\ &= 16 \end{aligned}$$

# Step-3

Note that in $R^4$, the volume of the hypercube is $2^4 = 16$. Since $\det H = 16$, therefore the determinant of Hadamard matrix will be same as the volume of the hypercube in $R^4$.