Step-1

A matrix A is not diagonalizable if the sum of the algebraic multiplicities of the eigen values is not equal to the sum of the geometric multiplicities of the respective eigen vectors.

Given that $A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(-2-\lambda) + 4 = 0$$

$$\Rightarrow -4 - 2\lambda + 2\lambda + \lambda^2 + 4 = 0$$
$$\Rightarrow \lambda^2 = 0$$

$\Rightarrow \lambda = 0, 0$ says that the eigen value 0 has algebraic multiplicity 2.

Step-2

To find the eigen vector corresponding to this eigen value, we solve the homogeneous system $(A - \lambda I)x = 0$

i.e, $\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Applying the row operation upon the coefficient matrix, $R_2 \to R_2 - R_1,\ R_1 / 2$ we get $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix and so, we rewrite the homogeneous system from this.

$$x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

Putting $x_1 = 1$, we get the solution set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is the eigen vector for $\lambda = 0, 0$.

We easily see that the number of eigen vectors = geometric multiplicity = 1 for $\lambda = 0, 0$

So, algebraic multiplicity is not equal to the geometric multiplicity and thus, $A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is not diagonalizable.

Step-3

$$A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

$$|A_2 - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2+\lambda) = 0$$

$\Rightarrow \lambda = -2, 2$ are the eigen values of A_2

Step-4

The eigen vector corresponding to $\lambda = -2$ is obtained by solving the homogeneous system $(A_2 - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying the row operations $R_2 \to 2R_2 - R_1$, $R_1 / 4$ on the coefficient matrix, it becomes $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix and so, we rewrite the homogeneous equations from this.

i.e., $x_1 = 0$

Replacing any real number in place of x_2, the system gets satisfied.

So, $x_1 = 0$, $x_2 = k$ is the solution where k is the parameter.

Putting $k = 1$, the solution is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is the eigen vector corresponding to $\lambda = -2$

Step-5

Similarly, for $\lambda = 2$, we solve $(A_2 - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 / 2 \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

This is the reduced matrix and so, the homogeneous equation is $x_1 - 2x_2 = 0$

In other words, $x_1 = 2x_2$

Putting $x_2 = 1$, we get $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as the solution set of the system which is the eigen vector corresponding to $\lambda = 2$.

The sum of the algebraic multiplicities of the eigen values = 2

The sum of the geometric multiplicities of the eigen vectors = 2

While both are equal, we can say that $A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$ is diagonalizable.

Step-6

$$A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$|A_3 - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda) = 0$$

$\Rightarrow \lambda = 2, 2$ are the eigen values of A_3

Observe that the algebraic multiplicity of the eigen value 2 is 2.

Step-7

To get the eigen vector corresponding to this eigen value, we solve $(A_3 - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying the row operation $R_1 \leftrightarrow R_2 / 2$ on the coefficient matrix, it becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the reduced matrix and so, we rewrite the homogeneous equations from this as

$$1 \cdot x_1 + 0 \cdot x_2 = 0$$

So, $x_1 = 0, x_2 = k$ satisfies this equation where k is any real number.

So, putting $k = 1$, the solution set is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is the eigen vector corresponding to the eigen value 2.

Step-8

We easily see that the number of eigen vectors is 1 and so the geometric multiplicity is 1.

But the algebraic multiplicity is 2.

So, $A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ is not diagonalizable.