Step-1

The number λ is an eigen value of A if and only if A - λI is singular

In other words, det (A - λI) = 0

This is the characteristic equation. Each λ is associated with eigen vector x such that

$Ax = \lambda x$.

Step-2

In view of this definition, let us consider $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

The characteristic equation of this matrix is det (A - λI) = $\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 9 - 16 = 0$

$\Rightarrow \lambda_1 = -5, \lambda_2 = 5$ are the roots of the characteristic equation or the eigen values of the given matrix.

Step-3

The eigen vector corresponding to the eigen value $\lambda_1 = -5$ satisfies $(A - \lambda_1 I)x_1 = 0$

$\Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Using the row operations, we reduce this as

$R_2 \to 2R_2 - R_1, R_1 / 4 \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 2t_1 + t_2 = 0$

$t_2 = -2t_1$

When $t_1 = 1$, we get $t_2 = -2$ and thus, the solution set is $x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda_1 = -5$

Step-4

Similarly, using $\lambda_2 = 5$, we get $(A-\lambda_2 I)x_2 = 0$ as $\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + 2R_1, R_1/-2 \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow t_1 - 2t_2 = 0$

When $t_2 = 1$, we get $t_1 = 2$ and thus, the solution set is $x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda_2 = 5$

Step-5

The characteristic equation of $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is $\det(A - \lambda \mathrm{I}) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda - a)^2 - b^2 = 0$
$\Rightarrow \lambda^2 - 2a\lambda + (a^2 - b^2) = 0$
$\Rightarrow \lambda = \dfrac{2a + \sqrt{4a^2 - 4(a^2 - b^2)}}{2}, \dfrac{2a - \sqrt{4a^2 - 4(a^2 - b^2)}}{2}$

$\Rightarrow \lambda_1 = a + b, \lambda_2 = a - b$

Step-6

Let the eigen vector corresponding to λ_1 is x_1

Then x_1 satisfies the equation $(A - \lambda_1 I)x_1 = 0$

$\Rightarrow \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Applying row operation $R_2 \to R_2 + R_1, R_2/-b$, we get $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow t_1 - t_2 = 0$

Putting $t_1 = 1$, we get $t_2 = 1$ and thus, the respective eigen vector is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Step-7

Similarly,

Let the eigen vector corresponding to λ_2 is x_2

Then x_2 satisfies the equation $(A-\lambda_2 I)x_2 = 0$

$$\Rightarrow \begin{bmatrix} b & b \\ b & b \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying row operation $R_2 \to R_2 - R_1, R_2 / b$, we get $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow t_1 + t_2 = 0$$

Putting $t_1 = 1$, we get $t_2 = -1$ and thus, the respective eigen vector is $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$