Step-1

We have to find all the 1 by 1 matrices that are Hermitian and also unitary.

Step-2

We know that a matrix A is Hermitian if $A^H = A$ and matrix B is unitary if $B^H B = I$.

Let us consider the 1 by 1 matrices $[1]$ and $[-1]$

Let $U = [1]$

Then

$$
\begin{aligned}
U^H U &= [1][1] \\
&= [1] \\
&= I
\end{aligned}
$$

Therefore $U = [1]$ is unitary.

And $U^H = [1] = U$

Hence U is Hermitian matrix.

Therefore, $U = [1]$ is both Hermitian and unitary.

Step-3

Let us consider $U = [-1]$

Now $U^H = [-1] = U$

Therefore, U is Hermitian.

Now

$$
\begin{aligned}
U^H.U &= [-1][-1] \\
&= [1] \\
&= I
\end{aligned}
$$

Therefore, U is unitary.

Hence U is Hermitian and unitary matrix.

Step-4

Now we have to find all the 2 by 2 matrices that are both Hermitian and unitary.

Let us consider 2×2 matrix

$U=\begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix}$ with $a^2+b^2+c^2=1$

Then

$$U^H=\begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix}$$

$$=U$$

Since $U^H=U$

So U is Hermitian.

Step-5

Now

$$\begin{aligned} U^H U &= \begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix}\begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix} \\ &= \begin{bmatrix} a(a)+(b+ic)(b-ic) & a(b+ic)-(b+ic)a \\ (b-ic)(a)-a(b-ic) & +(b+ic)(b-ic)+a(a) \end{bmatrix} \\ &= \begin{bmatrix} a^2+b^2+c^2 & ab+iac-ab-aic \\ ab-iac-ab+aic & b^2+c^2+a^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Since $U^H U=I$

So U is unitary.

Therefore $\boxed{\begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix}, a^2+b^2+c^2=1}$ is Hermitian and unitary matrix.