Step-1

Given that $P=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$, $Q=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $R=\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}$

We have to write *P, Q* and *R* in the form $\lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$.

Step-2

We find the eigenvalues of *P*.

The characteristic equation of *P* is $|P-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}-\lambda\end{vmatrix}=0$$

$$\Rightarrow \left(\frac{1}{2}-\lambda\right)^2-\frac{1}{4}=0$$
$$\Rightarrow \lambda^2-\lambda=0$$
$$\Rightarrow \lambda(\lambda-1)=0$$
$$\Rightarrow \lambda=0,1$$

Hence the eigenvalues of *P* are $\lambda_1=0, \lambda_2=1$.

Step-3

We know that $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is an eigenvector of *A* if and only if $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is the nonzero solution of $|P-\lambda I|x=0$

That is $\begin{bmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$ â€¦â€¦ (1)

Step-4

For $\lambda=0$, (1) becomes

$(P-0I)x=0$

$$\Rightarrow\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

The Augmented matrix is

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}$$

Step-5

Add (-1) times of row 1 to row 2, we get

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow\frac{1}{2}x_1+\frac{1}{2}x_2=0$$

Here x_1 is free variable.

Step-6

Let $x_1=k$, where k is a parameter.

$\Rightarrow x_2=-k$

Therefore, $x=\begin{bmatrix}k\\ -k\end{bmatrix}=k\begin{bmatrix}1\\ -1\end{bmatrix}$

Hence the eigenvector of P corresponding to $\lambda=0$ is $\begin{bmatrix}1\\ -1\end{bmatrix}$.

Therefore, x_1 with length scaled to 1 is $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$

Step-7

For $\lambda_2 = 1$, (1) becomes

$(P - 1I)x = 0$

$$\Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The Augmented matrix is

$$\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{bmatrix}$$

Step-8

Add row 1 to row 2, we get

$$\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \frac{-1}{2}x_1 + \frac{1}{2}x_2 = 0$$

Here x_2 is free variable.

Step-9

Let $x_2 = k$, where k is a parameter

$\Rightarrow x_1 = k$.

Therefore, $x = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence the eigenvector of P corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore, x_2 with length scaled to 1 is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-10

Hence P can be written as

$$
\begin{aligned}
P &= \lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H \\
&= 0\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + 1\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}
$$

Therefore, $\boxed{P = 0\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + 1\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}$

Step-11

Now we have to write Q as $\lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$

The characteristic equation of Q is $|Q - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$
\begin{aligned}
&\Rightarrow \lambda^2 - 1 = 0 \\
&\Rightarrow \lambda = \pm 1 \\
&\Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -1
\end{aligned}
$$

Therefore, the eigenvalues of Q are $\lambda_1 = 1 \text{ and } \lambda_2 = -1$.

Step-12

Now find the eigenvectors of Q.

We know that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the nonzero solution of $|Q - \lambda I| x = 0$

That is $\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ â€¦â€¦ (1)

For $\lambda_1 = 1$, (1) becomes

$(Q-I)x=0$

$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$

Step-13

The Augmented matrix is

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

Add row 1 to row 2, we get

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow -x_1+x_2=0$

Here x_2 is free variable

Step-14

Let $x_2=k$, where k is a parameter

Then $x_1=k$

Therefore, $x=\begin{bmatrix} k \\ k \end{bmatrix}=k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence the eigenvector of Q corresponding to $\lambda=1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore, x_1 with length scaled to 1 is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-15

For $\lambda_2=-1$, (1) becomes

$(Q+I)x=0$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Add (-1) times of row 1 to row 2, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

Here x_1 is free variable.

Let $x_2 = k$, where *k* is a parameter.

Then $x_2 = -k$

Step-16

Therefore, $x = \begin{bmatrix} k \\ -k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence the eigenvector of Q corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Therefore, x_2 with length scaled to 1 is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step-17

So we can write Q as

$$Q = \lambda_1 x_1 x_1^{H} + \lambda_2 x_2 x_2^{H}$$
$$= 1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - 1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Therefore, $\boxed{Q=1\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}-1\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}}$

Step-18

We have to write *R* as $\lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$

Given that $R=\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}$

The characteristic equation of *R* is $|R-\lambda I|=0$

$\Rightarrow\begin{vmatrix}3-\lambda & 4\\ 4 & -3-\lambda\end{vmatrix}=0$

$\Rightarrow(3-\lambda)(-3-\lambda)-16=0$

$\Rightarrow\lambda^2-25=0$

$\Rightarrow\lambda=\pm5$

Therefore the eigenvalues of *R* are $\boxed{\lambda_1=5 \text{ and } \lambda_2=-5}$.

Step-19

Now find the eigenvectors of *R*.

We know that $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is an eigenvector of *A* if and only if $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is the nonzero solution of $|R-\lambda I|x=0$

That is $\begin{bmatrix}3-\lambda & 4\\ 4 & -3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$ â€¦â€¦ (1)

Step-20

For $\lambda_1=5$, (1) becomes

$(R-5I)x=0$

$\Rightarrow\begin{bmatrix}-2 & 4\\ 4 & -8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

The Augmented matrix is

$$\begin{bmatrix} -2 & 4 & 0 \\ 4 & -8 & 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 4x_2 = 0$$

Here x_2 is free variable.

Step-21

Let $x_2 = k$, where k is a parameter

$$\Rightarrow x_1 = 2k$$

Therefore, $x = \begin{bmatrix} 2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Hence the eigenvector of R corresponding to $\lambda = 5$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Therefore, x_1 with length scaled to 1 is $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Step-22

For $\lambda_2 = -5$, (1) becomes

$$(R + 5I)x = 0$$

$$\Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The Augmented matrix is

$$\begin{bmatrix} 8 & 4 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

Add $\left(\frac{-1}{2}\right)$ times of row 1 to row 2, we get

$$\begin{bmatrix} 8 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow 8x_1 + 4x_2 = 0$

Here x_1 is free variable.

Step-23

Let $x_1 = k$, where k is a parameter

$\Rightarrow x_2 = -2k$

Therefore, $x = \begin{bmatrix} k \\ -2k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Hence the eigenvector of R corresponding to $\lambda = -5$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Therefore, x_2 with length scaled to 1 is $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Step-24

So we can write R as

$$R = \lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$$
$$= 5\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} - 5\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

Therefore, $\boxed{R = 5\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} - 5\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}}$