Step-1

We have to solve the system $F_4 c = y$, where

$$c_0 + c_1 + c_2 + c_3 = 2$$
$$c_0 + ic_1 + i^2 c_2 + i^3 c_3 = 0$$
$$c_0 + i^2 c_1 + i^4 c_2 + i^6 c_3 = -2$$
$$c_0 + i^3 c_1 + i^6 c_2 + i^9 c_3 = 0$$

And we have to verify that $c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix}$ takes the values 2, 0,-2, 0 at the points $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Step-2

Let $c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$

And $y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$

Step-3

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Fourier matrix

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$\begin{pmatrix} \text{since } i^2 = -1, i^3 = -i \\ i^4 = 1, i^6 = i^4 i^2 = -1 \\ i^9 = i^6 i^3 = i \end{pmatrix}$$

Step-4

Now

$$F_4 c = y$$
$$\Rightarrow c = F_4^{-1} y$$
But $F_n^{-1} = \frac{\overline{F_n}}{n} y, (n \text{ is the order of } F_n)$

Therefore $c = \frac{\overline{F_4}}{4} y$

Step-5

$$\Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$
$$= \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 0 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the solution of the system $F_4 c = y$ is $c = (c_0, c_1, c_2, c_3) = (0,1,0,1)$

Step-6

At $x = 0, c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix} = 2$

$\Rightarrow c_0+c_1+c_2+c_3=2\left(\text{since } e^0=1\right)$
$\Rightarrow 0+1+0+1=2$
$\Rightarrow \boxed{2=2}$

Step-7

At $x=\pi/2,\, c_0+c_1e^{i\pi/2}+c_2e^{2i\pi/2}+c_3e^{3i\pi/2}=0$

$\Rightarrow c_0+ic_1-1c_2-ic_3=0\left(\text{since } e^{i\pi/2}=i, e^{2i\pi/2}=-1, e^{3i\pi/2}=-i\right)$
$\Rightarrow 0+i+0-i=0$
$\Rightarrow \boxed{0=0}$

At $x=\pi,\, c_0+c_1e^{i\pi}+c_2e^{2i\pi}+c_3e^{3i\pi}=-2$

$\Rightarrow c_0-1c_1+1c_2-1c_3=-2\left(\text{ since } e^{i\pi}=-1, e^{2i\pi}=1, e^{3i\pi}=-1\right)$
$\Rightarrow 0-1(1)+0-1(1)=-2$
$\Rightarrow \boxed{-2=-2}$

Step-8

At $x=3\pi/2,\, c_0+c_1e^{3i\pi/2}+c_2e^{6i\pi/2}+c_3e^{9i\pi/2}=0$

$\Rightarrow c_0-ic_1-1c_2+ic_3=0\left(\text{since } e^{i\pi}=-1, e^{2i\pi}=1, e^{3i\pi}=-1\right)$
$\Rightarrow 0-i(1)+0+i(1)=0$
$\Rightarrow \boxed{0=0}$

Therefore the values of $c_0+c_1e^{ix}+c_2e^{2ix}+c_3e^{3ix}$ at $x=0,\dfrac{\pi}{2},\pi,\dfrac{3\pi}{2}$ are 2, 0,-2, 0.