南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 高等数学(上) A　　开课单位: 数 学 系

考试时长: 120 分钟　　命题教师: ______

题 号	1	2	3	4	5	6	7	8	9
分 值	15 分	15 分	10 分	10 分	10 分	10 分	10 分	16 分	4 分

1. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

   (1) The number of the real roots for the equation $x^3 - 3x + 3 = 0$ is

   (A) 0.　(B) 1.　(C) 2.　(D) 3.

   (2) If $f(x)$ is continuous on $(-\infty, +\infty)$, which of the following statements is **wrong** ?

   (A) $\int_0^1 f(x)\,dx = \int_0^1 f(t)\,dt$.　(B) $\int_0^1 f(x)\,dx = \int_0^1 f(\sin x)\,d(\sin x)$.

   (C) $d\left(\int_0^x f(t)\,dt\right) = f(x)\,dx$.　(D) $d\left(\int_0^{x^2} f(t)\,dt\right) = f(x^2)\,d(x^2)$.

   (3) Let
   $$f(x) = \begin{cases} x^4 \sin\frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$
   Then the largest positive integer n, for which $f^{(n)}(0)$ exists, is

   (A) 1.　(B) 2.　(C) 3.　(D) 4.

   (4) If $f(x)$ is twice-differentiable on $(-\infty, +\infty)$, and $g(x) = (1-x)f(0) + xf(1)$, then which of the following statements is **correct** on $(0, 1)$?

   (A) $f(x) > g(x)$ if $f'(x) > 0$.　(B) $f(x) > g(x)$ if $f'(x) < 0$.

   (C) $f(x) > g(x)$ if $f''(x) > 0$.　(D) $f(x) > g(x)$ if $f''(x) < 0$.

   (5) If the improper integral $\displaystyle\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k}\,dx$ converges, then the constant k must satisfy

   (A) $k < 1$.　(B) $k > 3$.

   (C) $1 < k < 2$.　(D) $1 < k < 3$.

   Solution: (1) B; (2) B; (3) B; (4) D; (5) D.

2. (15 pts) Fill in the blanks.

   (1) Function $f(x) = x^2$ has a tangent line $y = Kx - 1$ if $K =$ ______, or ______.

   (2) Assume that $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, and $f''(1) = -7$. Let $g(x) = f(\ln x)$. Then $g''(1) =$ ______.

(3) The average value for $f(x) = \sin^3 x$ on $[0, \pi]$ is ________.

(4) Let $y = (\cos x)^x$ for $0 < x < \frac{\pi}{2}$, then $y'(x) =$ ________.

(5) If $f''(a)$ exists, and $f'(a) \neq 0$, then $\lim\limits_{x \to a} \left(\dfrac{1}{f'(a)(x-a)} - \dfrac{1}{f(x) - f(a)} \right) =$ ________.

Solution: (1) 2, -2; (2) 2; (3) $\frac{4}{3\pi}$; (4) $(\cos x)^x (\ln(\cos x) - x \tan x)$; (5) $\frac{f''(a)}{2(f'(a))^2}$.

3. (10 pts) The region D is enclosed by the curve $y = \ln\sqrt{x-1}$, the straight line $x = 5$, and the x-axis.

(1) Find the area of the region D.

(2) Find the volumes generated by revolving the region D about the line $x = 5$.

Solution:

(1)
$$\int_2^5 \ln\sqrt{x-1}\, dx = \frac{1}{2}\int_1^4 \ln t\, dt = \frac{1}{2}\left(t(\ln t - 1)\right)\Big]_1^4 = 4\ln 2 - \frac{3}{2}.$$

(2)
$$\int_0^{\ln 2} \pi \left(5 - \left(e^{2y} + 1\right)\right)^2 dy = \pi \int_0^{\ln 2} \left(4 - e^{2y}\right)^2 dy = \pi\left(16\ln 2 - \frac{33}{4}\right).$$

4. (10 pts) Find the particular solution of

$$xy' + (x-2)y = 3x^3 e^{-x}, \quad x > 0,$$

satisfying $y(1) = 0$.

Solution: The integrating factor is $\frac{1}{x^2}e^x$.

$$\frac{d}{dx}\left(\frac{1}{x^2}e^x y\right) = 3$$

$$y = (3x + C)x^2 e^{-x}$$

Because $y(1) = 0$, we have $c = -3$. Therefore,

$$y = 3(x-1)x^2 e^{-x}$$

5. (10 pts) Evaluate the following limits.

(1) $\lim\limits_{n \to +\infty} \left(\dfrac{n}{2n^2 + 3n + 1^2} + \dfrac{n}{2n^2 + 6n + 2^2} + \cdots \dfrac{n}{2n^2 + 3nk + k^2} + \cdots + \dfrac{n}{2n^2 + 3n^2 + n^2} \right).$

(2) $\lim\limits_{x \to 0} \left(\dfrac{\ln(1+x)}{x} \right)^{\frac{1}{e^x - 1}}.$

Solution:

(1)
$$= \int_0^1 \frac{1}{(x+1)(x+2)}\, dx = 2\ln 2 - \ln 3$$

(2) Note $\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}} = e^{\frac{\ln(\ln(1+x))-\ln x}{e^x-1}}$.

$$\lim_{x\to 0}\frac{\ln(\ln(1+x))-\ln x}{e^x-1} = \lim_{x\to 0}\frac{\frac{1}{(1+x)\ln(1+x)}-\frac{1}{x}}{e^x} = \lim_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x\ln(1+x)}$$

$$= \lim_{x\to 0}\frac{-\ln(1+x)}{\ln(1+x)+\frac{x}{1+x}} = -\frac{1}{2}.$$

The final result is $= \frac{1}{\sqrt{e}}$.

6. (10 pts)

(1) For $y = \frac{x^2+1}{x+1}$, identify the coordinates of any local and absolute extreme points and inflection points that may exist.

(2) Sketch the graph of the above function. (Please identify all the asymptotes and some specific points, such as local maximum and minimum points, inflection points, and intercepts.)

Solution:

(1) $y' = 1 - \frac{2}{(x+1)^2}$, $y'' = \frac{4}{(x+1)^3}$.
local maximum point $(-1-\sqrt{2}, -2-2\sqrt{2})$, local minimum point $(-1+\sqrt{2}, 2\sqrt{2}-2)$.

(2) oblique asymptote is $y = x - 1$.

7. (10 pts) Find $\frac{dy}{dx}$ if

$$y = \int_{x^2+1}^{2x^2+3} t\ \tan\sqrt{x+t}\,dt.$$

Solution: Let $u = x + t$.

$$y = \int_{x^2+x+1}^{2x^2+x+3} (u-x)\tan\sqrt{u}\,du$$

$$\frac{dy}{dx} = -\int_{x^2+x+1}^{2x^2+x+3} \tan\sqrt{u}\,du + (4x+1)(2x^2+3)\tan\sqrt{2x^2+x+3}$$

$$+(2x+1)(x^2+1)\tan\sqrt{x^2+x+1}.$$

8. (16 pts) Evaluate the integrals.

(1) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^3 x\,dx$.

(2) $\displaystyle\int \sqrt{\frac{x}{x-2}}\,dx$, where $x > 2$.

(3) $\displaystyle\int_1^e \ln^3 x\,dx$.

(4) $\displaystyle\int_1^{+\infty} \frac{(x+2)\ln(x^2+1)}{x^3}\,dx$.

Solution:

(1)
$$= -\frac{1}{2}\csc x\cot x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\csc x\,dx = -\left[\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln(\csc x+\cot x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \sqrt{3} - \frac{1}{3} + \frac{1}{2}\ln\frac{2+\sqrt{3}}{\sqrt{3}}$$

(2)
$$= \int \frac{x}{\sqrt{x^2-2x}}\,dx$$

Let $u = x-1$, we have
$$= \int \frac{u+1}{\sqrt{u^2-1}}\,du = \sqrt{u^2-1} + \ln(u+\sqrt{u^2-1}) + C$$
$$= \sqrt{x^2-2x} + \ln(x-1+\sqrt{x^2-2x}) + C$$

(3) Let $u = \ln x$, we have
$$= \int_0^1 u^3 e^u du = (u^3-3u^2+6u-6)e^u\Big]_0^1 = 6-2e.$$

(4)
$$\int \frac{(x+2)\ln(x^2+1)}{x^3}\,dx = -\frac{1+x}{x^2}\ln(x^2+1) + 2\int\frac{x+1}{x(x^2+1)}\,dx$$
$$\int\frac{x+1}{x(x^2+1)}\,dx = \int\left(\frac{1}{x} - \frac{x-1}{x^2+1}\right)dx = \ln x - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

Therefore,
$$\int_1^{+\infty}\frac{(x+2)\ln(x^2+1)}{x^3}\,dx = 3\ln 2 + \frac{\pi}{2}.$$

9. (4 pts) Let $f(n) = \sum_{m=1}^{n}\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx$, here $\lfloor x+1\rfloor$ is the largest integer which is less than or equal to $x+1$. Evaluate $f(2021)$.

Solution:
$$\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx = \sum_{k=1}^{m}\int_{k-1}^{k}\cos\frac{2\pi nk}{m}\,dx = \sum_{k=1}^{m}\cos k\frac{2n\pi}{m}$$

If $m|n$, then $\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx = m$; otherwise, because
$$\sum_{k=1}^{m}\cos kt = \frac{\cos\left(\frac{m+1}{2}t\right)\sin\left(\frac{m}{2}t\right)}{\sin\frac{t}{2}},$$

when $t = \frac{2n\pi}{m}$, $\sin\left(\frac{m}{2}t\right) = 0$. Thus $\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx = 0$.

Note $2021 = 43 * 47$, therefore $f(2021) = 1+43+47+2021 = 2112$.