数学分析精讲期中考试题

考试日期: 2022.11.7  考试时长:   120  分钟

本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This exam paper contains <u>7</u> questions and the score is <u>100</u> in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

1. (30 points, 3 points each) True or false ?

   (共 30 分, 每小题 3 分) 对或错?

   (1) Compact metric spaces are complete.

   紧度量空间是完备的.

   (2) The intersection of a closed subset and a compact subset in a metric space $X$ is a compact subset of $X$.

   度量空间$X$中的一闭子集和一紧子集的交为$X$中的一紧子集.

   (3) Any Cauchy sequence is bounded.

   任何Cauchy点列都是有界的.

   (4) A continuous mapping from a compact metric space is uniformly continuous.

   从紧空间出发的连续映照是一致连续的.

   (5) The set of real numbers is countable.

   实数集是可数集.

   (6) Let $\{E_k\}_{k\in\Lambda}$ be a family of open sets in a metric space $X$. Then the set $E = \cap_{k=1}^{\infty}E_k$ is an open set in $X$.

   设$\{E_k\}_{k\in\Lambda}$为度量空间$X$中的一族开集. 则集合$E = \cap_{k\in\Lambda}E_k$ 是$X$中的开集.

   (7) Let $f : [a,b] \to \mathbf{R}$ be a differentiable map. If $f$ attains its minimum at a point $x \in (a,b)$, then $f'(x) > 0$.

   设 $f : [a,b] \to \mathbf{R}$ 为一可微映照. 如果 $f$ 在 $x \in (a,b)$ 达到最小值, 则 $f'(x) > 0$.

   (8) Let $f$ be a monotonically decreasing function defined on $(a,b)$. Then the set of points of (a, b) at which $f$ is discontinuous is finite.

   设 $f$ 为定义在 $(a,b)$ 上的一单调下降函数. 则 $f$ 在 $(a,b)$ 中的不连续点集是有限集.

   (9) A metric space $X$ is complete if and only if any Cauchy sequence in $X$ has a convergent subsequence.

   度量空间 $X$ 是完备的当且仅当 $X$ 中的任何 Cauchy 点列都有一收敛子列.

   (10) Let $X$ be a metric space. A subset $E$ of $X$ is closed if and only if for any sequence $\{p_n\} \subset E$ such that $\lim_{n\to\infty} p_n = p$, we have $p \in E$.

   设 $X$ 为度量空间. $X$的一子集 $E$ 是闭的当且仅当对任何点列$\{p_n\} \subset E$, 若 $\lim_{n\to\infty} p_n = p$, 则 $p \in E$.

2. (12 points) Calculate

$$\sum_{n=1}^{+\infty} \frac{(-1)^n + (-2)^{n+1}}{5^n}.$$

计算

$$\sum_{n=1}^{+\infty} \frac{(-1)^n + (-2)^{n+1}}{5^n}.$$

3. (12 points) Let $a_1, a_2, a_3, \cdots$ denote the sequence

$$-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \cdots, \frac{(-1)^n n}{n+1}, \cdots$$

Find $\limsup_{n \to +\infty} a_n$ and $\liminf_{n \to +\infty} a_n$.

设 $a_1, a_2, a_3, \cdots$ 表示数列

$$-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \cdots, \frac{(-1)^n n}{n+1}, \cdots$$

求 $\limsup_{n \to +\infty} a_n$ 和 $\liminf_{n \to +\infty} a_n$.

4. (12 points) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^3 \sin \frac{1}{2x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Calculate $f'(x)$.

设 $f : \mathbf{R} \to \mathbf{R}$ 由下式定义

$$f(x) = \begin{cases} x^3 \sin \frac{1}{2x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

求 $f'(x)$.

5. (12 points) Let $X$ be a compact metric space and let $E$ be a closed subset of $X$. Show that $E$ is a compact subset of $X$.

设 $X$ 为一紧度量空间, $E$ 为 $X$ 的一闭子集. 证明 $E$ 为 $X$ 的紧子集.

6. (12 points) Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces, $f : X \to Y$ be a continuous map. Let $U$ be an open subset of $Y$. Show that $f^{-1}(U)$ is an open subset of $X$.

设 $(X, d_1)$ 和 $(Y, d_2)$ 为度量空间, $f : X \to Y$ 为一连续映照. 设 $U$ 为 $Y$ 的一开子集. 证明 $f^{-1}(U)$ 为 $X$ 的一开子集.

7. (10 points) Show that the function $f : [1, \infty) \to \mathbf{R}$, $f(x) = \frac{1}{x^2}$ is uniformly continuous.

证明函数 $f : [1, \infty) \to \mathbf{R}$, $f(x) = \frac{1}{x^2}$ 是一致连续的.