

Step-1

It is clear that the product of V_i and V_j is zero when $j = i \pm 2$.

Consider $V_1 V_2$. This product is zero when $0 \leq x \leq \frac{1}{n+1}$ or $\frac{2}{n+1} \leq x \leq 1$.

Make two equal parts of the interval $\frac{1}{n+1} \leq x \leq \frac{2}{n+1}$. When $\frac{1}{n+1} \leq x \leq \frac{3}{2n+2}$, we get

$$\begin{aligned} V_1 V_2 &= (2 - (n+1)x)((n+1)x - 1) \\ &= 3(n+1)x - 2 - (n+1)^2 x^2 \end{aligned}$$

Step-2

Consider the following:

$$\begin{aligned} \int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} V_1 V_2 dx &= \int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} (3(n+1)x - 2 - (n+1)^2 x^2) dx \\ &= 3(n+1) \left[\frac{x^2}{2} \right]_{\frac{1}{n+1}}^{\frac{3}{2n+2}} - 2 \left[x \right]_{\frac{1}{n+1}}^{\frac{3}{2n+2}} - (n+1)^2 \left[\frac{x^3}{3} \right]_{\frac{1}{n+1}}^{\frac{3}{2n+2}} \\ &= \frac{3(n+1)}{2} \left[\frac{9}{(2n+2)^2} - \frac{1}{(n+1)^2} \right] - 2 \left[\frac{3}{2n+2} - \frac{1}{n+1} \right] \\ &\quad - \frac{(n+1)^2}{3} \left[\frac{27}{(2n+2)^3} - \frac{1}{(n+1)^3} \right] \\ &= \frac{3(n+1)}{2} \left[\frac{5}{(2n+2)^2} \right] - 2 \left[\frac{1}{2n+2} \right] - \frac{(n+1)^2}{3} \left[\frac{19}{(2n+2)^3} \right] \end{aligned}$$

$$\begin{aligned} \int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} V_1 V_2 dx &= \frac{3(n+1)}{2} \left[\frac{5}{4(n+1)^2} \right] - 2 \left[\frac{1}{2n+2} \right] - \frac{(n+1)^2}{3} \left[\frac{19}{8(n+1)^3} \right] \\ &= \frac{3}{2} \left[\frac{5}{4(n+1)} \right] - \left[\frac{1}{n+1} \right] - \frac{1}{3} \left[\frac{19}{8(n+1)} \right] \\ &= \frac{15}{8(n+1)} - \frac{1}{n+1} - \frac{19}{24(n+1)} \\ &= \frac{45 - 24 - 19}{24(n+1)} \end{aligned}$$

Step-3

Simplifying further, we get

$$\int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} V_1 V_2 dx = \frac{2}{24(n+1)} \\ = \frac{1}{12(n+1)}$$

Therefore, by symmetry, $\int_{\frac{1}{2n+2}}^{\frac{2}{3}} V_1 V_2 dx = \frac{1}{12(n+1)}$. Adding, we get $\int_{\frac{1}{n+1}}^{\frac{2}{n+1}} V_1 V_2 dx = \frac{1}{6(n+1)}$.

This gives us the idea that $\int_{\frac{1}{n+1}}^{\frac{i+1}{i}} V_i V_{i+1} dx = \frac{1}{6(n+1)}$. Therefore, $\int_0^1 V_i V_{i+1} dx = \frac{1}{6(n+1)}$ and $\int_0^1 V_{i-1} V_i dx = \frac{1}{6(n+1)}$, provided $1 < i < n$.

When $0 \leq x \leq \frac{1}{n+1}$, consider the following:

$$V_1^2 = V_1 V_1 \\ = ((n+1)x)((n+1)x) \\ = (n+1)^2 x^2$$

Therefore, we get

$$\int_0^{\frac{1}{n+1}} V_1^2 dx = \int_0^{\frac{1}{n+1}} ((n+1)^2 x^2) dx \\ = (n+1)^2 \left[\frac{x^3}{3} \right]_0^{\frac{1}{n+1}} \\ = \frac{(n+1)^2}{3(n+1)^3} \\ = \frac{1}{3(n+1)}$$

By symmetry, $\int_{\frac{1}{n+1}}^{\frac{2}{n+1}} V_1^2 dx = \frac{1}{3(n+1)}$. Adding, we get $\int_0^{\frac{2}{n+1}} V_1^2 dx = \frac{2}{3(n+1)}$. It should be clear that $\int_0^1 V_1^2 dx = \frac{2}{3(n+1)}$.

Step-4

Now, $h = \frac{1}{n+1}$.

Therefore,

$$\begin{aligned}\int_0^1 V_i V_{i+1} dx &= \frac{h}{6} \\ \int_0^1 V_{i-1} V_i dx &= \frac{h}{6} \\ \int_0^1 V_i^2 &= \frac{2h}{3} \\ &= \frac{4h}{6}\end{aligned}$$

Step-5

Thus, the mass matrix M_{ij} has all diagonal entries equal to $\frac{4h}{6}$ and the entries along the diagonals just above and below the main diagonal are $\frac{h}{6}$. If we take $\frac{h}{6}$ outside the matrix then all diagonal entries will be equal to 4 and the entries along the diagonals just above and below the main diagonal will be 1.

$$M_{ij} = \frac{h}{6} \begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}.$$

Therefore,