Step-1

Given matrix is $A=\begin{bmatrix}1 & 100\\ 0 & 1\end{bmatrix}$

We have to explain why the given matrix has norms between 100 and 101.

Step-2

The characteristic equation of A is

$$|A-\lambda I|=0$$
$$\Rightarrow\begin{vmatrix}1-\lambda & 100\\ 0 & 1-\lambda\end{vmatrix}=0$$
$$\Rightarrow(1-\lambda)(1-\lambda)=0$$
$$\Rightarrow\lambda=1,1$$

So, the eigenvalues of characteristic roots are $\lambda=1,1$

Therefore, $\lambda_{\max}=\lambda_{\min}=1$

Step-3

The corresponding eigenvector is obtained by solving the homogeneous system $(A-\lambda I)x=0$

That is $\begin{bmatrix}1-\lambda & 100\\ 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ â€¦â€¦ (1)

For $\lambda=1$, (1) becomes

$$\begin{bmatrix}1-1 & 100\\ 0 & 1-1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}0 & 100\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Step-4

From this, we get $0x + 100y = 0$

So, $y = 0$ and $x = k$ for any parameter k.

The solution set is $k\begin{bmatrix}1\\0\end{bmatrix}$

Putting k = 1, we get the eigenvector corresponding to the eigenvalue $\lambda = 1$ is $x = \begin{bmatrix}1\\0\end{bmatrix}$.

Step-5

While the given matrix has the repeated eigenvalues, the second eigenvector y is any vector linearly independent with the existing eigenvector satisfying $y = \begin{bmatrix}0\\1\end{bmatrix}$

The definition of the norm of a matrix A says $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

Step-6

Now, using the vectors x, y in this definition, we have

$$\frac{\|Ax\|}{\|x\|} = \frac{\left\|\begin{bmatrix}1 & 100\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right\|}{\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\|}$$

$$= \frac{\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\|}{\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\|}$$

$$= \frac{\sqrt{1^2 + 0}}{\sqrt{1^2 + 0}}$$

$$= 1$$

Step-7

And

$$\frac{\|Ay\|}{\|y\|} = \frac{\left\|\begin{bmatrix}1 & 100\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right\|}{\left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|}$$

$$= \frac{\left\| \begin{bmatrix} 100 \\ 1 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|}$$

$$= \frac{\sqrt{100^2 + 1^2}}{\sqrt{1^2 + 0^2}}$$

Clearly, this value lies between 100 and 101.

Step-8

Now

$$\|A\| = \max\left\{ \frac{\|Ax\|}{\|x\|}, \frac{\|Ay\|}{\|y\|} \right\}$$

$$= \max\{1, \sqrt{10001}\}$$

$$= \sqrt{10001}$$

Step-9

In the case of $A^{-1} = \begin{bmatrix} 1 & -100 \\ 0 & 1 \end{bmatrix}$, the above discussion repeats and ultimately, we get

$$\frac{\|Ay\|}{\|y\|} = \frac{\left\| \begin{bmatrix} 1 & -100 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|}$$

$$= \frac{\left\| \begin{bmatrix} -100 \\ 1 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|}$$

$$= \frac{\sqrt{(-100)^2 + 1^2}}{\sqrt{1^2 + 0^2}}$$

Therefore, $\|A^{-1}\| = \sqrt{10001}$

Clearly, $100 < \sqrt{10001} < 101$

Therefore, the norms of the given matrix and its inverse lies between 100 and 101.