MA327 Homework 6

Due on 26th May

1. Let $\varphi : S \to \bar{S}$ be an isometry and $\mathbf{x} : U \to S$ a parametrization at $p \in S$, then $\bar{\mathbf{x}} := \varphi \circ \mathbf{x}$ is a parametrization at $\varphi(p)$ and $E = \bar{E}, F = \bar{F}, G = \bar{G}$.

2. Show that a diffeomorphism $\varphi : S \to \bar{S}$ is an isometry if and only if the arc length of any parametrized curve in S is equal to the arc length of the image curve by φ.

3. Use the stereographic projection (cf. Question 9 in Homework 2) to show that the sphere is locally conformal to a plane.

4. Let S_1, S_2 and S_3 be regular surfaces. Prove that (a) If $\varphi : S_1 \to S_2$ is an isometry, then $\varphi^{-1} : S_2 \to S_1$ is also an isometry. (b) If $\varphi : S_1 \to S_2, \psi : S_2 \to S_3$ are isometries, then $\psi \circ \varphi : S_1 \to S_3$ is an isometry.

5. Let S be a surface of revolution. Prove that the rotations about its axis are isometries of S.

6. We say that a differentiable map $\varphi : S_1 \to S_2$ preserves the angles when for every $p \in S_1$ and every pair $v_1, v_2 \in T_p(S_1)$ we have

$$\cos(v_1, v_2) = \cos(d\varphi_p(v_1), d\varphi_p(v_2)).$$

Prove that φ is locally conformal if and only if it preserves angles.

7. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(x, y) = (u(x, y), v(x, y))$, where u and v are differentiable functions that satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Show that φ is a local conformal map from $\mathbb{R}^2 - Q$ into $\mathbb{R}^2$, where $Q := \{(x, y) \in \mathbb{R}^2 \mid u_x^2 + u_y^2 = 0\}$.

8. Show that if $\mathbf{x}$ is an orthogonal parametrization, that is, $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}}[(\frac{E_v}{\sqrt{EG}})_v + (\frac{G_u}{\sqrt{EG}})_u].$$

9. Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\Delta(\log \lambda),$$

where $\Delta f := \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$.

Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $K = 4c$, where c is a constant.

10. Verifies that the surfaces

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, \log u), u > 0,$$

and

$$\bar{\mathbf{x}}(u,v) = (u\cos v, u\sin v, v),$$

have equal Gaussian curvature at the points $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(u,v)$ but that the mapping $\bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.

11. Show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

12. Does there exist a surface $\mathbf{x}(u,v)$ with $E = 1, F = 0, G = \cos^2 u$ and $e = \cos^2 u, f = 0, g = 1$?