

## Step-1

From the relation  $x_1 + x_2 = x_3 + x_4 = x_5 + x_6$ , we see that there may not be any restriction on  $x_1, x_2, x_3$ , and  $x_5$ . Once, these values are fixed, the remaining values  $x_4$  and  $x_6$  are automatically fixed. That is, we have

$$x_4 = x_1 + x_2 - x_3$$

$$x_6 = x_1 + x_2 - x_5$$

Thus, we get the basis of the subspace as follows:

$$\left\{ (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbf{R}^6 \mid x_4 = x_1 + x_2 - x_3, x_6 = x_1 + x_2 - x_5 \right\}$$

## Step-2

(b) We need to obtain a matrix, whose nullspace contains the above subspace.

Suppose we contain the equality  $a = b = c$ , then observe the following:

$$a - b = 0$$

$$a - c = 0$$

$$b - c = 0$$

$$a + b - 2c = 0$$

$$a - 2b + c = 0$$

$$-2a + b + c = 0$$

## Step-3

The above 6 equations give us the idea of the required matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ -2 & -2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

## Step-4

Consider the following:

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ -2 & -2 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 + x_2 - x_3 \\ x_5 \\ x_1 + x_2 - x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

### Step-5

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ -2 & -2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Therefore, is a required matrix.

### Step-6

(c) We need to find a matrix, whose column space is the above subspace.

Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Any vector  $(x_1, x_2, x_3, x_4, x_5, x_6)$ , such that  $x_1 + x_2 = x_3 + x_4 = x_5 + x_6$ , can be expressed as a linear combination of the columns of the above matrix.