Step-1

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Step-2

Show that $e^A e^B$ is different from $e^B e^A$ and they both are different from e^{A+B}.

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = 0$$
$$(1-\lambda)(-\lambda) = 0$$
$$\lambda^2 - \lambda = 0$$

After solving following values are obtained:

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = 1$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-6

Therefore Eigen values are:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-7

Recall that $e^{At} = Se^{\Lambda t}S^{-1}$. Therefore,

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} e^t & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$$

Step-8

For $t = 1$:

$$e^{A} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

Consider the following matrix

$$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
$$B \cdot B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-9

Substitute B in the expansion of e^{Bt}.

$$e^{Bt} = I + Bt$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

Step-10

For $t = 1$:

$$e^{B} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Step-11

Now calculate the following:

$$e^A e^B = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} e & -1 \\ 0 & 1 \end{bmatrix}$$
$$e^B e^A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix}$$

Therefore, $\boxed{e^A e^B \neq e^B e^A}$.

Step-12

Consider the following matrix:

$$A+B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$(A+B)\cdot(A+B) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $(A+B)^2 = (A+B)$.

Step-13

Substitute $(A+B)$ in the expansion of $e^{(A+B)t}$.

$$
\begin{aligned}
e^{(A+B)t} &= I+(A+B)t+\frac{(A+B)^2t^2}{2!}+\ldots \\
&= I+(A+B)t+\frac{(A+B)t^2}{2!}+\ldots \\
&= I+(A+B)\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\right) \\
&= I+(A+B)\left(e^t-1\right) \\
&= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\left(e^t-1\right)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \\
&= \begin{bmatrix}1+\left(e^t-1\right) & 0\\ 0 & 1\end{bmatrix} \\
&= \begin{bmatrix}e^t & 0\\ 0 & 1\end{bmatrix}
\end{aligned}
$$

$$
e^{(A+B)} = \begin{bmatrix}e & 0\\ 0 & 1\end{bmatrix}
$$

Step-14

Therefore, $\boxed{e^Ae^B \neq e^Be^A}$ and e^Ae^B, e^Be^A are different from $\boxed{e^{A+B}}$.