Step-1

Solve the following matrices to show that matrix A has no square root:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=A}$$

Step-2

To solve the above matrices put different values of variables (a,b,c,d) to get the right side matrix. It can be seen clearly that right side matrix has maximum zeros and only element 1. So try putting each element as 1, one at a time, and solve the matrix multiplication.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\neq A}$$

Take another variable value to be 1 and solve the product.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\neq A}$$

Similarly, other two variables, value equal to 1, also does not make product equal to matrix A. Therefore, this shows that matrix A has no square root.

Step-3

Now change the diagonal entry of matrix A to 4 and then calculate the square root. Let the matrix A after putting 4 at diagonal is B.

$$B = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Step-4

As matrix B is upper diagonal matrix, Eigen values are $\lambda = (4,4)$. Repeated Eigen values shows that independent vector will not be sufficient to diagonalize matrix B. So, use Jordan form to calculate square root. Let $J = M^{-1}AM$, then to find the matrix square root of B from J.

Step-5

Eigen vectors corresponding to the Eigen values are calculated as follows.For $\lambda = 4$

$$(B-\lambda I)x_1 = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= \begin{bmatrix} y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Step-6

Other Eigen vector corresponding to $\lambda = 4$ will be $x_2 = (0,1)$.

Step-7

Matrix M is as follows:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-8

Now, do the following calculations to get J:

$$\begin{aligned} J &= M^{-1}AM \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

Next, compute the following:

$$\begin{aligned} J^{1/2} &= \begin{bmatrix} 4^{1/2} & (1/2)4^{1-1/2} \\ 0 & 4^{1/2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1/4 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} J^{1/2} \cdot J^{1/2} &= \begin{bmatrix} 2 & 1/4 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1/4 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \\ &= A \end{aligned}$$

Therefore, $\boxed{B = \left(J^{1/2}\right)^2}$. This shows that $J^{1/2}$ is a matrix square root of *B*.