

## Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node  $j$  to node  $k$ , then that row has -1 in column  $j$  and +1 in column  $k$ .

The incidence matrix  $A$  is  $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$  and its transpose is  $A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

Let the diagonal matrix  $C = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}$

## Step-2

We need to compute  $A^T C A$ . So,

$$A^T C A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} C_1 & -C_1 & 0 \\ 0 & C_2 & -C_2 \\ C_3 & 0 & -C_3 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 + C_3 & -C_1 & -C_3 \\ -C_1 & C_1 + C_2 & -C_2 \\ -C_3 & -C_2 & C_2 + C_3 \end{bmatrix}$$

$$A^T C A = \boxed{\begin{bmatrix} C_1 + C_3 & -C_1 & -C_3 \\ -C_1 & C_1 + C_2 & -C_2 \\ -C_3 & -C_2 & C_2 + C_3 \end{bmatrix}}.$$

Therefore, the matrix

## Step-3

After removing last column of  $A$  and last row of  $A^T$ , the 2 by 2 matrix is  $B = \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix}$

We need to show that this matrix is invertible.

So,

$$B = \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix}$$

Apply  $R_2 \rightarrow (C_1 + C_3)R_2$ ,

$$= \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1^2 - C_3C_1 & C_1^2 + C_1C_2 + C_1C_3 + C_2C_3 \end{bmatrix}$$

Apply  $R_2 \rightarrow R_2 + (C_1)R_1$

$$= \begin{bmatrix} C_1 + C_3 & -C_1 \\ 0 & C_1C_2 + C_1C_3 + C_2C_3 \end{bmatrix}$$

Apply  $R_2 \rightarrow \frac{R_2}{(C_1 + C_3)}$

$$= \begin{bmatrix} C_1 + C_3 & -C_1 \\ 0 & \frac{C_1C_2 + C_1C_3 + C_2C_3}{C_1 + C_3} \end{bmatrix}$$

So, the determinant of this matrix is not equals to zero. It is non-singular.

Therefore, the matrix  $B = \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix}$  is invertible and it has  $C_1 + C_3, \frac{C_1C_3 + C_1C_2 + C_2C_3}{C_1 + C_3}$  pivots.