

Step-1

Given $A = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$ and V is the nullspace of A .

(a) We have to find a basis for V and a basis for V^\top

V is the nullspace of A

Implies by definition, $Ax = 0$

Step-2

$$\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 - x_3 = 0$$

$$\text{Put } x_2 = k_1, x_3 = k_2$$

$$\Rightarrow 3x_1 = -k_1 + k_2$$

$$\Rightarrow x_1 = -\frac{1}{3}k_1 + \frac{1}{3}k_2$$

Step-3

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}k_1 + \frac{1}{3}k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$$

$$V = \left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis of

Step-4

Basis of V^\perp

=Row space of V

$$=\text{span}\left(\begin{bmatrix} 3 & 1 & -1 \end{bmatrix}^T\right)$$

Step-5

(b) We have to write an orthonormal basis for V^\perp and find the projection matrix P_1 that projects vectors in R^3 onto V^\perp .

$$\begin{aligned} a &= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \\ \Rightarrow \|a\| &= \sqrt{9+1+1} \\ &= \sqrt{11} \end{aligned}$$

Step-6

$$\begin{aligned} \alpha &= \frac{a}{\|a\|} \\ &= \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

$$V^\perp = \left\{ \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix} \right\}$$

Orthonormal basis for

Step-7

Projection of \vec{i} onto α

$$\begin{aligned}
&= \frac{a \cdot \bar{j}}{\|a\|} \alpha \\
&= \frac{3}{\sqrt{11}} \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 9/11 \\ 3/11 \\ -3/11 \end{bmatrix}
\end{aligned}$$

Step-8

Projection of \bar{j} onto

$$\begin{aligned}
&= \frac{a \cdot \bar{j}}{\|a\|} \alpha \\
&= \frac{1}{\sqrt{11}} \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 3/11 \\ 1/11 \\ -1/11 \end{bmatrix}
\end{aligned}$$

Step-9

Projection of \bar{k} onto

$$\begin{aligned}
&= \frac{a \cdot \bar{k}}{\|a\|} \alpha \\
&= \frac{-1}{\sqrt{11}} \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -3/11 \\ -1/11 \\ 1/11 \end{bmatrix}
\end{aligned}$$

$$P_1 = \begin{bmatrix} 9/11 & 3/11 & -3/11 \\ 3/11 & 1/11 & -1/11 \\ -3/11 & -1/11 & 1/11 \end{bmatrix}$$

Projection matrix

Step-10

(c) We have to find the projection matrix P_2 that projects vectors in R^3 onto V .

$$A = \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 10/9 & -1/9 \\ -1/9 & 10/9 \end{bmatrix}$$

Step-11

$$\begin{aligned} (A^T A)^{-1} &= \frac{81}{99} \begin{bmatrix} 10/9 & 1/9 \\ 1/9 & 10/9 \end{bmatrix} \\ &= \frac{1}{11} \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix} \end{aligned}$$

Step-12

$$\begin{aligned} A(A^T A)^{-1} &= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{11} \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix} \\ &= \frac{1}{11} \begin{bmatrix} -3 & 3 \\ 10 & 1 \\ 1 & 10 \end{bmatrix} \end{aligned}$$

Step-13

$$\begin{aligned} A(A^T A)^{-1} A^T &= \frac{1}{11} \begin{bmatrix} -3 & 3 \\ 10 & 1 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{11} \begin{bmatrix} 2 & -3 & 3 \\ -3 & 10 & 1 \\ 3 & 1 & 10 \end{bmatrix} \end{aligned}$$

Hence Projection matrix

$$P_2 = \begin{bmatrix} 2/11 & -3/11 & 3/11 \\ -3/11 & 10/11 & 1/11 \\ 3/11 & 1/11 & 10/11 \end{bmatrix}$$