Step-1

(a) We need to show that $\det A = 0$, provided $a+d=0$.

Consider the following:

$$\det A = \begin{vmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{vmatrix}$$

$$= \begin{vmatrix} 2a & c & b & 0 \\ b & 0 & 0 & b \\ c & 0 & 0 & c \\ 0 & c & b & 2d \end{vmatrix}$$

$$= 2a\begin{vmatrix} 0 & 0 & b \\ 0 & 0 & c \\ c & b & 2d \end{vmatrix} - c\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & b & 2d \end{vmatrix} + b\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & c & 2d \end{vmatrix}$$

Step-2

Note the following:

$$\begin{vmatrix} 0 & 0 & b \\ 0 & 0 & c \\ c & b & 2d \end{vmatrix} = 0$$

, because the first two columns are dependent

$$\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & b & 2d \end{vmatrix} = 0$$

, because the first two rows are dependent

$$\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & c & 2d \end{vmatrix} = 0$$

, because the first two rows are dependent

Therefore, $\boxed{\det A = 0}$.

Step-3

Since, $a+d=0$, we have $d=-a$. Consider the following:

$$\begin{bmatrix} 2a & c & b & 0 \\ b & 0 & 0 & b \\ c & 0 & 0 & c \\ 0 & c & b & -2a \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2au+cv+bw \\ bu+bz \\ cu+cz \\ cv+bw-2az \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives, $u=-z$. Therefore, the first and the fourth equations are as follows:

$$2au+cv+bw=0$$
$$cv+bw+2au=0$$

Step-4

The two equations that we obtained are the same. So, when we fix *u* and *v*, the other two variables *w* and *z* get fixed automatically.

Let $u=u$ and $v=v$.

Therefore, $z=-u$ and $w=\dfrac{-2au-cv}{b}$.

Therefore, the solution of the system $Ax=b$ is given by $\boxed{x=\begin{bmatrix} u \\ v \\ \dfrac{-2au-cv}{b} \\ -u \end{bmatrix}}$.

(b) Let us show that $\det A=0$, if $ad=bc$.

We have

$$\det A = \begin{vmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{vmatrix}$$

$$= 2a\begin{vmatrix} a+d & 0 & b \\ 0 & a+d & c \\ c & b & 2d \end{vmatrix} - c\begin{vmatrix} b & 0 & b \\ c & a+d & c \\ 0 & b & 2d \end{vmatrix} + b\begin{vmatrix} b & a+d & b \\ c & 0 & c \\ 0 & c & 2d \end{vmatrix}$$

$$= 2a\left(2d(a+d)^2 - 2(a+d)bc\right) - c\left(2bd(a+d)\right) + b\left(-2cd(a+d)\right)$$

$$\begin{aligned}\det A &= 2(a+d)\{a(2d(a+d)-2bc)-c(bd)-b(cd)\}\\ &= 2(a+d)\{2a^2d+2ad^2-2abc-2bcd\}\\ &= 2(a+d)\{2a^2d+2ad^2-2abc-2bcd\}\end{aligned}$$

Step-5

We have $ad = bc$. Putting *bc* for *ad*, we get

$$\begin{aligned}\det A &= 2(a+d)\{2a^2d+2ad^2-2abc-2bcd\}\\ &= 2(a+d)\{2abc+2bcd-2abc-2bcd\}\\ &= 0\end{aligned}$$

Thus, we have shown that $\boxed{\det A = 0}$.