

Step-1

(a)

Let us consider T is a triangular matrix of order 3 by 3.

$$T = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}$$

The characteristic equation $|T - \lambda I| = 0$

This implies;

$$\begin{vmatrix} a-\lambda & x & y \\ 0 & b-\lambda & z \\ 0 & 0 & c-\lambda \end{vmatrix} = 0$$
$$(a-\lambda)(b-\lambda)(c-\lambda) = 0$$

This implies

$$\lambda_1 = a$$

$$\lambda_2 = b$$

$$\lambda_3 = c$$

Are the Eigen values of T

Step-2

Now,

$$T - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & c-a \end{bmatrix}$$
$$T - \lambda_2 I = \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c-a \end{bmatrix} \text{ and}$$
$$T - \lambda_3 I = \begin{bmatrix} a-c & 0 & 0 \\ 0 & b-c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-3

Let us consider $(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I)$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & c-a \end{bmatrix} \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c-b \end{bmatrix} \begin{bmatrix} a-c & 0 & 0 \\ 0 & b-c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (c-a)(c-b) \end{bmatrix} \begin{bmatrix} a-c & 0 & 0 \\ 0 & b-c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

Step-4

Thus, it is shown that

$$\boxed{\begin{aligned} (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) &= 0 \\ &= |T - \lambda I| \end{aligned}}$$

Therefore, the products of the factors directly satisfy the characteristic equation of the triangular matrix T

(b)

Substitute $T = U^{-1}AU$ and get;

$$T^2 = U^{-1}A^2U$$

$$T^3 = U^{-1}A^3U$$

$$(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) = 0$$

Solve and get;

$$\begin{aligned} &= T^3 - (\lambda_1 + \lambda_2 + \lambda_3)T^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)T - \lambda_1\lambda_2\lambda_3I \\ &= U^{-1}A^3U - (\lambda_1 + \lambda_2 + \lambda_3)U^{-1}A^2U + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)U^{-1}AU - \lambda_1\lambda_2\lambda_3I \\ &= U^{-1}(A^3 - (\lambda_1 + \lambda_2 + \lambda_3)A^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)A - \lambda_1\lambda_2\lambda_3I)U \\ &= U^{-1}(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I)U \end{aligned}$$

Thus, it is shown that $(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = 0$ while $U \neq 0$

Therefore, A satisfies its own characteristic equation