Step-1

Objective is to prove that every unitary matrix A is diagonalizable. For this prove the following parts:

(a)

Suppose that A, U both are unitary matrix. Show that $T = U^{-1}AU$ is also unitary.

Since A, U are unitary matrix, therefore

$$\begin{aligned} UU^H &= U^H U \\ &= I \qquad \text{......(1)} \\ AA^H &= A^H A \\ &= I \qquad \text{......(2)} \end{aligned}$$

The matrix T will be unitary if

$$\begin{aligned} TT^H &= T^H T \\ &= I \end{aligned}$$

Step-2

Consider TT^H and use the above equations and solve as follows:

$$\begin{aligned} TT^H &= (U^{-1}AU)(U^{-1}AU)^H \\ &= (U^{-1}AU)(U^H A^H (U^{-1})^H) \\ &= U^{-1}A(UU^H)A^H(U^{-1})^H \\ &= U^{-1}AA^H(U^{-1})^H \qquad [UU^H = I] \end{aligned}$$

$$\begin{aligned} TT^H &= U^{-1}(U^{-1})^H \qquad [AA^H = I] \\ &= I \end{aligned}$$

Similarly, solve the right side as:

$$\begin{aligned} T^H T &= (U^{-1}AU)^H(U^{-1}AU) \\ &= (U^H A^H (U^{-1})^H)(U^{-1}AU) \\ &= U^H A^H((U^{-1})^H U^{-1})AU \\ &= U^H A^H AU \qquad [U^H = U^{-1}] \end{aligned}$$

$$\begin{aligned} T^H T &= U^H U \\ &= I \end{aligned}$$

Hence, $T = U^{-1}AU$ is an unitary matrix.

Step-3

(b)

Prove that upper triangular unitary matrix T is diagonal.

Note that the columns of an unitary matrices are always orthonormal. Let

$T = \left[a_{ij}\right]_{n\times n}$ where $1 \le i,\ j \le n$.

Since T is upper triangular unitary matrix, therefore the absolute value of its first entry is 1. The entry $a_{12} = 0$ and $\left|a_{22}\right| = 1$.

Similarly, the entry $a_{13} = 0,\ a_{23} = 0$ and $\left|a_{33}\right| = 1$ in T, in order to make the first three columns orthonormal.

Step-4

Proceed in the same manner and observe that the matrix T is nothing but the diagonal matrix where the absolute values of the diagonal entries are 1.

Thus, upper triangular unitary matrix $T = U^{-1}AU$ is diagonal and hence every unitary matrix A is diagonalizable.