Step-1

Let A be any matrix.

We have to prove that $\|A\| = \|A^T\|$ by comparing the eigenvalues of $A^T A$ and AA^T.

Step-2

We know that if B is a square matrix, then $\|B\| = \max_{\mathbf{x}_i \neq 0} \frac{\|B\mathbf{x}_i\|}{\|\mathbf{x}_i\|}$ â€¦â€¦ (1)

And also we know that $|P| = |P^T|$ whenever P is a square matrix.

And $|PP^T| = |P||P^T|$

$= |P^T||P|$

$= |P^T P|$

Therefore, $|A^T A| = |AA^T|$

From this, we follow that $|A^T A - \lambda I| = |AA^T - \lambda I|$

(Note that the addition of $-\lambda I$ to any matrix will result the change in the diagonal entries only which will not affect the transposing on the matrix and its determinant.)

Step-3

From this discussion, we can say that the eigenvalues of $A^T A$ and those of AA^T are one and the same.

Also by (1), the eigenvalues of A and those of A^T are one and the same whose squares are the eigenvalues of $A^T A$.

So, we have $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ and $A^T \mathbf{x}_i = \lambda_i \mathbf{x}_i$ for each eigenvalue λ_i and the corresponding eigenvector $\mathbf{x}_i$.

From these equations, we can write

$$\max_{\mathbf{x}_i \neq 0} \frac{\|A\mathbf{x}_i\|}{\|\mathbf{x}_i\|} = \max_{\mathbf{x}_i \neq 0} \frac{\|A^T\mathbf{x}_i\|}{\|\mathbf{x}_i\|}$$

$$\Rightarrow \|A\| = \|A^T\| \quad \text{(Since by (1))}$$

Hence $\boxed{\|A\| = \|A^T\|}$