Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node j to node k , then that row has -1 in column j and + 1 in column k .

The incidence matrix A is,

$$A=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$$

Step-2

Let us consider the matrix $b=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$.

To show directly from the columns that every vector b in the column space will satisfy $b_1+b_2-b_3=0$

So,

Let

$$[A:b]=\begin{bmatrix}1 & -1 & 0 & b_1\\ 0 & 1 & -1 & b_2\\ 1 & 0 & -1 & b_3\end{bmatrix}$$

Apply $R_3 \to R_3 - R_1$

$$=\begin{bmatrix}1 & -1 & 0 & b_1\\ 0 & 1 & -1 & b_2\\ 0 & 1 & -1 & b_3-b_1\end{bmatrix}$$

Step-3

So,

Apply $R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

Apply $R_3 \to (-1) R_3$

$$= \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 0 & 0 & b_1 + b_2 - b_3 \end{bmatrix}$$

Therefore, the two columns are independent and every vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the column space will satisfy $b_1 + b_2 - b_3 = 0$.

Step-4

To show directly from the rows that every vector b in the row space will satisfy $b_1 + b_2 - b_3 = 0$

Since, the row space of A is equal to the column space of A^T.

If $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

Let $\begin{bmatrix} A^T & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & b_1 \\ -1 & 1 & 0 & b_2 \\ 0 & -1 & -1 & b_3 \end{bmatrix}$

Apply $R_2 \to R_2 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 + b_1 \\ 0 & -1 & -1 & b_3 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 + b_2 + b_1 \end{bmatrix}$$

Therefore, the first two columns are independent and every vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the row space will satisfy $b_1 + b_2 - b_3 = 0$.

Step-5

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then,

$$AX = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix}$$

The potential difference at classâ€™s 1, 2, and 3 are $x_1 - x_2, x_2 - x_3$ and $x_1 - x_3$.

Step-6

The space of A is $x_1 = x_2$ and $x_2 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, this means that $\boxed{\text{equal potentials a cross each close.}}$