Step-1

First we have to show that every number is an Eigen value for $Tf(x) = df/dx$.

Suppose $f(x) = e^{ax}$ for a real number *a*, so by the above definition we have:

$$\begin{aligned} Tf(x) &= \frac{df}{dx} \\ &= \frac{d}{dx}\left(e^{ax}\right) \\ &= ae^{ax} \\ &= af(x) \end{aligned}$$

Therefore, $\boxed{\text{every real number } a \text{ is an eigenvalue for } Tf(x) = df/dx}$.

Step-2

Now, we have to show that the transformation $Tf(x) = \int_0^x f(t)dt$ has no eigenvalues, here $-\infty < x < \infty$.

Step-3

Let us take $Tf(x) = af(x)$, for some real number *a* and function *f*.

Step-4

We know that $Tf(x) = \int_0^x f(t)dt$, so we have

Step-5

$$\begin{aligned} \int_0^x f(t)dt &= Tf(x) \\ &= af(x) \end{aligned}$$

Step-6

Or we can write $\int_0^x f(t)dt = af(x)$.

Step-7

Differentiate both the sides of equation $\int_0^x f(t)dt = af(x)$.

$f(x) = af'(x)$

Rewrite the above equation as

$\frac{f'(x)}{f(x)} = \frac{1}{a}$

To solve for *f*, first integrate both the sides of equation:

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{1}{a}dx$$

$$\ln|f(x)| = \frac{1}{a}(x) + C$$

$$\ln|f(x)| = \frac{x}{a} + C$$

Step-8

Take exponential of both sides of equation $\ln|f(x)| = \frac{x}{a} + C$.

$$e^{\ln|f(x)|} = e^{\frac{x}{a}+C}$$

$$|f(x)| = e^{\frac{x}{a}+C}$$

$$|f(x)| = e^C e^{\frac{x}{a}}$$

Step-9

Substitute *A* for e^C (here *A* may be a negative).

Step-10

$$f(x) = Ae^{\frac{x}{a}}$$

Step-11

Use the function $f(x) = Ae^{\frac{x}{a}}$ and definition of T, $Tf(x) = \int_0^x f(t)dt$, so

$$Tf(x) = \int_0^x Ae^{\frac{t}{a}}dt$$

By integrating, we have

$$\begin{aligned} Tf(x) &= A\int_0^x e^{\frac{a}{t}}dt \\ &= \left[Aae^{\frac{t}{a}}\right]_0^x \\ &= Aae^{\frac{x}{a}} - Aa \\ &= a\left(Ae^{\frac{x}{a}} - A\right) \end{aligned}$$

Substitute $f(x)$ for $Ae^{\frac{x}{a}}$, so

$$\begin{aligned} Tf(x) &= a\left(Ae^{\frac{x}{a}} - A\right) \\ &= a(f(x) - A) \end{aligned}$$

We know that $Tf(x) = af(x)$, so we have

$$\begin{aligned} af(x) &= a(f(x) - A) \\ &= af(x) - aA \end{aligned}$$

This implies that either $a = 0$ or $A = 0$. So we get $f(x) = 0$ in either conditions,

Therefore, we have concluded that transformation $Tf(x) = \int_0^x f(t)dt$ has $\boxed{\text{no eigenvalues}}$.