Step-1

Given that the basis for the space of all 2 by 2 matrices contains the four vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We have to find the matrix A with respect to this basis for the linear transformation of transposing.

Step-2

Let $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Here T is a linear transformation of transposing.

Then

$$\begin{aligned} T(e_1) &= T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= e_1 \\ &= 1.e_1 + 0.e_2 + 0.e_3 + 0.e_4 \end{aligned}$$

Step-3

And

$$\begin{aligned} T(e_2) &= T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= e_3 \\ &= 0.e_1 + 0.e_2 + 1.e_3 + 0.e_4 \end{aligned}$$

Step-4

And

$$T(e_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= e_2$$
$$= 0.e_1 + 1.e_2 + 0.e_3 + 0.e_4$$

Step-5

And

$$T(e_4) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= e_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= e_4$$
$$= 0.e_1 + 0.e_2 + 0.e_3 + 1.e_4$$

Therefore, the matrix of T under the basis $\{e_1, e_2, e_3, e_4\}$ is $\boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$.

Step-6

We have $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Therefore,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence $\boxed{A^2 = I}$