MA327 Homework 5

April 20, 2022

1. Determine the asymptotic curves and the lines of curvature of the helicoid $x = v \cos u, y = v \sin u, z = cu$, and show that its mean curvature is zero.

2. Determine the asymptotic curves of the catenoid

$$\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v).$$

3. Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

and show that (a) The coefficients of the first fundamental form are $E = G = (1 + u^2 + v^2)^2$ and $F = 0$. (b) The coefficients of the second fundamental form are $e = 2, g = -2$ and $f = 0$. (c) The principal curvatures are $k_1 = \frac{2}{(1+u^2+v^2)^2}$ and $k_2 = -\frac{2}{(1+u^2+v^2)^2}$. (d) The lines of curvatures are the coordinate curves. (e) The asymptotic curves are $u + v = Const.$ and $u - v = Const.$.

4. (a) Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (see Q6 in Homework 1). (b) Rotate C about the line r; determine if the "surface" of revolution thus obtained is regular and find out a parametrization in a neighborhood of a regular point. (c) Show that the Gaussian curvature of any regular point of this surface is -1. (We call this surface **pseudo-sphere**.)

5. Consider the surface obtained by rotating the curve $y = x^3, -1 < x < 1$ about the line $x = 1$. Show that the points obtained by rotation of the origin $(0, 0)$ of the curve are planar points of the surface.

6. Show that a surface which is compact (i.e. it is bounded and closed in $\mathbb{R}^3$) has an elliptic point.

7. Prove that there are no compact minimal surfaces in $\mathbb{R}^3$.

8. When two differentiable functions $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann equations

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial v}, \quad \frac{\partial f}{\partial v} = -\frac{\partial g}{\partial u},$$

they are easily seen to be harmonic; in this situation, f and g are said to be harmonic conjugate. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugate; then $\mathbf{x}$ and $\mathbf{y}$ are called conjugate minimal surfaces. Prove that

(a) The helicoid and the catenoid are conjugate minimal surfaces.

(b) Given two conjugate minimal surfaces, $\mathbf{x}$ and $\mathbf{y}$, the surface

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$$

is again minimal for all $t \in \mathbb{R}$.

(c) All surfaces of the one-parameter family in (b) have the same first fundamental form: $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{y}_v, \mathbf{y}_v \rangle, F = 0, G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \langle \mathbf{y}_u, \mathbf{y}_u \rangle$.

Thus, any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces, and the first fundamental form of this family is independent of t.