Step-1

Given $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} AB=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$ and $BA=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$

Now $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

$$\begin{aligned}A-\lambda I &= \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\\ &= \begin{bmatrix}1-\lambda & 0\\ 1 & 1-\lambda\end{bmatrix}\end{aligned}$$

Step-2

Then

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}1-\lambda & 0\\ 1 & 1-\lambda\end{vmatrix}\\ &= (1-\lambda)^2\\ &= 1+\lambda^2-2\lambda\end{aligned}$$

Step-3

Becomes

$$\begin{aligned}\lambda^2-2\lambda+1 &= 0\\ \lambda^2-\lambda-\lambda+1 &= 0\\ \lambda(\lambda-1)-1(\lambda-1) &= 0\\ (\lambda-1)(\lambda-1) &= 0\\ \lambda &= 1,1\end{aligned}$$

Therefore the eigenvalues of A is 1, 1

Step-4

Now $B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}1&1\\0&1\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}1&1\\0&1\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}$$
$$=\begin{bmatrix}1-\lambda&1\\0&1-\lambda\end{bmatrix}$$

Step-5

Then

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&1\\0&1-\lambda\end{vmatrix}$$
$$=(1-\lambda)^2$$
$$=1+\lambda^2-2\lambda$$

Step-6

Becomes

$$\lambda^2-2\lambda+1=0$$
$$\lambda^2-\lambda-\lambda+1=0$$
$$\lambda(\lambda-1)-1(\lambda-1)=0$$
$$(\lambda-1)(\lambda-1)=0$$
$$\lambda=1,1$$

Therefore the eigenvalues of B is 1, 1

Step-7

Now $AB=\begin{bmatrix}1&1\\1&2\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}1&1\\1&2\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}1&1\\1&2\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}$$
$$=\begin{bmatrix}1-\lambda&1\\1&2-\lambda\end{bmatrix}$$

Step-8

Then

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$
$$= (1-\lambda)(2-\lambda)-1$$
$$= 2-\lambda-2\lambda+\lambda^2-1$$
$$= \lambda^2-3\lambda+1$$

$$\lambda^2-3\lambda+1=0$$

This is in the form of $ax^2+bx+c=0$

Then

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Now

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$$
$$= \frac{3 \pm \sqrt{5}}{2}$$

Therefore the eigenvalues of AB is $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$

Step-9

Now $BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$A-\lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

Step-10

Now

$$
\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= (2-\lambda)(1-\lambda)-1 \\
&= 2-\lambda-2\lambda+\lambda^2-1 \\
&= \lambda^2-3\lambda+1
\end{aligned}
$$

$$\lambda^2-3\lambda+1=0$$

Step-11

This is in the form of $ax^2+bx+c=0$

Then

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Now

$$
\begin{aligned}
\lambda &= \frac{3\pm\sqrt{9-4}}{2} \\
&= \frac{3\pm\sqrt{5}}{2}
\end{aligned}
$$

Therefore the eigenvalues of BA is $\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}$

Step-12

Eigen values of AB are not equal to eigenvalues of A times Eigen values of B

Eigenvlaues of AB are equal to eigenvalues of BA