Step-1

Given that $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^3 & \omega \\ 1 & \omega^3 & \omega & \omega^2 \end{bmatrix}$

We have to show that the columns of the matrix U are the eigenvectors of P.

Step-2

We have to find the eigenvalues of P.

The characteristic equation of P is

$|P - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

Step-3

Continuation to the above

$$\Rightarrow -\lambda\left\{-\lambda\left(\lambda^2\right)\right\} - 1\left\{-1(-1)\right\} = 0$$

$$\Rightarrow \lambda^4 - 1 = 0$$
$$\Rightarrow \lambda^4 = 1$$

The λ values are 4th roots of unity they are $\lambda = 1, \omega, \omega^2, \omega^3$

Hence the eigenvalues of P are $\lambda = 1, \omega, \omega^2, \omega^3$

Step-4

We know that $x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ is an eigenvector of P if and only if $x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ is the nonzero solution of the system $(P-\lambda I)x=0$.

That is $\begin{bmatrix}-\lambda & 1 & 0 & 0\\ 0 & -\lambda & 1 & 0\\ 0 & 0 & -\lambda & 1\\ 1 & 0 & 0 & -\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0$ â€¦â€¦ (1)

Step-5

For $\lambda=1$, (1) becomes

$(P-I)x=0$

$$\begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0$$

The Augmented matrix

$$\begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 1 & 0 & 0 & -1 & 0\end{bmatrix}=0$$

Step-6

Add row 1 to row 4

$$\begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & -1 & 0\end{bmatrix}$$

Add row 2 to row 4.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

Step-7

Add row 3 to row 4

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we get the equations

$-x_1 + x_2 = 0$

$-x_2 + x_3 = 0$
$-x_3 + x_4 = 0$

Here x_4 is free variable.

Step-8

Let $x_4 = k$, where *k* is a parameter.

Then $x_3 = k, x_2 = k, x_1 = k$

Therefore,

$$x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$
$$=\begin{bmatrix}k\\k\\k\\k\end{bmatrix}$$
$$=k\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda=1$ is $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

Step-9

For $\lambda=\omega$, (1) becomes

$(P-\omega I)x=0$

$$\begin{bmatrix}-\omega & 1 & 0 & 0\\ 0 & -\omega & 1 & 0\\ 0 & 0 & -\omega & 1\\ 1 & 0 & 0 & -\omega\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0$$

The Augmented matrix

$$\begin{bmatrix}-\omega & 1 & 0 & 0 & 0\\ 0 & -\omega & 1 & 0 & 0\\ 0 & 0 & -\omega & 1 & 0\\ 1 & 0 & 0 & -\omega & 0\end{bmatrix}=0$$

Step-10

Add row 1 to ω times of row 4

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 0 & 1 & 0 & -\omega^2 & 0 \end{bmatrix} = 0$$

Add row 2 to ω times of row 4

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 0 & 0 & 1 & -\omega^3 & 0 \end{bmatrix} = 0$$

Step-11

Add row 3 to ω times of row 4

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 0 & 0 & 0 & -\omega^4+1 & 0 \end{bmatrix} = 0$$

From this, we get the equations

$$-\omega x_1 + x_2 = 0$$

$$-\omega x_2 + x_3 = 0$$
$$-\omega x_3 + x_4 = 0$$
$$\left(-\omega^4 + 1\right) x_4 = 0$$

Step-12

Choose $x_4 = \omega^3 k$, where *k* is a parameter.

Then we get $x_3 = \omega^2 k, x_2 = \omega k, x_1 = k$

Therefore,

$$x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$
$$=\begin{bmatrix}k\\\omega k\\\omega^2 k\\\omega^3 k\end{bmatrix}$$
$$=k\begin{bmatrix}1\\\omega\\\omega^2\\\omega^3\end{bmatrix}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda=\omega$ is $\begin{bmatrix}1\\\omega\\\omega^2\\\omega^3\end{bmatrix}$.

For $\lambda=\omega^2$, (1) becomes

$$\left(P-\omega^2 I\right)x=0$$

$$\begin{bmatrix}-\omega^2 & 1 & 0 & 0\\0 & -\omega^2 & 1 & 0\\0 & 0 & -\omega^2 & 1\\1 & 0 & 0 & -\omega^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0$$

The Augmented matrix

$$\begin{bmatrix}-\omega^2 & 1 & 0 & 0 & 0\\0 & -\omega^2 & 1 & 0 & 0\\0 & 0 & -\omega^2 & 1 & 0\\1 & 0 & 0 & -\omega^2 & 0\end{bmatrix}=0$$

Step-13

Add row 1 to ω^2 times of row 4

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 1 & 0 & -\omega^4 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 1 & 0 & -\omega & 0 \end{bmatrix} = 0 \qquad \left(\text{Since } \omega^3 = 1\right)$$

Step-14

Add row 2 to ω^2 times of row 4

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & 0 & -\omega^3 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} = 0$$

Step-15

Add row 3 to ω^2 times of row 4

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & 0 & -\omega^2+1 & 0 \end{bmatrix} = 0$$

From this, we get the equations

$$-\omega^2 x_1 + x_2 = 0$$

$$-\omega^2 x_2 + x_3 = 0$$

$$-\omega^2 x_3 + x_4 = 0$$

$$\left(-\omega^2 + 1\right) x_4 = 0$$

Step-16

Choose $x_4 = \omega k$, where k is a parameter.

Then we get $x_3 = \omega^3 k, x_2 = \omega^2 k, x_1 = k$

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$
$$= \begin{bmatrix} k \\ \omega^2 k \\ \omega^3 k \\ \omega k \end{bmatrix}$$
$$= k \begin{bmatrix} 1 \\ \omega^2 \\ \omega^3 \\ \omega \end{bmatrix}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = \omega^2$ is $\begin{bmatrix} 1 \\ \omega^2 \\ \omega^3 \\ \omega \end{bmatrix}$.

Step-17

For $\lambda = \omega^3$, (1) becomes

$$\left(P - \omega^3 I\right)x = 0$$

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 \\ 0 & 0 & -\omega^3 & 1 \\ 1 & 0 & 0 & -\omega^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

The Augmented matrix

$$\left[\begin{array}{cccc|c} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 1 & 0 & 0 & -\omega^3 & 0 \end{array}\right] = 0$$

Step-18

Add row 1 to ω^3 times of row 4

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 1 & 0 & -\omega^6 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} = 0 \qquad \left(\text{Since } \omega^6 = \left(\omega^3\right)^2 = 1\right)$$

Step-19

Add row 2 to ω^3 times of row 4

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 0 & 1 & -\omega^3 & 0 \end{bmatrix} = 0$$

Add row 3 to ω^3 times of row 4

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 0 & 0 & -\omega^6+1 & 0 \end{bmatrix} = 0$$

From this, we get the equations

$$-\omega^3 x_1 + x_2 = 0$$

$$-\omega^3 x_2 + x_3 = 0$$
$$-\omega^3 x_3 + x_4 = 0$$
$$\left(-\omega^6 + 1\right) x_4 = 0$$

Step-20

Choose $x_4 = \omega^2 k$, where k is a parameter.

Then we get $x_3 = \omega k, x_2 = \omega^3 k, x_1 = k$

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$
$$= \begin{bmatrix} k \\ \omega^3 k \\ \omega k \\ \omega^2 k \end{bmatrix}$$
$$= k \begin{bmatrix} 1 \\ \omega^3 \\ \omega \\ \omega^2 \end{bmatrix}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = \omega^3$ is $\begin{bmatrix} 1 \\ \omega^3 \\ \omega \\ \omega^2 \end{bmatrix}$.

Hence the columns of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^3 & \omega \\ 1 & \omega^3 & \omega & \omega^2 \end{bmatrix}$ are the eigenvectors of the given permutation matrix.