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In other way, we may write it as Markov chain Y_n given by

$$Y_n = X_1 + X_2 + \cdots + X_n,$$

where $X_1,\ X_2,\ \cdots$ are i.i.d. and

$$X_i = \begin{cases} 1 & w.p.\ \frac{1}{2} \\ -1 & w.p.\ \frac{1}{2} \end{cases}$$

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$$X^{(\delta)}(t) = \left(X_1 + X_2 + \cdots + X_{[t/\delta]}\right) \Delta x,$$

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where $[x]$ denote the largest integer below x. Denote $[t/\delta] = n$. Then, $n \to \infty$ and $n\delta \to t$.

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- $\{X^{(\delta)}(t), t \geq 0\}$ has stationary increments, namely, the distribution of $X^{(\delta)}(t+s) - X^{(\delta)}(t)$ does not depend on t.

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Proposition

$(X(t_1), \cdots, X(t_n))$ *has joint pdf*

$$\begin{aligned} & f(x_1, \cdots, x_n) \\ = \; & (2\pi)^{-n/2}(t_1(t_2 - t_1)\cdots(t_n - t_{n-1}))^{-1/2} \\ & \times \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right)\right). \end{aligned}$$

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$$g(y_1, \cdots, y_n) = \Pi_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^{n} \frac{y_i^2}{2(t_i - t_{i-1})} \right).$$

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Note that the Jacobian determinant of the transformation $y_i = x_i - x_{i-1}$ is 1, and hence, the joint pdf of $(X(t_1), \cdots, X(t_n))$ is

$$f(x) = \Pi_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right).$$

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Proof:

$$f_{s|t}(x|b) = \frac{f(x,b)}{f_t(b)}$$

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f_{s|t}(x|b) &= \frac{f(x,b)}{f_t(b)} \\
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In a bicycle race between two competitors, let $Y(t)$ denote the amount of time (in seconds) by which the racer that started in the inside position is ahead when $100t$ percent of the race has been completed,

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$$\begin{aligned} P(N(\frac{1}{2}, \frac{1}{4}) > 0) &= P(Z > \frac{0 - \frac{1}{2}}{\frac{1}{2}}) \\ &= P(Z > -1) \end{aligned}$$

(b) We want to calculate

$$P(Y(1/2) > 0 | Y(1) = \sigma).$$

Let $X(t) = Y(t)/\sigma$. Then, $X(t)$ is a standard B.M. and we want to calculate

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$\square$

7.2. Hitting Times, Maximum Variable, and the Gambler's Ruin Problem

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The case for $a < 0$ can be obtained by symmetry. $\square$

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$\square$

Finally, we consider the gambler's ruin problem.

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$$P(T_a < T_{-b}) = \frac{b}{a+b}.$$

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Taking limit, we get our conclusion. $\square$

7.3. Variations on Brownian Motion

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The first variation is the Brownian motion with a drift:

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$$X(t) = \sigma B(t) + \mu t$$

where $B(t)$ is a standard Brownian motion, and $\sigma,\ \mu$ are two constants.

Another useful process is the geometric Brownian motion:

$$Y(t) = e^{X(t)},$$

where $X(t)$ is a Brownian motion with a drift.

Proposition

For any $t > s$, we have

$$E(Y(t)|Y(u),\ u \leq s) = Y(s)\exp\left((t-s)(\mu + \frac{1}{2}\sigma^2)\right).$$

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&= e^{X(s)}E\left(e^{X(t)-X(s)}\middle|X(u),\ u \le s\right) \\
&= Y(s)E\left(e^{\sigma(B(t)-X(s))+\mu(t-s)}\right)
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&= Y(s) E\left(e^{\sigma(B(t)-X(s))+\mu(t-s)}\right) \\
&= Y(s) e^{\mu(t-s)+\frac{1}{2}\sigma^2(t-s)}.
\end{aligned}$$

$\square$

HW: Ch10, 6, 7, 9, 16, 17, 27.