

Step-1

(a)

The objective is to construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

Consider the system of equation as shown below:

$$\begin{aligned}4y + 5z &= 1 \\2z &= 2 \\x + y + z &= 3\end{aligned}$$

Write the above system of equations in matrix form:

$$A = \begin{pmatrix} 0 & 4 & 5 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array} \right]$$

Augmented matrix of the above system is

Step-2

$$\left[\begin{array}{ccc|c} 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array} \right]$$

Consider the matrix

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array} \right] &\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & 5 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \end{array} \right] \end{aligned}$$

Here, we used only two row exchanges to get the triangular matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

Now the matrix is in triangular form.

Now write the equivalent system of equations as shown below

$$3z + y + z = 3$$

$$4y + 5z = 1$$

$$2z = 2$$

Now use back substitution to find the solution.

$$2z = 2 \Rightarrow z = 1$$

Substitute $z = 1$ in $4y + 5z = 1$.

$$4y + 5(1) = 1$$

$$4y + 5 = 1$$

$$4y = 1 - 5$$

$$4y = -4$$

$$y = -1$$

Substitute $z = 1$, $y = -1$ in $3z + y + z = 3$, then

$$3z + (-1) + (1) = 3$$

$$3z = 3$$

$$z = 1$$

Step-3

(b)

Consider the system of equations as shown below:

$$x + y + 2z = 1$$

$$x + z = 2$$

$$2y + z = 3$$

Write the above system of equations in matrix form:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array} \right]$$

Augmented matrix of the above system is

Consider the matrix $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array} \right]$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{array} \right]$$

Notice that if we change any row operations, then cannot get the triangular form.

Now use row operations to get the triangular form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array} \right] \xrightarrow{R_2: R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & 1 & 3 \end{array} \right]$$

$$\xrightarrow{R_3: R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 5 \end{array} \right]$$

Step-4

Now the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ is in triangular form.

Now write the equivalent system of equations as shown below

$$\begin{aligned} x + y + 2z &= 1 \\ -y - z &= 1 \\ -z &= 5 \end{aligned}$$

Now use back substitution to find the solution.

From the third equation $z = -5$.

Substitute $z = -5$ in $-y - z = 1$.

$$\begin{aligned} -y - (-5) &= 1 \\ -y + 5 &= 1 \end{aligned}$$

$$\begin{aligned} -y &= 1 - 5 \\ -y &= -4 \\ y &= 4 \end{aligned}$$

Now substitute $z = -5$, $y = 4$ in $x + y + 2z = 1$

$$x + y + 2z = 1$$

$$x + (4) + 2(-5) = 1$$

$$x + 4 - 10 = 1$$

$$x - 6 = 1$$

$$x = 1 + 6$$

$$x = 7$$

Therefore, solution of the system of equation is $x = 7, y = 4$ and $z = -5$.