Step-1

Given $a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

We have to express the Gram-Schmidt orthogonalization of a_1, a_2 as $A = QR$, and we have to find the shapes of A, Q, R if given *n* vectors a_i with *m* components.

Step-2

$$q_1 = \frac{a_1}{\|a_1\|}$$
$$= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Step-3

$$q_2 = \frac{\beta}{\|\beta\|} \text{ where}$$
$$\beta = a_2 - \left(q_1^T a_2\right) q_1$$

Step-4

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= \frac{1+6+2}{3}$$
$$= 3$$

Step-5

$$\left(q_1^T a_2\right) q_1 = 3\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Step-6

$$\beta = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\|\beta\| = \sqrt{0+1+1}$$
$$= \sqrt{2}$$

Step-7

Therefore

$$q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Step-8

$$q_1^T a_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$= \frac{1+4+4}{3}$$
$$= 3$$

Step-9

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= \frac{1+6+2}{3}$$
$$= 3$$

$$q_2^T a_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= \frac{0+3-1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}}$$
$$= \sqrt{2}$$

Step-10

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$
$$= QR$$

Therefore $\boxed{A = QR = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}}$

Step-11

In the above problem, there are two vectors with three components, then we have A has a matrix of order 3 by 2, Q is also a matrix of order 3 by 2, and R is a matrix of order 2 by 2.

$\boxed{\text{In the same way, if there are } n \text{ vectors } a_i \text{ with } m \text{ components then } A \text{ has a matrix of order } m \text{ by } n, Q \text{ is also a matrix of order } m \text{ by } n, \text{ and } R \text{ is a matrix of order } n \text{ by } n.}$