

Step-1

Given

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A_2) &= 0 - 1 \\ &= -1 \end{aligned}$$

Step-2

And

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A_3) &= (-1) \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= (-1)(-1) + 1(1) \\ &= 2 \end{aligned}$$

Step-3

Then

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ Then}$$

$$\det(A_4) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= -(4-1)$$

$$= -3$$

Step-4

Now

$$-3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -3 \left[\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \right]$$

$$= -3[-1+1+1]$$

$$= 3$$

$$= -3[\det(A_2) + \det(A_3)]$$

Step-5

And

$$A_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Expanding by 1st row

$$\det(A_5) = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Step-6

Observe that the determinants on R.H.S. Are each formed by one interchange of rows from the previous one so we can see that

$$\det(A_5) = -4 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$\left(\begin{array}{l} \text{if we call the determinants on R.H.S as } d_1, d_2, d_3, d_4 \text{ then } d_2 = -d_1, \\ d_3 = d_1, d_4 = -d_1 \end{array} \right)$$

Step-7

On solving

$$= -4 \left[\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \right] \quad (\text{working on similar lines})$$

$$= -4 [\det(A_3) + \det(A_4)]$$

$$= -4(2 - 3)$$

$$= 4$$

So, in general we can predict $\boxed{\det(A_n) = (-1)^{n-1} (n-1)}$

(so we get $\det A_2 = -(2-1) = -1$)

$$\det(A_3) = (3-1)$$

$$= 2$$

$$\det(A_4) = -(4-1)$$

$$= -3$$

$$\det(A_5) = 5-1$$

$$= 4 \text{ etc}$$