MA327 Homework 1

1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs **clockwise** around the circle with $\alpha(0) = (0, 1)$.

2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α **closest** to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

3. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

4. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line which is the intersection of two planes $y = 0$ and $z = x$.

5. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of circumference of the disk is called a cycloid (See Figure-1 in the file "Week-1-figures").

(a) Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.

(b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

6. Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = (\sin t, \cos t + \log \tan \frac{t}{2}),$$

where t is the angle that y axis makes with the vector $\alpha(t)$. The trace of α is called the tractrix (See Figure-2 in the file "Week-1-figures" and ignore the symbol t there). Show that:

(a) α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

7. Let $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t), t \in \mathbb{R}, a$ and b constants, $a > 0, b < 0$, be a parametrized curve.

(a) Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the logarithmic spiral, see Figure-3 in the file "Week-1-figures").

(b) Show that $\alpha'(t) \to (0, 0)$ as $t \to +\infty$ and that

$$\lim_{t \to +\infty} \int_{t_0}^{t} |\alpha'(\tau)| d\tau$$

is finite; that is, α has finite arc length in $[t_0, +\infty)$.

8. Given the parametrized curve (helix)

$$\alpha(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}), \quad s \in \mathbb{R}$$

where $c^2 = a^2 + b^2$.

(a) Show that the parameter s is the arc length.

(b) Determine the curvature and the torsion of α.

(c) Determine the osculating plane of α.

(d) Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\frac{\pi}{2}$.

(e) Show that the tangent lines to α make a constant angle with the z axis.

9. Show that the torsion τ of α (regular parametrized by arc length with $k \neq 0$ at all points) is given by

$$\tau(s) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|k(s)|^2}.$$

10. A translation by a vector v in $\mathbb{R}^3$ is the map $A : \mathbb{R}^3 \to \mathbb{R}^3$ that is given by $A(p) = p + v, p \in \mathbb{R}^3$. A linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in \mathbb{R}^3$. A rigid motion in $\mathbb{R}^3$ is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

(a) Demonstrate the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

(b) Show that the vector product of two vectors in invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

(c) Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

11. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve (not necessarily by arc length) and let $\beta : J \to \mathbb{R}^3$ be a reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$. Let $t = t(s)$ be the inverse function of s and set $\frac{d\alpha}{dt} = \alpha'$, $\frac{d^2\alpha}{dt^2} = \alpha''$, etc. Prove that

(a) $\frac{dt}{ds} = \frac{1}{|\alpha'|}$, $\frac{d^2t}{ds^2} = -\frac{\alpha' \cdot \alpha''}{|\alpha'|^4}$.

(b) The curvature of α at $t \in I$ is

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

(c) The torsion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

(d) If $\alpha : I \to \mathbb{R}^2$ is a plane curve $\alpha(t) = (x(t), y(t))$, the **signed** curvature of α at t is

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}}.$$

12. Assume that $\tau(s) \neq 0$ and $k(s), k'(s) \neq 0$ for all $s \in I$ (s is the arc length). Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a **sphere** is that

$$R^2 + (R')^2 T^2 = constant,$$

where $R = \frac{1}{k}$, $T = \frac{1}{\tau}$, and R' is the derivative of R relative to s.