Step-1

a) $M = \begin{bmatrix} 1 & 0 & x_1 & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & x_j & - & 0 \\ - & - & - & - & - \\ 0 & 0 & x_n & - & 1 \end{bmatrix}$

Determinant is M is $1\begin{vmatrix} 1 & 0 & x_2 & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & x_j & - & 0 \\ - & - & - & - & - \\ 0 & 0 & x_n & - & 1 \end{vmatrix} + (-1)^{1+j} x_1 \begin{vmatrix} 0 & 1 & - & - & 0 \\ 0 & 0 & 1 & - & 0 \\ 0 & 0 & - & - & 0 \\ - & - & - & - & 1 \\ 0 & 0 & 0 & - & 0 \end{vmatrix}$ â€¦â€¦ (1)

$= 1\begin{vmatrix} 1 & 0 & x_2 & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & x_j & - & 0 \\ - & - & - & - & - \\ 0 & 0 & x_n & - & 1 \end{vmatrix} + (-1)^{1+j} x_1 (0)$

Step-2

$= 1\begin{vmatrix} 1 & 0 & x_3 & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & x_j & - & 0 \\ - & - & - & - & - \\ 0 & 0 & x_n & - & 1 \end{vmatrix} + (-1)^{2+j} x_2 \begin{vmatrix} 0 & 1 & - & - & 0 \\ 0 & 0 & 1 & - & 0 \\ 0 & 0 & - & - & 0 \\ - & - & - & - & 1 \\ 0 & 0 & 0 & - & 0 \end{vmatrix}$

Continuing this, we get the determinant M is $(-1)^{j+j} x_j = x_j$

Step-3

Further, if $x_j = 0$, it directly follows from (1) that the j th column determinant on the left which is the multiple of 1 becomes zero and so, the determinant is zero.

Step-4

b) Suppose the non homogeneous system of m linear equations in n variables is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

It can be written as the product of matrices $Ax = b$ shown to be

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ . \\ . \\ b_m \end{bmatrix}$$

Step-5

$$\begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ - \\ - \\ b_m \end{bmatrix}_{m\times 1}$$

Step-6

We consider $AM = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix} \begin{bmatrix} 1 & 0 & x_1 & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & x_j & - & 0 \\ - & - & - & - & - \\ 0 & 0 & x_n & - & 1 \end{bmatrix}$ â€¦â€¦ (2)

$$= \begin{bmatrix} a_{11} & a_{12} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & - & a_{1n} \\ a_{21} & a_{22} & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & - & a_{mn} \end{bmatrix}$$

Step-7

Observe that the j^{th} column is nothing but the given system on the left side of the equations.

So, they can be replaced by the right side values of the system as

$$= \begin{bmatrix} a_{11} & a_{12} & b_1 & - & a_{1n} \\ a_{21} & a_{22} & b_2 & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & b_m & - & a_{mn} \end{bmatrix}$$ â€¦â€¦ (3)

When the j^{th} column of the coefficient matrix is changed by the column matrix *b*, the resultant is denoted by B_j.

Step-8

(c) Considering the determinant on both sides of (2) and (3), we get

$$\text{Det} \left\{ \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix} \begin{bmatrix} 1 & 0 & x_1 & - & 0 \\ 0 & 1 & - & - & 0 \\ 0 & 0 & x_j & - & 0 \\ - & - & - & - & - \\ 0 & 0 & x_n & - & 1 \end{bmatrix} \right\} = \det \begin{bmatrix} a_{11} & a_{12} & b_1 & - & a_{1n} \\ a_{21} & a_{22} & b_2 & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & b_m & - & a_{mn} \end{bmatrix}$$

Step-9

Using the result in (a), we follow that

$$\text{Det} \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix} \times x_j = \det \begin{bmatrix} a_{11} & a_{12} & b_1 & - & a_{1n} \\ a_{21} & a_{22} & b_2 & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & b_m & - & a_{mn} \end{bmatrix}$$

Therefore, $$x_j = \frac{\det \begin{bmatrix} a_{11} & a_{12} & b_1 & - & a_{1n} \\ a_{21} & a_{22} & b_2 & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & b_m & - & a_{mn} \end{bmatrix}}{\det \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}}$$

This is nothing but the Cramerâ€™s rule.