

## Step-1

(a)

Consider the eigen values of  $A$  are 1, 1, and 2

The objective is to verify that the matrix  $A$  is invertible or not.

We have three Eigen values, so the matrix  $A$  has order of  $3 \times 3$  with these eigen values.

The product of the eigen values is the determinant of that matrix.

Product of the eigen values is  $1 \cdot 1 \cdot 2 = 2 (\neq 0)$ .

Since determinant of matrix  $A$  is 0,  $A$  is non-singular matrix.

We know that every non-singular matrix is invertible.

Hence, the matrix  $A$  with eigen values 1, 1, and 2 is invertible is **true**.

## Step-2

(b)

Consider the eigen values of  $A$  are 1, 1, and 2

The statement is "  $A$  is diagonalizable "

The objective is to verify that the above statement is true or false.

The statement is **false**

**Counter example:**

Let 
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Here,  $A$  is a triangular matrix, the eigen values are the diagonal entries 1, 1, and 2 which are as given in the question.

Find the eigen vectors corresponding to the Eigen value  $\lambda = 1$

$$\begin{aligned}
 (A - \lambda I)\mathbf{x} &= 0 \\
 (A - \lambda I)\mathbf{x} &= 0 \quad (\text{Since } \lambda=1) \\
 \left( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

The system of equations for the above matrix form is,

$$\begin{aligned}
 x_2 + x_3 &= 0 \\
 x_2 &= 0 \\
 x_3 &= 0
 \end{aligned}$$

From the above equations, we have  $x_2 = x_3 = 0$  and let  $x_1 = k$

Therefore, eigen vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Therefore, the eigen space corresponding to the eigen value is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

Since the number of elements in eigen space is 1, geometric multiplicity for the eigen value  $\lambda = 1$  is 1.

The algebraic multiplicity of eigen value  $\lambda = 1$  is 2 which is not equals to the geometric multiplicity.

Therefore, the matrix  $A$  is **not diagonalizable**.

Therefore, the statement "A is diagonalizable" is **false**.

### Step-3

c)

Consider the eigen values of  $A$  are 1, 1, and 2

The statement is "A is not diagonalizable"

The objective is to verify that the above statement is true or false.

The statement is **false**.

**Counter example:**

Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

The eigen values are 1, 1, and 2

It is a diagonal matrix already.

So, it can be written as  $S^{-1}AS = \Lambda$  where  $S$  is the identity matrix and the diagonal matrix  $\Lambda$  is  $A$  itself.

This example shows that in every matrix with eigen values 1, 1, and 2 is diagonalizable also.

Therefore, the statement "A is not diagonalizable" is **false**.