Step-1

Schwarz inequality: a, b are any vectors in $\mathbf{R}^n$, then $|a^T b| \le \|a\|\|b\|$

(a) Given that x and y are positive numbers.

$b = \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix}$ and $a = \begin{pmatrix} \sqrt{y} \\ \sqrt{x} \end{pmatrix}$

In view of Schwarz inequality, we consider $|a^T b|$

$$\left| \left(\sqrt{y}, \sqrt{x}\right) \begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix} \right| = \left|\sqrt{xy} + \sqrt{xy}\right|$$

$= 2\left|\sqrt{xy}\right|$ â€¦â€¦ (1)

Step-2

On the other hand, we consider $\|a\|\|b\| = \left\|\left(\sqrt{y}, \sqrt{x}\right)\right\| \left\|\left(\sqrt{x}, \sqrt{y}\right)\right\|$

$$= \sqrt{\left(\sqrt{y}\right)^2 + \left(\sqrt{x}\right)^2} \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2}$$

$$= \sqrt{(x+y)^2}$$

$= x + y$ â€¦â€¦ (2)

Applying Schwarz inequality on (1) and (2), we get $2\left|\sqrt{xy}\right| \le x + y$

Or, $\sqrt{xy} \le \frac{1}{2}(x+y)$

Therefore, $\text{geometric mean} \le \text{arithmetic mean}$

Step-3

(b) We consider $\|x+y\|^2$

By definition, we get $= (x+y)^T (x+y)$

$$= \left(x^T + y^T\right)(x + y)$$
$$= \left(x^T x + x^T y + y^T x + y^T y\right)$$
$$= \|x\|^2 + x^T y + y^T x + \|y\|^2$$
$$= \|x\|^2 + 2\left|x^T y\right| + \|y\|^2 \text{ in } \mathbf{R}^2$$

Using the result in (a) here, we get $\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$

$$\leq \left(\|x\| + \|y\|\right)^2$$

While norm is a non negative quantity, we apply the square root on both sides, we get

$$\|x + y\| \leq \|x\| + \|y\|$$

This is the required triangular inequality.