Step-1

To find the limits as $k \to \infty$ for the following matrices:

(a) Consider the following matrix:

$$A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$$
$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then to find the limit for $A^k u_0$.

Step-2

To find the Eigen values calculate the following:

$$\det(A - \lambda I) = 0$$
$$\begin{bmatrix} 0.4-\lambda & 0.2 \\ 0.6 & 0.8-\lambda \end{bmatrix} = 0$$
$$5\lambda^2 - 6\lambda + 1 = 0$$
$$(5\lambda - 1)(\lambda - 1) = 0$$

After solving following values are obtained:

$$\lambda = 1$$
$$\lambda = \frac{1}{5}$$

Step-3

Calculate Eigen vectors for $\lambda = 1$:

$$(A-1I)x=0$$

$$\begin{bmatrix}0.4-1 & 0.2\\ 0.6 & 0.8-1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}\frac{-3}{5} & \frac{1}{5}\\ \frac{3}{5} & \frac{-1}{5}\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}$$

Step-4

Similarly, Eigen vectors for $\lambda=\frac{1}{5}$ are as follows:

$$\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

Therefore, Eigen vector matrix is as follows:

$$S=\begin{bmatrix}\frac{1}{3} & -1\\ 1 & 1\end{bmatrix}$$

$$S^{-1}=\begin{bmatrix}\frac{3}{4} & \frac{3}{4}\\ \frac{-3}{4} & \frac{1}{4}\end{bmatrix}$$

Step-5

Difference equation can be written as follows:

$$\begin{aligned}u_{k+1}&=A^k u_0\\ &=S\Lambda^k S^{-1}\cdot u_0\end{aligned}$$

Now, calculate the following:

$$\begin{aligned}u_{k+1}&=S\Lambda^k S^{-1}\cdot u_0\\ &=\begin{bmatrix}\frac{1}{3} & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1^k & 0\\ 0 & \left(\frac{1}{5}\right)^k\end{bmatrix}\begin{bmatrix}\frac{3}{4} & \frac{3}{4}\\ \frac{-3}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\end{aligned}$$

Step-6

After taking the limit $k \to \infty$ makes the element $(1/5)^k$ very small, so neglect it.

$$
\begin{aligned}
u_\infty &= \begin{bmatrix} \frac{1}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}
\end{aligned}
$$

Step-7

Therefore, the limit of $A^k u_0$ is as follows:

$$\boxed{\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}}$$

Step-8

(b) Consider the following matrix:

$$A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then to find the limit for $A^k u_0$.

Step-9

As the matrix A is same as in part (a) every step will be same only last step will change as value of u_0 is change. So, calculate the last step:

$$
\begin{aligned}
u_\infty &= \begin{bmatrix} \frac{1}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}
\end{aligned}
$$

Step-10

Therefore, the limit of $A^k u_0$ is as follows:

$$\boxed{\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}}$$

Step-11

(c) Consider the following matrix:

$$A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$$

Then to find the limit for A^k.

As the matrix A is same as in part (a) every step will be same only the last step will change. So, calculate the last step:

$$u_\infty = \begin{bmatrix} \frac{1}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}$$

Step-12

Therefore, the limit of A^k is as follows:

$$\boxed{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}}$$