Step-1

Let U be a complex matrix.

A complex matrix U is unitary if $UU^{*} = I$,

Where U^{*} is transpose of complex conjugate of U.

Clearly it can be written as $U\overline{U}^{T} = I$

Take the determinant on both sides then,

$$\left|U\overline{U}^{T}\right| = |I|$$

But $\left|U\overline{U}^{T}\right| = \left|\overline{U}^{T}\right||U|$

Thus, $\left|U\overline{U}^{T}\right| = |I|$ can be written as,

$$\left|\overline{U}^{T}\right||U| = 1$$

$\det\left(\overline{U}\right)\det U = 1$ Â Â Since U is a unitary matrix that is $\left|\overline{U}^{T}\right| = \left|\overline{U}\right|$

Step-2

By known condition, the determinant of conjugate of the matrix is equal to conjugate of determent of that matrix.

That is,

$$\det\overline{U} = \overline{\det(U)}$$

Thus,

$$\det\left(\overline{U}\right)\det U = 1$$

$$\overline{\det(U)}\det U = 1$$

$|\det U|^{2} = 1$ Â Â Â Â Â Â Since $\overline{\det(U)}\det(U) = |\det U|^{2}$ Â Â Â Â

$\boxed{|\det U| = 1}$. Â Â Â Â Â Â Â Â Â Â

Step-3

Suppose $U = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Then U^H = transpose of conjugatematrix of U

$$U^H = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

So $\det U = i$ and $\det U^H = -i$

Therefore, clearly $\det U \neq \det U^H$.

Step-4

Suppose $U = \begin{bmatrix} r_1 e^{i\theta_1} & r_3 e^{i\theta_3} \\ r_2 e^{i\theta_2} & r_4 e^{i\theta_4} \end{bmatrix}$ is unitary.

Then U has orthonormal columns,

So $r_1^2 + r_2^2 = 1$ and $r_3^2 + r_4^2 = 1$.

Let $r_1 = \sin\theta_1$ then $r_2 = \cos\theta_1$, and if $r_3 = \cos\theta_2$ then $r_4 = \sin\theta_2$

Where $0 \le \theta_1, \theta_2 \le \dfrac{\pi}{2}$.

By orthogonality of the column vectors,

$$\sin\theta_1 \cos\theta_2 e^{i(\theta_1+\theta_3)} + \cos\theta_1 \sin\theta_2 e^{i(\theta_2+\theta_4)} = 0.$$

This implies that $\theta_2 = \theta_1$ and $e^{\theta_1+\theta_3} = -e^{\theta_2+\theta_4}$

$$\theta_1 + \theta_3 = \theta_2 + \theta_4 + \pi.$$

So,

$U = \begin{bmatrix} \sin\theta e^{i\theta_1} & \cos\theta e^{i\theta_3} \\ \cos\theta e^{i\theta_2} & \sin\theta e^{i(\theta_1+\theta_3-\theta_2-\pi)} \end{bmatrix}$ for some $0 \le \theta \le \dfrac{\pi}{2}$.