Solutions for assignment 1

September 9, 2022

1 **(1)** $x \in B \setminus A \iff x \in B$ and $x \notin A \iff x \in B$ and $x \in A^c \iff x \in B \cap A^c$.

(2) $(A \setminus B) \cap C = A \cap B^c \cap C = (A \cap C) \setminus B = (A \cap C) \setminus (B \cap C)$.

(3) $x \in (\cup_{k=1}^{\infty} A_k)^c \iff x \notin \cup_{k=1}^{\infty} A_k \iff x \notin A_k, \forall k$, where $k \in \mathbb{Z} \iff x \in A_k^c, \forall k$ where $k \in \mathbb{Z}$

$\iff x \in \cap_{k=1}^{\infty} A_k^c$.

(4) $x \in (\cap_{k=1}^{\infty} A_k)^c \iff x \notin \cap_{k=1}^{\infty} A_k \iff \exists k_0 \in \mathbb{Z}, x \notin A_{k_0} \iff \exists k_0 \in \mathbb{Z}, x \in A_{k_0}^c$

$\iff x \in \cup_{k=1}^{\infty} A_k^c$.

(5) $x \in A \cup (\cap_{k=1}^{\infty} B_k) \iff x \in A$ or $x \in \cap_{k=1}^{\infty} B_k \iff x \in A$ or $\forall k, x \in B_k$, $k \in \mathbb{Z}_+$.

$\iff \forall k \in \mathbb{Z}_+, x \in A \cup B_k \iff x \in \cap_{k=1}^{\infty} (A \cup B_k)$.

(6) $x \in A \cap (\cup_{k=1}^{\infty} B_k) \iff x \in A$ and $x \in \cup_{k=1}^{\infty} B_k \iff \exists k_0 \in \mathbb{Z}_+, x \in A \cap B_{k_0}$

$\iff x \in \cup_{k=1}^{\infty} (A \cap B_k)$.

We only give a brief proof for problem **(7)** since others are similar and easy to check.

(7) Let $D = (\cup_{i \in I} A_i)^c$, $C = \cap_{i \in I} (A_i)^c$, we want to show that $D \subseteq C$, and $C \subseteq D$. Suppose $x \in D$, then $x \notin \cup_{i \in I} A_i$, namely $\forall i \in I, x \notin A_i$, i.e. $\forall i \in I, x \in (A_i)^c$, so $x \in \cap_{i \in I} (A_i)^c$, therefore $D \subseteq C$. Conversely, assume $x \in C$, i.e. $x \in \cap_{i \in I} (A_i)^c$, so $\forall i \in I, x \notin A_i$, hence $x \notin \cup_{i \in I} A_i$, it follows that $x \in (\cup_{i \in I} A_i)^c$, hence $C \subseteq D$.

2 Obviously.

3 Note that $A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n); \forall i = 1, 2, ..., n, a_i \in A_i\}$, let $B_1 = \{(a_1, a_2, ..., a_n);$ $a_1 \in A_1, a_2, ..., a_n \quad fixed\}$, clearly B_1 is countable, let $B_2 = \{(a_1, a_2, ..., a_n); a_1 \in A_1,$ $a_2 \in A_2, others\, fixed\}$, clearly B_2 is countable, by induction, $A_1 \times A_2 \times ... \times A_n$ is countable.

4 Simple, $A \subset B \subset C$, then $Card(A) \leq Card(B) \leq Card(C)$. Hence, $Card(A) = Card(C)$ implies $Card(A) = Card(B) = Card(C)$.

5 Now, suppose $[0,1]$ is countable, then it can be written as a sequence $\{x_1, x_2, x_3, \cdots\}$ say. Assume

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\cdots a_{1n}\cdots$$
$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\cdots a_{2n}\cdots$$
$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\cdots a_{3n}\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = 0.a_{n1}a_{n2}a_{n3}a_{n4}\cdots a_{nn}\cdots$$

(remember all of the numbers in [0,1] have been listed above, where a_{ij} are all one of the numbers

$\{0,1,2,3,4,5,6,7,8,9\}$. Now we define a number, x^*, say, as

$$x^* = 0.a_{*1}a_{*2}a_{*3}\cdots a_{*n}\cdots,$$

where $a_{*1} \neq a_{11}, a_{*2} \neq a_{22}, \ldots, a_{*n} \neq a_{nn}, \cdots$ and all $a_{*k}, k \geq 1$, take value in $\{0,1,2,\cdots,9\}$.

Surely $x^* \in [0,1]$, but x^* is not be listed. (since it equals neither of the x_n !) This is a contradiction!

6 $card((0,1)) = card(R)$, define $f : R \to (0,1)$, $f(x) = \frac{1}{\pi}arctanx + \frac{1}{2}$, then $f(x)$ is bijective.

7 (1) Clearly.

7 (2) For any $k \geqslant 1, \bigcup_{n=1}^{k} B_n = A_k$.(This is also clear). In fact, since $B_1 = A_1$ and $B_2 = A_2 \backslash A_1 = A_2 \backslash B_1$, and thus $B_1 \cup B_2 = A_2$. Now $B_3 = A_3 \backslash A_2$ i.e. $B_3 = A_3 \backslash (B_1 \cup B_2)$ and so $B_1 \cup B_2 \cup B_3 = A_3$, Assume, $\bigcup_{n=1}^{k} B_n = A_k$ is true for k, we show that , it is also true for $k+1$. Indeed, $B_{k+1} = A_{k+1} \backslash A_k$. i.e. $B_{k+1} = A_{k+1} \backslash \left(\bigcup_{n=1}^{k} B_n\right)$ and thus $\left(\bigcup_{n=1}^{k} B_n\right) \cup B_{k+1} = A_{k+1}$ That is that $\bigcup_{n=1}^{k+1} B_n = A_{k+1}$.

7 (3) Show that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

Proof. Since for any $n = 1, 2\ldots$ we have $B_n \subset A_n$ and thus $\bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$. Hence, we only need to show that $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$. To show this, assume that $x \in \bigcup_{n=1}^{\infty} A_n$. Then there must exist some k, such that $x \in A_k$, WLOG, assume that $x \notin A_1, x \in A_2, \cdots, x \notin A_{k-1}, x \in A_k$ (clearly such kind of k can bo found), in other words, $x \notin A_{k-1}$ and $x \in A_k$. Therefore $X \in A_k \backslash A_{k-1} = B_k$.Hene $x \in \bigcup_{n=1}^{\infty} B_n$. That is $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$. Hence, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

8 S is not countable. Actually, let $A = \{0,1\}$, then $S = A \times A \times A \times ...$, hence S is not countable.