Step-1

Given that $A=\begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}$

To find the Eigen value of A

To find λ value take $|A-\lambda I|=0$

This implies;

$$\begin{vmatrix} .6-\lambda & .4 \\ .4 & .6-\lambda \end{vmatrix}=0$$

This implies;

$$\begin{aligned} (.6-\lambda)^2-.16&=0 \\ \lambda^2-1.2\lambda+.36-.16&=0 \\ \lambda^2-\frac{12}{10}\lambda+\frac{2}{10}&=0 \\ 10\lambda^2-12\lambda+2&=0 \end{aligned}$$

This implies;

$$\begin{aligned} 10\lambda^2-10\lambda-2\lambda+2&=0 \\ 10\lambda(\lambda-1)-2(\lambda-1)&=0 \\ (10\lambda-2)(\lambda-1)&=0 \\ \lambda&=\frac{1}{5},1 \end{aligned}$$

Step-2

Now, to find Eigen vector corresponding to Eigen value $\lambda=\frac{1}{5}$

Take $\left(A-\frac{1}{5}I\right)x=0$

$$\begin{bmatrix} \frac{6}{10}-\frac{1}{5} & \frac{4}{10} \\ \frac{4}{10} & \frac{6}{10}-\frac{1}{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply the row operations on the coefficient matrix,

$$R_2 \to R_2 - R_1, R_1/0.4$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the reduced matrix, so, the homogeneous equation from this is $x_1 + x_2 = 0$

Put $x_1 = 1$, the solution set is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the Eigen vector corresponding to the Eigen value $\lambda = \frac{1}{5}$

Step-3

Similarly, when $\lambda = 1$, $(A-\lambda I)x = 0$ is $\begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, proceeding as above, the Eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

While the Eigen values are distinct, the corresponding Eigen vectors are linearly independent and so, matrix *S* whose columns are Eigen vectors is non singular and thus $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now, $A = S\Lambda S^{-1}$ Where $\Lambda = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}$ whose diagonal entries are the eigen values of *A*.

So, $A^k = S\Lambda^k S^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.2^k & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

As *k* approaches ∞, obtain $A^k = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This shows that A^k $\boxed{\text{does not}}$ approach 0 as k approaches ∞

Step-4

Now, consider another matrix

$$B = \begin{bmatrix} .6 & .9 \\ .9 & .6 \end{bmatrix}$$

The characteristic equation is $(\lambda - 0.6)^2 - 0.09 = 0$

This implies;

$$\begin{aligned} \lambda^2 - 1.2\lambda + 0.27 &= 0 \\ \lambda^2 - 0.9\lambda - 0.3\lambda + 0.27 &= 0 \\ (\lambda - 0.9)(\lambda - 0.3) &= 0 \end{aligned}$$

This implies;

$$\lambda_1 = 0.3$$
$$\lambda_2 = 0.9$$

Step-5

The Eigen vector corresponding to $\lambda_1 = 0.3$ is obtained by solving $(B - \lambda_1 I)x = 0$

This implies;

$$\begin{bmatrix} 0.3 & 0.9 \\ 0.1 & 0.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After the application of row operations on the coefficient matrix, it reduces to $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, the Eigen vector as above is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ by putting $x_2 = -1$

Step-6

Similarly, the Eigen vector corresponding to $\lambda_2 = 0.9$ is obtained by solving $(B - \lambda_2 I)x = 0$

This implies;

$$\begin{bmatrix} -0.3 & 0.9 \\ 0.1 & -0.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, the solution $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda_2 = 0.9$

Step-7

So, $B = S\Lambda S^{-1}$

Where

$$S = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.9 \end{bmatrix}$$
$$S^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix}$$

Applying the k^{th} power on both sides, it becomes $B^k = S\begin{bmatrix} (0.3)^k & 0 \\ 0 & (0.9)^k \end{bmatrix}S^{-1}$

Allowing k approach infinity, it becomes

$$= S\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}S^{-1}$$
$$= 0$$

This shows that $\boxed{B^k \to 0}$ as k approaches ∞

Step-8

Now, take up the actual question in view of the above results, that $A^k = S\Lambda^k S^{-1}$ approaches zero as $k \to \infty$ if and only if every λ has absolute value less than 1.