MA327 Homework 4

Due on 21th April

1. Let $P = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ be the xy-plane and let $\mathbf{x} : U \to P$ be a parametrization of P given by

$$\mathbf{x}(\rho, \theta) = (\rho \cos\theta, \rho \sin\theta),$$

where $U = \{(\rho, \theta) \in \mathbb{R}^2 \mid \rho > 0, 0 < \theta < 2\pi\}$. Compute the coefficients of the first fundamental form of P in this parametrization.

2. (Gradient on Surfaces) The gradient of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map $\nabla f : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\nabla f(p) \in T_p(S) \subset \mathbb{R}^3$ such that $\langle \nabla f(p), v\rangle_p = df_p(v)$ for all $v \in T_p(S)$. Show that:

(a) If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, then ∇f on $\mathbf{x}(U)$ is given by

$$\nabla f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v.$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y, $\nabla f = f_x e_1 + f_y e_2$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$.

(b) If you let $p \in S$ be fixed and v vary in the unit circle $|v| = 1$ in $T_p(S)$, then $df_p(v)$ is maximum if and only if $v = \frac{\nabla f}{|\nabla f|}$.

(c) If $\nabla f \neq 0$ at all points of the level curve $C := \{q \in S \mid f(q) = Const.\}$, then C is a regular curve on S and ∇f is normal to C at all points of C.

3. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

4. Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$k \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p.

5. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

7. Prove that (a) The image $N \circ \alpha$ by the Gauss map $N : S \to S^2$ of a parametrized regular curve $\alpha : I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the sphere S^2 (called the spherical image of α). (b) If $C = \alpha(I)$ is a line of curvature, and k is its curvature at p, then $k = |k_n \cdot k_N|$, where k_n is the normal curvature at p along the tangent line of C and k_N is the curvature of the spherical image $N(C) \subset S^2$ at $N(p)$.

8. Show that the meridians of a torus are lines of curvature.

9. Show that if the mean curvature is zero at a nonplanar point, then this point has two orthogonal asymptotic directions.

10. Show that if $H \equiv 0$ on S and S has no planar point, then the Gauss map $N : S \to S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2) \rangle = -K(p)\langle w_1, w_2 \rangle$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$. Show that the above condition implies that the angle of two intersecting curves on S^2 and the angle of their spherical images (see Exercise 7) are equal up to a sign.

11. Let $\lambda_1, \cdots, \lambda_m$ be the normal curvature at $p \in S$ along directions making angles $0, 2\pi/m, \cdots, (m-1)2\pi/m$ with a principal direction, $m > 2$. Prove that

$$\lambda_1 + \cdots + \lambda_m = mH,$$

where H is the mean curvature at p.

12. Show that at the origin $(0, 0, 0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.