## Step-1

Given that $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The objective is to solve $Ax = b$ by solving the triangular systems $Lc = b$ and $Ux = c$.

Let $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

We have $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

## Step-2

Solve the equation $Lc = b$.

$$Lc = b$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} c_1 \\ 4c_1 + c_2 \\ c_1 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From this, we get $c_1 = 0,\ 4c_1 + + c_2 = 0,\ c_1 + c_3 = 1$

Solving these equations, we get $c_1 = 0,\ c_2 = 0,\ c_3 = 1$

Therefore, $c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

## Step-3

Let $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Solve the equation $Ux=c$

$$\begin{aligned}Ux&=c\\ \begin{bmatrix}2&2&4\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}&=\begin{bmatrix}0\\0\\1\end{bmatrix}\\ \begin{bmatrix}2x_1+2x_2+4x_3\\x_2+3x_3\\x_3\end{bmatrix}&=\begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}$$

From this, we get $2x_1+2x_2+4x_3=0,\ x_2+3x_3=0$ and $x_3=1$

Substitute $x_3=1$ in $x_2+3x_3=0$, then

$x_2+3(1)=0\Rightarrow x_2=-3$

Substitute $x_3=1$ and $x_2=-3$ in $2x_1+2x_2+4x_3=0$

$$\begin{aligned}2x_1+2(-3)+4(1)&=0\\2x_1-6+4&=0\\2x_1-2&=0\\x_1&=1\end{aligned}$$

Therefore, $x=\begin{bmatrix}1\\-3\\1\end{bmatrix}$

Hence, the solution to the given system is $\boxed{x=\begin{bmatrix}1\\-3\\1\end{bmatrix}}$.

## Step-4

Notice that we have $b=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Since the last entry of $b$ is non-zero, we can find last column of the matrix $A^{-1}$ for this vector $b$.

This implies that,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & 2 & 4 \\ 8 & 9 & 19 \\ 2 & 2 & 5 \end{bmatrix}$$

Find the inverse of $A$

That is $$A^{-1} = \begin{pmatrix} \frac{7}{2} & -1 & 1 \\ -1 & 1 & -3 \\ -1 & 0 & 1 \end{pmatrix}$$

Notice that last column of $A^{-1}$ is same as the value of the $x = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$.

Therefore, **las column of** $A^{-1}$ has found with this particular $b$.