Step-1

Consider the equation, $3u^2 - 2\sqrt{2}uv + 2v^2 = 1$.

The objective is to reduce the equation to a sum of squares by finding the eigenvalues of the corresponding A, and sketch the ellipse.

Step-2

Consider the equation,

$$3u^2 - 2\sqrt{2}uv + 2v^2 = 1$$

$$\begin{bmatrix} u & v \end{bmatrix}\begin{bmatrix} 3u - \sqrt{2}v \\ -\sqrt{2}u + 2v \end{bmatrix} = 1$$

$$\begin{bmatrix} u & v \end{bmatrix}\begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 1$$

$\begin{bmatrix} u & v \end{bmatrix} A \begin{bmatrix} u \\ v \end{bmatrix} = 1$ Here $A = \begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$

Step-3

Compute eigenvalues of matrix A.

Eigenvalues of matrix A are roots of the equation $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 3-\lambda & -\sqrt{2} \\ -\sqrt{2} & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) - \left(-\sqrt{2}\right)\left(-\sqrt{2}\right) = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 5\lambda + 6 - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda^2 - \lambda - 4\lambda + 4 = 0$$

$$\lambda(\lambda - 1) - 4(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, 4$$

Therefore, eigenvalues of matrix A are $\lambda_1 = 1, \lambda_2 = 4$.

Step-4

Let $\mathbf{x}=(x_1,x_2)^{\mathrm{T}}$ be the eigenvector corresponding to $\lambda_1=1$. Then,

$$\begin{aligned}&(A-1\cdot I)\mathbf{x}=\mathbf{0}\\ &\begin{pmatrix}3-1 & -\sqrt{2}\\ -\sqrt{2} & 2-1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\\ &\underbrace{\begin{pmatrix}2 & -\sqrt{2}\\ -\sqrt{2} & 1\end{pmatrix}}_{\mathbf{M}}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\underbrace{\begin{pmatrix}0\\ 0\end{pmatrix}}_{\mathbf{0}}\end{aligned}$$

Augmented matrix associated with the above one is,

$$\begin{aligned}[\mathbf{M}\ |\ \mathbf{0}]&=\left[\begin{array}{cc|c}2 & -\sqrt{2} & 0\\ -\sqrt{2} & 1 & 0\end{array}\right]\\ &R_2\to\sqrt{2}R_2+R_1\\ &\approx\left[\begin{array}{cc|c}2 & -\sqrt{2} & 0\\ 0 & 0 & 0\end{array}\right]\\ &R_1\to-\frac{1}{\sqrt{2}}R_1\\ &\approx\left[\begin{array}{cc|c}-\sqrt{2} & 1 & 0\\ 0 & 0 & 0\end{array}\right]\end{aligned}$$

From the last matrix we have the system,

$$-\sqrt{2}x_1+x_2=0$$

Step-5

Observe that there are two variables (x_1,x_2) and one equation. So, there must be $2-1=1$ variable as free variable.

Suppose that x_1 be the free variable. That is, $x_1=r,\ \ r\in\mathbb{R}$.

From the equation $-\sqrt{2}x_1+x_2=0$ get $x_2=\sqrt{2}r$

So, eigenvector corresponding to the eigenvalue $\lambda_1=1$ is,

$$\mathbf{x} = \left\{ \left(r, \sqrt{2} r \right)^{\mathrm{T}} : \ r \in \mathbb{R} \right\}$$
$$= \left\{ r \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} : r \in \mathbb{R} \right\}$$

Step-6

Unit eigenvector corresponding to eigenvalue $\lambda_1 = 1$ is,

$$\mathbf{X} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$
$$= \frac{\left(1, \sqrt{2}\right)}{\sqrt{1^2 + \left(\sqrt{2}\right)^2}}$$
$$= \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \right)$$

Step-7

Let $\mathbf{y} = \left(y_1, y_2 \right)^{\mathrm{T}}$ be the eigenvector corresponding to $\lambda_2 = 4$. Then,

$$\left(A - 4 \cdot I \right) \mathbf{y} = \mathbf{0}$$
$$\begin{pmatrix} 3-4 & -\sqrt{2} \\ -\sqrt{2} & 2-4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\underbrace{\begin{pmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix}}_{\mathbf{N}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{0}}$$

Augmented matrix associated with the above one is,

$$[\mathbf{N} \mid \mathbf{0}] = \left[\begin{array}{cc|c} -1 & -\sqrt{2} & 0 \\ -\sqrt{2} & -2 & 0 \end{array} \right]$$
$$R_2 \to R_2 - \sqrt{2} R_1$$
$$\approx \left[\begin{array}{cc|c} -1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array} \right]$$

From the last matrix we have the system,

$$-y_1 - \sqrt{2} y_2 = 0$$

Step-8

Observe that there are two variables (y_1, y_2) and one equation. So, there must be $2-1=1$ variable as free variable.

Suppose that y_2 be the free variable. That is, $y_2 = s, \quad s \in \mathbb{R}$.

From the equation $-y_1 - \sqrt{2}y_2 = 0$ get $y_1 = -\sqrt{2}s$

So, eigenvector corresponding to the eigenvalue $\lambda_2 = 4$ is,

$$\begin{aligned}\mathbf{y} &= \left\{\left(-\sqrt{2}s, s\right)^{\mathrm{T}} : \; s \in \mathbb{R}\right\} \\ &= \left\{ s\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} : s \in \mathbb{R}\right\}\end{aligned}$$

Unit eigenvector corresponding to eigenvalue $\lambda_2 = 4$ is,

$$\begin{aligned}\mathbf{Y} &= \frac{\mathbf{y}}{\|\mathbf{y}\|} \\ &= \frac{\left(-\sqrt{2}, 1\right)}{\sqrt{\left(-\sqrt{2}\right)^2 + 1^2}} \\ &= \left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\end{aligned}$$

Step-9

Eigenvalues and the corresponding unit eigenvectors of the matrix A are,

$\lambda_1 = 1$, $\mathbf{X} = \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$

$\lambda_2 = 4$, $\mathbf{Y} = \left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Therefore, given equation can be written as the sum of squares in the following way.

$$3u^2 - 2\sqrt{2}uv + 2v^2 = \left(\frac{u}{\sqrt{3}} + \frac{\sqrt{2}v}{\sqrt{3}}\right)^2 + 4\left(-\frac{\sqrt{2}u}{\sqrt{3}} + \frac{v}{\sqrt{3}}\right)^2$$

Here $\lambda = 1$ and $\lambda = 4$ are outside the squares. The eigenvectors are inside.

Step-10

Sketch of the ellipse $3u^2 - 2\sqrt{2}uv + 2v^2 = 1$ is shown below.