Step-1

Given that $3x = 10, 4x = 5$

We have to find the least-squares solution $\hat{x}$ to $3x = 10, 4x = 5$.

Step-2

We know that the least-squares solution to a problem $ax = b$ is $\hat{x} = \frac{a^T b}{a^T a}$.

We can write $3x = 10, 4x = 5$ as $\begin{pmatrix} 3 \\ 4 \end{pmatrix} x = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$.

Take $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, b = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$

Now the least-squares solution to $3x = 10, 4x = 5$ is

$$\begin{aligned} a^T b &= \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 10 \\ 5 \end{pmatrix} \\ &= 3(10) + 4(5) \\ &= 30 + 20 \\ &= 50 \end{aligned}$$

Step-3

And

$$\begin{aligned} a^T a &= \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= 3(3) + 4(4) \\ &= 9 + 16 \\ &= 25 \end{aligned}$$

Step-4

Therefore,

$$\hat{x} = \frac{a^T b}{a^T a}$$
$$= \frac{50}{25}$$
$$= 2$$

Hence the least-square solution to $3x = 10, 4x = 5$ is $\boxed{\hat{x} = 2}$.

Step-5

We have to the error E^2 that is minimized.

Since $ax = b$ by minimizing $E^2 = \|ax - b\|^2$

$$\Rightarrow E^2 = (a_1 x - b_1)^2 + \dots\dots\dots + (a_m x - b_m)^2$$

Therefore $E^2 = (10 - 3x)^2 + (5 - 4x)^2$ is minimized.

Step-6

Let e = error vector

Then

$$e = (10 - 3\hat{x},\ 5 - 4\hat{x})$$
$$= (10 - 3(2),\ 5 - 4(2))$$
$$= (4, -3)$$

Step-7

Now

$$e^T a = (4, -3)\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
$$= 4(3) + (-3)4$$
$$= 12 - 12$$
$$= 0$$

Hence error vector $e = (10 - 3\hat{x},\ 5 - 4\hat{x})$ is perpendicular to $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.