SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 12 Solutions

1. The covariance between X and Y, denoted by $Cov(X, Y)$, is defined by

$$Cov(X, Y) = \mathrm{E}[(\mathrm{X} - \mathrm{E}(\mathrm{X}))(\mathrm{Y} - \mathrm{E}(\mathrm{Y}))].$$

Show that

$$Cov(X, Y) = \mathrm{E}[\mathrm{XY}] - \mathrm{E}[\mathrm{X}]\mathrm{E}[\mathrm{Y}].$$

Proof:

$$\begin{aligned} Cov(X, Y) &= \mathrm{E}[(\mathrm{X} - \mathrm{E}(\mathrm{X}))(\mathrm{Y} - \mathrm{E}(\mathrm{Y}))] \\ &= \mathrm{E}[(\mathrm{XY} - \mathrm{E}(\mathrm{X})\mathrm{Y} - \mathrm{XE}(\mathrm{Y}) + \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y}))] \\ &= \mathrm{E}[\mathrm{XY}] - \mathrm{E}[\mathrm{E}(\mathrm{X})\mathrm{Y}] - \mathrm{E}[\mathrm{XE}(\mathrm{Y})] + \mathrm{E}[\mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y})] \\ &= \mathrm{E}[\mathrm{XY}] - \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y}) - \mathrm{E}(\mathrm{Y})\mathrm{E}(\mathrm{X}) + \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y}) \\ &= \mathrm{E}[\mathrm{XY}] - \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y}). \end{aligned}$$

2. Let X be a discrete random variable with p.m.f as

$$\mathrm{P}\{X = 0\} = \mathrm{P}\{X = 1\} = \mathrm{P}\{X = -1\} = \frac{1}{3}.$$

Define

$$Y = \begin{cases} 0 & \text{if} \quad X \neq 0, \\ 1 & \text{if} \quad X = 0. \end{cases}$$

(i) Show that $Cov(X, Y) = 0$.

(ii) Write down the joint p.m.f of X and Y, and show that X and Y are not independent.

Solution:

(i) Firstly, notice that $XY \equiv 0$.
In fact, if $X(\omega) = 0$, then $Y(\omega) = 1$, hence

$$XY(\omega) = X(\omega) \cdot Y(\omega) = 0 \times 1 = 0;$$

If $X(\omega) \neq 0$, then $Y(\omega) = 0$, hence

$$XY(\omega) = X(\omega) \cdot Y(\omega) = X(\omega) \times 0 = 0;$$

In sum up, we get $XY \equiv 0$. So, $\mathrm{E}[\mathrm{XY}] = \mathrm{E}[0] = 0$.

$$\begin{aligned} \mathrm{E}(\mathrm{X}) &= -1 \times \mathrm{P}(X = -1) + -0 \times \mathrm{P}(X = 0) + 1 \times \mathrm{P}(X = 1) \\ &= -1 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} \\ &= 0. \end{aligned}$$

$$\mathrm{P}(Y = 0) = \mathrm{P}(X \neq 0) = \mathrm{P}(\{X = 1\} \cup X = -1) = \mathrm{P}(X = 1) + \mathrm{P}(X = -1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

$$\mathrm{P}(Y = 1) = \mathrm{P}(X = 0) = \frac{1}{3}.$$

so,

$$\begin{aligned}
\mathrm{E(Y)} &= 0 \times \mathrm{P}(Y=0) + 1 \times \mathrm{P}(Y=1) \\
&= 0 \times \frac{2}{3} + 1 \times \frac{1}{3} \\
&= \frac{1}{3}.
\end{aligned}$$

Therefore $\mathrm{C}ov(X,Y) = \mathrm{E[XY]} - \mathrm{E[X]} \cdot \mathrm{E[Y]} = 0 - 0 \times \frac{1}{3} = 0.$

(ii)

$$\begin{aligned}
&\mathrm{P}(X=-1, Y=0) = \mathrm{P}(X=-1, X \neq 0) = \mathrm{P}(X=-1) = \frac{1}{3}; \\
&\mathrm{P}(X=-1, Y=1) = \mathrm{P}(X=-1, X=0) = \mathrm{P}(\emptyset) = 0; \\
&\mathrm{P}(X=0, Y=0) = \mathrm{P}(X=0, X \neq 0) = \mathrm{P}(\emptyset) = 0; \\
&\mathrm{P}(X=0, Y=1) = \mathrm{P}(X=0, X=0) = \mathrm{P}(X=0) = \frac{1}{3}; \\
&\mathrm{P}(X=1, Y=0) = \mathrm{P}(X=1, X \neq 0) = \mathrm{P}(X=1) = \frac{1}{3}; \\
&\mathrm{P}(X=1, Y=1) = \mathrm{P}(X=1, X=0) = \mathrm{P}(\emptyset) = 0.
\end{aligned}$$

i.e., the joint p.m.f of (X,Y) is:

X \ Y	$Y=0$	$Y=1$
$X=-1$	$\frac{1}{3}$	0
$X=0$	0	$\frac{1}{3}$
$X=0$	$\frac{1}{3}$	0

Observed that

$$\mathrm{P}(X=0, Y=0) = 0 \neq \frac{1}{3} \times \frac{2}{3} = \mathrm{P}(X=0) \cdot \mathrm{P}(Y=0).$$

Hence X and Y are not independent.

3. Show that the following conclusions are true:

   (i) $\mathrm{C}ov(X,Y) = \mathrm{C}ov(Y,X)$;

   (ii) $\mathrm{C}ov(X,X) = \mathrm{V}ar(X)$;

   (iii) $\mathrm{C}ov(aX,Y) = a\mathrm{C}ov(X,Y)$, where a is a constant;

(iv) $\mathrm{Co}v(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{Co}v(X_i, Y_j)$;

(v) If X is a random variable and C is a constant, then $\mathrm{Co}v(X, C) = 0$.

(vi) Show that the following statements are true:

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i) + \sum\sum_{1\leq i\neq j\leq n} \mathrm{Co}v(X_i, X_j),$$

or, equivalently,

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i) + 2\sum\sum_{1\leq i<j\leq n} \mathrm{Co}v(X_i, X_j).$$

Further show that if $X_1, \ldots, X_n$ are pairwise independent (i.e. X_i and X_j are independent for $1 \leq i \neq j \leq n$), then we have

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i).$$

Proof:

(i) $\mathrm{Co}v(X, Y) = \mathrm{E[(X - E(X))(Y - E(Y))] = E[(Y - E(Y))(X - E(X))] = Cov(Y, X)}$.

(ii) $\mathrm{Co}v(X, X) = \mathrm{E[(X - E(X))(X - E(X))] = E[(X - E(X))^2] = Var(X)}$.

(iii)

$$\begin{aligned}\mathrm{Co}v(aX, Y) &= \mathrm{E[(aX - E(aX))(X - E(X))] = E[(aX - aE(X))(X - E(X))]} \\ &= \mathrm{E[a(X - E(X))(X - E(X))] = aE[(X - E(X))(X - E(X))]} \\ &= a\mathrm{Co}v(X, Y).\end{aligned}$$

(iv)

$$\begin{aligned}\mathrm{Co}v(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j) &= \mathrm{E}\left\{\Big[\sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{X_i} - \mathrm{E}(\sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{X_i})\Big]\Big[\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{Y_j} - \mathrm{E}(\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{Y_j})\Big]\right\} \\ &= \mathrm{E}\left\{\Big[\sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{X_i} - \sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{E(X_i)}\Big]\Big[\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{Y_j} - \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{E(Y_j)}\Big]\right\} \\ &= \mathrm{E}\left\{\Big[\sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{(X_i - E(X_i))}\Big]\Big[\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{(Y_j - E(Y_j))}\Big]\right\} \\ &= \mathrm{E}\left\{\sum_{\mathrm{i=1}}^{\mathrm{m}}\sum_{\mathrm{j=1}}^{\mathrm{n}} \Big[\mathrm{(X_i - E(X_i))(Y_j - E(Y_j))}\Big]\right\} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{E}\Big[\mathrm{(X_i - E(X_i))(Y_j - E(Y_j))}\Big] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{Co}v(X_i, Y_j).\end{aligned}$$

(v) $\mathrm{Co}v(X, C) = \mathrm{E[(X - E(X))(C - E(C))] = E[(X - E(X))(C - C)] = E[0] = 0}$.

(vi)

$$\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \mathrm{Co}v(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_i \mathrm{Co}v(X_i, X_j)$$

$$= \sum\sum_{1\le i=j\le n} \mathrm{Cov}(X_i, X_j) + \sum\sum_{1\le i\ne j\le n} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \mathrm{V}ar(X_i) + \sum\sum_{1\le i\ne j\le n} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \mathrm{V}ar(X_i) + 2\sum\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j)$$

If $X_1, \ldots, X_n$ are pairwise independent, then for $1 \le i \ne j \le n$,

$$\mathrm{Cov}(X_i, X_j) = 0.$$

so, $\mathrm{V}ar(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{V}ar(X_i) + \sum\sum_{1\le i\ne j\le n} \mathrm{Cov}(X_i, X_j) = \sum_{i=1}^{n} \mathrm{V}ar(X_i) + 0 = \sum_{i=1}^{n} \mathrm{V}ar(X_i)$.

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common (mean)expectation μ and common variance σ^2. Let $\overline{X}$ and S^2 be defined as follows.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

$$S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2.$$

The two random variables $\bar{X}$ and $\frac{S^2}{n-1}$ are called the sample mean and sample variance, respectively. Find

(i) $\mathrm{E}[\overline{\mathrm{X}}]$;

(ii) $\mathrm{V}ar(\overline{X})$;

(iii) $\mathrm{E}[\frac{\mathrm{S}^2}{\mathrm{n}-1}]$.

Solution:

(i) $\mathrm{E}[\overline{\mathrm{X}}] = \mathrm{E}[\frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{X_i}] = \frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{E}[\mathrm{X_i}]) = \frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mu = \frac{\mathrm{n}\cdot\mu}{\mathrm{n}} = \mu$;

(ii)

$$\mathrm{V}ar[\overline{X}] = \mathrm{V}ar[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n^2}\mathrm{V}ar[\sum_{i=1}^{n} X_i]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(\mathrm{V}ar[X_i]) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

(iii) <u>**Method 1 :**</u>

$$S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}(X_i - \mu + \mu - \overline{X})^2$$

$$= \sum_{i=1}^{n}\left[(X_i - \mu)^2 + 2(X_i - \mu)(\mu - \overline{X}) + (\mu - \overline{X})^2\right]$$

$$= \sum_{i=1}^{n}(X_i - \mu)^2 + 2\sum_{i=1}^{n}[(X_i - \mu)(\mu - \overline{X})] + \sum_{i=1}^{n}(\mu - \overline{X})^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(X_i-\mu)^2 + 2[\sum_{i=1}^{n}(X_i-\mu)](\mu-\overline{X}) + \sum_{i=1}^{n}(\mu-\overline{X})^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + 2[\sum_{i=1}^{n}X_i - \sum_{i=1}^{n}\mu)](\mu-\overline{X}) + \sum_{i=1}^{n}(\mu-\overline{X})^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + 2[n\overline{X} - n\mu](\mu-\overline{X}) + \sum_{i=1}^{n}(\mu-\overline{X})^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 - 2n(\overline{X}-\mu)^2 + \sum_{i=1}^{n}(\mu-\overline{X})^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 - 2n(\overline{X}-\mu)^2 + n(\overline{X}-\mu)^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 - n(\overline{X}-\mu)^2 \\
&= \sum_{i=1}^{n}(X_i-\mathrm{E}[\mathrm{X_i}])^2 - \mathrm{n}(\overline{\mathrm{X}}-\mathrm{E}[\overline{\mathrm{X}}])^2.
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathrm{E}[\frac{\mathrm{S}^2}{\mathrm{n}-1}] &= \frac{1}{n-1}\mathrm{E}[\mathrm{S}^2] = \frac{1}{\mathrm{n}-1}\mathrm{E}[\sum_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{X_i}-\mathrm{E}[\mathrm{X_i}])^2 - \mathrm{n}(\overline{\mathrm{X}}-\mathrm{E}[\overline{\mathrm{X}}])^2] \\
&= \frac{\sum_{i=1}^{n}\mathrm{E}[(\mathrm{X_i}-\mathrm{E}[\mathrm{X_i}])^2]}{n-1} - \frac{n}{n-1}\mathrm{E}[(\overline{\mathrm{X}}-\mathrm{E}[\overline{\mathrm{X}}])^2] \\
&= \frac{\sum_{i=1}^{n}\mathrm{V}ar[X_i]}{n-1} - \frac{n}{n-1}\mathrm{V}ar[\overline{X}] \\
&= \frac{\sum_{i=1}^{n}\sigma^2}{n-1} - \frac{n}{n-1}\cdot\frac{\sigma^2}{n} \\
&= \frac{n\sigma^2}{n-1} - \frac{\sigma^2}{n-1} = \sigma^2.
\end{aligned}
$$

<u>**Method 2 :**</u>

$$
\begin{aligned}
S^2 &= \sum_{i=1}^{n}(X_i-\overline{X})^2 = \sum_{i=1}^{n}[X_i^2 - 2X_i\overline{X} + (\overline{X})^2] \\
&= \sum_{i=1}^{n}X_i^2 - 2\sum_{i=1}^{n}[X_i\overline{X}] + \sum_{i=1}^{n}(\overline{X})^2 = \sum_{i=1}^{n}X_i^2 - 2\sum_{i=1}^{n}[X_i]\overline{X} + \sum_{i=1}^{n}(\overline{X})^2 \\
&= \sum_{i=1}^{n}X_i^2 - 2(n\overline{X})\overline{X} + \sum_{i=1}^{n}(\overline{X})^2 = \sum_{i=1}^{n}X_i^2 - 2n(\overline{X})^2 + \sum_{i=1}^{n}(\overline{X})^2 \\
&= \sum_{i=1}^{n}X_i^2 - 2n(\overline{X})^2 + n(\overline{X})^2 = \sum_{i=1}^{n}X_i^2 - n(\overline{X})^2
\end{aligned}
$$

Notice that for $1 \leq i \leq n$,

$$\mathrm{E}[\mathrm{X_i^2}] = \mathrm{V}ar(X_i) + (\mathrm{E}[\mathrm{X_i}])^2 = \sigma^2 + \mu^2;$$

$$\mathrm{E}[(\overline{\mathrm{X}})^2] = \mathrm{V}ar(\overline{X}) + (\mathrm{E}[\overline{\mathrm{X}}])^2 = \frac{\sigma^2}{\mathrm{n}} + \mu^2.$$

So,

$$
\begin{aligned}
\mathrm{E}[\frac{\mathrm{S}^2}{\mathrm{n}-1}] &= \frac{1}{n-1}\mathrm{E}[\mathrm{S}^2] = \frac{1}{\mathrm{n}-1}\mathrm{E}[\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{X}_\mathrm{i}^2 - \mathrm{n}(\overline{\mathrm{X}})^2] \\
&= \frac{\sum_{i=1}^{n}\mathrm{E}[\mathrm{X}_\mathrm{i}^2]}{n-1} - \frac{n}{n-1}\mathrm{E}[(\overline{\mathrm{X}})^2] \\
&= \frac{\sum_{i=1}^{n}(\sigma^2+\mu^2)}{n-1} - \frac{n}{n-1}(\frac{\sigma^2}{n}+\mu^2) \\
&= \frac{n(\sigma^2+\mu^2)}{n-1} - \frac{n}{n-1}\cdot(\frac{\sigma^2}{n}+\mu^2) \\
&= \frac{n\sigma^2}{n-1} - \frac{\sigma^2}{n-1} = \sigma^2.
\end{aligned}
$$

5. Let I_A and I_B be the indicator variables for the events A and B. That is,

$$
I_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A. \end{cases}
$$

$$
I_B(\omega) = \begin{cases} 1 & \omega \in B, \\ 0 & \omega \notin B. \end{cases}
$$

Show that

(i)

$$
\begin{aligned}
&\mathrm{E}[\mathrm{I_A}] = \mathrm{P(A)}; \\
&\mathrm{E}[\mathrm{I_B}] = \mathrm{P(B)}; \\
&\mathrm{E}[\mathrm{I_A I_B}] = \mathrm{P(AB)}.
\end{aligned}
$$

(ii)

$$
Cov(I_A, I_B) = P(AB) - P(A)P(B).
$$

Proof:

(i) Note that

$$
\{I_A = 1\} \triangleq \{\omega \in \Omega \colon I_A(\omega) = 1\} = A, \quad \{I_B = 1\} \triangleq \{\omega \in \Omega \colon I_B(\omega) = 1\} = B;
$$

$$
\{I_A = 0\} \triangleq \{\omega \in \Omega \colon I_A(\omega) = 0\} = A^c, \quad \{I_B = 0\} \triangleq \{\omega \in \Omega \colon I_B(\omega) = 0\} = B^c.
$$

So,

$$
\begin{aligned}
\mathrm{P}(I_A = 0) &= \mathrm{P}(A^c), \qquad \mathrm{P}(I_A = 1) = \mathrm{P}(A); \\
\mathrm{P}(I_B = 0) &= \mathrm{P}(B^c), \qquad \mathrm{P}(I_B = 1) = \mathrm{P}(B);
\end{aligned}
$$

i.e., the p.m.f of random variable I_A is:

I_A	0	1
P	$\mathrm{P}(A^c)$	$\mathrm{P}(A)$

the p.m.f of random variable I_B is:

I_B	0	1
P	$\mathrm{P}(B^c)$	$\mathrm{P}(B)$

Hence, take the expectation, then we have

$$\mathrm{E(I_A)} = 0 \times \mathrm{P(A^c)} + 1 \times \mathrm{P(A)} = \mathrm{P(A)};$$
$$\mathrm{E(I_B)} = 0 \times \mathrm{P(B^c)} + 1 \times \mathrm{P(B)} = \mathrm{P(B)}.$$

Method 1: We first show that for any $\omega \in \Omega$,

$$I_A(\omega) \cdot I_B(\omega) = I_{A\cap B}(\omega) \tag{$\star$}$$

In fact, we only discuss four cases:

1° If $\omega \in A$ and $\omega \in B$;
2° If $\omega \in A$ and $\omega \notin B$, i.e. $\omega \in A$ and $\omega \in B^c$;
3° If $\omega \notin A$ and $\omega \in B$, i.e. $\omega \in A^c$ and $\omega \in B$;
4° If $\omega \notin A$ and $\omega \notin B$, i.e. $\omega \in A^c$ and $\omega \in B^c$;

Proof:

1° If $\omega \in A$ and $\omega \in B$, then $\omega \in A \cap B$, yields

$$I_A(\omega) = 1, \quad I_B(\omega) = 1, \quad I_{A\cap B}(\omega) = 1.$$

Hence,

the left hand side of $(\star)$ is $I_A(\omega) \cdot I_B(\omega) = 1 \times 1 = 1$;
the right hand side of $(\star)$ is $I_{A\cap B}(\omega) = 1$.

So, $I_A(\omega) \cdot I_B(\omega) = 1 \times 1 = 1 = I_{A\cap B}(\omega)$, i.e., $(\star)$ is true.

2° If $\omega \in A$ and $\omega \notin B$, then $\omega \notin A \cap B$, yields

$$I_A(\omega) = 1, \quad I_B(\omega) = 0, \quad I_{A\cap B}(\omega) = 0.$$

Hence,

the left hand side of $(\star)$ is $I_A(\omega) \cdot I_B(\omega) = 1 \times 0 = 0$;
the right hand side of $(\star)$ is $I_{A\cap B}(\omega) = 0$.

So, $I_A(\omega) \cdot I_B(\omega) = 1 \times 0 = 0 = I_{A\cap B}(\omega)$, i.e., $(\star)$ is true.

3° If $\omega \notin A$ and $\omega \in B$, then $\omega \notin A \cap B$, yields

$$I_A(\omega) = 0, \quad I_B(\omega) = 1, \quad I_{A\cap B}(\omega) = 0.$$

Hence,

the left hand side of $(\star)$ is $I_A(\omega) \cdot I_B(\omega) = 0 \times 1 = 0$;
the right hand side of $(\star)$ is $I_{A\cap B}(\omega) = 0$.

So, $I_A(\omega) \cdot I_B(\omega) = 0 \times 1 = 0 = I_{A\cap B}(\omega)$, i.e., $(\star)$ is true.

4° If $\omega \notin A$ and $\omega \notin B$, then $\omega \notin A \cap B$, yields

$$I_A(\omega) = 0, \quad I_B(\omega) = 0, \quad I_{A\cap B}(\omega) = 0.$$

Hence,

$$\begin{aligned}&\text{the left hand side of } (\star) \text{ is } I_A(\omega) \cdot I_B(\omega) = 0 \times 0 = 0;\\ &\text{the right hand side of } (\star) \text{ is } I_{A\cap B}(\omega) = 0.\end{aligned}$$

So, $I_A(\omega) \cdot I_B(\omega) = 0 \times 0 = 0 = I_{A\cap B}(\omega)$, i.e., $(\star)$ is true.

In short, we have show that for any $\omega \in \Omega$,

$$I_A(\omega) \cdot I_B(\omega) = I_{A\cap B}(\omega).$$

Therefore

$$\mathrm{E[I_A \cdot I_B] = E[I_{A\cap B}] = P(A \cap B)}.$$

<u>**Method 2:**</u> Try to find the joint p.m.f of I_A and I_B and then use the definition.

$$\begin{aligned}
\mathrm{P}\left(I_A = 0, I_B = 0\right) &= \mathrm{P}\left(\{I_A = 0\} \cap \{I_B = 0\}\right) = \mathrm{P}(A^c \cap B^c) = \mathrm{P}(A^c B^c);\\
\mathrm{P}\left(I_A = 0, I_B = 1\right) &= \mathrm{P}\left(\{I_A = 0\} \cap \{I_B = 1\}\right) = \mathrm{P}(A^c \cap B) = \mathrm{P}(A^c B);\\
\mathrm{P}\left(I_A = 1, I_B = 0\right) &= \mathrm{P}\left(\{I_A = 1\} \cap \{I_B = 0\}\right) = \mathrm{P}(A \cap B^c) = \mathrm{P}(AB^c);\\
\mathrm{P}\left(I_A = 1, I_B = 1\right) &= \mathrm{P}\left(\{I_A = 1\} \cap \{I_B = 1\}\right) = \mathrm{P}(A \cap B) = \mathrm{P}(AB).
\end{aligned}$$

i.e.,

I_A \ I_B	$I_B = 0$	$I_B = 1$
$I_A = 0$	$\mathrm{P}(A^c B^c)$	$\mathrm{P}(A^c B)$
$I_A = 1$	$\mathrm{P}(AB^c)$	$\mathrm{P}(AB)$

Therefore, take the exception, then

$$\begin{aligned}
\mathrm{E[I_A I_B]} &= \sum_{y=0}^{y=1}\sum_{x=0}^{x=1} xy\mathrm{P}\left((I_A, I_B) = (x, y)\right) = \sum_{y=0}^{y=1}\sum_{x=0}^{x=1} xy\mathrm{P}(I_A = x, I_B = y)\\
&= 0 \times 0 \times \mathrm{P}(I_A = 0, I_B = 0) + 0 \times 1 \times \mathrm{P}(I_A = 0, I_B = 1)\\
&\quad + 1 \times 0 \times \mathrm{P}(I_A = 1, I_B = 0) + 1 \times 1 \times \mathrm{P}(I_A = 1, I_B = 1)\\
&= 0 \times 0 \times \mathrm{P}(A^c B^c) + 0 \times 1 \times \mathrm{P}(A^c B) + 1 \times 0 \times \mathrm{P}(AB^c) + 1 \times 1 \times \mathrm{P}(AB)\\
&= \mathrm{P}(AB).
\end{aligned}$$

(ii)

$$Cov(I_A, I_B) \xlongequal{see\ Tutorial12-01} \mathrm{E(I_A I_B) - E[I_A]E[I_B] = P(AB) - P(A)P(B)}.$$

6. Let$X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common variance σ^2. Show that for any fixed $i (1 \le i \le n)$,

$$Cov(X_i - \overline{X}, \overline{X}) = 0,$$

where $\overline{X}$ is the sample mean (i.e. $\overline{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i = \frac{1}{n}\sum\limits_{j=1}^{n} X_j$).

Proof: Note that $\{X_i\}_{i=1}^n$ are independent, then for $1 \le i \ne j \le n$,

$$\mathrm{C}ov(X_i, X_j) = 0.$$

Hence,

$$\begin{aligned}
\mathrm{C}ov(X_i - \overline{X}, \overline{X}) &= \mathrm{C}ov(X_i, \overline{X}) - \mathrm{C}ov(\overline{X}, \overline{X}) = \mathrm{C}ov(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j) - \mathrm{C}ov(\overline{X}, \overline{X}) \\
&= \frac{1}{n}\sum_{j=1}^{n} \mathrm{C}ov(X_i, X_j) - \mathrm{C}ov(\overline{X}, \overline{X}) = \frac{1}{n}[\mathrm{C}ov(X_i, X_i) + \sum_{j=1, j\ne i}^{n} \mathrm{C}ov(X_i, X_j)] - \mathrm{C}ov(\overline{X}, \overline{X}) \\
&\overset{Since\ \{X_i\}_{i=1}^n\ are\ independent}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=} \frac{1}{n}[\mathrm{C}ov(X_i, X_i) + 0] - \mathrm{C}ov(\overline{X}, \overline{X}) \\
&= \frac{1}{n}\mathrm{V}ar(X_i) - \mathrm{V}ar(\overline{X}) \\
&= \frac{1}{n} \cdot \sigma^2 - \frac{\sigma^2}{n} = 0.
\end{aligned}$$