Step-1

Let S be the subset of all polynomials with $\int_0^1 p(x)dx = 0$ in the vector space P_3 of all $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

The objective to show that S is the subspace and find its basis.

Consider the expression,

$$\int_0^1 p(x)dx = 0$$

$$\int_0^1 \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)dx = 0$$

$$\left[a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4\right]_0^1 = 0$$

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$$

$$a_0 = -\left(\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}\right)$$

So, subset of polynomials is $S = \left\{-\left(\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}\right) + a_1x + a_2x^2 + a_3x^3 \,/\, a_1, a_2, a_3 \in \mathrm{R}\right\}$

It can be written as,

$$S = \left\{\left(x - \frac{1}{2}\right)a_1 + \left(x^2 - \frac{1}{3}\right)a_2 + \left(x^3 - \frac{1}{4}\right)a_3 \,/\, a_1, a_2, a_3 \in \mathrm{R}\right\}$$

Step-2

Recollect that a subset S is a subspace of a vector space $\mathbf{V}$ if $p + q \in S$ for all $p, q \in S$ and $c \in V, p \in S$ implies $cp \in S$.

Let $p(x), q(x) \in S$

Then the polynomials are,

$$p(x) = \left(x - \frac{1}{2}\right)a_1 + \left(x^2 - \frac{1}{3}\right)a_2 + \left(x^3 - \frac{1}{4}\right)a_3,$$
$$q(x) = \left(x - \frac{1}{2}\right)b_1 + \left(x^2 - \frac{1}{3}\right)b_2 + \left(x^3 - \frac{1}{4}\right)b_3$$

Where $a_1, a_2, a_3, b_1, b_2, b_3 \in R$

Consider the expression,

$$\begin{aligned} p(x)+q(x) &= \left(x-\frac{1}{2}\right)a_1 + \left(x^2-\frac{1}{3}\right)a_2 + \left(x^3-\frac{1}{4}\right)a_3 \\ &\quad + \left(x-\frac{1}{2}\right)b_1 + \left(x^2-\frac{1}{3}\right)b_2 + \left(x^3-\frac{1}{4}\right)b_3 \\ &= \left(x-\frac{1}{2}\right)(a_1+b_1) + \left(x^2-\frac{1}{3}\right)(a_2+b_2) + \left(x^3-\frac{1}{4}\right)(a_3+b_3) \end{aligned}$$

Since sum of the real numbers is real number, $(a_1+b_1),(a_2+b_2),(a_3+b_3) \in R$

This implies $p(x)+q(x) = \left(x-\frac{1}{2}\right)c_1 + \left(x^2-\frac{1}{3}\right)c_2 + \left(x^3-\frac{1}{4}\right)c_3$,

Where $c_1 = (a_1+b_1), c_2 = (a_2+b_2), c_3 = (a_3+b_3)$

And

$$\begin{aligned} \int_0^1 \left[p(x)+q(x)\right]dx &= \int_0^1 \left[\left(x-\frac{1}{2}\right)c_1 + \left(x^2-\frac{1}{3}\right)c_2 + \left(x^3-\frac{1}{4}\right)c_3\right]dx \\ &= \left[\left(\frac{x^2}{2}-\frac{x}{2}\right)c_1 + \left(\frac{x^3}{3}-\frac{x}{3}\right)c_2 + \left(\frac{x^4}{4}-\frac{x}{4}\right)c_3\right]_0^1 \\ &= \left[\left(\frac{1^2}{2}-\frac{1}{2}\right)c_1 + \left(\frac{1^3}{3}-\frac{1}{3}\right)c_2 + \left(\frac{1^4}{4}-\frac{1}{4}\right)c_3\right] - 0 \\ &= 0 \end{aligned}$$

So, sum of the polynomials $p(x)+q(x) \in S$

Step-3

Let c be the polynomial and $p(x) \in S$

Then $p(x) = \left(x-\frac{1}{2}\right)a_1 + \left(x^2-\frac{1}{3}\right)a_2 + \left(x^3-\frac{1}{4}\right)a_3$, where $a_1, a_2, a_3 \in R$

The scalar multiple of polynomial is,

$$
\begin{aligned}
cp(x) &= c\left[\left(x-\frac{1}{2}\right)a_1+\left(x^2-\frac{1}{3}\right)a_2+\left(x^3-\frac{1}{4}\right)a_3\right] \\
&= \left(x-\frac{1}{2}\right)(ca_1)+\left(x^2-\frac{1}{3}\right)(ca_2)+\left(x^3-\frac{1}{4}\right)(ca_3)
\end{aligned}
$$

Since product of two real numbers is a real number, $ca_1, ca_2, ca_3 \in R$

And

$$
\begin{aligned}
\int_0^1 cp(x)\,dx &= \int_0^1\left[\left(x-\frac{1}{2}\right)(ca_1)+\left(x^2-\frac{1}{3}\right)(ca_2)+\left(x^3-\frac{1}{4}\right)(ca_3)\right]dx \\
&= \left[\left(\frac{x^2}{2}-\frac{x}{2}\right)(ca_1)+\left(\frac{x^3}{3}-\frac{x}{3}\right)(ca_2)+\left(\frac{x^4}{4}-\frac{x}{4}\right)(ca_3)\right]_0^1 \\
&= \left[\left(\frac{1^2}{2}-\frac{1}{2}\right)(ca_1)+\left(\frac{1^3}{3}-\frac{1}{3}\right)(ca_2)+\left(\frac{1^4}{4}-\frac{1}{4}\right)(ca_3)\right]-0 \\
&= 0
\end{aligned}
$$

So, the scalar multiple of polynomial $cp(x) \in R$

Therefore, S is a subspace of vector space P_3.

Step-4

The basis for S is $\left\{\left(x-\frac{1}{2}\right),\left(x^2-\frac{1}{3}\right),\left(x^3-\frac{1}{4}\right)\right\}$, because

$$
S=\left\{\left(x-\frac{1}{2}\right)a_1+\left(x^2-\frac{1}{3}\right)a_2+\left(x^3-\frac{1}{4}\right)a_3 \,/\, a_1, a_2, a_3 \in \mathrm{R}\right\}
$$

Since subspace S contains 3 elements, the dimension of subspace is $\boxed{3}$.