Step-1

(a)

Consider the following system of differential equations concerns the rabbit population r and the wolf population w.

$$\frac{dr}{dt} = 4r - 2w$$
$$\frac{dw}{dt} = r + w$$

Determine whether the system is stable or neutrally stable or unstable.

The differential equation $\frac{du}{dt} = Au$ is

Stable, when all $\operatorname{Re}\lambda_i < 0$

Neutrally stable, when all $\operatorname{Re}\lambda_i \leq 0$ and $\operatorname{Re}\lambda_1 = 0$

Unstable and e^{At} is unbounded if any eigenvalue has $\operatorname{Re}\lambda_i > 0$

Step-2

Consider $P(t)$ is the population vector, define as follows:

$$P(t) = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix}$$

Use the population vector, to write the system as,

$$P'(t) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} P(t)$$

Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

Determine eigenvalues of matrix A.

To find the eigenvalues of matrix A, solve for $\det(A - \lambda I) = 0$.

$$\det(A-\lambda I)=0$$
$$\begin{vmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{vmatrix}=0$$
$$(4-\lambda)(1-\lambda)-(-2)=0$$
$$(4-\lambda)(1-\lambda)+2=0$$

Step-3

By simplifying,

$$4-4\lambda-\lambda+\lambda^2+2=0$$
$$\lambda^2-5\lambda+6=0$$
$$(\lambda-3)(\lambda-2)=0$$

This gives $\lambda_1=3$ and $\lambda_2=2$.

Hence, the eigenvalues of matrix A are $\lambda_1=3$ and $\lambda_2=2$.

Since both the eigenvalues are greater than 0, so the system is $\boxed{\text{unstable}}$.

Step-4

(b)

Initially let $r=300$ and $w=200$.

Determine the populations at time t.

Since the Eigen values of A are distinct, so A is diagonalizable.

Therefore, there exist a diagonal matrix Λ so that

$$A=S\Lambda S^{-1}$$

Then, $\frac{dp}{dt}=Ap(t)$ has a solution of the form,

$$p(t)=e^{At}p(0)$$
$$=Se^{\Lambda t}S^{-1}p(0)$$

The eigenvalues of matrix A are $\lambda_1=3$ and $\lambda_2=2$.

Now, to diagonalizable the Matrix A, find the eigenvectors of A.

For that, for each eigenvalue; solve the equation $(A-\lambda I)\mathrm{x}=0$.

Step-5

Therefore, the eigenvector for $\lambda_1=3$ is given by,

$$(A-3I)\mathrm{x}=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}1 & -2\\1 & -2\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}=0$$
$$y-2z=0$$

Thus the eigenvector corresponding to the eigenvalue $\lambda_1=3$ is,

$$\boldsymbol{x}_1=\begin{bmatrix}2\\1\end{bmatrix}$$

The eigenvector for $\lambda_1=2$ is given by,

$$(A-2I)x=0$$
$$\begin{bmatrix}2 & -2\\1 & -1\end{bmatrix}\begin{bmatrix}y\\z\end{bmatrix}=0$$
$$2y-2z=0$$

Thus the eigenvector corresponding to the eigenvalue $\lambda_1=2$ is,

$$\boldsymbol{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$$

Step-6

The eigenvector matrix for A is given by,

$$S=\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$$
$$S^{-1}=\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}$$

The Eigen values matrix of A is given by

$$\Lambda=\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$$

Step-7

Therefore,

$$A = S\Lambda S^{-1}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Hence,

$$e^{At} = Se^{\Lambda t}S^{-1}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Step-8

By using the equation $p(t) = e^{At} p(0)$,

$$P(t) = e^{At} P(0)$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 300 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & -e^{3t} \\ -e^{2t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} 300 \\ 200 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 100e^{3t} \\ 100e^{2t} \end{bmatrix}$$

$$P(t) = 100 \begin{bmatrix} 2e^{3t} + e^{2t} \\ e^{3t} + e^{2t} \end{bmatrix}$$

Therefore, when $r = 300$ and $w = 300$, the rabbit population r at time t is

$\boxed{r(t) = 200e^{3t} + 100e^{2t}}$ and

The wolf population w at time t is,

$\boxed{w(t) = 100e^{3t} + 100e^{2t}}$

Step-9

(c)

For large values of time t, the value of e^{3t} is greater than the value of e^{2t}, so only consider the lager value.

So, for long time, the proportion vector

$$P(t) = \begin{bmatrix} 200e^{3t} + 100e^{2t} \\ 100e^{3t} + 100e^{2t} \end{bmatrix}$$

is can be taken approximately,

$$\begin{aligned} P(t) &\approx \begin{bmatrix} 200e^{3t} \\ 100e^{3t} \end{bmatrix} \\ &\approx e^{3t} \begin{bmatrix} 200 \\ 100 \end{bmatrix} \end{aligned}$$

Thus, for long time, the rabbit population r at time t is

$\boxed{r(t) = 200e^{3t}}$ And

The wolf population w at time t is $\boxed{w(t) = 100e^{3t}}$.

Therefore, the population of the rabbits is $\boxed{\text{twice}}$ to the population of wolves.