

## Step-1

Consider the matrix:

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}$$

Thus,

$$A^T = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 0 \\ 6 & 8 \end{bmatrix}$$

Therefore,  $A^T A$  is given by,

$$\begin{aligned} A^T A &= \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 0 \\ 6 & 8 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 100 & 60 \\ 60 & 100 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \end{aligned}$$

## Step-2

The eigenvalues of  $A^T A$  are given by,

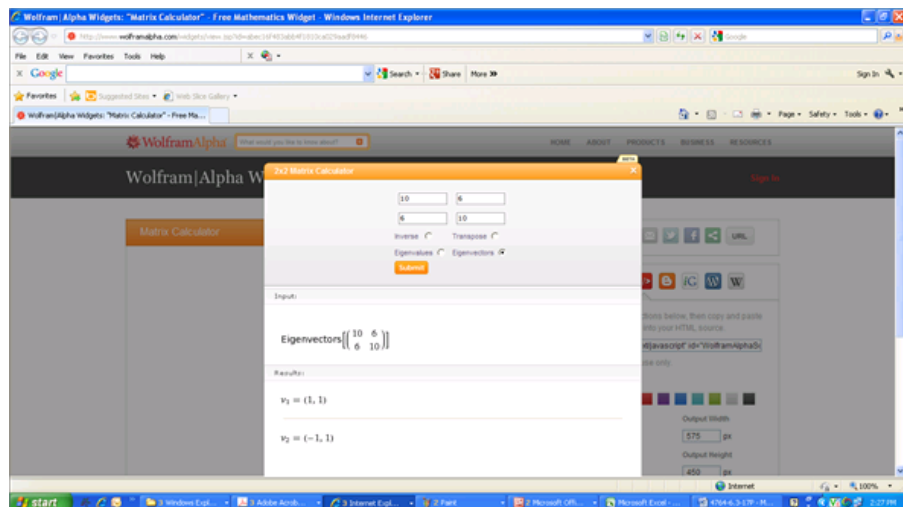
$$\begin{vmatrix} 10 - \lambda & 6 \\ 6 & 10 - \lambda \end{vmatrix} = 0$$
$$(10 - \lambda)(10 - \lambda) = 36$$

$$\begin{aligned} 100 - 20\lambda + \lambda^2 &= 36 \\ \lambda^2 - 20\lambda + 64 &= 0 \\ (\lambda - 16)(\lambda - 4) &= 0 \\ \lambda &= 16, 4 \end{aligned}$$

Therefore, the eigenvalues are 16 and 4.

## Step-3

By using matrix calculator (the screenshot is given below), the eigenvectors of  $AA^T$  are given by,



The eigenvectors of  $AA^T$  are given by,

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

## Step-4

Therefore,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

## Step-5

The positive definite square root  $S = V\Sigma^{\frac{1}{2}}V^T$  is given by,

## Step-6

$$\begin{aligned}
S &= V\Sigma^{\frac{1}{2}}V^T \\
&= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \\
&= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}
\end{aligned}$$

The inverse of  $S$  is given by,

$$S^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

Therefore, we get,

$$\begin{aligned}
Q &= AS^{-1} \\
&= \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix} \\
&= \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}
\end{aligned}$$

## Step-7

Thus,  $\boxed{S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}}$  and  $\boxed{Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}}$ .