



南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 高等数学(上) A

开课单位: 数学系

考试时长: 120 分钟

命题教师:

题号	1	2	3	4	5	6	7	8	9
分值	15 分	15 分	10 分	10 分	10 分	10 分	10 分	16 分	4 分

本试卷共 9 道大题, 满分 100 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

注意: 本试卷里的中文为直译 (即完全按英文字面意思直接翻译), 所有数学词汇的定义请参照教材 (Thomas' Calculus, 13th Edition) 中的定义. 如果其中有些数学词汇的定义不同于中文书籍 (比方说同济大学的高等数学教材) 里的定义, 以教材 (Thomas' Calculus, 13th Edition) 中的定义为准.

1. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

(1) The number of the real roots for the equation $x^3 - 3x + 3 = 0$ is

- (A) 0. (B) 1. (C) 2. (D) 3.

(2) If $f(x)$ is continuous on $(-\infty, +\infty)$, which of the following statements is **wrong** ?

- (A) $\int_0^1 f(x) dx = \int_0^1 f(t) dt$. (B) $\int_0^1 f(x) dx = \int_0^1 f(\sin x) d(\sin x)$.
(C) $d\left(\int_0^x f(t) dt\right) = f(x) dx$. (D) $d\left(\int_0^{x^2} f(t) dt\right) = f(x^2) d(x^2)$.

(3) Let

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Then the largest positive integer n , for which $f^{(n)}(0)$ exists, is

- (A) 1. (B) 2. (C) 3. (D) 4.

(4) If $f(x)$ is twice-differentiable on $(-\infty, +\infty)$, and $g(x) = (1-x)f(0) + xf(1)$, then which of the following statements is **correct** on $(0, 1)$?

- (A) $f(x) > g(x)$ if $f'(x) > 0$. (B) $f(x) > g(x)$ if $f'(x) < 0$.
(C) $f(x) > g(x)$ if $f''(x) > 0$. (D) $f(x) > g(x)$ if $f''(x) < 0$.

(5) If the improper integral $\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k} dx$ converges, then the constant k must satisfy

- (A) $k < 1$. (B) $k > 3$.
(C) $1 < k < 2$. (D) $1 < k < 3$.

2. (15 pts) Fill in the blanks.

- (1) Function $f(x) = x^2$ has a tangent line $y = Kx - 1$ if $K = \underline{\hspace{2cm}}$, or $\underline{\hspace{2cm}}$.
 - (2) Assume that $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, and $f''(1) = -7$. Let $g(x) = f(\ln x)$. Then $g''(1) = \underline{\hspace{2cm}}$.
 - (3) The average value for $f(x) = \sin^3 x$ on $[0, \pi]$ is $\underline{\hspace{2cm}}$.
 - (4) Let $y = (\cos x)^x$ for $0 < x < \frac{\pi}{2}$, then $y'(x) = \underline{\hspace{2cm}}$.
 - (5) If $f''(a)$ exists, and $f'(a) \neq 0$, then $\lim_{x \rightarrow a} \left(\frac{1}{f'(a)(x-a)} - \frac{1}{f(x) - f(a)} \right) = \underline{\hspace{2cm}}$.
3. (10 pts) The region D is enclosed by the curve $y = \ln \sqrt{x-1}$, the straight line $x = 5$, and the x -axis.
- (1) Find the area of the region D .
 - (2) Find the volumes generated by revolving the region D about the line $x = 5$.
4. (10 pts) Find the particular solution of

$$xy' + (x-2)y = 3x^3e^{-x}, \quad x > 0,$$

satisfying $y(1) = 0$.

5. (10 pts) Evaluate the following limits.

$$(1) \lim_{n \rightarrow +\infty} \left(\frac{n}{2n^2 + 3n + 1^2} + \frac{n}{2n^2 + 6n + 2^2} + \cdots + \frac{n}{2n^2 + 3nk + k^2} + \cdots + \frac{n}{2n^2 + 3n^2 + n^2} \right).$$

$$(2) \lim_{x \rightarrow 0} \left(\frac{\ln(1+x)}{x} \right)^{\frac{1}{e^x-1}}.$$

6. (10 pts)

- (1) For $y = \frac{x^2+1}{x+1}$, identify the coordinates of any local and absolute extreme points and inflection points that may exist.
- (2) Sketch the graph of the above function. (Please identify all the asymptotes and some specific points, such as local maximum and minimum points, inflection points, and intercepts.)

7. (10 pts) Find $\frac{dy}{dx}$ if

$$y = \int_{x^2+1}^{2x^2+3} t \tan \sqrt{x+t} dt.$$

8. (16 pts) Evaluate the integrals.

$$(1) \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^3 x dx.$$

$$(2) \int \sqrt{\frac{x}{x-2}} dx, \text{ where } x > 2.$$

$$(3) \int_1^e \ln^3 x dx.$$

$$(4) \int_1^{+\infty} \frac{(x+2)\ln(x^2+1)}{x^3} dx.$$

9. (4 pts) Let $f(n) = \sum_{m=1}^n \int_0^m \cos \frac{2\pi n \lfloor x+1 \rfloor}{m} dx$, here $\lfloor x+1 \rfloor$ is the largest integer which is less than or equal to $x+1$. Evaluate $f(2021)$.

一、 (15分) 单项选择题:

(1) 方程 $x^3 - 3x + 3 = 0$ 的实根个数为

- (A) 0. (B) 1. (C) 2. (D) 3.

(2) 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上连续, 则下列等式中**错误**的是

- (A) $\int_0^1 f(x) dx = \int_0^1 f(t) dt$. (B) $\int_0^1 f(x) dx = \int_0^1 f(\sin x) d(\sin x)$.
(C) $d\left(\int_0^x f(t) dt\right) = f(x) dx$. (D) $d\left(\int_0^{x^2} f(t) dt\right) = f(x^2) d(x^2)$.

(3) 设

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

那么使得 $f^{(n)}(0)$ 存在的最大的正整数 n 是

- (A) 1. (B) 2. (C) 3. (D) 4.

(4) 若 $f(x)$ 在 $(-\infty, +\infty)$ 上二阶可导, 且 $g(x) = (1-x)f(0) + xf(1)$, 则在开区间 $(0, 1)$ 里, 下列哪一个叙述是**正确**的?

- (A) 当 $f'(x) > 0$ 时, 必有 $f(x) > g(x)$. (B) 当 $f'(x) < 0$ 时, 必有 $f(x) > g(x)$.
(C) 当 $f''(x) > 0$ 时, 必有 $f(x) > g(x)$. (D) 当 $f''(x) < 0$ 时, 必有 $f(x) > g(x)$.

(5) 设 k 是常数, 若反常积分 $\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k} dx$ 收敛, 则必有

- (A) $k < 1$. (B) $k > 3$.
(C) $1 < k < 2$. (D) $1 < k < 3$.

二、 (15分) 填空题:

(1) 若 $y = Kx - 1$ 是曲线 $f(x) = x^2$ 的一条切线, 则 $K =$ _____ 或者 _____.

(2) 已知 $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, $f''(1) = -7$. 令 $g(x) = f(\ln x)$. 则 $g''(1) =$ _____.

(3) 函数 $f(x) = \sin^3 x$ 在 $[0, \pi]$ 上的平均值为 _____.

(4) 若 $y = (\cos x)^x$, 其中 $0 < x < \frac{\pi}{2}$, 则 $y'(x) =$ _____.

(5) 若 $f''(a)$ 存在, 且 $f'(a) \neq 0$, 则 $\lim_{x \rightarrow a} \left(\frac{1}{f'(a)(x-a)} - \frac{1}{f(x) - f(a)} \right) =$ _____.

三、 (10分) 设区域 D 是由曲线 $y = \ln \sqrt{x-1}$ 和直线 $x = 5$ 以及 x -轴围成.

(1) 求 D 的面积.

(2) 求 D 绕直线 $x = 5$ 旋转一周所成的旋转体的体积.

四、 (10分) 求解初值问题

$$xy' + (x-2)y = 3x^3 e^{-x}, \quad x > 0,$$

初始条件为 $y(1) = 0$.

五、 (10分) 求下列极限.

$$(1) \lim_{n \rightarrow +\infty} \left(\frac{n}{2n^2 + 3n + 1^2} + \frac{n}{2n^2 + 6n + 2^2} + \cdots + \frac{n}{2n^2 + 3nk + k^2} + \cdots + \frac{n}{2n^2 + 3n^2 + n^2} \right).$$

$$(2) \lim_{x \rightarrow 0} \left(\frac{\ln(1+x)}{x} \right)^{\frac{1}{e^x - 1}}.$$

六、 (10分)

(1) 设 $f(x) = \frac{x^2+1}{x+1}$, 求函数的所有(局部)极值、最值以及拐点.

(2) 给 $f(x)$ 画个草图. (请注明所有的极值、最值、拐点、渐近线以及与 x 轴和 y 轴的交点)

七、 (10分) 求 $\frac{dy}{dx}$, 这里

$$y = \int_{x^2+1}^{2x^2+3} t \tan \sqrt{x+t} dt.$$

八、 (16分) 计算积分.

$$(1) \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^3 x dx.$$

$$(2) \int \sqrt{\frac{x}{x-2}} dx, \text{ where } x > 2.$$

$$(3) \int_1^e \ln^3 x dx.$$

$$(4) \int_1^{+\infty} \frac{(x+2) \ln(x^2+1)}{x^3} dx.$$

九、 (4分) 设 $f(n) = \sum_{m=1}^n \int_0^m \cos \frac{2\pi n \lfloor x+1 \rfloor}{m} dx$, 这里 $\lfloor x+1 \rfloor$ 表示不超过 $x+1$ 的最大整数. 计算 $f(2021)$.