Step-1

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

We observe that the entries below the principal diagonal are zero.

So, this is the reduced form.

We write the homogeneous equations using this.

i.e., $$Ax=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x_2\\x_3\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x_2=0,\ x_3=0$$

From this, we can say that any real value k $=x_1$ can satisfy the system.

Therefore, the solution set is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}k\\0\\0\end{bmatrix}$

Therefore, the null space of the given matrix is spanned by $\{(k,0,0)\}$

In other words, the basis of the null space of A is $N(A)=\{(1,0,0)\}$

Further, dimension of N(A) = 1

Step-2

While the dimension of the given matrix is 3 and we have $\dim C(A)+\dim N(A)=3$

Therefore, dimension of C (A) = 2

The standard basis that spans the column space is $C(A)=\{(1,0,0),(0,1,0)\}$

Step-3

Similarly, $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Basis for column space of A^T or the row space of A $= \{(0,1,0),(0,0,1)\}$ and its dimension is 2

Basis for null space of $A^T = \{(1,0,0)\}$ and its dimension is 1

Therefore,

$$\boxed{\begin{array}{l} C(A) = \{(1,0,0),(0,1,0)\}, \dim C(A) = 2 \\ C(A^T) = \{(0,1,0),(0,0,1)\}, \dim C(A^T) = 2 \\ N(A) = \{(1,0,0)\}, \dim N(A) = 1 \\ N(A^T) = \{(1,0,0)\}, \dim N(A^T) = 1 \end{array}}$$

Step-4

On the other hand,

$$I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_3$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_1 \to R_1 - R_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The standard basis that spans column space $C(A) = \{(1,0,0),(0,1,0),(0,0,1)\}$ and its dimension is 3

The basis for row space $C(A^T) = \{(1,0,0),(0,1,0),(0,0,1)\}$ and its dimension is 3

Step-5

We write the homogeneous equations using this.

i.e., $$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0,\ x_2 = 0 \text{ and } x_3 = 0$

The basis of the null space of A is empty. i.e. $N(A) = \{(0,0,0)\}$ and its dimension is 0

The basis of the null space of A is empty. i.e. $N(A^T) = \{(0,0,0)\}$ and its dimension is 0

Therefore,

$$\boxed{\begin{array}{l} C(A) = \{(1,0,0),(0,1,0),(0,0,1)\}, \dim C(A) = 3 \\ C(A^T) = \{(1,0,0),(0,1,0),(0,0,1)\}, \dim C(A^T) = 3 \\ N(A) = \{(0,0,0)\}, \dim N(A) = 0 \\ N(A^T) = \{(0,0,0)\}, \dim N(A^T) = 0 \end{array}}$$