

Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Initial condition:

$$u(0) = (3, 1)$$

Find the general and specific solution that matches the initial condition. Also, find the steady state when $t \rightarrow \infty$.

Step-2

First step is to find the Eigen values and Eigen vectors of matrix A . To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (-1-\lambda)(-1-\lambda) - 1 &= 0 \\ \lambda^2 + 2\lambda &= 0 \end{aligned}$$

After solving following values are obtained:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -2 \end{aligned}$$

Step-3

Therefore, Eigen values are $0, -2$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} -1+2 & 1 \\ 1 & -1+2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = -2$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} -1-0 & 1 \\ 1 & -1-0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z corresponding to $\lambda = 0$ is as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step-6

Therefore Eigen values are:

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step-7

Recall that: $e^{At} = Se^{At}S^{-1}$

Here, Eigen value matrix is given as follows:

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-8

Therefore, the general solution of the differential equation is:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

Or

$$u(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, c_1 and c_2 are constants. Their values are determined by the following values:

$$c = S^{-1}u(0)$$

Step-9

So, the solution for differential equation can be written as follows:

$$\begin{aligned} u(t) &= Se^{At}S^{-1}u(0) \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-2t} & 1 \\ -e^{-2t} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-2t} + 1 & -e^{-2t} + 1 \\ -e^{-2t} + 1 & e^{-2t} + 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2e^{-2t} + 4 \\ -2e^{-2t} + 4 \end{bmatrix} \\ u(t) &= \begin{bmatrix} e^{-2t} + 2 \\ -e^{-2t} + 2 \end{bmatrix} \end{aligned}$$

Step-10

Therefore, specific solution of the differential equation is:

$$u(t) = \begin{bmatrix} e^{-2t} + 2 \\ -e^{-2t} + 2 \end{bmatrix}$$

Steady state when $t \rightarrow \infty$ will be:

$$u_{ss} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$