Step-1

Consider that a matrix A has eigen values 0 and 1, corresponding to the eigen vectors

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

The objective is to trace and determinant of the matrix and find the matrix A.

Step-2

The matrix A can be expressed as,

$$A = S\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}S^{-1}$$

Where $S = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Step-3

Find the matrix S^{-1} as,

$$\begin{aligned} \det(S) &= -1-4 \\ &= -5 \end{aligned}$$

Therefore,

$$S^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Step-4

Therefore, the matrix A is,

$$\begin{aligned} A &= S\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}S^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} \end{aligned}$$

Clearly the matrix A is the product of three symmetric matrices.

Therefore, the matrix A is symmetric.

Step-5

Since the matrix A is symmetric,

The determinant of matrix A is $\lambda_1\lambda_2$.

And the trace of the matrix A is $\lambda_1+\lambda_2$.

Here, the values $\lambda_1=0$ and $\lambda_2=1$.

Therefore,

$$\begin{aligned}\det(A)&=\lambda_1\lambda_2\\&=0\cdot 1\\&=0\end{aligned}$$

And

$$\begin{aligned}trace(A)&=\lambda_1+\lambda_2\\&=0+1\\&=1\end{aligned}$$

Step-6

Find the matrix A as,

$$\begin{aligned}A&=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{1}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{-1}{5}\end{bmatrix}\\&=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}0&0\\\frac{2}{5}&\frac{-1}{5}\end{bmatrix}\\&=\begin{bmatrix}\frac{4}{5}&\frac{-2}{5}\\\frac{-2}{5}&\frac{1}{5}\end{bmatrix}\end{aligned}$$