Suggested Solutions of Homework 4 MA327

Ex 1. E = 1, F = 0, G = ρ^2. Here we denote $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$ etc.

Ex 2.
(a) Assume $\nabla f = a\mathbf{x}_u + b\mathbf{x}_v$. Apply $v = \mathbf{x}_u, \mathbf{x}_v$ to the formula, we have

$$\langle a\mathbf{x}_u + b\mathbf{x}_v, \mathbf{x}_u \rangle = f_u$$
$$\langle a\mathbf{x}_u + b\mathbf{x}_v, \mathbf{x}_v \rangle = f_v$$

i.e.

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_u \\ f_v \end{pmatrix},$$

By calculation, there are

$$a = \frac{Gf_u - Ff_v}{EG - F^2}, \quad b = \frac{-Ff_u + Ef_v}{EG - F^2}.$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y, then $E = G = 1, F = 0$.
$\Rightarrow \nabla f = f_x e_1 + f_y e_2$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$.

(b) $df_p(v)$ is maximum $\Leftrightarrow$ v and ∇f are pointing to the same direction $\Leftrightarrow v = \frac{\nabla f}{|\nabla f|}$.

(c) (i)Let $p \in C$ be fixed. We can choose a parametrization $\mathbf{x}$ near p s.t. $\mathbf{x}(0,0) = p$, $E(0,0) = G(0,0) = 1, F(0,0) = 0$. We define $F : U \to \mathbb{R}^2$ by $F(u,v) = (u, f(u,v))$. Since $\nabla f(p) \neq 0$, we may assume $f_v(0,0) \neq 0$.

Then

$$dF_p = \begin{pmatrix} 1 & 0 \\ f_u(0,0) & f_v(0,0) \end{pmatrix} \Rightarrow det(dF_p) = f_v(0,0) \neq 0.$$

By the implict function theorem, on a neighbourhood V of p, F is a diffeomorphism. In particaluar, $v = v(u, a = const)$ is a differentiable function defined on an interval I whose graph is $V \cap C$ with $C = \{q \in S \big| f(q) = a\}$. Then $\alpha(t) = \mathbf{x}(t, v(t,a))$ defined on I is a parametrization of C near p. Moreover, $\alpha(t)$ is regular.

(c)(ii) Let $p \in C$ and $\alpha(t)$ be a parametrization of C near p. Then $f(\alpha(t)) = Const$.
$\frac{df}{dt}\big|_{t=0} = df_p(\alpha'(0)) = 0. \Rightarrow \langle \nabla f(p), \alpha'(0) \rangle = 0. \Rightarrow \nabla f$ is normal to C at p.

Ex 3 Because the surface is tangent to a plane, the normal vector $N(p)$ on the point p of this curve is invariant. Let $\alpha(t)$ be the parametrization of this curve with $\alpha(0) = p$, then $dN_p(\alpha'(0)) = 0$. That means that the kernel of dN_p is nontrivial, whence p is either parabolic or planar.

Ex 4 Let $\{e_1, e_2\}$ be the principal directions, $u = \cos\theta e_1 + \sin\theta e_2$ be. Applying Euler's formula, $k_n^u = k_1 \cos^2\theta + k_2 \sin^2\theta$. Then $k(p) \geq |k_n^u| = |k_1| \cos^2\theta + |k_2| \sin^2\theta \geq min\{|k_1|, |k_2|\}$. Here we used $K > 0$.

Ex 5 Applying the notations in **Ex 4**, $k_n(\theta) = k_1 \cos^2(\theta+\phi) + k_2 \sin^2(\theta+\phi)$, where the fixed direction is $\cos\phi e_1 + \sin\phi e_2$.
$\Rightarrow \frac{1}{\pi}\int_0^\pi k_1 \cos^2(\theta+\phi) + k_2 \sin^2(\theta+\phi) d\theta = \frac{k_1+k_2}{2} = H$.

Ex 6 Applying the notations in **Ex 4**.
$k_n(\theta) + k_n(\theta+\frac{\pi}{2}) = k_1 \cos^2(\theta) + k_2 \sin^2(\theta) + k_1 \cos^2(\theta+\frac{\pi}{2}) + k_2 \sin^2(\theta+\frac{\pi}{2}) = k_1 + k_2$.

Ex 7 (a) Since $\alpha' \neq 0$ and the fact that no planar or parabolic point implies $dN_{\alpha(t)}$ is injective, we then have $(N\circ\alpha)'(t) = dN_{\alpha(t)}(\alpha'(t)) \neq 0$.

(b)Because C is a line of curvature, there is $(N\circ\alpha)'(t) = \lambda(t)\alpha'(t) = -k_n\alpha'(t)$.
Recall the formula $k = \frac{|\alpha'\wedge\alpha''|}{|\alpha'|^3}$, there exists

$$\begin{aligned} k_N &= \frac{|(N\circ\alpha)' \wedge (N\circ\alpha)''|}{|N\circ\alpha'|^3} \\ &= \frac{1}{|k_n|}k \end{aligned}$$

$\Rightarrow \quad k = |k_n|k_N$.

Ex 8

$$\begin{aligned} \mathbf{x}(u,v) &= ((a+r\cos u)\cos v, (a+r\cos u)\sin v, r\sin u), \\ \mathbf{x}_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u), \\ \mathbf{x}_v &= (-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0), \\ \mathbf{x}_u\wedge\mathbf{x}_v &= (-(a+r\cos u)r\cos u\cos v, -(a+r\cos u)r\cos u\sin v, -r(a+r\cos u)\sin u), \\ \mathbf{N} &= (-\cos u\cos v, -\cos u\sin v, -\sin u), \\ \mathbf{N}_u &= (\sin u\cos v, \sin u\sin v, -\cos u) = -\frac{1}{r}\mathbf{x}_u. \end{aligned}$$

$\Rightarrow$ The meridians of a torus are lines of curvature.

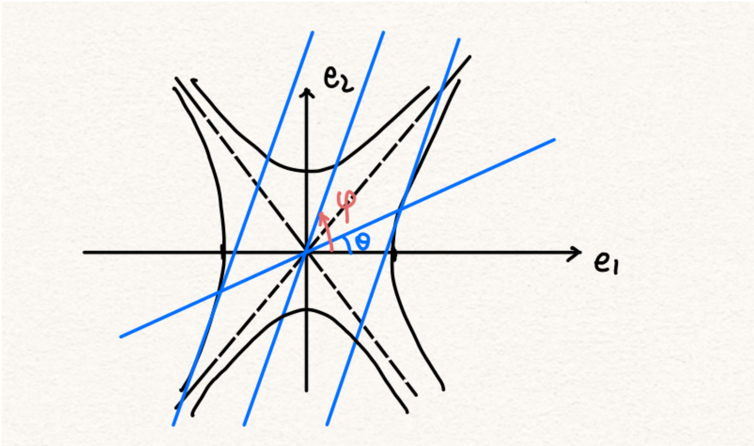
Ex 9
Let $k_1 = -k_2 = k \neq 0$.
Then $0 = k\cos^2(\pi/4) + (-k)\sin^2(\pi/4) = k\cos^2(3\pi/4) + (-k)\sin^2(3\pi/4)$.
$\theta = \pi/4, 3\pi/4$

Ex 10 It suffices to check whether two tangent lines satisfy $k_1\cos\theta\cos\phi + k_2\sin\theta\sin\phi = 0$.

Assume $k_1 > 0 > k_2$.
(i) $k_1x^2 + k_2y^2 = 1$.

$$\begin{aligned} \alpha(t) &= (\frac{1}{\sqrt{k_1}}\frac{1}{\cos t}, \frac{1}{\sqrt{-k_2}}\tan t), & \alpha'(t) &= (\frac{1}{\sqrt{k_1}}\frac{\sin t}{\cos^2 t}, \frac{1}{\sqrt{-k_2}}\frac{1}{\cos^2 t}) \\ \cos\theta &= \frac{\frac{1}{\sqrt{k_1}}\frac{1}{\cos t}}{\sqrt{\frac{1}{k_1\cos^2 t} - \frac{\tan^2 t}{k_2}}}, & \sin\theta &= \frac{\frac{1}{\sqrt{-k_2}}\tan t}{\sqrt{\frac{1}{k_1\cos^2 t} - \frac{\tan^2 t}{k_2}}} \\ \cos\phi &= \frac{\frac{1}{\sqrt{k_1}}\frac{\sin t}{\cos^2 t}}{\sqrt{\frac{\sin^2 t}{k_1\cos^4 t} - \frac{1}{k_2\cos^4 t}}}, & \sin\phi &= \frac{\frac{1}{\sqrt{-k_2}}\frac{1}{\cos^2 t}}{\sqrt{\frac{\sin^2 t}{k_1\cos^4 t} - \frac{1}{k_2\cos^4 t}}} \end{aligned}$$

$\Rightarrow k_1\cos\theta\cos\phi + k_2\sin\theta\sin\phi = 0$.



(ii) In similar, if $k_1x^2 + k_2y^2 = -1$, then we still have $k_1 \cos\theta \cos\phi + k_2 \sin\theta \sin\phi = 0$.

To sum up, if the line λ has intersections with hyperbola, then its conjugate direction is the tangent line at the intersection point; if there is no intersection, then the conjugate direction is itself.

Ex 11 Let $\{e_1, e_2\}$ be the unit vectors of principal directions. Suppose $w_1 = a_1e_1 + a_2e_2$, $w_2 = b_1e_1 + b_2e_2$, $k_1 = -k_2 = k$.
$\Rightarrow$

$$\begin{aligned}\langle dN_p(w_1), dN_p(w_2)\rangle &= k^2(a_1b_1 + a_2b_2) \\ -K(p)\langle w_1, w_2\rangle &= k^2(a_1b_1 + a_2b_2).\end{aligned}$$

Hence there is an equality.

(ii)

$$\begin{aligned}\frac{\langle dN_p(w_1), dN_p(w_2)\rangle}{|dN_p(w_1)| \cdot |dN_p(w_2)|} &= \frac{-K(p)\langle w_1, w_2\rangle}{-K(p)|w_1||w_2|} \\ &= \frac{\langle w_1, w_2\rangle}{|w_1| \cdot |w_2|}\end{aligned}$$

Hence the angle of their spherical images are equal up to a sign.

Ex 12 Since

$$\sum_{i=1}^{m} \cos^2(\frac{2\pi(i-1)}{m}) = \sum_{i=1}^{m-1} \sin^2(\frac{2\pi i}{m}) = \frac{m}{2}$$

$$\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} (k_1 \cos^2(\frac{2\pi(i-1)}{m}) + k_2 \sin^2(\frac{2\pi(i-1)}{m})) = m\frac{k_1 + k_2}{2} = mH.$$

Ex 13

$$\begin{aligned}
\mathbf{x}(u,v) &= (u,v,auv)\\
\mathbf{x}_u(u,v) &= (1,0,av), \quad \mathbf{x}_v(u,v) = (0,1,au)\\
\mathbf{x}_{uu}(0,0) &= (0,0,0), \quad \mathbf{x}_{uv}(0,0) = (0,0,a)\\
\mathbf{x}_{vv}(0,0) &= (0,0,0), \quad N(0,0) = (0,0,1)\\
\Rightarrow E(0,0) &= 1, F(0,0) = 0, G(0,0) = 1\\
e(0,0) &= 0, f(0,0) = a, g(0,0) = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow K &= \frac{eg - f^2}{EG - F^2} = -a^2\\
H &= \frac{eG + gE - 2fF}{2(EG - F^2)} = 0
\end{aligned}$$