Step-1

(a)

Consider the subspace of all vectors whose components are equal.

The objective is to find a basis for this subspaces of $\mathbf{R}^4$.

Step-2

The space is of all vectors of the form $(t,t,t,t)=t(1,1,1,1)$ that is all multiples of $(1,1,1,1)$.

This expresses that $\boxed{\{(1,1,1,1)\}}$ spans the space, and meanwhile it's a singleton, it is also linearly independent, and henceforth a basis.Â

Step-3

(b)

Consider the subspace of all vectors whose components add to equal.

The objective is to find a basis for this subspaces of $\mathbf{R}^4$.

Step-4

Now look for $(a,b,c,d)\in\mathbf{R}^4$ such that:Â

$a+b+c+d=0$

This implies that,

$d=-a-b-c$ That is,

$$\begin{aligned}(a,b,c,-a-\ b-\ c)&=(a,0,0,-a)+(0,b,0,-b)+(0,0,c,-c)\\&=a(1,0,0,-1)+b(0,1,0,-1)+c(0,0,1,-1)\end{aligned}$$

Thus, $\boxed{\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}}$ spans the space, and is provably linearly independent, consequently is a basis.

Step-5

(c)

Consider the subspace of all vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

The objective is to find a basis for this subspaces of $\mathbf{R}^4$.

Step-6

Two vectors perpendicular if their dot product is zero.

Now observe for $(a,b,c,d)\in\mathbf{R}^4$ such that,

$$(a,b,c,d)\cdot(1,1,0,0)=0$$

This implies that,

$$a+b=0$$

And,

$$(a,b,c,d)\cdot(1,0,1,1)=0$$

This implies that,

$$a+c+d=0$$

From this, $a=-c-d$ and from $a+b=0$;

$$\begin{aligned}-c-\ d+b&=0\\ b&=c+d\end{aligned}$$

Therefore, vectors of the form obtained are:Â

$$\begin{aligned}(a,b,c,d)&=(-c-d,\ c+d,c,d)\\ &=(-c,c,c,0)+(-d,d,0,d)\\ &=c(-1,1,1,0)+d(-1,1,0,1)\end{aligned}$$

The vectors $(-1,1,1,0)$ and $(-1,1,0,1)$ are linearly independent because first vector has fourth component zero and second vector has third component zero.

Hence, the set $\boxed{\{(-1,1,1,0),(-1,1,0,1)\}}$ is a basis for this space.

Step-7

(d)

Consider the below matrix;

$$U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$$

The objective is to find a basis for the column space (in $\mathbf{R}^2$) and null space (in $\mathbf{R}^5$) of U.

Step-8

The matrix U is already in Echelon form.

Therefore, the columns contain pivots are basis for the column space.

Thus, the basis for column space is $\boxed{\{(1,0),(0,1)\}}$.

Step-9

The null space of a matrix A is the solution x of the system $Ax=0$.

Consider $Ux=0$ where $x=(a,b,c,d,e)^T$.

So,

$$Ux=0$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives;

$$\begin{aligned} a+c+e&=0 \\ b+d&=0 \end{aligned}$$

This implies;

$$\begin{aligned} a&=-c-e \\ b&=-d \end{aligned}$$

So,

$$x = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

$$= \begin{bmatrix} -c-e \\ -d \\ c \\ d \\ e \end{bmatrix}$$

$$= c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the basis of null space is $\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$.