Step-1

(a)

If a 7 by 9 matrix has rank 5, find the dimensions of the subspaces: column space $\mathbf{C}(\mathbf{A})$, null space $\mathbf{N}(\mathbf{A})$, row space $\mathbf{C}(\mathbf{A}^{\mathbf{T}})$, and left null space $\mathbf{N}(\mathbf{A}^{T})$ of A, and find the sum of all four dimensions.

Step-2

Let A be a 7 by 9 matrix of rank 5, so of 9 columns, the number of linearly independent columns is 5, and the number of linearly dependent columns is 4.

This implies;

$\dim(\mathbf{C}(\mathbf{A}))=5$, and

$\dim(\mathbf{N}(\mathbf{A}))=4$ $\left(\text{because } \dim(C(A))+\dim(N(A))= \text{ Number of columns of } A=9\right)$

Step-3

A is 7 by 9 matrix has rank 5, so the number of linearly independent rows (or the number of linearly independent columns of A^{T}) is 5

Therefore,

$\boxed{\dim(\mathbf{C}(\mathbf{A}^{\mathbf{T}}))=5}$

$\boxed{\dim(\mathbf{N}(\mathbf{A}^{\mathbf{T}}))=2}$

$\left(\text{because } \dim\left(\mathbf{C}(\mathbf{A}^{\mathbf{T}}\right)+\dim(\mathbf{N}(\mathbf{A}^{\mathbf{T}}))=\text{Number of columns of } A^{T}=7)\right)$

Step-4

(b)

If a 3 by 4 matrix has rank 3, then find its column space $\mathbf{C}(\mathbf{A})$ and left null space $\mathbf{N}(\mathbf{A}^{T})$.

Let A be 3 by 4 matrix, which has rank 3, so of 4 columns, the number of linearly independent columns is 3.

This implies;

$\dim(\mathbf{C}(\mathbf{A}))=3$

Step-5

A is 3 by 4 matrix has rank 3, so the number of linearly independent rows (or the number of linearly independent columns of A^T) is 3.

Therefore,

$$\boxed{\dim\left(\mathbf{C}\left(\mathbf{A}^{\mathbf{T}}\right)\right)=3}$$

$$\boxed{\dim\left(\mathbf{N}\left(\mathbf{A}^{\mathbf{T}}\right)\right)=0}$$

$$\left(\begin{aligned}\text{because } \dim\left(\mathbf{C}\left(\mathbf{A}^{\mathbf{T}}\right)\right)+\dim\left(\mathbf{N}\left(\mathbf{A}^{\mathbf{T}}\right)\right) &= \text{ Number of columns of } A^T \\ &= 3\end{aligned}\right)$$