SUSTech

Midterm for Calculus II in Spring Semester, 2019

注意：本试卷里的中文为直译（即完全按英文字面意思直接翻译），所有数学词汇的定义请参照教材（Thomas' Calculus，13th Edition）中的定义。如果其中有些数学词汇的定义不同于中文书籍（比方说同济大学的高等数学教材）里的定义，以教材（Thomas' Calculus，13th Edition）中的定义为准。

1. (15 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

   (1) If $a_n < 0$ for $n > 1000$, and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ may diverge.

   (2) Suppose that the power series $\sum_{n=1}^{\infty} a_n(x-1)^n$ converges at $x = 0$, and diverges at $x = 2$, then the interval of convergence of this series is $[0, 2)$.

   (3) If $f(x, y)$ has partial derivatives $f_x(x, y)$ and $f_y(x, y)$ at (x_0, y_0), then
   $$\lim_{x \to x_0} f(x, y_0) = \lim_{y \to y_0} f(x_0, y) = f(x_0, y_0).$$

   (4) If a vector function $\mathbf{r}(t)$ is always perpendicular to its derivative $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, then $|\mathbf{r}(t)|$ must be constant.

   (5) The curvature of a unit circle is greater than the curvature of the parabola $y = x^2$ at the origin.

一、 （15分）判断题：

(1) 若当$n > 1000$时$a_n < 0$，并且已知$\sum_{n=1}^{\infty} a_n$收敛，则$\sum_{n=1}^{\infty} a_n^2$有可能发散.

(2) 若幂级数$\sum_{n=1}^{\infty} a_n(x-1)^n$在$x = 0$处收敛，而在$x = 2$ 处发散，那么该级数的收敛区间是$[0, 2)$.

(3) 若函数$f(x, y)$在(x_0, y_0)处存在偏导数$f_x(x, y)$和$f_y(x, y)$，那么
$$\lim_{x \to x_0} f(x, y_0) = \lim_{y \to y_0} f(x_0, y) = f(x_0, y_0).$$

(4) 若一个向量函数$\mathbf{r}(t)$总是垂直于其导数$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$，则$|\mathbf{r}(t)|$一定是常数.

(5) 在原点处，单位圆的曲率大于抛物线$y = x^2$的曲率.

2. (9pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

   (1) Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero orthogonal vectors, which of the following must be true?

   (A) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}|+|\mathbf{b}|$; (B) $|\mathbf{a}-\mathbf{b}|=|\mathbf{a}|-|\mathbf{b}|$;
   (C) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$; (D) $\mathbf{a}+\mathbf{b}=\mathbf{a}-\mathbf{b}$.

   (2) The equations of two lines are $l_1: x=t,\ y=2t,\ z=-t$, and $l_2: x=1-2t,\ y=t,\ z=-1+t$. Then l_1 and l_2 are

   (A) parallel; (B) orthogonal; (C) intersect with each other; (D) skew.

   (3) Suppose $0\le a_n<\frac{1}{n}$, $(n=1,2,\cdots)$, then which of the following series must converge?

   (A) $\sum\limits_{n=1}^{\infty}a_n$; (B) $\sum\limits_{n=1}^{\infty}(-1)^n a_n$; (C) $\sum\limits_{n=1}^{\infty}\sqrt{a_n}$; (D) $\sum\limits_{n=1}^{\infty}(-1)^n a_n^2$.

二、（9分）**单项选择题：**

   (1) 若向量a和b都是非零正交向量，下面哪一个结论是正确的？

   (A) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}|+|\mathbf{b}|$; (B) $|\mathbf{a}-\mathbf{b}|=|\mathbf{a}|-|\mathbf{b}|$;
   (C) $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$; (D) $\mathbf{a}+\mathbf{b}=\mathbf{a}-\mathbf{b}$.

   (2) 设两条直线的参数方程为$l_1: x=t, y=2t, z=-t$，以及$l_2: x=1-2t$, $y=t$, $z=-1+t$. 那么l_1 与l_2 的关系为

   (A) 平行; (B) 正交; (C) 相交; (D) 异面.

   (3) 若$0\le a_n<\frac{1}{n}$，$(n=1,2,\cdots)$，那么下面哪一个级数一定收敛?

   (A) $\sum\limits_{n=1}^{\infty}a_n$; (B) $\sum\limits_{n=1}^{\infty}(-1)^n a_n$; (C) $\sum\limits_{n=1}^{\infty}\sqrt{a_n}$; (D) $\sum\limits_{n=1}^{\infty}(-1)^n a_n^2$.

3. (9 pts) Does the following series absolutely converge, conditionally converge, or diverge? Give reasons for your answer.

   (1) $\sum\limits_{n=1}^{\infty}(-1)^n\left(1-\dfrac{1}{n}\right)^{n^2}$.

   (2) $\sum\limits_{n=1}^{\infty}(-1)^n e^{-\frac{1}{n}}$.

   (3) $\sum\limits_{n=2019}^{\infty}(-1)^n\dfrac{1}{\sqrt{n^2-2019n+1}}$.

三、（9分）下列级数是否绝对收敛、条件收敛或者发散？给出你的理由.

   (1) $\sum\limits_{n=1}^{\infty}(-1)^n\left(1-\dfrac{1}{n}\right)^{n^2}$.

   (2) $\sum\limits_{n=1}^{\infty}(-1)^n e^{-\frac{1}{n}}$.

(3) $\sum_{n=2019}^{\infty}(-1)^n\frac{1}{\sqrt{n^2-2019n+1}}$.

4. (10 pts)

   (1) Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2^nx^n}{\sqrt{n^2+n+1}}.$$

   (2) For what values of x does the series converge absolutely, or conditionally?

四、（10分）

   (1) 求级数

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2^nx^n}{\sqrt{n^2+n+1}}$$

   的收敛半径和收敛区间.

   (2) x取哪些值时级数绝对收敛，取哪些值时级数条件收敛？

5. (10 pts) Let $x=\cos^3 t$, $y=\sin^3 t$, where $0\le t\le\frac{\pi}{2}$, be a parametrization of a curve.

   (1) Find the length of the curve.

   (2) Find the area of the surface generated by revolving the curve about the x-axis.

五、（10分）设$x=\cos^3 t$，$y=\sin^3 t$，$0\le t\le\frac{\pi}{2}$，为一参数化曲线.

   (1) 求曲线的弧长.

   (2) 求曲线绕x轴旋转所得到的曲面的面积.

6. (10 pts) Find the equation of the plane through the points $(2,-1,-1)$ and $(1,0,-1)$ perpendicular to the plane $2x+3y-5z+6=0$.

六、（10分）如果一个平面通过两个点$(2,-1,-1)$和$(1,0,-1)$，并且与平面$2x+3y-5z+6=0$ 垂直，请写出平面方程.

7. (10 pts) A particle is located at the point $(1,0,-2)$. Its initial speed is $|\mathbf{v}(0)|=3$ at time $t=0$, and the direction of its initial velocity is toward the point $(2,-1,3)$. The particle moves with constant acceleration $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t.

七、（10分）一个物体在$t=0$时初始坐标为$(1,0,-2)$. 它在$t=0$时的初始速率为$|\mathbf{v}(0)|=3$，而且其初始速度的方向是指向点$(2,-1,3)$. 这个物体运动的加速度为常数向量，$\mathbf{i}+2\mathbf{j}+\mathbf{k}$. 请写出其位移向量$\mathbf{r}(t)$关于时间$t$的参数表示.

8. (10 pts) Is the following function, $f(x,y)$ continuous at $(0,0)$? Give reasons for your answer.

$$f(x,y)=\begin{cases}\frac{\sin(x^3+y^3)}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

八、 （10分）请问如下函数$f(x,y)$在原点处是否连续，给出你的理由.

$$f(x,y)=\begin{cases}\frac{\sin(x^3+y^3)}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

9. (9 pts) Let $f(u)$ be differentiable, $z=f(e^x y)$ $(y\neq 0)$, and $\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$. If $f(1)=0$, find $f(u)$.

九、 （9分）设函数$f(u)$可导，定义$z=f(e^x y)$ $(y\neq 0)$，而且有$\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$. 已知$f(1)=0$，请写出$f(u)$的表达式.

10. (8 pts) Find the Taylor series for $f(x)=\ln\left(x+\sqrt{x^2+1}\right)$ at $x=0$.

十、 （8分）请写出函数$f(x)=\ln\left(x+\sqrt{x^2+1}\right)$在$x=0$处的Taylor级数.