

## Step-1

Consider the equations,

$$\begin{aligned}-u'' &= x, \\ u(0) &= 0, \\ u(1) &= 1\end{aligned}$$

By using four intervals and two and an extra half-hat functions, with  $h = \frac{1}{3}$ , the matrix  $A$  (2 by 2) is given by,

$$A = 3 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Let

$$f(x) = x$$

Therefore, we get,

$$\begin{aligned}b &= hf(x) \\ &= \left(\frac{1}{3}\right)x \\ &= \frac{x}{3} \\ &= \frac{1}{3} \begin{bmatrix} x \\ x \end{bmatrix}\end{aligned}$$

## Step-2

By substituting  $A$ , and  $b$  into  $Ay = b$ , we get,

$$\begin{aligned}3 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} y &= \frac{1}{3} \begin{bmatrix} x \\ x \end{bmatrix} \\ \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} y &= \frac{1}{9} \begin{bmatrix} x \\ x \end{bmatrix} \\ y &= \frac{1}{9} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ x \end{bmatrix}\end{aligned}$$

The inverse matrix  $A$  is given by,

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

### Step-3

On substitution, we get,

$$y = \frac{1}{27} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 3x \\ 3x \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} x \\ x \end{bmatrix}$$

The linear finite element is given by,

$$U(x) = \frac{1}{9}V_1 + \frac{1}{9}V_2$$

### Step-4

Thus, the linear finite element is  $\boxed{U(x) = \frac{1}{9}V_1 + \frac{1}{9}V_2}$ .