Step-1

We have to find the vectors (b_1, b_2, b_3) for which the following systems have a solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Step-2

For the system,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow x_3 &= b_3 \\ x_2 + x_3 &= b_2 \\ x_1 + x_2 + x_3 &= b \end{aligned}$$

Step-3

$$\begin{aligned} \Rightarrow x_3 &= b_3 \\ x_2 &= b_2 - b_3 \\ x_1 &= b_1 - b_2 + b_3 - b_3 \end{aligned}$$

$$\begin{aligned} \Rightarrow x_3 &= b_3 \\ x_2 &= b_2 - b_3 \\ x_1 &= b_1 - b_2 \end{aligned}$$

Step-4

Or

$$\begin{aligned} b_3 &= x_3 \\ b_2 &= x_2 + x_3 \\ b_1 &= x_1 + x_2 + x_3 \end{aligned}$$

Step-5

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_2 - b_3 \\ b_3 \end{bmatrix}$$

Therefore for any (b_1, b_2, b_3) this system has a solution.

Step-6

For the system,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow b_1 = x_1 + x_2 + x_3 \\ &b_2 = x_2 + x_3 \\ &b_3 = 0 \end{aligned}$$

Therefore the given system has a solution if $(b_1, b_2, 0)$