SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Course Name: DG MA327 **Department:** Mathematics
Due date: 10th May **Exam Paper Setter:** HUANG Shaochuang

Question No.	1	2	3	4	5	6	7	8	9	10	11
Score	18	5	8	5	10	8	12	6	8	10	10

When you use some fact in the textbook or lecture notes, state it clearly first.

1. *(18 points)* (a)(6 points) Show that every regular parametrized differentiable curve can be reparametrized by arc length.

(b)(6 points) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized (by arc length s) curve with nowhere vanishing curvature. Show that α is a plane curve (i.e. $\alpha(I)$ is contained in a plane) if and only if its torsion τ is identically equal to 0.

(c)(6 points) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curve and let $[a, b] \subset I$. Show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|dt.$$

2. *(5 points)* Is there a simple closed curve in the plane with length equal to 6 meters and bounding an area of 3 square meters. (Write down yes or no and then explain your answer.)

3. *(8 points)* Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curves with curvature nowhere vanishing. Suppose every osculating plane along α passes through a fixed point, show that α is a plane curve.

4. *(5 points)* Construct a regular parametrized differentiable curve $\alpha : I \to \mathbb{R}^2$ which is injective and there exists $t_0 \in I$ such that for all small open disk $B(\alpha(t_0), \varepsilon)$ in $\mathbb{R}^2$ with center $\alpha(t_0)$ and radius ε, $B(\alpha(t_0), \varepsilon) \cap \alpha(I)$ is not homeomorphic to an open interval in $\mathbb{R}$. (You may draw some example with clear explanation or write down α explicitly by functions and then explain your example clearly.)

5. *(10 points)* (a)(2 points) Write down the definition of a regular surface clearly.

(b)(4 points) Write down or draw with explanation an example of some "two dimensional" object which is not a regular surface and explain why your example is not a regular surface without proof.

(c)(4 points) Write down an example of a regular surface and prove your example is a regular surface (you may use some facts in the textbook or lecture notes without proof).

6. *(8 points)* Prove that the tangent planes of a regular surface given by the graph of $z = x \cdot f(\frac{y}{x})$, $x \neq 0$, where f is a differentiable function, all pass through the origin $(0, 0, 0)$.

7. *(12 points)* (a)(6 points) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized differentiable curves with curvature nowhere vanishing which is parametrized by arc-length s. Consider the following parametrized surface

$$\mathbf{x}(s,v) = \alpha(s) + r(n(s)\cos v + b(s)\sin v), \quad s \in I.$$

Here r is some fixed positive constant, n is the normal of α and b is the bi-normal of α. Find the unit normal vector of $\mathbf{x}$ whenever it is regular. (Write down your computation clearly.)

(b)(6 points) Find the area of the above surface. (It depends on r and the length l of α. Write down your computation clearly.)

8. *(6 points)* Compute the first fundamental form of the following parametrization for the sphere:

$$\mathbf{x}(u,v) = (a\cos u\cos v, a\cos u\sin v, a\sin u),$$

where a is a positive constant. Consider two curves $u = v$ and $v = v_0$ with v_0 a fixed constant on the sphere, compute $\cos\beta$. Here β is the angle between these two curves where they intersect.

9. *(8 points)* Consider the paraboloid i.e. the graph of $z = x^2 + ky^2$ with k a positive constant and $p = (0,0,0)$. Prove that the unit vector of the x axis and the y axis are eigenvectors of dN_p, with eigenvalue 2 and $2k$, respectively (assuming that N is pointing outwards from the region bounded by the paraboloid).

10. *(10 points)* Prove that if all normal lines to a connected regular surface S meet a fixed point, then S is a piece of a sphere.

11. *(10 points)* (a)(8 points) Show that a surface which is compact (i.e. bounded and closed in $\mathbb{R}^3$) has an elliptic point.

(b)(2 points) Prove that there are no compact minimal surfaces in $\mathbb{R}^3$.