南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数 开课单位: 数学系
考试时长: 120 分钟 命题教师: 线性代数教学团队

题 号	1	2	3	4	5	6	7
分 值	15 分	25 分	10 分	20 分	10 分	10 分	10 分

本试卷共（7）大题, 满分（100）分. 请将所有答案写在答题本上.

This exam includes 7 questions and the score is 100 in total. **Write all your answers on the examination book.**

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1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共 15 分, 每小题 3 分) 选择题. 每题只有一个选项是正确的.

(1) Which of the following statements can guarantee that the following homogeneous system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

has a nonzero solution ?

(A) $m \le n$.

(B) $m = n$.

(C) $m > n$.

(D) The rank of the coefficient matrix is less than n.

当条件（ ）满足时, 下面的齐次线性方程组一定有非零解.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

(A) $m \le n$.

(B) $m = n$.

(C) $m > n$.

(D) 系数矩阵的秩小于 n.

(2) Which of the following matrices can be written as a product of elementary matrices ? ( )

(A) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(D) $\begin{bmatrix} -3 & 2 & 7 \\ -1 & 2 & 3 \\ 0 & -2 & -1 \end{bmatrix}$.

下列矩阵中可以化为有限个初等矩阵之积的矩阵是（ ）

(A) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(D) $\begin{bmatrix} -3 & 2 & 7 \\ -1 & 2 & 3 \\ 0 & -2 & -1 \end{bmatrix}$.

(3) Let $\beta_1, \beta_2, \beta_3$ be a basis of the null space $N(A)$ of some matrix A. Another basis of $N(A)$ is ( )

(A) $\beta_1 + \beta_2$, $\beta_2 + \beta_3$, $\beta_3 + \beta_1$.

(B) $\beta_1 + \beta_2$, $\beta_2 + \beta_3$, $\beta_3 - \beta_1$.

(C) $\beta_1 - \beta_2$, $\beta_2 - \beta_3$, $\beta_3 - \beta_1$.

(D) $\beta_1 + 2\beta_2$, $2\beta_2 + 3\beta_3$, $3\beta_3 - \beta_1$.

设 $\beta_1, \beta_2, \beta_3$ 为某矩阵 A 的零空间 $N(A)$ 的一组基. 则 $N(A)$ 的另一组基是 ( )

(A) $\beta_1 + \beta_2$, $\beta_2 + \beta_3$, $\beta_3 + \beta_1$.

(B) $\beta_1 + \beta_2$, $\beta_2 + \beta_3$, $\beta_3 - \beta_1$.

(C) $\beta_1 - \beta_2$, $\beta_2 - \beta_3$, $\beta_3 - \beta_1$.

(D) $\beta_1 + 2\beta_2$, $2\beta_2 + 3\beta_3$, $3\beta_3 - \beta_1$.

(4) Let $a, b \in \mathbb{R}$. The set

$$V = \{(x, y, z, w) : x + 2y + 3z + 4w = a + b + 1,\ x - 2y + 4z - w = a - 2b - 5\}$$

is a subspace of $\mathbb{R}^4$ if

(A) $a = -1, b = 1$.

(B) $a = -2, b = 1$.

(C) $a=1, b=-2$.

(D) $a=1, b=-1$.

设 $a, b \in \mathbb{R}$. 集合

$$V=\{(x,y,z,w): x+2y+3z+4w=a+b+1,\ x-2y+4z-w=a-2b-5\}$$

在 ( ) 成立时是 $\mathbb{R}^4$ 的子空间.

(A) $a=-1, b=1$.

(B) $a=-2, b=1$.

(C) $a=1, b=-2$.

(D) $a=1, b=-1$.

(5) Let u and v be unit vectors in $\mathbb{R}^3$. If the vectors $u+2v$ and $5u-4v$ are orthogonal, then the angle α between u and v is ( )

(A) $\alpha=\frac{\pi}{6}$.

(B) $\alpha=\frac{\pi}{4}$.

(C) $\alpha=\frac{\pi}{3}$.

(D) $\alpha=\frac{3\pi}{4}$.

设 u 和 v 均为 $\mathbb{R}^3$ 中的单位向量. 若向量 $u+2v$ 和 $5u-4v$ 正交, 则 u 和 v 之间的夹角 α 为 ( )

(A) $\alpha=\frac{\pi}{6}$.

(B) $\alpha=\frac{\pi}{4}$.

(C) $\alpha=\frac{\pi}{3}$.

(D) $\alpha=\frac{3\pi}{4}$.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let A be a 5×8 real matrix. If $\dim N(A)=3$, then $\dim(N(A^T))=$ ________.

设 A 为 5×8 实矩阵. 若 $\dim N(A)=3$, 则 $\dim(N(A^T))=$ ________.

(2) All the 2×2 matrices that commute with $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ can be written in the form ________.

所有和矩阵 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ 乘法可交换的 2×2 矩阵均可写成 ________ 的形式.

(3) Let $A=\begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & \lambda \\ 3 & 1 & -1 \end{bmatrix}$. If $AB=0$ for some nonzero matrix B, then $\lambda=$________.

设 $A=\begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & \lambda \\ 3 & 1 & -1 \end{bmatrix}$. 如果 $AB=0$ 对某个非零矩阵 B 成立, 则 $\lambda=$________.

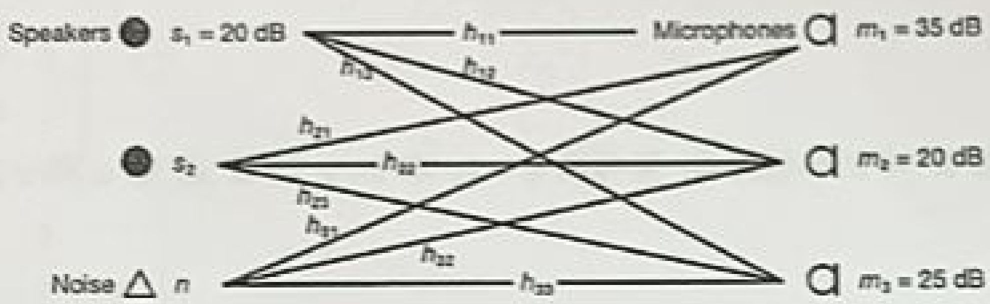
(4) An LU-factorization of $A=\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$ is $L=$________, $U=$________.

矩阵 $A=\begin{bmatrix}2 & 1\\ 3 & 7\end{bmatrix}$ 的一个 LU 分解是 $L=$__________, $U=$__________.

(5) Suppose b is a nonzero column vector. If η_1, η_2 are solutions to the system of linear equations $Ax=b$, and $\lambda_1\eta_1+\lambda_2\eta_2$ is another solution to $Ax=b$, then λ_1, λ_2 must satisfy __________.

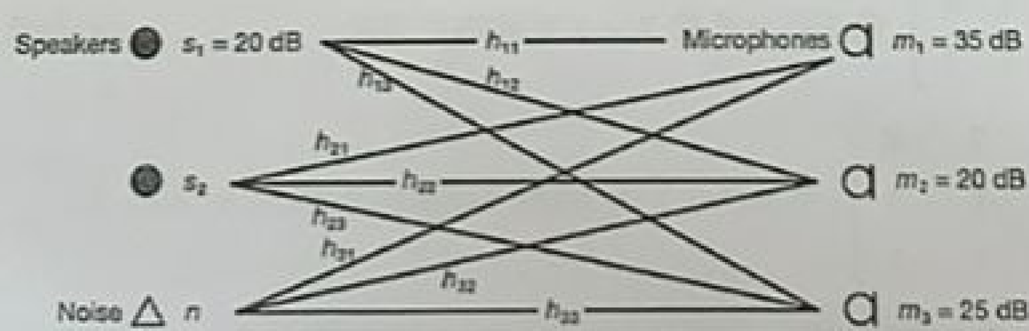
假设 b 为非零列向量. 已知 η_1, η_2 均是线性方程组 $Ax=b$ 的解, 若 $\lambda_1\eta_1+\lambda_2\eta_2$ 也是 $Ax=b$ 的解, 则 λ_1, λ_2 应满足__________.

3. (10 points) An audio processing company develops technology for mobile devices and is proud of the capacity of their products to filter surrounding noise. Here is a simplified model (a *single-layer neural network*) showing how it works. Let s_1, s_2 be the volumes of a pair of speakers and n denote that of noise. Use 3 microphones to receive signals with recorded volumes m_1, m_2, and m_3. All values are in decibels and shown in the following diagram, where the linear factor h_{ij} indicates the rate of decay along each channel. (When a sound of 100 dB is transmitted along a channel with rate of decay h, the volume received is $100h$ dB.)



Suppose we are given the matrix $[h_{ij}]=\begin{bmatrix}0.875 & 0.5 & 0.75\\ 0.25 & 0.5 & 0.5\\ 0.625 & 0.375 & 0.5\end{bmatrix}$. Estimate the volume of the unknown source speaker by solving a linear system for s_2.

(10 分) 某音频处理公司从事移动设备技术研发, 以其产品的过滤环境噪声能力而著称. 这里以一个简化模型 (单层神经网络) 来说明其技术原理. 设 s_1, s_2 为两个发声器 发出的音量, n 为噪音音量. 使用 3 个麦克风接收信号时得到的音量分别为 m_1, m_2 和 m_3. 这些值均以分贝为单位计算, 展示于下面的图表之中. 其中 h_{ij} 这些线性因子用于标示沿各个声道传输声音的衰减率. (当 100 分贝的声音经由衰减率为 h 的声道传输时, 接收到的音量为 $100h$ 分贝.)



假设我们得到的矩阵 $[h_{ij}]=\begin{bmatrix}0.875 & 0.5 & 0.75\\ 0.25 & 0.5 & 0.5\\ 0.625 & 0.375 & 0.5\end{bmatrix}$. 请通过求解一个线性方程组求出未知发声器发出的音量 s_2.

4. (20 points) Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T(\alpha_1) = \begin{bmatrix}1\\0\\0\end{bmatrix},\ T(\alpha_2) = \begin{bmatrix}1\\1\\2\end{bmatrix},\ T(\alpha_3) = \begin{bmatrix}0\\1\\1\end{bmatrix} \text{ where } \alpha_1 = \begin{bmatrix}1\\0\\1\end{bmatrix},\ \alpha_2 = \begin{bmatrix}2\\1\\3\end{bmatrix},\ \alpha_1 = \begin{bmatrix}0\\2\\1\end{bmatrix}.$$

(a) Show that $\alpha_1, \alpha_2, \alpha_3$ is a basis of $\mathbb{R}^3$.

(b) Find the representation matrix A of T (in the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$).

(c) Is the matrix A invertible? Why?

(20 分) 设 T 是 $\mathbb{R}^3$ 到 $\mathbb{R}^3$ 的线性变换, 它满足

$$T(\alpha_1) = \begin{bmatrix}1\\0\\0\end{bmatrix},\ T(\alpha_2) = \begin{bmatrix}1\\1\\2\end{bmatrix},\ T(\alpha_3) = \begin{bmatrix}0\\1\\1\end{bmatrix} \text{ 其中 } \alpha_1 = \begin{bmatrix}1\\0\\1\end{bmatrix},\ \alpha_2 = \begin{bmatrix}2\\1\\3\end{bmatrix},\ \alpha_1 = \begin{bmatrix}0\\2\\1\end{bmatrix}.$$

(a) 证明 $\alpha_1, \alpha_2, \alpha_3$ 是 $\mathbb{R}^3$ 的一组基.

(b) 求 T (在 $\mathbb{R}^3$ 的标准基 e_1, e_2, e_3 下) 的矩阵 A.

(c) 矩阵 A 是否可逆？为什么？

5. (10 points) Let L be the line of intersection of $x_1 + x_2 + x_3 = 0$ and $2x_1 - x_2 - 2x_3 = 0$ in $\mathbb{R}^3$.

Find the orthogonal projection of $b = \begin{bmatrix}1\\0\\1\end{bmatrix}$ onto L.

(10 分) 设 L 是 $x_1 + x_2 + x_3 = 0$ 和 $2x_1 - x_2 - 2x_3 = 0$ 在 $\mathbb{R}^3$ 中的交线. 求 $b = \begin{bmatrix}1\\0\\1\end{bmatrix}$ 到 L 上的正交投影.

6. (10 points) Let u, v be nonzero column vectors in $\mathbb{R}^n$ and $A = uv^T$.

(a) Prove that the rank of A is 1.

(b) What are the possible values of the rank of the matrix $\begin{bmatrix}u^Tv & 0\\0 & vu^T\end{bmatrix}$? Justify your answer.

(10 分) 设 u, v 为 $\mathbb{R}^n$ 中非零列向量, $A = uv^T$.

(a) 证明 A 的秩为 1.

(b) 矩阵 $\begin{bmatrix}u^Tv & 0\\0 & vu^T\end{bmatrix}$ 的秩可能取到哪些值？请解释理由.

7. (10 points) Let $\alpha_1, \cdots, \alpha_n$ be column vectors in $\mathbb{R}^n$. Suppose that the system $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ is linearly dependent, and that the system $\alpha_2, \alpha_3, \cdots, \alpha_n$ is linearly independent.

   Let $A = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

   (a) Show that α_1 can be written as a linear combination of $\alpha_2, \alpha_3, \cdots, \alpha_n$, i.e., there exist constants $k_2, k_3, \cdots, k_n$ so that $\alpha_1 = k_2\alpha_2 + k_3\alpha_3 + \cdots + k_n\alpha_n$.

   (b) Show that the linear system $Ax = \beta$ has infinitely many solutions.

   (c) Prove that if $n > 2$, then $A^2 \neq O$. Here O denotes the zero matrix of order n.

(10 分) 设 $\alpha_1, \cdots, \alpha_n$ 为 $\mathbb{R}^n$ 中的列向量. 假设向量组 $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ 线性相关, 而向量组 $\alpha_2, \alpha_3, \cdots, \alpha_n$ 线性无关.

令 $A = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ 以及 $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

(a) 证明 α_1 可由 $\alpha_2, \alpha_3, \cdots, \alpha_n$ 线性表出, 即, 存在常数 $k_2, k_3, \cdots, k_n$ 使得 $\alpha_1 = k_2\alpha_2 + k_3\alpha_3 + \cdots + k_n\alpha_n$.

(b) 证明线性方程组 $Ax = \beta$ 有无穷多个解.

(c) 证明: 若 $n > 2$, 则 $A^2 \neq O$. 这里 O 表示 n 阶零矩阵.