Step-1

Consider the matrices,

$$A=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix},\quad b=\begin{bmatrix}1\\2\\7\end{bmatrix}$$

The objective is to find the projection of b onto the column space of A.

Split b into $p+q$ with p in the column space and q perpendicular to that space.

Also determine the subspace that contains q.

Step-2

Projection of b onto the column space of A is $p=Pb$, here $P=A\left(A^TA\right)^{-1}A^T$

Compute matrix P as shown:

$$\begin{aligned}P&=A\left(A^TA\right)^{-1}A^T\\
&=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\left(\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}\\
&=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\left(\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}\end{aligned}$$

$$\begin{aligned}&=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\begin{bmatrix}6&-8\\-8&18\end{bmatrix}^{-1}\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}\\
&=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\left(\frac{1}{22}\begin{bmatrix}9&4\\4&3\end{bmatrix}\right)\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}\end{aligned}$$

Step-3

Continuation to the above steps,

$$P=\frac{1}{22}\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}\begin{bmatrix}9&4\\4&3\end{bmatrix}\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}$$
$$=\frac{1}{22}\begin{bmatrix}13&7\\5&1\\-2&4\end{bmatrix}\begin{bmatrix}1&1&-2\\1&-1&4\end{bmatrix}$$
$$=\frac{1}{22}\begin{bmatrix}20&6&2\\6&4&-6\\2&-6&20\end{bmatrix}$$

Step-4

Projection of b onto the column space of A is $p=Pb$

$$p=Pb$$
$$=\frac{1}{22}\begin{bmatrix}20&6&2\\6&4&-6\\2&-6&20\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix}$$
$$=\frac{1}{22}\begin{bmatrix}46\\-28\\130\end{bmatrix}$$
$$=\boxed{\begin{bmatrix}23/11\\-14/11\\65/11\end{bmatrix}}$$

Step-5

Split b into $p+q$ with p in the column space and q perpendicular to that space.

That is, $b=p+q$ where $p\in\mathrm{Col}(A)$ and $q\in\mathrm{Col}(A)^{\perp}=\mathrm{Nul}(A^{\mathrm{T}})$

As two vectors in columns of matrix $A=\begin{bmatrix}1&1\\1&-1\\-2&4\end{bmatrix}$ are linearly independent,

$$\mathrm{Col}(A)=\mathrm{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}1\\-1\\4\end{bmatrix}\right\}$$

Step-6

Find $\text{Nul}\left(A^{\text{T}}\right)$.

Let $\mathbf{x} \in \text{Nul}\left(A^{\text{T}}\right)$

$$A^{\text{T}}\mathbf{x} = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

Augmented matrix associated with the above notation is,

$$\left[\mathbf{M} \;\vdots\; \mathbf{0}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 4 & 0 \end{array}\right]$$

$$R_2 \to R_2 - R_1$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 6 & 0 \end{array}\right]$$

$$R_2 \to -\frac{1}{2}R_2$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

Step-7

From the above one we obtain the system,

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Suppose that $x_3 = s,\ s \in \mathbb{R}$

Then $x_1 = -s, x_2 = 3s$

Therefore,

$$\mathrm{Nul}\left(A^{\mathrm{T}}\right)=\left\{\begin{bmatrix}-s\\3s\\s\end{bmatrix}:s\in\mathbb{R}\right\}$$
$$=\left\{s\begin{bmatrix}-1\\3\\1\end{bmatrix}:s\in\mathbb{R}\right\}$$

Step-8

To express $b=p+q$ where $p\in\mathrm{Col}(A)$ and $q\in\mathrm{Col}(A)^{\perp}=\mathrm{Nul}\left(A^{\mathrm{T}}\right)$:

Observe that,

$$b=\begin{bmatrix}1\\2\\7\end{bmatrix}$$
$$=\begin{bmatrix}\frac{11}{11}\\\frac{22}{11}\\\frac{77}{11}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{23-12}{11}\\\frac{-14+36}{11}\\\frac{65+12}{11}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{23}{11}\\-\frac{14}{11}\\\frac{65}{11}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\\frac{36}{11}\\\frac{12}{11}\end{bmatrix}\quad(=p+q)$$
$$=\begin{bmatrix}\frac{9}{22}+\frac{37}{22}\\\frac{9}{22}-\frac{37}{22}\\-\frac{18}{22}+\frac{148}{22}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\\frac{36}{11}\\\frac{12}{11}\end{bmatrix}$$

Step-9

Continuation to the above step,

$$b=\begin{bmatrix}\frac{9}{22}+\frac{37}{22}\\ \frac{9}{22}-\frac{37}{22}\\ -\frac{18}{22}+\frac{148}{22}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\ \frac{36}{11}\\ \frac{12}{11}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{9}{22}\\ \frac{9}{22}\\ -\frac{18}{22}\end{bmatrix}+\begin{bmatrix}\frac{37}{22}\\ -\frac{37}{22}\\ \frac{148}{22}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\ \frac{36}{11}\\ \frac{12}{11}\end{bmatrix}$$

$$=\frac{9}{22}\begin{bmatrix}1\\1\\-2\end{bmatrix}+\frac{37}{22}\begin{bmatrix}1\\-1\\4\end{bmatrix}+\frac{12}{11}\begin{bmatrix}-1\\3\\1\end{bmatrix}$$

$$=\text{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}1\\-1\\4\end{bmatrix}\right\}+\text{span}\left\{\begin{bmatrix}-1\\3\\1\end{bmatrix}\right\}$$

$$=\text{Col}(A)+\text{Nul}\left(A^{T}\right)$$

Thus,

$b=p+q$ Here $p=\begin{bmatrix}\frac{23}{11}\\ -\frac{14}{11}\\ \frac{65}{11}\end{bmatrix}\in\text{Col}(A)$ and $q=\begin{bmatrix}-\frac{12}{11}\\ \frac{36}{11}\\ \frac{12}{11}\end{bmatrix}\in\text{Nul}\left(A^{T}\right)$

Therefore $q\in\text{Nul}\left(A^{T}\right)$