1. (15 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

   (1) If $f(x, y)$ has both partial derivatives $f_x(x, y)$, $f_y(x, y)$ at point (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

   (2) The curvature of a circle is the radius of the circle.

   (3) If both $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ converge, then $\sum_{n=1}^{+\infty} a_n b_n$ must also converge.

   (4) Let $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + xy\mathbf{k}$ represent the velocity of a gas flowing in space. The gas is neither expanding nor compressing at any point.

   (5) If $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is defined on an open region, and its component functions have continuous first partial derivatives and satisfy
   $$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
   Then $\mathbf{F}$ is conservative.

2. (12 pts) Please fill in the blank for the questions below.

   (1) If $\mathbf{r}$ is a differentiable vector function of t of constant length, then $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} =$ ______________.

   (2) The direction in which $f(x, y) = x^2 y + e^{xy} \sin y$ decreases most rapidly at the point $(1, 0)$ is ______________.

   (3) The equation for the tangent plane at the point $(1, -1, 3)$ on the surface $x^2 + 2xy - y^2 + z^2 = 7$ is ______________.

   (4) Suppose that $f(x, y)$ and its first and second partial derivatives are continuous, and $f(0, 0) = 1$, $f_x(0, 0) = 2$, $f_y(0, 0) = 3$, $f_{xx}(0, 0) = 2$, $f_{xy}(0, 0) = -1$, $f_{yy}(0, 0) = 4$. Then $f(x, y) \approx$ ______________ when both x and y are small (using Taylor's formula for $f(x, y)$ at $(0, 0)$) to find the quadratic approximation of f.

3. (3pts) Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$, $f_{xx}(a, b) = -2$, $f_{xy}(a, b) = 1$, $f_{yy}(a, b) = 2$. Then

   (A) f has a local maximum at (a, b); (B) f has a local minimum at (a, b);

   (C) f has a saddle point at (a, b); (D) the test is inconclusive.

4. (20 pts) Which of the following series converge absolutely, which converge conditionally, and which diverge? Give reasons for your answer.

$$(1) \sum_{n=1}^{+\infty} (-1)^n \frac{1}{\sqrt{n(n+1)}}; \quad (2) \sum_{n=2}^{+\infty} (-1)^n \frac{1}{n(\ln n)^3};$$
$$(3) \sum_{n=1}^{+\infty} (-1)^n \frac{n^2+1}{2n^2+n-1}; \quad (4) \sum_{n=1}^{+\infty} \frac{(-3)^n}{n!}.$$

5. (10 pts) Find the Maclaurin series for the function $f(x) = \frac{1}{(2-x)^2}$.

6. (10 pts) Find the length of the astroid

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi.$$

7. (10 pts) Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$. Show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = (f_x)^2 + (f_y)^2.$$

8. (10 pts) Find the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$, and the curvature κ for the plane curve

$$\mathbf{r}(t) = (2t+3)\mathbf{i} + (5 - t^2)\mathbf{j}.$$

9. (15 pts) Let

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

   (1) Show that $f(x, y)$ is continuous at $(0, 0)$.

   (2) Compute $f_y(0, 0)$.

   (3) Compute $f_{yx}(0, 0)$.

10. (10 pts) Use the Lagrange multipliers to find the minimal and maximal value of $f(x, y, z) = x^4 + y^4 + z^4$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

11. (10 pts) Consider

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx.$$

   (1) Sketch the region of integration.

   (2) Reverse the order of integration, and evaluate the integral.

12. (10 pts) Set up a triple integral in spherical coordinates that gives the volume of the solid bounded below by the xy-plane, on the sides by the sphere $x^2 + y^2 + z^2 = 4$, and above by the cone $z = \sqrt{x^2 + y^2}$, and then evaluate the integral.

13. (10 pts) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the **substitution in double integral** (please find the transformation by yourself) to evaluate the integral

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy.$$

14. (10 pts) Find the mass of a thin wire that lies along the curve

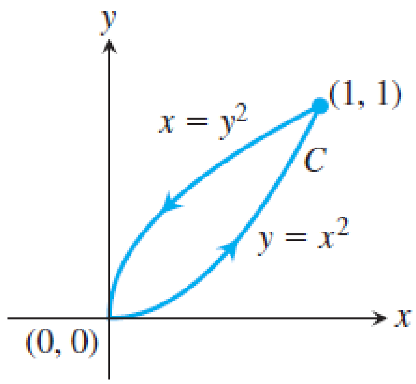
$$\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}, \quad 0 \le t \le 2,$$

if the density is $\delta(x, y, z) = 3\sqrt{25 + x + 2y}$.

15. (10 pts) Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

$$\mathbf{F} = (xy + y^2)\mathbf{i} + (x - y)\mathbf{j}\,;$$

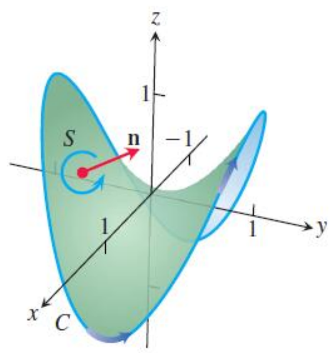
where C is shown in the figure below.



16. (10 pts) The surface S is formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the right circular cylinder of radius one around the z-axis. Let C be the boundary curve of S (see the figure below). Calculate

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$, and $\mathbf{n}$ is the unit normal vector of the surface S.



17. (15 pts) Consider the line integral

$$\int_{(1,1,1)}^{(1,2,3)} 3x^2\, dx + \frac{z^2}{y}\, dy + 2z \ln y\, dz\,.$$

(1) Show that the differential form in the integral is exact;

(2) Find a scalar function f such that $df = 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz$;

(3) Evaluate the integral.

18. (10 pts) Use the Divergence Theorem to find the outward flux of

$$\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$$

across the **boundary** of the solid sphere $D:\ x^2 + y^2 + z^2 \leq 4$.