## Step-1

A real symmetric matrix $A$ is given by

$$A^H = A$$

We know that the entries of a real symmetric matrix are symmetric to the diagonal.

$$a_{ij} = a_{ji}$$

Consider an $n$ by $n$ symmetric matrix, then there are $n$ entries on the diagonal and $(n-1)+\cdots+1$ entries above the diagonal that can be chosen arbitrary.

Since, other $1+\cdots+(n-1)$ entries below the diagonal are determined by the symmetry of the matrix.

So, there are only

$$n+(n-1)+\cdots+1=\frac{n(n+1)}{2}$$

degrees of freedom in the selection of the $n^2$ entries in an $n$ by $n$ real symmetric matrix $S$.

Therefore, dimension of an $n$ by $n$ symmetric matrix is $\boxed{\frac{n(n+1)}{2}}$.

## Step-2

If $A$ is a real symmetric matrix and $Q$ is an orthogonal matrix, then we can write:

$$\begin{aligned}
A &= A^H \\
&= QAQ^H \\
&= \begin{bmatrix} q_1 & \cdots & q_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix} \\
&= q_1 q_1^H \lambda_1 + \cdots + q_n q_n^H \lambda_n
\end{aligned}$$

Here matrix $q_n q_n^H$ is the projection matrix onto $q_n$, every symmetric matrix $A$ is a combination of $n$ projections matrices.

If matrix $A$ changes then the projections also changes.

Since the dimension exceeds $n$, there is no basis of $n$ fixed projection matrices, in the space $S$ of symmetric matrices.