Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node j to node k, then that row has -1 in column j and + 1 in column k.

The 6 by 4 incidence matrix A for the second graph in the figure is,

$$A = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Step-2

The dimensions of the four fundamental subspaces for the 6 by 4 incidence matrix and a basis for each subspace

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}$

In order to find the dimensions and a basis, we need to set the incidence matrix as $\begin{bmatrix} A & b \end{bmatrix}$.

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & b_1 \\ -1 & 0 & 1 & 0 & b_2 \\ 0 & -1 & 1 & 0 & b_3 \\ 0 & -1 & 0 & 1 & b_4 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Apply $R_1 \to R_1 + R_4, R_2 \to R_2 - R_5 \text{and} R_4 \to R_4 - R_6$

$$= \begin{bmatrix} -1 & 0 & 0 & 1 & b_1 + b_4 \\ 0 & 0 & 1 & -1 & b_2 - b_5 \\ 0 & -1 & 1 & 0 & b_3 \\ 0 & -1 & 1 & 0 & b_4 - b_6 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Step-3

Apply $R_1 \to R_1 - R_5$ and $R_3 \to R_3 - R_4$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & b_1 + b_4 - b_5 \\ 0 & 0 & 1 & -1 & b_2 - b_5 \\ 0 & 0 & 0 & 0 & b_3 - b_4 + b_6 \\ 0 & -1 & 1 & 0 & b_4 - b_6 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Apply $R_2 \to R_2 + R_6$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & b_1 + b_4 - b_5 \\ 0 & 0 & 0 & 0 & b_2 - b_5 + b_6 \\ 0 & 0 & 0 & 0 & b_3 - b_4 + b_6 \\ 0 & -1 & 1 & 0 & b_4 - b_6 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Therefore, $AX = b$ is solvable if,

$$b_1 + b_4 - b_5 = 0$$
$$b_3 - b_4 + b_6 = 0$$
$$b_2 - b_5 + b_6 = 0$$

Therefore, it is possible at;

$$\boxed{\begin{array}{l} b_1 + b_4 - b_5 = 0 \\ b_3 - b_4 + b_6 = 0 \text{and} \\ b_2 - b_5 + b_6 = 0 \end{array}}$$

Step-4

Now $AX = 0$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -x_2 + x_3 &= 0 \\ -x_1 + x_4 &= 0 \\ -x_3 + x_4 &= 0 \end{aligned}$$

Step-5

$$\begin{aligned} x_1 &= x_2 \\ &= x_3 \\ &= x_4 \end{aligned}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, null space of $\boxed{N(A) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$.

Step-6

Since, $\dim C(A) + \dim(N(A)) = 4$

$$\left(\text{As} \dim(N(A)) = 1\right)$$

$$\begin{aligned} \dim(C(A)) &= 4 - 1 \\ &= 3 \end{aligned}$$

Since, $\dim\left(C\left(A^T\right)\right) = \text{rank } r$ and $\dim\left(N\left(A^T\right)\right) = m - \text{rank}\left(r\right)$

So,

$$\left[\text{Since}, m = \text{no of rows}\left(6\right), \text{rank of} A = 3\right]$$

$$\begin{aligned}\dim\left(N\left(A^T\right)\right) &= m - r \\ &= 6 - 3 \\ &= 3\end{aligned}$$

And,

$$\left(\text{as rank of } A = 3\right)$$
$$\dim\left(C\left(A^T\right)\right) = 3$$

Therefore,

$$\boxed{\begin{array}{c}\dim\left(N\left(A\right)\right) = 1 \\ \dim\left(C\left(A\right)\right) = 3 \\ \dim\left(C\left(A^T\right)\right) = 3 \text{and} \\ \dim\left(N\left(A^T\right)\right) = 3\end{array}}$$