Step-1

If matrix A is invertible, then particular solution to following equation is $u_p = A^{-1}B$.

$$du/dt = Au - b$$

Let u_n be the general solution to the following differential equation:

$$du/dt = Au$$

Then the complete solution is $u_n + u_p$.

Step-2

(a) Find the complete solution to the following:

$$\frac{du}{dt} = 2u - 8$$

The above solution can be written as follows:

$$\frac{du}{dt} = Au - b$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Step-3

Particular solution will be:

$$\begin{aligned} u_p &= A^{-1}B \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot 8 \\ &= \frac{1}{2} \cdot 8 \\ &= 4 \end{aligned}$$

Step-4

Therefore, $u_p = 4$.

Step-5

General solution u_n can be calculated as follows:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

Calculate Eigen values of matrix A:

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = 0$$
$$(2-\lambda)(2-\lambda) = 0$$
$$(2-\lambda)^2 = 0$$

Therefore, repeated Eigen value is $\lambda = 2$.

Step-6

Let x be the Eigen vectors corresponding to Eigen value $\lambda = 2$. Then general solution u_n will be:

$$u_n = ce^{2t} x$$

Here, c is any constant.

Step-7

Therefore, the complete solution is $u_n + u_p$.

$$\boxed{u_n + u_p = ce^{2t} x + 4}$$

Step-8

(b) Find the complete solution to the following:

$$\frac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u - \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

The above solution can be written as follows:

$$\frac{du}{dt} = Au - B$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Step-9

Particular solution will be:

$$\begin{aligned} u_p &= A^{-1}B \\ &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \end{aligned}$$

Step-10

Therefore, particular solution is:

Step-11

$$u_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step-12

General solution u_n can be calculated as follows:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

Calculate Eigen values of matrix *A*:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (2-\lambda)(3-\lambda) &= 0 \end{aligned}$$

Therefore, Eigen value is $\lambda = 2,3$.

Step-13

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 3-2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda=2$ are as follows:

$$x_1=\begin{bmatrix} y \\ z \end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-14

To calculate another Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 2-3 & 0 \\ 0 & 3-3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda=3$ are as follows:

$$x_2=\begin{bmatrix} y \\ z \end{bmatrix}$$

$$=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step-15

Therefore, general solution is:

$$u_n=c_1e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+c_2e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step-16

Therefore, the complete solution is u_n+u_p.

$$\boxed{u_n + u_p = c_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix}}.$$