Step-1

Rule 1: $e^{\Lambda(t+T)} = e^{\Lambda t} e^{\Lambda T}$

Every diagonal entry of a diagonalize matrix satisfies the rule 1.

Step-2

(a) Explain why $e^{A(t+T)} = e^{At} e^{AT}$.

Recall the following;

$e^{At} = Se^{\Lambda t} S^{-1}$

Substitute this on the left hand side:

$$\begin{aligned} e^{A(t+T)} &= Se^{\Lambda(t+T)} S^{-1} \\ &= Se^{\Lambda t} e^{\Lambda T} S^{-1} \\ &= Se^{\Lambda t} S^{-1} Se^{\Lambda T} S^{-1} \\ &= e^{At} \cdot e^{AT} \end{aligned}$$

Step-3

Therefore, $\boxed{e^{A(t+T)} = e^{At} e^{AT}}$.

Step-4

(b) Let matrix A and B are defined as follows:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

To show the following:

Step-5

$e^{A+B} \neq e^{A} e^{B}$

Step-6

Recall the following:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots$$

Substitute $t = 1$:

$$e^{A} = I + A + \frac{(A)^2}{2!} + \cdots$$

Step-7

The calculation on matrix A and B shows the following:

$$A \cdot A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$B \cdot B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
$$B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-8

Substitute the above result in e^{A}:

$$\begin{aligned} e^{A} &= I + A + \frac{(A)^2}{2!} + \cdots \\ &= I + A \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} e^{B} &= I + B \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Multiplication of these two matrices is:

$$e^{A}\cdot e^{B}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$$

Step-9

Calculate $A+B$:

$$A+B=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

Since, in matrix $A+B$, trace is zero and determinant is 1. So, Eigen values are purely imaginary.

$$\det(A+B-\lambda I)=0$$
$$\begin{bmatrix}-\lambda & -1\\ 1 & -\lambda\end{bmatrix}=0$$
$$\lambda^2+1=0$$
$$\lambda=\pm i$$

Step-10

The Eigen vectors are $(1,i)$ and $(1,-i)$. The solution is:

$$e^{(A+B)t}=\frac{1}{2}e^{it}\begin{bmatrix}1\\ -i\end{bmatrix}+\frac{1}{2}e^{-it}\begin{bmatrix}1\\ i\end{bmatrix}$$

Substitute $\cos t\pm i\sin t$ for e^{it} and e^{-it}. Thus, $e^{(A+B)t}$ can be written as follows:

$$e^{(A+B)t}=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$$

For $t=1$:

$$e^{(A+B)}=\begin{bmatrix}\cos 1 & -\sin 1\\ \sin 1 & \cos 1\end{bmatrix}$$

Step-11

This matrix is not equal to the product matrix of e^{A} and e^{B}.

Step-12

Therefore, $\boxed{e^{A+B} \neq e^{A}e^{B}}$.