Step-1

a)

Consider the matrices,

$$A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \text{ and } \Lambda=\begin{bmatrix}0&0&0\\0&0&0\\0&0&3\end{bmatrix}.$$

The objective is to find an orthogonal matrix Q such that $Q^{-1}AQ=\Lambda.$

Step-2

Since the matrix Λ is diagonal matrix whose diagonal entries are the eigenvalues of the matrix A.

Therefore, the eigenvalues of the matrix A are $0,0,3.$

Now find the eigenvectors corresponding to the eigenvalues $0,0,3.$

By definition, $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if x is a nontrivial solution of $(A-\lambda I)x=0.$

That is, $$\begin{bmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \cdots\cdots (1).$$

Step-3

For $\lambda=0$, the system in (1) becomes as follows:

$$\begin{bmatrix}1-0&1&1\\1&1-0&1\\1&1&1-0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By doing row operations, the reduced row echelon form of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is obtained as $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, the system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Step-4

From this, the obtained equation is,

$x_1 + x_2 + x_3 = 0.$

Here, x_2, x_3 are free variables.

Choose $x_2 = s, x_3 = t$, where *s, t* are any parameters.

Then $x_1 = -s - t.$

The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be written as,

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} \\ &= s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, the eigenvectors corresponding to the eigenvalue $\lambda = 0$ are $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Step-5

For $\lambda = 3$, the system in (1) becomes as follows:

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-6

By doing row operations, the reduced row echelon form of the matrix $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ is obtained as $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, the system $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

From this, the obtained equation is,

$x_1 - x_3 = 0$, $x_2 - x_3 = 0$.

Here, x_3 is a free variable.

Choose $x_3 = t$, where t is any parameter.

Then $x_1 = t$ and $x_2 = t$.

Step-7

The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be written as,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$
$$= t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Step-8

Let the eigenvectors be $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note that the vector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to both the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Now find the orthogonal vectors from $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ using Gram Schmidt orthogonalisation.

Take $u_1 = v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Step-9

Compute the vector u_2 by using the formula $u_2 = v_2 - \frac{v_2^T v_1}{v_1^T v_1} v_1$.

First find the product $v_2^T v_1$.

$$v_2^T v_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
$$= (-1)(-1) + 0(1) + 1(0)$$
$$= 1 + 0 + 0$$
$$= 1$$

First find the product $v_1^T v_1$.

$$v_1^T v_1 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
$$= (-1)(-1) + 1(1) + 1(0)$$
$$= 1 + 1 + 0$$
$$= 2$$

Step-10

Substitute the known values in the formula $u_2 = v_2 - \dfrac{v_2^T v_1}{v_1^T v_1} v_1$.

$$u_2 = v_2 - \frac{v_2^T v_1}{v_1^T v_1} v_1$$
$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Step-11

Therefore, the orthonormal vectors that form the columns of the matrix Q are

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Write the matrix Q whose columns are the vectors $u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus, the matrix Q can be written as $Q = \begin{bmatrix} -1 & -1/2 & 1 \\ 1 & -1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The inverse of the matrix Q is $Q^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

Step-12

Verify whether $Q^{-1}AQ = \Lambda$ or not.

$$\begin{aligned} Q^{-1}AQ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1/2 & 1 \\ 1 & -1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \Lambda \end{aligned}$$

Hence, the required orthogonal matrix Q is $\boxed{Q = \begin{bmatrix} -1 & -1/2 & 1 \\ 1 & -1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$.

Step-13

b)

Consider the eigenvectors corresponding to the eigenvalue $\lambda = 0$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The objective is to show that $P = x_1 x_1^T + x_2 x_2^T$ is same for both the pairs of orthonormal eigenvectors for $\lambda = 0$.

First find the orthonormalization of the vectors $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

From part (a), the orthogonalisation of the $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is computed as

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Step-14

Now find the normalisation of the vectors $u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

$$\begin{aligned} x_1 &= \frac{u_1}{\|u_1\|} \\ &= \frac{1}{\sqrt{(-1)^2 + 1^2 + 0^2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \end{aligned}$$

Step-15

The normalization of the vector $u_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ is,

$$x_2 = \frac{u_2}{\|u_2\|}$$
$$= \frac{1}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$
$$= \frac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

Step-16

Now find the matrix $P = x_1 x_1^T + x_2 x_2^T$ for the pair of orthonormal vectors $x_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$.

$$P = x_1x_1^T + x_2x_2^T$$
$$= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} + \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}\begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$
$$P = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \cdots\cdots \ (1)$$

Step-17

Now compute the other pair of orthonormal vectors for the vectors $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$

Take $u_1 = v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

Compute the vector u_2 by using the formula $u_2 = v_2 - \frac{v_2^T v_1}{v_1^T v_1} v_1.$

First find the product $v_2^T v_1.$

$$v_2^T v_1 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
$$= (-1)(-1) + 1(0) + 0(1)$$
$$= 1 + 0 + 0$$
$$= 1$$

First find the product $v_1^T v_1.$

$$v_1^T v_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{aligned} &= (-1)(-1) + 1(0) + 1(1) \\ &= 1 + 0 + 1 \\ &= 2 \end{aligned}$$

Substitute the known values in the formula $u_2 = v_2 - \frac{v_2^T v_1}{v_1^T v_1} v_1$.

$$\begin{aligned} u_2 &= v_2 - \frac{v_2^T v_1}{v_1^T v_1} v_1 \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \end{aligned}$$

Therefore, the pair of orthogonal vectors are $u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$.

Step-18

The normalisation of the vectors $u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$ is as follows:

$$x_1 = \frac{u_1}{\|u_1\|}$$

$$= \frac{1}{\sqrt{(-1)^2 + 0^2 + 1^2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Step-19

The normalization of the vector $u_2 = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$ is,

$$x_2 = \frac{u_2}{\|u_2\|}$$

$$= \frac{1}{\sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2}} \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$= \frac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Step-20

Now find the matrix $P = x_1 x_1^T + x_2 x_2^T$ for the pair of orthonormal vectors $x_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and $x_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$.

$P = x_1 x_1^T + x_2 x_2^T$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} + \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \cdots\cdots \ (2)$$

From the matrices (1) and (2), observe that the matrix $P = x_1 x_1^T + x_2 x_2^T$ is same for both the pair of orthonormal eigenvectors x_1 and x_2.