Step-1

Given that $A=\begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$

We have to diagonalize the given matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S.

Step-2

First we find the eigenvalues of A.

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)(3-\lambda)-(1-i^2)=0$$

Step-3

Continuation to the above

$$\Rightarrow \lambda^2-5\lambda+6-2=0$$
$$\Rightarrow \lambda^2-9\lambda+4=0$$
$$\Rightarrow (\lambda-1)(\lambda-4)=0$$
$$\Rightarrow \lambda=1,4$$

Hence the eigenvalues of A are 1, 4

Therefore, the eigenvalue matrix is $\Lambda=\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Step-4

Now we find the eigenvectors of the given matrix.

We know that $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A if and only if $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the nonzero solution of the system $(A-\lambda I)x=0$

That is $\begin{bmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ â€¦â€¦ (1)

Step-5

For $\lambda = 1$, (1) becomes

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The Augmented matrix

$$\begin{bmatrix} 1 & 1-i & 0 \\ 1+i & 2 & 0 \end{bmatrix}$$

Add $(1-i)$ times of row 1 to row 2.

$$\begin{bmatrix} 1 & 1-i & 0 \\ 2 & 2-2i & 0 \end{bmatrix}$$

Step-6

Add (-2) times of row 1 to row 2.

$$\begin{bmatrix} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get

$$x_1 + (1-i)x_2 = 0$$

Here x_2 is free variable.

Let $x_2 = k$, where k is a free variable.

Then $x_1 = -(1-i)k$

Step-7

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} -(1-i)k \\ k \end{bmatrix}$$
$$= -k \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = 1$ is $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

Step-8

For $\lambda = 4$, (1) becomes

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The Augmented matrix is

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Add $(1(1-i))$ times of row 1 to 2 times of row 2

$$\begin{bmatrix} -2 & 1-i & 0 \\ 4 & -2+2i & 0 \end{bmatrix}$$

Step-9

Add 2 times of row 1 to row 2

$$\begin{bmatrix} -2 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get

$$-2x_1 + (1-i)x_2 = 0$$

Here x_2 is free variable.

Let $x_2 = k$, where k is a free variable.

Then $x_1 = \left(\frac{1-i}{2}\right)k$

Step-10

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{(1-i)}{2}k \\ k \end{bmatrix}$$
$$= \frac{k}{2}\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = 4$ is $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$.

Let $S = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}$

Then

$$S^{-1} = \frac{1}{2(1-i)+1(1-i)}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

$$= \frac{1}{2-2i+1-i}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$
$$= \frac{1}{2-2i+1-i}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

Step-11

Continuation to the above

$$=\frac{1}{3-3i}\begin{bmatrix}2 & -1+i\\ 1 & 1-i\end{bmatrix}$$
$$=\frac{3+3i}{18}\begin{bmatrix}2 & -1+i\\ 1 & 1-i\end{bmatrix}$$
$$=\frac{1+i}{6}\begin{bmatrix}2 & -1+i\\ 1 & 1-i\end{bmatrix}$$
$$=\frac{1}{6}\begin{bmatrix}2+2i & -2\\ 1+i & 2\end{bmatrix}$$

Therefore, $S^{-1}=\frac{1}{6}\begin{bmatrix}2+2i & -2\\ 1+i & 2\end{bmatrix}$

Step-12

Now

$$S\Lambda S^{-1}=\begin{bmatrix}1-i & 1-i\\ -1 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\frac{1}{6}\begin{bmatrix}2+2i & -2\\ 1+i & 2\end{bmatrix}$$
$$=\begin{bmatrix}1-i & 1-i\\ -1 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{6}(2+2i)+0(1+i) & \frac{1}{6}(-2)+0(2)\\ 0(2+2i)+\frac{4}{6}(1+i) & 0(-2)+\frac{4}{6}(2)\end{bmatrix}$$
$$=\begin{bmatrix}1-i & 1-i\\ -1 & 2\end{bmatrix}\begin{bmatrix}\frac{1+i}{3} & -\frac{1}{3}\\ \frac{2}{3}(1+i) & \frac{4}{3}\end{bmatrix}$$

Step-13

Continuation to the above

$$=\begin{bmatrix}(1-i)\left(\frac{1+i}{3}\right)+(1-i)\left(\frac{2}{3}(1+i)\right) & (1-i)\left(-\frac{1}{3}\right)+(1-i)\frac{4}{3}\\ -1\left(\frac{1+i}{3}\right)+2\left(\frac{2}{3}(1+i)\right) & (-1)\left(-\frac{1}{3}\right)+(2)\frac{4}{3}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{2}{3}+\frac{4}{3} & -(1-i)\frac{1}{3}+(1-i)\frac{4}{3}\\ -\frac{1}{2}(1+i)+\frac{4}{3}(1+i) & \frac{1}{3}+\frac{8}{3}\end{bmatrix}$$

Step-14

Continuation to the above

$$= \begin{bmatrix} \frac{6}{3} & (1-i)\frac{3}{3} \\ \frac{3}{3}(1+i) & \frac{9}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$$

Therefore, $A = S\Lambda S^{-1}$

Hence the matrix that diagonalizes the given matrix A is $\boxed{S = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}}$.