Step-1

The second order differential equation $y'' = 6y' - 9y$ can be written into a vector equation for $u(t) = (y(t), y'(t))$ and as a first order differential equation system by introducing y'.

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix}$$

Step-2

Above second order differential equation can also be written as follows:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$$

Step-3

Find the Eigen values and Eigen vector of matrix A. Substitute $y = e^{\lambda t}$ into the equation $y'' = 6y' - 9y$ to get the Eigen values again. Also, show that second solution is $y = te^{3t}$.

Step-4

First step is to find the Eigen values of matrix A. Do the following calculations:

$$A - \lambda I = \begin{bmatrix} 0-\lambda & 1 \\ -9 & 6-\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = 0$$
$$(-\lambda)(6-\lambda) + 9 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0$$

After solving following values are obtained:

$$\lambda_1 = 3$$
$$\lambda_2 = 3$$

Therefore, Eigen values are $\boxed{3,3}$

Step-5

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 0-\lambda & 1 \\ -9 & 6-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0-3 & 1 \\ -9 & 6-3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda=3$ are as follows:

$$x_1=\begin{bmatrix} y \\ z \end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore, Eigen vectors are as follows:

$$\boxed{x=\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

Step-6

Eigen values can also be obtained by substituting $y=e^{\lambda t}$ into the equation $y''=6y'-9y$. For the same do the following calculations:

$$y=e^{\lambda t}$$

$$y'=\lambda e^{\lambda t}$$

$$y''=\lambda^2 e^{\lambda t}$$

Step-7

Substitute the values in the following equation:

$$y''=6y'-9y$$

$$\lambda^2 e^{\lambda t}=6\lambda e^{\lambda t}-9e^{\lambda t}$$

$$\lambda^2-6\lambda+9=0$$

Step-8

Solve it further to get value of λ as 3, 3. These are the repeated roots.

To show that second solution is $y = te^{3t}$, do the following calculations:

$$\begin{aligned} y &= te^{3t} \\ y' &= 3te^{3t} + e^{3t} \\ y'' &= 9te^{3t} + 3e^{3t} + 3e^{3t} \\ &= 9te^{3t} + 6e^{3t} \end{aligned}$$

Step-9

Substitute the values in the following equation:

$$\begin{aligned} y'' &= 6y' - 9y \\ &= 6\left(3te^{3t} + e^{3t}\right) - 9\left(te^{3t}\right) \\ &= 18te^{3t} + 6e^{3t} - 9te^{3t} \\ &= 9te^{3t} + 6e^{3t} \end{aligned}$$

Solution $y = te^{3t}$ satisfies the equation $y'' = 6y' - 9y$. Therefore, second solution is $y = te^{3t}$.