Step-1

Given that, P is the projection matrix onto the line through a.

a) To find that, why is the inner product of x with Py equal to the inner product of Px with y.

The inner product of x with Py:

$$\begin{aligned} x^T(Py) &= (x^TP)y \\ &= (x^TP^T)y \qquad \text{Since the projection } P \text{ is symmetric, } P^T = P \\ &= (Px)^T y \end{aligned}$$

= inner product of Px with y.

Suppose $x^T(Py) = (Px)^T y$ â€¦â€¦ (1)

That is, the inner product of x with Py equal to the inner product of Px with y.

Step-2

b) From the data, $a = (1,1,-1)$, $x = (2,0,1)$, $y = (2,1,2)$,

If θ be the angle between x and Py, then $\cos\theta = \dfrac{x^TPy}{\|x\|\|Py\|} \Rightarrow \theta = \cos^{-1}\left(\dfrac{x^TPy}{\|x\|\|Py\|}\right)$.

Here, the projection matrix is $P = \dfrac{aa^T}{a^Ta}$.

Now,

$$\begin{aligned} aa^T &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

And also,

$$a^T a = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
$$= 1+1+1$$
$$= 3$$

So, the projection matrix onto the line through a is,

$$P = \frac{aa^T}{a^T a}$$
$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Step-3

The product of the matrices, P, y is,

$$Py = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 2+1-2 \\ 2+1-2 \\ -2-1+2 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

And also, the product of the matrices, x^T, Py is,

$$x^T (Py) = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{3}(2+0-1)$$
$$= \frac{1}{3}$$

Step-4

Now, find norm of the vectors as follows.

$$\begin{aligned}\|x\| &= \sqrt{4+0+1}\\ &= \sqrt{5}\end{aligned}$$

$$\begin{aligned}\|Py\| &= \frac{1}{3}\sqrt{1+1+1}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

Now, the angle is,

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{x^T Py}{\|x\|\|Py\|}\right)\\ &= \cos^{-1}\left(\frac{\left(\frac{1}{3}\right)}{\left(\sqrt{5}\right)\left(\frac{\sqrt{3}}{3}\right)}\right)\\ &= \cos^{-1}\left(\frac{1}{\sqrt{15}}\right)\end{aligned}$$

So, the angle is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right)$.

Step-5

If ϕ be the angle between Px with y, then $\cos\phi = \frac{(Px)^T y}{\|Px\|\|y\|} \Rightarrow \phi = \cos^{-1}\left(\frac{(Px)^T y}{\|Px\|\|y\|}\right)$.

Find the product of the matrices P, x.

$$Px = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 2+0-1 \\ 2+0-1 \\ -2-0+1 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

And also,

$$(Px)^T y = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$
$$= \frac{1}{3}(2+1-2)$$
$$= \frac{1}{3}$$

Step-6

Now, find norm of the vectors as follows.

$$\|y\| = \sqrt{4+1+4}$$
$$= 3$$

$$\|Px\| = \frac{1}{3}\sqrt{1+1+1}$$
$$= \frac{\sqrt{3}}{3}$$

Step-7

Now, the angle is,

$$\begin{aligned}\phi &= \cos^{-1}\left(\frac{(Px)^T y}{\|Px\|\|y\|}\right)\\ &= \cos^{-1}\left(\frac{\left(\frac{1}{3}\right)}{\left(\frac{\sqrt{3}}{3}\right)(3)}\right)\\ &= \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)\end{aligned}$$

So, the angle is $\phi = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)$.

Hence, the two angles are $\theta = \frac{1}{\sqrt{15}}$, $\phi = \frac{1}{3\sqrt{3}}$

That means the $\boxed{\text{two angles are not same}}$.

Step-8

c) Verify that why the inner product of Px with Py is same the inner product of x with Py and the inner product of Px with y, and also find that what is the angle between those two.

Step-9

Inner product of Px with Py is,

$$\begin{aligned}(Px)^T Py &= \left(x^T P^T\right) Py\\ &= x^T PPy && \text{Since } P^T = P\\ &= x^T P^2 y\\ &= x^T (Py) && \text{Since } P^2 = P\end{aligned}$$

Therefore from (1) and (4), $(Px)^T Py = (Px)^T y = x^T (Py)$

Hence the inner product of Px with Py is same the inner product of x with Py and the inner product of Px with y.

Step-10

Let θ be the angle between Px with Py

Step-11

Then $\cos\theta = \dfrac{(Px)^T Py}{\|Px\|\|Py\|}$

Here $Px = \dfrac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $Py = \dfrac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\begin{aligned}(Px)^T Py &= \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{9}(1+1+1) \\ &= \frac{1}{3}\end{aligned}$$

Step-12

From part (b), $\|Px\| = \dfrac{\sqrt{3}}{3}$ and $\|Py\| = \dfrac{\sqrt{3}}{3}$.

Now,

$$\begin{aligned}\cos\theta &= \frac{(Px)^T Py}{\|Px\|\cdot\|Py\|} \\ &= \frac{\left(\frac{1}{3}\right)}{\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}\right)} \\ &= \frac{\left(\frac{1}{3}\right)}{\left(\frac{3}{9}\right)} \\ &= 1\end{aligned}$$

So, $\cos\theta = 1 \Rightarrow \theta = 0$

Hence, the angle between Px, Py is $\boxed{\theta = 0}$.