CHAPTER

4

Higher Order Linear Equations

4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous everywhere. Hence solutions are valid on the entire real line.

2. Writing the equation in standard form, the coefficients are rational functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

3. Writing the equation in standard form, the coefficients are rational functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = \tan x/(x-1)$ is undefined, and hence not continuous, at $x_k = \pm(2k+1)\pi/2$, $k = 0, 1, 2, \ldots$. Hence solutions are defined on any interval that does not contain x_0 or x_k.

4. Writing the equation in standard form, the coefficients are rational functions with singularities at $x = \pm 2$. Hence the solutions are valid on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

5. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = -14$. Hence the functions are linearly independent.

7. Evaluating the Wronskian of the four functions, $W(f_1, f_2, f_3, f_4) = 0$. Hence the functions are linearly dependent. To find a linear relation among the functions,

we need to find constants c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) = 0\,.$$

Collecting the common terms, we obtain

$$(c_2 + 2c_3 + c_4)t^2 + (2c_1 - c_3 + c_4)t + (-3c_1 + c_2 + c_4) = 0\,,$$

which results in three equations in four unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2 + 2c_3 = 1$, $2c_1 - c_3 = 1$, $-3c_1 + c_2 = 1$, to find that $c_1 = 2/7,\ c_2 = 13/7,\ c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0\,.$$

8. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, t, \cos t, \sin t) = 1$.

10. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, x, x^3) = 6x$.

11. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

13. The operation of taking a derivative is linear, and hence $(c_1y_1 + c_2y_2)^{(k)} = c_1y_1^{(k)} + c_2y_2^{(k)}$. It follows that

$$L\,[c_1y_1 + c_2y_2] = c_1y_1^{(n)} + c_2y_2^{(n)} + p_1(c_1y_1^{(n-1)} + c_2y_2^{(n-1)}) + \ldots + p_n(c_1y_1 + c_2y_2).$$

Rearranging the terms, we obtain $L\,[c_1y_1 + c_2y_2] = c_1L\,[y_1] + c_2L\,[y_2]$. Since y_1 and y_2 are solutions, $L\,[c_1y_1 + c_2y_2] = 0$. The rest follows by induction.

15.(a) Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f,g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-by-3 determinant as

$$W(y_1,y_2,y_3) = y_1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

Differentiating, we obtain

$$\begin{aligned} W'(y_1,y_2,y_3) &= y_1'\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2'\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3'\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} + \\ &\quad + y_1\begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}. \end{aligned}$$

The second line follows from the observation above. Now we find that

$$W'(y_1,y_2,y_3) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hence the assertion is true, since the first determinant is equal to zero.

(b) Based on the properties of determinants,

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Adding the first two rows to the third row does not change the value of the determinant. Since the functions are assumed to be solutions of the given ODE, addition of the rows results in

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ -p_1\, y_1'' & -p_1\, y_2'' & -p_1\, y_3'' \end{vmatrix}.$$

It follows that $p_2(t)p_3(t)W' = -p_1(t)p_2(t)p_3(t)W$. As long as the coefficients are not zero, we obtain $W' = -p_1(t)W$.

(c) The first order equation $W' = -p_1(t)W$ is linear, with integrating factor $\mu(t) = e^{\int p_1(t)dt}$. Hence $W(t) = c\, e^{-\int p_1(t)dt}$. Furthermore, $W(t)$ is zero only if $c = 0$.

(d) It can be shown, by mathematical induction, that

$$W'(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}.$$

Based on the reasoning in part (b), it follows that

$$p_2(t)p_3(t)\ldots p_n(t)W' = -p_1(t)p_2(t)p_3(t)\ldots p_n(t)W,$$

and hence $W' = -p_1(t)W$.

16. Inspection of the coefficients reveals that $p_1(t) = 2$. Based on Problem 15, we find that $W' = -2W$, and hence $W = ce^{-2t}$.

19. Let $y(t) = y_1(t)v(t)$. Then $y' = y_1'v + y_1v'$, $y'' = y_1''v + 2y_1'v' + y_1v''$, and $y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. Substitution into the ODE results in

$$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v''' + p_1\left[y_1''v + 2y_1'v' + y_1v''\right] +$$

$$+p_2\left[y_1'v + y_1v'\right] + p_3y_1v = 0.$$

Since y_1 is assumed to be a solution, all terms containing the factor $v(t)$ vanish. Hence

$$y_1v''' + \left[p_1y_1 + 3y_1'\right]v'' + \left[3y_1'' + 2p_1y_1' + p_2y_1\right]v' = 0,$$

which is a second order ODE in the variable $u = v'$.

21. First write the equation in standard form:

$$y''' - 3\frac{t+2}{t(t+3)}y'' + 6\frac{t+1}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = 0\,.$$

Let $y(t) = t^2 v(t)$. Substitution into the given ODE results in

$$t^2 v''' + 3\frac{t(t+4)}{t+3}v'' = 0\,.$$

Set $w = v''$. Then w is a solution of the first order differential equation

$$w' + 3\frac{t+4}{t(t+3)}w = 0\,.$$

This equation is linear, with integrating factor $\mu(t) = t^4/(t+3)$. The general solution is $w = c(t+3)/t^4$. Integrating twice, $v(t) = c_1 t^{-1} + c_1 t^{-2} + c_2 t + c_3$. Hence $y(t) = c_1 t + c_1 + c_2 t^3 + c_3 t^2$. Finally, since $y_1(t) = t^2$ and $y_2(t) = t^3$ are given solutions, the third independent solution is $y_3(t) = c_1 t + c_1$.

4.2

1. The magnitude of $1+i$ is $R = \sqrt{2}$ and the polar angle is $\pi/4$. Hence the polar form is given by $1+i = \sqrt{2}\,e^{i\pi/4}$.

3. The magnitude of -3 is $R = 3$ and the polar angle is π. Hence $-3 = 3\,e^{i\pi}$.

4. The magnitude of $\sqrt{3} - i$ is $R = 2$ and the polar angle is $-\pi/6 = 11\pi/6$. Hence the polar form is given by $\sqrt{3} - i = 2\,e^{11\pi i/6}$.

5. Writing the complex number in polar form, $1 = e^{2m\pi i}$, where m may be any integer. Thus $1^{1/3} = e^{2m\pi i/3}$. Setting $m = 0, 1, 2$ successively, we obtain the three roots as $1^{1/3} = 1$, $1^{1/3} = e^{2\pi i/3}$, $1^{1/3} = e^{4\pi i/3}$. Equivalently, the roots can also be written as 1, $\cos(2\pi/3) + i\,\sin(2\pi/3) = (-1 + i\sqrt{3})/2$, $\cos(4\pi/3) + i\sin(4\pi/3) = (-1 - i\sqrt{3})/2$.

7. In polar form, $2(\cos\,\pi/3 + i\,\sin\,\pi/3) = 2\,e^{i(\pi/3 + 2m\pi)}$, in which m is any integer. Thus $[2(\cos\,\pi/3 + i\,\sin\,\pi/3)]^{1/2} = 2^{1/2}\,e^{i(\pi/6 + m\pi)}$. With $m = 0$, one square root is given by $2^{1/2}\,e^{i\pi/6} = (\sqrt{3} + i)/\sqrt{2}$. With $m = 1$, the other root is given by $2^{1/2}\,e^{i7\pi/6} = (-\sqrt{3} - i)/\sqrt{2}$.

8. The characteristic equation is $r^3 - r^2 - r + 1 = 0$. The roots are $r = -1, 1, 1$. One root is repeated, hence the general solution is $y = c_1 e^{-t} + c_2 e^t + c_3 t e^t$.

10. The characteristic equation can be written as $r^2(r^2 - 4r + 4) = 0$. The roots are $r = 0, 0, 2, 2$. There are two repeated roots, and hence the general solution is given by $y = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$.

12. The characteristic equation can be written as $(r^2-1)^3=0$. The roots are given by $r=\pm 1$, each with multiplicity three. Hence the general solution is $y=c_1e^{-t}+c_2te^{-t}+c_3t^2e^{-t}+c_4e^t+c_5te^t+c_6t^2e^t$.

13. The characteristic equation can be written as $r^2(r^4-1)=0$. The roots are given by $r=0,0,\pm 1,\pm i$. The general solution is $y=c_1+c_2t+c_3e^{-t}+c_4e^t+c_5\cos t+c_6\sin t$.

14. The characteristic equation can be written as $r(r^4-3r^3+3r^2-3r+2)=0$. Examining the coefficients, it follows that $r^4-3r^3+3r^2-3r+2=(r-1)(r-2)(r^2+1)$. Hence the roots are $r=0,1,2,\pm i$. The general solution of the ODE is given by $y=c_1+c_2e^t+c_3e^{2t}+c_4\cos t+c_5\sin t$.

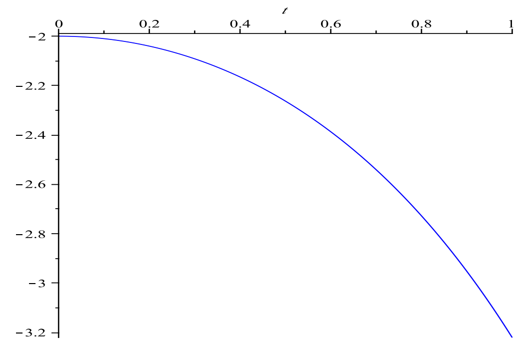
15. The characteristic equation can be written as $(r^4+4)^2=0$. The roots of the equation $r^4+4=0$ are $r=1\pm i$, $-1\pm i$. Each of these roots has multiplicity two. The general solution is $y=e^t\,[c_1\cos t+c_2\sin t\,]+te^t\,[c_3\cos t+c_4\sin t\,]+e^{-t}\,[c_5\cos t+c_6\sin t\,]+te^{-t}\,[c_7\cos t+c_8\sin t\,]$.

16. The characteristic equation can be written as $(r^2+1)^2=0$. The roots are given by $r=\pm i$, each with multiplicity two. The general solution is $y=c_1\cos t+c_2\sin t+t\,[c_3\cos t+c_4\sin t\,]$.

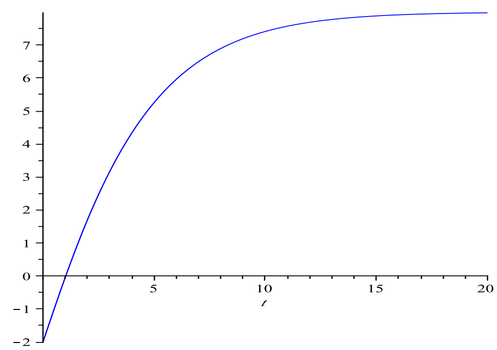
17. The characteristic equation is $r^3+5r^2+6r+2=0$. Examining the coefficients, we find that $r^3+5r^2+6r+2=(r+1)(r^2+4r+2)$. Hence the roots are deduced as $r=-1$, $-2\pm\sqrt{2}$. The general solution is $y=c_1e^{-t}+c_2e^{(-2+\sqrt{2})t}+c_3e^{(-2-\sqrt{2})t}$.

18. The characteristic equation is $r^4-7r^3+6r^2+30r-36=0$. By examining the first and last coefficients, we find that $r^4-7r^3+6r^2+30r-36=(r-3)(r+2)(r^2-6r+6)$. The roots are $r=-2,3,3\pm\sqrt{3}$. The general solution is $y=c_1e^{-2t}+c_2e^{3t}+c_3e^{(3-\sqrt{3})t}+c_4e^{(3+\sqrt{3})t}$.

23. The characteristic equation is $2r^4-r^3-9r^2+4r+4=0$, with roots $r=-1/2,1,\pm 2$. Hence the general solution is $y(t)=c_1e^{-t/2}+c_2e^t+c_3e^{-2t}+c_4e^{2t}$. Applying the initial conditions, we obtain the system of equations $c_1+c_2+c_3+c_4=-2$, $-c_1/2+c_2-2c_3+2c_4=0$, $c_1/4+c_2+4c_3+4c_4=-2$, $-c_1/8+c_2-8c_3+8c_4=0$, with solution $c_1=-16/15$, $c_2=-2/3$, $c_3=-1/6$, $c_4=-1/10$. Therefore the solution of the initial value problem is $y(t)=-(16/15)e^{-t/2}-(2/3)e^t-e^{-2t}/6-e^{2t}/10$. The solution decreases without bound.



25. The characteristic equation is $6\,r^3 + 5r^2 + r = 0$, with roots $r = 0\,,-1/3\,,-1/2$. The general solution is $y(t) = c_1 + c_2 e^{-t/3} + c_3 e^{-t/2}$. Invoking the initial conditions, we require that $c_1 + c_2 + c_3 = -2$, $-c_2/3 - c_3/2 = 2$, $c_2/9 + c_3/4 = 0$. The solution is $c_1 = 8\,,\ c_2 = -18\,,\ c_3 = 8$. Therefore the solution of the initial value problem is $y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}$. It approaches 8 as $t \to \infty$.



26.(a) Given $y = 3e^{-t} + (1/2)\cos t - \sin t$, we differentiate to find that

$$\begin{aligned} y' &= -3e^{-t} - \frac{1}{2}\sin t - \cos t, \\ y'' &= 3e^{-t} - \frac{1}{2}\cos t + \sin t, \\ y''' &= -3e^{-t} + \frac{1}{2}\sin t + \cos t, \\ y^{(4)} &= 3e^{-t} + \frac{1}{2}\cos t - \sin t \end{aligned}$$

Since $y^{(4)} = y$, the given function satisfies $y^{(4)} - y = 0$. To verify that the initial conditions are satisfied, we substitute $t = 0$ in the above derivatives, obtaining $y(0) = 3 + (1/2) - 0 = 7/2$, $y'(0) = -3 - 0 - 1 = -4$, $y''(0) = 3 - (1/2) + 0 = 5/2$, and $y'''(0) = -3 + 0 + 1 = -2$ as desired.

(b) To solve the initial value problem

$$y^{(4)} - y = 0, y(0) = \frac{7}{2}, y'(0) = -4, y''(0) = \frac{5}{2}, y'''(0) = -\frac{15}{8}$$

we find the characteristic polynomial $r^4 - 1 = 0$, which has roots $r = -1$, $r = 1$, $r = i$, and $r = -i$. The homogeneous equation has the general solution

$$y = a_1 e^{-t} + a_2 e^t + a_3 \cos t + a_4 \sin t$$

whose first three derivatives are

$$\begin{aligned} y' &= -a_1 e^{-t} + a_2 e^t - a_3 \sin t + a_4 \cos t, \\ y'' &= a_1 e^{-t} + a_2 e^t - a_3 \cos t - a_4 \sin t, \\ y''' &= -a_1 e^{-t} + a_2 e^t + a_3 \sin t - a_4 \cos t \end{aligned}$$

Evaluating these derivatives at $t = 0$ and applying the intial conditions yields the following system of equations:

$$\begin{aligned} a_1 + a_2 + a_3 &= 7/2 \\ -a_1 + a_2 + a_4 &= -4 \\ a_1 + a_2 - a_3 &= 5/2 \\ -a_1 + a_2 - a_4 &= -15/8 \end{aligned}$$

which may be solved to obtain $a_1 = 95/32$, $a_2 = 1/32$, $a_3 = 1/2$, and $a_4 = -17/16$. Thus the solution to the initial value problem is

$$y = \frac{95}{32} e^{-t} + \frac{1}{32} e^t + \frac{1}{2} \cos t - \frac{17}{16} \sin t$$

30.(a) Suppose that $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t} = 0$, and each of the r_k are real and different. Multiplying this equation by $e^{-r_1 t}$, we obtain that $c_1 + c_2 e^{(r_2 - r_1)t} + \ldots + c_n e^{(r_n - r_1)t} = 0$. Differentiation results in

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \ldots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0\,.$$

(b) Now multiplying the latter equation by $e^{-(r_2 - r_1)t}$, and differentiating, we obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3 - r_2)t} + \ldots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n - r_2)t} = 0\,.$$

(c) Following the above steps in a similar manner, it follows that

$$c_n(r_n - r_{n-1}) \ldots (r_n - r_1)e^{(r_n - r_{n-1})t} = 0\,.$$

Since these equations hold for all t, and all the r_k are different, we have $c_n = 0$. Hence $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_{n-1} e^{r_{n-1} t} = 0$, $\ -\infty < t < \infty$.

(d) The same procedure can now be repeated, successively, to show that $c_1 = c_2 = \ldots = c_n = 0$.

31.(a) Recall the derivative formula

$$\frac{d^n}{dx^n}(uv) = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \ldots + \binom{n}{n} \frac{d^n v}{dx^n} u\ .$$

Let $u=(r-r_1)^s$ and $v=q(r)$. Note that

$$\frac{d^n}{dr^n}\left[(r-r_1)^s\right]=s\cdot(s-1)\dots(s-n+1)(r-r_1)^{s-n}$$

and

$$\frac{d^s}{dr^s}\left[(r-r_1)^s\right]=s\,!\,.$$

Therefore

$$\frac{d^n}{dr^n}\left[(r-r_1)^s q(r)\right]\Big|_{r=r_1}=0$$

only if $n<s$, since it is assumed that $q(r_1)\neq 0$.

(b) Differential operators commute, so that

$$\frac{\partial}{\partial r}\left(\frac{d^k}{dt^k}e^{rt}\right)=\frac{d^k}{dt^k}\left(\frac{\partial\, e^{rt}}{\partial r}\right)=\frac{d^k}{dt^k}(t\,e^{rt}).$$

Likewise,

$$\frac{\partial^j}{\partial r^j}\left(\frac{d^k}{dt^k}e^{rt}\right)=\frac{d^k}{dt^k}\left(\frac{\partial^j\, e^{rt}}{\partial r^j}\right)=\frac{d^k}{dt^k}(t^j\;e^{rt}).$$

It follows that

$$\frac{\partial^j}{\partial r^j}L\left[e^{rt}\right]=L\left[t^j\;e^{rt}\right].$$

(c) From Eq. (i), we have

$$\frac{\partial^j}{\partial r^j}\left[e^{rt}\;\mathrm{Z}(r)\right]=L\left[t^j\;e^{rt}\right].$$

Based on the product formula in part (a),

$$\frac{\partial^j}{\partial r^j}\left[e^{rt}\;\mathrm{Z}(r)\right]\Big|_{r=r_1}=0$$

if $j<s$. Therefore $L\left[t^j\;e^{r_1t}\right]=0$ if $j<s$.

4.3

2. The general solution of the homogeneous equation is $y_c=c_1e^t+c_2e^{-t}+c_3\cos t+c_4\sin t$. Let $g_1(t)=3t$ and $g_2(t)=\cos t$. By inspection, we find that $Y_1(t)=-3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t)=t(A\cos t+B\sin t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A=0$ and $B=-1/4$. Hence the general solution of the nonhomogeneous problem is $y(t)=y_c(t)-3t-t\sin t/4$.

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1)=0$. The solution of the homogeneous equation is $y_c=c_1e^{-t}+c_2\cos t+c_3\sin t$. Let $g_1(t)=e^{-t}$ and $g_2(t)=4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t)=Ate^{-t}$. Substitution into the ODE results

in $A = 1/2$. Now let $Y_2(t) = Bt + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + te^{-t}/2 + 4(t-1)$.

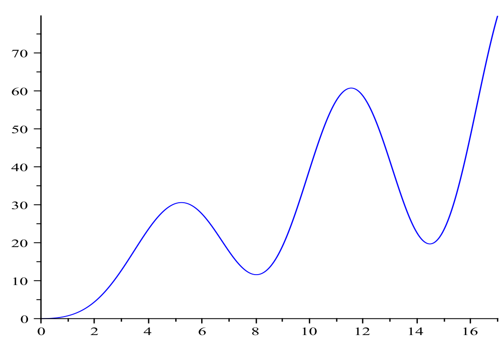
5. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1 \cos t + c_2 \sin t + t(c_3 \cos t + c_4 \sin t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + B\cos 2t + C \sin 2t$. Substitution into the ODE results in $A = 3$, $B = 1/9$, $C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \cos 2t/9$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

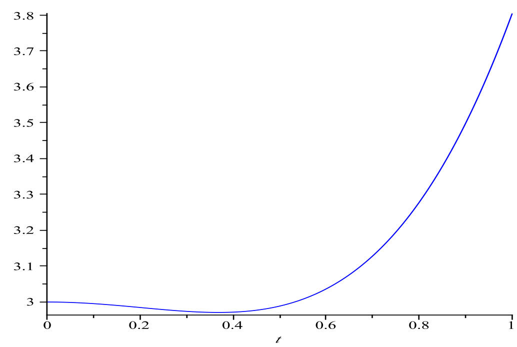
$$y_c = c_1 + c_2\, t + c_3 t^2 + c_4 e^{-t} + e^{t/2}\left[c_5 \cos(\sqrt{3}\, t/2) + c_5 \sin(\sqrt{3}\, t/2)\right].$$

Note the $g(t) = t$ is a solution of the homogenous problem. Consider a particular solution of the form $Y(t) = t^3(At + B)$. Substitution into the ODE gives us that $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. From Problem 16 in Section 4.2, the homogeneous solution is $y_c = c_1 \cos t + c_2 \sin t + t\,[c_3 \cos t + c_4 \sin t]$. Since $g(t)$ is not a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is $y(t) = (t-4)\cos t - (3t/2 + 4)\sin t + 3t + 4$.



9. The characteristic equation can be written as $(r-1)(r+3)(r^2+4) = 0$. Hence the homogeneous solution is $y_c = c_1 e^t + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + B\cos t + C \sin t$. Substitution into the ODE results in the values $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is $y(t) = c_1 e^t + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t + e^{-t}/20 - (2\cos t + 4 \sin t)/5$. Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



11. The homogeneous solution is $y_c = c_1 + c_2e^t + c_3e^{-t}$. Consider the terms $g_1(t) = te^{-t}$ and $g_2(t) = 2\cos t$. Note that since $r = -1$ is a simple root of the characteristic equation, we set $Y_1(t) = t(At + B)e^{-t}$. The function $2\cos t$ is not a solution of the homogeneous equation. We set $Y_2(t) = C\cos t + D\sin t$. Hence the particular solution has the form $Y(t) = t(At + B)e^{-t} + C\cos t + D\sin t$.

13. The characteristic equation can be written as $r^2(r^2 + 2r + 2) = 0$, with roots $r = 0$, with multiplicity two, and $r = -1 \pm i$. This means that the homogeneous solution is $y_c = c_1 + c_2t + c_3e^{-t}\cos t + c_4e^{-t}\sin t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt + C)e^{-t}$. Now $g_2(t) = e^{-t}\sin t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D\,e^{-t}\cos t + E\,e^{-t}\sin t)$. It follows that the particular solution has the form $Y(t) = Ae^t + (Bt + C)e^{-t} + t(D\,e^{-t}\cos t + E\,e^{-t}\sin t)$.

14. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^{n} \binom{n}{j} u^{(n-j)} v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \ldots, n$,

$$y^{(p)} = e^{\alpha t} \sum_{j=0}^{p} \binom{p}{j} \alpha^j\, u^{(p-j)}.$$

It follows that

$$L\left[e^{\alpha t}u\right] = e^{\alpha t} \sum_{p=0}^{n} \left[a_{n-p} \sum_{j=0}^{p} \binom{p}{j} \alpha^j\, u^{(p-j)} \right] \qquad (*).$$

It is evident that the right hand side of Eq. $(*)$ is of the form

$$e^{\alpha t}\left[k_0\, u^{(n)} + k_1\, u^{(n-1)} + \ldots + k_{\,n-1} u\,' + k_n\, u\right].$$

Hence the operator equation $L\,[e^{\alpha t} u] = e^{\alpha t}(b_0\, t^m + b_1\, t^{m-1} + \ldots + b_{\,m-1} t + b_m\,)$ can be written as

$$k_0\, u^{(n)} + k_1\, u^{(n-1)} + \ldots + k_{\,n-1} u\,' + k_n\, u = b_0\, t^m + b_1\, t^{m-1} + \ldots + b_{\,m-1} t + b_m\,.$$

The coefficients $k_{\,i}\,,\ i = 0, 1, \ldots, n$ can be determined by collecting the like terms in the double summation in Eq. $(*)$. For example, k_0 is the coefficient of $u^{(n)}$. The only term that contains $u^{(n)}$ is when $p = n$ and $j = 0$. Hence $k_0 = a_0$. On the other hand, k_n is the coefficient of $u(t)$. The inner summation in $(*)$ contains terms with u, given by $\alpha^p u$ (when $j = p$), for each $p = 0, 1, \ldots, n$. Hence

$$k_n = \sum_{p=0}^{n} a_{\,n-p}\, \alpha^p\,.$$

16.(a) Clearly, e^{2t} is a solution of $y\,' - 2y = 0$, and te^{-t} is a solution of the differential equation $y\,'' + 2y\,' + y = 0$. The latter ODE has characteristic equation $(r+1)^2 = 0$. Hence $(D-2)\left[3e^{2t}\right] = 3(D-2)\left[e^{2t}\right] = 0$ and $(D+1)^2\left[te^{-t}\right] = 0$. Furthermore, we have $(D-2)(D+1)^2\left[te^{-t}\right] = (D-2)\,[0] = 0$, and $(D-2)(D+1)^2\left[3e^{2t}\right] = (D+1)^2(D-2)\left[3e^{2t}\right] = (D+1)^2\,[0] = 0$.

(b) Based on part (a),

$$(D-2)(D+1)^2\left[(D-2)^3(D+1)Y\right] = (D-2)(D+1)^2\left[3e^{2t} - te^{-t}\right] = 0,$$

since the operators are linear. The implied operations are associative and commutative. Hence $(D-2)^4(D+1)^3 Y = 0$. The operator equation corresponds to the solution of a linear homogeneous ODE with characteristic equation $(r-2)^4(r+1)^3 = 0$. The roots are $r = 2$, with multiplicity 4 and $r = -1$, with multiplicity 3. It follows that the given homogeneous solution is $Y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t}$, which is a linear combination of seven independent solutions.

17. For Problem 13, observe that $(D-1)\,[e^t] = 0$, $(D+1)^2\,[te^{-t}] = 0$. The function $e^{-t}\sin t$ is a solution of a second order ODE with characteristic roots $r = -1 \pm i$. It follows that $(D^2 + 2D + 2)\,[e^{-t}\sin t] = 0$. Therefore the operator

$$H(D) = (D-1)(D+1)^2(D^2 + 2D + 2)$$

is an annihilator of $3e^t + 2te^{-t} + e^{-t}\sin t$. The operator corresponding to the left hand side of the given ODE is $D^2(D^2 + 2D + 2)$. It follows that

$$D^2(D-1)(D+1)^2(D^2 + 2D + 2)^2 Y = 0\,.$$

The resulting ODE is homogeneous, with solution $Y(t) = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + c_5 te^{-t} + e^{-t}(c_6\cos t + c_7\sin t) + te^{-t}(c_8\cos t + c_9\sin t\,)$. After examining the homogeneous solution of Problem 13, and eliminating duplicate terms, we have $Y(t) = c_3 e^t + c_4 e^{-t} + c_5 te^{-t} + te^{-t}(c_8\cos t + c_9\sin t\,)$.

4.4

2. The characteristic equation is $r(r^2-1)=0$. Hence the homogeneous solution is $y_c(t)=c_1+c_2e^t+c_3e^{-t}$. The Wronskian is evaluated as $W(1,e^t,e^{-t})=2$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{-t}\\ 0 & e^t & -e^{-t}\\ 1 & e^t & e^{-t}\end{vmatrix}=-2,\quad W_2(t)=\begin{vmatrix}1 & 0 & e^{-t}\\ 0 & 0 & -e^{-t}\\ 0 & 1 & e^{-t}\end{vmatrix}=e^{-t},$$

$$W_3(t)=\begin{vmatrix}1 & e^t & 0\\ 0 & e^t & 0\\ 0 & e^t & 1\end{vmatrix}=e^t.$$

The solution of the system of equations (10) is

$$u_1'(t)=\frac{t\,W_1(t)}{W(t)}=-t,\quad u_2'(t)=\frac{t\,W_2(t)}{W(t)}=te^{-t}/2,$$

$$u_3'(t)=\frac{t\,W_3(t)}{W(t)}=te^t/2.$$

Hence $u_1(t)=-t^2/2$, $u_2(t)=-e^{-t}(t+1)/2$, $u_3(t)=e^t(t-1)/2$. The particular solution becomes $Y(t)=-t^2/2-(t+1)/2+(t-1)/2=-t^2/2-1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t)=c_1+c_2e^t+c_3e^{-t}-t^2/2\,.$$

3. From Section 4.2, $y_c(t)=c_1e^{-t}+c_2e^t+c_3e^{2t}$. The Wronskian is evaluated as $W(e^{-t},e^t,e^{2t})=6\,e^{2t}$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{2t}\\ 0 & e^t & 2e^{2t}\\ 1 & e^t & 4e^{2t}\end{vmatrix}=e^{3t},\quad W_2(t)=\begin{vmatrix}e^{-t} & 0 & e^{2t}\\ -e^{-t} & 0 & 2e^{2t}\\ e^{-t} & 1 & 4e^{2t}\end{vmatrix}=-3e^t,$$

$$W_3(t)=\begin{vmatrix}e^{-t} & e^t & 0\\ -e^{-t} & e^t & 0\\ e^{-t} & e^t & 1\end{vmatrix}=2.$$

Hence $u_1'(t)=e^{5t}/6$, $u_2'(t)=-e^{3t}/2$, $u_3'(t)=e^{2t}/3$. Therefore the particular solution can be expressed as $Y(t)=e^{-t}\left[e^{5t}/30\right]-e^t\left[e^{3t}/6\right]+e^{2t}\left[e^{2t}/6\right]=e^{4t}/30.$

6. Based on the results in Problem 2, $y_c(t)=c_1+c_2e^t+c_3e^{-t}$. It was also shown that $W(1,e^t,e^{-t})=2$, with $W_1(t)=-2$, $W_2(t)=e^{-t}$, $W_3(t)=e^t$. Therefore we have $u_1'(t)=-\csc t$, $u_2'(t)=e^{-t}\csc t/2$, $u_3'(t)=e^t\csc t/2$. The particular solution can be expressed as $y_p(t)=[u_1(t)]+e^t\,[u_2(t)]+e^{-t}\,[u_3(t)]$. More specifically,

$$\begin{aligned}y_p(t)&=-\int_{t_0}^t\csc(s)ds+\frac{e^t}{2}\int_{t_0}^t e^{-s}\csc(s)ds+\frac{e^{-t}}{2}\int_{t_0}^t e^s\csc(s)ds\\ &=\ln\left|\frac{\csc(t)+\cot(t)}{\csc(t_0)+\cot(t_0)}\right|+\frac{e^t}{2}\int_{t_0}^t e^{-s}\csc(s)ds+\frac{e^{-t}}{2}\int_{t_0}^t e^s\csc(s)ds.\end{aligned}$$

and

$$
\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= c_1 + c_2 e^t + c_3 e^{-t} + \ln\left|\frac{\csc(t)+\cot(t)}{\csc(t_0)+\cot(t_0)}\right| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s}\csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s}\csc(s)ds.
\end{aligned}
$$

8. Based on the results in Problems 2 and 6, $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$ and $y_p(t) = u_1(t) + u_2(t)e^t + u_3(t)e^{-t}$, where $u_1'(t) = -\tan t$, $u_2'(t) = e^{-t}\tan t/2$, and $u_3'(t) = e^t \tan t/2$. Thus, with $t_0 = \frac{\pi}{4}$,

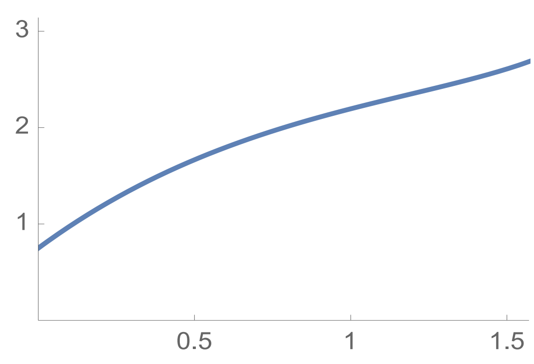
$$
\begin{aligned}
y(t) &= c_1 + c_2 e^t + c_3 e^{-t} - \int_{\pi/4}^{t}\tan(s)ds + \frac{e^t}{2}\int_{\pi/4}^{t} e^{-s}\tan(s)ds + \frac{e^{-t}}{2}\int_{\pi/4}^{t} e^{s}\tan(s)ds \\
&= c_1 + c_2 e^t + c_3 e^{-t} + \ln\left|\sqrt{2}\cos(t)\right| + \frac{e^t}{2}\int_{\pi/4}^{t} e^{-s}\tan(s)ds + \frac{e^{-t}}{2}\int_{\pi/4}^{t} e^{s}\tan(s)ds
\end{aligned}
$$

Note that $y(\pi/4) = y_c(\pi/4)$, $y'(\pi/4) = y_c'(\pi/4)$, and $y''(\pi/4) = y_c''(\pi/4)$. Therefore applying the intial conditions leads to the system of equations

$$
\begin{aligned}
c_1 + c_2 e^{\pi/4} + c_3 e^{-\pi/4} &= 2, \\
c_2 e^{\pi/4} - c_3 e^{-\pi/4} &= 1, \\
c_2 e^{\pi/4} + c_3 e^{-\pi/4} &= -1
\end{aligned}
$$

which can be solved to find that $c_1 = 3$, $c_2 = 0$, and $c_3 = -e^{\pi/4}$. Hence the solution of the initial value problem is

$$
y(t) = 3 - e^{-t+\pi/4} + \ln\left|\sqrt{2}\cos(t)\right| + \frac{e^t}{2}\int_{\pi/4}^{t} e^{-s}\tan(s)ds + \frac{e^{-t}}{2}\int_{\pi/4}^{t} e^{s}\tan(s)ds
$$



9. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$
W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3, \quad W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x,
$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$Y(x) = x\left[-x^3/3\right] + x^2\left[x^2/3\right] + \frac{1}{x}\left[x^5/15\right] = x^4/15\,.$$

11. The homogeneous solution is $y_c(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t$. The Wronskian is evaluated as $W(\cos t, \sin t, \cosh t, \sinh t) = 4$. Now the four additional determinants are given by $W_1(t) = 2 \sin t$, $W_2(t) = -2 \cos t$, $W_3(t) = -2 \sinh t$, $W_4(t) = 2 \cosh t$. If follows that

$$u_1'(t) = g(t)\,\sin(t)/2, \qquad u_2'(t) = -g(t)\,\cos(t)/2,$$
$$u_3'(t) = -g(t)\,\sinh(t)/2, \qquad u_4'(t) = g(t)\,\cosh(t)/2.$$

Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) \;&= \frac{\cos(t)}{2}\int_{t_0}^{t} g(s)\,\sin(s)\,ds - \frac{\sin(t)}{2}\int_{t_0}^{t} g(s)\,\cos(s)\,ds - \\ &- \frac{\cosh(t)}{2}\int_{t_0}^{t} g(s)\,\sinh(s)\,ds + \frac{\sinh(t)}{2}\int_{t_0}^{t} g(s)\,\cosh(s)\,ds\,. \end{aligned}$$

Using the appropriate identities, the integrals can be combined to obtain

$$Y(t) = \frac{1}{2}\int_{t_0}^{t} g(s)\,\sinh(t-s)\,ds - \frac{1}{2}\int_{t_0}^{t} g(s)\,\sin(t-s)\,ds\,.$$