Course Name: AppStocProc Exam Duration: 6/7, 8:00-10:00

Dept.: Mathematics Exam Paper Setter:

Qusetion No.	1	2	3	4	5	6	7	Total
Score								

This exam paper contains 7 questions and the score is 100 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

Note: All calculations and answers must be justified.

1. (15 points) Let $\{X_t : t \geq 0\}$ be a continuous time MC with state space $S = \{1, 2, 3, 4, 5\}$ and rate matrix

$$\begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

i) Is there is distribution such that the detailed balanced condition is satisfied?

ii) Is there a stationary distribution?

iii) What is the long time proportion for this MC to spend at state 3?

2. (15 points) Suppose that the times to finish a 5K race for two runners are independent and ideticallly distributed random variables T_1 and T_2, respectively. The common distribution is exponential with parameter λ.
i) Let $T = \max(T_1, T_2)$. Find $\mathbb{E}(T)$.
ii) If the first runner start at $t = 0$ and the second runner start at $t = \frac{1}{2\lambda}$. What is the probability that the second runner will win the race.

3. (16 points) Let $\{X_t : t \geq 0\}$ be a continuous time MC with rate matrix

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

i) Let $V_i = \min\{t \geq 0 : X_t = i\}$, $i = 1, 2, 3$. Find $\mathbb{E}_2 V_1$, where $\mathbb{E}_2 V_1 = \mathbb{E}(V_1 | X_0 = 2)$.
ii) Find $\mathbb{P}_1(V_2 < V_3)$ and $\mathbb{E}_1(V_2 | V_2 < V_3)$.

4. (15 points) Suppose that customers arrives at a sevice station with two servers according to a Poisson process with rate λ. The service times by server i are exponential with parameter μ_i, $i = 1, 2$. Assume that the service times and the Poisson process are all independent. The customers will wait in line until a server is free (choose one at random if both free). The customers leave the station after being served. Let X_t be the number of customers in the station at time t.
i) Is X_t a Markov chain? Why or why not?
ii) Let

$$Y_t = \begin{cases} X_t & \text{if } X_t > 1, \\ 1a & \text{if } X_t = 1 \text{ and being served by first server}, \\ 1b & \text{if } X_t = 1 \text{ and being served by second server}. \end{cases}$$

Is Y_t a Markov chain? If yes, what is the state space S?
iii) Write down the rate function $q(i, j)$, $i, j \in S$ for the MC above.

iv) Does detailed balanced condition hold? Why?

5 (16 p Let X_n be a discrete time MC with state space $S = \{1, 2, \cdots, 7, 8\}$ and the transition matrix

$$P = \begin{bmatrix} .4 & 0 & 0 & 0 & 0 & 0 & 0 & .6 \\ .4 & 0 & .6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & .4 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .8 & .2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & 0 & 0 & .4 & .6 \end{bmatrix}$$

i) Write S into the union of transient and irreducible recurrent classes.
ii) For each recurrent class, find its stationary distribution.

6. (12 points) Let $B(t)$ be a standard Brownian motion.
i) Find the joint distribution of $(B(1), B(2))$ by specifying the joint probability density function.
ii) Given $B(2) = a$, what is the conditional distribution of $B(1)$?
iii) Find $\mathbb{E}(B(2)^3 | B(1))$.

7. (11 points) Suppose the investors come to a stock exchange according to Poisson process $N(t)$ with rate λ. Each investor will bid on a stock up or down a proportion $\alpha \in (0, 1)$ with probability p and $1 - p$, respectively. More specifically, the proportions of up/down are given by i.i.d. random variables $X_1, X_2, \cdots$ such that

$$\mathbb{P}(X_1 = \alpha) = p, \qquad \mathbb{P}(X_1 = -\alpha) = 1 - p.$$

Then the price of the stock at time t is

$$S(t) = s_0 \prod_{i=1}^{N(t)} (1 + X_i)$$

where s_0 is a constant, the notation $\Pi_{i=1}^{n} a_i = a_1 a_2 \cdots a_n$ which is 1 if $n = 0$. Find $\mathbb{E}S(t)$ and $V(S(t))$. (Hint: Calculate $\mathbb{E}(S(t)|N(t) = n)$ first.)