Step-1

The matrix that connects r, θ to x, y where polar coordinates satisfy $x = r\cos\theta$ and $y = r\sin\theta$. Polar are $J\,drd\theta$ includes J:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta,\ \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

Step-2

And inverting the matrix we get

$$J^{-1} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

Step-3

So that

$$\frac{\partial r}{\partial x} = \cos\theta, \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}, \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Step-4

Therefore,

$$\frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= (\cos\theta)(\cos\theta) + (-r\sin\theta)\left(\frac{-\sin\theta}{r}\right)$$

$$= \cos^2\theta + \sin^2\theta$$

$$=1$$

$$= \frac{\partial x}{\partial x}$$

Step-5

Thus, the product $JJ^{-1} = J$ gives the chain rule

$$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}.\frac{\partial \theta}{\partial x}$$
$$= 1$$