Step-1

Consider the following orthogonal matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Eigen vectors of P are as follows:

$$(1,1,1), \left(1, e^{2\pi i/3}, e^{4\pi i/3}\right) \left(1, e^{4\pi i/3}, e^{8\pi i/3}\right)$$

Find the circulant matrix C which has the same Eigen vectors as orthogonal matrix P and satisfies the following relation:

$$C = 2I + 5P + 4P^2$$

Also, find Eigen values of the circulant matrix C.

Step-2

Circulant matrix C can be calculated as follows:

$$\begin{aligned} C &= 2I + 5P + 4P^2 \\ &= 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 5 \\ 5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} \end{aligned}$$

Therefore, circulant matrix is defined as follows:

$$\boxed{C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix}}$$

Step-3

To calculate the Eigen values of circulant matrix, do the following calculations:

$$C-\lambda I=\begin{bmatrix} 2-\lambda & 5 & 4 \\ 4 & 2-\lambda & 5 \\ 5 & 4 & 2-\lambda \end{bmatrix}$$

$$\det(C-\lambda I)=0$$

$$(2-\lambda)^3\left((2-\lambda)^2-20\right)-5(8-4\lambda-25)+4(6+\lambda)=0$$

$$\left(-\lambda^3+6\lambda^2+48\lambda+77\right)=0$$

After solving following values are obtained:

$$\lambda_1=11$$
$$\lambda_2=-\frac{5}{2}+i\frac{\sqrt{3}}{2}$$
$$\lambda_3=-\frac{5}{2}-i\frac{\sqrt{3}}{2}$$

Step-4

These Eigen values can also be written as a combination of Eigen values of matrix *P*:

$$\begin{aligned}\lambda_1&=11\\&=2+5+4\end{aligned}$$

$$\begin{aligned}\lambda_2&=-\frac{5}{2}+i\frac{\sqrt{3}}{2}\\&=2\cdot 1+5e^{2\pi i/3}+4e^{4\pi i/3}\end{aligned}$$

$$\begin{aligned}\lambda_3&=-\frac{5}{2}-i\frac{\sqrt{3}}{2}\\&=2\cdot 1+5e^{4\pi i/3}+4e^{8\pi i/3}\end{aligned}$$

Step-5

Therefore, following are the Eigen values of the circulant matrix:

$$\boxed{\begin{aligned}&\lambda_1=11\\&\lambda_2=-\frac{5}{2}+i\frac{\sqrt{3}}{2}\\&\lambda_3=-\frac{5}{2}-i\frac{\sqrt{3}}{2}\end{aligned}}$$