

Step-1

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

To find null space, row space, column space

Step-2

First, consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

By definition of null space, $Ax = 0$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-3

Therefore,

If $x_2 = a$, then

$$x_1 + 2x_2 = 0 \quad x_1 = -2a$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2a \\ a \end{bmatrix}$$

$$= a \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$x = (-2, 1)$ Therefore the null space spanned by the vector $(-2, 1)$

Step-4

The row space spanned by any row of A

Therefore row space spanned by the vector $(1,2)$

Step-5

The column space spanned by any column of A

Therefore column space spanned by the vector $(1,3)$

Step-6

By definition of left null space $yA = 0$

$$\begin{aligned} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ y_1 + 3y_2 &= 0 \\ y_2 &= a \\ y_1 &= -3a \end{aligned}$$

Step-7

Therefore,

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} -3a \\ a \end{bmatrix} \\ &= a \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ y &= (-3,1) \end{aligned}$$

And $\begin{bmatrix} 1 & 3 \end{bmatrix}^T \begin{bmatrix} 3 & -1 \end{bmatrix} = 0$

Therefore $N(A^T)$ is the perpendicular line through $(-3,1)$

Step-8

Consider the second matrix

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

By definition of null space, $Bx = 0$

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-9

Therefore,

$$x_1 = 0$$

Above system not depends on x_2 , \hat{A} therefore x_2 is any arbitrary.

Let $x_2 = a$

Hence,

Step-10

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} \\ = a \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$x = (0, 1)$ The null space spanned by the vector $(0, 1)$

Step-11

The row space spanned by any row of A

Therefore row space spanned by the vector $(1, 0)$

Step-12

The column space spanned by any column of A

Therefore column space spanned by the vector $(1, 3)$

Step-13

By definition of left null space $yA = 0$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$y_1 + 3y_2 = 0$$

$$\text{Put } y_2 = a$$

$$y_1 = -3a$$

Step-14

Therefore,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3a \\ a \end{bmatrix}$$

$$= a \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$y = (-3, 1)$$

Therefore $N(A^T)$ is the perpendicular line through $(-3, 1)$