SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 09 Solutions

1. Suppose that the continuous random variables X and Y have joint p.d.f

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Find the joint (cumulative) distribution function (joint c.d.f) of these two random variables X and Y.

(b) Find the marginal p.d.fs of X and Y.

(c) Find P(X > Y),

(d) Find P(X $\le$ 0.5).

Solutions:

(a) If either $x\le 0$ or $y\le 0$, it follows that $f(x,y)=0$, then

$$F(x,y)=\iint_{\substack{-\infty<s\le x\\-\infty<t\le y}} f(s,t)\mathrm{d}s\mathrm{d}t=\iint_{\substack{-\infty<s\le x\\-\infty<t\le y}} 0\mathrm{d}s\mathrm{d}t=0.$$

If $0<x<1$ and $0<y<1$, then we have

$$\begin{aligned}F(x,y)&=\iint_{\substack{-\infty<s\le x\\-\infty<t\le y}} f(s,t)\mathrm{d}s\mathrm{d}t=\iint_{\substack{0<s\le x\\0<t\le y}} f(s,t)\mathrm{d}s\mathrm{d}t=\iint_{\substack{0<s\le x\\0<t\le y}} (s+t)\mathrm{d}s\mathrm{d}t\\&=\int_0^y\int_0^x (s+t)\mathrm{d}s\mathrm{d}t=\int_0^y (\frac{s^2}{2}+st)\Big|_{s=0}^{s=x}\mathrm{d}t\\&=\int_0^y(\frac{x^2}{2}+xt)\mathrm{d}t=(\frac{x^2}{2}t+x\frac{t^2}{2})\Big|_{t=0}^{t=y}\\&=\frac{x^2}{2}y+x\frac{y^2}{2}=\frac{1}{2}xy(x+y).\end{aligned}$$

If $0<x<1$ and $y\ge 1$, then we have

$$\begin{aligned}F(x,y)&=\iint_{\substack{-\infty<s\le x\\-\infty<t\le y}} f(s,t)\mathrm{d}s\mathrm{d}t=\iint_{\substack{0<s\le x\\0<t\le 1}} f(s,t)\mathrm{d}s\mathrm{d}t=\iint_{\substack{0<s\le x\\0<t\le 1}} (s+t)\mathrm{d}s\mathrm{d}t\\&=\int_0^1\int_0^x (s+t)\mathrm{d}s\mathrm{d}t=\int_0^1 (\frac{s^2}{2}+st)\Big|_{s=0}^{s=x}\mathrm{d}t\\&=\int_0^1(\frac{x^2}{2}+xt)\mathrm{d}t=(\frac{x^2}{2}t+x\frac{t^2}{2})\Big|_{t=0}^{t=1}\end{aligned}$$

$$= \frac{x^2}{2} + \frac{x}{2} = \frac{1}{2}x(1+x).$$

If $x \geq 1$ and $0 < y < 1$, then we have

$$\begin{aligned}
F(x,y) &= \iint\limits_{\substack{-\infty<s\leq x\\ -\infty<t\leq y}} f(s,t)\mathrm{d}s\mathrm{d}t = \iint\limits_{\substack{0<s\leq 1\\ 0<t\leq y}} f(s,t)\mathrm{d}s\mathrm{d}t = \iint\limits_{\substack{0<s\leq 1\\ 0<t\leq y}} (s+t)\mathrm{d}s\mathrm{d}t \\
&= \int_0^y \int_0^1 (s+t)\mathrm{d}s\mathrm{d}t = \int_0^y (\frac{s^2}{2} + st)\Big|_{s=0}^{s=1}\mathrm{d}t \\
&= \int_0^y (\frac{1}{2} + t)\mathrm{d}t = (\frac{1}{2}t + \frac{t^2}{2})\Big|_{t=0}^{t=y} \\
&= \frac{1}{2}y + \frac{y^2}{2} = \frac{1}{2}y(1+y).
\end{aligned}$$

If $x \geq 1$ and $y \geq 1$,then we have

$$\begin{aligned}
F(x,y) &= \iint\limits_{\substack{-\infty<s\leq x\\ -\infty<t\leq y}} f(s,t)\mathrm{d}s\mathrm{d}t = \iint\limits_{\substack{0<s\leq 1\\ 0<t\leq 1}} f(s,t)\mathrm{d}s\mathrm{d}t = \iint\limits_{\substack{0<s\leq 1\\ 0<t\leq 1}} (s+t)\mathrm{d}s\mathrm{d}t = 1 \\
&= \int_0^1 \int_0^1 (s+t)\mathrm{d}s\mathrm{d}t = \int_0^1 (\frac{s^2}{2} + st)\Big|_{s=0}^{s=1}\mathrm{d}t \\
&= \int_0^1 (\frac{1}{2} + t)\mathrm{d}t = (\frac{1}{2}t + \frac{t^2}{2})\Big|_{t=0}^{t=1} \\
&= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

Hence, the joint cumulative distribution function of the random vector $(X,\ Y)$ is:

$$F(x,y) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } y \leq 0, \\ \frac{1}{2}xy(x+y) & \text{if } 0 < x < 1, 0 < y < 1, \\ \frac{1}{2}x(x+1) & \text{if } 0 < x < 1, y \geq 1, \\ \frac{1}{2}y(y+1) & \text{if } x \geq 1, 0 < y < 1, \\ 1 & \text{if } x \geq 1, y \geq 1. \end{cases}$$

(b) For $x < 0$ or $x > 1$, we have $f(x,y) = 0$, thus

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y = \int_{-\infty}^{+\infty} 0\mathrm{d}y = 0.$$

For $0 \leq x \leq 1$,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y \\
&= \int_{-\infty}^{0} f(x,y)\mathrm{d}y + \int_0^1 f(x,y)\mathrm{d}y + \int_1^{+\infty} f(x,y)\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} 0\mathrm{d}y + \int_{0}^{1}(x+y)\mathrm{d}y + \int_{1}^{+\infty} 0\mathrm{d}y \\
&= \int_{0}^{1}(x+y)\mathrm{d}y = (xy+\frac{y^2}{2})\Big|_{y=0}^{y=1} = x+\frac{1}{2}.
\end{aligned}
$$

Similarly, for $y<0$ or $y>1$, we have $f(x,y)=0$, thus

$$
f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x = \int_{-\infty}^{+\infty} 0\mathrm{d}x = 0.
$$

For $0 \le y \le 1$,

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x \\
&= \int_{-\infty}^{0} f(x,y)\mathrm{d}x + \int_{0}^{1} f(x,y)\mathrm{d}x + \int_{1}^{+\infty} f(x,y)\mathrm{d}x \\
&= \int_{-\infty}^{0} 0\mathrm{d}x + \int_{0}^{1}(x+y)\mathrm{d}x + \int_{1}^{+\infty} 0\mathrm{d}x \\
&= \int_{0}^{1}(x+y)\mathrm{d}x = (\frac{x^2}{2}+yx)\Big|_{x=0}^{x=1} = y+\frac{1}{2}.
\end{aligned}
$$

Hence, the marginal p.d.f of X is:

$$
f_X(x) = \begin{cases} x+\frac{1}{2} & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

The marginal p.d.f of Y is :

$$
f_Y(y) = \begin{cases} y+\frac{1}{2} & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

(c)

$$
\begin{aligned}
\mathrm{P}\{\mathrm{X}>\mathrm{Y}\} &= \iint\limits_{x>y} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{x>y\\-\infty<x<+\infty\\-\infty<y<+\infty}} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{x>y\\0\le x\le 1\\0\le y\le 1}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{x>y\\0\le x\le 1\\0\le y\le 1}} (x+y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^1\int_0^x (x+y)\mathrm{d}y\mathrm{d}x = \int_0^1 [xy+\frac{y^2}{2}]\Big|_0^x \mathrm{d}x = \int_0^1 (x^2+\frac{x^2}{2})\mathrm{d}x \\
&= \frac{3}{2}\int_0^1 x^2\mathrm{d}x = \frac{3}{2}[\frac{x^3}{3}]\Big|_0^1 = \frac{1}{2}.
\end{aligned}
$$

(d) **Method 1:**

$$
\begin{aligned}
\mathrm{P}\{\mathrm{X} \le \frac{1}{2}\} &= \int_{-\infty}^{\frac{1}{2}} f_X(x)\mathrm{d}x = \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_{0}^{\frac{1}{2}} f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} 0\mathrm{d}x + \int_{0}^{\frac{1}{2}} (x+\frac{1}{2})\mathrm{d}x \\
&= \int_{0}^{\frac{1}{2}} (x+\frac{1}{2})\mathrm{d}x = (\frac{x^2}{2}+\frac{x}{2})\Big|_0^{\frac{1}{2}} \\
&= \frac{1}{2}\cdot(\frac{1}{2})^2 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{8}.
\end{aligned}
$$

Method 2: Note that $\{X \le \frac{1}{2}\} = \{X \le \frac{1}{2}, -\infty < Y < +\infty\}$, and thus

$$
\begin{aligned}
&\mathrm{P}\{\mathrm{X} \le \frac{1}{2}\} = \mathrm{P}\{\mathrm{X} \le \frac{1}{2}, -\infty < \mathrm{Y} < +\infty\} \\
&= \iint\limits_{\substack{x \le \frac{1}{2} \\ -\infty < y < +\infty}} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{x \le \frac{1}{2} \\ 0 \le x \le 1 \\ 0 \le y \le 1}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{0 \le x \le \frac{1}{2} \\ 0 \le y \le 1}} (x+y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{\frac{1}{2}}\int_0^1 (x+y)\mathrm{d}y\mathrm{d}x = \int_0^{\frac{1}{2}} [xy+\frac{y^2}{2}]\Big|_0^1 \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} (x+\frac{1}{2})\mathrm{d}x = (\frac{x^2}{2}+\frac{x}{2})\Big|_0^{\frac{1}{2}} \\
&= \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\times\frac{1}{2} = \frac{3}{8}.
\end{aligned}
$$

Remark: Remember the following formula: for any $B \in \mathscr{B}(\mathbb{R}^2)$,

$$\mathrm{P}\left((\mathrm{X},\mathrm{Y}) \in \mathrm{B}\right) = \iint\limits_{\mathrm{B}} \mathrm{f(x,y)dxdy}.$$

The two marginal p.d.f's are:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y; \qquad f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x.$$

Moreover, we have the following more general conclusion:
Suppose $n \in \mathbb{N}_+$ and let $X = (X_1, \ldots, X_n)$ is a random vector. Then for any $B \in \mathscr{B}(\mathbb{R}^n)$, we have

$$\mathrm{P}\left(\mathrm{X} \in \mathrm{B}\right) = \int_{\mathrm{B}} \mathrm{f}(\mathrm{x}_1, \ldots, \mathrm{x}_n)\mathrm{dx}_1 \ldots \mathrm{dx}_n.$$

2. Find the joint probability density function of the two random variables X and Y whose joint (cumulative) distribution function (joint c.d.f)is given by

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x>0, y>0; \\ 0 & \text{otherwise.} \end{cases}$$

Also use the joint probability density to determine $\mathrm{P}(1 < \mathrm{X} < 3, 1 < \mathrm{Y} < 2)$.

Solution: The joint pdf is given by, if $x > 0$ and $y > 0$,

$$\begin{aligned}\frac{\partial^2}{\partial y \partial x}F(x,y) = \frac{\partial^2}{\partial x \partial y}F(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F(x,y)\right) \\ &= \frac{\partial}{\partial y}\left((-e^{-x}\cdot(-1))(1-\mathrm{e}^{-y})\right) \\ &= e^{-x}\cdot\frac{\partial}{\partial y}\left((1-\mathrm{e}^{-y})\right) = e^{-x}\cdot(-e^{-y}\cdot(-1)) \\ &= e^{-(x+y)}.\end{aligned}$$

and 0 elsewhere.

Hence we find the joint probability density of X and Y is:

$$f(x,y) = \begin{cases} e^{-(x+y)} & x>0, y>0; \\ 0 & \text{otherwise.} \end{cases}$$

Now, we can get that

$$\begin{aligned}P(1<X<3, 1<Y<2) &= \int_1^3\int_1^2 \mathrm{e}^{-(x+y)}\mathrm{d}y\mathrm{d}x \\ &= \int_1^3\int_1^2 \mathrm{e}^{-x}\cdot\mathrm{e}^{-y}\mathrm{d}y\mathrm{d}x = \int_1^3 \mathrm{e}^{-x}\mathrm{d}x\cdot\int_1^2 \mathrm{e}^{-y}\mathrm{d}y \\ &= (-\mathrm{e}^{-x})\Big|_1^3\cdot(-\mathrm{e}^{-y})\Big|_1^2 \\ &= (-\mathrm{e}^{-3}+\mathrm{e}^{-1})(-\mathrm{e}^{-2}+\mathrm{e}^{-1}) \\ &= \mathrm{e}^{-2}-\mathrm{e}^{-3}-\mathrm{e}^{-4}+\mathrm{e}^{-5}.\end{aligned}$$

3. Consider a circle of radius R and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, it follows, since (X,Y) is equally likely to be near each point in the circle, that the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} c & \text{if } x^2+y^2 \le R^2, \\ 0 & \text{if } x^2+y^2 > R^2. \end{cases}$$

for some value of c.

(a) Determine the constant c.

(b) Find the marginal density functions of X and Y.

(c) Compute the probability that the distance from the origin of the point selected is not greater that a. $(0 \le a \le R.)$

(d) Are X and Y independent? Specify your reasons clearly.

Solution:

(a) By the property of joint pdf, we have

$$
\begin{aligned}
1 &= \iint\limits_{\mathbb{R}^2} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{x^2+y^2\le R^2} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{x^2+y^2\le R^2} c\mathrm{d}x\mathrm{d}y \\
&= c \iint\limits_{x^2+y^2\le R^2} \mathrm{d}y\mathrm{d}x = c\pi R^2 \\
&= c\int_0^{2\pi}\int_0^R r\mathrm{d}r\mathrm{d}\mathrm{d}\theta = c\int_0^{2\pi} \frac{r^2}{2}\Big|_{r=0}^{r=R}\mathrm{d}\theta \\
&= c\int_0^{2\pi}\frac{R^2}{2}\mathrm{d}\theta = c\frac{R^2}{2}\cdot 2\pi = c\pi R^2.
\end{aligned}
$$

Hence $c = \frac{1}{\pi R^2}$.
So, the joint density function of X and Y is given by

$$
f(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & \text{if } x^2+y^2 \le R^2, \\ 0 & \text{if } x^2+y^2 > R^2. \end{cases}
$$

(b) Notice that for fixed $x_0 \in \mathbb{R}$, $x_0^2 + y^2 \le R^2 \Leftrightarrow y^2 \le R^2 - x_0^2$
If $x_0^2 > R^2$, i.e. $|x_0| > R$, then $f(x_0, y) = 0$, so

$$
f_X(x_0) = \int_{-\infty}^{+\infty} f(x_0,y)\mathrm{d}y = \int_{-\infty}^{+\infty} 0\mathrm{d}y = 0.
$$

If $x_0^2 \le R^2$, i.e. $|x_0| \le R$, then

$$
\begin{aligned}
f_X(x_0) &= \int_{-\infty}^{+\infty} f(x_0,y)\mathrm{d}y \\
&= \int_{-\infty}^{-\sqrt{R^2-x_0^2}} f(x_0,y)\mathrm{d}y + \int_{-\sqrt{R^2-x_0^2}}^{\sqrt{R^2-x_0^2}} f(x_0,y)\mathrm{d}y + \int_{\sqrt{R^2-x_0^2}}^{+\infty} f(x_0,y)\mathrm{d}y \\
&= \int_{-\infty}^{-\sqrt{R^2-x_0^2}} 0\mathrm{d}y + \int_{-\sqrt{R^2-x_0^2}}^{\sqrt{R^2-x_0^2}} \left(\frac{1}{\pi R^2}\right)\mathrm{d}y + \int_{\sqrt{R^2-x_0^2}}^{+\infty} 0\mathrm{d}y \\
&= \frac{2\sqrt{R^2-x_0^2}}{\pi R^2}.
\end{aligned}
$$

Hence the marginal density of X is:

$$f_X(x) = \begin{cases} \dfrac{2\sqrt{R^2 - x^2}}{\pi R^2} & |x| \le R, \\ 0 & |x| > R. \end{cases}$$

Similarly, the marginal density of Y is:

$$f_Y(x) = \begin{cases} \dfrac{2\sqrt{R^2 - y^2}}{\pi R^2} & |y| \le R, \\ 0 & |y| > R. \end{cases}$$

(c) Let $Z = \sqrt{X^2 + Y^2}$, then for $0 \le a \le R$, we have

$$\begin{aligned} \mathrm{P}\{\sqrt{\mathrm{X}^2+\mathrm{Y}^2} \le \mathrm{a}\} &= \mathrm{P}(\mathrm{Z} \le \mathrm{a}) \overset{\text{since } \mathrm{Z} \ge 0}{=\!=\!=\!=} \mathrm{P}(|\mathrm{Z}| \le \mathrm{a}) \overset{\text{since } \mathrm{a} \ge 0}{=\!=\!=\!=} \mathrm{P}(|\mathrm{Z}| \le |\mathrm{a}|) \\ &= \mathrm{P}\{\mathrm{Z}^2 \le \mathrm{a}^2\} = \mathrm{P}\{\mathrm{X}^2 + \mathrm{Y}^2 \le \mathrm{a}^2\} \\ &= \iint_{x^2+y^2 \le a^2} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \iint_{x^2+y^2 \le a^2} \frac{1}{\pi R^2}\mathrm{d}x\mathrm{d}y = \frac{1}{\pi R^2} \iint_{x^2+y^2 \le a^2} \mathrm{d}x\mathrm{d}y \\ &= \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}. \end{aligned}$$

(d) X and Y are not independent, since for $x^2 + y^2 \le R^2$,

$$f(x,y) = \frac{1}{\pi R^2} \ne \frac{2\sqrt{R^2 - x^2}}{\pi R^2} \cdot \frac{2\sqrt{R^2 - y^2}}{\pi R^2} = f_X(x) \cdot f_Y(y).$$

4. A man and a woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 12 noon and 1 p.m., find the probability that the first to arrive has to wait longer than 10 minutes.

Solution: Let X and Y denote, respectively, the time past 12 that the man and the woman arrives. Then X and Y are independent random variables, each of which is uniformly distributed over $(0, 60)$.

Hence the desired probability is $\mathrm{P}(|\mathrm{X}-\mathrm{Y}| > 10) = \mathrm{P}\{\mathrm{X}+10 < \mathrm{Y}\} + \mathrm{P}\{\mathrm{Y}+10 < \mathrm{X}\}$, which by symmetry equals $2P\{X + 10 < Y\}$. The latter probability can be obtained as follows:

$$\begin{aligned} 2P\{X + 10 < Y\} &= 2 \iint_{x+10<y} f(x,y)dxdy \\ &= 2 \iint_{x+10<y} f_X(x) f_Y(y) dxdy \end{aligned}$$

$$
\begin{aligned}
&= 2\int_{10}^{60}\int_{0}^{y-10}\left(\frac{1}{60}\right)^2 dxdy \\
&= \frac{2}{(60)^2}\int_{10}^{60}(y-10)dy \\
&= \frac{25}{36}.
\end{aligned}
$$

We can also obtain the desired probability as follows:

$$
\begin{aligned}
\mathrm{P}(|\mathrm{X}-\mathrm{Y}|>10) &= \iint\limits_{|x-y|>10} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{|x-y|>10\\ -\infty<x<+\infty\\ -\infty<y<+\infty}} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{|x-y|>10\\ -\infty<x<+\infty\\ -\infty<y<+\infty}} f_X(x)f_Y(y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{|x-y|>10\\ 0\le x\le 60\\ 0\le y\le 60}} \left(\frac{1}{60}\cdot\frac{1}{60}\right)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{3600}\iint\limits_{\substack{|x-y|>10\\ 0\le x\le 60\\ 0\le y\le 60}} \mathrm{d}x\mathrm{d}y = \frac{1}{3600}\cdot\left(\frac{1}{2}\times 50\times 50\times 2\right) = \frac{25}{36} \\
&= \frac{1}{3600}\left[\int_0^{10}\int_{x+10}^{60}\mathrm{d}y\mathrm{d}x + \int_{10}^{50}\left(\int_0^{x-10}\mathrm{d}y + \int_{x+10}^{60}\mathrm{d}y\right)\mathrm{d}x + \int_{50}^{60}\int_0^{x-10}\mathrm{d}y\mathrm{d}x\right] \\
&= \frac{1}{3600}\left[\int_0^{10}(50-x)\mathrm{d}x + \int_{10}^{50}\left[(x-10)+(50-x)\right]\mathrm{d}x + \int_{50}^{60}(x-10)\mathrm{d}x\right] \\
&= \frac{1}{3600}\left[\left(50x-\frac{x^2}{2}\right)\Big|_0^{10} + 40\times 40 + \left(\frac{x^2}{2}-10x\right)\Big|_{50}^{60}\right] \\
&= \frac{2500}{3600} = \frac{25}{36}.
\end{aligned}
$$