Step-1

Here, $A=3\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$. Let us obtain *M*. We have, $M_{ij}=\int V_iV_j dx$.

Therefore,

$$M_{12}=\int_0^1 V_1V_2 dx$$

Now, it should be clear that $V_1V_2=0$, when $0\le x\le\frac{1}{3}$ or $\frac{2}{3}\le x\le 1$.

Further note that, when $\frac{1}{3}\le x\le\frac{2}{3}$, we have

$$\begin{aligned}V_1V_2&=(3x-1)(2-3x)\\&=9x-2-9x^2\end{aligned}$$

Step-2

Therefore,

$$\begin{aligned}\int_0^1 V_1V_2 dx&=\int_{\frac{1}{3}}^{\frac{2}{3}}\left(9x-2-9x^2\right)dx\\&=9\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{2}{3}}-2[x]_{\frac{1}{3}}^{\frac{2}{3}}-9\left[\frac{x^3}{3}\right]_{\frac{1}{3}}^{\frac{2}{3}}\\&=\frac{9}{2}\left(\frac{1}{3}\right)-2\left(\frac{1}{3}\right)-\frac{9}{3}\left(\frac{7}{27}\right)\\&=\frac{1}{18}\end{aligned}$$

Step-3

Thus, $M_{12}=\frac{1}{18}$.

It should be clear that $M_{21}=\frac{1}{18}$.

Step-4

Now obtain M_{11}. Note that $M_{11} = M_{22}$.

$$M_{11} = \int_0^1 (V_1)^2\, dx$$

Consider

$$\int_0^{\frac{1}{3}} (V_1)^2\, dx = \int_0^{\frac{1}{3}} 9x^2 dx$$
$$= 9\left[\frac{x^3}{3}\right]_0^{\frac{1}{3}}$$
$$= 3\left[\frac{1}{27}\right]$$
$$= \frac{1}{9}$$

Similarly, $\int_{\frac{1}{3}}^{\frac{2}{3}} (V_1)^2\, dx = \frac{1}{9}$, by symmetry.

Therefore, $M_{11} = \frac{2}{9}$ and hence, $M_{22} = \frac{2}{9}$.

Therefore,

$$M = \begin{bmatrix} \frac{2}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{2}{9} \end{bmatrix}$$

Step-5

Consider the equation $Ax = \lambda Mx$

This is same as $(A - \lambda M)x = 0$.

Therefore, $\det(A - \lambda M) = 0$

Step-6

Thus, we get

$$
\begin{aligned}
0 &= \begin{vmatrix} 6-\dfrac{2\lambda}{9} & -3-\dfrac{\lambda}{18} \\ -3-\dfrac{\lambda}{18} & 6-\dfrac{2\lambda}{9} \end{vmatrix} \\
&= \left(6-\frac{2\lambda}{9}\right)^2-\left(3+\frac{\lambda}{18}\right)^2 \\
&= 36-\frac{24\lambda}{9}+\frac{4\lambda^2}{81}-\left(9+\frac{6\lambda}{18}+\frac{\lambda^2}{324}\right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= 27-\frac{54\lambda}{18}+\frac{15\lambda^2}{324} \\
&= \frac{5\lambda^2}{108}-3\lambda+27
\end{aligned}
$$

Step-7

Therefore,

$$
\begin{aligned}
\lambda &= \frac{9\pm\sqrt{9-5}}{\frac{5}{54}} \\
&= \frac{54}{5}(9\pm 2) \\
&= \frac{378}{5} \text{ or } \frac{594}{5}
\end{aligned}
$$

Step-8

Therefore, the eigenvalues are $\boxed{\frac{378}{5}}$ and $\boxed{\frac{594}{5}}$.