Step-1

Consider the matrix:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The objective is to find the column space and null space of the matrix *A*.

Column space of $A = \left\{ b \mid b = Ax, \text{for } x \in \mathbf{R}^2 \right\}$

Let $x = \begin{bmatrix} u \\ v \end{bmatrix} \in R^2$

Now,

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u - v \\ 0 \end{bmatrix}$$

Therefore,

Column space of matrix A is;

$$\mathbf{C}(A) = \left\{ (u - v, 0) \mid u, v \in \mathbf{R} \right\}$$

Thus, $\boxed{\mathbf{C}(A) \text{ is a line in } \mathbf{R}^2}$.

Step-2

The definition of null space is as follows:

$$\mathbf{N}(A) = \left\{ x \mid Ax = 0 \right\}.$$

Let $x = \begin{bmatrix} u \\ v \end{bmatrix} \in R^2$

Now,

$$A\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}1&-1\\0&0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}u-v\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$
$$u-v=0$$

This implies,

$u=v$

Therefore, $\mathbf{N}(A)=\left\{\begin{bmatrix}u\\v\end{bmatrix}\mid u=v\right\}$

Hence, $\boxed{\mathbf{N}(A)\text{is a line } u=v \text{ in } \mathbf{R}^2}$.

Thus, for every vector $\begin{bmatrix}u\\u\end{bmatrix}$ gives a zero vector to construct a null space.

Step-3

Consider the matrix:

$$B=\begin{bmatrix}0&0&3\\1&2&3\end{bmatrix}$$

Let $x=\begin{bmatrix}u\\v\\w\end{bmatrix}\in R^3$

Then column space of B is;

$$B=\begin{bmatrix}0&0&3\\1&2&3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}$$
$$=\begin{bmatrix}3w\\u+2v+3w\end{bmatrix}$$

Therefore, $\mathbf{C}(B)=\{(3w,u+2v+3w)\mid u,v,w\in\mathbf{R}\}$

Therefore, $\boxed{\mathbf{C}(B)=\mathbf{R}^2}$.

Step-4

Now, $\mathbf{N}(B)=\{x\in\mathbf{R}^3 \mid Bx=0\}$

Now,

$Bx=0$

$$\begin{bmatrix}0&0&3\\1&2&3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{aligned}3w&=0\\u+2v+3w&=0\\u&=-2v\end{aligned}$$

Therefore, $\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}-2v\\v\\0\end{bmatrix}$

Therefore, $\mathbf{N}(B)=\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}v \mid v\in\mathbf{R}\right\}$

Therefore, $\boxed{\mathbf{N}(B)\text{is the line passing through}\begin{bmatrix}-2\\1\\0\end{bmatrix}.}$

Step-5

Consider a matrix,

$$C=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

Let $x=\begin{bmatrix}u\\v\\w\end{bmatrix}\in R^3$

Column space is;

$\mathbf{C}(C) = \{b \in \mathbf{R}^2 \mid b = Cx, \text{for } x \in \mathbf{R}^3\}$

$$Cx = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the column space is $\boxed{\mathbf{C}(C) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}}$.

Calculate the null space as follows:

$\mathbf{N}(C) = \{x \mid Cx = 0\}$

$$Cx = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, every vector (u, v, w) satisfy the above equation

Therefore, $\boxed{\mathbf{N}(c) = \mathbf{R}^3}$.