Step-1

Consider the following equation:

$$\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix} u$$

The objective is to find the eigenvalues λ and frequencies ω, and the general solution of the above equation.

Step-2

Find the eigenvalues for matrix $A = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix}$ by using the equation,

$$\det(A - \lambda I) = 0$$
$$\det\begin{bmatrix} -5-\lambda & 4 \\ 4 & -5-\lambda \end{bmatrix} = 0$$
$$(-5-\lambda)(-5-\lambda) - (16) = 0$$
$$(-5-\lambda)^2 - 16 = 0$$

$$(-5-\lambda)^2 = 16$$
$$-5-\lambda = \pm\sqrt{16}$$
$$-5-\lambda = \pm 4$$
$$-5-\lambda = -4 \ (\text{or}) \ -5-\lambda = 4$$

$$\lambda = -1 \ (\text{or}) \ \lambda = -9$$

Therefore, the eigenvalues for matrix A are $\lambda_1 = \boxed{-1}$ and $\lambda_2 = \boxed{-9}$.

Step-3

The eigenvector for $\lambda_1 = -1$ can be evaluated as follows:

$$(A - \lambda_1 I)\mathrm{x} = 0$$
$$\begin{bmatrix} -5+1 & -1 \\ -1 & -5+1 \end{bmatrix}\mathrm{x} = 0$$

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\mathrm{x} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \left(\text{By } R_2 + R_1 \to R_2, -\frac{1}{4}R_1 \to R_1\right)$$

The reduced system is,

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_1 &= x_2 \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \\ &= x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

The eigenvector for $\lambda_1 = -1$ is $\mathrm{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-4

The eigenvector for $\lambda_2 = -9$ can be evaluated as follows:

$$(A - \lambda_2 I)\mathrm{x} = 0$$

$$\begin{bmatrix} -5+9 & 4 \\ 4 & -5+9 \end{bmatrix}\mathrm{x} = 0$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\mathrm{x} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \left(\text{By } R_2 - R_1 \to R_2, \frac{1}{4}R_1 \to R_1\right)$$

The reduced system is,

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 &= -x_2 \end{aligned}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The eigenvector for $\lambda_2 = -9$ is $\mathrm{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step-5

If the matrix A has negative eigenvalues $\lambda_1, \ldots, \lambda_n$ and if $\omega_j = \sqrt{-\lambda_j}$, then general solution of $\frac{d^2u}{dt^2} = Au$ is,

$$u(t) = (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t)\mathrm{x}_1 + \cdots + (a_n \cos \omega_n t + b_n \sin \omega_n t)\mathrm{x}_n$$

Here, the frequencies $\omega_j = \sqrt{-\lambda_j}$.

Step-6

Substitute -1 for λ_1 in the equation $\omega_1 = \sqrt{-\lambda_1}$.

$$\omega_1 = \sqrt{-(-1)}$$
$$= 1$$

Substitute -9 for λ_2 in equation $\omega_2 = \sqrt{-\lambda_2}$.

$$\omega_2 = \sqrt{-(-9)}$$
$$= \sqrt{9}$$
$$= 3$$

Therefore, the frequencies are $\omega_1 = \boxed{1}$ and $\omega_2 = \boxed{3}$.

Step-7

Put the values $\mathrm{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathrm{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \omega_1 = 1, \omega_2 = 3$ in the following equation to get the general solution.

$$u(t) = (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t)\mathrm{x}_1 + \cdots + (a_n \cos \omega_n t + b_n \sin \omega_n t)\mathrm{x}_2$$

$$\boxed{u(t) = (a_1 \cos t + b_1 \sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (a_2 \cos 3t + b_2 \sin 3t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$