

Step-1

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

The objective is to compute $A^T A$ and AA^T .

Step-2

Compute $A^T A$.

$$\begin{aligned} A^T &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ A^T A &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Compute AA^T .

$$\begin{aligned} AA^T &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

Step-3

Find the eigenvalues of AA^T .

$$AA^T - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

The eigenvalues of AA^T are $\lambda_1 = 1, \lambda_2 = 3$.

Step-4

Find an eigenvector of AA^T corresponding to the eigenvalue $\lambda_1 = 1$.

Solve the system $(AA^T - I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$

$$= -x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

An eigenvector of AA^T corresponding to the eigenvalue $\lambda_1 = 1$ is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step-5

Find an eigenvector of AA^T corresponding to the eigenvalue $\lambda_2 = 3$.

Solve the system $(AA^T - 3I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \\ = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

An eigenvector of AA^T corresponding to the eigenvalue $\lambda_2 = 3$ is $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-6

Find the eigenvalues of $A^T A$.

$$A^T A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(1-\lambda)-1]-(1-\lambda)=0 \\ \lambda(\lambda-1)(\lambda-3)=0$$

$$\lambda = 0, 1, 3$$

Therefore, the eigenvalues of $A^T A$ are $\lambda_1 = 1, \lambda_2 = 3$, and $\lambda_3 = 0$

Step-7

Find an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_1 = 1$.

Solve the system $[A^T A - I]\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} \\ = -x_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Therefore, an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_1 = 1$ is

Step-8

Find an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_2 = 3$.

Solve the system $[A^T A - 3I]\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix for the system is $\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right]$.

The reduced row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} \\ = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_2 = 3$ is $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Step-9

Find an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_3 = 0$.

Solve the system $A^T A \mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix for the system is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$.

The reduced row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} \\ = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore, an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_3 = 0$ is

Step-10

Eigenvectors of AA^T are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The vectors \mathbf{x}_1 and \mathbf{x}_2 are orthogonal.

Find the orthonormal eigenvectors of AA^T .

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} \\ = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \\ = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\mathbf{u}_1, \mathbf{u}_2$ are orthonormal eigenvectors of AA^T .

The matrix U is

$$U = [\mathbf{u}_1 \quad \mathbf{u}_2]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step-11

Eigenvectors of $A^T A$ are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The vectors $\mathbf{x}_1, \mathbf{x}_2$ and \mathbf{x}_3 are orthogonal.

Find the orthonormal eigenvectors of $A^T A$.

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 are orthonormal eigenvectors of $A^T A$.

The matrix V is

$$V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] \\ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Eigenvalues of AA^T are $\lambda_1 = 1, \lambda_2 = 3$.

Singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} \\ = 1$$

$$\sigma_2 = \sqrt{\lambda_2} \\ = \sqrt{3}$$

The matrix Σ is

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \\ \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

Step-12

Find $U\Sigma V^T$.

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step-13

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Therefore, $U\Sigma V^T = A$.