CHAPTER

8

Numerical Methods

8.1

2. The following pseudo-code can be used to produce a table of approximate values of the solution to $y' = 1 - t + 4y$, $y(0) = 1$ on the interval $0 \le t \le 2$ using the backward Euler's method with $h = 0.01$.

```
h = 0.01
n = 2/h
t = 0
y = 1
print (t, y)
for j from 1 to n do
    t = t + h
    y = (y + h(1 - t))/(1 - 4h)
    print (t, y)
end do
```

Note that the formula for updating the value of y is determined by solving the equation $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$ for y_{n+1}: $y_{n+1} = \dfrac{y_n + h(1 - t_{n+1})}{1 - 4h}$. To obtain the corresponding tables for other values of h, just change the value of h in the first line of the pseudo-code and recalculate the other commands in order.

3. The Euler formula for this problem is $y_{n+1} = y_n + h(5\,t_n - 3\sqrt{y_n}\,)$, in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write $y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}$ with $y_0 = 2$.

(a) Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.59980	1.29288	1.07242	0.930175

(b) Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.61124	1.31361	1.10012	0.962552

The backward Euler formula is $y_{n+1} = y_n + h(5\,t_{n+1} - 3\sqrt{y_{n+1}}\,)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write $y_{n+1} = y_n + 5(n+1)h^2 - 3h\sqrt{y_{n+1}}$, with $y_0 = 2$. Solving for y_{n+1}, and choosing the positive root, we find that

$$y_{n+1} = \left[-\frac{3}{2}h + \frac{1}{2}\sqrt{(20n+29)h^2 + 4y_n}\right]^2 .$$

(c) Backward Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.64337	1.37164	1.17763	1.05334

(d) Backward Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.63301	1.35295	1.15267	1.02407

5. The Euler formula is $y_{n+1} = y_n + h\,[2\,t_n + e^{-t_n\,y_n}]$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write $y_{n+1} = y_n + 2nh^2 + h\,e^{-nh\,y_n}$, with $y_0 = 1$.

(a) Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.10244	1.21426	1.33484	1.46399

(b) Euler method with $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	1.10365	1.21656	1.33817	1.46832

The backward Euler formula is $y_{n+1} = y_n + h\,[2\,t_{n+1} + e^{-t_{n+1}\,y_{n+1}}]$. Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1} = y_n + 2h^2(n+1) + h\,e^{-(n+1)h\,y_{n+1}},$$

with $y_0 = 1$. This equation cannot be solved explicitly for y_{n+1}. At each step, given the current value of y_n, the equation must be solved numerically for y_{n+1}.

(c) Backward Euler method with $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.10720	1.22333	1.34797	1.48110

(d) Backward Euler method with $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	1.10603	1.22110	1.34473	1.47688

7. The Euler formula for this problem is $y_{n+1} = y_n + h(t_n^2 - y_n^2)\sin y_n$. Here $t_0 = 0$ and $t_n = nh$. So that $y_{n+1} = y_n + h(n^2h^2 - y_n^2)\sin y_n$, with $y_0 = -1$.

(a) Euler method with $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.920498	-0.857538	-0.808030	-0.770038

(b) Euler method with $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	-0.922575	-0.860923	-0.82300	-0.774965

The backward Euler formula is $y_{n+1} = y_n + h(t_{n+1}^2 - y_{n+1}^2)\sin y_{n+1}$. Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1} = y_n + h\left[(n+1)^2h^2 - y_{n+1}^2\right]\sin y_{n+1}\,,$$

with $y_0 = -1$. Note that this equation cannot be solved explicitly for y_{n+1}. Given y_n, the transcendental equation

$$y_{n+1} + h\,y_{n+1}^2\sin y_{n+1} = y_n + h(n+1)^2h^2$$

must be solved numerically for y_{n+1}.

(c) Backward Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.928059	-0.870054	-0.824021	-0.788686

(d) Backward Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	-0.926341	-0.867163	-0.820279	-0.784275

9. The Euler formula $y_{n+1} = y_n + h(5\,t_n - 3\sqrt{y_n}\,)$, in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write $y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}$ with $y_0 = 2$.

(a) Euler method with $h = 0.025$:

	$n=20$	$n=40$	$n=60$	$n=80$
t_n	0.5	1.0	1.5	2.0
y_n	0.891830	1.25225	2.37818	4.07257

(b) Euler method with $h = 0.0125$:

	$n=40$	$n=80$	$n=120$	$n=160$
t_n	0.5	1.0	1.5	2.0
y_n	0.908902	1.26872	2.39336	4.08799

The backward Euler formula is $y_{n+1} = y_n + h(5\,t_{n+1} - 3\sqrt{y_{n+1}}\,)$, in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write $y_{n+1} = y_n + 5(n+1)h^2 - 3h\sqrt{y_{n+1}}$ with $y_0 = 2$. Solving for y_{n+1}, and choosing the positive root, we find that

$$y_{n+1} = \left[-\frac{3}{2}h + \frac{1}{2}\sqrt{(20n+29)h^2 + 4y_n} \right]^2 .$$

(c) Backward Euler method with $h = 0.025$:

	$n=20$	$n=40$	$n=60$	$n=80$
t_n	0.5	1.0	1.5	2.0
y_n	0.958565	1.31786	2.43924	4.13474

(d) Backward Euler method with $h = 0.0125$:

	$n=40$	$n=80$	$n=120$	$n=160$
t_n	0.5	1.0	1.5	2.0
y_n	0.942261	1.30153	2.24389	4.11908

10. The Euler formula is $y_{n+1} = y_n + h\,[2\,t_n + e^{-t_n\,y_n}]$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write $y_{n+1} = y_n + 2nh^2 + h\,e^{-nh\,y_n}$, with $y_0 = 1$.

(a) Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	1.60729	2.46830	3.72167	5.45963

(b) Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	1.60996	2.47460	3.73356	5.47774

The backward Euler formula is $y_{n+1} = y_n + h\,[2\,t_{n+1} + e^{-t_{n+1}\,y_{n+1}}]$. Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1} = y_n + 2h^2(n+1) + h\,e^{-(n+1)h\,y_{n+1}},$$

with $y_0 = 1$. This equation cannot be solved explicitly for y_{n+1}. At each step, given the current value of y_n, the equation must be solved numerically for y_{n+1}.

(c) Backward Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	1.61792	2.49356	3.76940	5.53223

(d) Backward Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	1.61528	2.48723	3.75742	5.51404

12. The Euler formula is $y_{n+1} = y_n + h(y_n^2 + 2\,t_n\,y_n)/(3 + t_n^2)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write $y_{n+1} = y_n + (h\,y_n^2 + 2\,nh^2\,y_n)/(3 + n^2h^2)$, with $y_0 = 0.5$.

(a) Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.587987	0.791589	1.14743	1.70973

(b) Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.589440	0.795758	1.15693	1.72955

The backward Euler formula is $y_{n+1} = y_n + h(y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1})/(3 + t_{n+1}^2)$. Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1}\left[3 + (n+1)^2h^2\right] - h\,y_{n+1}^2 = y_n\left[3 + (n+1)^2h^2\right] + 2(n+1)h^2\,y_{n+1},$$

with $y_0 = 0.5$. Note that although this equation can be solved explicitly for y_{n+1}, it is also possible to use a numerical equation solver. At each time step, given the current value of y_n, the equation may be solved numerically for y_{n+1}.

(c) Backward Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.593901	0.808716	1.18687	1.79291

(d) Backward Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.592396	0.804319	1.17664	1.77111

17. Given that $\phi(t)$ is a solution of the initial value problem, the local truncation error for the Euler method, on the interval $t_n \le t \le t_{n+1}$, is

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

where $t_n < \bar{t}_n < t_{n+1}$. Based on the ODE, $\phi'(t) = \sqrt{t + \phi(t)}$, and hence

$$\phi''(t) = \frac{1 + \phi'(t)}{2\sqrt{t + \phi(t)}} = \frac{1}{2\sqrt{t + \phi(t)}} + \frac{1}{2}$$

and so

$$e_{n+1} = \frac{1}{4}\left[1 + \frac{1}{\sqrt{\bar{t}_n + \phi(\bar{t}_n)}}\right]h^2$$

where $t_n < \bar{t}_n < t_{n+1}$.

18. Let $\phi(t)$ be a solution of the initial value problem. The local truncation error for the Euler method, on the interval $t_n \le t \le t_{n+1}$, is given by

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'(t) = 2t + e^{-t\,\phi(t)}$, it follows that

$$\phi''(t) = 2 - [\phi(t) + t\,\phi'(t)] \cdot e^{-t\,\phi(t)} = 2 - \left[\phi(t) + 2t^2 + t\,e^{-t\,\phi(t)}\right] \cdot e^{-t\,\phi(t)}.$$

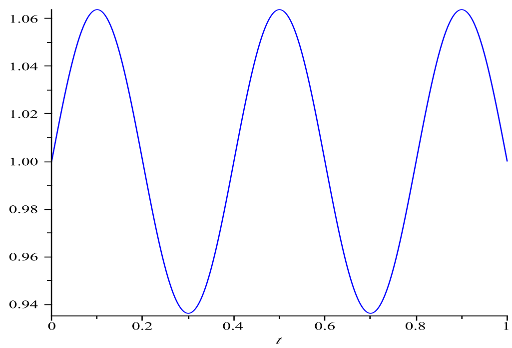
Hence

$$e_{n+1} = h^2 - \frac{h^2}{2}\left[\phi(\bar{t}_n) + 2\,\bar{t}_n^{\,2} + \bar{t}_n e^{-\bar{t}_n \phi(\bar{t}_n)}\right] \cdot e^{-\bar{t}_n \phi(\bar{t}_n)}.$$

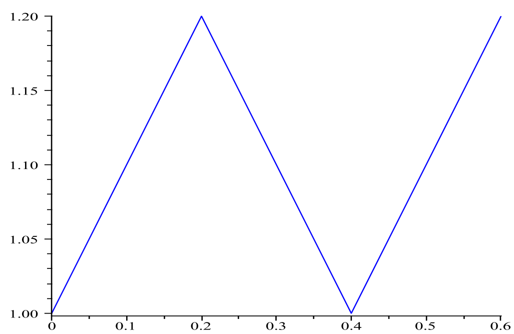
19. (a)

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.2	0.4	0.6
y_n	1.0	1.2	1.0	1.2

(b) Direct integration yields $\phi(t) = (1/5\pi)\sin 5\pi t + 1$.



(c)



(d)

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.0	0.1	0.2	0.3	0.4
y_n	1.0	1.1	1.1	1.0	1.0

(e) Since $\phi''(t) = -5\pi \sin 5\pi t$, the local truncation error for the Euler method, on the interval $t_n \le t \le t_{n+1}$, is given by

$$e_{n+1} = -\frac{5\pi h^2}{2} \sin 5\pi \,\bar{t}_n .$$

(f) In order to satisfy

$$|e_{n+1}| \le \frac{5\pi h^2}{2} < 0.05\,,$$

it is necessary that

$$h < \frac{1}{\sqrt{50\pi}} \approx 0.08\,.$$

23.(a)

$$1000 \cdot \begin{vmatrix} 6.0 & 18 \\ 2.0 & 6.0 \end{vmatrix} = 1000 \cdot (0) = 0\,.$$

(b)

$$1000 \cdot \begin{vmatrix} 6.01 & 18.0 \\ 2.00 & 6.00 \end{vmatrix} = 1000(0.06) = 60\,.$$

(c)

$$1000 \cdot \begin{vmatrix} 6.010 & 18.04 \\ 2.004 & 6.000 \end{vmatrix} = 1000(-0.09216) = -92.16\,.$$

24. Rounding to three digits, $a(b-c) \approx 0.224$. Likewise, to three digits, $ab \approx 0.702$ and $ac \approx 0.477$. It follows that $ab - ac \approx 0.225$.

8.2

5. The improved Euler formula is

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{2(3+t_n^2)} + h\,\frac{K_n^2 + 2\,t_{n+1}K_n}{2(3+t_{n+1}^2)},$$

in which

$$K_n = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{3+t_n^2}.$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2nh\,y_n}{2(3+n^2h^2)} + h\,\frac{K_n^2 + 2(n+1)hK_n}{2\,[3+(n+1)^2h^2]},$$

with $y_0 = 0.5$.

(a) $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	0.510164	0.524126	0.54083	0.564251

(b) $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	0.510168	0.524135	0.542100	0.564277

(c) $h = 0.0125$:

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
t_n	0.1	0.2	0.3	0.4
y_n	0.510169	0.524137	0.542104	0.564284

6. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}(t_n^2 - y_n^2)\sin y_n + \frac{h}{2}(t_{n+1}^2 - K_n^2)\sin K_n,$$

in which

$$K_n = y_n + h\,(t_n^2 - y_n^2)\sin y_n \;.$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}(n^2h^2 - y_n^2)\sin y_n + \frac{h}{2}\left[(n+1)^2h^2 - K_n^2\right]\sin K_n,$$

with $y_0 = -1$.

(a) $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924650	-0.864338	-0.816642	-0.780008

(b) $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924550	-0.864177	-0.816442	-0.779781

(c) $h = 0.0125$:

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924525	-0.864138	-0.816393	-0.779725

8. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(5\,t_n - 3\sqrt{y_n}\,) + \frac{h}{2}(5\,t_{n+1} - 3\sqrt{K_n}\,),$$

in which $K_n = y_n + h(5\,t_n - 3\sqrt{y_n}\,)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}(5\,nh - 3\sqrt{y_n}\,) + \frac{h}{2}\left[5\,(n+1)h - 3\sqrt{K_n}\right],$$

with $y_0 = 2$.

(a) $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.926139	1.28558	2.40898	4.10386

(b) $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.925815	1.28525	2.40869	4.10359

9. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{t_n + y_n} + \frac{h}{2}\sqrt{t_{n+1} + K_n}\,,$$

in which $K_n = y_n + h\sqrt{t_n + y_n}$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{nh + y_n} + \frac{h}{2}\sqrt{(n+1)h + K_n}\,,$$

with $y_0 = 3$.

(a) $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	3.96217	5.10887	6.43134	7.92332

(b) $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	3.96218	5.10889	6.43138	7.92337

11. The improved Euler formula is

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{2(3+t_n^2)} + h\,\frac{K_n^2 + 2\,t_{n+1}K_n}{2(3+t_{n+1}^2)},$$

in which

$$K_n = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{3+t_n^2}.$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2nh\,y_n}{2(3+n^2h^2)} + h\,\frac{K_n^2 + 2(n+1)hK_n}{2\left[3+(n+1)^2h^2\right]},$$

with $y_0 = 0.5$.

(a) $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.590897	0.799950	1.16653	1.74969

(b) $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.590906	0.799988	1.16663	1.74992

14. The exact solution of the initial value problem is $\phi(t) = \frac{1}{2} + \frac{1}{2}e^{2t}$. Based on the result in Problem 12(c), the local truncation error for a linear differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3,$$

where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'''(t) = 4\,e^{2t}$, the local truncation error is

$$e_{n+1} = \frac{2}{3}e^{2\,\bar{t}_n}h^3.$$

Furthermore, with $0 \le \bar{t}_n \le 1$,

$$|e_{n+1}| \le \frac{2}{3}\,e^2h^3.$$

It also follows that for $h = 0.1$,

$$|e_1| \le \frac{2}{3}\,e^{0.2}(0.1)^3 = \frac{1}{1500}\,e^{0.2}.$$

Using the improved Euler method, with $h = 0.1$, we have $y_1 \approx 1.11000$. The exact value is given by $\phi(0.1) = 1.1107014$.

15. The exact solution of the initial value problem is given by $\phi(t) = \frac{1}{2}t + e^{2t}$. Using the modified Euler method, the local truncation error for a linear differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3,$$

where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'''(t) = 8\,e^{2t}$, the local truncation error is

$$e_{n+1} = \frac{4}{3}e^{2\,\bar{t}_n}h^3.$$

Furthermore, with $0 \le \bar{t}_n \le 1$, the local error is bounded by

$$|e_{n+1}| \le \frac{4}{3}\,e^2h^3.$$

It also follows that for $h = 0.1$,

$$|e_1| \le \frac{4}{3}\,e^{0.2}(0.1)^3 = \frac{1}{750}\,e^{0.2}.$$

Using the improved Euler method, with $h = 0.1$, we have $y_1 \approx 1.27000$. The exact value is given by $\phi(0.1) = 1.271403$.

16. Using the Euler method, $y_1 = 1 + 0.1(0.5 - 0 + 2\cdot 1) = 1.25$. Using the improved Euler method, $y_1 = 1 + 0.05(0.5 - 0 + 2) + 0.05(0.5 - 0.1 + 2.5) = 1.27$. The estimated error is $e_1 \approx 1.27 - 1.25 = 0.02$. The step size should be adjusted by a factor of $\sqrt{0.0025/0.02} \approx 0.354$. Hence the required step size is estimated as $h \approx (0.1)(0.36) = 0.036$.

18. Using the Euler method, $y_1 = 3 + 0.1\sqrt{0+3} = 3.173205$. Using the improved Euler method,

$$y_1 = 3 + 0.05\sqrt{0+3} + 0.05\sqrt{0.1 + 3.173205} = 3.177063\,.$$

The estimated error is $e_1 \approx 3.177063 - 3.173205 = 0.003858$. The step size should be adjusted by a factor of $\sqrt{0.0025/0.003858} \approx 0.805$. Hence the required step size is estimated as $h \approx (0.1)(0.805) = 0.0805$.

19. Using the Euler method,

$$y_1 = 0.5 + 0.1\frac{(0.5)^2 + 0}{3 + 0} = 0.508334$$

Using the improved Euler method,

$$y_1 = 0.5 + 0.05\frac{(0.5)^2 + 0}{3+0} + 0.05\frac{(0.508334)^2 + 2(0.1)(0.508334)}{3 + (0.1)^2} = 0.510148\,.$$

The estimated error is $e_1 \approx 0.510148 - 0.508334 = 0.0018$. The local truncation error is less than the given tolerance. The step size can be adjusted by a factor of $\sqrt{0.0025/0.0018} \approx 1.1785$. Hence it is possible to use a step size of

$$h \approx (0.1)(1.1785) \approx 0.117\,.$$

20. Assuming that the solution has continuous derivatives at least to the third order,

$$\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)h + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3,$$

where $t_n < \bar{t}_n < t_{n+1}$. Suppose that $y_n = \phi(t_n)$.

(a) The local truncation error is given by

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1}.$$

The modified Euler formula is defined as

$$y_{n+1} = y_n + h\, f\left[t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}h\, f(t_n, y_n)\right].$$

Observe that $\phi'(t_n) = f(t_n\,, \phi(t_n)) = f(t_n\,, y_n)$. It follows that

$e_{n+1} = \phi(t_{n+1}) - y_{n+1} =$

$$= h\, f(t_n\,, y_n) + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3 - h\, f\left[t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}h\, f(t_n, y_n)\right].$$

(b) As shown in Problem 12(b),

$$\phi''(t_n) = f_t(t_n\,, y_n) + f_y(t_n\,, y_n) f(t_n\,, y_n)\,.$$

Furthermore,

$$\begin{aligned} f\left[t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}h\, f(t_n, y_n)\right] &= f(t_n\,, y_n) + f_t(t_n\,, y_n)\frac{h}{2} + f_y(t_n\,, y_n)\, k + \\ &+ \frac{1}{2!}\left[\frac{h^2}{4} f_{tt} + hk\, f_{ty} + k^2 f_{yy}\right]_{t=\xi\,,\, y=\eta}, \end{aligned}$$

in which $k = \frac{1}{2}h\, f(t_n, y_n)$ and $t_n < \xi < t_n + h/2\,,\ y_n < \eta < y_n + k$. Therefore

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)}{3!}h^3 - \frac{h}{2!}\left[\frac{h^2}{4} f_{tt} + hk\, f_{ty} + k^2 f_{yy}\right]_{t=\xi\,,\, y=\eta}.$$

Note that each term in the brackets has a factor of h^2. Hence the local truncation error is proportional to h^3.

(c) If $f(t, y)$ is linear, then $f_{tt} = f_{ty} = f_{yy} = 0$, and

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)}{3!}h^3.$$

21. The modified Euler formula for this problem is

$$\begin{aligned} y_{n+1} &= y_n + h\left\{3 + t_n + \frac{1}{2}h - \left[y_n + \frac{1}{2}h(3 + t_n - y_n)\right]\right\} \\ &= y_n + h(3 + t_n - y_n) + \frac{h^2}{2}(y_n - t_n - 2). \end{aligned}$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h(3 + nh - y_n) + \frac{h^2}{2}(y_n - nh - 2) \ ,$$

with $y_0 = 1$. Setting $h = 0.05$, we obtain the following values:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.19512	1.38120	1.55909	1.72956

23. The modified Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + h\left[2y_n - 3t_n - \frac{3}{2}h + h(2y_n - 3t_n)\right] \\ &= y_n + h(2y_n - 3t_n) + \frac{h^2}{2}(4y_n - 6t_n - 3). \end{aligned}$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h(2y_n - 3nh) + \frac{h^2}{2}(4y_n - 6\,nh - 3) \ ,$$

with $y_0 = 1$. Setting $h = 0.05$, we obtain:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.20526	1.42273	1.65511	1.90570

24. The modified Euler formula for this problem is

$$y_{n+1} = y_n + h\left[2t_n + h + e^{-(t_n + \frac{h}{2})K_n}\right],$$

in which $K_n = y_n + (h/2)\left[2t_n + e^{-t_n y_n}\right]$. Now $t_n = t_0 + nh$, with $t_0 = 0$ and $y_0 = 1$. Setting $h = 0.05$, we obtain the following values :

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.10485	1.21886	1.34149	1.47264

25. Let $f(t, y)$ be linear in both variables. The improved Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}h\left[f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))\right] \\ &= y_n + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf\left[h, hf(t_n, y_n)\right] \\ &= y_n + hf(h, y_n) + \frac{1}{2}hf\left[h, hf(t_n, y_n)\right]. \end{aligned}$$

The modified Euler formula is

$$y_{n+1} = y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right] = y_n + hf(t_n, y_n) + hf\left[\frac{1}{2}h, \frac{1}{2}hf(t_n, y_n)\right].$$

Since $f(t,y)$ is linear in both variables,

$$f\left[\frac{1}{2}h, \frac{1}{2}hf(t_n,y_n)\right] = \frac{1}{2}f\left[h, hf(t_n,y_n)\right].$$

8.3

1. The following pseudo-code can be used to produce a table of approximate values of the solution to $y' = 1 - t + 4y$, $y(0) = 1$ on the interval $0 \le t \le 2$ using the Runge-Kutta method with $h = 0.02$.

```
h = 0.025
n = 2/h
t = 0
y = 1
print (t, y)
for j from 1 to n do
    k1 = 1 - t + 4 * y
    k2 = 1 - (t + h/2) + 4(y + h * k1/2)
    k3 = 1 - (t + h/2) + 4(y + h * k2/2)
    k4 = 1 - (t + h/2) + 4(y + h * k3/2)
    t = t + h
    y = y + h * (k1 + 2 * k2 + 2 * k3 + k4)/6
    print (t, y)
end do
```

To obtain the corresponding tables for other values of h, just change the value of h in the first line of the pseudo-code and recalculate the other commands in order.

2. The ODE is linear, with $f(t,y) = 3 + t - y$. The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}) \\
k_{n3} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a) For $h = 0.1$:

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.19516	1.38127	1.55918	1.72968

(b) For $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.19516	1.38127	1.55918	1.72968

The exact solution of the IVP is $y(t) = 2 + t - e^{-t}$.

3. In this problem, $f(t, y) = 5t - 3\sqrt{y}$. At each time step, the Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}) \\
k_{n3} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a) For $h = 0.1$:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
t_n	0.1	0.2	0.3	0.4
y_n	1.62231	1.33362	1.12686	0.993839

(b) For $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.62230	1.33362	1.12685	0.993826

The exact solution of the IVP is given implicitly as

$$\frac{1}{(2\sqrt{y} + 5t)^5(t - \sqrt{y})^2} = \frac{\sqrt{2}}{512}\,.$$

5. In this problem, $f(t, y) = (y^2 + 2ty)/(3 + t^2)$. The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}) \\
k_{n3} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a) For $h = 0.1$:

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	0.510170	0.524138	0.542105	0.564286

(b) For $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	0.520169	0.524138	0.542105	0.564286

The exact solution of the IVP is $y(t) = (3 + t^2)/(6 - t)$.

6. In this problem, $f(t, y) = (t^2 - y^2)\sin y$. At each time step, the Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}) \\
k_{n3} &= f(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a) For $h = 0.1$:

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924517	-0.864125	-0.816377	-0.779706

(b) For $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924517	-0.864125	-0.816377	-0.779706

7.(a) For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	2.96825	7.88889	20.8349	55.5957

(b) For $h = 0.05$:

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	2.96828	7.88904	20.8355	55.5980

The exact solution of the IVP is $y(t) = e^{2t} + t/2$.

8.(a) For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	0.925725	1.28516	2.40860	4.10350

(b) For $h = 0.05$:

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	0.925711	1.28515	2.40860	4.10350

9.(a) For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	3.96219	5.10890	6.43139	7.92338

(b) For $h = 0.05$:

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	3.96219	5.10890	6.43139	7.92338

The exact solution is given implicitly as

$$\ln\left[\frac{2}{y+t-1}\right] + 2\sqrt{t+y} - 2\,\mathrm{arctanh}\,\sqrt{t+y} = t + 2\sqrt{3} - 2\,\mathrm{arctanh}\,\sqrt{3}\,.$$

11. See Problem 5 for the exact solution.

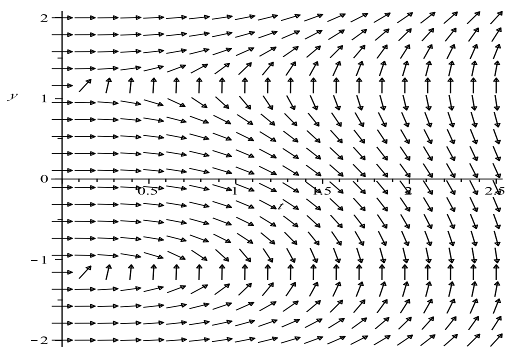
(a) For $h = 0.1$:

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
t_n	0.5	1.0	1.5	2.0
y_n	0.590909	0.800000	1.166667	1.75000

(b) For $h = 0.05$:

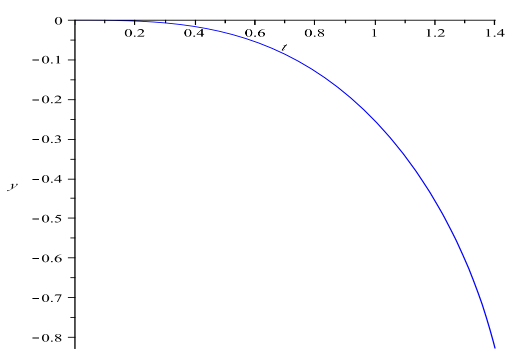
	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.590909	0.800000	1.166667	1.75000

12.(a)



(b) For the integral curve starting at $(0, 0)$, the slope turns infinite near $t_M \approx 1.5$. Note that the exact solution of the IVP is defined implicitly as

$$y^3 - 4y = t^3.$$



(c) Using the classic Runge-Kutta algorithm with the given values of h, we obtain the values

$h = 0.1$

	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$
t_n	1.2	1.3	1.4	1.5	1.6
y_n	-0.45566	-0.60448	-0.82806	-1.73807	-1.56618

$h = 0.05$

	$n = 26$	$n = 28$	$n = 30$	$n = 32$	$n = 34$
t_n	1.3	1.4	1.5	1.6	1.7
y_n	−0.60447	−0.82786	−1.06266	−1.42862	−1.17608

$h = 0.025$

	$n = 54$	$n = 56$	$n = 58$	$n = 60$	$n = 62$
t_n	1.35	1.4	1.45	1.5	1.55
y_n	−0.70134	−0.82783	−1.05986	−1.49336	−1.30986

$h = 0.01$

	$n = 142$	$n = 143$	$n = 144$	$n = 145$	$n = 146$
t_n	1.42	1.43	1.44	1.45	1.46
y_n	−0.89513	−0.93617	−0.98653	−1.05951	−0.76554

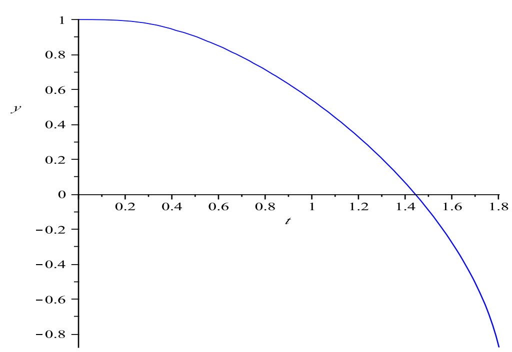
Based on the direction field, the solution should decrease monotonically to the limiting value $y = -2/\sqrt{3}$. In the following table, the value of t_M corresponds to the approximate time in the iteration process that the calculated values begin to increase.

h	t_M
0.1	1.55
0.05	1.65
0.025	1.525
0.01	1.455

(d) Numerical values will continue to be generated, although they will not be associated with the integral curve starting at $(0, 0)$. These values are approximations to nearby integral curves.

(e) We consider the solution associated with the initial condition $y(0) = 1$. The exact solution is given by

$$y^3 - 4y = t^3 - 3\,.$$



For the integral curve starting at $(0\,, 1)$, the slope becomes infinite near $t_M \approx 2.0$. In the following table, the values of t_M corresponds to the approximate time in the iteration process that the calculated values begin to increase.

h	t_M
0.1	1.85
0.05	1.85
0.025	1.86
0.01	1.835

8.4

1.(a) Using the notation $f_n = f(t_n\,, y_n)$, the predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.1	0.2	0.3
y_n	1.0	1.19516	1.38127	1.55918

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
t_n	0.4	0.4	0.5	0.5
y_n	1.72967690	1.72986801	1.89346436	1.89346973

(b) With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

In this problem, $f_{n+1} = 3 + t_{n+1} - y_{n+1}$. Since the ODE is linear, we can solve for

$$y_{n+1} = \frac{1}{24 + 9\,h}\left[24\,y_n + 27\,h + 9h\,t_{n+1} + h(19\,f_n - 5\,f_{n-1} + f_{n-2})\right].$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	1.7296800	1.8934695

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12hf_{n+1}\right].$$

In this problem, $f_{n+1} = 3 + t_{n+1} - y_{n+1}$. Since the ODE is linear, we can solve for

$$y_{n+1} = \frac{1}{25 + 12\,h}\left[36\,h + 12h\,t_{n+1} + 48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3}\right].$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	1.7296805	1.8934711

The exact solution of the IVP is $y(t) = 2 + t - e^{-t}$.

2.(a) Using the notation $f_n = f(t_n\,, y_n)$, the predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.1	0.2	0.3
y_n	2.0	1.62231	1.33362	1.12686

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
t_n	0.4	0.4	0.5	0.5
y_n	0.993751	0.993852	0.925469	0.925764

(b) With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

In this problem, $f_{n+1} = 5t_{n+1} - 3\sqrt{y_{n+1}}$. Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[45t_{n+1} - 27\sqrt{y_{n+1}} + 19\,f_n - 5\,f_{n-1} + f_{n-2}\right]$$

at each time step. We obtain the approximate values:

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	0.993847	0.925746

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h f_{n+1}\right].$$

Since the ODE is nonlinear, an equation solver is used to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h(5t_{n+1} - 3\sqrt{y_{n+1}}\,)\right]$$

at each time step.

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	0.993869	0.925837

The exact solution of the IVP is given implicitly by

$$\frac{1}{(2\sqrt{y} + 5t)^5 (t - \sqrt{y}\,)^2} = \frac{\sqrt{2}}{512}\,.$$

4.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

Using the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.1	0.2	0.3
y_n	0.5	0.51016950	0.52413795	0.54210529

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
t_n	0.4	0.4	0.5	0.5
y_n	0.56428532	0.56428577	0.59090816	0.59090918

(b) With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

In this problem,

$$f_{n+1} = \frac{y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}}{3 + t_{n+1}^2}\,.$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[9\,\frac{y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}}{3 + t_{n+1}^2} + 19\,f_n - 5\,f_{n-1} + f_{n-2}\right]$$

at each time step.

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	0.56428578	0.59090920

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h f_{n+1}\right].$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h\frac{y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}}{3 + t_{n+1}^2}\right]$$

at each time step. We obtain the approximate values:

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	0.56428588	0.59090952

The exact solution of the IVP is $y(t) = (3 + t^2)/(6 - t)$.

5.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.1	0.2	0.3
y_n	-1.0	-0.924517	-0.864125	-0.816377

	$n = 4$ (pre)	$n = 4$ (cor)	$n = 5$ (pre)	$n = 5$ (cor)
t_n	0.4	0.4	0.5	0.5
y_n	-0.779832	-0.779693	-0.753311	-0.753135

(b) With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

In this problem, $f_{n+1} = (t_{n+1}^2 - y_{n+1}^2)\sin y_{n+1}$. Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = y_n + \frac{h}{24}\left[9(t_{n+1}^2 - y_{n+1}^2)\sin y_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}\right].$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	-0.779700	-0.753144

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12hf_{n+1}\right].$$

Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h(t_{n+1}^2 - y_{n+1}^2)\sin y_{n+1}\right].$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	−0.779680	−0.753089

7.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.05	0.1	0.15
y_n	2.0	1.7996296	1.6223042	1.4672503

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.9257133	1.285148	2.408595	4.103495

(b) Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[45t_{n+1} - 27\sqrt{y_{n+1}} + 19\,f_n - 5\,f_{n-1} + f_{n-2}\right]$$

at each time step. We obtain the approximate values:

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.9257125	1.285148	2.408595	4.103495

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12hf_{n+1}\right].$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h(5t_{n+1} - 3\sqrt{y_{n+1}}\,)\right]$$

at each time step.

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	0.9257248	1.285158	2.408594	4.103493

The exact solution of the IVP is given implicitly by

$$\frac{1}{(2\sqrt{y}+5t)^5(t-\sqrt{y}\,)^2}=\frac{\sqrt{2}}{512}\,.$$

8.(a) The predictor formula is

$$y_{n+1}=y_n+\frac{h}{24}(55\,f_n-59\,f_{n-1}+37\,f_{n-2}-9\,f_{n-3}).$$

With $f_{n+1}=f(t_{n+1}\,,y_{n+1})$, the corrector formula is

$$y_{n+1}=y_n+\frac{h}{24}(9\,f_{n+1}+19\,f_n-5\,f_{n-1}+f_{n-2}).$$

Using the starting values generated by the Runge-Kutta method:

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.05	0.1	0.15
y_n	3.0	3.087586	3.177127	3.268609

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	3.962186	5.108903	6.431390	7.923385

(b) With $f_{n+1}=f(t_{n+1}\,,y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1}=y_n+\frac{h}{24}(9\,f_{n+1}+19\,f_n-5\,f_{n-1}+f_{n-2}).$$

In this problem, $f_{n+1}=\sqrt{t_{n+1}+y_{n+1}}$. Since the ODE is nonlinear, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1}=y_n+\frac{h}{24}\left[9\sqrt{t_{n+1}+y_{n+1}}+19\,f_n-5\,f_{n-1}+f_{n-2}\right]$$

at each time step.

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	3.962186	5.108903	6.431390	7.923385

(c) The fourth order backward differentiation formula is

$$y_{n+1}=\frac{1}{25}\left[48\,y_n-36\,y_{n-1}+16\,y_{n-2}-3\,y_{n-3}+12hf_{n+1}\right].$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\sqrt{t_{n+1} + y_{n+1}}\,\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	3.962186	5.108903	6.431390	7.923385

The exact solution is given implicitly by

$$\ln\left[\frac{2}{y+t-1}\right] + 2\sqrt{t+y} - 2\,\mathrm{arctanh}\,\sqrt{t+y} = t + 2\sqrt{3} - 2\,\mathrm{arctanh}\,\sqrt{3}\,.$$

9.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.05	0.1	0.15
y_n	1.0	1.051230	1.104843	1.160740

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	1.612622	2.480909	3.7451479	5.495872

(b) With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2}).$$

In this problem, $f_{n+1} = 2\, t_{n+1} + e^{-t_{n+1}\, y_{n+1}}$. Since the ODE is nonlinear, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left\{9\left[2\, t_{n+1} + e^{-t_{n+1}\, y_{n+1}}\right] + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right\}$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	1.612622	2.480909	3.7451479	5.495872

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h f_{n+1}\right].$$

Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = \frac{1}{25}\left\{48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\left[2\, t_{n+1} + e^{-t_{n+1}\, y_{n+1}}\right]\right\}.$$

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	1.612623	2.480905	3.7451473	5.495869

10.(a) The predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2}).$$

We use the starting values generated by the Runge-Kutta method:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.05	0.1	0.15
y_n	0.5	0.5046218	0.5101695	0.5166666

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.5909091	0.8000000	1.166667	1.750000

(b) With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order Adams-Moulton formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2}).$$

In this problem,

$$f_{n+1} = \frac{y_{n+1}^2 + 2\, t_{n+1}\, y_{n+1}}{3 + t_{n+1}^2}\,.$$

Since the ODE is nonlinear, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[9\,\frac{y_{n+1}^2 + 2\, t_{n+1}\, y_{n+1}}{3 + t_{n+1}^2} + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.5909091	0.8000000	1.166667	1.750000

(c) The fourth order backward differentiation formula is

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h f_{n+1}\right].$$

Since the ODE is nonlinear, we obtain the implicit equation

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\, \frac{y_{n+1}^2 + 2\, t_{n+1}\, y_{n+1}}{3 + t_{n+1}^2}\right].$$

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	0.5909092	0.8000002	1.166667	1.750001

The exact solution of the IVP is $y(t) = (3+t^2)/(6-t)$.

11. Both Adams methods entail the approximation of $f(t\,,y)$, on the interval $[t_n\,, t_{n+1}]$, by a polynomial. Approximating $\phi'(t) = P_1(t) \equiv A$, which is a constant polynomial, we have

$$\phi(t_{n+1}) - \phi(t_n) = \int_{t_n}^{t_{n+1}} A\, dt = A(t_{n+1} - t_n) = Ah\,.$$

Setting $A = \lambda\, f_n + (1-\lambda) f_{n-1}$, where $0 \le \lambda \le 1$, we obtain the approximation

$$y_{n+1} = y_n + h\left[\lambda\, f_n + (1-\lambda) f_{n-1}\right].$$

An appropriate choice of λ yields the familiar Euler formula. Similarly, setting

$$A = \lambda\, f_n + (1-\lambda) f_{n+1}\,,$$

where $0 \le \lambda \le 1$, we obtain the approximation

$$y_{n+1} = y_n + h\left[\lambda\, f_n + (1-\lambda) f_{n+1}\right].$$

13. For a third order Adams-Bashforth formula, we approximate $f(t\,,y)$, on the interval $[t_n\,, t_{n+1}]$, by a quadratic polynomial using the points (t_{n-2}, y_{n-2}), (t_{n-1}, y_{n-1}) and (t_n, y_n). Let $P_3(t) = At^2 + Bt + C$. We obtain the system of equations

$$\begin{aligned} At_{n-2}^2 + Bt_{n-2} + C &= f_{n-2} \\ At_{n-1}^2 + Bt_{n-1} + C &= f_{n-1} \\ At_n^2 + Bt_n + C &= f_n\,. \end{aligned}$$

For computational purposes, assume that $t_0 = 0$, and $t_n = nh$. It follows that

$$\begin{aligned} A &= \frac{f_n - 2f_{n-1} + f_{n-2}}{2\,h^2} \\ B &= \frac{(3-2n)f_n + (4n-4)f_{n-1} + (1-2n)f_{n-2}}{2\,h} \\ C &= \frac{n^2 - 3n + 2}{2} f_n + (2n - n^2) f_{n-1} + \frac{n^2 - n}{2} f_{n-2}\,. \end{aligned}$$

We then have

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} \left[At^2 + Bt + C\right] dt = Ah^3(n^2 + n + \frac{1}{3}) + Bh^2(n + \frac{1}{2}) + Ch,$$

which yields

$$y_{n+1} - y_n = \frac{h}{12}(23\, f_n - 16\, f_{n-1} + 5\, f_{n-2})\,.$$

8.5

1. In vector notation, the initial value problem can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + t \\ 4x - 2y \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) The Euler formula is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h\begin{pmatrix} x_n + y_n + t_n \\ 4x_n - 2y_n \end{pmatrix}.$$

That is,

$$x_{n+1} = x_n + h(x_n + y_n + t_n)$$
$$y_{n+1} = y_n + h(4x_n - 2y_n).$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.26	1.7714	2.58991	3.82374	5.64246
y_n	0.76	1.4824	2.3703	3.60413	5.38885

(b) The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = (x_n + y_n + t_n\,, 4x_n - 2y_n)^T$$

$$\mathbf{k}_{n2} = \left[x_n + \frac{h}{2}k_{n1}^1 + y_n + \frac{h}{2}k_{n1}^2 + t_n + \frac{h}{2}\,, 4(x_n + \frac{h}{2}k_{n1}^1) - 2(y_n + \frac{h}{2}k_{n1}^2)\right]^T$$

$$\mathbf{k}_{n3} = \left[x_n + \frac{h}{2}k_{n2}^1 + y_n + \frac{h}{2}k_{n2}^2 + t_n + \frac{h}{2}\,, 4(x_n + \frac{h}{2}k_{n2}^1) - 2(y_n + \frac{h}{2}k_{n2}^2)\right]^T$$

$$\mathbf{k}_{n4} = \left[x_n + hk_{n3}^1 + y_n + hk_{n3}^2 + t_n + h\,, 4(x_n + hk_{n3}^1) - 2(y_n + hk_{n3}^2)\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.32493	1.93679	2.93414	4.48318	6.84236
y_n	0.758933	1.57919	2.66099	4.22639	6.56452

(c) With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.32489	1.9369	2.93459	4.48422	6.8444
y_n	0.759516	1.57999	2.66201	4.22784	6.56684

The exact solution of the IVP is

$$x(t) = e^{2t} + \frac{2}{9}e^{-3t} - \frac{1}{3}t - \frac{2}{9}$$
$$y(t) = e^{2t} - \frac{8}{9}e^{-3t} - \frac{2}{3}t - \frac{1}{9}.$$

2.(a) The Euler formula is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h\begin{pmatrix} -t_n\, x_n - y_n - 1 \\ x_n \end{pmatrix}.$$

That is,

$$x_{n+1} = x_n + h(-t_n\, x_n - y_n - 1)$$
$$y_{n+1} = y_n + h(x_n).$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	0.582	0.117969	−0.336912	−0.730007	−1.02134
y_n	1.18	1.27344	1.27382	1.18572	1.02371

(b) The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = (-t_n\, x_n - y_n - 1, x_n)^T$$
$$\mathbf{k}_{n2} = \left[-(t_n + \frac{h}{2})(x_n + \frac{h}{2}k^1_{n1}) - (y_n + \frac{h}{2}k^2_{n1}) - 1\, , x_n + \frac{h}{2}k^1_{n1}\right]^T$$
$$\mathbf{k}_{n3} = \left[-(t_n + \frac{h}{2})(x_n + \frac{h}{2}k^1_{n2}) - (y_n + \frac{h}{2}k^2_{n2}) - 1\, , x_n + \frac{h}{2}k^1_{n2}\right]^T$$
$$\mathbf{k}_{n4} = \left[-(t_n + h)(x_n + hk^1_{n3}) - (y_n + hk^2_{n3}) - 1\, , x_n + hk^1_{n3}\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	0.568451	0.109776	−0.32208	−0.681296	−0.937852
y_n	1.15775	1.22556	1.20347	1.10162	0.937852

(c) With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	0.56845	0.109773	-0.322081	-0.681291	-0.937841
y_n	1.15775	1.22557	1.20347	1.10161	0.93784

3.(a) The Euler formula gives

$$x_{n+1} = x_n + h(x_n - y_n + x_n\, y_n)$$
$$y_{n+1} = y_n + h(3x_n - 2y_n - x_n\, y_n).$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	-0.198	-0.378796	-0.51932	-0.594324	-0.588278
y_n	0.618	0.28329	-0.0321025	-0.326801	-0.57545

(b) Given

$$f(t, x, y) = x - y + x\, y$$
$$g(t, x, y) = 3x - 2y - x\, y\,,$$

the Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T$$
$$\mathbf{k}_{n2} = \left[f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1}), g(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1})\right]^T$$
$$\mathbf{k}_{n3} = \left[f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2}), g(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2})\right]^T$$
$$\mathbf{k}_{n4} = \left[f(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3}), g(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3})\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	-0.196904	-0.372643	-0.501302	-0.561270	-0.547053
y_n	0.630936	0.298888	-0.0111429	-0.288943	-0.508303

(c) With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	-0.196935	-0.372687	-0.501345	-0.561292	-0.547031
y_n	0.630939	0.298866	-0.0112184	-0.28907	-0.508427

4.(a) The Euler formula gives

$$x_{n+1} = x_n + h\,[x_n(1 - 0.5\,x_n - 0.5\,y_n)]$$
$$y_{n+1} = y_n + h\,[y_n(-0.25 + 0.5\,x_n)]\,.$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	2.96225	2.34119	1.90236	1.56602	1.29768
y_n	1.34538	1.67121	1.97158	2.23895	2.46732

(b) Given

$$f(t,x,y) = x(1 - 0.5\,x - 0.5\,y)$$
$$g(t,x,y) = y(-0.25 + 0.5\,x)\,,$$

the Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T$$
$$\mathbf{k}_{n2} = \left[f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1}), g(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1})\right]^T$$
$$\mathbf{k}_{n3} = \left[f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2}), g(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2})\right]^T$$
$$\mathbf{k}_{n4} = \left[f(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3}), g(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3})\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	3.06339	2.44497	1.9911	1.63818	1.3555
y_n	1.34858	1.68638	2.00036	2.27981	2.5175

(c) With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	3.06314	2.44465	1.99075	1.63781	1.35514
y_n	1.34899	1.68699	2.00107	2.28057	2.51827

5.(a) The Euler formula gives

$$x_{n+1} = x_n + h\left[e^{-x_n + y_n} - \cos\,x_n\right]$$
$$y_{n+1} = y_n + h\,[\sin(x_n - 3\,y_n)]\,.$$

With $h = 0.1$, we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.42386	1.82234	2.21728	2.61118	2.9955
y_n	2.18957	2.36791	2.53329	2.68763	2.83354

(b) The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T$$
$$\mathbf{k}_{n2} = \left[f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1}), g(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1})\right]^T$$
$$\mathbf{k}_{n3} = \left[f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2}), g(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2})\right]^T$$
$$\mathbf{k}_{n4} = \left[f(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3}), g(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3})\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.41513	1.81208	2.20635	2.59826	2.97806
y_n	2.18699	2.36233	2.5258	2.6794	2.82487

(c) With $h = 0.1$, we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.41513	1.81209	2.20635	2.59826	2.97806
y_n	2.18699	2.36233	2.52581	2.67941	2.82488

6. The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [x_n - 4\,y_n, -x_n + y_n]^T$$
$$\mathbf{k}_{n2} = \left[x_n + \frac{h}{2}k^1_{n1} - 4(y_n + \frac{h}{2}k^2_{n1}), -(x_n + \frac{h}{2}k^1_{n1}) + y_n + \frac{h}{2}k^2_{n1}\right]^T$$
$$\mathbf{k}_{n3} = \left[x_n + \frac{h}{2}k^1_{n2} - 4(y_n + \frac{h}{2}k^2_{n2}), -(x_n + \frac{h}{2}k^1_{n2}) + y_n + \frac{h}{2}k^2_{n2}\right]^T$$
$$\mathbf{k}_{n4} = \left[x_n + hk^1_{n3} - 4(y_n + hk^2_{n3}), -(x_n + hk^1_{n3}) + y_n + hk^2_{n3}\right]^T.$$

Using $h = 0.05$, we obtain the following values:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$	$n = 20$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.3204	1.9952	3.2992	5.7362	10.227
y_n	−0.25085	−0.66245	−1.3752	−2.6434	−4.9294

Using $h = 0.025$, we obtain the following values:

	$n = 8$	$n = 16$	$n = 24$	$n = 32$	$n = 40$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.3204	1.9952	3.2992	5.7362	10.227
y_n	−0.25085	−0.66245	−1.3752	−2.6435	−4.9294

The exact solution is given by

$$\phi(t) = \frac{e^{-t} + e^{3t}}{2}\,, \quad \psi(t) = \frac{e^{-t} - e^{3t}}{4}\,,$$

and the associated tabulated values:

	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$
t_n	0.2	0.4	0.6	0.8	1.0
$\phi(t_n)$	1.3204	1.9952	3.2992	5.7362	10.227
$\psi(t_n)$	−0.25085	−0.66245	−1.3752	−2.6435	−4.9294

8. The predictor formulas are

$$x_{n+1} = x_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3})$$
$$y_{n+1} = y_n + \frac{h}{24}(55\,g_n - 59\,g_{n-1} + 37\,g_{n-2} - 9\,g_{n-3}).$$

With $f_{n+1} = x_{n+1} - 4\,y_{n+1}$ and $g_{n+1} = -x_{n+1} + y_{n+1}$, the corrector formulas are

$$x_{n+1} = x_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})$$
$$y_{n+1} = y_n + \frac{h}{24}(9\,g_{n+1} + 19\,g_n - 5\,g_{n-1} + g_{n-2}).$$

We use the starting values from the exact solution:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0	0.1	0.2	0.3
x_n	1.0	1.12883	1.32042	1.60021
y_n	0.0	−0.11057	−0.250847	−0.429696

One time step using the predictor-corrector method results in the approximate values:

	$n = 4$ (pre)	$n = 4$ (cor)
t_n	0.4	0.4
x_n	1.99445	1.99521
y_n	−0.662064	−0.662442

8.6

3. The solution of the initial value problem is $\phi(t) = e^{-100\,t} + t$.

(a,b) Based on the exact solution, the local truncation error for both of the Euler methods is

$$|e_{loc}| \le \frac{10^4}{2} e^{-100\,\bar{t}_n} h^2.$$

Hence $|e_n| \le 5000\,h^2$, for all $0 < \bar{t}_n < 1$. Furthermore, the local truncation error is greatest near $t = 0$. Therefore $|e_1| \le 5000\,h^2 < 0.0005$ for $h < 0.0003$. Now the truncation error accumulates at each time step. Therefore the actual time step should be much smaller than $h \approx 0.0003$. For example, with $h = 0.00025$, we obtain the data

	Euler	B.Euler	$\phi(t)$
$t = 0.05$	0.056323	0.057165	0.056738
$t = 0.1$	0.100040	0.100051	0.100045

Note that the total number of time steps needed to reach $t = 0.1$ is $N = 400$.

(c) Using the Runge-Kutta method, comparisons are made for several values of h; $h = 0.005$ is sufficient.

$h = 0.1$:

	$\phi(t)$	y_n	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.057416	0.000678
$t = 0.1$	0.100045	0.100055	0.000010

$h = 0.005$:

	$\phi(t)$	y_n	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.056766	0.000027
$t = 0.1$	0.100045	0.100046	0.0000004

6.(a) Using the method of undetermined coefficients, it is easy to show that the general solution of the ODE is $y(t) = c\,e^{\lambda t} + t^2$. Imposing the initial condition, it follows that $c = 0$ and hence the solution of the IVP is $\phi(t) = t^2$.

(b) Using the Runge-Kutta method, with $h = 0.01$, numerical solutions are generated for various values of λ:

$\lambda = 1$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	0.0624999	2×10^{-11}
$t = 0.5$	0.25	0.25	0
$t = 0.75$	0.5625	0.5625	0
$t = 1.0$	1.0	1.0	0

$\lambda = 10$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	0.0624998	2.215×10^{-7}
$t = 0.5$	0.25	0.249997	2.920×10^{-6}
$t = 0.75$	0.5625	0.562464	3.579×10^{-5}
$t = 1.0$	1.0	0.999564	4.362×10^{-4}

$\lambda = 20$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	0.062489	1.10×10^{-5}
$t = 0.5$	0.25	0.248342	1.658×10^{-3}
$t = 0.75$	0.5625	0.316455	0.246045
$t = 1.0$	1.0	-35.5143	36.5143

$\lambda = 50$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	-0.044803	0.107303
$t = 0.5$	0.25	-28669.55	28669.804
$t = 0.75$	0.5625	-7.66014×10^{9}	7.66014×10^{9}
$t = 1.0$	1.0	-2.04668×10^{15}	2.04668×10^{15}

The following table shows the calculated value, y_1, at the first time step:

$\phi(t)$	$y_1\ (\lambda = 1)$	$y_1\ (\lambda = 10)$	$y_1\ (\lambda = 20)$	$y_1\ (\lambda = 50)$
10^{-2}	9.99999×10^{-5}	9.99979×10^{-5}	9.99833×10^{-5}	9.97396×10^{-5}

(c) Referring back to the exact solution given in part (a), if a nonzero initial condition, say $y(0) = \varepsilon$, is specified, the solution of the IVP becomes

$$\phi_\varepsilon(t) = \varepsilon\, e^{\lambda t} + t^2.$$

We then have $|\phi(t) - \phi_\varepsilon(t)| = |\varepsilon|\ e^{\lambda t}$. It is evident that for any $t > 0$,

$$\lim_{\lambda \to \infty} |\phi(t) - \phi_\varepsilon(t)| = \infty\,.$$

This implies that virtually any error introduced early in the calculations will be magnified as $\lambda \to \infty$. The initial value problem is inherently unstable.