Step-1

Thus, we have

$$\begin{aligned}0 &= \begin{vmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}\\ &= (1-\lambda)^2 - 1\\ &= \lambda^2 - 2\lambda\end{aligned}$$

Thus, one eigenvalue of A is positive and the other is zero.

Step-2

Consider the matrix $C^{\mathrm{T}}AC$.

We have

$$\begin{aligned}C^{\mathrm{T}}AC &= \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}2 & 2\\ -1 & -1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}\end{aligned}$$

Step-3

To obtain the eigenvalues of $C^{\mathrm{T}}AC$, we solve $\det\left(C^{\mathrm{T}}AC-\lambda I\right)=0$. This gives,

$$\begin{aligned}0 &= \begin{vmatrix}4-\lambda & -2\\ -2 & 1-\lambda\end{vmatrix}\\ &= (4-\lambda)(1-\lambda)-4\\ &= \lambda^2-5\lambda+4-4\\ &= \lambda^2-5\lambda\end{aligned}$$

Thus, here also one eigenvalue of $C^{\mathrm{T}}AC$ is positive and the other is zero.

Step-4

Let $Q=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

Consider the following chain of matrices:

$$
\begin{aligned}
C(t) &= tQ+(1-t)C \\
&= t\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+(1-t)\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \\
&= \begin{bmatrix}t & 0\\ 0 & -t\end{bmatrix}+\begin{bmatrix}2(1-t) & 0\\ 0 & t-1\end{bmatrix} \\
&= \begin{bmatrix}2-t & 0\\ 0 & -1\end{bmatrix}
\end{aligned}
$$

Step-5

Note that

$$
\begin{aligned}
C(0) &= \begin{bmatrix}2-0 & 0\\ 0 & -1\end{bmatrix} \\
&= \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \\
C(1) &= \begin{bmatrix}2-1 & 0\\ 0 & -1\end{bmatrix} \\
&= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}
\end{aligned}
$$

Step-6

Thus, $\boxed{C(t)=tQ+(1-t)C}$ is the required chain of non-singular matrices.

Since C has one positive and one zero eigenvalue and the identity matrix I has both positive eigenvalues, it is impossible to have such a chain of matrices from C to I.