Chapter 4. Markov chain: Limits and stationarity

4.1. Stationary distribution

Suppose X_0 has a distribution π,i.e.,

$$\mathbb{P}(X_0 = i) = \pi(i), \qquad i \in S.$$

Then,

$$\begin{aligned}\mathbb{P}(X_1 = i) &= \sum_j \mathbb{P}(X_1 = i | X_0 = j)\mathbb{P}(X_0 = j) \\ &= \sum_j \pi(j)p(j,i) = (\pi P)(i).\end{aligned}$$

Definition

π is a *stationary distribution* if $\pi P = \pi$.

Example: (Social mobility)

$$P = \begin{bmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{bmatrix}$$

Find its stationary distribution.
Solution: Do it on board!

Theorem

Suppose that the $k \times k$ transition matrix is irreducible. Then, there is a unique solution to $\pi P = \pi$ with $\sum_{i=1}^{k} \pi_i = 1$. Further, $\pi_i > 0, \forall\ i$.

Proof: Omit

Proposition

All states y *s.t.* $\pi(y) > 0$ *are recurrent.*

Proof: Do it on board!

4.2. Special examples

a) Double stochastic chains

Definition

Transition matrix P is *double stochastic* if its each column sums to 1,i.e.

$$\sum_x p(x, y) = 1, \qquad \forall\ y.$$

Theorem

If P is double stochastic with N states, then $\pi(x) = \frac{1}{N}$, $\forall x$ is a stationary distribution.

Proof: Do it on board!

Example

Symmetric reflecting random walk on $\{0, 1, \cdots, L\}$.
Take $L = 4$.

$$P = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & .5 & .5 \end{bmatrix}$$

It is clearly double stochastic.

b) Detailed balance condition.

Definition

A distribution π satisfies the *detailed balance condition* if

$$\pi(x)p(x,y) = \pi(y)p(y,x), \qquad \forall\ x, y \in S.$$

Proposition

If π satisfies the DBC, then π is a stationary distribution.

Proof: Do it on board!

Example

Consider a random walk on $0, 1, \cdots, 9$ given by

$$p(i, i+1) = \frac{2}{3},\ p(i, i-1) = \frac{1}{3}, \qquad 1 \le i \le 8.$$

$$p(0,1) = 1, \quad p(9,8) = 1.$$

Does the DBC hold?

Solution: Do it on board!

Example

$$P = \begin{bmatrix} .5 & .5 & 0 \\ .3 & .1 & .6 \\ .2 & .4 & .4 \end{bmatrix}$$

Does the DBC hold?

Solution: Do it on board!

c) Reversibility.
Let $p(i,j)$ be a transition probability with stationary distribution $\pi(i)$, $i \in S$.
Let X_n be the MC starting from the stationary distribution,

$$\mathbb{P}(X_0 = i) = \pi(i).$$

Let n be fixedand

$$Y_m = X_{n-m}, \qquad 0 \leq m \leq n$$

(look at the chain backward). Namely, we consider the process

$$X_n,\ X_{n-1},\ \cdots,\ X_1,\ X_0.$$

Theorem

(Y_m) is a MC with transition probability

$$\hat{p}(i,j) = \mathbb{P}(Y_{m+1} = j | Y_m = i) = \frac{\pi(j)p(j,i)}{\pi(i)}.$$

Proof: Do it on board!

Definition

The MC is *reversible* if $\hat{p}(i,j) = p(i,j),\ \forall\ i,j \in S$.

Proposition

(X_n) is reversible iff DBC holds.

Proof:

$$\begin{aligned} \hat{p}(i,j) = p(i,j) \quad &\Leftrightarrow \quad \frac{\pi(j)p(j,i)}{\pi(i)} = p(i,j) \\ &\Leftrightarrow \quad \pi(j)p(j,i) = \pi(i)p(i,j). \end{aligned}$$

$\square$

4.3. Limit behavior

Condition:

(I): P is irreducible.

(R): All states are recurrent.

(S): There is a stationary distribution π.

Recall: If y recurrent, then $N(y) = \infty$ a.s.
Let

$$N_n(y) = \sum_{i=1}^{n} 1_{X_i = y}$$

be the # of visits to y before time n.
How fast it goes to ∞?

Theorem

{I}&{R}

$$\lim_{n\to\infty}\frac{N_n(y)}{n}=\frac{1}{\mathbb{E}_y T_y}, \qquad a.s.$$

Proof: Do it on board!

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

As a consequence, we have

Theorem

Under (I) and (S), we have

$$\pi(y) = \frac{1}{\mathbb{E}_y T_y},$$

and hence, the stationary distribution is unique.

Proof: Do it on board!

Finally, we generalize Theorem **??**.Recall

$$\frac{1}{n}\sum_{i=1}^{n} 1_{X_i=y} \to \pi(y).$$

Rewrite

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \to \sum_{x} f(x)\pi(x)$$

with $f(x) = 1_{x=y}$.

Theorem

Under (I), (S) and $\sum_x |f(x)|\pi(x) < \infty$, *we have*

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x), \qquad a.s.$$

Namely, long term average equals spatial average,i.e., ergodicity.

Proof: Do it on board!

Example (Inventory chain)

Electronic store that sells a game system,with a potential for sales of 0, 1, 2, 3 of these units each day with prob. .3, .4, .2, .1.Each night, the store will bring inventory up to 3 items if it is x or less.Suppose the store makes \$12 for each unit sold and cost \$2 a day if it is stored overnight.Which x is the best?

Solution: Do it on board!

4.4. Infinite state spaces

Example (Reflecting random walk)

A particle moves on $\{0, 1, 2, \cdots\}$ with prob. p to the right and $1 - p$ to the left(at 0, it stays at 0 with prob. $1 - p$ instead of jumping to the left).

Solution:

Transition:

$$\begin{cases} p(i, i+1) = p, & i \geq 0, \\ p(i, i-1) = 1 - p, & i \geq 1, \\ p(0, 0) = 1 - p. \end{cases}$$

Stationary distribution π: Do it on board! $(p < .5)$

Next, we consider the case of $p > .5$.

Proposition

Let

$$V_x = \inf\{n \geq 0 : \ X_n = x\}.$$

If $p \neq .5$ *and* $\theta = \frac{p}{1-p} \neq 1$, *then*

$$\mathbb{P}_x(V_N < V_0) = \frac{\theta^{-x} - 1}{\theta^{-N} - 1}.$$

Proof: Do it on board!

We come back to our example for $p > .5$. Do it on board!
Finally, we consider $p = .5$.
Do it on board!

Proposition

Suppose $p = .5$. *Then,*

$$\mathbb{E}_x(T_0 \wedge T_N) = x(N - x), \qquad x \neq 0, N.$$

Proof: Do it on board!

Note that

$$\mathbb{E}_x(T_0 \wedge T_N) = x(N-x), \qquad x \neq 0, N.$$

So

$$\mathbb{E}_1 T_0 = \lim_{N\to\infty}(N-x) = \infty.$$

Thus,

$$\mathbb{E}_0 T_0 = \frac{1}{2} \times 1 + \frac{1}{2}\mathbb{E}_1(1+T_0) = \infty.$$

Take a long time to come back to 0.

Definition

A recurrent state x is *positive recurrent* if $\mathbb{E}_x T_x < \infty$; it is *null recurrent* if $\mathbb{E}_x T_x = \infty$.

$$0 \text{ is } \begin{cases} \text{positive recurrent} & p < .5 \\ \text{null recurrent} & p = .5 \\ \text{transient} & p > .5 \end{cases}$$

Let $\mu_x(y)$ be the expected # of visits to y before returning to the initial position x.Namely

$$\begin{aligned}\mu_x(y) &= \mathbb{E}_x \sum_{0\le n\le T_x-1} 1_{X_n=y} \\ &= \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y,\ T_x > n).\end{aligned}$$

Moving one-step in time (i.e. $1 \le n \le T_x$), the # of visits remains the same.

Theorem

Suppose (I) and (R) hold. Then, μ_x is a finite stationary measure.

Proof: Omit.

Recall

$$\pi(x) = \frac{1}{\mathbb{E}_x T_x} > 0 \text{ if } x \text{ is positive recurrent.}$$

Theorem

For any irreducible chain the following are equivalent:
i) There is a positive recurrent state.
ii) There is a stationary distribution π.
iii) All states are positive recurrent.

Proof: Do it on board!

Example (Branching processes)

Recall $X_n =\#$ of individuals in nth generation. $Y_i =\#$ of children of ith individual.

$$\mathbb{P}(Y_i = k) = p_k, \quad k = 0, 1, 2, \cdots .$$

$$\mu = \mathbb{E}Y_i = \sum_{k=0}^{\infty} kp_k.$$

Find extinction probability.

Solution: Do it on board!

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Lemma

ρ is the smallest root of $x = \phi(x)$, $0 \leq x \leq 1$.

Proof: Do it on board!

Example (Binary branching)

$$p_0 = 1 - a, \qquad p_2 = a.$$

Do it on board!

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Proposition

If $\mu > 1$, then $\rho < 1$ (there is a chance of avoiding extinction).

Proof: Do it on board!

HW: Ch4, 18, 20, 28, 36, 42, 46, 56, 64, 74.