Step-1

(a) Singular Value Decomposition (SVD) for any *m* by *n* matrix *A* is as follows

$$A = U\Sigma V^T$$
$$= \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m\times m \end{pmatrix}\begin{pmatrix} m\times n \text{ matrix } \Sigma \\ \sigma_1\cdots\sigma_r \text{ on diagonal} \end{pmatrix}\begin{pmatrix} \text{orthogonal} \\ V \text{ is } n\times n \end{pmatrix}$$

Here eigenvectors of AA^T are in *U*, eigenvectors of A^TA are in *V*.

The *r* singular-values on the diagonal of Σ are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Step-2

We know that the eigenvalue equation for AA^T is given by

$$AA^T x = \lambda x$$

Multiply both sides of above equation by a constant *c*.

$$cAA^T x = c\lambda x$$

So, if we multiple the above equation by any constant the eigenvectors remain same.

But the eigenvalues changes to c times of λ.

So, if we change *m* by *n* matrix *A* to $4A$, *U* and *V* remain same in the SVD for $4A$.

Step-3

We have *m* by *n* matrix as $4A$, so

$$(4A)(4A^T) = 16AA^T$$

The eigenvalue of $16AA^T$ is give by

$$16AA^T x = 16\lambda x$$

This implies eigenvalue of $16AA^T$ is 16λ.

Step-4

We know that the diagonal of Σ are the square roots of the nonzero eigenvalues of AA^T

$$\sigma = \sqrt{\lambda}.$$

Since the eigenvalue of $16AA^T$ is 16λ, so

$$\begin{aligned}\sigma &= \sqrt{16\lambda}\\ &= 4\sqrt{\lambda}\end{aligned}$$

The diagonal matrix for $16AA^T$ is 4Σ.

Step-5

The SVD for $4A$ is as follows:

$$\begin{aligned}4A &= U(4\Sigma)V^T\\ &= 4U\Sigma V^T\end{aligned}$$

Therefore, SVD for $4A$ is $\boxed{4A = 4U\Sigma V^T}$.

Step-6

(b) Consider the SVD for m by n matrix A.

$$A = U\Sigma V^T$$

The transpose of A is as follows:

$$\begin{aligned}A^T &= \left(U\Sigma V^T\right)^T\\ &= V\Sigma^T U^T\end{aligned}$$

Here Σ^T is n by m matrix, with r nonzero entries σi.

Therefore, SVD for $\boxed{A^T = V\Sigma^T U^T}$.

Step-7

If matrix A non-singular matrix, then the inverse of A is as follows:

$$\begin{aligned}A &= U\Sigma V^T\\ A^+ &= \left(U\Sigma V^T\right)^{-1}\\ &= V\Sigma^{-1}U^{-1}\\ &= V\Sigma^{-1}U^T\end{aligned}$$

Here Σ^{-1} is n by m matrix, with r nonzero entries $\frac{1}{\sigma i}$.

If matrix A is square and invertible matrix then $A^{+} = A^{-1}$.

Therefore, when A is square and invertible then SVD for $\boxed{A^{-1} = V\Sigma^{-1}U^{T}}$.