Step-1

Given unit vector is $u=\left(\frac{1}{6},\frac{1}{6},\frac{3}{6},\frac{5}{6}\right)$

P is the projection matrix given by $P=uu^T$

$$=\begin{bmatrix}\frac{1}{6} & \frac{1}{6} & \frac{3}{6} & \frac{5}{6}\end{bmatrix}\begin{bmatrix}\frac{1}{6}\\ \frac{1}{6}\\ \frac{3}{6}\\ \frac{5}{6}\end{bmatrix}$$

$$=\left[\frac{1}{36}+\frac{1}{36}+\frac{9}{36}+\frac{25}{36}\right]$$
$$=[1]$$

Clearly, this is a matrix of size 1×1, with determinant 1

So, the rank of this matrix is 1.

Step-2

a) $Pu=\left(uu^T\right)u$

$=u\left(u^Tu\right)$ by the associative property

$=u\cdot I$ while u is the unit vector

$=u$

We know that if λ is the eigen value of a matrix A and x is the corresponding eigen vector, then it follows that $Ax=\lambda x$.

In view of this, we can write $Pu = u$ as $Pu=1u$ and thus, $\lambda = 1$ is the corresponding eigen value of u.

Step-3

b) Suppose v is a perpendicular vector to u.

Then $u^Tv=0$

$Pv = (uu^T)v$ while $P = uu^T$

$= u(u^T v)$ by associate

$= u(0)$ since $u^T v = 0$

$= 0$

This can be written as $Pv = \lambda v$ where $\lambda = 0$

Therefore, the eigen value of *P* corresponding to eigen vector *v* is 0.

Step-4

c) We consider

$$x_1 = (-1,1,0,0)$$
$$x_2 = (-3,0,1,0)$$
$$x_3 = (-5,0,0,1)$$

Then

$$u^T x_1 = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \\ \frac{5}{6} \end{bmatrix} (-1,1,0,0)$$
$$= \left[\frac{-1}{6} + \frac{1}{6} + 0 + 0 \right]$$
$$= 0$$

Step-5

$$u^T x_2 = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \\ \frac{5}{6} \end{bmatrix} [-3,0,1,0]$$

$$= 0$$

$$u^T x_3 = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \\ \frac{5}{6} \end{bmatrix} [-5 \quad 0 \quad 0 \quad 1]$$

$$= \left[\frac{-5}{6} + 0 + 0 + \frac{5}{6} \right]$$

$$= 0$$

Therefore x_1, x_2, x_3 are orthogonal to *u*.

Step-6

So, in view of the result (b), we confirm that the eigen value of *P* with respect to the eigen vectors x_1, x_2, x_3 is $\lambda = 0$.