Step-1

We have to apply elimination with the extra column to reach $Rx = 0$ and $Rx = d$:

$$[U \;\; 0] = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}, \text{ and } \quad [U \;\; c] = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$$

We have to find the solutions to $Rx = 0$ and $Rx = d$.

Step-2

Given that $[U \;\; 0] = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-3

Now this is reduced row echelon form x_1, x_3 are pivots, x_2 is free variable.

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 = 0$$
$$x_3 = 0$$

$$\Rightarrow x_1 = 0, x_3 = 0$$

Step-4

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore the solution to $Rx = 0$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Step-5

Now $\begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & 9 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

This is reduced row echelon form.

Step-6

$$Rx = d$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

All entries last row in R are zeros, but in $Rx = d$ last entry is 5, therefore there are no solutions for $Rx = d$.