

## Step-1

Given that  $A$  commutes with every 2 by 2 matrix ( $AB = BA$ )

$$\text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in (i)$$

We have to show that  $a = d$  and  $b = c = 0$

## Step-2

Since  $A$  commutes with  $B_1$  and  $B_2$ , we have  $AB_1 = B_1A$  and  $AB_2 = B_2A$

$$\begin{aligned} AB_1 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} B_1A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} AB_1 &= B_1A \\ \Rightarrow \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} &= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \\ \Rightarrow \boxed{b=0 \text{ and } c=0} &\in (i) \end{aligned}$$

## Step-3

$$\begin{aligned} AB_2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \end{aligned}$$

$$B_2A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

$$AB_2 = B_2A$$

$$\Rightarrow \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{c=0 \text{ and } a=d} \quad \text{â€} (ii)$$

From (i) and (ii), we have  $\boxed{a=d \text{ and } b=c=0}$ .

## Step-4

Suppose that  $AB = BA$  for all matrices  $A$  and  $B$ , then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= aI$$

Hence  $A$  is a multiple of the identity.