

Step-1

The objective is to find that the trace of the matrix is equals the sum of the eigenvalues.

Step-2

Consider that a $n \times n$ matrix A .

Let $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$ be the eigenvalues of the matrix A

Recall that characteristic equation is kept to zero to get the eigenvalues. This means that determinant value of $A - \lambda I$ must be equal to zero.

That is,

The characteristic equation of the matrix is given by,

$$\begin{aligned} P(\lambda) &= |\lambda I - A| \\ &= \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0 \end{aligned} \quad \dots(1)$$

Since the set $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are all eigenvalues of the matrix A

Hence, the characteristic equation factors as $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

Step-3

From the equation (1) the coefficient of λ^{n-1} is c_{n-1}

The coefficient of the λ^{n-1} calculated in two different ways as follows:

First By expanding the characteristic equation $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

Hence, by expansion the λ^{n-1} term will be

$$-\lambda_1\lambda^{n-1} - \lambda_2\lambda^{n-1} \dots - \lambda_n\lambda^{n-1} = -(\lambda_1 + \lambda_2 + \dots + \lambda_n)\lambda^{n-1}$$

By comparing the λ^{n-1} term with equation (1) obtain $c_{n-1} = -(\lambda_1 + \lambda_2 + \dots + \lambda_n)$ (2)

Step-4

Second the coefficients can be calculated by expanding $\det(A - \lambda I)$ as,

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}$$

From the determinant of the matrix $\det(A - \lambda I)$ one of the product is $(\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn})$ and every other possible products can contain at most $n-2$ λ 's.

Hence, the product of all other possible products will be a polynomial of degree at most $n-2$.

Let that polynomial is denoted by $q(\lambda)$

Thus, by expanding $\det(A - \lambda I)$ obtain,

The characteristic equation $p(\lambda) = (\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn}) + q(\lambda)$

Since the degree of the polynomial $q(\lambda)$ is $n-2$ it has no λ^{n-1} term.

Hence, the λ^{n-1} term of the characteristic equation $p(\lambda)$ is in the form $(\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn})$

By expanding the product $(\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn})$ obtain,

$$-(a_{11} + a_{22} \dots + a_{nn}) \lambda^{n-1} \quad \text{--- (3)}$$

Compare the equation (1) and (3) obtain,

$$c_{n-1} = -(a_{11} + a_{22} + \dots + a_{nn}) \quad \text{--- (4)}$$

From equation (2) and (4) obtain,

$$\begin{aligned} -(\lambda_1 + \lambda_2 + \dots + \lambda_n) &= -(a_{11} + a_{22} + \dots + a_{nn}) \\ (\lambda_1 + \lambda_2 + \dots + \lambda_n) &= (a_{11} + a_{22} + \dots + a_{nn}) \end{aligned}$$

Hence, trace of the matrix is equal to the sum of the eigenvalues.