南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目:	高等数学(下) A	开课单位:	数 学 系
考试时长:	180 分钟	命题教师:	王融、吴纪桃 等

题 号	1	2	3	4	5	6	7	8	9	10
分 值	9 分	15 分	9 分	7 分	7 分	7 分	7 分	7 分	7 分	7 分
题 号	11	12	13							
分 值	7 分	7 分	4 分							

1. (9 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

(1) If $a_n > 0, \forall n$, and $\lim\limits_{n\to\infty} na_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(2) The plane $x + y - 2z = 1$ is perpendicular to the plane $x + y + z = 1$.

(3) If $f(x, y)$ has two local maxima, then f must have a local minimum.

Solution: (1) F; (2) T; (3) F.

2. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

(1) Which one of the following series diverges?

(A) $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$.　　(B) $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

(C) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$.　　(D) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n(3+(-1)^n\cdot 2)^n}{6^n}$.

(2) The iterated integral $\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} f(r\cos\theta, r\sin\theta)r\,drd\theta$ can be written as

(A) $\int_0^1\int_0^{\sqrt{y-y^2}} f(x,y)\,dxdy$.　　(B) $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy$.

(C) $\int_0^1\int_0^1 f(x,y)\,dydx$.　　(D) $\int_0^1\int_0^{\sqrt{x-x^2}} f(x,y)\,dydx$.

(3) For the function, $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0), \end{cases}$ which of the following statements is correct?

(A) f is not continuous at $(0, 0)$.

(B) f is continuous at $(0,0)$, but its partial derivative f_x and f_y do not exist at $(0,0)$.

(C) Both partial derivatives f_x and f_y exist everywhere and are also continuous at $(0,0)$.

(D) f is not differentiable at $(0,0)$.

(4) For the critical points of the function $f(x,y) = 2x^4 + y^4 - 2x^2 - 2y^2$, which one of the following statements is correct?

(A) $(0,0)$ is a local minima.

(B) $(0,1)$ is a local maxima.

(C) $(0,-1)$ is a saddle point.

(D) There are no local maxima among all the critical points.

(5) If the function $f(x,y)$ has the continuous first partial derivatives $\frac{\partial f}{\partial x} > 0$ and $\frac{\partial f}{\partial y} < 0$, $\forall (x,y) \in \mathbf{R}^2$, which one of the following statements is correct?

(A) $f(0,0) > f(1,1)$.　　(B) $f(0,0) < f(1,1)$.

(C) $f(0,1) > f(1,0)$.　　(D) $f(0,1) < f(1,0)$.

Solution: (1) B; (2) D; (3) D; (4) C; (5) D.

3. (9 pts) Please fill in the blank for the questions below.

(1) Compute the limit: $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2} =$ ________________.

(2) The direction (unit vector) in which the function $f(x,y) = x^2 + xy + y^2 - y$ increases most rapidly at the point $(-1,2)$ is ________________.

(3) $\displaystyle\int_0^1 \int_y^1 \frac{\tan x}{x}\, dxdy =$ ________________.

Solution: (1) $\frac{1}{2}$; (2) $(0,1)$; (3) $\ln(\sec 1)$.

4. (7 pts)

(1) Find the interval of convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n (x-1)^{2n+1}}{\sqrt{n + 9012 \ln n}}$.

(2) For what values of x does the series converge absolutely, or conditionally?

Solution:

(1) $[0,2]$.

(2) The series converges absolutely for $0 < x < 2$, and converges conditionally for $x = 0, 2$.

5. (7 pts) The region D is bounded by $z = \sqrt{x^2+y^2}$ and $z = \sqrt{1-x^2-y^2}$. Consider the following integral

$$\iiint\limits_D (x+z)\, dxdydz,$$

(1) Convert the above integral to an equivalent iterated integral in cylindrical coordinates;

(2) Convert the above integral to an equivalent iterated integral in spherical coordinates.

Solution:

(1) $\displaystyle\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}}(r\cos\theta+z)r\,dzdrd\theta.$

(2) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1(\rho\sin\varphi\cos\theta+\rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta.$

6. (7 pts) Assume we can put a cuboid into the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Use the method of **Lagrange multipliers** to find the length, width and height of the cuboid such that it achieve the maximum volume.

Solution:

$$\begin{cases} yz=\lambda\frac{2x}{a^2} \\ xz=\lambda\frac{2y}{b^2} \\ xy=\lambda\frac{2z}{c^2} \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \end{cases}$$

$x=\frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$, $z=\frac{c}{\sqrt{3}}$. Thus the length is $\frac{2a}{\sqrt{3}}$, the width is $\frac{2b}{\sqrt{3}}$, and the height is $\frac{2c}{\sqrt{3}}$.

7. (7 pts) Find the equation of the osculating circle for the parabola $y=x^2$ at $x=1$.

Solution:

$$\mathbf{r}(t)=(t,t^2),\qquad \mathbf{v}(t)=(1,2t),|\mathbf{v}|=\sqrt{1+4t^2}.$$

$$\mathbf{T}=\left(\frac{1}{\sqrt{1+4t^2}},\frac{2t}{\sqrt{1+4t^2}}\right)$$

$$\frac{d\mathbf{T}}{dt}=\left(-\frac{4t}{(1+4t^2)^{\frac{3}{2}}},\frac{2}{(1+4t^2)^{\frac{3}{2}}}\right)$$

$$\kappa(1)=\frac{2}{5\sqrt{5}}$$

The radius of the osculating circle is $\frac{5\sqrt{5}}{2}$, and the center is

$$(1,1)+\frac{5\sqrt{5}}{2}\cdot\frac{1}{\sqrt{5}}(-2,1)=\left(-4,\frac{7}{2}\right)$$

The equation is

$$(x+4)^2+\left(y-\frac{7}{2}\right)^2=\frac{125}{4}.$$

8. (7 pts) A solid in the first octant is bounded by the planes $y=0$ and $z=0$ and by the surfaces $z=4-x^2$ and $x=y^2$ (see the figure below). Its density function is $\delta(x,y,z)=xy$. Find the center of the mass for the solid.

Solution:

$$M=\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2}xy\,dzdydx=\frac{32}{15}$$

$$M_{yz}=\int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2}x^2y\,dzdydx=\frac{8}{3}$$

$$M_{xz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} xy^2 \, dzdydx = \frac{256\sqrt{2}}{231}$$

$$M_{yz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} xyz \, dzdydx = \frac{256}{105}$$

$$\bar{x} = \frac{5}{4}, \qquad \bar{y} = \frac{40\sqrt{2}}{77}, \qquad \bar{z} = \frac{8}{7}$$

9. (7 pts) Use the **substitution in double integral** (please find the transformation by yourself) to evaluate the integral

$$\iint_D e^{\frac{y-x}{y+x}} \, dxdy,$$

here D is the triangular region bounded by the lines $x = 0$, $y = 0$, and $x + y = 2$.

Solution: Let $u = y - x$, and $v = y + x$.

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}.$$

The boundaries in uv-plane is a triangle enclosed by $u = v$, $u = -v$, and $v = 2$.

$$\int_0^2 \int_{-v}^{v} e^{\frac{u}{v}} \frac{1}{2} \, dudv = e - \frac{1}{e}.$$

10. (7 pts) Consider the line integral

$$\int_{(1,1,1)}^{(1,3,\pi)} e^x \ln y \, dx + \left(\frac{e^x}{y} + \sin z\right) dy + y \cos z \, dz.$$

(1) Show that the differential form in the integral is exact.

(2) Evaluate the integral.

Solution:

(1) The potential function is $f(x, y, z) = e^x \ln y + y \sin z + C$.

(2)
$$\int_{(1,1,1)}^{(1,3,\pi)} e^x \ln y \, dx + \left(\frac{e^x}{y} + \sin z\right) dy + y \cos z \, dz = e \ln 3 - \sin 1.$$

11. (7 pts) Evaluate

$$\iint_S \nabla \times (4x\mathbf{j}) \cdot \mathbf{n} \, d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$. Use the normal vectors pointed away from the origin.

Solution:

$$\mathbf{r}(t) = 4 \cos t \, \mathbf{i} + 4 \sin t \, \mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

$$\iint_S \nabla \times (4x\mathbf{j}) \cdot \mathbf{n} d\sigma = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 32(1+\cos 2t)dt = 64\pi.$$

12. (7 pts) Find the outward flux of $\mathbf{F} = (6x+y)\mathbf{i} - (x+z)\mathbf{j} + 4yz\mathbf{k}$ across the boundary of D, where D is the region in the first octant bounded by the cone $z = \sqrt{x^2+y^2}$, the cylinder $x^2+y^2=1$, and the coordinate planes.

Solution:

$$\nabla \cdot \mathbf{F} = 6 + 4y.$$

$$\begin{aligned} \text{Flux} &= \iiint_D (6+4y)\, dV \\ &= \int_0^{\pi/2} \int_0^1 \int_0^r (6+4r\sin\theta)\, dz r dr d\theta \\ &= \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin\theta)\, dr d\theta \\ &= \int_0^{\pi/2} (2+\sin\theta)\, d\theta \\ &= \pi + 1. \end{aligned}$$

13. (4 pts) The sequences $\{a_n\}$ and $\{b_n\}$ satisfy $0 < a_n < \frac{\pi}{2}$, $0 < b_n < \frac{\pi}{2}$, and $\cos a_n - a_n = \cos b_n$, $n = 1, 2, 3, \cdots$. The series $\sum_{n=1}^{\infty} b_n$ converges. Show that $\lim_{n\to\infty} a_n = 0$.

Solution:

$$a_n = \cos a_n - \cos b_n = -2\sin\frac{a_n+b_n}{2}\sin\frac{a_n-b_n}{2}$$

$$0 < a_n < b_n$$

Use the Sandwich theorem to complete the proof.