Step-1

Compute the Eigen values and Eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Solve the following:

$$\frac{du}{dt} = Au$$

Initial conditions are as follows:

$$u(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Step-2

First step is to find the Eigen values and Eigen vectors of matrix *A*. To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 6 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(3-\lambda)(4-\lambda) = 0$$

After solving following values are obtained:

$$\lambda_1 = 4$$
$$\lambda_2 = 3$$
$$\lambda_3 = 1$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}1-4 & 2 & 1\\ 0 & 3-4 & 6\\ 0 & 0 & 4-4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-3 & 2 & 1\\ 0 & -1 & 6\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

On solving, values of x, y and z corresponding to $\lambda=4$ are as follows:

$$x_1=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$=\begin{bmatrix}13/3\\ 6\\ 1\end{bmatrix}$$

Step-4

Similarly, Eigen vectors corresponding to Eigen value $\lambda=3$ is as follows:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}1-3 & 2 & 1\\ 0 & 3-3 & 6\\ 0 & 0 & 4-3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-2 & 2 & 1\\ 0 & 0 & 6\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_2=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 1$ is as follows:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-1 & 2 & 1 \\ 0 & 3-1 & 6 \\ 0 & 0 & 4-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Step-6

Recall that $e^{At} = Se^{\Lambda t}S^{-1}$. Therefore,

$$\begin{aligned} u(t) &= e^{At}u(0) \\ &= Se^{\Lambda t}S^{-1}u(0) \\ &= \begin{bmatrix} 1 & 1 & 13/3 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 5/3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{t} & e^{3t} & (13/3)e^{4t} \\ 0 & e^{3t} & 6e^{4t} \\ 0 & 0 & e^{4t} \end{bmatrix} \begin{bmatrix} (8/3) \\ -6 \\ 1 \end{bmatrix} \end{aligned}$$

$$u(t) = \begin{bmatrix} (8/3)e^{t} - 6e^{3t} + (13/3)e^{4t} \\ -6e^{3t} + 6e^{4t} \\ e^{4t} \end{bmatrix}$$

Step-7

Therefore, solution is as follows:

$$\boxed{u(t) = \begin{bmatrix} (8/3)e^{t} - 6e^{3t} + (13/3)e^{4t} \\ -6e^{3t} + 6e^{4t} \\ e^{4t} \end{bmatrix}}$$