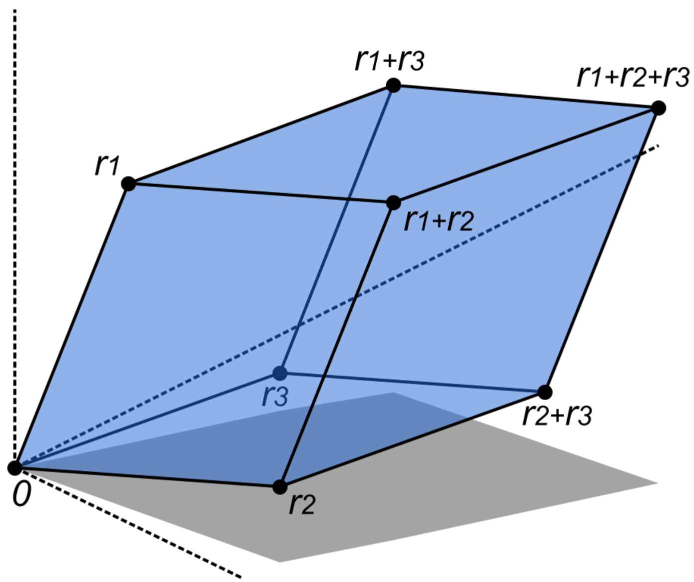
4 Determinants (行列式)

4.3

FORMULAS FOR THE DETERMINANT

- Formula from pivots
- A Big formula
- Expansion Rule
- *Calculations (Some Tricks)*



I. Formula from Pivots (行列式的计算公式：主元)

Theorem 1 *If* $\boldsymbol{A}$ *is invertible, then* $\boldsymbol{PA} = \boldsymbol{LDU}$ *and* $|\boldsymbol{P}| = \pm 1$. *The product rule gives*

$$|A| = \pm\, |L|\, |D|\, |U| = \pm\, |D| = \pm(\textit{product of the pivots})$$

The sign ± 1 depends on whether the number of row exchanges is even (偶) or odd (奇). ($+1$ for even; -1 for odd)

For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & (ad-bc)/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix},$$

the product of the pivots is $ad - bc = |\boldsymbol{A}| = |\boldsymbol{D}|$.

If there is a row exchange, then

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix},$$ and $|\boldsymbol{A}| = -|\boldsymbol{D}|$.

Example 1 The $-1, 2, -1$ second difference matrix

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \frac{2}{1} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

Its determinant is the product of its pivots.

$$|A_4| = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right) = 5.$$

In general, for A_n in this pattern, we have $|A_n| = n + 1$.

II. A Big formula – An equivalent definition (行列式的等价定义)

For $n = 2$, we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c+0 & 0+d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0+d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad - bc.$$

To get nonzero terms: Suppose 1st row has a nonzero term in column α, 2nd row is nonzero in column β, …, and finally the n-th row in column υ.

Then the column numbers $\alpha, \beta, \dots, \upsilon$ are all different.

They are a reordering, or *permutation* (排列), of the numbers $1, 2, \dots, n$.

For $n = 3$, we have

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

There are $n!$ ways to permute numbers $1,2,\dots,n$.

Column numbers: α, β, υ= (1,2,3), (2,3,1), (3,1,2), (3,2,1), (2,1,3), (1,3,2).

$|\boldsymbol{A}|$

$$= a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

- Every term is a product of $n = 3$ entries a_{ij}, with ***each row and column represented once***.
- If the columns come in the order $(\alpha, \dots, \upsilon)$, that term is the product $a_{1\alpha} \dots a_{n\upsilon}$ *times* the determinant of a permutation matrix $\boldsymbol{P}$.
- The determinant of the whole matrix $\boldsymbol{A}$ is the sum of these $n!$ terms.

*This leads a **'big formula'** for computing the determinant of* $\boldsymbol{A} = [a_{ij}]_{n\times n}$.

Theorem 2 (Big formula)

$$|\boldsymbol{A}| = \sum_{all\ \boldsymbol{P}'s} (a_{1\alpha} a_{2\beta} \dots a_{n\upsilon})\, |\boldsymbol{P}|.$$

Notes:

$|\boldsymbol{P}| = 1\ or\ -1$ for an *even* or *odd* number of row exchanges.

For example,

$$\boldsymbol{P} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix},$$

with determinants equal to $1, 1, -1$, respectively.

Equivalently, it depends on the permutation of the n numbers being an even permutation (偶排列) *or odd permutation* (奇排列).

(2,3,1), (3,1,2): even; (1,3,2): odd.

(1,3,2) requires one exchange to recover (1,2,3);

(2,3,1), (3,1,2) requires two exchanges to recover (1,2,3).

Example 2 Let $f(x) = \begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 2x & 1 \end{vmatrix}$.

Find the coefficient of x^3.

Solution

$$f(x) = \begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 2x & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} a_{11}a_{22}a_{33}a_{44} = x^3,$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{vmatrix} a_{11}a_{22}a_{34}a_{43} = -2x^3.$$

The coefficient of x^3 is -1.

III. Expansion of det$A$ in cofactors (使用代数余子式展开行列式)

No row or column can be used twice in the same term.

For $n = 3$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

Theorem 3 *The determinant of* $\boldsymbol{A}$ *is a combination of any row* i *times its cofactors:*

$$|\boldsymbol{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor C_{ij} is the determinant of $\boldsymbol{M}_{ij}$ with the correct sign:

$$C_{ij} = (-1)^{i+j}|\boldsymbol{M}_{ij}|.$$

The submatrix $\boldsymbol{M}_{ij}$ is formed by *throwing away row i and column j.*

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \quad \boldsymbol{M}_{23}=\begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix},$$

$$C_{23}=(-1)^{2+3}|\boldsymbol{M}_{23}|=-|\boldsymbol{M}_{23}|.$$

注意： 一个元素的代数余子式只与该元素所处位置有关，而与该元素等于多少无关.

思考 设 $D=\begin{vmatrix} 3 & -5 & 2 & 1 \\ 1 & 1 & 0 & -5 \\ -1 & 3 & 1 & 3 \\ 2 & -4 & -1 & -3 \end{vmatrix}$，求 $2C_{11}-4C_{12}-C_{13}-3C_{14}$.

结论：某一行元素依次乘以另一行元素的代数余子式再求和，其结果等于0.

Example 1 (Continued) The $-1, 2, -1$ second difference matrix (4×4)

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

$$C_{11} = |A_3| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = |A_2|.$$

$$|A_4| = 2C_{11} - C_{12} = 2|A_3| - |A_2| = 2(4) - 3 = 5.$$

The same idea applies to every A_n: $|A_n| = 2|A_{n-1}| - |A_{n-2}|.$

By recursion (递推), we can get $|A_n| = 2(n) - (n-1) = n+1.$

IV. Some Tricks for Computing Determinants

例1 计算 $D=\begin{vmatrix} a_1+\lambda & a_2 & \cdots & a_n \\ a_1 & a_2+\lambda & \cdots & a_n \\ & \cdots & \cdots & \\ a_1 & a_2 & \cdots & a_n+\lambda \end{vmatrix}$.

例2 证明Vandermonde行列式

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (x_i - x_j).$$

例3 计算

$$D_n = \begin{vmatrix} \alpha+\beta & \alpha\beta & & & \\ 1 & \alpha+\beta & \alpha\beta & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha+\beta & \alpha\beta \\ & & & 1 & \alpha+\beta \end{vmatrix}.$$

例4 计算 $D=\begin{vmatrix} a_0 & 1 & 2 & \cdots & n \\ 1 & a_1 & 0 & \cdots & 0 \\ 2 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n & 0 & 0 & \cdots & a_n \end{vmatrix}$.

爪形行列式
(箭头形行列式)