Solution for Assignment 07

Fan Zeteng

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Problem 1. Find the following values by using the Statistical Tables:

(a) $F(-1.72), F(-1.723), F(0.48)$ and $F(1.234)$, where F(x) is the c.d.f. of the standard normal random variable.

(b) Find x such that $F(x) = 0.546$, where $F(x)$ is the c.d.f. of the standard normal random variable. Similarly find y such that $F(y) = 0.258$.

Solution. $F(-1.72) = 1 - F(1.72) \approx 1 - 0.9573 = 0.0427$

$F(-1.723) = 1 - F(1.723) \approx 1 - F(1.72) \approx 0.0427$

(a) $\cancel{F(1.72) = 0.9573, F(1.73) = 0.9582}, F(0.48) = 0.6844, F(1.23) = 0.8888, \cancel{F(1.24) = 0.8907}.$

$F(1.234) \approx F(1.23) \approx 0.8907$

(b) $F(0.11) = 0.5438, F(0.12) = 0.5478,$

$1 - F(y) = 0.742, F(0.65) = 0.7422, y = -0.65.$

Problem 2. Assume that heights of children in a certain age group average are normally distributed, i.e., $X \sim N(\mu, \sigma^2)$, where $\mu = 58.4$ inches and with $\sigma = 2.9$ inches.

(a) What proportion of children are between 57 and 61 inches tall?

(b) What is the number c such that 90% of the children's height in a certain age group average is less than c?

SOLUTION.

(a) As $Y = \frac{X-58.4}{2.9} \sim N(0,1)$, we have

$$\begin{aligned}
P(57 \leq X \leq 61) =& P(\frac{57-58.4}{2.9} \leq \frac{X-58.4}{2.9} \leq \frac{61-58.4}{2.9}) \\
=& P(-0.48 \leq X \leq 0.90) \\
=& F(0.90) - F(-0.48) \\
=& F(0.90) + F(0.48) - 1 \\
=& 0.8159 + 0.6844 - 1 \\
=& 0.5003 = 0.50.
\end{aligned}$$

(b) It equals to find c such that $F(\frac{c-58.4}{2.9}) = 0.9$.

From the table, we know $F(1.028) = 0.8997, F(1.29) = 0.9015$,so we may take $\frac{c-58.4}{2.9} = 1.29$ and get $c = 62.141$.

PROBLEM 3. Suppose $X \sim N(\mu, \sigma^2)$ and let $Y = exp(X) = e^X$

(a) What are all possible values of Y ?

(b) Obtain the probability density function of Y.

SOLUTION.

(a) X takes values on $\mathbb{R}$, so $Y = e^X$ can take any positive number, i.e.,$Y(\Omega) = (0, \infty)$.

(b) Let $g(x) = e^x$, Since $Y = g(X)$ is strictly increasing about X, we can take the formula on class and get

$$\begin{aligned}
f_Y(y) =& f_X[g^{-1}(y)] * |\frac{d}{dy}g^{-1}(y)| \\
=& f_X(lny) * |\frac{1}{y}| \\
=& \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}},
\end{aligned}$$

for any $y > 0$.And $f_Y(y) = 0$ for $y \leq 0$.

PROBLEM 4. Suppose $X \sim N(\mu, \sigma^2)$ and let $Y = aX + b$ where a and b are two constants and the constant a is not zero.

(a) What are all possible values of Y?

(b) Obtain the probability density function of Y.

(c) Explain Y is also normally distributed. What are the parameters of Y?

SOLUTION.

(a) Since $a \neq 0$, $Y(\Omega) = \mathbb{R}$.

(b) Same as problem 3, when $a \neq 0$, $g(x) = ax + b$ is strictly monotone, so

$$\begin{aligned}
f_Y(y) =& f_X[g^{-1}(y)] * |\frac{d}{dy}g^{-1}(y)| \\
=& f_X(\frac{y-b}{a}) * |\frac{1}{a}| \\
=& \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{(y-b-a\mu)^2}{2(a\sigma)^2}}
\end{aligned}$$

for any $y \in \mathbb{R}$.

(c) From (b) and definition for normal r.v., we know $Y \sim N(b + a\mu, a^2\sigma^2)$

PROBLEM 5. Suppose $X \sim N(0, 1)$ and let $Y = X^2$.

(a) What are all possible values of Y?

(b) We have argued the problem in 3.7.3 of the lecture notes, and

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}.$$

for any $y > 0$. And $f_Y(y) = 0$ for $y \leq 0$.

SOLUTION.

(a) $Y(\Omega) = [0, \infty)$.

(b) Obtain the probability density function of Y.

PROBLEM 6. Suppose $Y \sim N(0,1)$. Let $-\infty < a < b < +\infty$ and $m = \frac{1}{2}max\{a^2, b^2\}$. Show that

$$(b-a)e^{-m} \leq \sqrt{2\pi} P\{a \leq Y \leq b\} \leq b - a.$$

SOLUTION.

$$\sqrt{2\pi} P\{a \leq Y \leq b\} = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Since $-m \leq -\frac{x^2}{2} \leq 0$, we have

$$(b-a)e^{-m} \leq \sqrt{2\pi} P\{a \leq Y \leq b\} \leq b - a.$$

PROBLEM 7. Suppose $Y \sim N(0,1)$. Show that for any $y > 0$, we have

$$\frac{1}{y} - \frac{1}{y^3} \leq \sqrt{2\pi} e^{\frac{y^2}{2}} P\{Y \geq y\} \leq \frac{1}{y}.$$

SOLUTION. The nonequation is equal to

$$e^{-\frac{y^2}{2}} (\frac{1}{y} - \frac{1}{y^3}) \leq \int_y^\infty e^{-\frac{x^2}{2}} dx \leq e^{-\frac{y^2}{2}} \frac{1}{y}.$$

We prove it by two equtions:

$$\begin{aligned}
\int_y^\infty \frac{1}{x^2} e^{-\frac{x^2}{2}} dx = & \int_y^\infty e^{-\frac{x^2}{2}} d(-\frac{1}{x}) \\
= & -\frac{1}{x} e^{-\frac{x^2}{2}} |_y^\infty + \int_y^\infty \frac{1}{x} d e^{-\frac{x^2}{2}} \\
= & \frac{1}{y} e^{-\frac{y^2}{2}} - \int_y^\infty e^{-\frac{x^2}{2}} dx,
\end{aligned}$$

i.e.

$$\int_y^\infty e^{-\frac{x^2}{2}} dx = \frac{1}{y} e^{-\frac{y^2}{2}} - \int_y^\infty \frac{1}{x^2} e^{-\frac{x^2}{2}} dx \le \frac{1}{y} e^{-\frac{y^2}{2}}.$$

And

$$\begin{aligned}
\int_y^\infty \frac{3}{x^4} e^{-\frac{x^2}{2}} dx = & \int_y^\infty e^{-\frac{x^2}{2}} d(-\frac{1}{x^3}) \\
= & -\frac{1}{x^3} e^{-\frac{x^2}{2}} |_y^\infty + \int_y^\infty \frac{1}{x^3} d e^{-\frac{x^2}{2}} \\
= & \frac{1}{y^3} e^{-\frac{y^2}{2}} - \int_y^\infty \frac{1}{x^2} e^{-\frac{x^2}{2}} dx \\
= & (\frac{1}{y^3} - \frac{1}{y}) e^{-\frac{y^2}{2}} + \int_y^\infty e^{-\frac{x^2}{2}} dx,
\end{aligned}$$

i.e.

$$\int_y^\infty e^{-\frac{x^2}{2}} dx = (\frac{1}{y} - \frac{1}{y^3}) e^{-\frac{y^2}{2}} + \int_y^\infty \frac{3}{x^4} e^{-\frac{x^2}{2}} dx \ge (\frac{1}{y} - \frac{1}{y^3}) e^{-\frac{y^2}{2}}.$$