Step-1

Similar matrices: Matrices A and B are similar if $A = M^{-1}BM$ for some invertible matrix M.

State the reason for following true statements:

(a) If matrix A is similar to B, then matrix A^2 is similar to B^2.

If matrix A is similar to matrix B then following must be true:

$$\begin{aligned} A &= M^{-1}BM \\ A \cdot A &= \left(M^{-1}BM\right)\cdot\left(M^{-1}BM\right) \\ &= \left(M^{-1}B\right)I\left(BM\right) \\ &= M^{-1}B^2M \end{aligned}$$

Therefore, $\boxed{A^2 = M^{-1}B^2M}$ shows that the statement, matrix A^2 is similar to B^2 when matrix A is similar to B, is true.

Step-2

(b) Matrix A^2 can be similar to B^2 even when matrix A is not similar to B.

For this consider two matrices A and B.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix} \\ &= -A \end{aligned}$$

Step-3

If these matrices are similar then they must belong to same family. For this do the following calculations:

$$\begin{aligned} M^{-1}AM &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= J_1 \end{aligned}$$

$$
\begin{aligned}
M^{-1}BM &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= J_2 \\
&\neq J_1
\end{aligned}
$$

Step-4

Above calculations shows that matrix A is not similar to matrix B.

Step-5

Now, check for matrices A^2 and B^2.

$$
\begin{aligned}
M^{-1}A^2M &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= J_1
\end{aligned}
$$

$$
\begin{aligned}
M^{-1}B^2M &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= J_1
\end{aligned}
$$

As matrices are equal so they will belong to same family.

Step-6

Therefore, this could be possible that matrix A^2 can be similar to B^2 even when matrix A is not similar to B.

Step-7

(c) Matrix A is similar to matrix B defined below:

$$
A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}
$$

It can be seen that matrix B is upper triangular matrix with different Eigen values $\lambda = (3,4)$. This implies that it can be diagonalize into a matrix Λ. Matrix S will contain Eigen vectors of matrix B.

$$\begin{aligned} S^{-1}BS &= \Lambda \\ &= A \end{aligned}$$

Step-8

Therefore, $\boxed{S^{-1}BS = A}$ implies that matrix A is similar to matrix B.

(d) Matrix A is not similar to matrix B defined below:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

It can be seen that matrix B is upper triangular matrix with repeated Eigen values $\lambda = (3,3)$. This implies that it can not be diagonalize into a matrix Λ. However matrix A is equal to Eigen value matrix of B.

$$\begin{aligned} S^{-1}BS &\neq \Lambda \\ &\neq A \end{aligned}$$

Step-9

Therefore, $\boxed{S^{-1}BS \neq A}$ implies that matrix A is not similar to matrix B.

Step-10

(e) If we exchange rows 1 and 2 of matrix A, and then exchange columns 1 and 2, the Eigen values stay the same.

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Now exchange rows 1 and 2.

$$A_1 = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Exchange columns 1 and 2.

$$A_2 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Step-11

It can be seen that the position of the Eigen values in matrix A_2 are interchanged however, Eigen values remain the same as in matrix A. Therefore, exchanging rows 1 and 2 and then exchanging columns 1 and 2 makes no change in the Eigen values.