

Step-1

Given that $S = \{0 = (0, 0, 0, 0)\}$

We know that this is the trivial subspace of \mathbf{R}^4

Suppose $S^\perp = \{v = (x, y, z, w) : v \in \mathbf{R}^4\}$

Then by definition of orthogonal complement, we get $v^T 0 = 0$

$$\text{i.e.,} \quad \begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

We know that this condition is satisfied by every $v = (x, y, z, w) : v \in \mathbf{R}^4$

Therefore, $\mathbf{R}^4 \subseteq S^\perp$

Since \mathbf{R}^4 is the linear space, S is the subspace, we follow that $S^\perp \subseteq \mathbf{R}^4$

Putting these observations together, we get $S^\perp = \mathbf{R}^4$

Step-2

Suppose $w = (0, 0, 0, 1)$ spans the subspace S .

Then $S = \{x = (0, 0, 0, k)\}$ where k is any real number.

Suppose $S^\perp = \{v = (x, y, z, w) : v \in \mathbf{R}^4\}$

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix} = 0$$

Then by definition, we get

In other words, $x \cdot 0 + y \cdot 0 + z \cdot 0 + w \cdot k = 0$

We easily see that the first three summands obviously become zero with any real numbers x, y, z .

But k is any real number such that $w \cdot k = 0$ is possible only when $w = 0$

Therefore, we can write $S^\perp = \{v = (x, y, z, 0) : v \in \mathbf{R}^4\}$

Observe that S is of dimension 1 and S^\perp is of 3 such that their sum is the dimension of \mathbf{R}^4 .

Step-3

Assuming $S^\perp = U$, we follow that U is of dimension 3 and so, U^\perp is of dimension 1 and thus, $U^\perp \subseteq (S^\perp)^\perp$

In other words, $S \subseteq (S^\perp)^\perp$ (1)

On the other hand, S spans all the vectors of the form $(0, 0, 0, k)$ and $(S^\perp)^\perp$ contains vectors of the form $(0, 0, 0, k)$

Therefore, $(S^\perp)^\perp \subseteq S$ (2)

Putting (1) and (2) together, we get $(S^\perp)^\perp = S$