Step-1

If A is an m by n matrix, using row operations, we can reduce A to r non zero rows and

m â€“ r zero rows. Then

1. $C(A) =$ Column space of A; dimension r

2. $N(A)$ = null space of A; dimension n â€“ r

3. $C(A^T)$ = row space of A; dimension r

4. $N(A^T)$ = left null space of A; dimension m â€“ r

Step-2

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \overset{R_2-3R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, column 1 is independent

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ â€¦â€¦ (1)

To find null space $Ax = 0 \Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The basis for $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ â€¦â€¦ (2)

Step-3

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \overset{R_2-2R_1}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, column 1 is independent

Basis for $C\left(A^T\right)=\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ â€¦â€¦ (3)

To find null space $A^T,\ A^TY=0$

$$\Rightarrow \begin{bmatrix}1&3\\0&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow\ y_1+3y_2=0$$

$$\Rightarrow\ y_1=-3y_2$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}-3y_2\\y_2\end{bmatrix}=y_2\begin{bmatrix}-3\\1\end{bmatrix}$$

Basis for null space of $A^T=\left\{\begin{bmatrix}-3\\1\end{bmatrix}\right\}$ â€¦â€¦ (4)

Step-4

(b) Basis for $C(B)=\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$

$$x_1+2x_2=0$$

$$\Rightarrow\ x_1=-2x_2$$

$$N(B)=\left\{\begin{bmatrix}-2x_2\\x_2\end{bmatrix}/\,x_2\in\mathrm{R}\right\}$$

Basis for $N(B)=\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}$

$B^T=\begin{bmatrix}0&1\\0&2\end{bmatrix}$, column 2 is independent

Basis $C\left(B^T\right)=\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ row space of B

Step-5

To find $N\left(B^T\right)$, we use $B^T y = 0$

$\Rightarrow\ y_2 = 0, 2y_2 = 0$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A basis for $N\left(B^T\right) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Step-6

(c) $C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$1,2$ Columns are independent

Hence a basis for $C(C) = \{(1,0),(1,1)\}$

To find null space $Cx = 0$

$\Rightarrow x_1 + x_2 = 0$

$x_2 + x_4 = 0$

$\Rightarrow x_1 = -x_2$

Step-7

$x_4 = -x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis for null space of $N(C)=\left\{\begin{bmatrix}-1\\1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$

$$C^T=\begin{bmatrix}1&0\\1&1\\0&0\\0&1\end{bmatrix}\overset{R_2-R_1}{\sim}\begin{bmatrix}1&0\\0&1\\0&0\\0&1\end{bmatrix}$$

Columns 1, 3 are independent

A basis of $C(C^T)=\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$

$\dim N(C(T))=0$

$N(C(T))=\{(0,0)\}$, basis is empty for $N(C(T))$