Step-1

Given that the vectors $a_1 = (1,1,0), a_2 = (1,1,1)$ span a plane in $\mathbf{R}^3$. We have to find the projection matrix P onto the plane, and we have to find a nonzero vector b that is projected to zero.

Step-2

Write

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Required projection matrix $P = A\left(A^T A\right)^{-1} A^T$

Step-3

Now

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\left(A^T A\right)^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

Step-4

Then

$$A\left(A^T A\right)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -2 & 2 \end{bmatrix}$$

Step-5

Therefore

$$P = A\left(A^T A\right)^{-1} A^T$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence $$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step-6

Verification:

$$P^2 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = P$$

$$\Rightarrow P^2 = P$$

Step-7

$$P^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= P$$

$$\Rightarrow P^2 = P$$

Hence P is a projection matrix

Step-8

Let $$b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Consider $Pb = 0$

$$\frac{1}{2}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Step-9

$x+y=0$

$2z=0$

$\Rightarrow z=0,$

$x=-y$

Put $y=k$

$\Rightarrow x=-k$

$$b=\begin{bmatrix}-k\\ k\\ 0\end{bmatrix}$$

$$=k\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$$

Required non zero vector $b=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$