

Step-1

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Suppose

The corresponding $Ax = 0$ is given as,

$$\begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply the row operation $R_2 \rightarrow R_2 - iR_1$ to get,

$$\begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve the above system as,

$$x + iy = 0$$

$$y + z = 0$$

This can be rewritten as,

$$x = -iy, z = -y$$

Therefore, the null space of A is given as,

$$N(A) = (-iy, y, -y \mid y \in \mathbb{C})$$

Step-2

(b)

The objective is to verify that the null space calculated in last part is orthogonal to $C(A^H)$ and is not orthogonal to $C(A^T)$.

First determine A^H

Since, A^H is computed by taking the conjugate transpose of the matrix A , therefore A^H is,

$$\begin{aligned}
A^H &= (\overline{A})^T \\
&= \begin{bmatrix} 1 & -i & 0 \\ -i & 0 & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Assume an arbitrary vector $X \in N(A)$.

Now, the aim is to verify that $N(A)$ is orthogonal to columns of A^H that is find $X^H \cdot c_i = 0$ where $c_i, i=1,2$ denote the columns of A^H

First compute $X^H \cdot c_1$

$$\begin{aligned}
X^H \cdot c_1 &= \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \cdot \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \quad (\text{Apply conjugate transpose}) \\
&= iy - iy + 0 \\
&= 0
\end{aligned}$$

Step-3

Now, compute $X^H \cdot c_2$

$$\begin{aligned}
X^H c_2 &= \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \quad (\text{Apply conjugate transpose}) \\
&= -i^2 y + 0 - y \quad (i^2 = -1) \\
&= y - y \\
&= 0
\end{aligned}$$

As, $X^H \cdot c_1 = 0$, $X^H \cdot c_2 = 0$, $N(A)$ is orthogonal to $C(A^H)$

Hence, $N(A)$ is orthogonal to $C(A^H)$

Step-4

Now, check whether null space is orthogonal to $C(A^T)$.

Determine A^T

Since, A^T is computed by taking the transpose of the matrix A , therefore A^T is,

$$A^T = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}^T$$

$$A^T = \begin{bmatrix} 1 & i \\ i & 0 \\ 0 & 1 \end{bmatrix}$$

First compute $X^H \cdot d_i$ where $d_i, i=1,2$ denote the columns of A^T .

$$X^H \cdot d_1 = \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \cdot \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad (\text{Apply conjugate transpose})$$

$$= iy + iy + 0$$

$$= 2iy \neq 0 \quad (\text{for } y \neq 0)$$

Now, compute $X^H \cdot d_2$

$$X^H d_2 = \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \quad (\text{Apply conjugate transpose})$$

$$= i^2 y + 0 - y \quad (i^2 = -1)$$

$$= -y - y$$

$$= -2y \neq 0 \quad (\text{for } y \neq 0)$$

As, the arbitrary vector $X \in N(A)$ is not orthogonal for any of the column vectors of A^T .

Hence, $N(A)$ is not orthogonal to $C(A^T)$