Step-1

(a)

The objective is to compute AA^T and its Eigen values $\sigma_1^2, 0$ and its unit Eigen vector u_1, u_2

Step-2

Consider the matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

Thus,

$$A^T = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

Therefore, AA^T is given by,

$$\begin{aligned} AA^T &= \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1+16 & 2+32 \\ 2+32 & 4+64 \end{bmatrix} \\ &= \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix} \end{aligned}$$

The Eigen values are given by,

$$\begin{vmatrix} 17-\lambda & 34 \\ 34 & 68-\lambda \end{vmatrix} = 0$$

$$(17-\lambda)(68-\lambda) - 34\times 34 = 0$$

This implies;

$$\begin{aligned} 17\times 68 - 17\lambda - 68\lambda + \lambda^2 &= 34\times 34 \\ \lambda^2 - 85\lambda + 17\times 17(4-4) &= 0 \\ \lambda(\lambda - 85) &= 0 \\ \lambda &= 85, 0 \end{aligned}$$

Step-3

Therefore, the Eigen values are;

$\sigma_1^{\,2} = 85$ And 0

The eigenvectors are given by,

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The unit eigenvectors are given by,

$$\begin{aligned} u_1 &= \left(\frac{1}{\sqrt{1^2+2^2}}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{5}}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \\ u_1 &= \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \end{aligned}$$

And,

$$\begin{aligned} u_2 &= \left(\frac{1}{\sqrt{(-2)^2+1^2}}\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{5}}\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \\ u_2 &= \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \end{aligned}$$

Step-4

(b)

The objective is to show the signs so that $Av_1 = \sigma_1 u_1$ and verify the SVD.

Step-5

Consider the equation:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$$

Step-6

Now take the unit Eigen vector v_1, v_2 of the matrix $A^T A$,

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ -\frac{4}{\sqrt{17}} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}$$

Therefore,

$$\begin{aligned}
\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \end{bmatrix}^T \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 4\sqrt{5} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}
\end{aligned}$$

$\boxed{\text{Hence verified}}$

Thus, $\boxed{AA^T = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}}$, $\boxed{v_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}}$, and $\boxed{u_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, u_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}}$

Step-7

(c)

The objective is to determine the four vectors which give orthonormal bases for $C(A), N(A), C(A^T), N(A^T)$.

Step-8

Vector give Orthonormal bases for $C(A)$ is $\boxed{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}}$

Vector give Orthonormal bases for $C(A^T)$ is $\boxed{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \end{bmatrix}}$

Vector give Orthonormal bases for $N(A)$ is $\boxed{\begin{bmatrix} \frac{-4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}}$

Vector give Orthonormal bases for $N(A^T)$ is $\boxed{\begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}}$