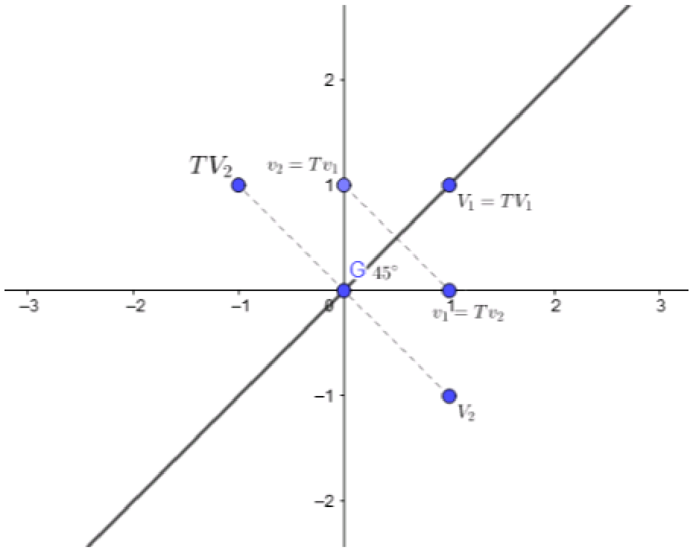
Step-1

Consider a transformation T that is a reflection across the 45° line in the plane. Objective is to determine corresponding matrix with respect to the basis $v_1 = (1,0), v_2 = (0,1)$ and $V_1 = (1,1), V_2 = (1,-1)$.

For the sake of understanding of transformation T consider the following figure. At the reflection of 45°, T maps the standard basis vectors as:

$$\begin{aligned} Tv_1 &= T(1,0) \\ &= (0,1) \\ Tv_2 &= T(0,1) \\ &= (1,0) \end{aligned}$$



Step-2

Also (see figure) the vectors $V_1 = (1,1), V_2 = (1,-1)$ will get map as:

$$\begin{aligned} TV_1 &= T(1,1) \\ &= (1,1) \\ TV_2 &= T(1,-1) \\ &= (-1,1) \end{aligned}$$

Then their linear combinations will be:

$$Tv_1 = 0v_1 + 1v_2$$
$$Tv_2 = 1v_1 + 0v_2$$
$$TV_1 = 1V_1 + 0V_2$$
$$TV_2 = 0V_1 - 1V_2$$

Let A denote the matrix corresponding to T with respect to standard basis and B denote the matrix corresponding to T with respect to $V_1 = (1, 1), V_2 = (1, -1)$. Then the matrix will be:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Step-3

Next find the change of basis from vector $V_1 = (1, 1), V_2 = (1, -1)$ to standard vectors $v_1 = (1, 0),\, v_2 = (0, 1)$. Let the corresponding matrix is P. Then

$$\begin{aligned} V_1 &= (1, 1) \\ &= (1, 0) + (0, 1) \\ &= v_1 + v_2 \end{aligned}$$

$$\begin{aligned} V_2 &= (1, -1) \\ &= (1, 0) - (0, 1) \\ &= v_1 - v_2 \end{aligned}$$

An thus, $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Step-4

Two matrices A and B is said to be similar if there exists an invertible matrix P such that $P^{-1}AP = B$.

Inverse of matrix P is given as:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Consider the left side and solve as:

$$P^{-1}AP = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= -\frac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= B$$

Step-5

Hence, both the matrices A and B are similar.