Step-1

Given that $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.8 \end{bmatrix}$ and $A^{\infty} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

We have to find the eigenvalues and eigenvectors for both the given Markov matrices.

Step-2

Now we find the eigenvalues of A.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0.6-\lambda & 0.2 \\ 0.4 & 0.8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (0.6-\lambda)(0.8-\lambda) - 0.08 = 0$$

$$\Rightarrow \lambda^{-2} 1.4\lambda + 0.4 = 0$$
$$\Rightarrow \lambda^2 - \lambda - 0.4\lambda + 0.4 = 0$$
$$\Rightarrow \lambda(\lambda - 1) - 0.4(\lambda - 1) = 0$$
$$\Rightarrow \lambda = 0.4, 1$$

Therefore, eigenvalues are $\boxed{\lambda = 0.4 \text{ and } 1}$.

Step-3

We know that x is the eigenvector of A if and only if x is the nonzero solution of $(A - \lambda I)x = 0$

That is $\begin{bmatrix} 0.6-\lambda & 0.2 \\ 0.4 & 0.8-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ â€¦â€¦ (1)

For $\lambda = 0.4$, (1) becomes

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Step-4

The augmented matrix is

$$\begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.4 & 0.4 & 0 \end{bmatrix}$$

Add (-2) times of row 1 to row 2, we get

$$\begin{bmatrix} 0.2 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get $0.2x_1 + 0.2x_2 = 0$

Here x_2 is free variable.

Step-5

Let $x_2 = k$, where *k* is a parameter

Then $x_1 = -k$

Therefore,

$$\begin{aligned} x &= \begin{bmatrix} -k \\ k \end{bmatrix} \\ &= k\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, the eigenvector corresponding to $\lambda = 0.4$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step-6

For $\lambda = 1$, (1) becomes

$$\begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Step-7

The augmented matrix is

$$\begin{bmatrix} -0.4 & 0.2 & 0 \\ 0.4 & -0.2 & 0 \end{bmatrix}$$

Add row 1 to row 2, we get

$$\begin{bmatrix} -0.4 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get $-0.4x_1 + 0.2x_2 = 0$

Here x_1 is free variable.

Step-8

Let $x_1 = k$, where k is a free variable.

Then $-0.4x_1 = -0.2x_2$

$\Rightarrow x_2 = 2k$

Therefore,

$$x = \begin{bmatrix} k \\ 2k \end{bmatrix}$$
$$= k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Step-9

Let $S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ and

Then

$A = S\Lambda S^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

Step-10

Now

$A^{\infty} = S\Lambda^{\infty}S^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (.4) & 0 \\ 0 & 1 \end{bmatrix}^{\infty} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (.4)^{\infty} & 0 \\ 0 & 1^{\infty} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

Step-11

Since $|\lambda| < 1$

So as $n \to \infty$, $\lambda^n \to 0$

Hence the eigenvalues of A^{∞} are 0 and 1

And the eigenvectors of A^{∞} are $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$.

Step-12

We have to explain why A^{100} is close to A^{∞}.

Now

$A^{100} = S\Lambda^{100}S^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}^{100} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (0.4)^{100} & 0 \\ 0 & (1)^{100} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

Since $|\lambda| < 1$

So $n \to 100$, $\lambda^{100} \to 0$

Therefore A^{100} close to A^{∞}.