Step-1

Consider the following matrices:

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$$

$$B=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

Let following are the classes of matrices:

Orthogonal, invertible, projection, permutation, Hermitian, rank-1, diagonalizable and Markov matrices. Determine A and B belongs to which classes of matrices. Also find the Eigen values of A and B.

Step-2

Orthogonal: If $PP^T=I$, then P is said to be orthogonal matrix.

$$\begin{aligned}AA^T&=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\cdot\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\\&=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\\&=I\end{aligned}$$

$$\begin{aligned}BB^T&=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\\&=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\\&\neq I\end{aligned}$$

Therefore, only matrix A $\boxed{\text{is orthogonal}}$.

Invertible: Orthogonal matrices are always invertible. As $A^{-1} = A^T$. Therefore matrix *A* is invertible, but matrix *B* is not invertible.

Step-3

Projection: If $P^2 = P$, then *P* is said to be projection matrix.

$$\begin{aligned} A \cdot A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &\neq A \end{aligned}$$

$$\begin{aligned} B \cdot B &= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= B \end{aligned}$$

Therefore, only matrix *B* $\boxed{\text{is a projection}}$ matrix.

Step-4

Permutation matrix: Every row and column contains a single 1 with 0â€™s everywhere else.

Therefore, only matrix *A* $\boxed{\text{is a permutation}}$ matrix.

Step-5

Rank-1 matrix: One row is non-zero after reduction.

Matrix *A* cannot be reduced in one non-zero row. Matrix *A* has rank 4. But matrix *B* can be reduced into one non-zero row.

Therefore, $\boxed{\text{matrix B is rank-1 matrix}}$.

Step-6

Hermitian: If $P^H = P$, then P is said to be Hermitian matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A^H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\neq A$$

$$B = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$B^H = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= B$$

Therefore, matrix A $\boxed{\text{is not Hermitian}}$ while matrix B $\boxed{\text{is Hermitian}}$.

Step-7

Diagonalizable: Sufficient Eigen vectors make the matrix diagonalizable.

Matrix B has only two Eigen values and corresponding to that it has two Eigen vectors, which are not sufficient to diagonalize the matrix.

Therefore, matrix B $\boxed{\text{is not diagonalizable}}$.

Whereas matrix A has sufficient Eigen vectors corresponding to all Eigen values. Therefore, matrix A $\boxed{\text{is diagonalizable.}}$

Step-8

Markov matrix: In Markov matrix every element of the matrix is positive and sum of the column elements equals to 1.

Every element of both matrices is positive and individual sum of the columns equal to 1. Therefore, both matrices A $\boxed{\text{and}}$ B $\boxed{\text{are Markov}}$ matrices.

Step-9

Eigen values: To calculate the Eigen values do the following calculations:

$$A-\lambda I=\begin{bmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 1 & 0 \\ 0 & 0 & 0-\lambda & 1 \\ 1 & 0 & 0 & 0-\lambda \end{bmatrix}$$

$$\det\left(A-\lambda I\right)=0$$

$$\left(\lambda\right)^4-1=0$$

After solving following Eigen values are obtained:

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \\ \lambda_3 &= i \\ \lambda_4 &= -i \end{aligned}}$$

Step-10

Similarly, Eigen values of matrix *B* will be:

$$B-\lambda I=\frac{1}{4}\begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{bmatrix}$$

$$\det\left(B-\lambda I\right)=0$$

$$\lambda^4-\lambda^3=0$$

After solving following Eigen values are obtained:

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 0 \end{aligned}}$$