Step-1

We need to explain why $\left|\frac{\lambda_n}{\lambda_{n-1}}\right|$ controls the convergence of the usual power method. Construct a matrix A for which this method does not converge.

The power method with the initial guess u_0 can be seen as $u_{k+1} = Au_k$

Also, if λ_k is the Eigen value and u_k is the respective Eigen vector, then $Au_k = \lambda_k u_k$

Using $u_1 = Au_0, u_2 = Au_1 = A^2u_0, \ldots, u_{k+1} = A^k u_0$,

Step-2

If $x_1, x_2, \ldots, x_n$ are the Eigen vectors corresponding to the Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $u_k = c_1\lambda_1^{\ k} x_1 + c_2\lambda_2^{\ k} x_2 + \ldots + c_n\lambda_n^{\ k} x_n$ such that $|\lambda_1| \le |\lambda_2| \le \ldots \le |\lambda_n|$.

Dividing throughout by $\lambda_n^{\ k}$,

So,

$$\begin{aligned}\frac{u_k}{\lambda_n^{\ k}} &= c_1\frac{\lambda_1^{\ k}}{\lambda_n^{\ k}}x_1 + c_2\frac{\lambda_2^{\ k}}{\lambda_n^{\ k}}x_2 + \ldots + c_{n-1}\frac{\lambda_{n-1}^{\ k}}{\lambda_n^{\ k}}x_{n-1} + c_n x_n \\ &= c_1\left(\frac{\lambda_1}{\lambda_n}\right)^k x_1 + c_2\left(\frac{\lambda_2}{\lambda_n}\right)^k x_2 + \ldots + c_{n-1}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^k x_{n-1} + c_n x_n\end{aligned}$$

The vectors u_k point more and more accurately towards the direction of x_n.

Step-3

Their convergence factor is the ratio,

If $r = \frac{|\lambda_{n-1}|}{|\lambda_n|}$ is nearly equal to 1, then the convergence of Eigen value will be very slow.

If $\frac{|\lambda_i|}{|\lambda_1|} < 1$ for every i, then $\frac{u_k}{\lambda_n^{\ k}} = c_1x_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + c_3\left(\frac{\lambda_3}{\lambda_1}\right)^k x_3 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k x_n$ converges to c_1x_1.

The largest ratio controls the convergence when k is large. it is nothing but $\frac{\lambda_n}{\lambda_{n-1}}$.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the suitable matrix such that $|\lambda_1| = |\lambda_2|$ and has no convergence.

Therefore,

$$\boxed{\begin{array}{l} \dfrac{u_k}{{\lambda_n}^k} = c_1 x_1 + c_2 \left(\dfrac{\lambda_2}{\lambda_1}\right)^k x_2 + c_3 \left(\dfrac{\lambda_3}{\lambda_1}\right)^k x_3 + \ldots + c_n \left(\dfrac{\lambda_n}{\lambda_1}\right)^k x_n \to c_1 x_1 \text{ if } \dfrac{|\lambda_i|}{|\lambda_1|} < 1 \text{ for every } i. \\ \text{The largest ratio controls the convergence when } k \text{ is large.} \\ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ has } |\lambda_1| = |\lambda_2| \text{ and no convergence.} \end{array}}$$