Step-1

A is the 4×4 matrix of ones

$$A=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

$$A-I=B=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$$

Step-2

The characteristic equation of $A-I=B$ is $|B-\lambda\mathrm{I}|=0$

i.e., $$\begin{vmatrix}-\lambda&1&1&1\\1&-\lambda&1&1\\1&1&-\lambda&1\\1&1&1&-\lambda\end{vmatrix}=0$$

we apply the elementary operations to reduce this matrix to the echelon form and find the determinant.

Step-3

$$R_1\to R_1+R_2+R_3+R_4\Rightarrow\begin{vmatrix}3-\lambda&3-\lambda&3-\lambda&3-\lambda\\1&-\lambda&1&1\\1&1&-\lambda&1\\1&1&1&-\lambda\end{vmatrix}=0$$

$$\Rightarrow(3-\lambda)\begin{vmatrix}1&1&1&1\\1&-\lambda&1&1\\1&1&-\lambda&1\\1&1&1&-\lambda\end{vmatrix}=0$$

Step-4

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1 \Rightarrow \quad (3-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda-1 & 1 & 1 \\ 0 & 0 & -\lambda-1 & 0 \\ 0 & 0 & 0 & -\lambda-1 \end{vmatrix} = 0$$

While this is the determinant of the upper triangular matrix, it is equal to the product of the diagonal entries.

$$= (3-\lambda)(-1-\lambda)^3 = 0$$

Therefore, the eigen values are -1, -1, -1, and 3.

Step-5

Also, the determinant of $A - I$ is $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3 + R_4$, we get $\begin{vmatrix} 3 & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$

Step-6

$$= 3\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1 \Rightarrow \quad 3\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= 3(-1)^3$$
$$= \boxed{-3}$$