Step-1

Suppose $A+iB$ is a unitary matrix, where *A, B* are real.

We have to show that $Q=\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is orthogonal.

Step-2

Since $A+iB$ is unitary

So $(A+iB)^H(A+iB)=I$

Now $(A+iB)^H(A+iB)=I$

$$\Rightarrow (A+iB)^H=(A+iB)^{-1}$$
$$\Rightarrow (A^H-iB^H)=\frac{1}{A+iB}$$
$$\Rightarrow A^T-iB^T=\frac{1}{A^2+B^2}[A-iB] \qquad \begin{pmatrix} \text{Since } A \text{ and } B \text{ are real} \\ \text{So } A^H=A^T \text{ and } B^H=B^T \end{pmatrix}$$

Step-3

Comparing the real and imaginary parts on both sides, we get

$$A^T=\frac{A}{A^2+B^2}, B^T=\frac{B}{A^2+B^2}$$

Now we have $Q=\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

Therefore,

$$\begin{aligned} Q^TQ &= \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix}\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \\ &= \begin{bmatrix} A^TA+B^TB & -A^TB+B^TA \\ -B^TA+BA^T & BB^T+A^TA \end{bmatrix} \end{aligned}$$

Step-4

Continuation to the above

$$=\begin{bmatrix} \frac{A}{A^2+B^2}+\frac{B}{A^2+B^2} & \frac{-AB}{A^2+B^2}+\frac{AB}{A^2+B^2} \\ \frac{-AB}{A^2+B^2}+\frac{AB}{A^2+B^2} & \frac{B^2}{A^2+B^2}+\frac{A^2}{A^2+B^2} \end{bmatrix} \qquad \text{(Since by (1))}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=I$$

Since $Q^T Q = I$

Therefore Q is an orthogonal matrix.