SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 14 Solutions

1. Suppose the m.g.f of X is $M^X(t) = \frac{2}{\sqrt{4-t}}, \ (t < 4)$.

(i) Find E(X), E(X^2) and Var(X).

(ii) If X and Y are independent and both with this m.g.f(i.e. $M_X(t) = M_Y(t) = \frac{2}{\sqrt{4-t}}$.). Then find the m.g.f of of $X+Y$, and also identify the distribution of $X+Y$.

Solution:

(a) Since $M_X(t) = \frac{2}{\sqrt{4-t}}$, then we have

$$M'_X(t) = (-\frac{1}{2}) \cdot 2(4-t)^{-\frac{1}{2}-1} \cdot (-1) = (4-t)^{-\frac{3}{2}};$$
$$M''_X(t) = \frac{\mathrm{d}}{\mathrm{d}t} M'_X(t) = (-\frac{3}{2}) \cdot (4-t)^{-\frac{3}{2}-1} \cdot (-1) = \frac{3}{2}(4-t)^{-\frac{5}{2}}.$$

So, we can obtain

$$\mathrm{E(X) = M'_X(0) = M'_X(t)|_{t=0}} = (4-0)^{-\frac{3}{2}} = \frac{1}{8};$$
$$\mathrm{E(X^2) = M''_X(0) = M''_X(t)|_{t=0}} = \frac{3}{2}(4-0)^{-\frac{5}{2}} = \frac{3}{64}.$$

Hence, $\mathrm{Var}(X) = \mathrm{E(X^2)} - (\mathrm{E(X)})^2 = \frac{3}{64} - (\frac{1}{8})^2 = \frac{2}{64} = \frac{1}{32}$.

(b) By using the properties of m.g.f, then we have for $t < 4$,

$$M_Z(t) = M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \frac{2}{\sqrt{4-t}} \cdot \frac{2}{\sqrt{4-t}} = \frac{4}{4-t}$$

So, $X + Y \sim \mathrm{Exp}(4)$.

2. (a) If the m.g.f of X is $M_X(t) = \frac{a^2}{a^2-t^2}$, then find the k'th moment $\mathrm{E(X^k)}, k \in \mathbb{N}_+$.

(b) Suppose the m.g.f of X can be expressed as a power series:

$$M_X(t) = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots ,$$

and assume further that $a_0 = 1$, $a_1 = 3$ and $a_2 = 7$. Find E(X) and Var(X).

Solution:

(a) Observed that

$$M_X(t) = \frac{a^2}{a^2 - t^2} = \frac{a^2}{(a+t)(a-t)} = \frac{a}{2}(\frac{1}{a+t} + \frac{1}{a-t}),$$

Then, we can obtain*

$$M_X^{(k)}(t) = \frac{a}{2}\left(\frac{(-1)^k k!}{(a+t)^{k+1}} + \frac{(-1)^k \cdot (-1)^k k!}{(a-t)^{k+1}}\right) = \frac{a}{2}\left(\frac{(-1)^k k!}{(a+t)^{k+1}} + \frac{k!}{(a-t)^{k+1}}\right).$$

*Indeed, you can use mathematical induction to prove, here we omitted.

Hence, for $k \in \mathbb{N}_+$,[†]

$$\mathrm{E}[\mathrm{X}^\mathrm{k}] = \mathrm{M}_\mathrm{X}^{(\mathrm{k})}(0) = \mathrm{M}_\mathrm{X}^{(\mathrm{k})}(\mathrm{t})|_{\mathrm{t}=0} = \frac{\mathrm{a}}{2}\left(\frac{(-1)^\mathrm{k}\mathrm{k}!}{(\mathrm{a}+0)^{\mathrm{k}+1}} + \frac{\mathrm{k}!}{(\mathrm{a}-0)^{\mathrm{k}+1}}\right) = \frac{\mathrm{a}}{2}\left(\frac{(-1)^\mathrm{k}\mathrm{k}!}{\mathrm{a}^{\mathrm{k}+1}} + \frac{\mathrm{k}!}{\mathrm{a}^{\mathrm{k}+1}}\right) = \frac{\mathrm{k}!}{2\mathrm{a}^\mathrm{k}}((-1)^\mathrm{k}+1).$$

(b) Since $M_X(t)$ can be expressed as a power series:

$$M_X(t) = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots.$$

Then, we have $a_k = \frac{M_X^{(k)}(0)}{k!}$, also we know $\mathrm{E}(\mathrm{X}^\mathrm{k}) = \mathrm{M}_\mathrm{X}^{(\mathrm{k})}(0)$, yields

$$\begin{aligned}\mathrm{E}(\mathrm{X}) &= \mathrm{M}'_\mathrm{X}(0) = \mathrm{a}_1 = 3;\\ \mathrm{E}(\mathrm{X}^2) &= \mathrm{M}''_\mathrm{X}(0) = 2\mathrm{a}_2 = 14.\end{aligned}$$

Hence $\mathrm{Var}(X) = \mathrm{E}(\mathrm{X}^2) - (\mathrm{E}(\mathrm{X}))^2 = 14 - 3^2 = 14 - 9 = 5$.

3. Suppose the m.g.f of X has the Maclaurin series

$$M_X(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots.$$

Find the variance and the third central moment of X in terms of a_1, a_2, and a_3.

Solution: Since $M_X(t)$ can be expressed as a Maclaurin series:

$$M_X(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots.$$

Then, we have $a_k = \frac{M_X^{(k)}(0)}{k!}$, also we know $\mathrm{E}(\mathrm{X}^\mathrm{k}) = \mathrm{M}_\mathrm{X}^{(\mathrm{k})}(0)$, yields

$$\begin{aligned}\mathrm{E}(\mathrm{X}) &= \mathrm{M}'_\mathrm{X}(0) = \mathrm{a}_1;\\ \mathrm{E}(\mathrm{X}^2) &= \mathrm{M}''_\mathrm{X}(0) = 2\mathrm{a}_2;\\ \mathrm{E}(\mathrm{X}^3) &= \mathrm{M}'''_\mathrm{X}(0) = 6\mathrm{a}_3.\end{aligned}$$

Hence $\mathrm{Var}(X) = \mathrm{E}(\mathrm{X}^2) - (\mathrm{E}(\mathrm{X}))^2 = 6\mathrm{a}_2 - \mathrm{a}_1^2$.

$$\begin{aligned}\mathrm{E}[(\mathrm{X}-\mathrm{E}(\mathrm{X}))^3] &= \mathrm{E}[\mathrm{X}^3 - 3\mathrm{X}^2\mathrm{E}(\mathrm{X}) + 3\mathrm{X}(\mathrm{E}(\mathrm{X}))^2 - (\mathrm{E}(\mathrm{X}))^3]\\ &= \mathrm{E}(\mathrm{X}^3) - 3\mathrm{E}[\mathrm{X}^2\mathrm{E}(\mathrm{X})] + 3\mathrm{E}[\mathrm{X}(\mathrm{E}(\mathrm{X}))^2] - \mathrm{E}[\mathrm{E}(\mathrm{X})^3]\\ &= \mathrm{E}(\mathrm{X}^3) - 3\mathrm{E}(\mathrm{X}^2)\mathrm{E}(\mathrm{X}) + 3\mathrm{E}(\mathrm{X})(\mathrm{E}(\mathrm{X}))^2 - (\mathrm{E}(\mathrm{X}))^3\\ &= \mathrm{E}(\mathrm{X}^3) - 3\mathrm{E}(\mathrm{X}^2)\mathrm{E}(\mathrm{X}) + 2\mathrm{E}(\mathrm{X})(\mathrm{E}(\mathrm{X}))^2\\ &= 6a_3 - 3\times(2a_2)\times a_1 + 2(a_1)^3\\ &= 6a_3 - 6a_1a_2 + 2a_1^3.\end{aligned}$$

4. Let $X_1, X_2, \cdots, X_n$ be i.i.d., each having the normal distribution with parameters μ and σ^2.

(i) Find the m.g.fs of the sample sum($S_n = \sum_{i=1}^{n} X_i$) and sample average($\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$).

[†] If $k(= 2m+1, m \in \mathbb{N})$ is odd, then $\mathrm{E}(\mathrm{X}^\mathrm{k}) = \frac{\mathrm{k}!}{2\mathrm{a}^\mathrm{k}}((-1)^\mathrm{k}+1) = \frac{\mathrm{k}!}{2\mathrm{a}^\mathrm{k}}((-1)+1) = 0$;
while if $k(= 2m, m \in \mathbb{N}_+)$ is even, then $\mathrm{E}(\mathrm{X}^\mathrm{k}) = \frac{\mathrm{k}!}{2\mathrm{a}^\mathrm{k}}((-1)^\mathrm{k}+1) = \frac{\mathrm{k}!}{2\mathrm{a}^\mathrm{k}}(1+1) = \frac{\mathrm{k}!}{\mathrm{a}^\mathrm{k}}(= \frac{(2\mathrm{m})!}{\mathrm{a}^{2\mathrm{m}}})$.

(ii) What are the distributions of these two random variables?

Solution:

(i) Since $X_1, X_2, \cdots, X_n$ be i.i.d., each having the normal distribution with parameters μ and σ^2. Then for $i = 1, 2, \cdots, n$, we have

$$M_{X_i}(t) = \mathrm{e}^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Hence,

$$\begin{aligned}
\mathrm{M}_{S_n}(t) &= \mathrm{E}[\mathrm{e}^{\mathrm{tS_n}}] = \mathrm{E}[\mathrm{e}^{\mathrm{t}\sum\limits_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{X_i}}] = \mathrm{E}[\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{e}^{\mathrm{tX_i}}] \\
&\overset{\underline{Since\ \{X_i\}_{i=1}^n\ are\ independent}}{=\!=\!=\!=\!=} \prod_{i=1}^{n} \mathrm{E}[\mathrm{e}^{\mathrm{tX_i}}] = \prod_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{M_{X_i}(t)} \\
&= \prod_{i=1}^{n} (\mathrm{e}^{\mu t + \frac{\sigma^2 t^2}{2}}) = \mathrm{e}^{n\mu t + \frac{n\sigma^2}{2} t^2}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{M}_{X_n}(t) &= \mathrm{E}[\mathrm{e}^{\mathrm{tX_n}}] = \mathrm{E}[\mathrm{e}^{\mathrm{t}(\frac{1}{\mathrm{n}}\sum\limits_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{X_i})}] = \mathrm{E}[\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{e}^{\frac{\mathrm{t}}{\mathrm{n}}\mathrm{X_i}}] \\
&\overset{\underline{Since\ \{X_i\}_{i=1}^n\ are\ independent}}{=\!=\!=\!=\!=} \prod_{i=1}^{n} \mathrm{E}[\mathrm{e}^{\frac{\mathrm{t}}{\mathrm{n}}\mathrm{X_i}}] = \prod_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{M_{X_i}}(\frac{\mathrm{t}}{\mathrm{n}}) \\
&= \prod_{i=1}^{n} (\mathrm{e}^{\mu\frac{t}{n} + \frac{\sigma^2(\frac{t}{n})^2}{2}}) = \mathrm{e}^{\mu t + \frac{\sigma^2}{2n} t^2}.
\end{aligned}$$

(ii) As m.g.f and distribution is one to one correspondence, then we know

$$S_n \sim \mathrm{N}(n\mu, n\sigma^2), \quad \overline{X}_n \sim \mathrm{N}(\mu, \frac{\sigma^2}{n}).$$

5. Suppose X is a discrete random variable taking values of non-negative integers(or subset of non-negative integers) with probability mass function(p.m.f) $\{p_k; k \geq 0\}$. Define the probability generating function (p.g.f) of X, denoted by $\prod_X(t)$, as

$$\prod\nolimits_X(t) = \mathrm{E}[\mathrm{t^X}].$$

(i) Write down the form $\prod_X(t)$ in terms of the p.m.f $\{p_k; k \geq 0\}$.

(ii) Investigate the problem as how to get E(X) and Var(X) by using the p.g.f $\prod_X(t)$.

(iii) Find the p.g.f of the Binomial Random Variable X with parameter n and p.

(iv) Find the p.g.f of the Poisson Random Variable X with parameter λ.

Solution:

(i) $\prod_X(t) = \sum\limits_{k=0}^{\infty} p_k t^k$ or $\prod_X(t) = \sum\limits_{k=0}^{n} p_k t^k$(if X takes values of $\{0, 1, \cdots, n\}$).

(ii) Since $\prod_X(t) = \sum\limits_{k=0}^{\infty} p_k t^k$, then differentiating and exchange summation order, we obtain

$$\prod\nolimits_X^{'}(t) = \sum_{k=1}^{\infty} p_k k t^{k-1}, \quad \prod\nolimits_X^{''}(t) = \sum_{k=2}^{\infty} p_k k(k-1) t^{k-2}.$$

Letting $t = 1$,[‡] yields

$$\prod_X^{\prime}(1) = \sum_{k=1}^{\infty} k p_k = \sum_{k=0}^{\infty} k p_k = \mathrm{E(X)};$$

$$\prod_X^{\prime\prime}(1) = \sum_{k=2}^{\infty} k(k-1) p_k = \sum_{k=0}^{\infty} k(k-1) p_k = \mathrm{E[X(X-1)]} = \mathrm{E[X^2 - X]} = \mathrm{E(X^2) - E(X)}.$$

Hence,

$$\mathrm{E(X)} = \prod_{\mathrm{X}}^{\prime}(1) \quad \text{and} \quad \mathrm{E(X^2) - E(X)} = \prod_{\mathrm{X}}^{\prime\prime}(1).$$

It follows that,

$$\mathrm{E(X^2) = E(X^2) - E(X) + E(X)} = \prod_{\mathrm{X}}^{\prime\prime}(1) + \prod_{\mathrm{X}}^{\prime}(1).$$

Hence,

$$\mathrm{Var}(X) = \mathrm{E(X^2)} - \big(\mathrm{E(X)}\big)^2 = \prod_{\mathrm{X}}^{\prime\prime}(1) + \prod_{\mathrm{X}}^{\prime}(1) - \big(\prod_{\mathrm{X}}^{\prime}(1)\big)^2$$

(iii) Recall the p.m.f of Binomial random variable with parameter p and n is:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots, n.$$

Since $X \sim \mathrm{Binomial}(n, p)$, then the p.g.f of the Binomial random variable with parameter p and n is:

$$\begin{aligned}\prod_X(t) &= \sum_{k=0}^{n} p_k t^k = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} t^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (pt)^k (1-p)^{n-k} = (pt + 1 - p)^n.\end{aligned}$$

(iv) Recall the p.m.f of Poisson random variable with parameter λ is:

$$p_X(k) = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \cdots.$$

Since $X \sim \mathrm{Poisson}(n, p)$, then the p.g.f of the Binomial random variable with parameter λ is:

$$\begin{aligned}\prod_X(t) &= \sum_{k=0}^{\infty} p_k t^k = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} t^k \\ &= \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda} \frac{(\lambda t)^k}{k!} = \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \\ &= \mathrm{e}^{-\lambda} \cdot \mathrm{e}^{\lambda t} = \mathrm{e}^{\lambda(t-1)}.\end{aligned}$$

[‡]Probability generating functions obey all the rules of power series with non-negative coefficients. In particular, $\prod_X(1-) = 1$, where $\prod_X(1-) = \lim_{t \to 1} \prod_X(t)$ from below, since the probabilities must sum to one. So the radius of convergence of any probability generating function must be at least 1, by Abel's theorem for power series with non-negative coefficients.