Step-1

Consider the second-order equation

$$\frac{d^2u}{dt^2} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$$

With the following conditions

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$u'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We know that the if A has negative eigenvalues $\lambda_1, \dots, \lambda_n$ and if $\omega_j = \sqrt{-\lambda_j}$, then solution of $\frac{d^2u}{dt^2} = Au$ is given by

$u(t) = (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t) x_1 + \cdots + (a_n \cos \omega_n t + b_n \sin \omega_n t) x_n$. â€¦â€¦(1)

Step-2

To find the eigenvalues for matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, by using the equation

$\det(A - \lambda I) = 0$.

$$\det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0$$
$$(-2-\lambda)(-2-\lambda) - (1) = 0$$
$$(-2-\lambda)^2 - 1 = 0$$

Step-3

Solve the $(-2-\lambda)^2 - 1$, to find the value of λ.

$$(-2-\lambda)^2 = 1$$
$$-2-\lambda = \pm\sqrt{1}$$
$$-2-\lambda = \pm 1$$
$$-2-\lambda = -1 \text{ or } -2-\lambda = 1$$

So, we get λ_1 and λ_2 as

$\lambda_1 = -1$
$\lambda_2 = -3$

Step-4

The eigenvector for $\lambda_1 = -1$ is given by

$$\begin{aligned}(A-\lambda_1 I)x_1 &= \begin{bmatrix}-2+1 & 1\\ 1 & -2+1\end{bmatrix}x_1\\ &\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}x_1\\ &= 0\end{aligned}$$

The eigenvector for $\lambda_1 = -1$ is given by $x_1 = \begin{bmatrix}1\\1\end{bmatrix}$.

Step-5

The eigenvector for $\lambda_2 = -3$ is given by

$$\begin{aligned}(A-\lambda_2 I)x_2 &= \begin{bmatrix}-2+3 & 1\\ 1 & -2+3\end{bmatrix}x_2\\ &\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}x_2\\ &= 0\end{aligned}$$

The eigenvector for $\lambda_2 = -3$ is given by $x_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

Step-6

We know that $\omega_j = \sqrt{-\lambda_j}$.

Substituting -1 for λ_1 in equation $\omega_1 = \sqrt{-\lambda_1}$, we get

$$\begin{aligned}\omega_1 &= \sqrt{-(-1)}\\ &= 1\end{aligned}$$

Similarly substituting -3 for λ_2 in equation $\omega_2 = \sqrt{-\lambda_2}$, we get

$$\omega_2 = \sqrt{-(-3)}$$
$$= \sqrt{3}$$

To find the values of *a*â€™s, substitute $t=0$ into the equation (1).

$$u(0) = (a_1\cos(0)+b_1\sin(0))x_1 + \cdots + (a_n\cos(0)+b_n\sin(0))x_n$$

so all $b_n\sin(0)=0$ and $a_n\cos(0)=1$, this gives:

$$a = S^{-1}u(0)$$

We have:

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So, the values of *a*â€™s are:

$$a = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-7

Thus, $a_1 = 0$ and $a_2 = 0$.

Step-8

To find the values of *b*â€™s, first differentiating $u(t)$ and then substituting $t=0$ we get:

$$u'(t) = b_1\omega_1 x_1 + \cdots + b_n\omega_n x_n$$

We have:

$$u'(0)=\begin{bmatrix}1\\0\end{bmatrix}$$
$$x_1=\begin{bmatrix}1\\1\end{bmatrix}$$
$$x_2=\begin{bmatrix}-1\\1\end{bmatrix}$$

And

$$\omega_1=1$$
$$\omega_2=\sqrt{3}$$

So, the values of *b*â€™s are:

$$\begin{aligned}\begin{bmatrix}1\\0\end{bmatrix}&=b_1\times1\times\begin{bmatrix}1\\1\end{bmatrix}+b_2\times\sqrt{3}\times\begin{bmatrix}-1\\1\end{bmatrix}\\&=\begin{bmatrix}b_1-\sqrt{3}b_2\\b_1+\sqrt{3}b_2\end{bmatrix}\end{aligned}$$

Step-9

Solving the following system of equation, we get the values of *b*â€™s.

$$b_1-\sqrt{3}b_2=1$$
$$b_1+\sqrt{3}b_2=0$$

Thus, $b_1=\frac{1}{2}$ and $b_2=-\frac{1}{2\sqrt{3}}$.

Step-10

Substituting the values of *a*â€™s and *b*â€™s into the equation (1), we get

$$\begin{aligned}u(t)&=\left(0\times\cos(1)t+\frac{1}{2}\sin(1)t\right)\begin{bmatrix}1\\1\end{bmatrix}+\left(0\times\cos\sqrt{3}t+\left(\frac{1}{-2\sqrt{3}}\right)\sin\sqrt{3}t\right)\begin{bmatrix}-1\\1\end{bmatrix}\\&=\frac{1}{2}\sin t\begin{bmatrix}1\\1\end{bmatrix}-\frac{1}{2\sqrt{3}}\sin\sqrt{3}t\begin{bmatrix}-1\\1\end{bmatrix}\end{aligned}$$

Therefore, the motion $\boxed{u(t)=\frac{1}{2}\sin t\begin{bmatrix}1\\1\end{bmatrix}-\frac{1}{2\sqrt{3}}\sin\sqrt{3}t\begin{bmatrix}-1\\1\end{bmatrix}}$.