Step-1

The number λ is an eigen value of A if and only if A - λI is singular

In other words, det (A - λI) = 0

This is the characteristic equation. Each λ is associated with eigen vector x such that

$Ax = \lambda x$.

Further, the eigen vectors corresponding to the distinct eigen values are linearly independent.

Considering the linearly independent eigen vectors as the columns of a matrix S, we can see that $A = S\Lambda S^{-1}$ where Λ is the diagonal matrix whose diagonal entries are nothing but the eigen values of A.

Now, raising both sides to the power n, we get
$$\begin{aligned} A^n &= \left(S\Lambda S^{-1}\right)^n \\ &= \left(S\Lambda S^{-1}\right)\left(S\Lambda S^{-1}\right)\ldots\left(S\Lambda S^{-1}\right) \quad n \text{ times} \end{aligned}$$

$$\begin{aligned} &= S\Lambda\left(S^{-1}S\right)\Lambda\left(S^{-1}S\right)\Lambda\ldots\Lambda S^{-1} \\ &= S\Lambda I\Lambda\ldots\Lambda S^{-1} \\ &= S\Lambda^n S^{-1} \end{aligned}$$

Step-2

In view of the above discussion, we get A^{100} for the given matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ as

The characteristic equation of A is det (A - λI) = 0

$$\begin{aligned} |A - \lambda I| = \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} &= 0 \\ &= (4-\lambda)(2-\lambda) - 3 \\ &= 8 - 4\lambda - 2\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 6\lambda + 5 \end{aligned}$$

$$\Rightarrow (\lambda - 5)(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = 5$$

Step-3

Let x_1 is the vector such that $\left(A - \lambda_1 \mathrm{I}\right)x_1 = 0$

i.e., $\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Using the row operation $R_2 \to 3R_2 - R_1$, we get $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Rewriting the system from this, we get $t_1 + t_2 = 0$

Putting $t_1 = 1$, we get $t_2 = -1$ and thus, $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda_1 = 1$.

Step-4

Let x_2 is the vector such that $(A - \lambda_2 \mathrm{I}) x_2 = 0$

i.e., $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Using the row operation $R_2 \to R_2 + R_1$ on this, we get $\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Rewriting the system, we get $-t_1 + 3t_2 = 0$

Putting $t_2 = 1$, we get $t_1 = 3$ and thus, $x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda_2 = 5$

Step-5

Considering the eigen vectors as the columns of *S*, we get $S = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

$S^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

So, $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

The above notes provides $A^{100} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}^{100}\frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$$