Step-1

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The matrix that rotates the x - y plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We have to verify that $A(\theta_1)A(\theta_2) = A(\theta_1+\theta_2)$ from the identities for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$. Also we have to find $A(\theta).A(-\theta)$.

Step-2

Given $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Hence $A(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ and

$$A(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Now $\mathrm{A}(\theta_1).\mathrm{A}(\theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} . \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \end{bmatrix} + \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} \begin{bmatrix} \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Step-3

Continuation to the above

$$= \begin{pmatrix} \cos\theta_1.\cos\theta_2 & -\cos\theta_1.\sin\theta_2 \\ \sin\theta_1.\cos\theta_2 & -\sin\theta_1.\sin\theta_2 \end{pmatrix} + \begin{pmatrix} -\sin\theta_1.\sin\theta_2 & -\sin\theta_1.\cos\theta_2 \\ \cos\theta_1.\sin\theta_2 & \cos\theta_1.\cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2 & -\cos\theta_1.\sin\theta_2 - \sin\theta_1.\cos\theta_2 \\ \sin\theta_1.\cos\theta_2 + \cos\theta_1.\sin\theta_2 & -\sin\theta_1.\sin\theta_2 + \cos\theta_1.\cos\theta_2 \end{pmatrix}$$

$$=\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$
$$= A(\theta_1+\theta_2)$$

Therefore $\boxed{A(\theta_1)A(\theta_2)=A(\theta_1+\theta_2)}$

Step-4

$$A(-\theta)=\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$A(\theta).A(-\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\left(\because \sin(-\theta)=-\sin\theta \text{ and } \cos(-\theta)=\cos\theta\right)$

Step-5

Continuation to the above

$$=\begin{pmatrix} \cos^2\theta+\sin^2\theta & \cos\theta.\sin\theta-\sin\theta.\cos\theta \\ \sin\theta.\cos\theta-\cos\theta.\sin\theta & \cos^2\theta+\sin^2\theta \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$=I$$

Therefore $A(\theta).A(-\theta)=\boxed{I}$