南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Applied stochastic processes

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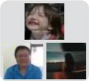
Math 208, Spring 2023



群聊：应随 2023

该二维码7天内(2月20日前)有效，重新进入将更新

Outline for this semester

1. Revisiting of probability

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Chapter 1. Revisiting of probability

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P: probability measure which assign a number to each event $E \subset S$.

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, \ E_2, \ \cdots$ are disjoint, then

$$P\left(\cup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).$$

Random variable (r.v.): $X : S \to \mathbb{R}$.

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Joint distribution function of $(X_1, \cdots, X_d)$:

$$F(x_1, \cdots, x_d) = P(X_1 \leq x_1, \cdots, X_d \leq x_d).$$

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Another formula

$$\sigma^2 = \mathbb{E}(X^2) - \mu^2.$$

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for a suitable function f, which is called the probability density function (p.d.f.) of X.

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$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

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$$P(A_{i_1} \cdots A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

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- Continuous r.v.: $f(x, y) = f_X(x)f_Y(y)$.

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Thus,

$$\mathbb{E}(X^n) = \phi^{(n)}(0).$$

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Solution: Do it on board!

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Definition

A stochastic process is a family of random variables $\{X_t : t \in \mathbb{T}\}$ with time set $\mathbb{T}$ and state space S.

Example

Suppose that you will bid \$1 in each play. You will win \$1 with probability p *and lose \$1 with probability* $q = 1 - p$. *You will stop if you have nothing to bid or you played 3 bids. If you start with \$1, what is the probability that you will end up bankrupted?*

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Solution: Do it on board!

HW: Ch1, 8, 10, 13, 36. Ch2, 1, 4, 5, 30, 33, 43, 46, 67, 69.
Note: Ch2 means Chapter 2 in the textbook.