Step-1

Consider the following matrices.

$A=\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}$, and $B=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$

The objective is to reduce the matrices *A* and *B* to echelon form, to find their ranks and also find the free variables. Find the special solutions to $Ax=0,\ Bx=0$ and also find all solutions.

Step-2

Consider the matrix,

$$A=\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}$$

Let $x=\begin{bmatrix}u\\v\\w\\y\end{bmatrix}, b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

The augmented matrix is,

$$[A\ b]=\begin{bmatrix}1&2&0&1&b_1\\0&1&1&0&b_2\\1&2&0&1&b_3\end{bmatrix}$$

$$\underline{R_3-R_1}\begin{bmatrix}1&2&0&1&b_1\\0&1&1&0&b_2\\0&0&0&0&b_3-b_1\end{bmatrix}$$

$$\underline{R_1-2R_2}\begin{bmatrix}1&0&-2&1&b_1-2b_2\\0&1&1&0&b_2\\0&0&0&0&b_3-b_1\end{bmatrix}$$ â€¦â€¦ (1)

Step-3

The last equation shows the solvability condition as $b_3-b_1=0$

The matrix $\left[A,b\right]$ is converted as $\left[R\,d\right]$

The rank of A is 2 because there are two pivot columns, the first, second columns are pivot columns.

Pivot variables are *u, v*, and the free variables are *w, y*.

Step-4

Now find special solutions to $Ax=0$.

Consider, $Ax=0$

From matrix (1),

$$\begin{bmatrix}1 & 0 & -2 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\ v\\ w\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{aligned}u-2w+y&=0\\ v+w&=0\end{aligned}$$

So,

$$\begin{aligned}u&=2w-y\\ v&=-w\end{aligned}$$

Step-5

Therefore,

$$\begin{aligned}\begin{bmatrix}u\\ v\\ w\\ y\end{bmatrix}&=\begin{bmatrix}2w-y\\ -w\\ w\\ y\end{bmatrix}\\ &=w\begin{bmatrix}2\\ -1\\ 1\\ 0\end{bmatrix}+y\begin{bmatrix}-1\\ 0\\ 0\\ 1\end{bmatrix}\end{aligned}$$

Therefore, the special solutions are, $\boxed{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}$.

Step-6

Now, $Ax=b$ is solvable if $b_1=b_3$.

Let $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$

For $Ax=\begin{bmatrix}1\\2\\1\end{bmatrix}$,

Consider the matrix, $Ax=b$.

$$\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}\begin{bmatrix}u\\v\\w\\y\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$\underline{R_3-R_1}\begin{bmatrix}1&2&0&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}u\\v\\w\\y\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\underline{R_1-2R_2}\begin{bmatrix}1&0&-2&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}u\\v\\w\\y\end{bmatrix}=\begin{bmatrix}-3\\2\\0\end{bmatrix}$$

$$u-2w+y=-3$$
$$v+w=2$$

$$u=-3+2w-y$$
$$v=2-w$$

Step-7

Now

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3+2w-y \\ 2-w \\ w \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step-8

Therefore, the particular solution is,

$$x_p = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Hence, all solutions of $Ax = b$ are $x = x_p + x_n$

Therefore $\boxed{x = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$

Step-9

Consider the matrix,

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Let $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The augmented matrix is,

$$[B\ b]=\begin{bmatrix}1 & 2 & 3 & b_1\\ 4 & 5 & 6 & b_2\\ 7 & 8 & 9 & b_3\end{bmatrix}$$

Step-10

Simplify, by using row operations

$$\xrightarrow{\substack{R_2-4R_1,\\ R_3-7R_1}}\begin{bmatrix}1 & 2 & 3 & b_1\\ 0 & -3 & -6 & (b_2-4b_1)\\ 0 & -6 & -12 & (b_3-7b_1)\end{bmatrix}$$

$$\xrightarrow{R_3-2R_2}\begin{bmatrix}1 & 2 & 3 & b_1\\ 0 & -3 & -6 & (b_2-4b_1)\\ 0 & 0 & 0 & b_3+b_1-2b_2\end{bmatrix}$$ â€¦â€¦ (2)

The system $Bx=b$ is solvable if $b_1-2b_2+b_3=0$

The rank of *B* is two, since there are two pivot columns. First, second columns are pivot columns, *u, v* are pivot variables and *w* is a free variable.

Step-11

Now find special solutions to $Bx=0$

Consider, $Bx=0.$

From the matrix (2),

$$\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{aligned}u+2v+3w&=0\\ 3v+6w&=0\end{aligned}$$

So,

$$\begin{aligned}u&=-2v-3w\\ 3v&=-6w\\ v&=-2w\end{aligned}$$

Now substitute $v=-2w$ in $u=-2v-3w$

$$u = -2(-2w) - 3w \quad (\text{Since } v = -2w)$$
$$= 4w - 3w$$
$$= w$$

Therefore,

$$v = -2w$$
$$u = w$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} w \\ -2w \\ w \end{bmatrix}$$
$$= w\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Hence, $x_n = w\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Step-12

To find particular solution, take

$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (Since $b_1 - 2b_2 + b_3 = 0$)

Now consider $Bx = b$,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\xrightarrow[R_3 - 7R_1]{R_2 - 4R_1,} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -6 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$u+2v+3w=1$

$-3v-6w=-3$

From the second equation,

$$\begin{aligned}-3v-6w&=-3\\ -3v&=-3+6w\\ v&=\frac{3-6w}{3}\\ v&=1-2w\end{aligned}$$

Substitute, $v=1-2w$ in $u+2v+3w=1$

$$\begin{aligned}u+2(1-2w)+3w&=1\\ u+2-4w+3w&=1\\ u-w&=1-2\\ u-w&=-1\\ u&=w-1\end{aligned}$$

Step-13

The particular solution is,

$$\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}w-1\\1-2w\\w\end{bmatrix}$$
$$=\begin{bmatrix}-1\\1\\0\end{bmatrix}+w\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

$$x_p=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

All solutions for $BX=b$ are $x=x_n+x_p$.

Hence, the complete solutions is,

$$x=\begin{bmatrix}-1\\1\\0\end{bmatrix}+w\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$