

Step-1

Consider the following orthogonal matrix:

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Diagonalize it to reach $Q = U \Lambda U^H$.

Step-2

First step is to find the Eigen values and Eigen vectors of matrix Q . To calculate the Eigen values do the following calculations;

$$Q - \lambda I = \begin{bmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{bmatrix}$$

$$\det(Q - \lambda I) = 0$$

$$(\cos \theta - \lambda)(\cos \theta - \lambda) + \sin^2 \theta = 0$$

$$(\lambda^2 - 2\lambda \cos \theta + 1) = 0$$

After solving following values are obtained:

$$\lambda_1 = \cos \theta + i \sin \theta$$

$$\lambda_2 = \cos \theta - i \sin \theta$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(Q - \lambda I)x = 0$$

$$\begin{bmatrix} \cos \theta - \cos \theta - i \sin \theta & -\sin \theta \\ \sin \theta & \cos \theta - \cos \theta - i \sin \theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i \sin \theta & -\sin \theta \\ \sin \theta & -i \sin \theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = \cos \theta + i \sin \theta$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Step-4

Similarly, Eigen vectors corresponding to Eigen value $\lambda = \cos \theta - i \sin \theta$ is as follows:

$$(Q - \lambda I)x = 0$$

$$\begin{bmatrix} \cos \theta - \cos \theta + i \sin \theta & -\sin \theta \\ \sin \theta & \cos \theta - \cos \theta + i \sin \theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i \sin \theta & -\sin \theta \\ \sin \theta & i \sin \theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Step-5

To put the Eigen vectors in unitary matrix make them orthonormal by dividing the length of the vector.

$$\|x\|^2 = |(1)^2| + |(-i)^2|$$

$$= |1| + |1|$$

$$= 2$$

Let the length be L . So $L = \sqrt{2}$.

Step-6

Now, the diagonalization of the matrix can be written as follows:

$$Q = U \Lambda U^H$$

$$= \frac{1}{L} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \cos \theta + i \sin \theta & 0 \\ 0 & \cos \theta - i \sin \theta \end{bmatrix} \frac{1}{L} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Here, $L = \sqrt{2}$

Step-7

Therefore, orthogonal matrix Q is diagonalize to reach $\boxed{Q = U \Lambda U^H}$. All Eigen values of unitary matrix is $\boxed{|\lambda| = 1}$.