Solution for Assignment 05

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PROBLEM 1. A r.v. X is said to follow the logarithmic distribution with parameter $p(0 < p < 1)$, if X has probability mass function

$$P(X = k) = A(\frac{p^k}{k}) \ \ for \ k = 1, 2, 3, \ldots$$

for some appropriate constant A. Find the value of the constant A in terms of the parameter p.

SOLUTION. We know that $\ln(1-x)$ has Taylor expansion:

$$\ln(1-x) = -x - x^2/2 - x^3/3 - \cdots - x^k/k - \cdots$$

So for $p(0 < p < 1)$,

$$\begin{aligned}\sum_{k=1}^{\infty} P(X = k) =& A\sum_{k=1}^{\infty} \frac{p^k}{k} \\ =& A \ln \frac{1}{1-x}\end{aligned}$$

By the property that $\sum_{k=1}^{\infty} P(X = k) = 1$, $A = \ln(1-x)$

PROBLEM 2. Suppose that independent trials, each having a probability $p, 0 < p < 1$, of being a success, are performed until a success occurs. Let X equal the number of trials required. What are the possible values of X? Write down the p.m.f. of X.

SOLUTION. X can be any positive integer.

Let A_k be the event that i-th trial is success, then

$$\begin{aligned}P(X = k) =& P(A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c \cap A_n)\\ =& \prod_{i=1}^{n-1} P(A_i^c) * P(A_n)\\ =& p(1-p)^{n-1}\end{aligned}$$

PROBLEM 3. Suppose that independent trials, each having a probability $p, 0 < p < 1$, of being a success, are performed until a total of r successes is accumulated, where $r > 1$ is a positive integer. Let X equal the number of trials required. What are the possible values of X? Write down the p.m.f. of X.

SOLUTION. X can be any integer k that satisfies $k \geq r$.

For an event $\{X = k\}$, it means the total success number for the first $k-1$ trials, if we denote it as Y_{k-1}, which we know follows the distribution of a binomial r.v. $B(n-1, p)$, must be $r-1$, and the k-th trial must be a success, so

$$\begin{aligned}P(X = k) =& P(A_k) * P(Y_{k-1} = r-1)\\ =& p * C_{n-1}^{r-1} p^{r-1} (1-p)^{n-r}\\ =& C_{n-1}^{r-1} p^r (1-p)^{n-r}\end{aligned}$$

PROBLEM 4. Assuming that each dart has probability 0.2 of hitting its target, give the c.d.f. of the number of darts one should throw at the target to get the first successful hit. What is the c.d.f. of the number of throws required to get two hits? Finally what is the probability of at least one hit in n throws, and what is the smallest value of n for which this is greater than 0.9?

SOLUTION. Denote X as the number of first time the dart hit the target.We have calculated the p.m.f of X in problem 2:

$$P(X=k)=p(1-p)^{k-1}=0.2*0.8^{k-1}.$$

So the c.d.f of X for $k\leq x<k+1\in\mathbb{N}$ is:

$$\begin{aligned}F(x)=&P(X\leq x)\\=&P(X\leq k)\\=&\sum_{i=1}^{k}0.2*0.8^{k-1}\\=&0.2*\frac{1-0.8^k}{1-0.8}\\=&1-0.8^k.\end{aligned}$$

As for c.d.f of number of throws required to get two hits,denote by Y, by problem 3, the p.m.f is

$$P(Y=k)=C_{k-1}^{r-1}p^r(1-p)^{k-r}=C_{k-1}^{r-1}0.2^r0.8^{k-r}.$$

As for $2\leq k\leq x<k+1\in\mathbb{N}$, $F_2(x)=P(Y\leq x)=P(Y\leq k)$, the event $\{Y\leq k\}$ means in the first k trials, it has no less than two success, so if Z_k denotes the success in the first k trials, $\{Y\geq k\}=\{Z_k\geq 2\}$, use the argument in problem 3, we know that $Z_kB(k,0.2)$, so

$$\begin{aligned}F_2(x)=&P(Y\leq x)\\=&P(Y\leq k)\\=&P(Z_k\geq 2)\\=&1-P(Z_k=1)-P(Z_k=0)\\=&1-0.8^k-0.2*0.8^{k-1}k\end{aligned}$$

We know $F(k)$ is a increasing function about x, $F(k)\geq 0.9$ equals to $0.8^k\leq 0.1$, $k\geq\ln_{0.8}0.1$. Just do some easy calculation and get $0.8^11<0.1,0.8^10>0.1$, so the smallest $x=k$ should be 11.

PROBLEM 5. The number of phone calls received at a certain residence in any period of c hours is a Poisson random variable with parameter $\lambda = 0.5c$.

(i) What is the probability that the phone rings during a given 15 minute period?

(ii) How long a period must one wait for the probability of at least one phone call during that period to be at least 0.5?

SOLUTION.

(i) For a given 15 minute period, the parameter $\lambda = 0.5 * 0.25 = 0.125$, let X be the total number of the calls received in this period. Then $X Poisson(0.125)$, so

$$\begin{aligned} P(\{the\ phone\ rings\}) =& P(X \geq 1) \\ =& 1 - P(X = 0) \\ =& 1 - e^{-0.125} \end{aligned}$$

(ii) From the argument in (i), we know for a c hour period,

$$P\{the\ phone\ rings\}) = P(X \geq 1) = 1 - e^{-0.5c}.$$

So $P\{the\ phone\ rings\}) \geq 0.5$ equals to $e^{0.5c} \geq 2$,i.e. $c \geq 2 * ln2$.