

## Step-1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5$$

Suppose

$$a \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On scalar multiplication and vector addition, we get

$$\begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 0 & c & 0 \\ c & 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & 0 & d \\ 0 & d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & e \\ e & 0 & 0 \\ 0 & e & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d & a+b & c+e \\ a+e & c & b+d \\ b+c & d+e & a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow a=b=c=d=e=0$$

## Step-2

Thus, we have shown that whenever

$$a \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ we get } a=b=c=d=e=0$$

Therefore, the matrices  $p_1, p_2, p_3, p_4, p_5$  are linearly independent

$$\begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 0 & c & 0 \\ c & 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & 0 & d \\ 0 & d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & e \\ e & 0 & 0 \\ 0 & e & 0 \end{bmatrix} = \begin{bmatrix} d & a+b & c+e \\ a+e & c & b+d \\ b+c & d+e & a \end{bmatrix}$$

We see that the row sums of the linear combination

$$a+b+c+d+e$$

$$a+b+c+d+e$$

$$a+b+c+d+e$$

Similarly, the column sums are

$$a+b+c+d+e$$

$$a+b+c+d+e$$

$$a+b+c+d+e$$

We easily see that all the row sums are equal to  $a+b+c+d+e$

Similarly, column sums are equal to  $a+b+c+d+e$

Therefore, the matrices  $\{p_1, p_2, p_3, p_4, p_5\}$  is a basis for the subspace of 3 by 3 matrices with row and column sum are all equal.