Step-1

Given

$$S_1 = |3|, \quad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}, \quad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

Step-2

Now

$$\begin{aligned} S_1 &= |3| \\ &= 3 \end{aligned}$$

$$\begin{aligned} S_2 &= \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\ &= 9 - 1 \\ &= 8 \end{aligned}$$

Step-3

Then

$$\begin{aligned} S_3 &= \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} \\ &= 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \text{ expanding I row} \end{aligned}$$

$$\begin{aligned} &= 3S_2 - 3 \\ &= 3S_2 - S_1 \\ &= 24 - 3 \\ &= 21 \end{aligned}$$

Step-4

Cofactors of 1, 3, 1 tridiagonal matrices gives the relation

$$S_n = 3S_{n-1} - S_{n-2}$$

We have Fibonacci numbers

$$\begin{array}{cccccccc} 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21 \\ & & & | & & & & | \\ & & & 4th & & & & 8th \end{array}$$

Are given by the rule $F_K = F_{K-1} + F_{K-2}$

Step-5

Therefore

$$\begin{aligned} F_{2n+2} &= F_{2n+2-1} + F_{2n+2-2} \\ &= F_{2n+1} + F_{2n} \\ &= F_{2n} + F_{2n-1} + F_{2n} \\ &= 2F_{2n} + \left(F_{2n} - F_{2n-2}\right) \end{aligned}$$

$\left(\because\ F_{2n} = F_{2n-1} + F_{2n-2}\right)$

$$F_{2n+2} = 3F_{2n} - F_{2n-2} \qquad \ldots(i)$$

Step-6

Observe that

$$\begin{aligned} S_1 &= 3 \\ &= F_{2+2} \\ &= 3 \end{aligned}$$

$$\begin{aligned} S_2 &= F_{4+2} = F_6 \\ &= 8 \end{aligned}$$

$$\begin{aligned} S_3 &= F_{6+2} = F_8 \\ &= 21 \end{aligned}$$

Step-7

Therefore, by induction

$$S_n = 3S_{n-1} - S_{n-2}$$

$$\begin{aligned} &= 3.F_{2(n-1)+2} - F_{2(n-2)+2} \\ &= 3F_{2n} - F_{2n-2} \end{aligned}$$

$$= F_{2n+2} \ \left(from(i)\right)$$