Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$$

Step-2

Initial condition:

$$u(0) = (5, -2)$$

Find the Eigen values and Eigen vectors, so that $u = e^{\lambda t}x$ solves the differential equation. Also, compute the general solution that starts from the initial value.

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = 0$$
$$(4-\lambda)(1-\lambda) = 0$$

After solving following values are obtained:

$$\lambda_1 = 4$$
$$\lambda_2 = 1$$

Step-4

Therefore, Eigen values are $\boxed{4,1}$

Step-5

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-4 & 3 \\ 0 & 1-4 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda=4$ are as follows:

$$x_1=\begin{bmatrix} y \\ z \end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-6

Similarly, Eigen vectors corresponding to Eigen value $\lambda=1$ is as follows:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-1 & 3 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2=\begin{bmatrix} y \\ z \end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-7

Therefore Eigen values are:

$$\boxed{\begin{aligned} x_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}}$$

Step-8

Therefore, values of $u = e^{\lambda t}x$ corresponding to both Eigen values are as follows:

$$\boxed{\begin{aligned} u_1 &= e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ u_2 &= e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}}$$

Step-9

Recall that: $e^{At} = Se^{\Lambda t}S^{-1}$

Here, Eigen value matrix is given as follows:

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the general solution of the differential equation is:

$$\begin{aligned} u(t) &= c_1e^{\lambda_1 t}x_1 + c_2e^{\lambda_2 t}x_2 \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} e^{4t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Here, c_1 and c_2 are constants. Their values are determined by the following values:

$$c = S^{-1}u(0)$$

Step-10

So, the solution for differential equation can be written as follows:

$$
\begin{aligned}
u(t) &= Se^{\Lambda t}S^{-1}u(0) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\end{aligned}
$$

Step-11

Therefore, general solution of the differential equation is:

$$
\boxed{u(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}
$$