MATH 131P WINTER 2025 FINAL PROJECT

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Deriving and Solving the Hanging Chain Equation

1. Introduction

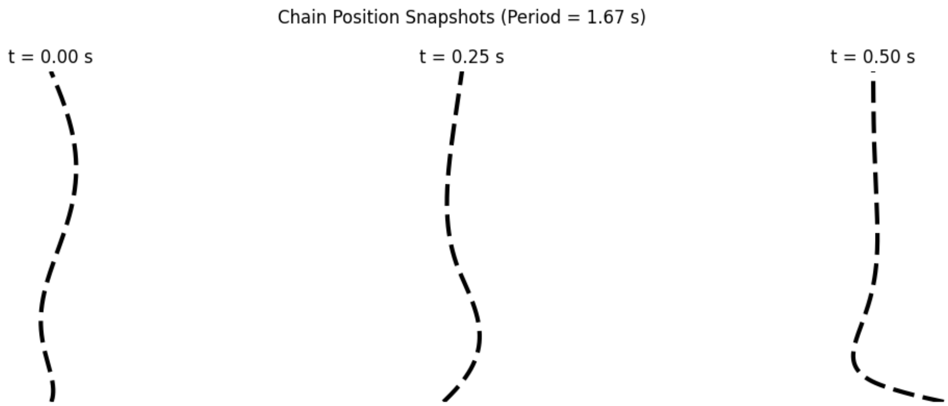

Figure 1. Hanging chain initialized with a sine wave displacement. See code for modeling.

The goal of this paper is to understand the movement and oscillations of a hanging chain with one end fixed as shown in Figure 1. The oscillations of a hanging chain is a foundational differential equations problem which was first solved by Jacob Bernoulli (1655-1705) and was the first use of Bessel functions [3, Wilson (1908)]. Since its discovery, the hanging chain equation has been applied to many modern systems, some of which include satellite structures, suspension cables, and robotic arm mechanics. In this paper, we will begin by deriving the hanging chain equation through mechanics. Then we will find an equation to describe the general motion of a hanging chain.

2. Background: Deriving Hanging Chain Equation Through Mechanics

In order to understand the motion of a hanging chain, we must first derive an equation that models the chain's motion. Consider a chain of length L with uniform linear density, which means the mass of the chain is distributed evenly along its length. This chain is pinned at an endpoint such that the chain is not touching the ground.

First we will examine the tension forces on the chain. The mass (m) at a given point along the chain is $m = \rho \cdot dx$ where ρ is the mass per unit length of the chain and dx is an infinitesimal distance along the chain. Since the chain has uniform linear density we know $\rho = 1$. As such $m = dx$, so we can represent the tension on the chain at rest as

$$T(x) = \rho g x = gx,$$

where g represents gravity. When points along the chain have been displaced at an angle θ, we can separate tension at that point in terms of vertical (T_y) and horizontal (T_x) components

$$T(x) = \begin{bmatrix} T_x(x) \\ T_y(x) \end{bmatrix} = \begin{bmatrix} T(x)\cos(\theta) \\ T(x)\sin(\theta) \end{bmatrix}.$$

Let $u(x,t)$ represent the position of the chain such that $\frac{\partial u}{\partial x}$ represents the change in displacement relative to x. For small displacements from the chain's resting state, $\frac{\partial u}{\partial x} = \tan(\theta)$ since both the difference in displacement and the tangent function can be thought of as a slope. Moreover for small angles we know that both $\sin\theta$ and $\tan\theta$ approximate to θ so

$$\sin(\theta) \approx \theta \approx \tan(\theta),$$
$$\sin(\theta) \approx \frac{\partial u}{\partial x}.$$

As such, the vertical component of the tension becomes $T_y(x,\theta) \approx T(x)\frac{\partial u}{\partial x}$.

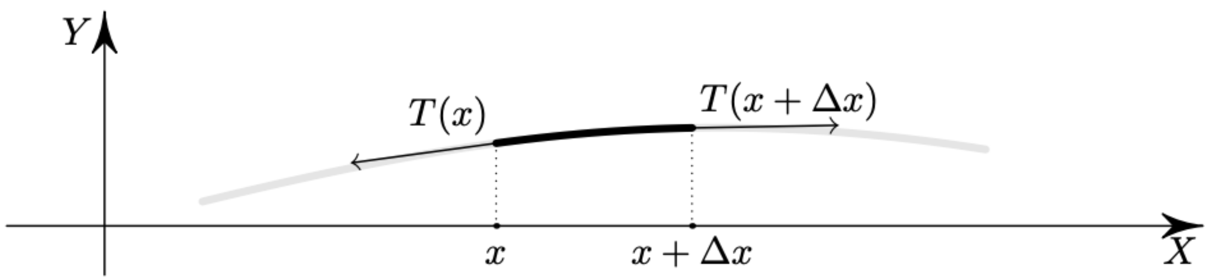

FIGURE 2. Segment of chain with tension forces [2, Cutillas (2018)]

Next we will examine the net vertical forces acting on the chain as shown in Figure 2. By Newton's second law, the net vertical force on a point in the chain is given by the sum of the vertical forces acting on it. $\frac{\partial^2 u}{\partial t^2}$ represents the acceleration along the chain so we can model net vertical force

$$\sum F_y(x) = m \cdot \frac{\partial^2 u}{\partial t^2} \Longrightarrow \sum F_y(x) = dx\frac{\partial^2 u}{\partial t^2}.$$

We can also derive the net vertical forces acting on a segment in the chain as the difference in tension forces acting on its endpoints. Consider a segment on the chain of infinitesimal length dx such that its endpoints are x and $x + dx$. We can represent the vertical forces acting on it as the difference in tension forces

$$\sum F_y(x) = T_y(x+dx) - T_y(x),$$

where $T_y(x) = T(x)\frac{\partial u}{\partial x}$ and $T_y(x+dx) = T(x+dx)\frac{\partial u}{\partial x}$. Since we are taking the difference of an infinitesimal difference, we can use linear approximation

$$T_y(x+dx) - T_y(x) \approx \frac{d}{dx}(T_y(x))dx.$$

We can apply our linear approximation of the differences in tension to our net vertical force equation

$$\begin{aligned}
\sum F_y(x) &= T_y(x+dx) - T_y(x), \\
&= \frac{\partial}{\partial x}(T_y(x))dx, && \text{(Infinitesimal difference is the derivative)} \\
&= \frac{\partial}{\partial x}(T(x)\frac{\partial u}{\partial x})dx. && \text{(Apply definition of } T_y\text{)}
\end{aligned}$$

When we combine these two equations, we get

$$\begin{aligned}
\sum F_y(x) = dx\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}(T(x)\frac{\partial u}{\partial x})dx, \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}(T(x)\frac{\partial u}{\partial x}), && \text{(Cancel out dx)} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}(gx\frac{\partial u}{\partial x}), && \text{(Apply definition of tension)} \\
\frac{\partial^2 u}{\partial t^2} &= g(\frac{\partial}{\partial x}(x)\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}), && \text{(Apply product rule)} \\
\frac{\partial^2 u}{\partial t^2} &= g(\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}). && \text{(Simplify)}
\end{aligned}$$

Now we have an equation modeling our hanging chain which we can use to solve for $u(t, x)$. Lastly, we must apply boundary conditions to fit our problem's constraints. Since the chain is hanging from one endpoint at a fixed point, we will set $u(L, t) = 0$. The rest of the chain will start an initial shape $u(x, 0) = f(x)$ where $f(x)$ represents its initial displacement and at an initial velocity $u_t(x, 0) = v(x)$ where $v(x)$ represents its initial velocity. Thus, our hanging chain equation is

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2} = g\left[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right], \quad 0 < x < L, \quad t > 0, \\
u(L,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = v(x).
\end{gathered} \tag{1}$$

3. Main Result: Solving the Hanging Chain Equation

3.1. **Finding Separable ODEs.** From our hanging chain equation (1), we will now find a solution by separating it into two ODEs and solving each ODE. We will assume that $u(x, y)$ is separable such that

$$u(x,t) = X(x)T(t). \tag{2}$$

As such, we can apply $u(x,t) = X(x)T(t)$ to our hanging chain equation (1)

$$\begin{aligned}
\frac{\partial^2 X(x)T(t)}{\partial t^2} &= g(x\frac{\partial^2 X(x)T(t)}{\partial x^2} + \frac{\partial X(x)T(t)}{\partial x}), && \text{(Apply } u \text{ equation)}\\
X(x)\frac{\partial^2 T}{\partial t^2} &= gT(t)(x\frac{\partial^2 X}{\partial x^2} + \frac{\partial X}{\partial x}), && \text{(Simplify terms)}\\
X(x)T''(t) &= gT(t)(xX''(x) + X'(x)), && \text{(Change derivative syntax)}\\
\frac{T''(t)}{gT(t)} &= \frac{xX''(x) + X'(x)}{X(x)}. && \text{(Separate variables)}
\end{aligned}$$

Since we now have two expressions with respect to x and t that are independent of each other, they must be equal to a constant λ for the equation to hold:

$$\frac{T''(t)}{gT(t)} = \lambda \quad \text{and} \quad \frac{xX''(x) + X'(x)}{X(x)} = \lambda.$$

In the case where $\lambda \geq 0$, our equation for T would grow linearly or exponentially which does not reflect real-world conditions. As such we will consider λ to be negative as the only possible real-world case. In order to simplify notation, we will replace λ with $-\lambda^2$ such that our separated equations are

$$\frac{T''(t)}{gT(t)} = -\lambda^2 \quad \text{and} \quad \frac{xX''(x) + X'(x)}{X(x)} = -\lambda^2. \tag{3}$$

Now that we have two separated ODEs for X and T, we can solve them individually.

3.2. **Solving our Equation for T.** For our equation involving T (3), we can isolate its terms

$$\begin{aligned}
\frac{T''(t)}{gT(t)} &= -\lambda^2,\\
T''(t) &= -\lambda^2 gT(t), && \text{(Rearrange terms)}\\
T''(t) + \lambda^2 gT(t) &= 0. && \text{(Consolidate terms to one side)}
\end{aligned}$$

Since our equation for T is a second-order homogeneous linear ODE, we can use the characteristic polynomial by assuming a solution of the form $T(t) = e^{rt}$ where r is the root of our characteristic polynomial. Applying this to our ODE gives

$$
\begin{aligned}
r^2 + \lambda^2 g &= 0, \\
r^2 &= -\lambda^2 g, && \text{(Isolate } r\text{)} \\
r &= \pm\sqrt{-\lambda^2 g}, && \text{(Apply square root)} \\
r &= \pm\sqrt{g}\lambda i. && \text{(Cancel out exponent)}
\end{aligned}
$$

Since our characteristic polynomial has imaginary roots as shown above, our general solution is

$$T(t) = A\cos\left(\sqrt{g}\,\lambda t\right) + B\sin\left(\sqrt{g}\,\lambda t\right). \tag{4}$$

3.3. **Solving our Equation for X.** For our equation involving X (3), we can rewrite our equation such that it is in Sturm-Liouville form

$$
\begin{aligned}
\frac{xX''(x) + X'(x)}{X(x)} &= \lambda, \\
xX''(x) + X'(x) &= -\lambda^2 X(x), && \text{(Rearrange terms)} \\
xX''(x) + X'(x) + \lambda^2 X(x) &= 0, && \text{(Consolidate terms to one side)} \\
(xX'(x))' + \lambda^2 X(x) &= 0. && \text{(Rewrite with product rule)}
\end{aligned}
$$

Now that our equation is in Sturm-Liouville form, we can recognize its terms compared to the general form

$$\frac{d}{dx}\left[p(x)\frac{dX}{dx}\right] + \left[\lambda\, w(x) - q(x)\right] X = 0, \quad a \leq x \leq b.$$

As such, $p(x) = x$, $w(x) = 1$, and $q(x) = 0$. Since $p(0) = 0$, our equation for X is a singular Sturm-Liouville problem. We will use a change of variables to see that Bessel's equations can be applied to this equation. Let $s = 2\sqrt{x}$ such that by chain rule

$$
\begin{aligned}
X' &= \frac{dX}{dx} = \frac{dX}{ds}\frac{ds}{dx} = \frac{dX}{ds} \cdot \frac{1}{\sqrt{x}}, \\
X'' &= \frac{d^2X}{dx^2} = \frac{d}{dx}\left[\frac{dX}{ds} \cdot \frac{1}{\sqrt{x}}\right] = \frac{1}{x}\frac{d^2X}{ds^2} - \frac{1}{2x^{3/2}}\frac{dX}{ds}.
\end{aligned}
$$

Substituting this into our ODE, we find

$$
\begin{aligned}
(xX')' + \lambda^2 X &= 0, \\
xX'' + X' + \lambda^2 X &= 0, && \text{(Expand product rule)} \\
x\left[\frac{1}{x}\frac{d^2X}{ds^2} - \frac{1}{2x^{3/2}}\frac{dX}{ds}\right] + \frac{dX}{ds}\cdot\frac{1}{\sqrt{x}} + \lambda^2 X &= 0, && \text{(Apply expressions for } X' \text{ and } X''\text{)} \\
\frac{d^2X}{ds^2} - \frac{1}{2\sqrt{x}}\frac{dX}{ds} + \frac{dX}{ds}\cdot\frac{1}{\sqrt{x}} + \lambda^2 X &= 0, && \text{(Distribute } x\text{)} \\
\frac{d^2X}{ds^2} - \frac{1}{2\sqrt{x}}\frac{dX}{ds} + \lambda^2 X &= 0, && \text{(Combine terms)} \\
\frac{d^2X}{ds^2} + \frac{1}{s}\frac{dX}{ds} + \lambda^2 X &= 0, && \text{(Apply expression for } x\text{)} \\
s^2\left[\frac{d^2X}{ds^2} + \frac{1}{s}\frac{dX}{ds} + \lambda^2 X\right] &= 0\cdot s^2. && \text{(Multiply by } s^2\text{).}
\end{aligned}
$$

This leaves us with our final equation for the change of variables applied to the hanging chain

$$s^2\frac{d^2X}{ds^2} + s\frac{dX}{ds} + \lambda^2 s^2 X = 0. \tag{5}$$

We should also apply our change of variables to our boundary conditions (1). For $u(s,t)$, our boundary conditions are

$$0 < s < 2\sqrt{L} \quad \text{and} \quad u(2\sqrt{L}, t) = 0.$$

Our change of variables equation (5) is the same as the parametric form of Bessel's equation of order 0, which means the zero's of the Bessel function are λ and X can be represented as a Bessel function [1, Theorem 4.8.3]. Thus

$$\lambda = \lambda_j = \frac{\alpha_j}{2\sqrt{L}} \quad \text{and} \quad X(s) = X_j(s) = J_0\left(\frac{\alpha_j}{2\sqrt{L}}s\right) \tag{6}$$

where $j = 1, 2, 3, \ldots$ and α_j denotes the jth zero of the Bessel function J_0. Substituting back x for s, we see

$$X_j(2\sqrt{x}) = J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right). \tag{7}$$

3.4. **Solving for u.** Now that we have equations for X_j (7), λ_j (6) and T (4), we can can apply them to our initial equation for u (2). First, consider u_j as

$$u_j(x,t) = X_j(x)T_j(t),$$

$$= J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right)\left[A_j\cos\left(\sqrt{g}\,\lambda_j t\right) + B_j\sin\left(\sqrt{g}\,\lambda_j t\right)\right], \qquad \text{(Applying } X \text{ and } T\text{)}$$

$$u_j(x,t) = J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right)\left[A_j\cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t\right) + B_j\sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t\right)\right]. \qquad \text{(Applying } \lambda_j\text{)}$$

Since Bessel functions are orthogonal on the interval $0 \leq x \leq L$ with respect to the weight function $2\sqrt{x}$, we can sum the terms of u_j to get u:

$$u(x,t) = \sum_{j=1}^{\infty} J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right)\left[A_j\cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t\right) + B_j\sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t\right)\right]. \tag{8}$$

We can apply our boundary conditions for $u(x,0)$ (1) in order to find A_j

$$u(x,0) = \sum_{j=1}^{\infty} A_j J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right),$$

$$f(x) = \sum_{j=1}^{\infty} A_j J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right). \qquad \text{(Substituting } f\text{)}$$

By the orthogonality property of Bessel functions we can apply Bessel series expansion to isolate A_j

$$A_j = \frac{1}{LJ_1^2(\alpha_j)}\int_0^L f(x)J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right)dx. \tag{9}$$

Similarly, we can apply our boundary conditions for $\frac{\partial u}{\partial t}(x,0)$ (1) in order to find B_j

$$\frac{\partial u}{\partial t}(x,0) = \sum_{j=1}^{\infty} B_j\sqrt{\frac{g\alpha_j}{L}}J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right),$$

$$v(x) = \sum_{j=1}^{\infty} B_j\sqrt{\frac{g\alpha_j}{L}}J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right). \qquad \text{(Substituting } v\text{)}$$

By the orthogonality property of Bessel functions we can apply Bessel series expansion to isolate B_j

$$B_j = \frac{2}{\alpha_j J_1^2(\alpha_j)}\frac{1}{\sqrt{gL}}\int_0^L v(x)J_0\left(\alpha_j\sqrt{\frac{x}{L}}\right)dx. \tag{10}$$

4. Conclusion

Thus we have derived the hanging chain equation from mechanics and solved it using differential equations. From our differential equation representing the hanging chain problem (1), we have solved for $u(x,t)$ (8) and its corresponding constants A_j (9) and B_j (10). As a computer science student with an interest in space technology, this problem had particular relevance to my future career ambitions. I am a member of the Stanford Space Initiative satellite team where we are building a satellite that will launch in October. The hanging chain problem is relevant and important to my learning because it shows up in many applications of satellites and in coordinating structures in space. As such I plan to use my learning from this class in my future career in space technology.

References

[1] N. H. Asmar. *Partial Differential Equations with Fourier Series and Boundary Value Problems*. 3rd ed. Dover Publications, 2017.

[2] Philippe Cutillas. "Hanging Chain Vibration MATH 485". In: *University of Arizona* (2018).

[3] E. B. Wilson. "The equilibrium of a heavy homogeneous chain in a uniformly rotating plane". In: *Annals of Mathematics* 9.3 (1908), pp. 99–115.