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Abstract

History and research on proof by contradiction suggests proof by contradiction is difficult for students in a number of ways. Students' *comprehension* of already-written proofs by contradiction is one such aspect that has received relatively little attention. Applying the cognitive lens of **A**ction-**P**rocess-**O**bject-**S**chema (APOS) Theory to proof by contradiction, we constructed and tested a cognitive model that describes how a student might construct the concept 'proof by contradiction' in an introduction to proof course. Data for this study was collected from students in a series of five teaching interventions focused on proof by contradiction. This paper will report on two participants as case studies to illustrate that our cognitive trajectory for proof by contradiction is a useful model for describing how students may come to understand the proof method.

Keywords: Proof by Contradiction, Proof Comprehension, APOS Theory, Genetic Decomposition

1. Background of the problem

Proof plays a pivotal role in undergraduate mathematics education. While some courses briefly introduce the concept of a proof and how to write one (e.g., Geometry and Calculus), it is not until after the Calculus sequence that undergraduate students in the United States truly focus on reading, reproducing, and writing

5   mathematical proofs. This shift in focus normally occurs in a Transition-to-Proof course that uses Elementary Number Theory as a sandbox for students to practice proofs in a *somewhat* familiar content setting (David & Zazkis, 2020). While the curriculum for this course is not ubiquitous (see David & Zazkis (2020) for a characterization of curriculum in Transition-to-Proof courses in the United States), it always includes an introduction to various proof methods and some mathematical content to practice these methods. This

10   paper will primarily focus on how these students develop an understanding of a particular proof method: Proof by Contradiction.

Proof by Contradiction is based on the *law of excluded middle*: either a statement is true or the negation of

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the statement is true. By showing the negation of the statement cannot be true (i.e., leads to a contradiction), the statement must be true. In symbolic notation, proof by contradiction is of the form $(\sim P \rightarrow (Q \wedge \sim Q)) \rightarrow P$, where P is the statement to be proved and Q is any other statement (Smith et al., 2015). Proof by contradiction is one of two[1] prominent proof methods that rely on **valid but not sound proof** arguments. An argument is *valid* when the truth of the conclusion necessarily follows from the truth of the premises, but the premises need not be true. An argument is *sound* if it is valid and the premises are true. Thus proof by contradiction requires a student argue with a statement that is either false or of indeterminate truth-value, in conjunction with their logical congruences and other proof methods, to arrive at some contradiction and thus prove the statement true indirectly. Research studies on proof by contradiction in particular remain scarce (Antonini, 2019). Therefore, we set out to examine the history, research results, and cognitive models on proof by contradiction to determine how students in a transition-to-proof course develop an understanding of the concept. Our efforts focused on a single research question: How do students develop an understanding of proof by contradiction?

2. Literature Review

Historically, proof by contradiction has been flat-out rejected as a valid proof method as it relies on the law of excluded middle. Even when the method was accepted as valid, sixteenth and seventeenth century mathematicians argued to eliminate its use as it is not based on true premises. Arguments against proof by contradiction were eventually dropped with the acceptance of Hilbert's formalism in the nineteenth century (Krantz, 2010) yet the method's status as "peculiar" lived on (Antonini & Mariotti, 2008).

Various theories have since been posited as to why the method is still not readily accepted by students. One prevalent theory is the lack of construction leads students to less conviction when using proof by contradiction. Leron (1985) mused the following:

> We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards 'destroying' this world, proving it is indeed false and impossible. We are thus involved in an act of *mathematical destruction*, not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered (p. 323).

Leron describes a dichotomy between the construction of (and thus evidence for) a mathematical object versus the deconstruction of an impossible mathematical object. He suggested re-formalizing this decon-

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[1]The other being proof by mathematical induction.

struction as a positive construction *separate from the negative assumption* which would both alleviate the amount of time a student would spend in the "false" part of the proof and provide a construction for evidence

45 of the truth of the statement. Leron's theory that students crave constructive evidence for a theorem was provided credence by Harel & Sowder (1998), who found students would rely on constructions to convince themselves of the truth of a statement.

Brown (2017) examined the effect of constructive vs nonconstructive proofs on a student's conviction in a pair of studies. Her results found that the constructive characteristic of a proof did not correlate to their level

50 of conviction. In fact, she reported students were more often convinced by the nonconstructive version of a proof than the constructive alternative. These results came with a caveat: a majority of students interpreted nonconstructive existence proofs as constructive. Regardless, Brown provided evidence that a student's lack of conviction may not necessarily be due to the nonconstructive nature of a proof and thus there may be alternative explanations for why a student lacks conviction for a particular proof by contradiction.

55 Other theories suggest student issues with proof by contradiction are caused by:
- Notions prevalent in proof by contradiction (contradiction, absurd, negation) are not developed enough before a transition-to-proof course (Thompson, 1996).
- Construction difficulties germane to proof by contradiction, such as difficulty interpreting or constructing the negation of a mathematical statement (Dubinsky & Yiparaki, 2000).
60 - Negating intuitive statements and/or dealing with impossible objects (Antonini & Mariotti, 2008).

The first two issues relate to proof construction difficulties while the third relates to difficulties with how students understand the proofs they read. Note the distinction between issues with writing and understanding specific proofs by contradiction and understanding the method in general. These issues may impact how students come to understand the proof method but do not provide a cognitive learning trajectory for how

65 students may develop their understanding.

Two models related to student understanding of proof by contradiction are available: one model describing how students develop understanding and another model for analyzing how students understand the contradiction proofs they read. We briefly describe each model and discuss its benefits and limitations.

Lin et al. (2003) are the only researchers to present a model of how students develop an understanding

70 of proof by contradiction. This model is based on a few assumptions:

Understanding proof by contradiction shall mean to have both the procedural and conceptual knowledge of proof by contradiction. The procedure knowledge is: negating the conclusion q, and then inferring a mathematical fact or assertion that is contradicted to p. The conceptual knowledge is: "if $\sim q$ then $\sim p$" implies "if p then q". This step, which is the principle of proof
75 by contradiction, is based on the law of contrapositive (p. 443).

The authors developed a questionnaire to address the three major components of their assumptions on student understanding of proof by contradiction: negating a statement, recognizing the procedure, and recognizing the law of contrapositive. After analyzing their questionnaire data, Lin et al. (2003) proposed the following model for how students develop an understanding of proof by contradiction:

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The first step of proof by contradiction is to negate the conclusion. After a student is able to negate a basic statement, he/she can begin to learn the procedure knowledge of proof by contradiction. However, only until a student understands the law of contrapositive, he/she will know why the procedure is finished. The ability of negating a statement might be developed unrelated to the understanding of the procedural knowledge of proof by contradiction (p. 448).

Lin et al. (2003) drew the attention of the research community to the need for a model of student understanding of proof by contradiction. However, limitations of the authors' study mitigate the usefulness of this model in the context of students in a transition-to-proof course. First and foremost, the authors assumed "the conceptual knowledge of proof by contradiction is the law of contrapositive" (p.446). While both methods start by negating a statement, proof by contradiction is significantly different than proof by contraposition in both construction process and cognitive underpinning. It is not appropriate to use one model to describe a student's understanding of both proof methods.

Antonini & Mariotti (2008) presented a model through which student understanding of proof by contradiction can be identified, analyzed, and interpreted. This model is the culmination of numerous studies (Antonini, 2003, 2004; Antonini & Mariotti, 2006, 2007). While this model is for indirect proof and not solely for proof by contradiction, Antonini & Mariotti (2008) stated:

[. . .] what seems to be psychologically meaningful in proof by contradiction is the starting point that is the negation of the thesis. This characteristic is also shared by proof by contraposition. In spite of significant differences, we can point out some important commonalities of these types of proof. Therefore, in this paper, we deal with both proof by contradiction and proof by contraposition, referring to them through the term *indirect proof* (p. 402).

Antonini & Mariotti (2008) examined indirect proof using a didactic notion of mathematical theorem. This notion states "a mathematical theorem consists in the system of relations between a statement, its proof, and the theory within which the proof makes sense" (p. 404). This triplet is referred to as (S, P, T), where S is the statement, P is the proof, and T is the theory.

For indirect proofs, students move from a principal statement S to a secondary statement S^*, which is tautologically equivalent to S. It is through proving the secondary statement S^* that the primary statement S is proved, i.e. $S^* \to S$. This secondary statement will have its own proof within the same mathematical theory and so has its own triplet (S^*, P^*, T). In this view, students should now prove two statements: S^* and the meta-statement $S^* \to S$. Consider the following examples of principal and secondary statements presented in Antonini & Mariotti (2008):

Principal Statement S: Let a and b be two real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.
Secondary Statement S^*: Let a and b be two real numbers. If $ab = 0, a \neq 0, b \neq 0$, then $1 = 0$
(p. 404).

Note the secondary statement can now be proved directly. Using a didactic notion of mathematical theorem, the authors can identify nearly all difficulties students have with proof by contradiction as student difficulties with the proof of the meta-statement $S^* \to S$.

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This analysis model is useful in a few ways. By separating the proof of a secondary statement and of the meta-statement $S^* \to S$, students may become aware of the underlying role of the law of excluded middle for a proof by contradiction. The authors conclude that analyzing the secondary statement and meta-statement
[120] "[...] can also be an efficient didactical tool for designing teaching/learning situations aimed to introduce indirect proofs." (p. 411). It also explains why students have issues dealing with impossible mathematical objects; when transitioning to the secondary statement S^*, a student may decide that the previous theory T is no longer adequate for proving S^*. However, Antonini & Mariotti (2008) do not provide a cognitive trajectory for how students could develop an understanding of proof by contradiction. Thus there is still a
[125] need in the literature for a model that describes how a student in a transition-to-proof course could develop an understanding of proof by contradiction. We now present the theoretical framework that drives our proposed model.

3. Theoretical Framework

We chose a cognitive framework specific to mathematics: APOS Theory. This framework posits "Individ-
[130] uals make sense of mathematical concepts by building and using certain mental *structures* (or *constructions*) which are considered in APOS Theory to be *stages* in the learning of mathematical concepts." (Arnon et al., 2014, p. 17). The abbreviation APOS Theory stands for these mental structures: **A**ctions, **P**rocess, **O**bjects, and **S**chemas[2]. We briefly introduce these constructions below.

An *Action* is a transformation of mathematical objects by the individual requiring external, step-by-step
[135] instructions on how to perform the transformation. As an individual reflects on an Action, they may stop relying on the external, step-by-step instructions of an Action to perform and describe the transformation. The transformation of mathematical objects without external instructions is referred to as a *Process*. As an individual reflects on a Process, they may think of the dynamic Process as a totality and can now perform transformations on this previously dynamic Process. This new mathematical totality is referred to as an
[140] *Object*. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principals to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001). We can use students' exhibited construction of these mental structures to describe a hierarchy of understanding for a particular concept. For example, if a student interacts with a mathematical concept primarily by relying on external, step-by-step procedures, we say the student exhibits an Action
[145] conception of that particular mathematical concept[3]. In order to analyze student data, we made the following

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[2]Actions, Processes, Objects, and Schemas are capitalized to differentiate between the mental constructions of APOS Theory and the colloquial use of these terms.
[3]Again, we can only make claims about a student's *observed* understanding that may or may not reflect their *true* under-

conjectures (before collecting data) on the observable actions students would exhibit at the various stages of understanding proof by contradiction.

3.1. Mental structures for proof by contradiction

A student with an *Action* conception of proof by contradiction would have to work with a concrete example and perform memorized or specifically given instructions for steps when using this proof method. At this level, a student would not be able to prove a statement that requires a deviation from these steps. For example, a student may have memorized the following steps that need to be done to prove an implication $(P \rightarrow Q)$ statement by contradiction:

1. Convert the statement into symbolic notation;
2. Identify the premise (P), the conclusion (Q), and write the statement in the 'If P, then Q' form $(P \rightarrow Q)$;
3. Write the negation of the conclusion $(\sim Q)$;
4. Assume the negation of the conclusion and the premise are both true $(P \wedge \sim Q)$;
5. Symbolically manipulate the assumption $(P \wedge \sim Q)$ to get a statement that contradicts this assumption; and
6. State that a contradiction has been made and thus the proof is done.

In this example, the student would not be able to prove a statement that requires a deviation from these steps, such as if the statement needs more than symbolic manipulation to reach a contradiction or if the initial statement was not an implication. The student would also need to deal with an explicit proof to apply these memorized steps to and could not talk about these steps in general.

A student with a *Process* conception of proof by contradiction would have written many different particular proofs, repeating the above steps. When the student can reflect on these Actions and no longer needs to work with concrete examples or external cues in order to explain how a proof by contradiction works, the student is said to have interiorized the Action into a Process. At this level, a student is able to describe a general procedure to prove a statement by contradiction without referring to a particular example *and* without a need to perform the procedure. For example, consider how a student may interiorize the example step-by-step procedure presented above. After reflecting on the specific procedure of an implication statement over some particular proofs, the student would then be able to state that the following general steps could be used to prove any[4] statement by contradiction:

1. Assume the premise and the negation of the conclusion are true;
2. Show that step 1 leads to a mathematical absurdity, i.e. a contradiction; and

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standing.

[4]It is important to note that the student may believe this Process encompasses all proofs by contradiction until they are presented with a statement that their Process does not work for (e.g. to prove a uniqueness statement). At this point, a student may rephrase the first step and thus refine their Process conception of proof by contradiction.

3. Conclude the statement to be proved is true.

Note the generality of steps 1 and 2. In step 1, the student no longer relies on the symbolic representations of statements to begin the proof. In step 2, the contradiction can be of any form, rather than being restrained to arising from symbolic manipulation (as in the example Action conception). The generality of these two steps encompasses steps 1 through 5 in the Action conception example. Moreover, the student would be able to describe *why* they follow the general steps above in order to complete a proof by contradiction. For example, the student could describe that the assumption leading to a contradiction means the assumption is not true and so the statement to be proved is "not not" true, and thus is true. The ability to describe *why* the student follows the general procedure is critical to a Process conception.

A student with an *Object* conception of proof by contradiction would have encapsulated the Process described above into a static Object by performing some Action on the contradiction proof method itself. This Action could be proving the validity of the proof method itself by utilizing the *law of excluded middle*. Alternatively, this Action could be comparing the contradiction proof method to other proof methods, such as a proof by contraposition or a proof by cases. Regardless, the proof method itself is transformed from a dynamic procedure into a static structure that can be acted on. The student can de-encapsulate their Object conception back into a Process it came from in order to prove a particular statement by contradiction when necessary.

While these stages are hierarchical, they are not necessarily linear. That is, an individual may construct and reconstruct many mental structures before being considered to have developed a conception at that stage. For example, a student may construct multiple Processes for specific types of statements (e.g., implications, nonexistence, uniqueness) that they must then restructure and coordinate together in order to develop a single Process, at which point the student would be considered to have developed a Process conception of proof by contradiction. Indeed, Piaget & Garcia (1989) argued "that cognitive development is never linear, but generally requires the reconstruction of what had been acquired during the preceding stages in order to advance to a higher level" (p. 109). APOS Theory models this development in two ways: genetic decomposition and Schema development. Generally speaking, a genetic decomposition describes the mental structures (Actions, Process, and Objects) the student constructs as they develop an understanding of a concept while Schema development describes the relations between Schemas within a particular concept. This paper will focus solely on describing a genetic decomposition for proof by contradiction. Utilizing our own knowledge of proof by contradiction (as presented in the introduction) and the available literature on the method (including pilot studies Chamberlain Jr. & Vidakovic, 2016, 2017, 2018), we present our

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preliminary[5] genetic decomposition for proof by contradiction.

3.2. Preliminary Genetic Decomposition for proof by contradiction

²¹⁰ A *preliminary genetic decomposition* outlines the hypothetical constructions students should make in order to understand a concept and describes how these constructs would be made. There may be more than one distinct genetic decomposition for a concept as there is more than one way to develop understanding of a concept. Every genetic decomposition is developed based on an analysis of the historical development of the concept in question, a literature review, and the conception of the instructor or researcher. It may ²¹⁵ also include a description of prerequisite knowledge a student should possess in order to begin developing a concept[6] (Arnon et al., 2014). We outline this prerequisite knowledge with regard to our specific preliminary genetic decomposition below.

First, a student should have a general understanding of the propositional and predicate logic that underscores all proofs. In particular, a student should be able to perform and understand negations of the ²²⁰ following types of statements: statements without quantification, implications, and single-level quantified statements. This is necessary as every proof by contradiction requires a negation of some statement. General understanding of mathematical logic will allow students to outline proofs as a proof framework (Selden & Selden, 2017) and avoid focusing on superficial formulae that could impede understanding of the proof (Inglis & Alcock, 2012). Secondly, a student should be able to move between semantic, symbolic, and algebraic ²²⁵ representations of mathematical statements. For example, a student should be able to represent the statement 'If a is even, then $a + 1$ is odd' as the propositional implication '$P \to Q$', as the predicate statement $(\forall a \in \mathbb{Z})(2 \mid a \to 2 \nmid (a + 1))$, and as the algebraic representation 'If $a = 2n$ for $n \in \mathbb{Z}$, then $a + 1 = 2m + 1$ for $m \in \mathbb{Z}$'. Moving between representations of mathematical statements (consciously or subconsciously) is useful for making sense of mathematical statements and proofs (Alcock & Inglis, 2008; Mejía-Ramos et al., ²³⁰ 2012; Selden & Selden, 1995; Inglis & Alcock, 2012) and thus is suggested before developing a notion of proof by contradiction. Finally, a student should be familiar with the direct proof method in a mathematical context as this forms the basis of understanding for a proof by contradiction. This prerequisite knowledge covers some of the most common difficulties students have when constructing proofs by contradiction, both in our own teaching experiences and as identified in the literature. Again, this prerequisite knowledge is ²³⁵ specific to our preliminary genetic decomposition and another preliminary genetic decomposition on proof by contradiction may not include any prerequisite logic knowledge.

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[5]Preliminary in that it is a theoretical model not based on data.

[6]Begin developing a concept at the appropriate level of the individual. In our specific case, for an undergraduate in a Transition-to-Proof course

Our preliminary genetic decomposition for proof by contradiction is presented below:

1. Action conception of propositional or predicate logic statements consists of following a given or memorized specific step-by-step instructions to construct proofs by contradiction for the following types of statements: (I) implication [e.g., $P \to Q$], (II) single-level quantification [e.g., $\exists x P(x)$], and (III) property claim [e.g., element a has property P].
2. Interiorization of each Action in step 1 individually as general steps to writing a proof by contradiction for statements of the form (I), (II), and (III).
3. Coordination of the Processes from step 2 as general steps to writing a proof by contradiction.
4. Encapsulate the Process in step 3 as an Object by utilizing the law of excluded middle to show proof by contradiction is a valid proof method or by comparing the contradiction proof method to other proof methods.
5. De-encapsulate the Object in step 4 into a Process similar to step 3 that then coordinates with a Process conception of quantification to prove multi-level quantified statements.

Note that the first constructions a student should make are with the underlying logic of proof by contradiction. This allows the student to construct a specific step-by-step procedure for proof by contradiction in symbolic form for each type of statement encountered [step 1] similar to the proof frameworks suggested by Selden & Selden (2017) that avoid focusing on superficial aspects of the proof (Alcock & Inglis, 2008). For example, the student could have memorized procedures for proof by contradiction based on the structure of the statement, such as the procedures included in Table 1

Table 1: Possible memorized procedures for implication statements (left) and non-existence statements (right).

Statement: $P \to Q$	Statement: $(\nexists x)(P(x))$
1. Assume $\sim (P \to Q)$	1. Assume $\sim (\nexists x)(P(x))$
2. $P \wedge \sim Q$	2. $(\exists x)(P(x))$
3. $\sim Q$	3. $P(n)$
4. $(\sim Q \wedge P) \to Q$	4. Using $P(n)$, get to a contradiction.
5. Q	5. $\sim (\sim (\nexists x)(P(x)))$
6. $Q \wedge \sim Q$	6. $(\nexists x)(P(x))$
7. $\sim (\sim (P \to Q))$	
8. $P \to Q$	

At this stage, a student relies on translating mathematical statements into their symbolic logic form in order to use the step-by-step procedure for each specific type of statement. In the example above, a student with an Action conception would not be able to describe the logical reasoning behind the procedure but would have memorized the steps and applied them in close to identical situations.

As a student reflects on a particular step-by-step procedure, the student interiorizes the memorized procedure into general steps to write a proof. At this stage, a student no longer needs to translate statements into their underlying symbolic logic in order to prove a mathematical statement or could think of the statements in their head without the need to write them step-by-step. Instead students could attend to semantic proof strategies where they do not need to stay within a symbolic representation system to make sense of the proof (Alcock & Inglis, 2008).

As a student reflects on these different general steps to prove different types of statements, the student can coordinate these general steps into a single series of general steps that can be used to prove a variety of

statements. For example, these general steps could be: (1) negate the statement, (2) find a contradiction, and (3) conclude the proof. At this stage, a student has a unifying structure to how a proof by contradiction
₂₇₀ is written, which allows the student to determine whether a presented proof is by contradiction or not. At this stage, a student would be considered to have a Process conception of proof by contradiction as their procedure is now flexible enough to work for many types of statements.

A student may show they have encapsulated the Process in step 3 of the genetic decomposition as a proof method by utilizing the *law of excluded middle* to show proof by contradiction is a valid proof method
₂₇₅ [step 4]. By encapsulating the Process in step 3, a student can think of proof by contradiction as a totality, or an entity that he or she can compare to other proof techniques, rather than a dynamic procedure. In this case, we say that a student has constructed an Object level of understanding of proof by contradiction. Furthermore, such a constructed Object can subsequently be de-encapsulated into a Process that then can be coordinated with a Process conception of quantification to prove multi-level quantified statements.

₂₈₀ To test our preliminary genetic decomposition, we developed a series of aligned instructional materials that were used to teach students in a transition-to-proof course. The following section briefly describes how we conducted these teaching sessions.

4. Methodology and Data Collection

This study was conducted with students enrolled in a Transition-to-Proof course at a research university
₂₈₅ in the Southeastern region of the US during Fall 2016. To test our preliminary genetic decomposition for proof by contradiction, we conducted a teaching experiment (Steffe & Thompson, 2000) utilizing an instructional approach aligned with APOS Theory: the ACE teaching cycle (Arnon et al., 2014). This approach consists of three phases (**A**ctivities, **C**lassroom discussion, and **E**xercises)[7] that ideally induce the construction of mental structures and relations as described by the preliminary genetic decomposition.

₂₉₀ This teaching experiment consisted of 5 short, consecutive teaching sessions over a period of 5 weeks that each mimicked the ACE teaching cycle. That is, each session consisted of: students working on an Activity worksheet focusing on a particular component of the genetic decomposition for proof by contradiction (**A**); a discussion about the concepts from the worksheet (**C**); and a typical series of proof comprehension questions related to the content of the worksheet (**E**). Two instructors, who were not associated with the study, allowed
₂₉₅ one of the authors to conduct the first two teaching sessions during class time. A short description of the first two episodes is provided below while the associated worksheets are provided in the Appendix.

Episode 1 was designed to introduce a set of step-by-step instructions for students to use to construct

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[7]The Activities, Classroom discussion notes, and Exercises used for this study are provided in the Supplemental Materials.

proofs by contradiction for implication statements, which aligns with step 1 for developing an Action conception for implication statements (type I statements) in the preliminary genetic decomposition. In addition, focusing students' attention on the roles of specific lines in the proof and having students identify the key steps of the proof provided students with the opportunity to interiorize the step-by-step instructions for implication into a Process, which aligns with step 2 in the preliminary genetic decomposition.

Episode 2 was designed to introduce a set of step-by-step instructions for students to use to construct proofs by contradiction for nonexistence statements, which aligns with step 1 for developing an Action conception for single-level quantified statements (type II statements) in the preliminary genetic decomposition. In addition, focusing students' attention on the roles of specific lines in the proof and having students identify the key steps of the proof provided students with the opportunity to interiorize the step-by-step instructions for nonexistence statements into a Process, which aligns with step 2 in the preliminary genetic decomposition. Encouraging students to compare the Processes for implication and nonexistence statements would allow students to coordinate these Processes into a single Process for constructing a proof by contradiction, which aligns with step 3 in the preliminary genetic decomposition.

A total of 27 students participated in the first two teaching sessions. The other three teaching sessions were conducted by an author outside of the students' class period. Five of the original 27 students volunteered to participate in teaching session 3, three of those students volunteered to participate in teaching session 4, and only two (Wesley and Yara) volunteered for all five teaching sessions. Due to the low number of volunteers and class scheduling conflicts, Wesley and Yara completed teaching sessions 3-5 individually.

Five short, consecutive ACE teaching cycles (as opposed to a single cycle) were necessary for a number of reasons. Firstly, the unaffiliated instructors allowed for only two class periods, 50 minutes each, to be used for the study. The first two episodes were thus developed to provide an adequate introduction to proof by contradiction without the need for further episodes[8]. Secondly, students could not dedicate more than an hour at a time to meet outside of class. Each episode was thus developed to be conducted entirely within an hour. Thirdly, proof comprehension is recognized as distinct from proof construction (Inglis & Mejia-Ramos, 2009; Samkoff & Weber, 2015) and that explicit instruction on proof comprehension can improve student's cognitive engagement with a proof (Hodds et al., 2014). Neither instructor explicitly taught proof comprehension in class and thus, out of necessity, more than one concept needed to be covered during the teaching experiment. Thus, two distinct ideas were developed during the teaching experiment: proof comprehension and proof by contradiction. Finally, reading a proof for comprehension is a time-intensive

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[8]The first two episodes provided an adequate introduction to *constructing* proofs by contradiction for the most commonly encountered statement types in the course. This satisfied the instructors' expectation that students know how to reproduce and construct familiar proofs by contradiction.

Episode	1	2	3	4	5
Statement type	Implication $P \rightarrow Q$	Nonexistence $(\nexists x)(P(x))$	Uniqueness $(\exists! x)(P(x))$	Infinitely many $|\{x : P(x)\}| = |\mathbb{N}|$	Property claim Element a has property P
Focal Proof(s)	Strong Goldbach Conjecture implies Weak Goldbach Conjecture.	There is no odd integer of the form $4j - 1$ and $4k + 1$.	[1] The equation $5x - 4 = 1$ has a unique solution. [2] The multiplicative inverse of a non-zero real number r is unique.	[1] The set of Natural numbers is infinite. [2] The set of primes is infinite.	[1] Validation of proof by contradiction. [2] $\sqrt{2}$ is an irrational number.
Activity	Logical outline; define proof by contradiction.	Logical outline; define proof by contradiction; compare outlines.	Logical outline of [1]; define proof by contradiction; compare and contrast all outlines.	Logical outline of [1]; define proof by contradiction; compare and contrast all outlines.	Define proof by contradiction; compare validation to definition.
Classroom Discussion	Discuss Activity answers.	Discuss Activity answers – focus on compare outlines.	Discuss Activity answers – focus on compare outlines.	Discuss Activity answers – focus on compare outlines.	Generally discuss proof by contradiction.
Exercise	Comprehension questions on proof.	Comprehension questions on proof.	Comprehension questions on proof [2].	Comprehension questions on proof [2].	Comprehension questions on proof [2].
Goal of Episode	Specific procedure for Implication statements.	Specific procedure for Nonexistence; possible shared procedure.	Specific procedure for Uniqueness; Develop internal, general procedure.	Specific procedure for Infinitely many; Incorporate specific procedure into internal, general procedure.	Specific procedure for Property claim; Incorporate specific procedure into internal, general procedure; Validate proof method.

Figure 1: An overview of each teaching episode for the teaching experiment.

task. The pilot studies suggested students could read and answer comprehension questions on a single proof in about 30 minutes (Chamberlain Jr. & Vidakovic, 2017, 2018). As there are a variety of statements

330 that can be proved by contradiction, each statement type required a separate episode to be introduced and subsequently compared. Together, these reasons necessitated a sequence of five ACE teaching cycles to assist students in developing a robust understanding of proof by contradiction. Each of the five teaching episodes were developed based on the proposed preliminary genetic decomposition. An overview of each teaching episode is provided in Figure 1 while the details of each cycle can be found in the Supplementary Materials.

335 Note that examples of every sub-statement type were not included in the activities (e.g., $\forall x P(x)$ was not included for single-level quantification). However, students who develop a Process conception for proof by contradiction would be able to incorporate these excluded sub-structures into their Schema and thus we believed it was not necessary to include every single sub-structure in the teaching episodes.

The primary source of data for this paper consists of audio/video recordings and written work during

340 teaching episodes with two students: Wesley and Yara. These students were chosen as *case studies* - detailed examinations of a single subject's understanding (Bogdan & Biklen, 2007) - for a variety of reasons: (i) they were enrolled in different sections of the Transition-to-Proof course (ii) they completed all five teaching episodes, (iii) each student's evolution of understanding was distinct and representative of other participants' understanding during teaching episodes 3-5, and (iv) these students provided the most data through which

345 to analyze and support how their understanding of proof by contradiction evolved.

Each video recording was first transcribed with special attention to any gestures or extended pauses the students made while speaking. We then used the audio recording to validate the transcription of the teaching episode. Once all transcriptions were completed and verified for accuracy, they were uploaded to MAXQDA

- a qualitative analysis software[9]. The focus of analyzing the transcripts and written responses during the
₃₅₀ teaching experiment was thus to interpret and categorize how students' understanding evolved during each of the five teaching episodes. Any responses made by the participant were coded as potentially exhibiting an Action, Process, or Object conception as described in section 3.1.

5. Results

We will provide an overview of each student's conceptual development followed by examples of observable
₃₅₅ actions that were indicative of the mental structures they built to understand proof by contradiction due to space limitations. We start with Wesley.

5.1. Wesley

Wesley was a sophomore Mathematics major returning to the classroom after many years in the work force. His overall conceptual development is summarized in Figure 2 and Figure 3. Initially during Teaching
₃₆₀ Episode 1, Wesley's group described proof by contradiction as "[...] showing that a statement or equation contradicts itself, or is false. So therefore the *[did not finish statement]*." This group initially conceptualized proof by contradiction as a proof that contradicts *the statement to be proved*. Wesley later wrote down the agreed-upon definition crowd-sourced and refined during the Classroom Discussion section of the first episode. It appeared he interiorized this definition for himself with his verbal definitions provided in Teaching
₃₆₅ Episodes 2 and 3. However, we see this initial conception return in his generalized logical outlines of proof by contradiction in Teaching Episodes 4 and 5. Based on his responses during episodes 3, 4, and 5, Wesley's initial conception that the contradiction **must necessarily be related to the statement being proved** proved resistant to change. As he was able to complete similar proofs by contradiction during the teaching experiment but could not explain how the proofs worked in general (due to the resilient misconception),
₃₇₀ Wesley exhibited an Action conception of proof by contradiction throughout the teaching experiment. What follows is some of the most exemplary evidence of his Action conception throughout the teaching experiment, both with and without the resilient misconception.

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[9]To be clear, MAXQDA was used to organize the data analysis performed and was not used to analyze the data.

Teaching Episode 1:	Prove statement is true by assuming the negation of the statement is true and proving the assumption is false w/ a contradiction. *[Written after Classroom Discussion]*
Teaching Episode 2:	You assume the contradiction is true then disprove the contradiction → the original statement must be true.
Teaching Episode 3:	Have your statement and you assume the opposite and then prove it false.
Teaching Episode 4:	To prove a statement by assuming its contradiction is false.
Teaching Episode 5:	Well that's *[Outline of general proof for proof by contradiction]* like, perfect example right there, isn't it?

Figure 2: Wesley's definitions for proof by contradiction during each Teaching Episode.

Task 1 (CD2)	Task 2 (CD3)	Task 3 (CD4)	Task 4 (CD5)
Statement: S	**Statement:** S	**Statement:** S	**Statement:** S
1. Assume $\sim S$	1. Assume $\sim S$	1. Assume $\sim S$ is true	1. Assume $\sim S$ is true
2. $\vdots$	2. $\vdots$	2. Attempt to prove	2. Attempt to prove
3. $\rightarrow\leftarrow$	3. $\rightarrow\leftarrow$	$\sim S$ is true	$\sim S$ is true
4. $\sim(\sim S)$	4. S	3. $\rightarrow\leftarrow$ of $S = \sim S$	3. $\rightarrow\leftarrow$ of $S = \sim S$
5. S		4. $\sim(\sim S)$	4. $\sim(\sim S)$
		5. S	5. S

Figure 3: Wesley's comparisons of the logical outlines constructed during Classroom Discussions (CD) 2-5.

5.1.1. Wesley during Teaching Episode 4

Wesley was unable to outline the key steps of the presented proof that the set of Natural numbers was infinite. After being provided the logical representation $P(\mathbb{N})$ as the set of natural numbers having the property "infinitely many elements", he was able to outline the key steps of the provided proof. Even then Wesley could not provide justification for the outlined steps nor could he describe, in general, the purposes of these lines. Details of how he constructed the outline in Figure 4 follow.

Statement: $P(\mathbb{N})$
1. Assume $\sim P(\mathbb{N})$
2. $\mathbb{N} = \{n_1, n_2, n_3, \ldots, n_k\}$
3. $n = n_k + 1$ is a natural number
4. n is not an element of $\mathbb{N}$
5. $\sim(\sim P(\mathbb{N}))$
6. $P(\mathbb{N})$

Figure 4: Wesley's outline of presented proof during Activity 4.

After being provided the representation $P(\mathbb{N})$, he stated "So... so the proof would be, if you are going to prove that by contradiction it would be... you are going to assume that it's not [pause] $P(N)$." The phrase "assume it's not $P(N)$" is a desired first step of the proof and suggests he followed a general procedure that began by negating the assumption, which is indicative of an Action conception of proof by contradiction. He then stated:

Then let N equal the set n_1, n_2, n_3, n_k are natural numbers. *[long pause]* So how do you... how...

okay *[pause]* so I have not a set of... I have not a set of natural numbers and we're... I'm assuming that [sic] how I am going to contradict that to itself.

The phrase "I have not a set of natural numbers" suggests he interpreted $P(\mathbb{N})$ as $\mathbb{N}$, and thus suggests he would not have interpreted $\sim (P(\mathbb{N}))$ as $\mathbb{N} = \{n_1, n_2, n_3, \ldots, n_k\}$ without the presented proof. This undesired representation strengthens the interpretation that he followed an external procedure for the method as he does not understand what $\sim P(\mathbb{N})$ represents. It is also important to note the phrase "I'm assuming that [sic] how I am going to contradict that to itself". Here, we see the return of the assumption contradicting itself that would later manifest as his 3rd generalized step to proof by contradiction: $\rightarrow\leftarrow$ of $S = \sim S$.

Wesley then read the next line of the proof (n is not an element of $\mathbb{N}$) and exhibited difficulty reconciling the contradiction caused by the construction. In particular, he stated:

And let's see... and n is not an element of N, which is a contradiction. *[long pause, mumbling and reading next lines to self]* But that's *[pause]* but how do you... how do you map then $n = n_k + 1$ is a natural number and n is not an element in N?

He recognized that the proof stated there was a contradiction ($n \in \mathbb{N} \wedge n \notin \mathbb{N}$) and wrote this down. However, the phrase "how do you map then $n = n_k + 1$ is a natural number and n is not an element in N?" suggests he could not justify why n is not an element in $\mathbb{N}$ and thus reconcile how n is simultaneously both in and not in the set. In other words, he represented the contradiction desirably without understanding why the proof arrived at a contradiction, which is indicative of an Action conception of proof by contradiction.

Finally, after arriving at a contradiction, he returned to his procedure for proof by contradiction and stated "So then you would have not not *[pause]*." The phrase "not not", with no other description of the assumption, suggests he focused on key words to describe his procedure for proof by contradiction. That is, after arriving at a contradiction, the next step is "not not" the statement. Describing a key step in the proof procedure using cue words is indicative of an Action conception of proof by contradiction.

When asked to compare the four outlines to construct a new list of steps, Wesley instead described a general procedure (see Figure 5) similar to the one he developed during Classroom Discussion 3.

Statement: S
1. Assume $\sim S$ is true
2. Attempt to prove $\sim S$ is true
3. $\rightarrow\leftarrow$ of $S = \sim S$
4. $\sim (\sim S)$
5. S

Figure 5: Wesley's general procedure for proof by contradiction during Activity 4.

This outline described the two key steps of a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 3). However, lines 2 and 3 are not desirable as they both indicate circular reasoning. In addition, he previously represented a contradiction as $Q = \sim Q$ during

Exercise 3 in response to Question 8, which asked for the key steps of a particular proof. The teacher thus focused on line 3 in order to formalize the procedure.

First, the teacher pointed out that $S = \sim S$ was not present in any of the four outlines Wesley was presented to compare, to which he responded:

> So... and I don't see S equals not S, so I understand that. How many times, or I shouldn't say how many times... how *[pause]* often is that kind of, are you going to come to this kind of conclusion where you are not coming to a... you get your contradiction but it's not really just an S doesn't equal not S?

Wesley is in a state of cognitive dissonance. His first comment suggests that none of the proofs seen up until this point had a contradiction represented as $S = \sim S$. His second comment then suggests a contradiction represented as $S = \sim S$ is common with the phrase "but it's not really just an S doesn't equal not S?" The teacher then focused Wesley's attention specifically on the contradiction in the presented proof from Activity 2 (that a number was both even an odd) to clarify that the contradiction was not directly related to the statement and thus not representable as $S \wedge \sim S$. He then responded "But you got an odd and an even. *[...]* I'm used to seeing a comparison in that step." This response suggests that the representation $S = \sim S$ does not refer to the original statement and instead refers to a general comparison of two statements that cannot both be true. Yet, his previous use of phrases such as "contradict itself" suggest Wesley is attempting to explain his resilient misconception.

5.1.2. Wesley during Teaching Episode 5

Wesley focused on his previous procedure for proof by contradiction when asked to outline the new proof (see Figure 6).

Statement: P
1. Assume $\sim P$
2. $\sim P = P$ $(\rightarrow\leftarrow)$
3. $\sim (\sim P)$
4. P

Figure 6: Wesley's outline of presented proof during Activity 5.

This outline does not represent the proof as written. Instead, the outline focused on his previously constructed procedure for proof by contradiction, which is most evident by the inclusion of the improper representation $\sim P = P$. This equation was meant to represent the statement "If $\sim P$ leads to a contradiction, then $\sim P$ must be false" as illustrated by his comment:

> If not P leads to a contradiction, then P [sic] must be false. So I guess it would be *[long pause]* not P and just put arrows going to each other. *[gestures that the arrows point toward each other as $\rightarrow\leftarrow$]* So I guess it would be *[pause]* if not P leads *[pause]* so it would be not P equals P. And then you'd have the arrows going.

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Wesley first recited the line he planned to represent. He then represented the phrase "not P leads to a contradiction" as $\sim P \rightarrow\leftarrow$, which suggests he changed the representation for the new context of the presented proof. However, his phrase "so it would be not P equals P" changes his initial representation to the improper registration $\sim P = P$ from his procedure in Activity 4 (Figure 4 on page 14). The teacher then followed up on this representation, as illustrated below.

> Teacher: So, in this proof here, we didn't *[pause]* get a direct opposite of the statement we were trying to prove. It was kind of like, a little piece, a little side thing of that. So if we are trying to make this even more general, we get rid of not P equals P and we just say 'some contradiction'.
>
> Wesley: So *[long pause]* this is the only one we've done like this where you are using an approach to the original statement that's kind of off to the side.

As in Activity 4, the teacher prompted Wesley to recognize that the contradiction in this specific proof was not a direct negation of the statement (i.e., the contradiction was not $P \wedge \sim P$) by using non-mathematical jargon such as "direct opposite" and "little side thing" as the previous intervention was not successful. Wesley then retorted "this is the only one we've done like this" and implied all previously presented proofs had a contradiction that was directly related to the statement proved. He was made aware of two such proofs during Activity 4 though. This suggests he did not recognize a need to change his improper representation of a contradiction, $\sim P = P$, even when presented contradictory evidence. Moreover, this response suggests the representation $\sim P = P$ was not solely an improper representation of a contradiction (i.e., improperly representing the general contradiction $\sim P \wedge P$). In other words, the step $\sim P = P$ was an external rule that he focused on to the exclusion of other relevant evidence, referred to as *centration* (Piaget & Garcia, 1989). The focus on this particular step prevented him from properly conceptualizing what a contradiction can be in a proof by contradiction, which is indicative of an Action conception of the proof method and appears to be what prevents Wesley from developing a Process conception.

5.2. Yara

Yara was a senior[10] Mathematics major with a minor in Educational Psychology. Beyond the required prerequisite courses for Transition-to-Proof, she had already taken Mathematical Statistics, Methods of Regression and Analysis of Variance, Foundations of Numbers and Operations, and Applied Combinatorics. However, none of these courses required proof writing and thus Transition-to-Proof was her first experience with formal proofs. Yara's progression of understanding proof by contradiction reflected the preliminary genetic decomposition, as illustrated in Figure 7 and Figure 8. In particular, we see both her verbal definition and logical outline of proof by contradiction grow from relying on specific examples to describing the method

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[10]Senior by credits, but not prepared to graduate as a Mathematics major.

in general. We present evidence how Yara's understanding developed throughout the teaching experiment.

Teaching Episode 1:	Assuming something is not true, proving that it is.
Teaching Episode 2:	Because like we assumed *[long pause]* in our assumption, we assumed that our statement was true. So that negated that and then like, as you are doing the proof, we got to the contradiction, proving that our assumption was wrong and so our statement was true.
Teaching Episode 3:	Because you assume the statement wasn't true and then it would be *[pause]* you reached a contradiction.
Teaching Episode 4:	Assuming the negation of the statement and getting a contradiction.
Teaching Episode 5:	It's when you assume the statement isn't *[pause]* the original *[pause]* assuming the negation of the statement and getting a contradiction.

Figure 7: Yara's definitions for proof by contradiction during each Teaching Episode.

Task 1 (CD2)	Task 2 (CD3)	Task 3 (CD4)	Task 4 (CD5)
Statement: S	**Statement:** P	**Statement:** S	**Statement:** P
1. Assume $\sim S$	1. Assume $\sim P$	1. Assume $\sim S$	1. Assume $\sim P$
2. Rewrite $\sim S$	2. Negate P (Rewrite $\sim P$)	2. Do work	2. $\rightarrow\leftarrow$
3. Look at specific value of step 2.	3. Use math skills to get to a contradiction.	3. $\rightarrow\leftarrow$	3. P
4. Work (Algebra).	4. $\sim$ Assumption	4. S	
5. Get Contradiction.	5. P		
6. $\sim$ Assumption.			
7. S			

Figure 8: Yara's comparisons of the logical outlines constructed during Classroom Discussions 2-5.

5.2.1. Yara during Teaching Episode 3

Yara's outline (Figure 9) described the two key steps of a proof by contradiction (assuming the statement is false and arriving at a contradiction). In addition, she represented nearly every line in the proof, after which she then generalized the purposes of particular lines or sets of lines.

Statement: $\exists!x$ s.t. $P(x)$

1. Assume $\sim (\exists!x$ s.t. $P(x))$
2. $R \vee Q$
3. $\sim R$
4. Q
5. $5y - 4 = 1 \wedge 5z - 4 = 1$ (Algebra)
6. More algebra $(y = z)$
7. $\sim Q$
8. $\sim (R \vee Q)$
9. $\sim (\sim (\exists!x$ s.t. $P(x)))$
10. $\exists!x$ s.t. $P(x)$

Figure 9: Yara's logical outline of the presented proof during Activity 3.

Yara provided a desired representation of the statement as $(\exists!x)(P(x))$ and first line of the outline as $\sim (\exists!x)(P(x))$. Then, she switched to propositional logic and represented the statement "then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$"

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as $R \vee Q$. She successfully leveraged the propositional logic to understand the relation between lines in the proof. For example, when she reached the contradiction line in the presented proof, she stated:

> Then... this is a contradiction as we assumed that there is either no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. So it would be Q and not Q? Or would we not have to put that because we have it $[R \vee Q]$... It's already labeled out. *[...]* Okay, so then not... I was just trying to make sure I had it in my head right like, that $[R \vee Q]$ would go into not R and not Q.

Her first sentence quoted the line from the presented proof. She then immediately considered the representation $Q \wedge \sim Q$: the standard representation of a contradiction. Representing a statement by focusing on cue words (i.e., contradiction means $Q \wedge \sim Q$) is indicative of an Action conception of proof by contradiction. However, she then recognized that she would not represent this particular contradiction in this way as she already represented part of this contradiction as $R \vee Q$. Indeed, her final comment "I was just trying to make sure I had it in my head right like, that $[\sim (R \vee Q)]$ would go into not R and not Q" suggests that she recognized the logical equivalence $\sim (R \vee Q) \cong \sim R \wedge \sim Q$ and thus recognized that a contradiction of the representation $R \vee Q$ was reached. That is, she recognized and verbally described the logical reasoning behind how a contradiction was reached in this particular proof, which is indicative of a Process conception of proof by contradiction. In addition, she generalized lines 5 and 6 in her outline as "algebra" and thus described the purpose of the algebraic manipulations in the overall argument. Therefore, Yara exhibited a Process conception of proof by contradiction as she described the logical relation between lines in her outline and generalized specific lines to describe their purpose in the overall argument. It should be noted that a more obvious contradiction, $Q \wedge \sim Q$, could have been formed for step 8. The teacher did not probe Yara to explain why she choose the contradiction $\sim (R \vee Q)$.

When asked to define proof by contradiction, Yara stated "Because you assume the statement wasn't true and then it would be *[pause]* you reached a contradiction" to justify that the presented proof was a contradiction. She then stated "Because we assumed our statement was not *[pause]* not true, then we, then we brought, arrived at a contradiction, proving that the statement was true" as a general definition. This definition included the two key steps of the proof method: assuming the statement is false and arriving at a contradiction. When asked to explain why arriving at a contradiction would prove the statement is true, she stated "Because you get a contradiction, you negate the assumption, which gives you the statement." This explanation, listed as a series of steps, logically relates the two key steps of a proof by contradiction with the statement to be proved. Therefore, Yara exhibited a Process conception of proof by contradiction.

When prompted to compare the outlines she had seen in episodes 1 and 2, Yara grouped lines together and described a general purpose for each group of lines (see Figure 10).

Classroom Discussion 2 Statement: S	Classroom Discussion 3 Statement: P
1. Assume $\sim S$ 2. Rewrite $\sim S$ 3. Look at specific value of step 2. 4. Work (Algebra) 5. Get contradiction 6. $\sim$ Assumption 7. S	1. Assume $\sim P$ 2. Negate P (Rewrite $\sim P$) 3. Use math skills to get to a contradiction. 4. $\sim$ Assumption 5. P

Figure 10: Yara's Side-by-side procedures for proof by contradiction constructed during Classroom Discussion 2 (left) and Classroom Discussion 3 (right).

Yara's procedure for proof by contradiction contained both the key steps of a proof by contradiction and descriptions of how these key steps are logically related (e.g., that lines 3 and 7 logically implied line 8). In addition, it appears that Yara incorporated the procedure from Classroom Discussion 2 to help her deal with the presented proof during Activity 3. Consider the following exchange as Yara compared the desired outlines from Activities 2 and 3 to generalize the contradiction step.

> Yara: So I guess it just, maybe it likes, depends on the proof, and what you are trying to prove? Whether you do algebra or... umm... *[pause]*
> Teacher: So what do we do in that one *[outline during Activity 2]*?
> Yara: In this one, it says to use $P(x)$, get a contradiction. So we did algebra, right? So this one you do... which math skills do you use? Because math skills could mean plenty of things. It could be, like, one of them induction whatever...

Yara decided that the phrase 'math skills' included algebra as well as possibly other proof methods, such as induction. This is noteworthy as the presented proof does not include another proof method as a sub-proof, yet Yara generalized the step to include utilizing other proof methods. In doing so, Yara produced a minimal change on her previous procedure for proof by contradiction and thus incoporated the new outline into her Schema for proof by contradiction. Therefore, Yara exhibited an enhanced Process conception of proof by contradiction of her previous outline during Activity 3.

5.2.2. Yara during Teaching Episode 5

When asked to outline the proof, Yara focused on an abbreviated version of her previous procedure for proof by contradiction (see Figure 11).

Statement: P
1. Assume $\sim P$
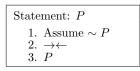2. $\rightarrow\leftarrow$
3. P

Figure 11: Yara's outline of presented proof during Activity 5.

This outline described the two key steps of a proof by contradiction in lines 1 and 2, after which the procedure was completed and thus the statement is true (line 3). It focused on the key steps of her previous procedure and did not represent statements such as "A statement P is either true or false," which is indicative of an

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Action conception of proof by contradiction. However, after outlining the proof, Yara stated "It's a proof for how to get a proof by contradiction?" This response suggests she considered acting on the proof method (i.e., proving the proof method), which is indicative of an Object conception of the proof by contradiction method. When asked to clarify what she meant, she instead stated it was slightly helpful "because it, like, kind of breaks down how you can get a proof by contradiction." The phrase "how you can get" may be a vague reference to validating the procedure, though it is not explicit enough to state she can consider acting on a proof procedure as a whole. Therefore, these two comments suggest Yara was developing toward an Object conception of proof by contradiction and could not yet describe how to validate the proof method.

When Yara was asked to compare her general procedure to the outline of the presented proof, she stated "Yeah, it's uh... *[long pause]* For the most part, it's the same. Minus one little section in between the assume not P and the contradiction we had 'Do work' to get the contradiction." The phrase "minus one little section" suggests she thought of the procedure as 'sections' and the line described the role of the 'section', which is indicative of a Process conception of proof by contradiction. It is possible Yara is de-encapsulating her Object conception of the proof by contradiction method into the Process the Object came from in order to compare her general procedure to the outline of the presented proof. In short, Yara is exhibiting *at least* a Process conception of the proof method.

5.3. Obstacles Along Cognitive Trajectory

Two prominent cognitive obstacles emerged from the case study analysis of Wesley and Yara's responses to tasks during the teaching experiment: the contradiction and considering valid (but not sound) arguments. The first cognitive obstacle, the contradiction, has been identified as one of the main cognitive difficulties of the proof method in that students exhibit difficulty identifying the contradiction of a proof, especially when it does not directly relate to the primary statement (Barnard & Tall, 1997; Antonini & Mariotti, 2009). However, another difficulty emerged from an analysis of Wesley's conception of proof by contradiction: **the contradiction as a direct negation of the statement to be proved**. That is, if S is the statement to be proved, then the contradiction would be $S \wedge \sim S$. To be clear, it is sometimes the case that a proof by contradiction will arrive at a contradiction that is directly related to the statement to be proved, such as when proving uniqueness claims. The belief that the contradiction will *always* be related to the statement, however, can appear to students as a form of circular reasoning and inhibit their conception of proof by contradiction as a valid proof method.

As noted previously when describing Wesley's responses to the Comparison tasks, he did not modify his general procedure for proof by contradiction throughout the entire teaching experiment. He remained steadfast in describing the contradiction in the procedure as directly related to the statement to be proved.

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There is some potential evidence Yara exhibited this same proclivity but was able to avoid fixating on it. During Teaching Episode 3, Yara rewrote the proof using her own propositional logic. We noted that a more obvious contradiction, $Q \wedge \sim Q$, could have been formed. The teacher did not probe Yara to explain why she choose the contradiction $\sim (R \vee Q)$. However, it is possible Yara chose this as it was the contradiction to her initial assumption that the statement to be proved was false. During Teaching Episodes 4 and 5, Yara made no attempts to relate the contradiction to the initial assumption. In this way, it appears there was some initial conception of the contradiction needing to relate to the initial assumption that was later refined.

The second cognitive obstacle, valid (and not sound) arguments, relates to a student's mathematical logic Schema. As noted in the introduction, proof by contradiction is one of two[11] prominent proof methods that rely on **valid but not sound proof** arguments. Antonini & Mariotti (2006) suggested that when students use either a false or indeterminate truth-valued statement, they no longer know what theorems, congruences, and proofs methods are available to be used. Yet, this would not happen if students conceptualized theorems, congruences, and proof methods as *valid* and not sound, as determining the truth-value of the premises is not necessary in a valid argument. Instead, this suggests students consider theorems, congruences, and proof methods as sound arguments that must then be questioned when dealing with non-true premises.

As noted previously when describing Wesley's responses to the Comparison tasks, he included the following two steps: (1) Assume $\sim S$ is true and (2) Attempt to prove $\sim S$ is true. This was explained as a form of circular reasoning as it 'attempts to prove' a statement that was already assumed true. This need to prove an assumption may also be indicative of viewing proof by contradiction as a sound argument. This would explain the motivation to prove a statement true that was already assumed to be true as a sound argument requires the premise to be true. Both Wesley and Yara initially attempted to verify the truth value of every statement in the presented proofs, such as confirming that the negation of the statement to prove was false with examples. This verification even occurred with statements of unknown truth value and so the study was not able to closely examine how students understand proofs with statements of unknown truth value. Therefore, this cognitive difficulty remains a theoretical explanation of difficulties exhibited by Wesley and eventually overcome by Yara.

6. Conclusion and Implications

Returning to the research question, the preliminary genetic decomposition provided a reasonable cognitive trajectory for proof by contradiction. Students appear to first memorize an externally-provided list of steps for the method that are eventually described in their own words as the key steps of a proof by contradiction.

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[11]The other being proof by mathematical induction.

Wesley's attempts to interiorize the procedure resulted in a fixation on the contradiction directly relating to the statement to be proved. Yara was able to compare multiple examples of contradiction proofs to eventually describe the three key steps of a proof by contradiction and how they logically relate to one another. The other five students who completed teaching episode 3 followed similar trajectories to these two students, though it is unknown whether the trajectories would have diverged in teaching episodes 4 and 5. Thus, the preliminary genetic decomposition shows promise for describing how students could develop an understanding of proof by contradiction. This understanding would be expanded by encountering different types of proofs through the rest of their undergraduate studies, such as the multi-quantified statements common in Analysis or proof by contradiction as a sub-proof.

6.1. Revised Genetic Decomposition

While the results of this study support the constructions called for by the preliminary genetic decomposition, they also suggest that some steps need to be refined and that a new step should be added. In particular, the results suggest additional prerequisite knowledge students should possess in order to begin developing an understanding of proof by contradiction: valid (and not sound) arguments. This knowledge is necessary to understand the validity of the proof method and, without it, may result in a cognitive rejection of the method.

The steps that are new or revised in the following genetic decomposition are indicated in bold:

1. Action conception of propositional or predicate logic statements as specific step-by-step instructions to construct proofs by contradiction for the following types of statements: (I) implication, **(II) non-existence, and (III) uniqueness;**
2. Interiorization of each Action in step 1 individually as general steps to writing a proof by contradiction for statements of the form (I), (II), and (III).
3. Coordination of the Processes from step 2 as general steps to writing a proof by contradiction.
4. **Incorporate new proof by contradiction procedures (e.g., for infinity, property claim, or single-level quantified statements) into one's Process conception from step 3 to enhance the general steps to writing a proof by contradiction.**
5. Encapsulate the Process in **step 4** as an Object by utilizing the law of excluded middle to show proof by contradiction is a valid proof method or by comparing the contradiction proof method to other proof methods.
6. De-encapsulate the Object in **step 5** into a Process similar to **step 4** that then coordinates with a Process conception of **other proof methods to prove statements that require two or more proof methods**.

In step 1, the types of statements to develop step-by-step instructions for were reduced to only implication, non-existence, and uniqueness. This was changed as students in this study were able to construct a general procedure for any type of statement after constructing and comparing these three specific procedures. To be clear - students' conception of proof by contradiction should be reinforced through reflection on single-level and multi-level quantified statements. Our results suggest this is not a necessary component to a

generalized proof by contradiction procedure. This hypothesis would need to be tested in any follow-up study by presenting students with universal and existential single-quantified proofs by contradiction.

A new step was added between constructing a general procedure for any type of proof by contradiction and encapsulating these general steps into a static Object that can be acted on. This step described how students incorporate the specific procedures for infinity and property claim statements into their existing general procedures for proof by contradiction.

Finally, step 5 was changed to describe how students could prove statements that required two or more proof methods. This change more accurately described how students attempted to write a proof for the statement "If a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$."

6.2. Limitations and Future Research

Four major limitations were identified for this study, leading to ways the study could be extended for future research. First, mental processes such as understanding are impossible to observe directly. While this study used a cognitive framework to help explain how students' observable actions illustrated their understanding of particular proofs and of proof by contradiction in general, there is no way to make definitive statements about their conceptions. This was most prevalent with Wesley, as it initially appeared he re-conceptualized proof by contradiction as not necessarily being a contradiction involving the statement to be proved. The data used in this study could be re-examined utilizing a different theoretical framework to make stronger claims about the exhibited conceptions of the participants.

Related to the first limitation, another major limitation of this study was the sample size. Only two students (Wesley and Yara) completed all 5 teaching episodes and provided data for how a student might develop an understanding of proof by contradiction over time. More data would be necessary to continue refining the ACE teaching cycle activities and the genetic decomposition. More data would also allow us to provide more evidence for evaluating students' conceptions of proof by contradiction and potential cognitive obstacles. However, Yara provided evidence that the proposed genetic decomposition has the potential to describe how students might develop a robust understanding of proof by contradiction.

The authors used a proof that the Strong Goldbach Conjecture implies the Weak Goldbach Conjecture to examine the effects of an unknown truth-value statement within a proof. The effect appeared lost on students as they assumed both conjectures were true. In another iteration, we would present the conjectures alone first to analyze how valid and sound arguments contribute to understanding proof by contradiction.

Finally, the authors had no control over the curriculum and instruction of the introduction to proof course beyond the first two teaching episodes completed in class. It is possible students' understanding was developed based on in-class instructional tasks and not the tasks in teaching episodes 3-5. However, there

was evidence that Yara developed and refined her understanding of proof by contradiction *within* some of the episodes - especially episodes 3 and 5. To alleviate this concern in a future study, the authors would need to be able to complete the entire teaching experiment in-class.

A related avenue for future study is how students develop an understanding of proof by induction. Within this method, students must suspend their perceived knowledge of the statement to be proved and prove that the k term of the statement being true implies the $k + 1$ term of the statement being true. We see two similarities with proof by contradiction: the assumed truth-value directly related to the statement and a valid (but not sound) argument being used. As these similarities coincide with two major cognitive obstacles observed by the participants of this study, a more robust understanding of one method may be beneficial to developing an understanding of the other.

7. Funding Sources

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8. Appendix

The following instruments were used in Fall 2016. Response space was removed to conserve space.

Activity 1

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \rightarrow \boldsymbol{Q}]$.

Proof 1: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\sim (\boldsymbol{P} \rightarrow \boldsymbol{Q})]$. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \wedge \sim \boldsymbol{Q}]$. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k $[\sim \boldsymbol{Q}]$. Then $k = 2n + 1$. Since $k > 5$, $k - 3 > 2$. Thus $k - 3 = 2n - 2 > 2$ and $k - 3$ is even. By our assumption, $k - 3$ is then the sum of two primes: p and q. Thus $k - 3 = p + q$. Solving for k, we get $k = p + q + 3$ $[\boldsymbol{Q}]$. This is a contradiction, as we assumed k was not the sum of three primes $[\boldsymbol{Q} \wedge \sim \boldsymbol{Q}]$. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\sim (\sim (\boldsymbol{P} \rightarrow \boldsymbol{Q}))]$. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \rightarrow \boldsymbol{Q}]$.

1. Using the symbols in the statement and proof above, outline the proof.

2. In your own words, state a definition of "proof by contradiction". Then, explain why Proof 1 is a proof by contradiction.

Discussion 1

Proof by Contradiction for Implications $(P \rightarrow Q)$:

1. Assume $\sim (P \rightarrow Q)$
2. $P \wedge \sim Q$
3. $\sim Q$
4. Get to Q *this may take more than one step
5. $Q \wedge \sim Q$
6. $\sim (\sim (P \rightarrow Q))$
7. $P \rightarrow Q$

Things to discuss:

- Structure above - Students should agree on something similar to the structure above. Students may include the specific algebraic manipulations - try to have them generalize these specifics to the "Get to Q" step.

- Logical flow of structure - In particular, focus on how step 1, 2, and 3 logically follow but how 4 is separate from these 3. The focus on how steps 6 and 7 are logically equivalent.

- Definition of contradiction - Students seem to conflate negation and contradiction. Be sure to have students describe the difference.

- Definition of proof by contradiction - Try to get students to recognize the important steps of a proof by contradiction: (1) assume not statement, (2) get contradiction, (3) thus statement.

Exercise 1

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

Proof 1: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k. Then $k = 2n + 1$. Since $k > 5$, $k - 3 > 2$. Thus $k - 3 = 2n - 2 > 2$ and $k - 3$ is even. By our assumption, $k - 3$ is then the sum of two primes: p and q. Thus $k - 3 = p + q$. Solving for k, we get $k = p + q + 3$. This is a contradiction, as we assumed k was not the sum of three primes. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

1. Please give an example of a prime number and explain why it is prime.

2. Why is every even natural number greater than 2 the sum of two primes?

3. Why exactly can one conclude that $k - 3$ is the sum of two primes?

4. What is the purpose of the statement "Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes."?

5. Which of the following **best** summarizes the main idea of this proof? Explain your choice.

   (a) The main idea of the proof is to show that if there exists an odd natural number greater than 5 that is not the sum of three primes, one could find three primes that sum to be that number, contradicting the assumption.

   (b) The main idea of the proof is to assume that every even natural number greater than 2 is the sum of two primes and there exists an odd natural number greater than 5 that is not the sum of three primes, and then to show that the odd natural number can be written as three primes, which is impossible.

6. What do you think are the key steps of this proof?

7. In this proof, we subtracted 3 and worked with $k-3$. Would the proof still work if we instead subtracted 5 and worked with $k - 5$? Why or why not?

8. Using the method of this proof, show that: if every odd natural number greater than 5 is the sum of three primes and one of those primes is 3, then every even natural number greater than 2 is the sum of two primes.

Activity 2

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

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Statement 1: There is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. $(\not\exists \boldsymbol{x})[\boldsymbol{P(x)} \wedge \boldsymbol{Q(x)}]$

Proof 1: Assume it is not true that there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{Z}$ such that $n = 4j - 1$ and $n = 4k + 1$ for $j, k \in \mathbb{Z}$. Then $4j - 1 = 4k + 1$ and so $2j = 2k + 1$. Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.

2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

3. Read the following statement:

   Statement 2: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of 3 primes.

   Compare Statement 1 and Statement 2. How are they similar? How are they different?

4. Read the following proof:

   Proof 2: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k. Then $k = 2n + 1$. Since $k > 5$, $k - 3 > 2$. Thus $k - 3 = 2n - 2 > 2$ and $k - 3$ is even. By our assumption, $k - 3$ is then the sum of two primes: p and q. Thus $k - 3 = p + q$. Solving for k, we get $k = p + q + 3$. This is a contradiction, as we assumed k was not the sum of three primes. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

   Compare Proof 1 and Proof 2. How are they similar? How are they different?

Discussion 2

Proof by contradiction for nonexistence $(\not\exists x)[P(x) \wedge Q(x)]$:

1. Assume $\sim ((\not\exists x)[P(x) \wedge Q(x)])$

2. $(\exists x)[P(x) \wedge Q(x)]$

3. $P(n)$

4. Using $P(n)$, get a contradiction.

5. $\sim (\sim (\not\exists x)(P(x)))$

6. $(\not\exists x)(P(x))$

Things to discuss:

- Structure above - Students should agree on something similar to the structure above. Students should be encouraged to use quantification symbols in order to contrast the structures for nonexistence and uniquneness (the next type of statement).
- Logical flow of structure - In particular, focus on how step 1, 2, and 3 logically follow but how 4 is separate from these 3. Then focus on how steps 5 and 6 are logically equivalent.
- Definition of proof by contradiction - Try to get students to recognize the important steps of a proof by contradiction: (1) assume not statement, (2) get contradiction, (3) thus statement. Associate these parts with the steps in the proof.

Exercise 2

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: There is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k.

Proof 1: Assume it is not true that there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{N}$ such that $n = 4j - 1$ and $n = 4k + 1$ for $j, k \in \mathbb{Z}$. Then $4j - 1 = 4k + 1$ and so $2j = 2k + 1$. Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k.

1. Please give an example of an integer that is odd and explain why it is odd.

2. What kinds of numbers can be expressed in the form $4j - 1$?

3. Why exactly can we assume "there is an odd integer n such that $n = 4j - 1$ and $n = 4k + 1$ for integers j and k."?

4. What is the purpose of the statement "Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number."?

5. Which of the following **best** summarizes the main idea of this proof? Explain your choice.

   (a) The main idea of the proof is to show that if an odd integer could be expressed in the form 4j-1 and 4k+1, one could find a number that is both even and odd, contradicting the assumption.

   (b) The main idea of the proof is to assume that an odd integer could be written in the form 4j-1 and 4k+1, and then to show 2j is both even and odd, which is impossible.

6. What do you think are the key steps of the proof?

7. In the statement, we have $4j - 1$ and $4k + 1$. Would the proof still work if we instead say no odd integer can be expressed in the form $4j - 3$ and in the form $4k + 3$? Why or why not?

8. Using the method of this proof, show that there is no odd integer that can be expressed in the form $8j - 1$ and in the form $8k + 1$ for integers j and k.

Activity 3

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The equation $5x - 4 = 1$ has a unique solution.

Proof 1: Assume the equation $5x - 4 = 1$ does not have a unique solution. Then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Note $x = 1$ is a solution of $5x - 4 = 1$. Thus there are at least two distinct solutions to the equation $5x - 4 = 1$, call them y and z. As both y and z are solutions of the equation $5x - 4 = 1$, $5y - 4 = 1$ and $5z - 4 = 1$. Then $5y - 4 = 5z - 4$ and so $y = z$. Therefore it is not true that there are at least two distinct solutions to the equation $5x - 4 = 1$. This is a contradiction, as we assumed that either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Therefore it is not true that the equation $5x - 4 = 1$ does not have a unique solution. In other words, the equation $5x - 4 = 1$ does have a unique solution.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.
3. Look at your responses for question 2 in Activities 2 and 3. Can you write a list of steps to prove *any* type of statement by contradiction?
4. Using the steps in question 3, try to write an outline proof for the following statement:

   Statement 2: The multiplicative inverse of a non-zero real number x is unique.

Discussion 3

Proof by contradiction for uniqueness $(\exists! x)(P(x))$:

1. Assume $\sim (\exists! x)(P(x))$
2. $(\forall x)(\sim P(x)) \ \lor (\exists x)(\exists y)(P(x) \ \land P(y) \ \land x \neq y)$
3. $\sim (\forall x)(\sim P(x))$ by finding a such that $P(a)$
4. $P(m) \ \land P(n) \ \land m \neq n$
5. $\sim ((\exists x)(\exists y)(P(x) \ \land P(y) \ \land x \neq y))$ by showing $m = n$
6. $\sim ((\forall x)(\sim P(x)) \ \lor (\exists x)(\exists y)(P(x) \ \land P(y) \ \land x \neq y))$
7. $\sim (\sim (\exists! x)(P(x)))$
8. $(\exists! x)(P(x))$

Proof by contradiction (in general) for statement P:

1. Assume $\sim P$.
2. Rewrite $\sim P$ to a form we can use
3. Use $\sim P$ to get to a contradiction $Q \land \sim Q$.
4. Negate our assumption: $\sim (\sim P)$.
5. Therefore P.

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Things to discuss:

- Structure above - Students may get bogged down in the quantification for uniqueness (especially line 2). If this happens, have students write the statement without quantifiers. While this will make the comparison with nonexistence weaker, it may help the student recognize the general structure for proof by contradiction.

- Logical flow of structure - In particular, focus on how steps 1, 2, and 3 logically follow but how 4 is separate from these 3. Then focus on how steps 5 and 6 are logically equivalent.

- Definition of proof by contradiction - Try to get students to recognize the important steps of a proof by contradiction: (1) assume not statement, (2) get contradiction, (3) thus statement. Associate these parts with the steps in the proof.

- Outline of the proof - This encourages students to actively access their conception of proof by contradiction in order to construct a proof.

Exercise 3

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 2: The multiplicative inverse of a non-zero real number r is unique.

Proof 2: Assume the multiplicative inverse of an arbitrary non-zero real number r is not unique. Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Note $x = \frac{1}{r}$ is a multiplicative inverse of r. Thus there are at least two distinct multiplicative inverses of r, call them x and y. As both x and y are both multiplicative inverses of r, $rx = 1$ and $ry = 1$. Then $rx = ry$ and so $x = y$. Therefore it is not true that there are at least two distinct multiplicative inverses of r. This is a contradiction, as we assumed that either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Therefore it is not true that the multiplicative inverse of an arbitrary non-zero real number r is not unique. In other words, the multiplicative inverse of a non-zero real number r is unique.

1. Compare the outline of your proof in question 4 to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach to the proof.

2. Please give an example of a multiplicative inverse of a non-zero real number and explain why it is a multiplicative inverse.

3. Why does r have to have a multiplicative inverse?

4. Why exactly can one conclude that $x = y$?

5. What is the purpose of the statement "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r."?

6. Summarize in your own words the main idea of this proof.

7. What do you think are the key steps of the proof?

8. Would the proof still work if we instead say the multiplicative inverse of a real number x is unique? Why or why not?

9. Using the method of this proof, show that: if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Activity 4

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The set of natural numbers is not finite.

Proof 1: Assume there are not infinitely many natural numbers. Then there are finitely many natural numbers. Let $N = \{n_1, n_2, n_3, \ldots, n_k\}$ be all the natural numbers, where $n_1 < n_2 < n_3 < \ldots < n_k$. Then $n = n_k + 1$ is a natural number and $n \notin N$, which is a contradiction. Therefore there are infinitely many natural numbers.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.
3. Look at your responses for question 2 in Activities 2-4. Can you write a list of steps to prove *any* type of statement by contradiction?
4. Using the steps in question 3, try to write an outline proof for the following statement:

   Statement: The set of primes is infinite.

Discussion 4

Proof by contradiction for infinitely many x such that $P(x)$:

1. Assume (there are not infinitely many x such that $P(x)$)
2. There are finitely many x such that $P(x)$
3. Using the previous step, find a contradiction
4. There are infinitely many x such that $P(x)$

Proof by contradiction (in general) for a statement P:

1. Assume $\sim P$.
2. Find a contradiction $Q \wedge \sim Q$.
3. Conclude P.

Things to discuss:

- Structure above - Students may not know how to write infinitely many as a quantified symbol. This should encourage students to avoid the quantification and write the general structure without it.
- Logical flow of structure - Without the symbolic structure, students may not recognize the logical connections between steps. In particular, the negation of the "assume" statement has been removed. Ask students whether it was necessary.
- Definition of proof by contradiction - The structure should now highly resemble the structure of a general proof by contradiction. Ask students how an "infinitely many" statement differs from the other types of statements they have proved so far.
- Outline of the proof - This encourages students to actively access their conception of proof by contradiction in order to construct a proof. It also helps students recognize the most difficult aspect of proof by contradiction (knowing what the contradiction will be).

Exercise 4

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The set of primes is infinite.

Proof 1: Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$. Let n be one more than the product of all of them. That is, $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Then n is a natural number greater than 1, so n has a prime divisor q.

Since q is prime, $q > 1$. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are all the primes, q is one of the p_i in this list. Thus, q divides the product $p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$. Since q divides n, q divides the difference $n - (p_1 p_2 p_3 \ldots p_k)$. But this difference is 1, so $q = 1$. From the contradiction, $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

1. Compare the outline of your proof in question 4 to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach to the proof.

2. Please give an example of a set that is infinite and explain why it is infinite.

3. Why does n have to have a prime divisor?

4. Why exactly can one conclude that q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k)$?

5. What is the purpose of the statement "Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$."?

6. Summarize in your own words the main idea of this proof.

7. What do you think are the key steps of the proof?

8. In the proof, we define $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Would the proof still work if we instead defined $n = (p_1 p_2 p_3 \ldots p_k) + 31$? Why or why not?

9. Define the set $S_k = \{2, 3, 4, \ldots, k\}$ for any $k > 2$. Using the method of this proof, show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in S_k.

Activity 5

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: Statement P is true.

Proof 1: A statement P is either true or false. Assume P is false; that is, the negation of P, $\sim P$, is true. If $\sim P$ leads to a contradiction, then $\sim P$ must be false. This implies our initial assumption was not true; that is, it is not true that P is false. Since P is either true or false and P is not false, P is true.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.

2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

3. Look at your response for question 2 in Activity 4. Does the proof above follow the steps you wrote for a proof by contradiction? If so, explain why. If not, revise your steps and explain why the previous steps did not work.

4. Using the steps in question 3, try to write a proof of the following statement:

   Claim: $\sqrt{2}$ is an irrational number.

Discussion 5

Proof by Contradiction (General Form)
Statement: P

1. Assume $\sim P$

2. Show $\sim P$ leads to a contradiction $Q \wedge \sim Q$

3. Claim $\sim (\sim P)$

4. Conclude P

Things to discuss:

- This proof illustrates why proof by contradiction is a valid proof method. Try to get students to recognize this.

- Logical flow of structure - Try to get students to describe the logic of the proof method *as a whole* rather than line-by-line.

- Definition of proof by contradiction - Try to get students to recognize the important steps of a proof by contradiction: (1) assume not statement, (2) get contradiction, (3) thus statement. Associate these parts with the steps in the proof.

- Writing a proof - Encourage students to first use the structure of a proof by contradiction and then fill in the details.

Exercise 5

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement: $\sqrt{2}$ is an irrational number.

Proof: Suppose $\sqrt{2}$ is a rational number. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}, q \neq 0$, and $\gcd(p, q) = 1$. This implies $p^2 = 2q^2$ and so p^2 is even. Now, if p were odd, then p^2 would be odd. Thus p must be an even number and so $p = 2r$ where $r \in \mathbb{Z}$. Since $p = 2r$ and $p^2 = 2q^2$, $4r^2 = 2q^2$ and so $2r^2 = q^2$. By the same argument as above, q must also be even number. Since both p and q are even, $\gcd(p, q) \neq 1$. From the contradiction that $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$, we conclude that the assumption that $\sqrt{2}$ is a rational number is false. Therefore, $\sqrt{2}$ is an irrational number.

1. Compare your proof in question 4 to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach ot the proof.

2. Please given an example of a number that is irrational and explain why it is irrational.

3. What does $\gcd(p, q) = 1$ mean and why can we conclude $\gcd(p, q) = 1$?

4. Why exactly can one conclude that p^2 is an even number?

5. What is the purpose of the statement "Now, if p were odd, then p^2 would be odd."?

6. How exactly can one conclude that q is an even number?

7. Summarize in your own words the main idea of this proof.

8. What do you think are the key steps of the proof?

9. Using the method of this proof, show that for any prime number n, $\sqrt{n}$ is an irrational number.

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