A Framework for Analyzing Asynchronous Discussion Activities

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Abstract

While discussion activities are widely utilized in face-to-face modalities, little is yet known about how students develop and leverage mathematical meanings in asynchronous online mathematics courses. We report on the results of an iterative design experiment aimed at analyzing the cognitive and social dynamics of students' co-construction of knowledge in asynchronous mathematics discussion activities. From this iterative research, we have developed an analytical framework that enables flexible exploration of the complex dynamics of an unfolding discussion activity. We elaborate our analytical process, highlight ways that instructors might benefit from such analysis, and conclude by posing questions for retrospective analysis of asynchronous discussions from the lens of an instructor, a course coordinator, and a course designer.

Introduction

Classroom discourse is a powerful tool for facilitating students' development of robust mathematical meanings and is an often implemented in face-to-face instruction, along with other active-learning pedagogies. We are motivated to explore the learning within asynchronous discussions in more detail. Discussion activities, if implemented correctly, can serve as a primary setting in which connections between students, their peers, and their instructor can be leveraged to create a robust learning environment. Towards this end, we have conducted research on the learning taking place within the context of asynchronous discussion activities. While our initial analyses have focused on asynchronous calculus courses, we find that the implemented design principles, and more importantly the insights gained from our research, have broader implications for mathematics courses in asynchronous formats.

Despite the growing number of students enrolling in asynchronous mathematics courses, we still know very little about the ways that learning occurs in this modality (Trenholm, Peschke, & Chinnappan, 2019). The need for further examination has become more necessary with the growth of online courses in the wake of the COVID-19 pandemic. Within the small corpus of literature on teaching and learning in online formats, researchers have identified mathematics courses as among the more difficult to teach asynchronously (e.g. Engelbrecht & Harding, 2005). To address this difficulty, we focus specifically on discussion activities as opportunities for students to develop robust mathematical meanings within a supportive and collaborative community of discourse. We will also present a framework to rigorously assess the social and cognitive contributions of students and instructors over a period of mathematical discourse. This Analytical Framework enables educators to engage in exploratory analysis around the discourse in their online discussion boards.

We will begin by considering what is already known regarding asynchronous discussions and ways to measure aspects of student discourse. We continue with ways others might utilize our Analytical Framework and give an example of an exploratory analysis from our own work. We conclude by discussing ways that instructors, course coordinators, and course developers might benefit from employing such analysis.

Literature Review

Engaging students in discourse in a classroom setting can powerfully provide opportunities to gain mathematical knowledge (Erath et al., 2021). Researchers agree that among other beneficial active learning pedagogies, interactive in-class discussions can enable students to develop robust and productive ways of understanding and reasoning in mathematics (Howe et al., 2019). In fact, numerous learning trajectories relying on group discussions have been developed and have been seen to benefit students (Borji & Martinez-Planell, 2019). Discussion activities provide opportunities for students to gain crucial mathematical understanding, as well as critical thinking, communication, and problem-solving skills. Despite this, there is a gap in the literature concerning the development of meaningful discussion activities for students learning mathematics in an asynchronous format.

Little is known about the ways students develop mathematical conceptions in online courses (Trenholm, Perschke, & Chinnappan, 2019). From what has been explored thus far, some have considered mathematics courses among the more difficult to teach in an online format (Engelbrecht & Harding, 2005). While discussion activities are commonly utilized as a face-to-face active learning pedagogy, discussions occur less often in asynchronous mathematics courses. In a survey examining the design and implementation of asynchronous courses, 39% of instructors surveyed used at least 1 discussion (Trenholm et al., 2019)[1]. This situates the learning taking place at an individual level, where students interact with the course materials in isolation. If implemented carefully, online students may benefit greatly from application of the teaching methods developed in face-to-face settings.

Much of the research thus far focusing on asynchronous discussion activities has given descriptions of activity type and format (e.g. Gao et al., 2013), general best practices for instructor actions (e.g. De Noyelles et al., 2014), quantitative aspects of students' behaviors such frequency and length of posts (e.g., Yang et. al., 2011), and content-agnostic studies of discourse

[1] Note, the aforementioned studies all took place before the onset of the COVID-19 pandemic, which resulted in a grater perception of the viability of online instruction and likely resulted in a lasting increase to the numbers of students taking online mathematics classes, further suggesting a need to improve the quality of online teaching methods.

dynamics (e.g. Lee & Recker, 2022). Some of the research specifically focusing on the learning taking place in asynchronous discussion activities has utilized Weinberger and Fischer's (2006) Argumentative Knowledge Construction (AKC) framework. Thus far, studies of asynchronous discussions drawing from the AKC framework have examined learning in non-mathematical courses (Schrire, 2006; Clark & Sampson, 2008; Dubovi & Tabak, 2020). While some mathematics-focused preliminary research utilizing AKC has explored students' learning face-to-face mathematics settings (Williams, Lopez Torres, & Keene, 2019) little is yet known about the mechanisms of students learning mathematics in asynchronous discussion formats.

One shortcoming of the AKC framework and other similar frameworks is their prescriptive nature, which requires some particular analytical method paired with a particular theoretical framework. This is perfectly reasonable when operationalizing such frameworks in research but limits their usefulness to practitioners. Instead, we provide an analytical method usable as a tool for flexible, exploratory, and interest-driven analysis that can be paired with various theoretical frameworks. Our hope is that our Analytical Framework might be utilizable by instructors, course mentors, and course designers desiring to employ various educational lenses to examine their own targeted avenues of inquiry about the knowledge being constructed in their discussion activities in a systematic way.

Theoretical Lens and Frameworks

There are two frameworks at play in our research: a *theoretical framework* that describes how one identifies student knowledge and an *Analytical Framework* that describes a systematic way to explore identified student knowledge. The primary contribution of this chapter is the *Analytical Framework*, which we find to have great potential for use in instruction, coordination, and course design. One prepares to use the Analytical Framework by first enacting an initial coding process on the discussion data at a micro-level (i.e. post-by-post or sentence-by-sentence). One then employs the Analytical Framework by aggregating the corpus of micro-level codes into a network that allows for visually based meta-analyses of the discourse.

The methods by which one generates and applies codes to the discussion data are quite flexible, and we do not prescribe a specific theory for use of our Analytical Framework. We will, however, describe our coding process, which was enacted primarily through application of our *theoretical framework*, which we refer to as Mathematical Argumentation and Meanings Interaction (MAMI) and is based on applying the Argumentative Knowledge Construction (AKC) framework to the mathematics context. As there are multiple frameworks and lenses at play in our chapter, we provide, in Table 1, a breakdown of the acronyms and terms used throughout the paper and a brief description of their role in our work.

Before describing the theoretical framework itself, we provide a brief summary of our epistemological lens motivating the development of MAMI.

Background Epistemological Lens Motivating the Research

Our development of the Analytical Framework is situated within the theoretical backdrop of a *meanings-oriented* lens for instructional design (Reed, Tallman, & Oehrtman, 2023). Here, we unpack the tenets of this instructional design lens.

Adopting a *meanings-oriented* lens for instruction first entails an underlying recognition that students' knowledge of the content in mathematics courses comprises an interwoven collection of personal and unique mathematical meanings and ways of thinking that are brought to bear in the solving of mathematics problems[2]. This lens is an outworking of constructivist

[2] This contrasts the perspective that one's content understanding is a collection of partitioned groupings of declarative knowledge and set behavioral procedures to be rehearsed and

epistemologies (Piaget, 1971; Glasersfeld, 1990), and the more recent work of Harel (2008) and Thompson (2014) to delineate one's understanding from a "normative understanding" or a "correct understanding".

Table 1:Mathematical Argumentation and Meaning Interaction (MAMI) Theoretical Framework.

Framework or Lens	**Brief Description**
Meanings-Oriented Instruction	The lens through we engage in instructional design. This lens heavily draws from an epistemological foundation of radical constructivism (Paiget, 1970; Glasersfeld, 1991) applied to mathematics education.
Mathematical Argumentation and Meanings Interaction (MAMI)	Our particular framework for applying micro-level codes to discussion data. The framework identifies instances of conveyed mathematical meaning, mathematical justifications, and argumentative moves made by discussion participants.
MAMI: Argumentation Dimension (A)	A decomposition of argument formation taken from Toulmin's (1964) framework. Applied at both sentence-level (micro) and to the argumentative role of the post in the discussion (macro).
MAMI: Meanings and Justification Dimension (MJ)	Mathematical meanings, derived from the mathematics education research literature. Meanings coded as expressed in students' sentences (micro) and as expressed in present justifications within the post (macro).
Analytical Framework	The main contribution and ultimate focus of this chapter. An aggregation of micro-level codes into a network structure enabling meta-analyses of emergent discursive themes.

memorized during instruction and recreated on an assessment, commonly derived from behaviorist epistemologies.

To operationalize this for instruction, we must recognize that "…knowledge persists because it has proved viable in the experience of the knower" (Thompson, 2008, p. 44). It is thus the responsibility of educators to provide students with experiences where target knowledge is useful (practically or academically) rather than expecting students to learn because the instructor said to do so.

The words *meaning, understanding,* and *way of thinking* are widely used throughout education without precise definitions. Our use of the terms, based on Thompson et. al. (2014), is provided here to describe exactly what we intend *meanings* to be in *meanings-oriented instruction*. When experiencing an event, a thinker filters their interpretation of (and possible actions taken in response to) the event through a lens formed by mental organizations of processed past events[3]. There are various possible implications of an experiential event in terms of actions to take or other inferences to make about present related experiential events. For instance, implications of experiencing "warmness" might be "The sun must be out today", "I should move further from the fire", "It's nice being inside while it rains outside", or "I'll set the temperature when I get home". The collection of implications afforded a thinker by their lens is what we call a *meaning* of the experiential event, and the organization of processed past events comprising a thinker's lens is what we call an *understanding*.

A thinker has a *way of thinking* when they habitually operationalize certain meanings when reasoning about an idea (Thompson et al., 2014). Note that one's understandings and ways of thinking co-mingle and overlap, and identification and delineation of meanings are more of a

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[3] This is our very loose description of what we would formally call an *assimilation* to a *scheme* (Piaget, 1971).

useful tool to educators and researchers than they are a description of discretized isolated collections of knowledge.

While individual's understandings are unique and inaccessible by another, we as educators and researchers can infer features of one's meanings by observing what is said or done, and then use a model comprising these features for use in research or teaching (Steffe & Thompson, 2000; Thompson, 2008). For instance, one model - that we utilize in instruction and research - describing commonly inferred features of students' meanings for constant rate of change (CROC) is a *perceivable geometric property* meaning[4]. Students might describe CROC in terms of the "straightness" or "steepness" of a line, or perhaps whether a line is "tilted up" or "tilted down" corresponding to a "positive slope" or a "negative slope" respectively. Actions and implications mobilized by this meaning entail making associations between the perceivable features of the graph of a line in Cartesian coordinates and other aspects of the problem in which the line is being applied, perhaps as related to features of the standard formula for such a line.

Similarly, students might understand constant rates of change (CROCs) in terms of *geometric, contextual, or algebraic ratios* (Nagle et al., 2013). The CROC itself (e.g. speed, slope, etc.) can be understood as a ratio of two related quantities, perhaps the related measures of changes in distance and time of a moving object. Students might also put other aspects of the context in ratio, such as the values of the quantities. We might infer such meanings to students who numerically or symbolically compute or refer to such ratios in the context of solving or describing the solutions to problems involving CROC.

[4] Derived from Nagle, Moore-Russo, Viglietti and Martin's (2013) notion of a *physical property* conceptualization of slope.

In teaching, we regularly infer students' meanings as we observe what they say, do, or write. Traditional methods of instruction that lean on a correct versus incorrect dichotomy can limit the ability of an instructor to attend to student thinking when teaching. Consider traditional correct/incorrect grading on assessments – while instructors might infer top-level meanings from the *final product* of student thinking, it provides no insight on the process of student thinking.

Employing a meanings-oriented perspective in instructional design, then, entails being `` cognizant of the meanings for which [one seeks] to hold students accountable…", building tasks that require students to adopt or develop one's targeted meanings for successful completion, and to generate curricular activities "…that *elicit observable products* of students' application of these meanings" (Reed, Tallman, & Oehrtman, 2023, emphasis added). Note we emphasize products of student thinking *during* the problem-solving process.

Theoretical Framework: Mathematical Argumentation and Meaning Interaction (MAMI)

Now that we have described and defined *meanings-oriented* instruction, we can turn to what meaning is in a mathematical context. More specifically in our case, what meaning is in an asynchronous mathematics discussion. While we leveraged the Mathematical Argumentation and Meaning Interaction (MAMI) framework in initial coding, investigators should feel free to utilize whatever framework they find to be appropriate for describing mathematical understanding in their context.

MAMI utilizes two dimensions to analyze the content and discourse in an asynchronous mathematics discussion. These theoretical dimensions capture the dynamic between the discussion as a record of the meanings being communicated by a group of individuals and the social situating of the activity within a discursive community functioning according to a collection of social and mathematical norms. MAMI supports analysis at an individual level, coding the cognitive and argumentative content of an individual's post. An investigator's models of student thinking and discourse developed at this individual level are then later aggregated in our Analytical Framework.

MAMI comprises two analytical dimensions: the *Meanings and Justifications* (MJ) dimension and the *Argumentative* (A) dimension. The MJ dimension consists of codes applied at multiple grain sizes representing constructs for tracking students' meanings and ways of thinking about the mathematical concepts central to the discussion, inferable through the descriptions students employ in their posts. We primarily refer to two grain sizes: the *micro-level* made up of individual sentences, images, or tables, and the *macro-level* which consists of the entire post or comment at a holistic level.

Analyzing along the MJ dimension most heavily relies on one's adoption of the *meanings-oriented instruction* lens. The codes being applied to the records of student discussion posts are words or phrases that the researcher (or instructor) utilizes to infer features of a student's meanings from their observable actions, utterances, or inscriptions within the posts. This allows the MJ dimension to accommodate prominent mathematics education research results specific to the context of the asynchronous discussion. For example, our use of 'Chunky Covariational Reasoning' in Figure 1 was derived from Byerley and Thompson (2017). In the appendix, we provide the entirety of the codes used for the MJ dimension, along with citations of studies in mathematics education that directly influenced our specific code categories. The interested reader can follow these citations to further examine the teaching and learning of precalculus and calculus.

The *A* dimension consists of applying Toulmin's (1964) argumentation scheme to the individual components (e.g., sentences, images, tables) of students' posts. Toulmin decomposed the discursive aspects of argument formation into three main components (*claim, data, warrant)* with three further supportive argumentative moves (*qualifier, rebuttal, and backing)* as seen in Figure 1.

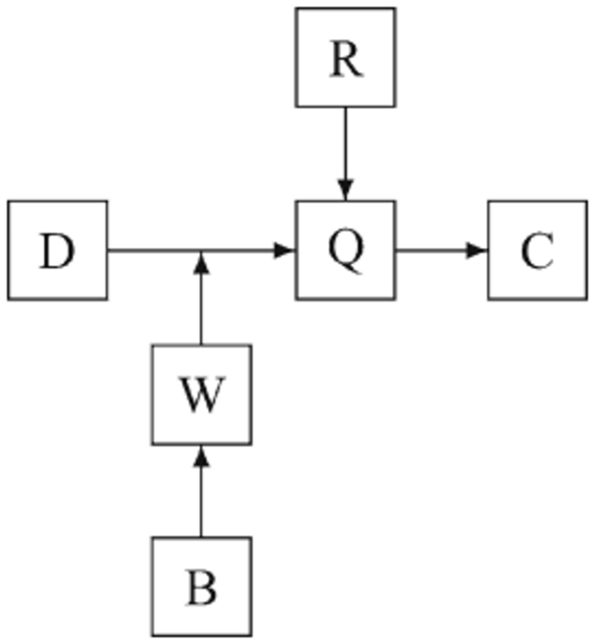


Figure 1: Toulmin's model of a general argument, taken from Ingles, Mejia-Ramos & Simpson (2007).

We then identify the argumentative role of the post in relation to other posts in the discussion (e.g., counterarguments, integrations, isolated arguments, etc.) to consider the holistic argumentative aspect of the post.

Table 2:Mathematical Argumentation and Meaning Interaction (MAMI) Theoretical Framework.

Argumentation	**Meanings and Justifications**
Macro-Arguments	**Macro–Justifications**
• Argument (A)	• Prompt – Specific Justifications and Ways of Thinking
• Counterargument (K)	
• Integration (I)	

• Evaluation (E) • Non-Argumentative Moves (X)	○ Ex: Chunky Covariational Reasoning ○ Ex: Geometric Justification
Micro-Arguments • Claim (C) • Data (Graphical, Symbolic, Tabular) (D) • Modal Qualifier (Q) • Warrant, Backing, Rebuttal (W, B, R) • Non-Argumentative Statements (X)	**Micro–Meanings** • Prompt-Specific Understandings ○ Ex: Algebraic Ratio ○ Ex: Geometric Ratio ○ Ex: AROC – Linear Substitute

Analytical Framework

To engage in the initial analysis process using MAMI, one applies codes from both the *MJ* and *A* dimensions at the desired analytical grain size (i.e. sentence-by-sentence, post-by-post) across the entirety of the data set. The individual codes are then aggregated by the network analysis tool provided in this paper to enable a visual meta-level analysis of the discourse at play in the discussion activity. The network analysis consists of a thread-by-thread examination wherein each post is considered a node in a network, and the codes from each dimension are attributed to the node through multiple representational forms (i.e. shape, color, text).

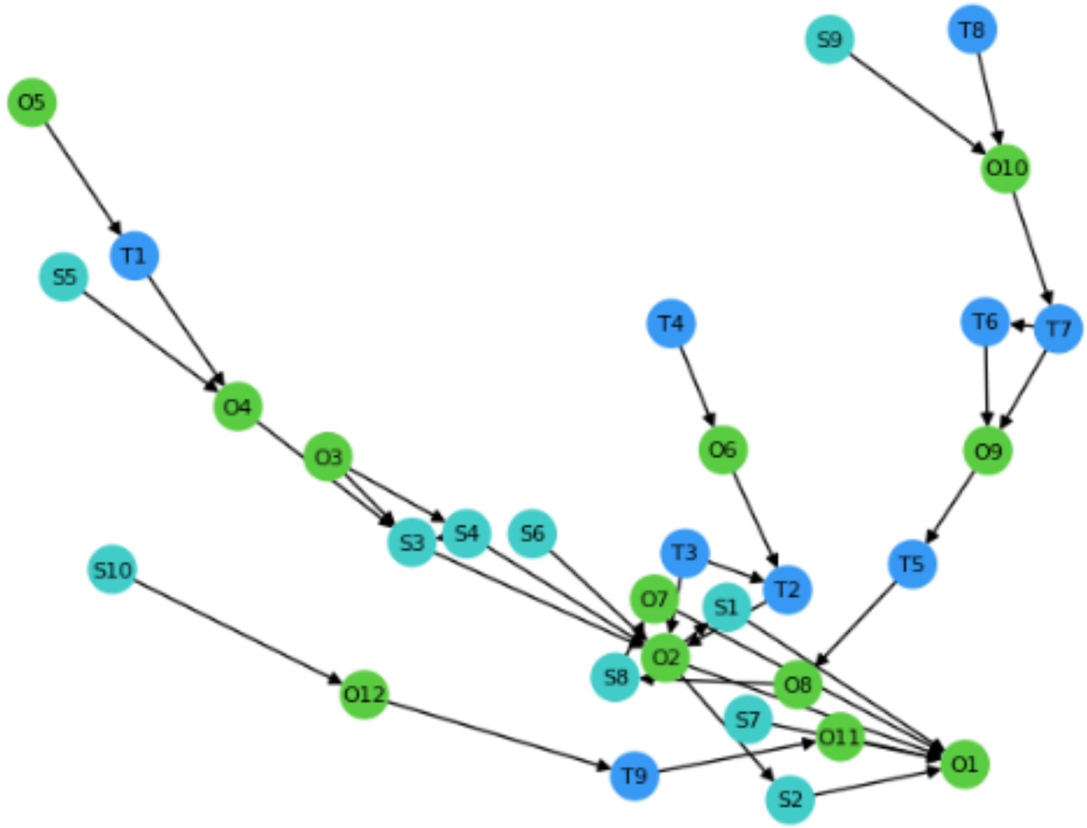


Figure 2: Social interactions to a single post with nodes colored by participant.

Figure 2 provides an example network analysis coloring based on participant type. Colors are used here to emphasize speakers: green for the original poster (O), blue for the instructor (I), and teal for other students in the class (S). Numbers are then used to correlate between a node and a specific post by a speaker. For example, 'O7' would be the 7th post in the discussion chain (as read in a linear way by scrolling on the page) by the original poster. This directed graph provides insight into *how* the original poster, other students, and the instructor are interacting in discussion. With the node labels identifying speakers, we can then investigate the types of argumentative moves made in these chains as well as the types of knowledge that proliferated by populating the nodes with the codes taken from the initial analysis of individual students.

Analysis of the network structure enables an investigator to engage in flexible and robust question-driven exploration. Some possible avenues of exploration include:

- **Meaning Origin:** The presence and emergence of particular meanings within the discourse;
- **Meaning Proliferation:** Tracking the spread of mathematical meanings along threads (or across threads) from a particular origin source;
- **Argument Stagnation:** Social and argumentative explanations for stagnation within a group's progress towards a solution;
- **Argument Habits:** Social and argumentative patterns that enabled or hindered group advancement towards adoption of productive meanings; and
- **Linking Meaning and Argumentation:** More general insights that link conceptual progression to argumentative moves made by the discussion participants.

Overview of the Theoretical and Analytical Frameworks

As we have progressively laid out in this Theoretical Section, our Analytical Framework provides a systematic way for one to observe patterns, relationships, and to discern emergent themes from the temporal, relational, and other features of the codes applied in the initial stage. As these codes all represent the investigator's own models for the meanings, ways of thinking, and argumentation of the individual student, this network analysis is akin to viewing a roadmap of one's own micro analyses. This enables a post-analysis record in a manner that elicits further, socially oriented, observations. While we find MAMI to be a useful theoretical framework for the coding process, users of our Analytical Framework might seek out other means of coding the cognitive and social dynamics of the conversants on a post-by-post basis. Also note this network analysis provides insights into the *process* of student thinking as it developed toward the final *product* of the mathematical problem at hand.

We provide examples of our entire analysis process (both the theoretical and analytical analyses) in the Results Sections so that the theory can be observed in action, and then in the Discussion we indicate what insights analyzers from three perspectives (instructor, designer, course mentor) might seek to gain from application of our Analytical Framework.

Data Collection

The Theoretical and Analytical Frameworks were developed through an iterative design methodology (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003), first utilizing small group discussions and then whole-class asynchronous discussion data. The discussion data was taken from multiple online sections of Calculus 1 (differential calculus) taught at a primarily undergraduate university. The university's student population is non-traditional in many ways:

- **Predominantly male:** 79.3% of students identify as male;
- **Predominantly military-affiliated**: 60.3% of students are active military and 19.5% of students are veterans; and
- **Predominantly older students:** Average student age is 33, median student age is 32, and 93.2% of students are over age 23.

Data analyzed for this qualitative study is comprised of textual records of small-group discussion activities taken from the Spring 2022 and Spring 2023 terms.

The data consisted of two discussion activities: one given to a whole class (25-30 students) and one given to small groups (4-8 students). We present only the analysis and findings from the whole class discussion. However, the small group analyses were integral to the iterative design of both frameworks. Students could not edit nor delete their posts, and as such the data represents the authentic, unedited contributions of students to the discussions.

Discussion Prompt

Recall that a key goal of engaging in *meanings-oriented* instruction is the design of tasks that enable an instructor to make inferences about students' meanings and ways of thinking in route to a finished product. This requires that some tasks, particularly the tasks for which one wishes to make inferences of students' meanings, will involve multi-step coordination of performed mathematical actions in pursuit of some grander goal for which a simple solution is not readily accessible nor quickly discernible[5]. Such is the discussion task around which our presented data is centered. The discussion activity spans the first two weeks of instruction and focuses on the key concepts of rate of change while also requiring that students construct personal graphical models of the situation below.

> **Situation:** *Torty and Harry are competing in a 100m footrace. Torty's average speed on any 5-second interval is always less than Harry's average speed on any 5-second interval, but Torty wins the race!*
>
> **Additional Details:** *The race will be at least 10 seconds long, but you can make the race last longer if you wish. For consistency, let's say Torty and Harry keep running after 100 meters, so their speeds can always be calculated by looking forward in time even though*

[5] See, for instance, the features of exam items requiring application of understanding given in Reed, Tallman, Oehrtman and Carlson (2021).

the race ends at 100 meters. Also, neither Torty nor Harry ever moves backward during the race.

The task situation elicits mathematical discourse in that it is easily stated and understood, but producing a fully justified solution to the initially surprising result requires significant coordination of robust meanings for key calculus concepts such as constant and average rate of change, and the clever intentional variation of instantaneous rates (though students don't have this vocabulary at the time of the discussion) within intervals of time. The following task around the situation requires that students produce (and describe) graphical models of the two racers.

> **Task:** *Upload a final, neatly-drawn (or computer-generated) graph that shows both Torty and Harry's distance-time relationships satisfying the race constraints. You will each need to include a description (in words) of why the graphs visualize the constraints of the race, and a few drawn examples of Torty and Harry's AROCs that show the constraints of the race being met. Your final post will be graded for accuracy and for a thorough description.*

Requiring that students justify the fit of their graphs to the race is one key strategy for inferring meanings from responses. This is similar to a feature for exam items requiring application of understanding identified by Reed et al. (2021). Asking for justification makes it more likely that student responses enable instructors to reasonably infer meanings.

In this case, the students were required to coordinate the contextually kinematic instantiation of average rates of change with the graphical manifestation of average rate of change as providing the slope for a linear substitute of a nonlinear curve over a set interval. Moreover, the students were required to describe the features of their graphs such that the 5-second AROC requirement was not simply met on set 5-second intervals (such as [0,5], [5,10], etc.), but further why it was reasonable for the AROC requirement be met on *any* 5-second interval. In constructing such descriptions, students are required to describe the graphical and contextual connections in a way that an instructor might infer aspects of the students' meanings for rate of change. Moreover, such discourse is more likely to be observable within the discussion activity itself as students make similar statements to each other and the instructor during the process of solution construction.

Many possible solutions exist; however, common solution methods define Torty or Harry's distance vs time functions as linear, and define the other racer's function as a linearly-adjusted periodic function with period 5. The following two solutions are recreations of students' GeoGebra submissions, uploaded to a blinded GeoGebra account to ensure student anonymity: Student 1, Student 2. The second re-created student solution is provided in Figure 3.

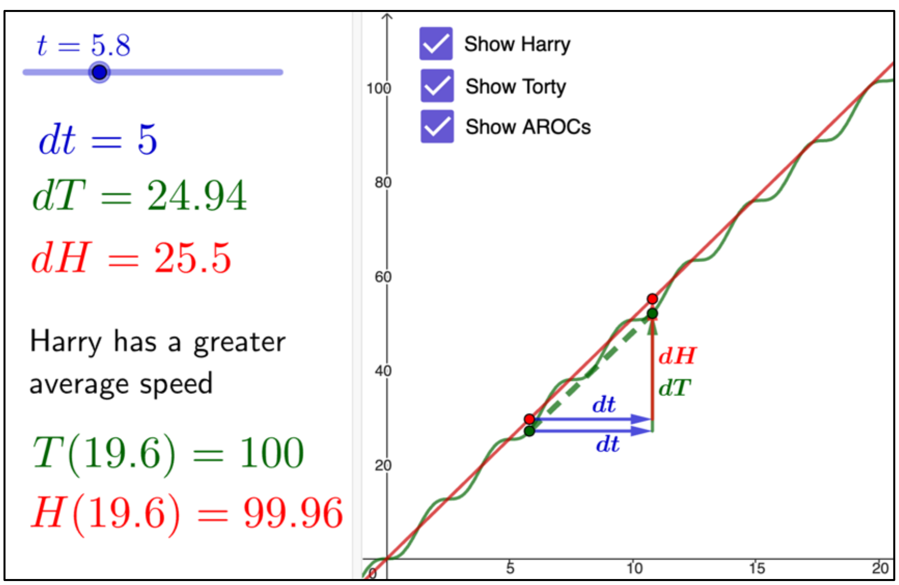


Figure 3: Solution to the Harry and Torty problem.

One common justification involves the recognition that during the first 5-second interval, Harry travels a greater distance than Torty, resulting in a higher average rate of change. That Torty's distance-time-relationship is a linear adjustment of a periodic function (period 2.5 in Figure 3) function guarantees that the growth in any 5-second interval will be no more than the constant rate of change of the linear term, which can be set to less than Harry's constant rate of change. Another common alteration includes a similar periodic nature with a diminishing amplitude so that Torty's 5-second AROC is itself decreasing with each period. Keep in mind that students are only required to draw solution curves, so their descriptions will be less formal than the solution just given.

Data Analysis Method

Iterative Design Research

While much of the paper is devoted to operationalizing the Analytical Framework, we take a moment here to provide the research method by which both frameworks came about. Production of the frameworks occurred within an iterative design experiment methodology (Cobb et al., 2003), where the overall instructional goal of the design was the development of engaging and meanings-eliciting discussion activities in asynchronous mathematics courses. In line with design experiment methodologies, the analyzed discussions in these asynchronous settings are "instances of broader classes of phenomena, thereby opening them to theoretical analysis" (pg. 10). Engaging in retrospective theoretical analyses of initial experimental design forms a dynamic process of iteration in which theoretical insights form new means of and processes for instructional experimentation and refinement. Such instructional refinement then further informs development of the theory, in a cycle that enables researchers to both make progress in instructional design and in knowledge generation (Cobb et al., 2003).

The theoretical and analytical frameworks are the result of four cycles of experimentation and retrospective analysis, which has yielded the progressive production of subtlety-revealing theory production and a refinement of analytical methods. As a result of the most recent retrospective analysis, we see the Analytical Framework as a useful tool for not just researchers, but instructional designers as well. Below, we first give an account of our analytical process, and then recommend a practical implementation of the MAMI framework by educators.

Method of Rigorous Data Analysis - Coding

Discussion posts and reply data were collected and entered into an Excel sheet. This included varying types of response data (textual, graphical, syntactic, and programmatic) as well as positional data (post number and depth) to identify the general interactive structure of the discussion. Response data was categorized by speaker, then split into units: single sentence, single graph, single equation, or single program script.

The researchers individually coded the response units according to the MJ and A dimensions, then met to negotiate the final coding of each response unit for both dimensions. Figure 4 gives example units of data analysis from the whole-class discussion and shows the granularity of the various codes that we applied in our codebooks for the MJ and A dimensions, as well as the codes themselves. The MJ codes are given in green, and the A codes are given in orange.

The Argumentative dimension includes Toulmin's model applied at a micro level, identifying the specific components of a post that build up the argument of the poster. The MJ codebook for this particular discussion attended to students' engagement with quantities (*QR,* in the sense of Thompson, 1990), their meanings for constant, average, and any rate of change (CROC, AROC, and ROC, respectively), along with their meanings for derivative (*D*) and function (F). We would only expect students repeating calculus to mobilize derivative meanings, and students did not demonstrate attention to quantities as measurable attributes of the mathematical concepts at play (Thompson & Thompson, 1992).

Speaker ID	Analysis Unit	C, D Claim, Data	W, B Warrant, Backing Warrant	Q, R Qualifier, Rebuttal	E Evaluative	X Non-arg	A, K, I, X, E Macro Arg	QR Quantative Reasoning	CROC Constant Rate of Change	AROC Average Rate of Change	ROC Any Rate of Change	D Derivative	F Function	J Justification	Coord Coordination
Student 1	However, after going through the process, I figured that Harry was running in 5 second intervals and slowing down.	C							CP						
Student 1	Good thinking!						I								
Student 2	Hello Class,														
Student 2	The following graph shows that Harry had a faster speed than Torty by the end of every 5 second interval.	C													
Student 2	When you place a line at y= 0 and 29 to calculate the slope for Harry's line, you get a slope of 29.	D-S							GR						
Student 2	This also represents the average rate of change of every 5 second interval, which is greater than Torty's average rate of 28.6 meters.	D-S	W						GR	LIN					
Student 2		D-G, D-T					A							Geo-Chunky	Tot

Figure 4: Example units of data analysis.

Finally, we attended to the justifications provided by the student, specifically whether the meanings conveyed in their justification coordinated the necessary components of context, focused attention to the graphical nature of an answer, and consideration of time intervals.

The coding process largely unfolded as described in the paragraphs above. The authors first individually coded sections of the discussion data according to one or both dimensions, and then met to negotiate differences in the codes (or to develop new codes to explain a commonly observed phenomenon). This process continued until each response unit was coded, at which point the method of inquiry transitioned to the Analytical Framework.

Analytical Framework (Network Analysis)

Response units were then aggregated back to the whole post level and, utilizing each post's positional data, incorporated into a network visualization. This network visualization was achieved through the Python open-source programming language and the *networkx* package that simplifies the programming requirements to create network visualizations. Note the *networkx* package does not allow for changing the shape of a node, which is why the authors relied on network coloring to convey information.

After the initial creation of functions to systematically apply certain patterns of coloring (such as whether for two codes one, another, both, or neither appeared in the discussion) and converting the format of the data from the Excel sheet used by the researchers to apply qualitative codes to one that *networkx* required, the coding of network visualization was primarily two parts: creating the underlying graph and applying some desired coloring. An example block of code for both parts is provided below. Note the green text are comments and provide description for the function of each line of code.

```
G=nx.DiGraph() # G is the name of our graph that we will add info to
G.add_nodes_from(list_of_nodes) # Adds nodes from our organized data
G.add_edges_from(list_of_edges) # Adds connections between nodes based on our positional data
H = nx.relabel_nodes(G, alt_node_label_dict) # Converts to the desired O#, I#, S# labeling for nodes
my_H_pos = nx.spring_layout(H, seed = 100) # Choose a seed so the orientation of nodes and edges
remains fixed
```

By defining the types of colorings beforehand, the process of creating each network visualization is a few lines of code. For example, to color nodes that included a particular MJ code (AROC) green and keep all other nodes gray, we run the following lines of code:

```
aroc_color_map = present_absent_coloring(test_xl['AROC\nAverage Rate of Change']) # test_xl is
our data and 'AROC\nAverage Rate of Change' the name of a column in that data
nx.draw(H, pos=my_H_pos, font_size=8, with_labels=True, node_color=aroc_color_map)
```

All author-defined functions for this network analysis, along with code used to collect and modify discussion data, are available on GitHub at: https://github.com/Darryl-Chamberlain-Jr/Analyzing_Async_Discussions. After generating and evaluating numerous network colorings, the researchers met to develop inter- and intra-dimensional insights for each discussion based on insights gained from the network organization and visualization.

Data Analysis Examples

Initial analysis example

Here, we begin with an example of the initial coding process, and then give examples of the exploratory analysis enabled by our Analytical Framework. Figure 5 provides an example of our structured data to explain how we coded each post.

In each dimension, codes were applied both on a micro grain-size level (i.e. sentence-by-sentence and graphical unit) and at a macro grain-size (i.e. a code applied to the post as a whole). One distinction that we found useful for analyzing the Torty and Harry discussion was delineating between data types for those students that provided data in their posts. The markers *D-S* (symbolic or formulaic), *D-G* (graphical or graphical reference) and *D-T* (tabular data) can provide useful information at the aggregate level should an investigator care to link data types to other discursive or content features of the discourse.

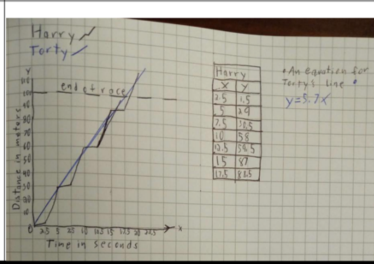
Speaker ID	Analysis Unit	C, D Claim, Data	W, B Warrant, Backing Warrant	Q, R Qualifier, Rebuttal	E Evaluative	X Non-arg	A, K, I, X, E Macro Arg	QR Quantative Reasoning	CROC Constant Rate of Change	AROC Average Rate of Change	ROC Any Rate of Change	D Derivative	F Function	J Justification	Coord Coordination
Student 1	However, after going through the process, I figured that Harry was running in 5 second intervals and slowing down.	C							CP						
Student 1	Good thinking!						I								
Student 2	Hello Class,														
Student 2	The following graph shows that Harry had a faster speed than Torty by the end of every 5 second interval.	C													
Student 2	When you place a line at y= 0 and 29 to calculate the slope for Harry's line, you get a slope of 29.	D-S							GR						
Student 2	This also represents the average rate of change of every 5 second interval, which is greater than Torty's average rate of 28.6 meters.	D-S	W						GR	LIN					
Student 2		D-G, D-T					A							Geo-Chunky	Tot

Figure 5: Example coding.

The Macro-Argumentative codes marked the more relational aspects of the post. Did the poster posit a new argument (*A*), provide a counterargument against (*K*) or integrate (*I*) an argument from a prior post, evaluate (*E*) the merits or structural aspects of another argument, or make a non-argumentative statement (*X*)?

In Figure 5, Student 2 claims that his graph visualizes “Harry [having] a faster speed than Torty by the end of every 5 second interval.” Student 2 provides symbolic, graphical, and tabular data in support of this claim, and specifically links the data to the claim in a warrant connecting the graphical data to the contextual situation. This was the beginning of a new thread that did not reference past posts, and hence was a new argument.

As with Student 1, we took vague references to speeds and speed variation as indication that the student considered the constant rate of change as a contextual property (*CP*) but was not in that moment addressing a particular conceptualization of the mathematical structure of the rate. Student 2 specifically attended to slope and its relation to two points on a line, hence in the moment considering a graphical ratio (*GR*) to be at play. They then also attempted to link the slope of Harry’s line to Harry’s AROC for comparison of Torty’s AROC taken from the tabular data, hence considering AROC to involve substitution of linear variation amidst nonlinear variation (*LIN*). Note that, consistent with the intention to create models of students’ mathematics, we are not claiming that Student 2 would use phrases such as “linear substitution” or “nonlinear variation” or “mathematical structure of rate”, but rather these phrases are our ways of describing what cognitive features we observe or infer from the statements and work of Students 1 and 2. Nor are we claiming that Student 2 is linking the graphical and contextual registers “normatively”, but rather we are attending to the actions of the student and making attempts at descriptive inference, and our observation that an attempt was made to link the registers is captured in the *LIN* code.

Student 2's predominant discourse and data centered on geometric presentation, and the given variation is in discrete 5-second segments, aligning with Carlson et al.'s (2002) notion of "chunky" reasoning (*Geo-chunky*). That Student 2 is attempting to coordinate the graphical, variation, and contextual elements of the task leads us to code this post as involving total (*Tot*) coordination of the necessary elements to advance towards a solution.

Example of Exploratory Inquiry using the Analytical Framework

As noted previously, the Analytical Framework we use provides a mechanism to systematically explore the data for holistic patterns. Based on the coding, we identified six patterns within the argumentative and meanings-justification dimensions for a noteworthy student discussion.

1. *Micro-argumentative use of Data (D) in chain*
2. *Macro-argumentative moves in chain*
3. *Overlap of Data (D) and Integrative (I) statements in chain*
4. *Macro-meanings inclusion of Non-Referential Symbolic (NRS) Function in chain*
5. *Macro-meanings inclusion of Average Rate of Change (AROC) in chain*
6. *Macro-justification moves in chain.*

These patterns together follow one line of inquiry, as observations from one led to questions answered by the next. This analysis, therefore, is an example of the flexible analysis by which one might engage the data. For each, we will provide a network coloring and provide commentary on what pattern of student thinking we believe the coloring illuminates.

1. Micro-argumentative use of data in chain

Recall that Toulmin's scheme calls for identifying when a student provides data as part of their argument. When coding individual sentences, we attended to the types of data students provided as this can be related to the types of meaning they are arguing and justifying. The inclusion of data in each reply is provided in Figure 6.

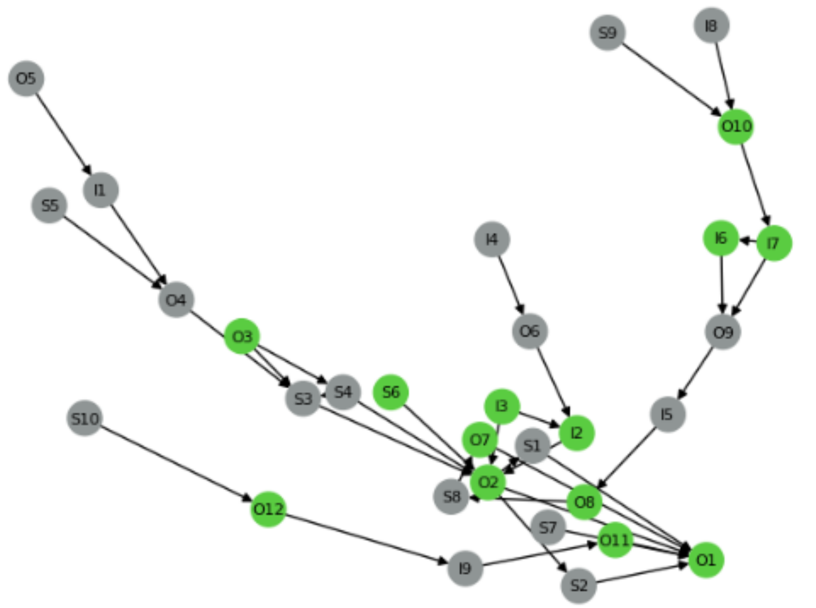


Figure 6: Coloring of micro-argumentative code Data (D).

First note that data was present in each of the three major chains of replies. However, the initial poster (O) and Instructor (I) are the primary arguers with data outside of a single instance (S6). This suggests students were not providing their own data to argue with the original poster O and instead were performing some other function in the discussion.

2. Macro-argumentative moves in chain

To further explore the argumentative function other students were making in the discussion chain, we color the macro-argumentative moves by post. This coloring is provided in Figure 7.

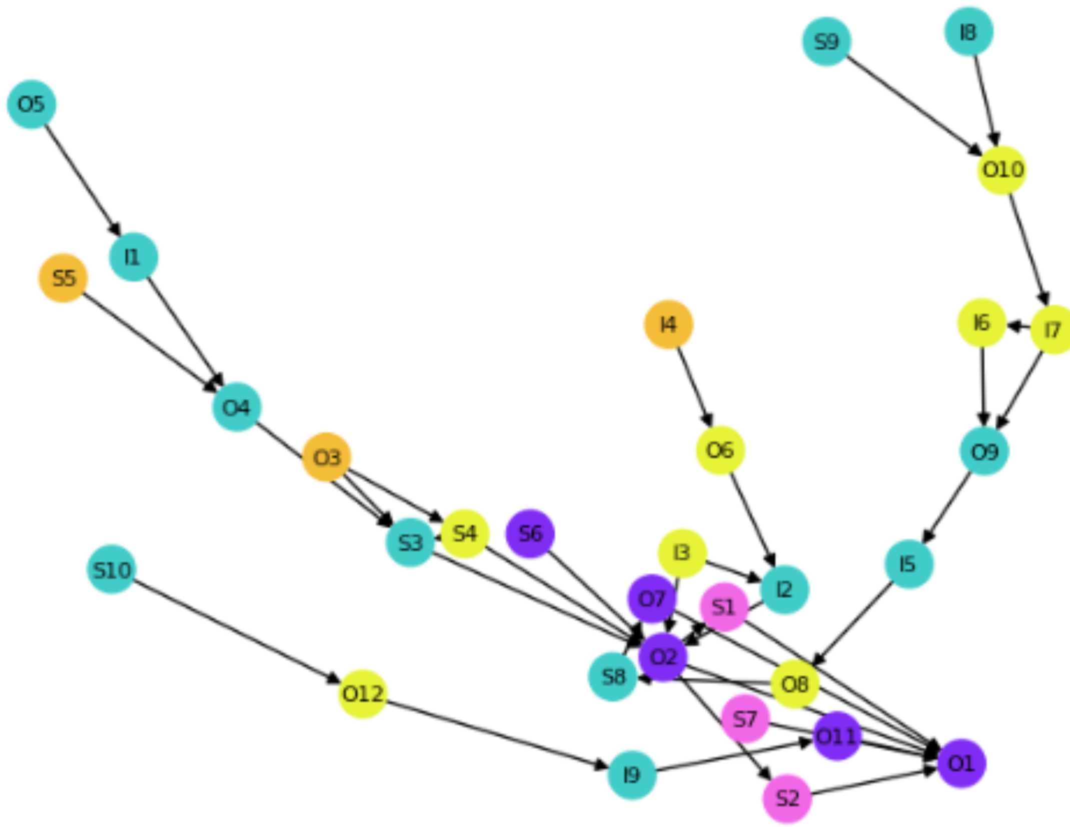


Figure 7: Macro-argumentative moves in chain. purple=A, pink=K, yellow=I, teal=E, orange=X.

Here we see that O is providing most of the argumentative claims (purple) for the discussion chain. The three offshoots of the discussion are primarily evaluative statements (teal) and integrative statements (yellow). The student responses that do not produce long chains of replies are non-argumentative statements (orange). One overall pattern we notice is that evaluative statements lead to integrative statements and thus foster argument in the discussion. Non-argumentative statements (orange) lead to an end of a reply chain and thus do not foster argument in the discussion, though they may play some other non-argumentative role in the discussion such as confirmation or resolution.

3. Overlap of Data (D) and Integrative (I) statements in chain

Based on a side-by-side look at Figure 6 and Figure 7, it may appear that Integrative statements (I) feature Data (D). To better examine this, Figure 8 provides an overlapping view of these micro- and macro-argumentative codes.

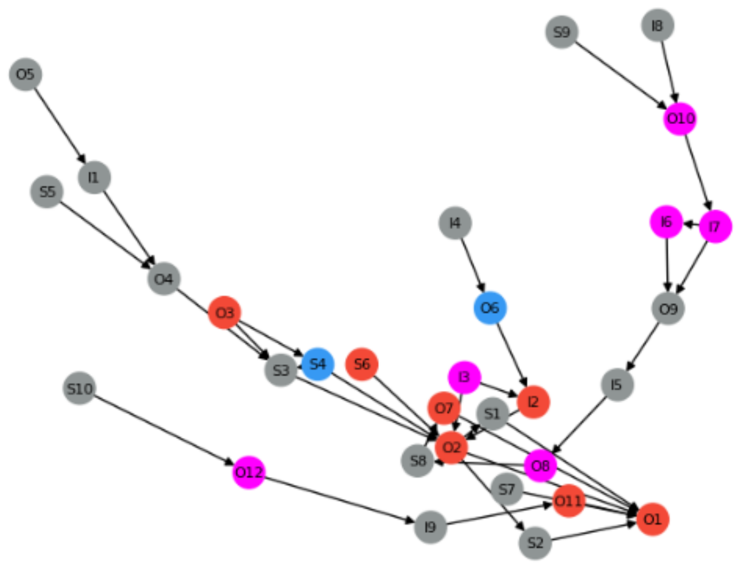


Figure 8: Coloring of micro-argumentative code Data (D) and macro-argumentative code Integration (I). Red=D only, Blue=I only, Magenta= Both D and I, Gray=Neither D nor I.

Note the majority of the Integrative statements (8 of 10) include Data in the post. We can consider these high-quality posts as they both consider data while attempting to combine an external idea with the previous poster to create a new idea. We also note that the Integrative statements with Data are direct responses with either a post with Data initially or an Evaluation of a post with Data.

4. Macro-meanings inclusion of Non-Referential Symbolic (NRS) Function in chain

After considering the patterns seen in the argumentative side of the discussion, we can focus on particular concepts and consider how they propagated. First, we consider the most frequent meaning inclusion: Non-Referential Symbolic (NRS) Function (Harel, 2008). The inclusion of this code in the discussion chain is provided in Figure 9.

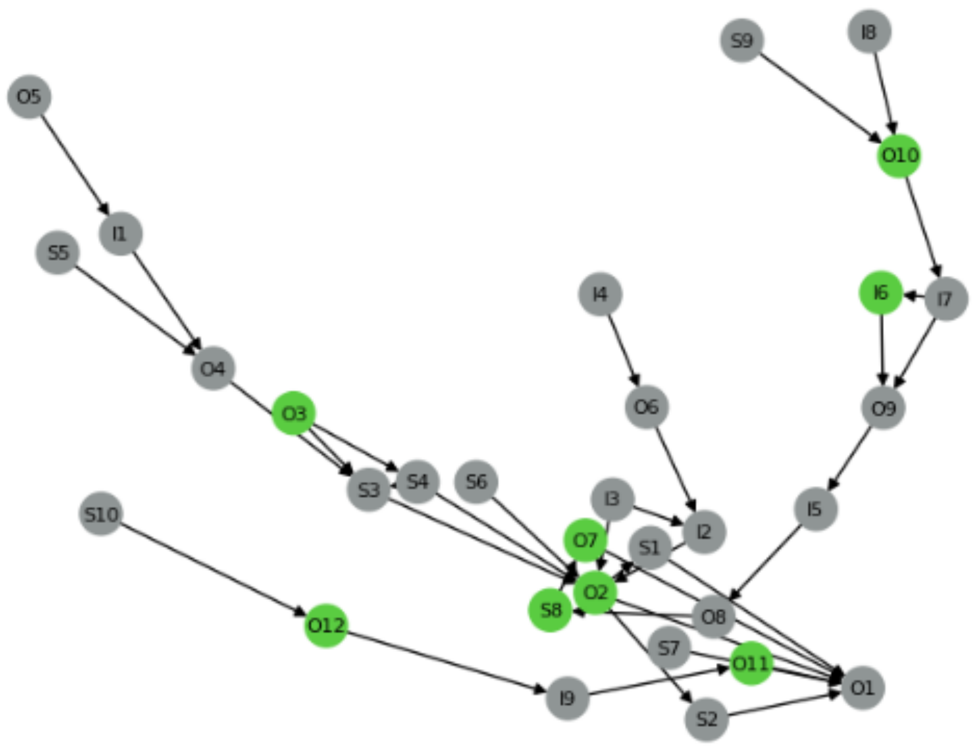


Figure 9: Coloring for inclusion of Function code Non-Referential Symbolic (NRS).

Outside of one instance of the code appearing with the Instructor (I6) and another with a student (S8), the majority were present in the original poster's replies. This would suggest the use of Non-Referential Symbolic (Harel, 2008) reasoning was not picked up by repliers and did not propagate through the discussion chain even though it was the most common meanings code.

5. Macro-meanings inclusion of Average Rate of Change (AROC) in chain

Next, we can consider the second most common meanings code applied in the discussion post: Average Rate of Change (AROC). Figure 10 provides the presence or absence of any type of Average Rate of Change in each reply.

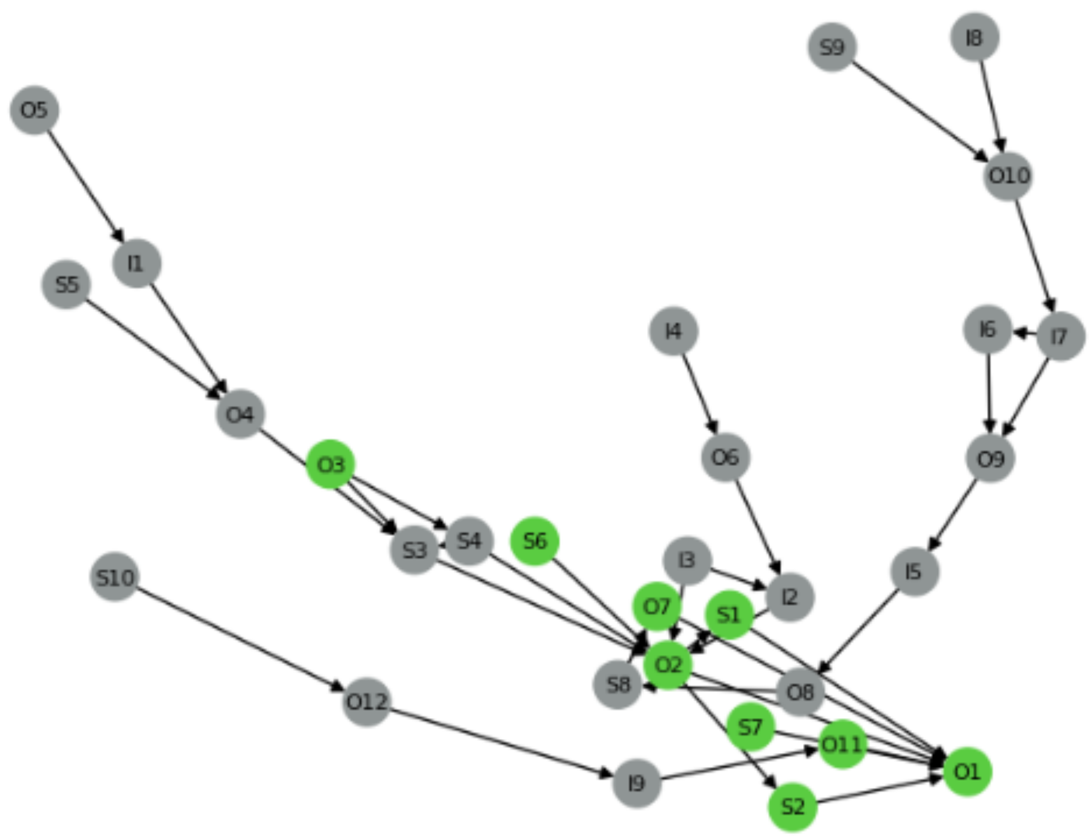


Figure 10: Coloring for Average Rate of Change (AROC).

Note that unlike the function concept Non-Referential Symbolic (NRS), Average Rate of Change (AROC) does propagate from the original poster to other students. We can then consider the types of AROC students presented in their posts by Figure 11.

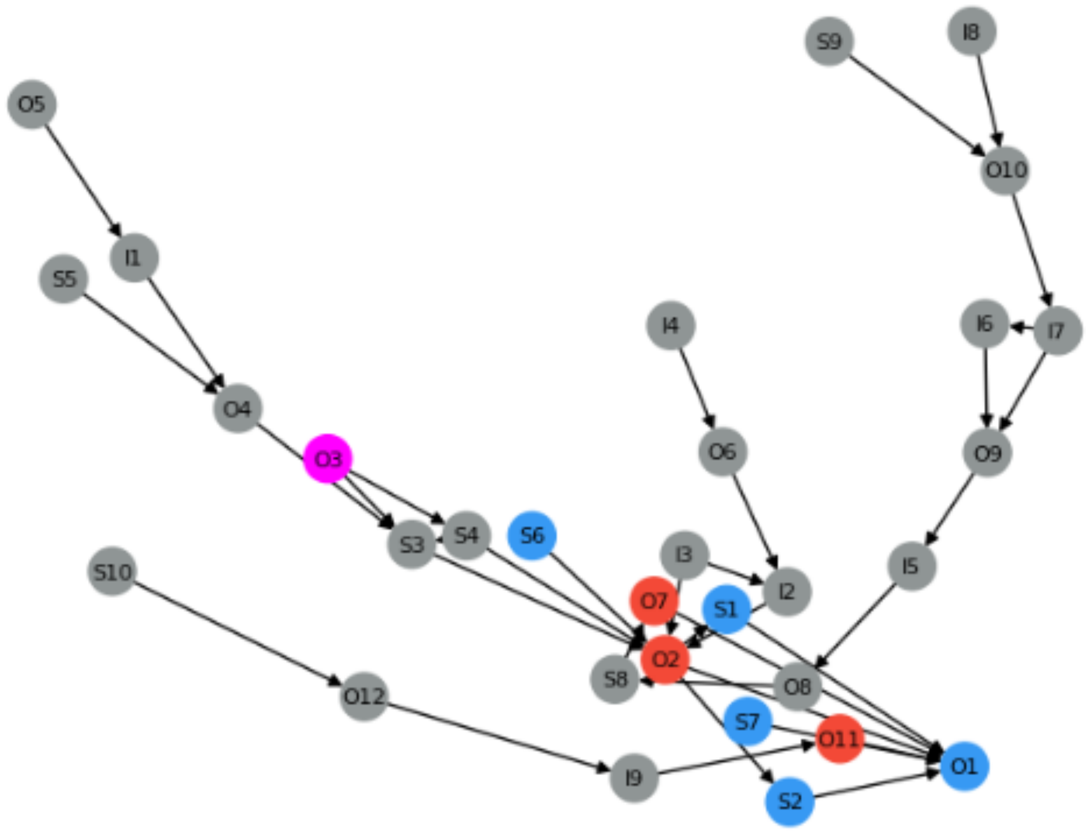


Figure 11: Coloring for Average Rate of Change (AROC). Red=DIST, Blue=VAR, Magenta=Both DIST and VAR, Gray=Neither DIST nor VAR.

Splitting the ways to represent Average Rate of Change (AROC) provides a more refined picture of the initial discussion. While the original poster initially describes AROC as a varying intermittent speed (VAR) like “speeding up and slowing down” in an interval, they later provide follow-up replies describing AROC as distance changes over intervals of time (DIST) like “Traveling D meters every S seconds”. Other students who talked about AROC only described it as VAR and thus the spread of the concept Average Rate of Change is limited to just the VAR description.

6. Macro-justification moves in chain

Finally, we can consider the ways students justified their responses. Figure 12 provides a complete coloring for all types of justifications used in this discussion chain.

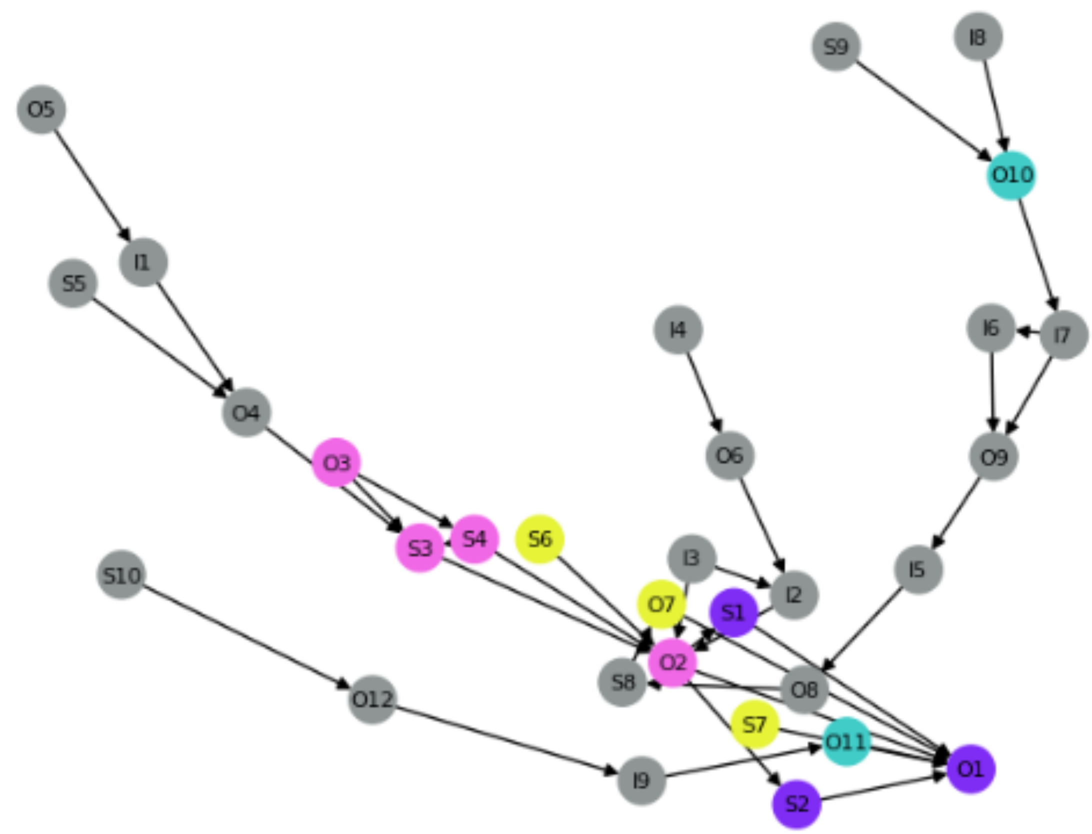


Figure 12: Coloring for Justification codes. CTXT=purple, GEO-CHUNKY=pink, GEO-SCALE=yellow, GEO-CONT=teal.

Initially, the original poster justifies their response with the context of the question. As the prompt has a strong colloquial story associated to the problem, this isn't surprising. However, we then see two ways to justify the responses based on the geometric properties of the solution graphs: by comparing rates of change of the two functions (Torty and Harry) over contiguous intervals (GEO-CHUNKY) and comparing rates of change of the two functions at significantly different scales of interval sizes (GEO-SCALE) such as zooming in at the end to show Torty's function has a slight bump for the win. Later in the discussion chain, the original poster and other students shift to comparing rates of changes over a variety of intervals (GEO-CONT). This progressive sophistication of how students justified their solutions shows how the idea of rate of change was abstracted through collective student interaction with the prompt.

Overall Takeaways from Exploratory Inquiry

By examining a single post through multiple lenses (argumentative and meaning), levels within each lens (micro- vs macro-), and comparing presence or absence of one or more codes between a lens and level, we identified patterns of student thinking as they propagated through the discussion. We noted argumentative moves that encouraged responses from others such as evaluative statements that led to integrative statements. We also noted data concurrence with integrative statements. Within meanings, we showed that some (Non-Referential Symbolic - NRS) were widespread in the discussion but mainly used by a single student while others (Average Rate of Change – AROC) had evidence of being spread through the discussion among other students. We finished with evidence for an increasing sophistication of how students justified their responses.

While these results may not be generalizable patterns across all discussions, they illustrate how utilizing using the analytic framework by leveraging technology can allow for visual exploration of data for global patters. Moreover, it shows the potential for checking on temporal propagation of arguments, concepts, and justification that cannot be easily achieved in a top-down reading of a discussion as it is presented in Learning Management Systems such as Canvas.

Discussion

Recommended Method of Data Analysis - Coding

Here, we detail the ways in which an instructor or course mentor might engage in a network analysis. As described in the Network Analysis Section, the current primary grain size examined in the network is at the level of post. This would allow investigators to speed up collaborative coding by focusing solely on the MJ and A dimensions during the initial coding process, without considering individual sentence-level moves.

The aggregation of codes only attends to the presence of argumentative and meanings-oriented codes within individual posts. This will enable the investigator to ask various global and cross-dimensional questions. However, the sentence-by-sentence analysis holds no implications for the actual aggregation process nor for the analytical method enabled by the network structure. Although a finer-grained analysis allows for a more careful coding process, it's important to consider whole-post dynamics even during sentence-by-sentence coding. Often, one must consider the relationship of each sentence to the greater argument or to the other concepts at play. For instance, in Student 2's work in the above data excerpt, one can only infer the warrant in Line 4 by its relationship to the claim in Line 2.

With these considerations in mind, we recommend that investigators read and analyze the discussion data at the whole-post grain size, and in their analysis record the presence of particular meanings, argumentative moves, and social argumentative moves. This might lead to the alteration provided in Figure 13 of the above example.

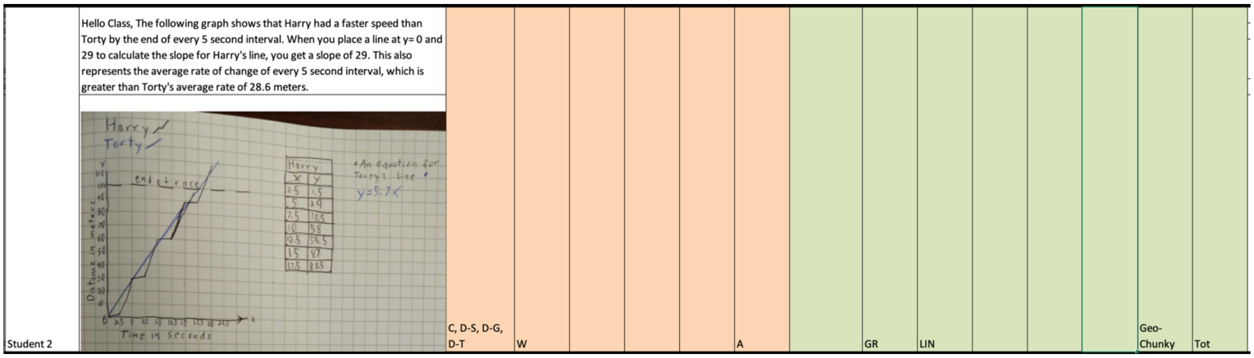


Figure 13: Unit of measure for analysis as entire post rather than individual sentences.

While only a slight alteration, this can expedite the coding process and admits a slight alteration to the analytical method of the investigator. Rather than take each line in isolation and in relation to the other lines of the post, the investigator need only ask the question "What meanings and argumentative moves might I infer from the poster's work?". We consider that this seemingly insignificant change of scope can hold significant implications for the total time it takes to engage in the dimensional analysis, as well as for the cognitive load required by the investigator during this phase of analysis.

Questions from Different Perspectives

As demonstrated by the unfolding inquiries elaborated in the example network data analysis, there are many possible insights to be gained from attending to observable patterns and emergent themes from the network aggregated response units. As indicated throughout the chapter, we consider our method of analysis to flexibly enable question-driven exploration, rather than to prescriptively set the constraints by which one might approach the data and extract findings. In this discussion section, we posit questions that one might ask of the data from the lenses of instructor, course mentor, and course designer. We also give some initial recommendations for pursuing such questions so that interested educators might have a starting point for examination. Finally, we conclude with the limitations of the current analysis method and software development and suggest ways that the framework and its technological structure can be adapted and improved upon in the future to further enable efficient inquiry.

Possible Questions to Guide an Instructor's Inquiry

While evaluation might be the most straightforward application of the Analytical Framework, we would caution an instructor to determine whether the evaluative gains from this in-depth analysis would outweigh the costs of the extra effort to employ this analysis above and beyond typical student-by-student evaluative practices. Rather, we consider two possible introspective questions instructors might bring to the table when employing the Analytical Framework.

Firstly, instructors might benefit from periodic application of the Analytical Framework on a randomly (or intentionally) selected discussion data set for the purpose of determining their own pedagogical satisfaction with the discourse and learning that took place within the activities of a course. Said more plainly, the instructor might retrospectively want to ask "Did the students learn what I wanted them to?", or "Am I satisfied with the level of student engagement throughout the discussion?". Such questions, if answered in retrospective analysis, might have different answers than if answered in the immediate wake of a discussion activity.

Instructors might not have engaged with every student over the course of a discussion, and discourses occurring in between instructor teaching actions might also go unnoticed over the course of a discussion activity. This is to be expected, as the instructor is not solely responsible for the learning taking place in discussion activities, in fact in some instances it is more incumbent on students to collaboratively learn than it is for them to be all taught by an instructor.

As such, an instructor necessarily has a limited understanding of the actual discussion that unfolded over the activity period and might benefit from analyzing the discourse in its entirety so that the students' meanings and ways of thinking might be more readily inferred. This meanings-oriented practice can serve as an initial measure of self-evaluation and self-reflection.

Relatedly, an instructor might want to utilize the Analytical Framework to examine their own teaching practices and to hone their abilities to run such discussions. This is related to the coordinator-focused inquiries discussed below, but in essence we consider that instructors might want to critically analyze their role in the unfolding discussion, perhaps in conjunction with the aforementioned retrospective examinations, and seek opportunities to either identify teaching moves made that were particularly effective discursively or conceptually, or to identify patterns of interaction that might stifle the proliferation of meanings.

Possible Questions to Guide a Coordinator's Inquiry

We see course coordinators as primarily interested in offering mentorship to instructors and determining best practices for implementation of the intended curriculum. A coordinator might want to examine the strategies and modes of interaction between different instructors and the students in a discussion for the extrapolation of patterns that might provide actionable feedback.

For instance, a coordinator might observe differences between instructors' interactions in terms of the distribution of instructors' posts throughout the discussion, the content of instructors' posts, the timing of instructors' posts, or other such differences between instructors' interactions in the discussions. If an instructor consistently yields high engagement and it seems that students make good conceptual progress, further examination might reveal patterns and themes disseminable to the instructor team at large.

Similarly, instructors might structure their interactions differently, enabling examination of the kinds of discussions that unfold. For instance, one instructor might primarily respond to each student individually, offering conceptual and technical recommendations at the micro-level, and offer very little in the way of general feedback to the discussion group as a whole. In contrast, instructors might largely offer general feedback in the way of hints and concepts and discussion-furthering questions, and then interact at a micro level to only address technical issues of computation or format.

Analyzing the discourse unfolding within multiple discussions from these two epistemic instructors might reveal features of their strategies that enable or hinder a diverse and robust conversation to unfold between students at the broad discussion level. For instance, the former strategy might yield many instructor-student conversations that are largely interactions between just two individuals, while the latter strategy might yield more one-sided conversations with students interacting largely on their own posts and less visibly engaging with the instructor's thread.

These hypothetical examples do not point to definitive general outcomes of various strategies, but rather serve as examples of functional questions a coordinator might ask in the context of an instructor group and given a history of discussion data, for the purpose of professional development. Furthermore, guiding instructors through a data analysis exercise throughout a term could enhance their professional development, fostering awareness of and responsiveness to students' cognitive and discursive needs during asynchronous discussions.

Questions for a Course Designer

We imagine that the role of course designer and course coordinator will overlap on the same individual in many contexts, however it is important even for individuals to functionally adopt multiple lenses even when analyzing the same discussion. Designers have more direct control over the curriculum itself and can affect change on the levels of (for instance) task, structure, grading, and intervention. Designers will ask broader questions than the mentor or the instructor and think structurally across multiple discussions.

For instance, analyzing the monolithic progression of meanings throughout various threads might yield questions about the task, the posting requirements, or the interventions recommended throughout a discussion. Observing student-dependent argumentative variety and idiosyncratic cognitive progression may suggest the need for curricular adjustments. As an example, noting little argumentative variety from students other than an original poster might suggest the need for altered requirements on the structure of a percentage of the posts. One might impose a required format for the initial responses students make to an original poster, such as requiring that students specify a claim, provide evidence of the claim, and justify the claim by linking the evidence to the claim and concepts of the course (i.e. Evidence – Claim – Justification format). In conjunction with this, one might add a posting requirement that students reference the meanings of main concepts (speed, CROC, AROC, etc.) whenever a mathematical idea is brought up.

A designer would likely benefit from looking at both the network analyses of the MJ codes and the A codes side-by-side to examine the social dynamics contributing to the progression of meaning (or lack thereof) throughout the discussions.

Conclusion

As is seen by looking at research published in undergraduate mathematics education, as well as the huge addition of online asynchronous mathematics teaching in the last 20 years, we believe that understanding how asynchronous learning may happen is a significant contribution to research in undergraduate mathematics education. We feel that the use of networks, based on coding by some theoretical framework, to visually explore student posts and replies as a web of interactions will play an important role in analyzing knowledge as it propagates through a discussion post.

This work is part of a larger design project to engage students and faculty in online mathematical learning. We will use our results and altered framework to support revisions and development of STEM courses in asynchronous environment. We also believe the new framework will be applicable to face-to-face student discussions in mathematics. Understanding how student communication impacts mathematics learning may have significant implications for both online and in-person instruction. One method to extend this work would be to develop a machine learning algorithm to perform the qualitative coding of responses according to some theoretical framework, then automate building large sets of social interaction webs for practical uses in the classroom.

Limitations to Consider for Application

The kinds of data collected set natural limitations on the inferences allowable from analysis. For instance, data of student work that is largely symbolic and not infused with comment and discourse will yield very little insight, as very few meanings (if any at all) are discernible from a record of symbolic procedural steps. This is one reason for the necessity of instructional design to begin from a meanings-oriented perspective, as the content of the students' posts dictate the types of coding enabled. To potentially elicit more robust argumentative discourse, we provided students with a short instructional video on engagement in

STEM discourse but made no requirement or formalization of the communication principles established in the video. Even with this intervention, the instances of argumentation were interspersed and argument-driven (as opposed to integration, counterargument, etc.). As such, finding ways to elicit varied and robust argumentation will only increase the usability of discussion data.

Another limitation, specifically to automating the qualitative analysis using a machine learning algorithm, is the non-textual nature of some data types. Supervised machine learning methods require the creator to identify features that may be indicative of a code. In mathematics education qualitative research, features are both implicitly and explicitly identified by individual coders in a potentially non-systematic way. This is further exacerbated by the inability of automated tools (for now?) to describe the salient mathematical features of a student's graph in the context of their answer. We anticipate the need for expert qualitative coders to textually describe students' graphs, tables, and other non-textual work so that textual feature analysis can be performed, and a supervised machine learning algorithm can be employed. This would necessarily limit the widespread use of theoretical frameworks to identify student thinking of graphs but would allow for partial automated coding of discussions.

One limitation of the current network analysis is the lack of timestamp data. Inclusion of time data would enable more flexible interpretation of the unfolding network in each thread. It would also better enable cross-thread analysis (as discourse also occurs between posts) and even more invisibly as students observe content in a discussion and then integrate the ideas within other threads without reference. Another limitation with the current network analysis is the reliance on coloring of nodes to convey information about whether one or more codes appear. While *networkx*, the most frequently used graph visualization Python package, does not allow users to change the shapes of nodes, other less user-friendly packages may. More effort could be made to address the inaccessible nature of relying on colors to convey information and provide templates for others to create more accessible network analysis graphs.

Link to Analytical Framework GitHub

https://github.com/Darryl-Chamberlain-Jr/Analyzing_Async_Discussions.

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Appendix A: MJ Codebook for Torty and Harry and Relevant Literature on Student Thinking and Learning of Rate of Change

We provide an elaborated and cited codebook both as an example of the ways one might attempt to capture expressed meanings in the context of a discussion, but also for those readers who are interested in further exploring the research on student thinking and learning, specifically the learning of the rate of change concept. This codebook was provided for (and subsequently updated by) Author 3 as she went through the coding process.

For the interested reader, we have placed citations within the code boxes that reference studies from the math ed literature base from which we developed certain codes. For instance, the *Constant Rate of Change Described as an Algebraic Ratio* code was adapted from Nagle et al. (2013), and the relevant row in the codebook will be given as follows:

Meaning Description	**Abbreviation**	**Description/Commentary**
Constant Rate of Change Described as an Algebraic Ratio (Nagle et al., 2013)	CROC-AR	In contrast to a graphical referent, the student now attends to some calculation or some algebraic formula, and specifies division by numerical or algebraic objects. “change in y over change in x” or a ratio of symbolic/algebraic objects. While the referents can also be quantitative or geometric, it is attention to the algebraic computation of the ratio and change measures that constitutes ROC-AR.

Other meanings are less directly attributable to certain authors, and so the given citations might not specifically cover student thinking on rate of change, however we applied the results of the study to the case of constant and average rate of change. For instance, our code *Function Described Non-Referential Symbolically* was adapted from Harel’s (2008) notion of a non-referential symbolic way of thinking, where students operate on symbols without attention to any quantitative (or otherwise meaningful) referent for the symbol. The associated row in the codebook is as follows:

Meaning Description	**Abbreviation**	**Description/Commentary**
Function Described as Nonreferential Symbolic (Harel, 2008)	F-NRS	Functions accompany symbols (like x^2 or e^x) that can be manipulated according to a learned set of rules to solve some problem. Not attending to quantities, graphs, or context, just the symbols and the rules of manipulation.

A Brief Conceptualization of Rate of Change

While the literature on rate of change is vast, and has been considered from many perspectives (e.g. Quantitative and Covariational Reasoning (Thompson & Carlson, 2017), APOS theory (Nagle et al., 2019), metonymy and metaphor (Zandieh & Knapp, 2006), etc.) we particularly attend to students' construction of mathematical meaning (in the sense of Thompson, Carlson, Byerley & Hatfield (2014)) as it relates to supporting students' development of quantitative and covariational (Thompson, 2011; Thompson & Carlson, 2017) ways of thinking in calculus and pre-calculus. We give a brief conceptualization of various rates of change below, and encourage the interested reader to further examine the works of Thompson and Carlson and colleagues (1992; 2002; 2011; 2014; 2017) for more in-depth and empirically-driven treatments.

Quantitatively understanding rates of change requires attention to the multiplicative comparisons of two related change quantities' measures as their underlying quantities co-vary within defined domains. Rather than just attending to the static ratio of the measures of two quantities at a single instance, one must link the two quantities (such as distance and time) together and anticipate that the quantities' measures smoothly vary in-tandem (we say co-vary). One particularly conceives of a constant rate of change by attending to the invariance of the multiplicative comparisons of the change measures for the two related quantities amidst variation in the intervals over which the change quantities are measured.

One conceives of non-constant rates of change by noting the existence of variation in the multiplicative comparison of related change quantities' measures, and then an average rate of change is an imputing of invariance on the multiplicative comparisons within a specified interval. Instantaneous rates find their meaning via the quantitative construction through approximations from average rates across shrinking intervals wherein the error between upper and lower bounds on average rates within the interval is made as small as desired, essentially linearizing the variation on small intervals.

Codebook

We now provide the annotated codebook for the MJ dimension, with reference to the related mathematics education studies.

Micro – Codes

These codes are to be applied in the following cases when the student's language or written work indicates an expression of meaning for the concepts of task (i.e. "speed", "slope", "rate of change").

Meaning Title	**Abbreviation**	**Description**
Constant Rate of Change Described Quantitatively (Thompson & Thompson, 1992; Thompson & Carlson, 2017)	CROC-QR	Student must indicate that the rate of change is the invariant multiplicatively comparative measure between related changes to two quantities.

Constant Rate of Change Described as a Geometric Ratio (Nagle et al., 2013)	CROC-GR	Specific to a graphical context, and the student is referring to the properties of some graph (as opposed to talking about some context). "Rise over run" or some indication of tracking related vertical and horizontal displacements.
Constant Rate of Change Described as an Algebraic Ratio (Nagle et al., 2013)	CROC-AR	In contrast to a graphical referent, the student now attends to some calculation or some algebraic formula and specifies division of numerical or algebraic objects. "change in y over change in x" or a ratio of symbolic/algebraic objects. While the referents can also be quantitative or geometric, it is attention to the algebraic computation of the ratio and change measures that constitutes ROC-AR.
Constant Rate of Change Described as a Contextual Ratio (Thompson & Thompson, 1992, Nagle et al.'s, (2013) notions of *Physical Property* and *Real-world situation*).	CROC-CR	Similar to GR, and AR, but contextual. "change in distance over change in time" a ratio of two changes of the contextual quantities.
Rate of Change Described as a "Derivative" (Adapted from Nagle et al.'s, (2013) notion of *Calculus Conception*).	ROC-D	Reference to limit of secant lines, or tangent line, or value of the derivative. Essentially, any rate of change "constant, average or instantaneous" described in terms of the derivative.
Constant Rate of Change Described as a Perceivable Geometric Property	CROC-GP	Here, CROC is described as a geometric property of a graph. Typically, students describe a physical/geometric feature such as "slantiness, steepness, slope up, slope down".

(Adapted from Nagle et al.'s, (2013) notions of *Geometric Ratio* and *Physical Property*).		
Constant Rate of Change Described as determining amount of increase or decrease. (Adapted from Carlson et al.'s (2002) *MA 2* and *MA 3* mental actions)	CROC-INC	i.e. reference to how high or low the graph goes.
Constant Rate of Change Described as a Perceivable Contextual Property (Nagle et al.'s (2013) *Real-World Situation*)	CROC-CP	Here, CROC is described as a physical property of a situation. Typically students describe a contextual feature such as "speed" without further reference to the distance/time dynamic. Also "sped up", "slow down", etc.
Constant Rate of Change Described as the "slope constant" m in $y = mx + b$. (Nagle et al.'s (2013) notion of *parametric coefficient*; Harel (2008))	CROC-NRS	Very formula driven. The role that CROC plays here is just as the "m" part of the formula, to perhaps be algebraically solved for (or used in the formula).
Average Rate of Change Described as an arithmetic mean of instantaneous rates (Eluded to by Tallman et al., (2016))	AROC-MEAN	Here, rather than describing a ratio of change quantities, the student appeals to the arithmetic mean (i.e. the "average") of a list of numbers, where here the numbers are the "rates" at any moment. "Take an average of all of the rates of change over an interval"
Average Rate of Change Described as distance changes over intervals of time	AROC-DIST	Here, the average rate of change is referenced as prescribing distance amounts traveled over set intervals of time. The rate is implicit here,

(Carlson et al., 2022; Thompson & Carlson, 2017)		as it is more of a correspondence between distance intervals and time intervals. “Traveling D meters every S seconds”
Average Rate of Change Described as slope of secant line.	AROC-SEC	The AROC here is specifically referenced as the slope of the secant line between two points on a graph. It might be accompanied with other codes, but the student should either reference (or have indicated that they considered) the graphical slope of a tangent line.
Average Rate of Change Described as linear substitute (Tallman et al., 2016)	AROC-LIN	Here we’re thinking just in a graphical context. AROC described as the slope of a line substituting the variation in the function over the set interval. “The function is curvy between x=1 and x=3, but we pretend that it’s a line of slope 5 instead”.
Average Rate of Change Described as Varying Intermittent Instantaneous Speeds	AROC-VAR	The opposite of AROC-LIN, and is context-agnostic (i.e. graphical, physical, speed, etc). Students specifically talk about “speeding up and slowing down” within an interval (as if referring to the instantaneous speeds, but without the particular language).
Function decontextualized and described as a class of functions	F-CL	Function or graph represents a class of functions (x^2 is a quadratic that curves up or down). Rather than thinking about the properties of a graph, the student associates a graph with a known function or function descriptor (exponential, quadratic, logarithmic).
Function described as nonreferential symbolic (Harel, 2008)	F-NRS	Functions accompany symbols (like x^2 or e^x) that can be manipulated according to a learned set of rules to solve some problem. Not attending to quantities, graphs, or context, just the symbols and the rules of manipulation.
	F-G	

Function geometrically described. (Associated with Moore & Thompson's (2015) *Static Shape Thinking*)		Function represents a familiar geometric object. Here, the graph does have some geometric property that is being referred to (in contrast with F-CLS).
Function described as input output	F-IO	When students think about *both* input and output ("you put in an x, you get out an x^2"). Or ("you put in a t, and you get out 5 miles). So both input output, which might interpreted graphically or physically.
Function described contextually	F-Ctxt	Function has a contextual meaning. "D(t)=t^2 means that at time t, Torty has traveled t^2 meters"
Derivative described as giving increase/decrease	D-INC	Appealing to the derivative as determining where a function increases or decreases "Since the derivative is positive, the function must increase"
Derivative described as giving Slope	D-S	Appealing to the derivative as determining the slope of a tangent line or of a linear approximation "At t=3 the slope of the tangent line is 2*t=6"
Derivative described as giving Rate of Change	D-ROC	Appealing to the derivative as giving a rate of change. "The derivative gives an instantaneous rate of change."

Macro – Codes

Successful determination (and specifically justification) of two graphs whose comparative AROCs satisfy the race requirements entails appealing to the functions' rates of change over various intervals of time, and moreover appealing to the persistence of the rate comparisons over all appropriate intervals (constituting a reflective abstraction of a ratio into a rate (Thompson & Carlson, 2017)).

The following macro codes specifically track the justifications the students might employ to qualify their graphs as satisfying the requirements. These codes should be applied to the posts as a whole (rather than on each line).

Macro Justification	**Abbreviation**	**Description**
Justification by Derivative	J-Der	Student might calculate derivatives to say things about the graphs' rates of change.
Justification by Geometry, Chunky (Thompson & Carlson, 2017; Castillo-Garsow et al., 2013)	J-Geo-Chunky	Student focuses on the comparative ROCs of the two graphs over contiguous intervals,
Justification by Geometry – Scaling (Ellis et al., 2020)	J-Geo-Scale	Student appeals to ROCS at significantly different scales of interval sizes (e.g. "zooms in" an gives Torty a small bump at the end of the race)
Justification by Geometry, continuous (Thompson & Carlson, 2017)	J-Geo-Cont	Student focuses on comparative ROCs over a variety of intervals,
Justification by context	J-Ctxt	Student appeals to possible physical aspects of the race (e.g. Torty moving in "spurts" or "going back and forth" or "winding around")
Coordination	Coord-Tot	Student coordinates the rate of change equation, geometry, and/or context

Partial Coordination	Coord-Part	Student coordinates two or more ideas while pursuing a solution, but not all of the necessary ideas.
Isolation	Coord-Iso	Student's activity attends to one particular focus in isolation