Objective 2 - Left and Right Limits

Interpret the notation for limits.

Link to section in online textbook.

Intro video for left/right limits.

Introduction to Notation

Let's look back at our notation:

Symbol	Meaning
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right
$x \to \infty$	x approaches infinity
$x \to -\infty$	x approaches negative infinity

This gives us a way to talk about the limits of functions **when the limits on either side do not match**. Let's look back at our Desmos link of $f(x) = \dfrac{1}{x}$ and try to evaluate the left and right limit at $x = 0$.

Question 1 *Evaluate the following limits:*

$$\lim_{x \to 0^-} \left(\frac{1}{x}\right) = \boxed{??}$$

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = \boxed{??}$$

This allows us to refine our definition of a limit:

Theorem 1. *Relating one-sided and two-sided limits*

Learning outcomes:

Author(s): Darryl Chamberlain Jr.

$$\lim_{x \to a}(f(x)) = L$$

if and only if

$$\lim_{x \to a^-}(f(x)) = L = \lim_{x \to a^+}(f(x))$$

In other words, if the limit is equal to something, the left and right limits agree (and if the left/right limits agree, the limit is equal to something). *Note: We say the limit **exists** if L is a Real number. The limit can be equal to ∞ or $-\infty$, but we would not say the limit **exists**.*

Evaluating One-Sided Limits - Graphically and Analytically

In Calculus I, you will learn a few tricks to evaluate more difficult limits. We will focus on evaluating limits of our elementary functions: polynomials, rational, radical, logarithmic, and exponential.

Graphical Evaluation

When we can graph a function, it is intuitive to evaluate limits *(especially limits that go to $\pm\infty$)*. We will scan along the function until we get *very close* to the value we are looking at. Try to evaluate the one-sided limits below.

Question 2

Graph of $f(x) = 1/x$

$\lim_{x \to 0^-} f(x) = \boxed{\textbf{??}}$

$\lim_{x \to 0^+} f(x) = \boxed{\textbf{??}}$

$\lim_{x \to 1^-} f(x) = \boxed{??}$

$\lim_{x \to -1^+} f(x) = \boxed{??}$

Question 3

Graph of $f(x) = 1/(x-3)^2$

$\lim_{x \to 3^-} f(x) = \boxed{??}$

$\lim_{x \to 3^+} f(x) = \boxed{??}$

$\lim_{x \to 2^-} f(x) = \boxed{??}$

$\lim_{x \to 2^+} f(x) = \boxed{??}$

Analytical Evaluation

If we cannot graph a function, we may want to analytically evaluate the one-sided limit. We can do this by plugging in numbers very close to the value and see what happens with the function.

Example 1. $\lim_{x \to 1^-} \dfrac{\frac{1}{x} - 1}{x - 1} =$

If we try $f(1)$*, we get* $\dfrac{0}{0}$*. This doesn't help us...*

Since we don't know what this graph looks like, let's try evaluating the one-sided limit analytically. We'll try the values $0.9, 0.99.0.999, 0.9999$ *since we are looking for the limit as we approach 1 from the left.*

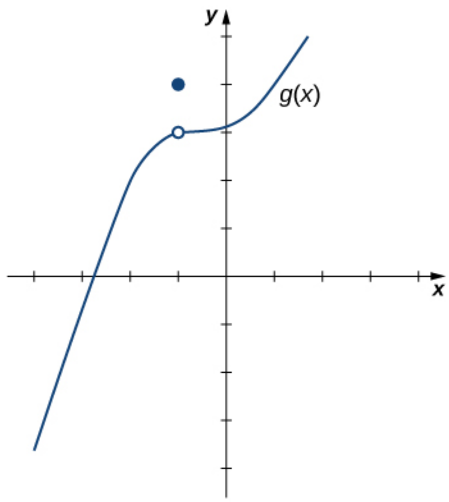

Figure 1: Function with a hole at $x = -1$.

x	$f(x)$
0.9	$\boxed{-1.11}$
0.99	$\boxed{-1.01}$
0.999	$\boxed{-1.001}$
0.9999	$\boxed{-1.0001}$

We can keep going, but based on the chart above the limit appears to be $\boxed{-1}$*!*

With technology, we can become more confident with this analytical approach. For us, this will be our best way to approximate limits of complicated functions.

Caution!

Limits do not care what the value is at the point!

The left- and right-sided limits of this function are both 3, so

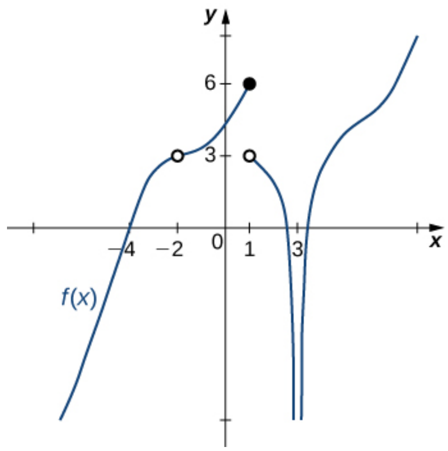

Figure 2: Piecewise function to evaluate.

$$\lim_{x\to -1^+} g(x) = 3$$
$$\lim_{x\to -1^-} g(x) = 3$$
and
$$\lim_{x\to -1} g(x) = 3$$
BUT
$$g(-1) = 4$$

Practice Evaluating One-Sided Limits

For the rest of this section, we will practice evaluating one-sided limits. You can graph these functions or use the analytical approach.

Question 4 *Based on the graph, evaluate the following one-sided limits.*

$\lim_{x\to -4^-} g(x) = \boxed{0}$

$\lim_{x\to -4^+} g(x) = \boxed{0}$

$$\lim_{x\to -2^-} g(x) = \boxed{3}$$

$$\lim_{x\to -2^+} g(x) = \boxed{3}$$

$$\lim_{x\to 1^-} g(x) = \boxed{6}$$

$$\lim_{x\to 1^+} g(x) = \boxed{3}$$

$$\lim_{x\to 3^-} g(x) = \boxed{\textbf{??}}$$

$$\lim_{x\to 3^+} g(x) = \boxed{\textbf{??}}$$

Question 5 *Let* $f(x) = (1+x)^{1/x}$*. Evaluate the following one-sided limits below.*

$$\lim_{x\to 0^-} = \boxed{2.7183}$$

$$\lim_{x\to 0^+} = \boxed{2.7183}$$

$$\lim_{x\to 1^-} = \boxed{2}$$

$$\lim_{x\to 1^+} = \boxed{2}$$
