Rigid Body Mechanics and Tensors

1 The Rigid Body Equations of Motion

1.1 Angular Momentum and Kinetic Energy of Motion about a Point

We consider two position vectors $\mathbf{R}_1$, $\mathbf{R}_2$ and a difference vector $\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$.

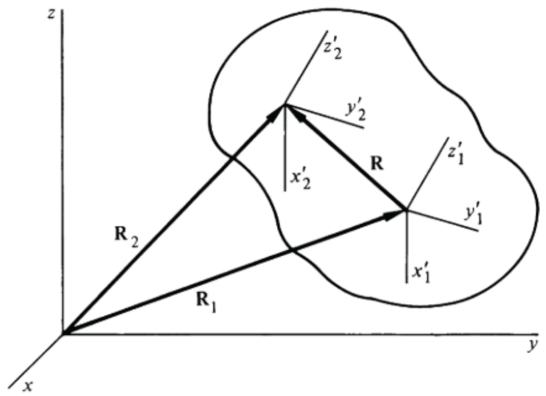

FIGURE 5.1 Vectorial relation between sets of rigid body coordinates with different origins.

*The center of gravity of course coincides with the center of mass in a uniform gravitational field.

Figure 1:

The rate of change of $\mathbf{R}_2$ relative to the space axes is given by:

$$\left(\frac{d\mathbf{R}_2}{dt}\right)_s = \left(\frac{d\mathbf{R}_1}{dt}\right)_s + \left(\frac{d\mathbf{R}}{dt}\right)_s = \left(\frac{d\mathbf{R}_1}{dt}\right)_s + \omega \times \mathbf{R}$$

where the term $\left(\frac{d\mathbf{R}}{dt}\right)_s$ is relative to the fixed-space axes.

Relative to the body axes:

$$\left(\frac{d\mathbf{R}}{dt}\right)_s = \left(\frac{d\mathbf{R}}{dt}\right)_b + \omega \times \mathbf{R}$$

If $\mathbf{R}$ is fixed in the body, then $\left(\frac{d\mathbf{R}}{dt}\right)_b = 0$.

When a rigid body moves with one fixed point, the angular momentum $\mathbf{L}$ is:

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i)$$

where $\mathbf{v}_i = \omega \times \mathbf{r}_i$.

Hence:

$$\begin{aligned}\mathbf{L} &= \sum_i m_i[\mathbf{r}_i \times (\omega \times \mathbf{r}_i)] \\ &= \sum_i m_i[\omega r_i^2 - \mathbf{r}_i(\mathbf{r}_i \cdot \omega)]\end{aligned}$$

Expand the x-component:

$$L_x = \omega_x \sum_i m_i(y_i^2 + z_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i$$

Similarly, we can write the components of the angular momentum vector as:

$$\begin{aligned}L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{aligned}$$

where I_{xx}, etc. are the moment of inertia coefficients or products of inertia.

$$\begin{aligned}I_{xx} &= \sum_i m_i(r_i^2 - x_i^2) = \sum_i m_i(y_i^2 + z_i^2) \\ I_{xy} &= -\sum_i m_i x_i y_i\end{aligned}$$

By a volume integration:

$$I_{xx} = \int_V \rho(\mathbf{r})(r^2 - x^2)dV$$

or

$$I_{jk} = \int_V \rho(\mathbf{r})(r^2\delta_{jk} - x_j x_k)dV$$

In sum:

$$\mathbf{L} = \mathbf{I}\omega$$

1.2 Tensors

1.2.1 Definition in Cartesian Space

In Cartesian three-dimensional space, a tensor $\mathbf{T}$ of the Nth rank may be defined as a quantity having 3^N components $T_{ijk\ldots}$ that transform under an orthogonal transformation of coordinates.

$$T'_{ijk\ldots}(x') = \sum_{l,m,n,\ldots} a_{il}a_{jm}a_{kn}\ldots T_{lmn\ldots}(x)$$

By definition:

- a tensor of zero rank is a scalar.
- a tensor of first rank is a vector: $T'_i = \sum_j a_{ij}T_j$.
- a tensor of second rank: $T'_{ij} = \sum_{k,l} a_{ik}a_{jl}T_{kl}$.

1.2.2 Transformation Rules

Under a linear change of coordinate:

$$\mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^{-1}$$

For an orthogonal transformation, $\mathbf{A}^{-1} = \mathbf{A}^T$:

$$\mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^T$$

or in component form:

$$T'_{ij} = \sum_{k,l} a_{ik} a_{jl} T_{kl}$$

1.2.3 Constructing Tensors

We can construct a tensor of second rank from two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{T} = \mathbf{A}\mathbf{B}^T \quad \text{or} \quad T_{ij} = A_i B_j$$

For example:

$$\mathbf{T} = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} = \begin{pmatrix} A_x B_x & A_x B_y \\ A_y B_x & A_y B_y \end{pmatrix}$$

By definition, the transformation is:

$$\begin{aligned} T'_{ij} &= \sum_{k,l=1}^{3} a_{ik} a_{jl} T_{kl} = \sum_{k,l} a_{ik} a_{jl} (A_k B_l) \\ &= (\sum_k a_{ik} A_k)(\sum_l a_{jl} B_l) = A'_i B'_j \end{aligned}$$

1.2.4 Tensor Operations

The dot product on the right of a tensor $\mathbf{T}$ with a vector $\mathbf{C}$ is defined as the vector $\mathbf{D}$ by:

$$\mathbf{D} = \mathbf{T} \cdot \mathbf{C} \quad \text{where} \quad D_i = \sum_{j=1}^{3} T_{ij} C_j = T_{ij} C_j$$

On the left with a vector $\mathbf{F}$:

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{T} \quad \text{where} \quad E_j = \sum_{i=1}^{3} F_i T_{ij} = F_i T_{ij}$$

A scalar being constructed by a double dot product:

$$S = \mathbf{F} \cdot \mathbf{T} \cdot \mathbf{C} \quad \text{where} \quad S = \sum_{i,j=1}^{3} F_i T_{ij} C_j = F_i T_{ij} C_j$$

If $T_{ij} = A_i B_j$, then:

$$\mathbf{T} \cdot \mathbf{C} = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) = (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

and

$$\mathbf{F} \cdot \mathbf{T} = (\mathbf{F} \cdot \mathbf{A})\mathbf{B}^T = (\mathbf{A} \cdot \mathbf{F})\mathbf{B}^T$$

2 The Inertia Tensor and the Moment of Inertia

2.1 Kinetic Energy of Rotation

The kinetic energy of motion about a point is:

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i v_i^2 \\
&= \frac{1}{2}\sum_i m_i(\omega \times \mathbf{r}_i) \cdot (\omega \times \mathbf{r}_i) \\
&= \frac{1}{2}\sum_i m_i \omega \cdot [\mathbf{r}_i \times (\omega \times \mathbf{r}_i)] \\
&= \frac{1}{2}\omega \cdot \left(\sum_i m_i[\mathbf{r}_i \times (\omega \times \mathbf{r}_i)]\right) \\
&= \frac{1}{2}\omega \cdot \mathbf{L}
\end{aligned}$$

Since $\mathbf{L} = \mathbf{I}\omega$, where $\mathbf{I}$ is the inertia tensor, the kinetic energy can be written as:

$$T = \frac{1}{2}\omega \cdot (\mathbf{I}\omega)$$

The kinetic energy $T_{rotation}$ is a bilinear form in the components of ω. Using index notation:

$$T_{rotation} = \frac{1}{2}\sum_i m_i(\omega \times \mathbf{r}_i)^2 = \frac{1}{2}\sum_{\alpha,\beta}\omega_\alpha\omega_\beta\left(\sum_i m_i(\delta_{\alpha\beta}r_i^2 - r_{i\alpha}r_{i\beta})\right)$$

It can be written as:

$$T_{rotation} = \frac{1}{2}\sum_{\alpha,\beta} I_{\alpha\beta}\omega_\alpha\omega_\beta$$

where $I_{\alpha\beta}$ is the moment of inertia tensor:

$$I_{\alpha\beta} = \sum_i m_i(\delta_{\alpha\beta}r_i^2 - r_{i\alpha}r_{i\beta})$$

For a continuous body, this becomes a volume integral:

$$I_{\alpha\beta} = \int_V \rho(\mathbf{r})(\delta_{\alpha\beta}r^2 - r_\alpha r_\beta)dV$$

2.2 Moment of Inertia about an Axis

Let $\mathbf{n}$ be a unit vector along the axis of rotation, so $\omega = \omega\mathbf{n}$. Then the kinetic energy is:

$$T = \frac{1}{2}\omega^2(\mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n}) = \frac{1}{2}I\omega^2$$

where I is a scalar, the moment of inertia about the axis of rotation defined by $\mathbf{n}$:

$$I = \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n} = \sum_i m_i[r_i^2 - (\mathbf{r}_i \cdot \mathbf{n})^2]$$

It can also be written as:

$$
\begin{aligned}
I &= \sum_i m_i(\mathbf{r}_i \times \mathbf{n})^2 \\
&= \frac{m_i(\omega \times \mathbf{r}_i)\cdot(\omega \times \mathbf{r}_i)}{\omega^2} \\
&= \frac{2T}{\omega^2}
\end{aligned}
$$

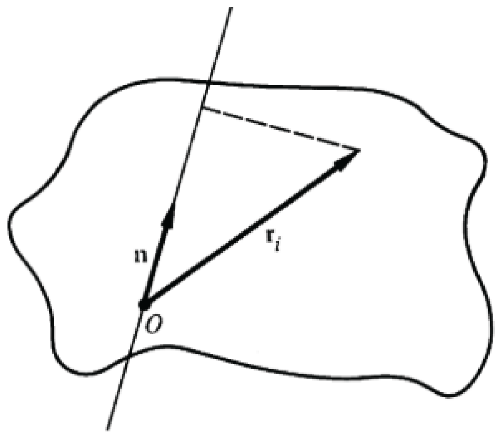

FIGURE 5.2 The definition of the moment of inertia.

Figure 2:

2.3 Parallel Axis Theorem

We have the relation $\mathbf{r}_i = \mathbf{R} + \mathbf{r'}_i$, where $\mathbf{R}$ is the vector from the origin O to the center of mass, and $\mathbf{r'}_i$ is the position vector of the particle i relative to the center of mass.

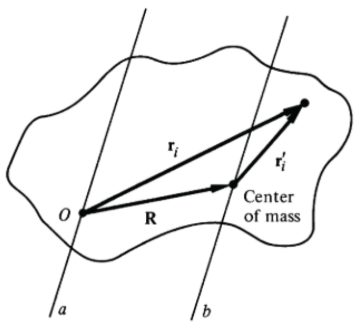

FIGURE 5.3 The vectors involved in the relation between moments of inertia about parallel axes.

Figure 3:

Therefore, the moment of inertia about an axis through the origin is:

$$\begin{aligned} I_a &= \sum_i m_i(\mathbf{r}_i \times \mathbf{n})^2 = \sum_i m_i[(\mathbf{R} + \mathbf{r'}_i) \times \mathbf{n}]^2 \\ &= \sum_i m_i[(\mathbf{R} \times \mathbf{n}) + (\mathbf{r'}_i \times \mathbf{n})]^2 \\ &= \sum_i m_i(\mathbf{R} \times \mathbf{n})^2 + \sum_i m_i(\mathbf{r'}_i \times \mathbf{n})^2 + 2\sum_i m_i(\mathbf{R} \times \mathbf{n}) \cdot (\mathbf{r'}_i \times \mathbf{n}) \end{aligned}$$

Since $\sum_i m_i\mathbf{r'}_i = 0$ by the definition of the center of mass, the last term vanishes.

$$\sum_i m_i(\mathbf{R} \times \mathbf{n}) \cdot (\mathbf{r'}_i \times \mathbf{n}) = (\mathbf{R} \times \mathbf{n}) \cdot \left(\left(\sum_i m_i\mathbf{r'}_i \right) \times \mathbf{n} \right) = 0$$

Hence,

$$I_a = I_0 + M(\mathbf{R} \times \mathbf{n})^2 = I_0 + MR^2 \sin^2 \theta$$

where $I_0 = \sum_i m_i(\mathbf{r'}_i \times \mathbf{n})^2$ is the moment of inertia about a parallel axis through the center of mass, M is the total mass, and θ is the angle between $\mathbf{R}$ and $\mathbf{n}$.

2.4 Inertial Ellipsoid

We define a vector $\mathbf{n} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$. The moment of inertia $I = \mathbf{n} \cdot \mathbf{I} \cdot \mathbf{n}$ can be expanded as:

$$I = I_{xx}\alpha^2 + I_{yy}\beta^2 + I_{zz}\gamma^2 + 2I_{xy}\alpha\beta + 2I_{yz}\beta\gamma + 2I_{zx}\gamma\alpha$$

We define a point $\mathbf{p}$ on the line of $\mathbf{n}$ such that $\mathbf{p} = \frac{\mathbf{n}}{\sqrt{I}}$. Then $n_i = p_i\sqrt{I}$, so $\alpha = p_x\sqrt{I}$, $\beta = p_y\sqrt{I}$, etc. Substituting this into the equation for I gives the equation for the inertial ellipsoid:

$$1 = I_{xx}p_x^2 + I_{yy}p_y^2 + I_{zz}p_z^2 + 2I_{xy}p_xp_y + 2I_{yz}p_yp_z + 2I_{zx}p_zp_x$$

This can be transformed into a simpler form by choosing the principal axes.

3 Eigenvalues of the Inertia Tensor and Principal Axis Transformation

By definition, we have $I_{xy} = I_{yx}$, etc. This means the inertia tensor will in general have nine components, but only six of them will be independent (three along the diagonal plus three of the off-diagonal elements).

The components of $\mathbf{L}$ are $L_i = \sum_j I_{ij}\omega_j$, and the kinetic energy is $T = \frac{1}{2}\omega \cdot \mathbf{L} = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j$.

For a vector $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ whose magnitude is $\sqrt{V_x^2 + V_y^2 + V_z^2}$, we consider a transformation $\mathbf{I}_D = \mathbf{R}\mathbf{I}\mathbf{R}^T$. We choose a coordinate system (the principal axes x', y', z') such that the inertia tensor is diagonal:

$$\mathbf{I}_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

where I_1, I_2, I_3 are the eigenvalues of $\mathbf{I}$, called the principal moments of inertia. In this basis, the equation for the inertial ellipsoid becomes:

$$1 = I_1 p_x'^2 + I_2 p_y'^2 + I_3 p_z'^2$$

For the transformation defined by Euler angles, $\mathbf{I} = \mathbf{S}\mathbf{I}_0\mathbf{S}^{-1}$. To find the eigenvalues, we solve the characteristic equation $\det(\mathbf{I} - \lambda\mathbf{1}) = 0$:

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0$$

As an example, we consider a homogeneous cube of density ρ, mass M, and side a. For a coordinate system with its origin at one corner and axes along the edges, the inertia tensor is:

$$\mathbf{I} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

(Note: The second matrix form with $b = Ma^2$ from the provided image has been transcribed. The first form with a factor of $Ma^2/12$ is the standard result for this setup.)

3.1 Radius of Gyration

We also define the radius of gyration k_0, defined by:

$$I = Mk_0^2$$

where I is the moment of inertia about a given axis and M is the total mass.

4 Solving Rigid Body Problems and the Euler Equations of Motion

4.1 General Formulation

We can write the kinetic energy of a rigid body as the sum of its translational and rotational kinetic energy:

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

For holonomic conservative systems, the Lagrangian $L(q, \dot{q})$ can be separated into two parts:

$$L(q, \dot{q}) = L_c(q_c, \dot{q}_c) + L_b(q_b, \dot{q}_b)$$

where L_c is the part involving the generalized coordinates of the center of mass, and L_b is the part relating to the orientation of the body about the center of mass.

For the rotational motion of a rigid body, we have the Euler Angles. Whether the rotation is about a fixed point or the center of mass, we have the fundamental equation relating the external torque $\mathbf{N}$ to the rate of change of the angular momentum vector $\mathbf{L}$:

$$\left(\frac{d\mathbf{L}}{dt}\right)_s = \mathbf{N}$$

The subscript 's' denotes the derivative in a fixed (space) reference frame. Using the transformation to a rotating (body) frame, this can be written as:

$$\left(\frac{d\mathbf{L}}{dt}\right)_s = \left(\frac{d\mathbf{L}}{dt}\right)_b + \omega \times \mathbf{L}$$

Thus, Euler's equation of motion in the body frame is:

$$\frac{d\mathbf{L}}{dt} + \omega \times \mathbf{L} = \mathbf{N}$$

Or, using index notation with the Levi-Civita symbol ϵ_{ijk}:

$$\frac{dL_i}{dt} + \epsilon_{ijk}\omega_j L_k = N_i$$

In the principal axis frame of the body, the components of angular momentum are related to the angular velocity components by the principal moments of inertia I_1, I_2, I_3:

$$L_i = I_i\omega_i \quad \text{(no summation over } i\text{)}$$

Substituting this into the equation of motion, we obtain Euler's equations in component form:

$$I_i\frac{d\omega_i}{dt} + \epsilon_{ijk}\omega_j(I_k\omega_k) = N_i$$

The expansions for $i = 1, 2, 3$ are:

$$\begin{cases} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3 \end{cases}$$

4.2 Torque-Free Motion of a Rigid Body

In the absence of any net torques ($\mathbf{N} = 0$), Euler's equations of motion are reduced to:

$$\begin{cases} I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) \\ I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) \end{cases}$$

From these equations, we can see that if the components of ω are to be constant (i.e., $\dot{\omega}_i = 0$), then the right-hand side of all three equations must be zero. This requires ω to be directed along only one of the principal axes.

4.3 Geometric Interpretation: Poinsot's Construction

We consider a coordinate system oriented along the principal axes of the body. Let a vector $\mathbf{P}$ be defined along the instantaneous axis of rotation:

$$\mathbf{P} = \frac{\omega}{\omega}$$

We define a function $F(\mathbf{P})$:

$$F(P) = \mathbf{P} \cdot \mathbf{I} \cdot \mathbf{P} = \sum_i P_i^2 I_i$$

The surfaces of constant F are ellipsoids. Specially, when $F = 1$, we get the *inertia ellipsoid*:

$$\sum_i P_i^2 I_i = 1$$

By definition, the gradient of F is given by $\nabla_p F = 2\mathbf{I} \cdot \mathbf{P}$. We can also show that:

$$\nabla_p F = \frac{2\mathbf{L}}{\sqrt{2T}}$$

where T is the rotational kinetic energy. We have two conserved quantities in torque-free motion:

1. **Kinetic Energy:** $T = \frac{1}{2} \sum_i I_i \omega_i^2 = \text{constant}$.
2. **Angular Momentum:** $\mathbf{L} = \text{constant}$ in the space frame.

The equation $\mathbf{P} \cdot \mathbf{L} = \sqrt{2T}$ defines a fixed plane in the space frame, known as the *invariable plane*. The point of contact between the inertia ellipsoid and the invariable plane traces a curve on the ellipsoid called the *polhode*, while the curve traced on the invariable plane is the *herpolhode*.

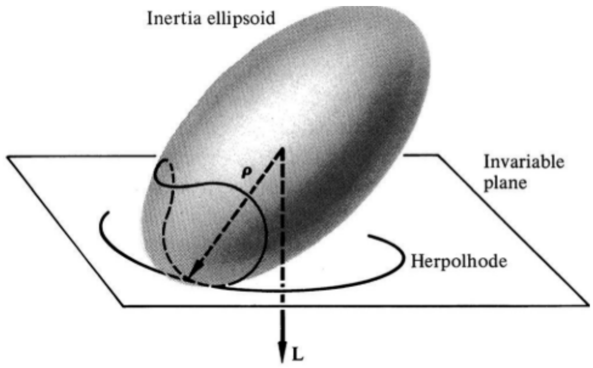

FIGURE 5.4 The motion of the inertia ellipsoid relative to the invariable plane.

Figure 4:

4.4 Binet's Construction

The conservation of kinetic energy T can be expressed in terms of the angular momentum vector $\mathbf{L}$ (since $L_i = I_i \omega_i$):

$$T = \sum_{i=1,2,3} \frac{L_i^2}{2I_i} = \text{constant}$$

This equation describes an ellipsoid in angular momentum space, referred to as the Binet ellipsoid. We can adopt the convention $I_3 \leq I_2 \leq I_1$. The equation for the ellipsoid is:

$$\sum_i \frac{L_i^2}{2I_i T} = 1$$

The conservation of the magnitude of angular momentum, $L^2 = |\mathbf{L}|^2$, gives:

$$\sum_i L_i^2 = L^2 = \text{constant}$$

This equation describes a sphere in angular momentum space. These two surfaces, the energy ellipsoid and the momentum sphere, intersect. The path of the tip of the angular momentum vector $\mathbf{L}$ in the body frame is the curve of intersection.

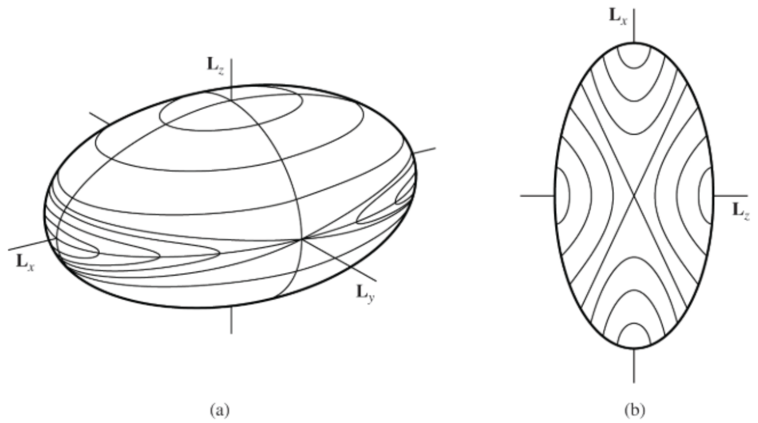

FIGURE 5.5 (a) The kinetic energy, or Binet, ellipsoid fixed in the body axes, and some possible paths of the **L** vector in its surface. (b) Side view of Binet ellipsoid.

Figure 5:

4.5 Torque-Free Motion of a Symmetric Top

By symmetry, we choose the symmetry axis as the principal axis 3, so that $I_1 = I_2$. Euler's equations then reduce to:

$$\begin{cases} I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3 \\ I_1\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 = 0 \end{cases}$$

From the third equation, we immediately see that ω_3 is a constant.

$$\dot{\omega}_3 = 0 \implies \omega_3 = \text{constant}$$

The first two equations become:

$$\dot{\omega}_1 = -\Omega\omega_2, \quad \dot{\omega}_2 = \Omega\omega_1$$

where Ω is the precession frequency, defined as:

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

This system of equations describes simple harmonic motion, with the solution:

$$\omega_1(t) = A\cos(\Omega t), \quad \omega_2(t) = A\sin(\Omega t)$$

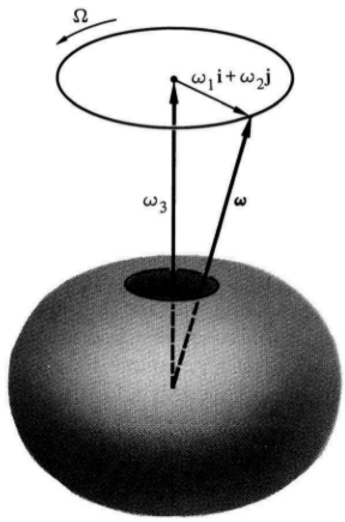

FIGURE 5.6 Precession of the angular velocity about the axis of symmetry in the force-free motion of a symmetrical rigid body.

Figure 6:

This implies that the vector component (ω_1, ω_2) has a constant magnitude $A = \sqrt{\omega_1^2 + \omega_2^2}$ and rotates uniformly about the body's z-axis (axis 3) with the angular frequency Ω.

In terms of the constant amplitude A and constant component ω_3, the kinetic energy and squared angular momentum are:

$$T = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \omega_3^2$$
$$L^2 = I_1^2 A^2 + I_3^2 \omega_3^2$$

5.7 The Heavy Symmetrical Top with one Point Fixed

We have three Euler's angles:

- θ gives the inclination of the z axis from the vertical.
- ϕ measures the azimuth of the top about the vertical.
- ψ is the rotation angle of the top about its own z axis.

$$\begin{cases} \dot{\psi} = \text{rotation/spinning of the top about its own figure axis, } z \\ \dot{\phi} = \text{precession/rotation of the figure axis } z \text{ about the vertical axis } z' \\ \dot{\theta} = \text{nutation/bobbing up and down of the } z \text{ figure axis relative to the vertical space axis } z' \end{cases}$$

We have $\dot{\psi} \gg \dot{\theta} \gg \dot{\phi}$, and $I_1 = I_2 \neq I_3$.

Euler's equations become:

$$\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1) &= N_1 \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= N_2 \\ I_3\dot{\omega}_3 &= N_3 \end{aligned}$$

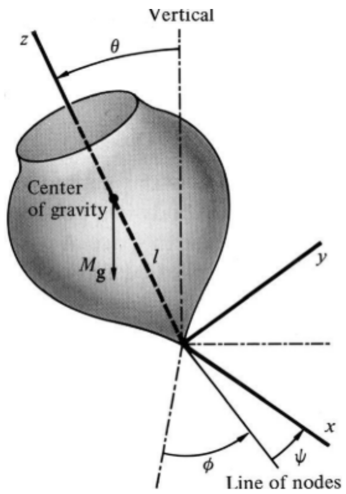

FIGURE 5.7 Euler's angles specifying the orientation of a symmetrical top.

Figure 7:

We consider initially, $N_3 = N_2 = 0$, $N_1 \neq 0$, $\omega_1 = \omega_2 = 0$, $\omega_3 \neq 0$. The ω_3 is constant. By symmetry, the kinetic energy:

$$T = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

In terms of Euler's angles

$$T = \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

In a constant gravitational field the potential energy is the same as if the body were concentrated at the center of mass.

$$V = -m\mathbf{g} \cdot \mathbf{r_G} = -mgL\cos\theta$$

Thus,

$$L = \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta$$

Note that ϕ and ψ are cyclic coordinates. The corresponding generalized momenta:

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_3 := I_1 a$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = I_1 b$$

where a, b are constants.

The system is conservative, E is constant in time.

$$E = T + V = \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3\omega_3^2 + Mgl\cos\theta$$

where $I_3\omega_3 = I_1 a - I_3\dot{\phi}\cos\theta$.

$$\begin{cases} I_1\dot{\phi}\sin^2\theta + I_1 a\cos\theta = I_1 b \\ \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \\ \dot{\psi} = \frac{I_1 a}{I_3} - \dot{\phi}\cos\theta = \frac{I_1 a}{I_3} - \frac{b - a\cos\theta}{\sin^2\theta}\cos\theta \end{cases}$$

Thus, $E' = E - \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1\dot{\theta}^2 + \frac{I_1(b-a\cos\theta)^2}{2\sin^2\theta} + Mgl\cos\theta$ is a constant of the motion.

And the effective potential is given by

$$V'(\theta) = Mgl\cos\theta + \frac{I_1}{2}\left(\frac{b-a\cos\theta}{\sin\theta}\right)^2$$

We define four normalized constants

$$\alpha = \frac{2E'}{I_1}, \quad \beta = \frac{2Mgl}{I_1}, \quad a = \frac{p_\psi}{I_1}, \quad b = \frac{p_\phi}{I_1}$$

$$\implies \alpha = \dot{\theta}^2 + \frac{(b-a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta$$

Using $u = \cos\theta$, it'll be written as

$$\dot{u}^2 = (1-u^2)(\alpha-\beta u) - (b-au)^2$$

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

Point 2

We designate the function

$$f(u) = \beta u^3 - (\alpha+a^2)u^2 + (2ab-\beta)u + (\alpha-b^2)$$

where the roots are u_1, u_2, u_3 such that $u_1 < u_2 < u_3$.

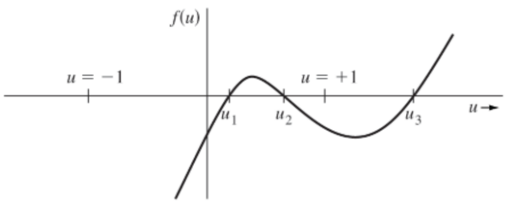

FIGURE 5.8 Illustrating the location of the turning angles of θ in the motion of a heavy symmetric top supported on a horizontal plane. A point support could allow one of the roots to be negative.

Figure 8:

The curve is the locus of the figure axis whose shape is determined by the value of the root of $b - au \implies u' = \frac{b}{a}$.

For u' lying between u_1 and u_2, the locus exhibits loops. (b)

For $u' = u_2$, the locus is tangent to the bounding circles such that $\dot{\phi}$ is in the same direction at both θ_1 and θ_2.

For $u' = u_3$ or $u' = u_1$, (c, c') $\dot{\phi}$ and $\dot{\theta}$ vanish, the locus have cusps touching the circle.

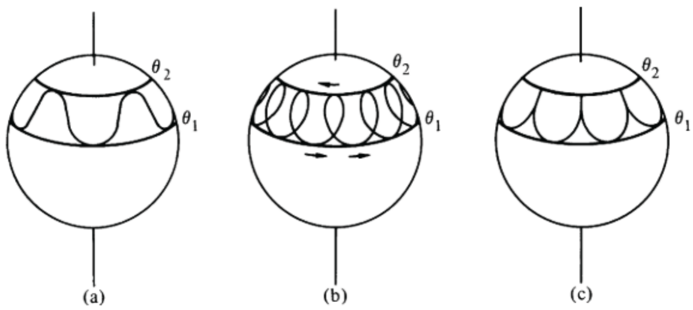

FIGURE 5.9 The possible shapes for the locus of the figure axis on the unit sphere.

Figure 9:

Point 3

For nutation: $E' = Mgl\cos\theta_0$, $c \gg a \implies \dot{u} = 0$.

$$f(u) = (u_0 - u)[\beta(1 - u^2) - \alpha^2(u_0 - u)]$$

The root of $f(u)$, u, satisfies

$$(1 - u_0^2) - \beta^2(u_0 - u) = 0$$

Denoting $u_0 - u$ by x and $u_0 - x_1$ by x_1.

$$\implies x_1^2 + px_1 - q = 0$$

where $p = a^2 - 2\cos\theta_0$, $q = \sin^2\theta_0$. We assume that $\frac{1}{2}I_3\omega_3^2 \gg 2Mgl$ which implies that $p \gg q$.

$$\frac{a^2}{B} = \left(\frac{I_3}{I_1}\right)\frac{I_3\omega_3^2}{2Mgl}$$

and the root is then $x_1 = \frac{q}{p}$.

Point 4

Neglecting $2\cos\theta_0$, $x_1 = \frac{\beta\sin^2\theta_0}{a^2} = \frac{I_1}{I_3}\frac{2Mgl}{I_3\omega_3^2}\sin^2\theta_0$.

For the fast top, $1 - u^2$ can be replaced by $\sin^2\theta_0$.

$$\implies f(u) = x^2 = a^2x(x_1 - x)$$

By changing variable to $y = x - \frac{x_1}{2}$

$$\implies \dot{y}^2 = a^2(\frac{x_1^2}{4} - y^2) \implies \ddot{y} = -a^2y$$

In the condition $x = 0$ at $t = 0$, the solution $x = \frac{x_1}{2}(1 - \cos at)$.

The angular frequency of nutation of the figure axis between θ_0 and θ_1 is $a = \frac{I_3}{I_1}\omega_3$ which increases the faster the top is spun initially.

Finally, $\dot{\phi} = \frac{a(u_0 - u)}{\sin^2\theta_0} \approx \frac{ax}{\sin^2\theta_0} = \frac{B}{2a}(1 - \cos at)$.

The average of which is $\bar{\dot{\phi}} = \frac{B}{2a} = \frac{Mgl}{I_3\omega_3}$.

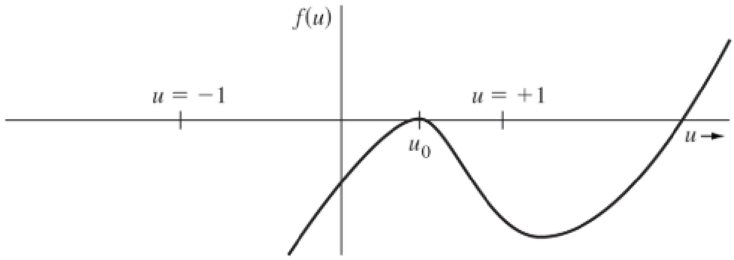

FIGURE 5.10 Appearance of $f(u)$ for a regular precession.

Figure 10:

Point 5

We consider a case where θ remains constant at θ_0, $\implies \dot{\theta}_1 = \ddot{\theta}_1 = 0$.
$f(u) = u^2 = 0$. $\frac{df}{du} = 0$. $u = u_0$. with $\dot{u} = 0$.

$$\implies f(\alpha - \beta u_0) = \frac{(b - au_0)^2}{1 - u_0^2}$$

$$\frac{B}{2} = a(b - au_0) - \frac{u_0(\alpha - \beta u_0)}{1 - u_0^2} = \frac{u_0(a - \beta u_0)}{1 - u_0}$$

$$\implies \frac{B}{2} = a\dot{\phi} - \dot{\phi}^2 \cos\theta_0$$

In terms of ω_3 or $\psi, \dot{\phi}$

$$Mgl = \dot{\phi}(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0)$$

or

$$Mgl = \dot{\phi}(I_3(\dot{\psi} - \dot{\phi}) - (I_1 - I_3)\dot{\phi}\cos\theta_0)$$

whose discriminant must be positive:

$$I_3^2\omega_3^2 > 4MglI_1\cos\theta_0$$

$$\theta_0 > \frac{\pi}{2} : \omega_3 \text{ leads to uniform precession}$$

$$\theta_0 < \frac{\pi}{2} : \omega_3 > \omega_{3_{min}} = \frac{2}{I_3}\sqrt{MglI_1\cos\theta_0}$$

$$\begin{cases} \dot{\phi} \approx \frac{B}{2a} = \frac{MgI}{I_3\omega_3} & \text{(slow)} \\ \dot{\psi} \approx \frac{I_3\omega_3}{I_1\cos\theta_0} & \text{(fast)} \end{cases}$$

Point 6

At time $t = 0$. $E' = E - \frac{1}{2}I_3\omega_3^2 = Mgl$. By definitions of α and $\beta \implies \alpha = \beta$. Therefore $\dot{u}^2 = (1 - u^2)\beta(1 - u) - a^2(1 - u)^2$ or $\dot{u}^2 = (1 - u)^2[\beta(1 + u) - a^2]$. where $u = 1$ is a double root and the third root is $u_3 = \frac{a^2}{\beta} - 1$.

If $a^2/\beta > 2$ (fast) $\implies u_3 > 1$. If $a^2/\beta < 2$ (slow) $\implies u_3 < 1$. Specially, $\frac{a^2}{\beta} = \left(\frac{I_3}{I_1}\right)^2 \frac{I_1\omega_3^2}{2Mgl} = 2$ or $\omega^2 = \frac{4MglI_1}{I_3^2}$.

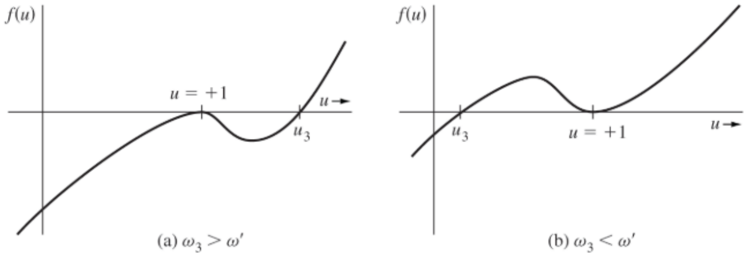

FIGURE 5.11 Plot of $f(u)$ when the figure axis is initially vertical.

Figure 11:

5.8 Precession of the Equinoxes and of Satellite Orbits

Mutual Gravitational Potential

The mutual gravitational potential between two bodies is given by:

$$V = -\frac{Gm_1M}{r_1} - \frac{Gm_2m_1}{|\mathbf{r_1} - \mathbf{r_2}|}$$

$$V = -\frac{GMm}{r}\frac{1}{\sqrt{1 + (r'/r)^2 - 2(r'/r)\cos\psi}}$$

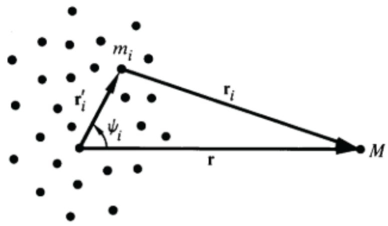

FIGURE 5.12 Geometry involved in gravitational potential between an extended body and a mass point.

Figure 12:

By the generating function for Legendre polynomials:

$$V = -\frac{GMm}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(\cos\psi)$$

where r is the distance from the origin to M, and

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

For a continuous spherical body with only a radial variation of density, only the first term survives. For a body with spherical symmetry and $\rho(\mathbf{r'})$,

$$\iiint d^3r'\rho(\mathbf{r'})(\frac{r'}{r})^n P_n(\cos\psi)$$

Using spherical polar coordinates, with the polar axis along $\mathbf{r}$, this becomes:

$$2\pi \int r'^2 dr' \rho(r') (\frac{r'}{r})^n \int_{-1}^{+1} d(\cos\psi) P_n(\cos\psi)$$

which vanish except at $n = 0$. Since $n = 1$:

$$-\frac{GM}{r^2} m r' \cos\psi_i = -\frac{GM}{r^2} \mathbf{r} \cdot m_i \mathbf{r_i}$$

which is zero by the choice of the center of mass. For $n = 2$, it can be written as:

$$-\frac{GM}{2r^3} m_i r_i^2 (1 - 3\cos^2\psi_i)$$

By tensor manipulation:

$$V = -\frac{GMm}{r} + \frac{GM}{2r^3}(3I_{rr} - \mathrm{Tr}\boldsymbol{I})$$

where I_{rr} is the moment of inertia about $\mathbf{r}$ and $\boldsymbol{I}$ is the moment of inertia tensor.

From the diagonal representation:

$$V = -\frac{GmM}{r} + \frac{GM}{2r^3}[3(I_1\alpha^2 + I_2\beta^2 + I_3\gamma^2) - (I_1 + I_2 + I_3)]$$

which is the MacCullagh's formula. We take the axis of symmetry to be the third principal axis, so that $I_1 = I_2$. If α, β, γ are the direction cosines of $\mathbf{r}$ relative to the principal axes,

$$I_{rr} = I_1(\alpha^2 + \beta^2) + I_3\gamma^2 = I_1 + (I_3 - I_1)\gamma^2$$

Thus,

$$\begin{aligned} V &= -\frac{GMm}{r} + \frac{GM(I_3 - I_1)}{2r^3}(3\gamma^2 - 1) \\ &= -\frac{GMm}{r} + \frac{GM(I_3 - I_1)}{r^3} P_2(\gamma) \end{aligned}$$

whose terms that could give rise to the torques is

$$V_2 = \frac{GM(I_3 - I_1)}{r^3} P_2(\gamma)$$

For the example of Earth's precession, γ is the direction cosine between the figure axis of Earth and the radius vector from Earth's center to the Sun or Moon.

We have $\gamma = \sin\theta\cos\psi$. Hence:

$$V_2 = \frac{GM(I_3 - I_1)}{2r^3}(3\sin^2\theta\cos^2\psi - 1)$$

The average of which is

$$\bar{V_2} = \frac{GM(I_3 - I_1)}{2r^3}(\frac{3}{2}\sin^2\theta - 1) = \frac{GM(I_3 - I_1)}{2r^3}(\frac{3}{2} - \frac{3}{2}\cos^2\theta)$$

$$= -\frac{GM(I_3 - I_1)}{2r^3} P_2(\cos\theta)$$

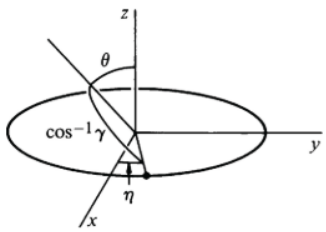

FIGURE 5.13 Figure axis of Earth relative to orbit of mass point.

Figure 13:

The Lagrangian

The Lagrangian is

$$L = \frac{1}{2}I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - V(\cos\theta)$$

We only assume uniform precession, i.e., $\ddot\theta, \dot\theta$ are zero. Thus:

$$\frac{\partial L}{\partial\dot\phi} = I_1\dot\phi\sin^2\theta - I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta - \frac{\partial V}{\partial\dot\phi} = 0$$

or

$$I_3\omega_3\dot\phi - I_1\dot\phi^2\cos\theta = \frac{\partial V}{\partial(\cos\theta)}$$

For slow precession, $\dot\phi \ll \omega_s$. $\dot\phi^2$ can be neglected.

$$\Rightarrow \dot\phi = \frac{1}{I_3\omega_3}\frac{\partial V}{\partial(\cos\theta)}$$

$$= \frac{-3GM}{2\omega_3 r^3}\frac{I_3 - I_1}{I_3}\cos\theta$$

For the case of the precession due to the Sun, we take the semi-major axis of Earth's orbit and using Kepler's law:

$$\omega_0^2 = (\frac{2\pi}{T})^2 = \frac{GM}{r^3}$$

$$\Rightarrow \frac{\dot\phi}{\omega_0} = -\frac{3}{2}\frac{\omega_0}{\omega_3}\frac{I_3 - I_1}{I_3}\cos\theta$$

$\dot\phi$ can be written as:

$$\dot\phi = \frac{T}{2\pi r^2}\frac{\partial V}{\partial(\cos\theta)}$$

$$= -\frac{T}{2\pi}\frac{3}{2}\frac{G(I_3 - I_1)}{r^5}\cos\theta$$

5.9 Precession of Systems of Charges in a Magnetic Field

Magnetic Moment and Torque

The magnetic moment of a system of moving charges is

$$\mathbf{M} = \frac{1}{2}\sum_i q_i(\mathbf{r_i} \times \mathbf{v_i}) \to \frac{1}{2}\int dV \rho_e(\mathbf{r})(\mathbf{r} \times \mathbf{v})$$

The angular moment is

$$\mathbf{L} = \sum_i m_i(\mathbf{r_i} \times \mathbf{v_i}) \to \int dV \rho_m(\mathbf{r})(\mathbf{r} \times \mathbf{v})$$

$$\Rightarrow \mathbf{M} = \gamma \mathbf{L} \quad \text{where the gyromagnetic ratio is } \gamma = \frac{q}{2m}$$

The potential:

$$V = -(\mathbf{M} \cdot \mathbf{B})$$

The torque:

$$\mathbf{N} = \mathbf{M} \times \mathbf{B}$$

Thus:

$$\frac{d\mathbf{L}}{dt} = \mathbf{L} \times \gamma \mathbf{B}$$

For the classical gyromagnetic ratio the precession angular velocity is

$$\omega_\mathbf{l} = -\frac{q}{2m}\mathbf{B} \quad \text{known as the Larmor frequency}$$

Lagrangian Formulation

For a system

$$L = \frac{1}{2}m_i v_i^2 + \frac{q}{m}m_i \mathbf{v_i} \cdot \mathbf{A}(\mathbf{r_i}) - V(|\mathbf{r_i} - \mathbf{r_j}|)$$

where

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$$

In terms of $\mathbf{B}$,

$$L = \frac{1}{2}m_i v_i^2 + \frac{qB}{2m}(\mathbf{r_i} \times m_i \mathbf{v_i}) \cdot \hat{k} - V(|\mathbf{r_i} - \mathbf{r_j}|)$$

The interaction with the magnetic field:

$$\frac{qB}{2m}\hat{k} \cdot \mathbf{L} = \mathbf{M} \cdot \mathbf{B} = -\omega_\mathbf{l} \cdot (\mathbf{r_i} \times m_i \mathbf{v_i})$$

The velocities relative to the new axes has the relation

$$\mathbf{v_i} = \mathbf{v_i}' + \omega_\mathbf{l} \times \mathbf{r_i}$$

The two terms in the Lagrangian affected by the transformation are:

$$\frac{m_i}{2}v_i^2 = \frac{m_i}{2}v_i'^2 + m_i \mathbf{v_i}' \cdot (\omega_\mathbf{l} \times \mathbf{r_i}) + \frac{m_i}{2}(\omega_\mathbf{l} \cdot \mathbf{r_i}) \cdot (\omega_\mathbf{l} \times \mathbf{r_i})$$

$$-\omega_{\mathbf{l}} \cdot \mathbf{r_i} \times m_i \mathbf{v_i}' = -\omega_{\mathbf{l}} \cdot (\mathbf{r_i} \times m_i \mathbf{v_i}') - \omega_{\mathbf{l}} \cdot (\mathbf{r_i} \times m_i(\omega_{\mathbf{l}} \times \mathbf{r_i}))$$

where

$$-\frac{m_i}{2}(\omega_{\mathbf{l}} \times \mathbf{r_i}) \cdot (\omega_{\mathbf{l}} \times \mathbf{r_i}) = -\frac{1}{2}\omega_l \cdot \boldsymbol{I} \cdot \omega_l = -\frac{1}{2} I_l \omega_l^2$$

where I_l denotes the moment of inertia about $\omega_{\mathbf{l}}$. Thus,

$$L = \frac{1}{2} m_i v_i'^2 - V(|\mathbf{r_i}' - \mathbf{r_j}'|) - \frac{1}{2} I_l \omega_l^2$$