Chapter VI: The Kinematics of Rigid Body Motion

4.1 The Independent Coordinates of a Rigid Body

1. For a rigid body with N particles, the degrees of freedom are reduced by the constraints $r_{ij} = c_{ij}$. To fix a rigid body, we just need to specify any three non-collinear points, thus the degrees of freedom can't be more than 9, and with $r_{12} = c_{12}$, $r_{23} = c_{23}$, $r_{13} = c_{13}$, they're reduced to 6.

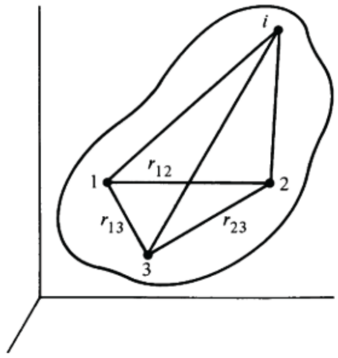

FIGURE 4.1 The location of a point in a rigid body by its distances from three reference points.

Figure 1:

2. For two Cartesian coordinates, the direction cosines are defined by

$$\begin{cases} \cos\theta_{11} = \cos(\mathbf{i}', \mathbf{i}) = \mathbf{i}' \cdot \mathbf{i} \\ \cos\theta_{12} = \cos(\mathbf{i}', \mathbf{j}) = \mathbf{i}' \cdot \mathbf{j} \\ \cos\theta_{21} = \cos(\mathbf{j}', \mathbf{i}) = \mathbf{j}' \cdot \mathbf{i}, \text{etc.} \end{cases}$$

   In terms of the unit vectors of the unprimed system, the unit vector in the primed system can be expressed

$$\begin{aligned} \mathbf{i}' &= \cos\theta_{11}\mathbf{i} + \cos\theta_{12}\mathbf{j} + \cos\theta_{13}\mathbf{k} \\ \mathbf{j}' &= \cos\theta_{21}\mathbf{i} + \cos\theta_{22}\mathbf{j} + \cos\theta_{23}\mathbf{k} \\ \mathbf{k}' &= \cos\theta_{31}\mathbf{i} + \cos\theta_{32}\mathbf{j} + \cos\theta_{33}\mathbf{k} \end{aligned}$$

   Thus, we can write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'$$

   where $x' = (\mathbf{i}' \cdot \mathbf{r})$, $y' = (\mathbf{j}' \cdot \mathbf{r})$, $z' = (\mathbf{k}' \cdot \mathbf{r})$. Note that $\mathbf{i}' \cdot \mathbf{j}' = \mathbf{j}' \cdot \mathbf{k}' = \mathbf{k}' \cdot \mathbf{i}' = 0$ and $\mathbf{i}' \cdot \mathbf{i}' = \mathbf{j}' \cdot \mathbf{j}' = \mathbf{k}' \cdot \mathbf{k}' = 1$.

   We can use the Kronecker symbol

$$\sum_{l=1}^{3} \cos\theta_{li'} \cos\theta_{lj'} = \delta_{ij}$$

4.2 Orthogonal Transformations

1. As showed before, for $x \to x_1, y \to x_2, z \to x_3$, we write $a_{ij} = \cos\theta_{ij}$, thus

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}$$

   We use the summation convention introduced by Einstein

$$x_i' = \sum_j a_{ij}x_j, \quad i = 1, 2, 3$$

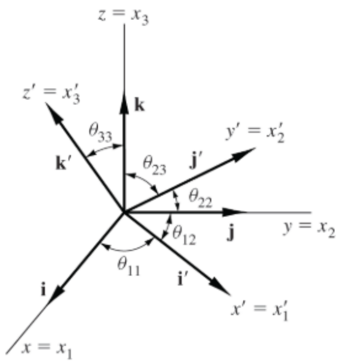

FIGURE 4.3 Direction cosines of the body set of axes relative to an external set of axes.

Figure 2:

and $\sum_i x'_i x'_i$ can be written as $x'_i x'_i$.
Thus,

$$x'_i x'_i = x_j x_j = a_{ij} a_{ik} x_j x_k$$
$$\Rightarrow a_{ij} a_{ik} = \delta_{jk}, \quad j,k = 1,2,3$$

which is the orthogonal condition.
The matrix of transformation is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

such that $\mathbf{x}' = A\mathbf{x}$.

4.3 Formal Properties of the Transformation Matrix

1. We consider two successive transformations

$$\mathbf{x}' = B\mathbf{x} \quad \text{and} \quad \mathbf{x}'' = A\mathbf{x}'$$

or

$$x'_k = b_{kj} x_j \quad \text{and} \quad x''_i = a_{ik} x'_k$$

Thus, we have

$$x''_i = a_{ik} b_{kj} x_j \Rightarrow x''_i = c_{ij} x_j$$

where $C = AB$ where $c_{ij} = a_{ik} b_{kj}$.
or $C = BA$ is defined by $d_{ij} = b_{ik} a_{kj}$.

2.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad x' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$
$$(Ax)_i = a_{ij} x_j = x'_i \Rightarrow x' = Ax$$

3. $(AB)C = A(BC)$

4. $C = A + B \Rightarrow c_{ij} = a_{ij} + b_{ij}$

5. For A^{-1} where $x_i = a_{ij} x'_j$

$$\Rightarrow a_{ki} a_{ij} = \delta_{kj} \quad \text{and} \quad AA^{-1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

where $x = Ix$.

6. Let's consider $a_{kl}a_{ki}a_{ij}$ or $c_{lj}a_{ij}$ with $c_{li} = a_{kl}a_{ki}$ or $a_{ki}d_{kj}$ with $d_{kj} = a_{ki}a_{ij}$.
   $a_{kl}\delta_{kj} = a_{lj}$
7. $A \to A^{-1} : a'_{ij} = a_{ji}$. For orthogonal matrices, $A^{-1} = \tilde{A}$ where $\tilde{A}$ is the transposed matrix, and thus we have
$$a_{ki}a_{ji} = \delta_{kj} \Rightarrow A\tilde{A} = I$$
8. $A_{ij}x_j = x_j(A)_{ji} \Rightarrow Ax = x\tilde{A}$
9. If $A_{ij} = A_{ji}$, the matrix is symmetric.
   If $A_{ij} = -A_{ji}$, the matrix is antisymmetric.
10. $|AB| = |A||B|$
    $A\tilde{A} = I \Rightarrow |A||\tilde{A}| = |A|^2 = 1$
    $|A^{-1}||A| = 1$

4.4 The Euler Angles

We consider a transformation

$$xyz \xrightarrow[\text{about } z]{\text{rotate by } \phi \text{ counterclockwise}} \xi\eta\zeta \xrightarrow[\text{about } \xi]{\text{rotate by } \theta \text{ counterclockwise}} \xi'\eta'\zeta'$$

$$\xrightarrow[\text{about } \zeta']{\text{rotate by } \psi \text{ counterclockwise}} x'y'z'$$

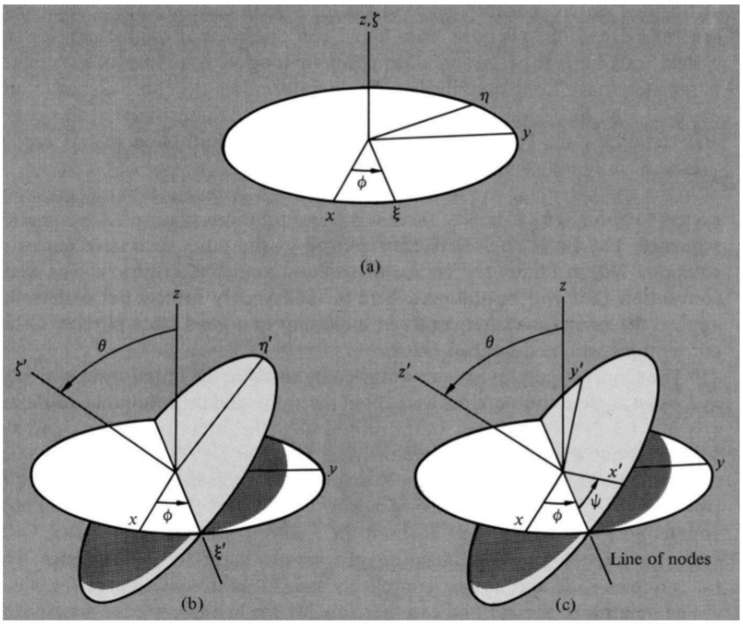

FIGURE 4.7 The rotations defining the Eulerian angles.

Figure 3:

We have

$$\begin{cases} ① & \xi = Dx \\ ② & \xi' = C\xi \\ ③ & x' = B\xi' \end{cases} \qquad \text{where } \xi \text{ and } x \text{ are column matrices}$$

Hence, $A = BCD$ where

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.5 The Cayley-Klein Parameters and Related Quantities

We consider four parameters $\alpha, \beta, \gamma, \delta$ with the constraints that $\beta = -\gamma^*$ and $\delta = \alpha^*$.

The transformation matrix

$$A = \begin{bmatrix} \frac{1}{2}(\alpha^2 - \gamma^2 + \delta^2 - \beta^2) & \frac{i}{2}(\gamma^2 - \alpha^2 + \delta^2 - \beta^2) & \gamma\delta - \alpha\beta \\ \frac{1}{2}(\alpha^2 + \gamma^2 - \beta^2 - \delta^2) & \frac{1}{2}(\alpha^2 + \gamma^2 + \beta^2 + \delta^2) & -i(\alpha\beta + \gamma\delta) \\ \beta\delta - \alpha\gamma & i(\alpha\gamma + \beta\delta) & \alpha\delta + \beta\gamma \end{bmatrix}$$

And we choose the Euler parameters e_0, e_1, e_2, e_3 where $\sum_{i=0}^{3} e_i^2 = 1$ and

$$\alpha = e_0 + ie_3$$
$$\beta = e_2 + ie_1$$

4.6 Euler's Theorem on the Motion of a Rigid Body

1. **Euler's Theorem:** The general displacement of a rigid body with one point fixed is a rotation about some axis.

   $$\Rightarrow R' = AR = R\lambda$$

   where λ is a complex constant where we have the eigenvalue equations

   $$(A - \lambda I)R = 0 \Rightarrow |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

   which is the characteristic equation.

   The eigenvalue equation can also be written:

   $$\sum_j a_{ij} x_{jk} = \lambda_k x_{ik} = \sum_j x_{ij} \delta_{jk} \lambda_k$$

4.6 Euler's Theorem (Continued)

We designate the matrix by

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Hence: $AX = X\Lambda \Rightarrow X^{-1}AX = \Lambda$.

2. Consider the expression $(A - I)\tilde{A} = I - \tilde{A}$ (orthogonal matrices).

   $$\Rightarrow |A - I||\tilde{A}| = |I - \tilde{A}| = |I - A|$$

   Since $|-B| = (-1)^n |B|$, we have $|I - A| = 0$.

3. **Chasles' Theorem:** The most general displacement of a rigid body is a translation plus a rotation.

4.7 Finite Rotations

1. For $\vec{r} = \vec{OP}$, $\vec{r'} = \vec{OQ}$, $\vec{ON} = \vec{n}(\vec{n} \cdot \vec{r})$, $\vec{NP} = \vec{r} - \vec{n}(\vec{n} \cdot \vec{r})$.

   $$\vec{NQ} = \vec{r} \times \vec{n}, \quad \vec{r'} = \vec{ON} + \vec{NV} + \vec{VQ}$$

   $$\Rightarrow \vec{r'} = \vec{r}\cos\Phi + \vec{n}(\vec{n} \cdot \vec{r})(1 - \cos\Phi) + (\vec{r} \times \vec{n})\sin\Phi$$

   which is the rotation formula.

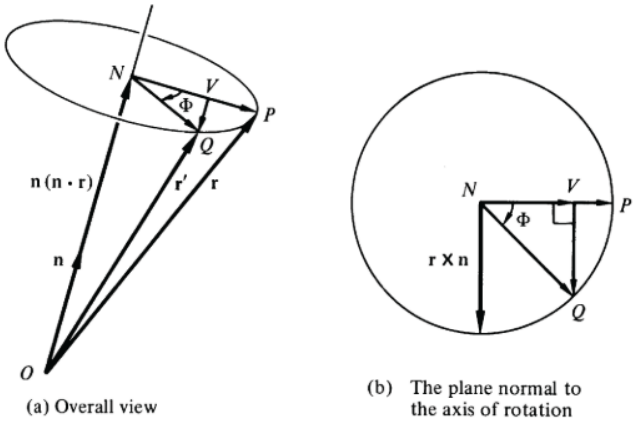

FIGURE 4.8 Vector diagrams for derivation of the rotation formula.

Figure 4:

4.8 Infinitesimal Rotations

1. From $\vec{x}$ to $\vec{x'}$, we consider

$$x'_i = x_i + \epsilon_{i1}x_1 + \epsilon_{i2}x_2 + \epsilon_{i3}x_3 \Rightarrow x'_i = (\delta_{ij} + \epsilon_{ij})x_j$$

where $\epsilon_{11}, \epsilon_{12}$ are infinitesimals.

$$\Rightarrow x' = (I + \epsilon)x$$

In fact, $(I + \epsilon_1)(I + \epsilon_2) = I + \epsilon_1 + \epsilon_2$ where $\epsilon_1\epsilon_2$ is neglected.

$A = I + \epsilon$ and $A^{-1} = I - \epsilon$ since $AA^{-1} = I$.

Further, as for orthogonality, $\tilde{A} = I + \tilde{\epsilon} = A^{-1} = I - \epsilon \Rightarrow \tilde{\epsilon} = -\epsilon$.

2. For Euler infinitesimal rotation:

$$A = \begin{bmatrix} 1 & d\phi & 0 \\ -d\phi & 1 & d\theta \\ 0 & -d\theta & 1 \end{bmatrix} \text{ and } d\Omega = \hat{i}d\theta + \hat{k}(d\phi + d\psi)$$

$$\epsilon = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

we write the form: $\vec{r'} - \vec{r} = d\vec{r} = \epsilon\vec{r}$

$$\Rightarrow \begin{cases} dx_1 = x_2 d\Omega_3 - x_3 d\Omega_2 \\ dx_2 = x_3 d\Omega_1 - x_1 d\Omega_3 \\ dx_3 = x_1 d\Omega_2 - x_2 d\Omega_1 \end{cases} \Longrightarrow d\vec{r} = \vec{r} \times d\vec{\Omega}$$

3. In another fashion: $\vec{r'} - \vec{r} = d\vec{r} = \vec{r} \times \vec{n}d\Phi$ where $d\vec{\Omega} = \vec{n}d\Phi$.

The magnitude is: $dr = r\sin\theta d\Phi$.

4.9 Rate of Change of a Vector

1. For a general vector $\vec{G}$, we have

$$(d\vec{G})_{\text{space}} = (d\vec{G})_{\text{body}} + (d\vec{G})_{\text{rot}}$$

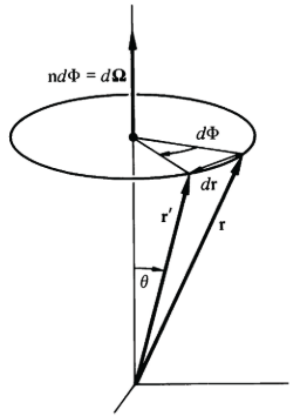

FIGURE 4.11 Change in a vector produced by an infinitesimal clockwise rotation of the vector.

Figure 5:

where $(d\vec{G})_{\text{rot}} = d\vec{\Omega} \times \vec{G}$.

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{G}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{G}$$

where $\vec{\omega}dt = d\vec{\Omega}$ and $\vec{\omega}$ is along the instantaneous axis of rotation.

We have $dG_i = a_{ji}dG'_j + da_{ji}G'_j$.

where $a_{ji}G'_j = dG_i$ and $da_{ji} = (\tilde{\epsilon})_{ij} = -\epsilon_{ij}$.

By antisymmetry of ϵ, $-\epsilon_{ij} = -\epsilon_{ijk}d\Omega_k = \epsilon_{ikj}d\Omega_k$.

$$\Rightarrow dG_i = dG'_i + \epsilon_{ikj}d\Omega_k G_j$$

$$dG_i = dG'_i + (d\vec{\Omega} \times \vec{G})_i$$

2. We consider the transformation $A = BCD$.

   where $\omega_\phi = \dot{\phi}$, $\omega_\theta = \dot{\theta}$, $\omega_\psi = \dot{\psi}$.

$$\Rightarrow \begin{cases} (\omega_\phi)x' = \dot{\phi}\sin\theta\sin\psi, \quad (\omega_\phi)y' = \dot{\phi}\sin\theta\cos\psi, \quad (\omega_\phi)z' = \dot{\phi}\cos\theta \\ (\omega_\theta)x' = \dot{\theta}\cos\psi, \quad (\omega_\theta)y' = -\dot{\theta}\sin\psi, \quad (\omega_\theta)z' = 0 \\ (\omega_\psi)z' = \dot{\psi} \end{cases}$$

$$\Rightarrow \begin{cases} \omega_{x'} = (\omega_\phi)x' + (\omega_\theta)x' \\ \omega_{y'} = (\omega_\phi)y' + (\omega_\theta)y' \\ \omega_{z'} = (\omega_\phi)z' + (\omega_\psi)z' \end{cases}$$

4.10 The Coriolis Effect

1. We have $\vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r}$ where $\vec{v}_s, \vec{v}_r$ are the velocities of the particle relative to the space and rotating set of axes.

$$\Rightarrow \left(\frac{d\vec{v}_s}{dt}\right)_s = \vec{a}_s = \left(\frac{d}{dt}(\vec{v}_r + \vec{\omega} \times \vec{r})\right)_s + \vec{\omega} \times \vec{v}_s$$
$$= \vec{a}_r + 2(\vec{\omega} \times \vec{v}_r) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

   The equation of motion $\vec{F} = m\vec{a}_s$.

$$\Rightarrow \vec{F} - 2m(\vec{\omega} \times \vec{v}_r) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\vec{a}_r = \vec{F}_{\text{eff}}$$

   where $-2m(\vec{\omega} \times \vec{v}_r)$ is the Coriolis effect and $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ $\{m\omega^2 r\sin\theta\}$ is the centrifugal force.

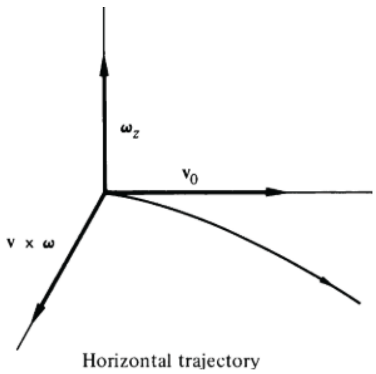

FIGURE 4.13 Direction of Coriolis deflection in the northern hemisphere.

Figure 6:

2. Example: the equation of motion:

$$m\frac{d^2x}{dt^2} = -2m(\vec{\omega}\times\vec{v}_r)_x = -2m\omega v_z\sin\theta$$

where $v_z = -gt$, $t = \sqrt{2z/g}$.
The deflection caused by Coriolis effect on v_z is

$$x = \frac{\omega g}{3}t^3\sin\theta = \frac{\omega}{3}\sqrt{\frac{8z^3}{g}}\sin\theta$$

Normally, for the equator ($\theta = \pi/2$) and $z = 100$m,

$$x \approx 2.2 \text{ cm}$$