# The Hamilton Equations of Motion

## 1 8.1 Legendre Transformations and the Hamilton Equations of Motion

### 1.1 From Lagrangian to Hamiltonian Formalism

For a system with $n$ degrees of freedom, we have $n$ equations of motion of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

We require $2n$ independent first-order equations of motion in terms of $2n$ independent variables to describe the behavior of the system point in a phase space.

The conjugate momenta $p_i$ is defined by

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_j} \quad \text{(no sum on } j\text{)}$$

The quantities $(q, p)$ are known as the canonical variables.

### 1.2 The Legendre Transformation

We consider a function of only two variables $f(x, y)$. The differential of $f$ has the form

$$df = u\,dx + v\,dy$$

where $u = \frac{\partial f}{\partial x}$, $v = \frac{\partial f}{\partial y}$.

Let $g$ be a function of $u$ and $y$ defined by

$$g = f - ux$$

whose differential has the form

$$dg = df - u\,dx - x\,du$$

or

$$dg = v\,dy - x\,du$$

The quantities $x$ and $v$ are given by

$$x = -\frac{\partial g}{\partial u}, \quad v = \frac{\partial g}{\partial y}$$

## 1.3 Example: Thermodynamics

The Legendre transformation is used in thermodynamics. For example: The differential change in energy $dU$ whose corresponding change in heat content $dQ$ and the work done $dW$:

$$dU = dQ - dW$$

For a gas undergoing a reversible process,

$$dU = T\,dS - P\,dV$$

where $U(S, V)$ is a function of entropy $S$, and the volume $V$. And $T, P$ are given by

$$T = \frac{\partial U}{\partial S}, \quad P = -\frac{\partial U}{\partial V}$$

The enthalpy $H(S, P)$ is generated by the Legendre transformation

$$H = U + PV$$

which gives: $dH = T\,dS + V\,dP$ where $T = \frac{\partial H}{\partial S}$, $V = \frac{\partial H}{\partial P}$.

Additional Legendre Transformations generate the Helmholtz free energy $F(T, V)$ and the Gibbs free energy $G(T, P)$:

$$F = U - TS$$
$$G = H - TS$$

## 1.4 The Transformation to Hamiltonian Mechanics

The transformation from $(q, \dot{q}, t)$ to $(q, p, t)$. The total differential of the Lagrangian $L(q_i, \dot{q}_i, t)$ is

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

The canonical momentum $p_i = \partial L / \partial \dot{q}_i$ gives:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

So, $dL$ can be written as

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

The Hamiltonian $H(q, p, t)$ is generated by the Legendre transformation

$$H(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t)$$

whose differential is

$$\begin{aligned}
dH &= \sum_i \dot{q}_i dp_i + \sum_i p_i d\dot{q}_i - dL \\
&= \sum_i \dot{q}_i dp_i + \sum_i p_i d\dot{q}_i - \left( \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \right) \\
&= \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt
\end{aligned}$$

Since $H = H(q, p, t)$, its differential is also given by

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

We thus obtain Hamilton's canonical equations:

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} & 1 \\ -\dot{p}_i = \frac{\partial H}{\partial q_i} & 2 \\ -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} & 3 \end{cases}$$

## 1.5 Hamiltonian in Terms of Energy

$H$, in terms of the generalized velocities of degree 1, 2, can be written as:

$$H = \sum_i p_i \dot{q}_i - L = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

If $L = L_2 + L_1 + L_0$, where $L_k$ are homogeneous functions of $\dot{q}_i$ of degree $k$, then by Euler's theorem on homogeneous functions:

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial (L_2 + L_1 + L_0)}{\partial \dot{q}_i} = 2L_2 + L_1$$

So the Hamiltonian becomes

$$H = (2L_2 + L_1) - (L_2 + L_1 + L_0) = L_2 - L_0$$

Where $L_2 = T$ is the kinetic energy, and if the forces are derivable from a conservative potential $V$, then $L_0 = -V$. Then

$$H = T + V = E$$

## 1.6 Hamiltonian for Quadratic Lagrangians

If $L$ is a quadratic function of the generalized velocities and $L_1$ is a linear function of the same variables with the following dependencies:

$$L(q, \dot{q}, t) = L_2(q, \dot{q}, t) + L_1(q, \dot{q}, t) + L_0(q, t)$$

where

$$L_2 = \frac{1}{2} \tilde{\dot{q}} T \dot{q}, \quad L_1 = \tilde{a} \dot{q}$$

where $T_{ij}$'s and $a_i$'s are functions of $q$'s and $t$. We can form the $\dot{q}_i$'s into a single column matrix $\dot{q}$. Then:

$$L(q, \dot{q}, t) = \frac{1}{2} \tilde{\dot{q}} T \dot{q} + \tilde{a} \dot{q} + L_0(q, t)$$

where the single row matrix $\tilde{\dot{q}}$ is the transpose of $\dot{q}$ and $a$ is a column matrix, and $T$ is a square $n \times n$ matrix.

We consider the special case where $q_i = \{x, y, z\}$ and $T$ is diagonal. Then

$$\frac{1}{2}\tilde{\dot{q}}T\dot{q} = \frac{1}{2}\begin{pmatrix}\dot{x} & \dot{y} & \dot{z}\end{pmatrix}\begin{pmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and

$$\tilde{a}\dot{q} = \begin{pmatrix}a_x & a_y & a_z\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = a_x\dot{x} + a_y\dot{y} + a_z\dot{z} = \mathbf{a}\cdot\dot{\mathbf{q}}$$

In this notation, the conjugate momenta is (since $T$ is symmetric):

$$p = T\dot{q} + a$$

or

$$\dot{q} = T^{-1}(p - a) \quad (\text{if } T^{-1} \text{ exists})$$

The corresponding equation for $\tilde{\dot{q}}$ is

$$\tilde{\dot{q}} = (\tilde{p} - \tilde{a})(T^{-1})^T$$

Then the final form for the Hamiltonian, using $H = L_2 - L_0 = \frac{1}{2}\tilde{\dot{q}}T\dot{q} - L_0$, is

$$\begin{aligned}H(q,p,t) &= \frac{1}{2}\left[(\tilde{p} - \tilde{a})(T^{-1})^T\right] T \left[T^{-1}(p - a)\right] - L_0(q,t)\\ &= \frac{1}{2}(\tilde{p} - \tilde{a})(T^{-1})^T T T^{-1}(p - a) - L_0(q,t)\end{aligned}$$

Since T is symmetric, $T^{-1}$ is also symmetric, so $(T^{-1})^T = T^{-1}$.

$$H(q,p,t) = \frac{1}{2}(\tilde{p} - \tilde{a})T^{-1}(p - a) - L_0(q,t)$$

# Advanced Mechanics Notes

## Matrix Inverse and Tensors

where $T^{-1} = \frac{\tilde{T}^c}{|T|}$, with $\tilde{T}^c$ the cofactor matrix whose elements $(\tilde{T}^c)_{jk} = (-1)^{j+k}$(j-th row of $T$)(k-th colur

Let a tensor $T$ be defined as:

$$T = \begin{bmatrix}m & m\\ m & m\end{bmatrix}, \quad T^{-1} = \begin{bmatrix}m & m\\ m & m\end{bmatrix}, \quad T_c = \begin{bmatrix}m & m\\ m & m\end{bmatrix}$$

The determinant is $|T| = m^2$.

## 6. Central Force Field

We consider the spatial motion of a particle in a central force field, using $(r, \theta, \phi)$. The kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\phi\,\dot{\theta}^2 + r^2\dot{\phi}^2)$$

The Hamiltonian is the total energy $T+V$:

$$H(r,\theta,\phi,p_r,p_\theta,p_\phi) = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}) + V(r)$$

We choose the Cartesian coordinates:

$$T = \frac{1}{2}mv^2 = \frac{m}{2}\dot{x}_i\dot{x}_i$$

So,

$$\begin{aligned} H(x_i,p_i) &= \frac{p_ip_i}{2m} + V(r) \\ &= \frac{p\cdot p}{2m} + V(\sqrt{x_ix_i}) \end{aligned}$$

Whenever a vector is used from here on to represent canonical momenta, it will refer to the momenta conjugate to Cartesian position coordinates.

## 7. Electromagnetic Field

We consider a single particle of mass $m$ and charge $q$ moving in an electromagnetic field. The Lagrangian is:

$$\begin{aligned} L = T - V &= \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v} \\ &= \frac{m}{2}\dot{x}_i\dot{x}_i + qA_i\dot{x}_i - q\phi \end{aligned}$$

where $\phi, \vec{A}$ are functions of $x_i$ and time.

The canonical momenta are:

$$p_i = m\dot{x}_i + qA_i$$

The Hamiltonian is:

$$H = \frac{(p_i - qA_i)(p_i - qA_i)}{2m} + q\phi$$

By vector $\vec{p}$, we write that:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi$$

where $\vec{p}$ refers only to momenta conjugate to $x_i$.

## 8. System with n Degrees of Freedom

For a system of $n$ degrees of freedom, we construct a column matrix $\eta$ with $2n$ elements such that $y_i = q_i$ and $y_{i+n} = p_i$ for $i \leq n$. The column matrix $\frac{\partial H}{\partial \eta}$ has the elements:

$$\left(\frac{\partial H}{\partial \eta}\right)_i = \frac{\partial H}{\partial q_i}, \quad \left(\frac{\partial H}{\partial \eta}\right)_{i+n} = \frac{\partial H}{\partial p_i}; \quad i \leq n$$

Let $J$ be the $2n\times 2n$ square matrix:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

with the inverse

$$\tilde{J} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

which means $\tilde{J} = \tilde{J}^{-1} = J$. So $J\tilde{J} = I$ and $J^2 = -I$. Thus $|J| = \pm 1$. Hence, Hamilton's equations of motion are:

$$\dot{\eta} = J\frac{\partial H}{\partial \eta}$$

For two coordinate variables:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial H/\partial q_1 \\ \partial H/\partial q_2 \\ \partial H/\partial p_1 \\ \partial H/\partial p_2 \end{bmatrix} = \begin{bmatrix} \partial H/\partial p_1 \\ \partial H/\partial p_2 \\ -\partial H/\partial q_1 \\ -\partial H/\partial q_2 \end{bmatrix}$$

## 8.2 Cyclic Coordinates and Conservation Theorems

We see that $\dot{p}_j = -\frac{\partial H}{\partial q_j}$. Hence a cyclic coordinate will be absent from the Hamiltonian and conversely, if a generalized coordinate does not occur in $H$, the conjugate momentum is conserved.

If $L$ and $H$ are not an explicit function of $t$, then $H$ is a constant of motion.

$$\frac{dH}{dt} = \sum_i \left( \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \right) + \frac{\partial H}{\partial t}$$

Since $\frac{\partial H}{\partial q_i} = -\dot{p}_i$ and $\frac{\partial H}{\partial p_i} = \dot{q}_i$, we can write that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$.

We define the generalized coordinates $r_m = r_m(q_1, \ldots, q_n; t)$ if they don't depend explicitly upon the time and if the potential is velocity independent.

$$H = T + V$$

We consider a one-dimensional system . Suppose a point mass $m$ is attached to a spring of force constant $k$, the other end of which is fixed on a massless cart that is being moved uniformly by an external device with velocity $v_0$.

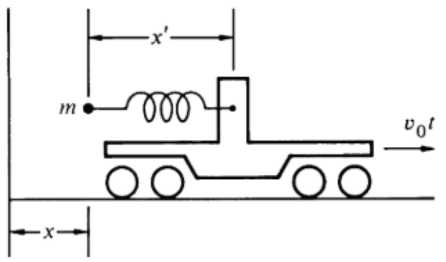


**FIGURE 8.1** A harmonic oscillator fixed to a uniformly moving cart.

Figure 1:

The Lagrangian is:

$$L(x, \dot{x}, t) = T - V = \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2$$

The corresponding equation of motion:

$$m\ddot{x} = -k(x - v_0 t)$$

We change the unknown to $x'(t)$ defined as:

$$x' = x - v_0 t$$

Noting that $\ddot{x}' = \ddot{x}$, thus:

$$m\ddot{x}' = -kx'$$

We see that $x'$ is the displacement of the particle relative to the cart. The equation exhibits simple harmonic motion. Since the potential energy doesn't involve generalized velocity, the Hamiltonian relative to $x$ is the sum of kinetic and potential energies.

$$H(x, p, t) = T + V = \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t)^2$$

which is explicitly a function of $t$, it's not conserved.

In terms of $x'$:

$$L(x', \dot{x}') = \frac{m\dot{x}'^2}{2} + m\dot{x}' v_0 + \frac{mv_0^2}{2} - \frac{kx'^2}{2}$$

$$H'(x', p') = \frac{(p' - mv_0)^2}{2m} + \frac{kx'^2}{2} - \frac{mv_0^2}{2}$$

which is the new conserved Hamiltonian and leads to the same motion for the particle.

## 8.3 Routh's Procedure

1. We consider some coordinate, say $q_n$, is cyclic. The Lagrangian is a function of $q, \dot{q}$:

   $$L = L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t)$$

   In this condition, $p_n$ is some constant. $H$ has the form:

   $$H = H(q_1, \ldots, q_{n-1}; p_1, \ldots, p_{n-1}; \alpha; t)$$

   We obtain that $\dot{q}_n = \frac{\partial H}{\partial \alpha}$.

2. We consider the cyclic coordinates are $q_{s+1}, \ldots, q_n$, and a function $R$ (Routhian), defined as:

   $$R(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s; p_{s+1}, \ldots, p_n; t) = \sum_{k=s+1}^{n} p_k \dot{q}_k - L$$

   or

   $$R = H_{cycl}(p_k) - L_{noncycl}(q_i, \dot{q}_i, \ddot{q}_i)$$

   The Lagrange equations for the $s$ coordinates:

   $$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, \quad i = 1, \ldots, s$$

   while for the $n - s$ coordinates:

   $$\frac{\partial R}{\partial q_i} = -\dot{p}_i = 0, \quad \frac{\partial R}{\partial p_i} = \dot{q}_i, \quad i = s + 1, \ldots, n$$

3. We consider the Kepler problem, where the inverse-square central force $f(r)$ derived from the potential $V(r) = -k/r$. Then:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

The ignorable coordinate is $\theta$. The Routhian is:

$$R(r, \dot{r}, p_\theta) = \frac{p_\theta^2}{2mr^2} - \frac{1}{2}m\dot{r}^2 - \frac{k}{r}$$

The equation of motion corresponding is:

$$\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{k}{mr^2} = 0$$

and

$$p_\theta = 0, \quad \frac{p_\theta}{mr^2} = \dot{\theta} \Rightarrow p_\theta = mr^2\dot{\theta} = \text{constant}$$

# 8.4 The Hamiltonian Formulation of Relativistic Mechanics

## 1. Single-particle Lagrangian and Hamiltonian

For a single-particle Lagrangian:

$$L = -mc^2\sqrt{1-\beta^2} - V \tag{1}$$

and the Hamiltonian:

$$H = T + V \tag{2}$$

The energy $T$ can be expressed in terms of $p_i$:

$$T^2 = p^2c^2 + m^2c^4 \tag{3}$$

So,

$$H = \sqrt{p^2c^2 + m^2c^4} + V \tag{4}$$

When the system consists of an electromagnetic field:

$$L = -mc^2\sqrt{1-\beta^2} + q\vec{A}\cdot\vec{v} - q\phi \tag{5}$$

Thus:

$$H = T + q\phi \tag{6}$$

The canonical momenta conjugate to the Cartesian $P^i = mu^i + qA^i$. Thus the final form:

$$H = \sqrt{(\vec{P} - q\vec{A})^2c^2 + m^2c^4} + q\phi \tag{7}$$

We note that $(H - q\phi)/c$ is the zeroth component of the 4-vector $mu^\nu + qA^\nu$.

## 2. Covariant Formulation

The Lagrangian is in the $(q_1, \ldots, q_n, t)$ configuration space.

$$\Lambda(q, \dot{q}', t, t') = t' L(q, \frac{\dot{q}'}{t'}, t) \tag{8}$$

The momentum conjugate to $t$ is

$$P_t = \frac{\partial \Lambda}{\partial t'} = L + t' \frac{\partial L}{\partial t'} \tag{9}$$

Since $\dot{q} = \dot{q}'/t'$, the relation becomes

$$P_t = L - \frac{\dot{q}_i'}{t'^2} \frac{\partial L}{\partial \dot{q}_i} = L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = -H \tag{10}$$

**1) For a single free particle**

The covariant Lagrangian

$$\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu \tag{11}$$

also leads to the equations of motion.

The Hamiltonian is

$$H_c = \frac{p_\mu p^\mu}{2m} \tag{12}$$

**2) In an electromagnetic field**

A covariant Lagrangian is:

$$\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu + q u_\mu A^\mu(x_\nu) \tag{13}$$

with the canonical momenta

$$p_\mu = m u_\mu + q A_\mu \tag{14}$$

and the corresponding Hamiltonian

$$H_c' = \frac{(p_\mu - qA_\mu)(p^\mu - qA^\mu)}{2m} \tag{15}$$

Both $H_c'$ and $H_c$ are constant and equal to $-\frac{mc^2}{2}$.

## The equations of motion for one particle

which is due to the covariant Hamiltonian:

$$\frac{dx^\mu}{d\tau} = \frac{\partial H_c'}{\partial p_\mu} g^{\mu\nu} \tag{16}$$

and

$$\frac{dp^\mu}{d\tau} = -\frac{\partial H_c'}{\partial x_\mu} g^{\mu\nu} \tag{17}$$

One of the $\nu = 0$ equations is the constitutive equation for $p^0$:

$$u^0 = \frac{\partial H'_c}{\partial p_0} = \frac{1}{m}(P^0 - qA^0) \tag{18}$$

or

$$p^0 = \frac{1}{c}(T + q\phi) = \frac{H'_c}{c} \tag{19}$$

The other can be written as

$$\frac{1}{\sqrt{1-\beta^2}}\frac{dp^0}{dt} = -\frac{1}{c}\frac{\partial H_c}{\partial t} \tag{20}$$

or

$$\frac{dH}{dt} = \sqrt{1-\beta^2}\frac{\partial H_c}{\partial t} \tag{21}$$

## 8.5 Derivation of Hamilton's Equations from a Variational Principle

We see that, from Hamilton's principle,

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0$$

$$= \delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q,p,t))\, dt = 0$$

which is the modified Hamilton's principle. We rewrite that, for a space of $2n$ dimensions

$$\delta I = \delta \int_{t_1}^{t_2} f(q,\dot{q},p,\dot{p},t)\, dt = 0$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) - \frac{\partial f}{\partial q_j} = 0, \quad j = 1,\ldots,n$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_j}\right) - \frac{\partial f}{\partial p_j} = 0, \quad j = 1,\ldots,n$$

Since $f$ contains $\dot{q}_j$ only through $p_i\dot{q}_i$ and $q_j$ only in $H$

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0$$

Similarly,

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0$$

We can subtract or add the total time derivative of an arbitrary function $F(q,p,t)$ to the integrand without affecting the validity. For example: $\frac{d}{dt}(F_1)$

$$\Rightarrow \delta \int_{t_1}^{t_2} (-p_i\dot{q}_i - H(q,p,t))\, dt = 0$$

## 8.6 The Principle of Least Action

1. We define the possible varied paths as

$$q_i(t,\alpha) = q_i(t,0) + \alpha\eta_i(t)$$

where $\alpha$ is an infinitesimal parameter that goes to zero for the correct path.

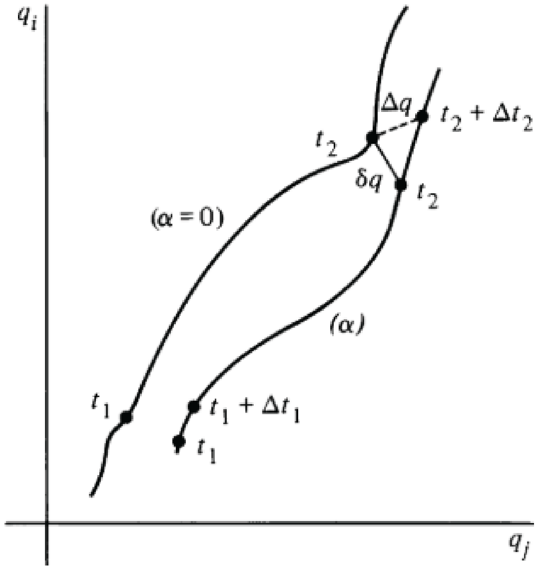


**FIGURE 8.3** The $\Delta$-variation in configuration space.

Figure 2:

We evaluate the $\Delta$ variation of the action integral

$$\Delta \int_{t_1}^{t_2} L\,dt = \int_{t_1+\delta t_1}^{t_2+\delta t_2} L(\alpha)\,dt - \int_{t_1}^{t_2} L(0)\,dt$$

$$= [L(t_2)\Delta t_2 - L(t_1)\Delta t_1 + \int_{t_1}^{t_2} \delta L\,dt]$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i\,dt + \delta L \delta q_i\Big|_{t_1}^{t_2}$$

By Lagrange's equations:

$$\Delta \int_{t_1}^{t_2} L\,dt = [L\Delta t + p_i \delta q_i]\Big|_{t_1}^{t_2}$$

We see that $\Delta q_i = \delta q_i + \dot{q}_i \Delta t$. Hence

$$\Delta \int_{t_1}^{t_2} L\,dt = [L\Delta t - p_i \dot{q}_i \Delta t + p_i \Delta q_i]\Big|_{t_1}^{t_2}$$

$$= (p_i \Delta q_i - H\Delta t)\Big|_{t_1}^{t_2}$$

We restrict that

(a) Only consider systems where $L, H$ are not explicit functions of time $\rightarrow H$ is conserved.

(b) The variation is such that $H$ is conserved on both the varied and actual path.

(c) The varied paths are limited by requiring $\Delta q_i$ vanish at the end points.

Then

$$\Delta \int_{t_1}^{t_2} L\,dt = -H(\Delta t_2 - \Delta t_1)$$

and

$$\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} p_i \dot{q}_i\,dt - H(t_2 - t_1)$$

$$\Rightarrow \Delta \int_{t_1}^{t_2} L\,dt = \Delta \int p_i \dot{q}_i\,dt - H(\Delta t_2 - \Delta t_1)$$

We obtain the principle of least action

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i\,dt = 0$$

If the trajectory is described by $\theta$, the modified Hamilton's principle is

$$\delta \int_0^{\theta} (p_i \dot{q}_i' - H)'\,d\theta = 0$$

Since $p_i$ don't change under the shift from $t$ to $\theta$ and $q_i' = \dot{q}_i'$ and the momentum conjugate to $t$ is $-H$. It can be written as

$$\delta \int_0^{\theta} \sum_{i=1}^{n+1} p_i q_i'\,d\theta = 0$$

2. The kinetic energy is a quadratic function of $\dot{q}_i$'s if the generalized coordinates don't involve $t$.

$$T = \frac{M_{jk}(\underline{q})}{2}\dot{q}_j\dot{q}_k \quad \text{and} \quad p_i\dot{q}_i = 2T$$

The principle of least action is

$$\Delta \int_{t_1}^{t_2} T\,dt = 0$$

Further, if there are no external forces, we can also write that since $T$ and $H$ are conserved

$$\Delta(t_2 - t_1) = 0$$

3. We consider a curvilinear space whose metric is $g_{\mu\nu}$ and the interval traversed for displacements given by $dx^{\mu} \ldots ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ is the interval. For a configuration space whose $M_{jk}$ coefficients form the metric tensor. The element of path length in the space is

$$|dP|^2 = M_{jk}\,dq_j\,dq_k$$

Thus,

$$T = \frac{1}{2}\left(\frac{dP}{dt}\right)^2 \quad \text{or} \quad dt = \frac{dP}{\sqrt{2T}}$$

The principle of least action

$$\Delta \int_{t_1}^{t_2} T\, dt = 0 = \Delta \int_{P_1}^{P_2} \sqrt{T/2}\, dP$$

$$\Rightarrow \Delta \int_{P_1}^{P_2} \sqrt{H - V(q)}\, dP = 0$$

which is the Jacobi's form.