Boundary-Value Problems in Electrostatics (II)

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1.1 Laplace Equation in Spherical Coordinates

In spherical coordinates, the Laplace equation is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0 \tag{1}$$

We assume a solution of the form $\Phi(r,\theta,\phi)=U(r)P(\theta)Q(\phi)$.

$$\frac{1}{U}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{1}{Q\sin^2\theta}\frac{d^2Q}{d\phi^2}=0 \tag{2}$$

ϕ is isolated, hence we let $\frac{1}{Q}\frac{d^2Q}{d\phi^2}=-m^2$, whose solutions are $Q=e^{\pm im\phi}$, where m is an integer.

Similarly, we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \tag{3}$$

and

$$\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)-l(l+1)U=0 \tag{4}$$

The solutions of the second equation are $U=Ar^l+Br^{-l-1}$, where A is a constant.

1.2 Legendre Equation and Legendre Polynomials

In terms of $x=\cos\theta$, we have

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0 \tag{5}$$

which is the generalized Legendre equation. When $m=0$, we obtain the Legendre equation.

The solution is assumed to be a power series:

$$P(x)=x^\alpha\sum_{j=0}^{\infty}a_jx^j \tag{6}$$

$$\Rightarrow\sum_{j=0}^{\infty}\{(\alpha+j)(\alpha+j-1)a_jx^{\alpha+j-2}-[(\alpha+j)(\alpha+j+1)-l(l+1)]a_jx^{\alpha+j}\}=0 \tag{7}$$

We find that if $a_0\neq 0$, then $\alpha(\alpha-1)=0$. If $a_1\neq 0$, then $(\alpha+1)\alpha=0$. And the recurrence relation is:

$$a_{j+2}=\frac{(\alpha+j)(\alpha+j+1)-l(l+1)}{(\alpha+j+1)(\alpha+j+2)}a_j \tag{8}$$

The solutions $P_l(x)$ are the Legendre polynomials.

Rodrigues' formula for Legendre polynomials is:

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l,\quad |x|<1 \tag{9}$$

To prove the orthogonality, we need to show:

$$\int_{-1}^{1} P_l(x)\left\{\frac{d}{dx}\left[(1-x^2)\frac{dP_{l'}}{dx}\right]+l'(l'+1)P_{l'}(x)\right\}dx=0 \tag{10}$$

Integrating by parts, we obtain:

$$\int_{-1}^{1}\left[(x^2-1)\frac{dP_l}{dx}\frac{dP_{l'}}{dx}+l'(l'+1)P_lP_{l'}\right]dx=0 \tag{11}$$

Interchange l and l':

$$[l(l+1)-l'(l'+1)]\int_{-1}^{1} P_lP_{l'}dx=0 \tag{12}$$

For $l\neq l'$, $\int_{-1}^{1} P_lP_{l'}dx=0$.

For $l=l'$, the normalization constant is:

$$N_l=\int_{-1}^{1}[P_l(x)]^2dx=\frac{1}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{d^l}{dx^l}(x^2-1)^l\frac{d^l}{dx^l}(x^2-1)^l dx \tag{13}$$

(Integration by parts, l times)

$$=\frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^{1}(x^2-1)^l\frac{d^{2l}}{dx^{2l}}(x^2-1)^l dx \tag{14}$$

$$=\frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(1-x^2)^l dx \tag{15}$$

Since $(1-x^2)^l=(1-x)^l(1-x^2)^{-l+1}=(1-x)^l+\frac{x}{2l}\frac{d}{dx}(1-x^2)^l$. Thus we have

$$N_l=\frac{(2l-1)!!}{2^l l!}N_{l-1}+\frac{(-1)^l}{(2^l l!)^2}\int_{-1}^{1}x\frac{d}{dx}[(1-x^2)^l]dx \tag{16}$$

$$=\frac{(2l-1)!!}{2^l l!}N_{l-1}-\frac{1}{2l}N_l \tag{17}$$

Anyway,

$$\int_{-1}^{1}P_{l'}(x)P_l(x)dx=\frac{2}{2l+1}\delta_{l'l} \tag{18}$$

Thus, the orthonormal functions are

$$U_l(x)=\sqrt{\frac{2l+1}{2}}P_l(x) \tag{19}$$

The Legendre series representation is

$$f(x)=\sum_{l=0}^{\infty}A_lP_l(x) \tag{20}$$

where

$$A_l=\frac{2l+1}{2}\int_{-1}^{1}f(x)P_l(x)dx \tag{21}$$

Example:

$$f(x)=\begin{cases}1, & x>0\\ -1, & x<0\end{cases} \tag{22}$$

Then

$$A_l=\frac{2l+1}{2}\left[\int_0^1 P_l(x)dx-\int_{-1}^{0}P_l(x)dx\right] \tag{23}$$

$$=(2l+1)\int_0^1 P_l(x)dx \quad \text{(By Rodrigues' formula)} \tag{24}$$

$$=\left(-\frac{1}{2}\right)^l\frac{(2l+1)(l-2)!!}{2(l+1)!!} \tag{25}$$

Thus,

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \dots \tag{26}$$

From Rodrigues' formula, we can prove the recurrence relation:

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0 \tag{27}$$

Combined with the Legendre equation, we obtain

$$(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0 \tag{28}$$

$$\frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx} - (l+1)P_l = 0 \tag{29}$$

$$(x^2-1)\frac{dP_l}{dx} - lxP_l + lP_{l-1} = 0 \tag{30}$$

We consider the integral

$$I_1 = \int_{-1}^{1} xP_{l'}(x)P_l(x)dx \tag{31}$$

$$= \frac{1}{2l+1}\int_{-1}^{1} P_{l'}(x)[(l+1)P_{l+1}(x) + lP_{l-1}(x)]dx \tag{32}$$

$$= \begin{cases} 0, & l' \neq l \pm 1 \\ \frac{2(l+1)}{(2l+1)(2l+3)}, & l' = l+1 \\ \frac{2l}{(2l-1)(2l+1)}, & l' = l-1 \end{cases} \tag{33}$$

Similarly,

$$\int_{-1}^{1} x^2 P_{l'}(x)P_l(x)dx = \begin{cases} \frac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)}, & l' = l+2 \\ \frac{2(l^2+l-1)}{(2l-1)(2l+3)}, & l' = l \\ \frac{2l(l-1)}{(2l+1)(2l-1)(2l-3)}, & l' = l-2 \end{cases} \tag{34}$$

1.3 Boundary-Value Problems with Azimuthal Symmetry

We have the general solution for the Laplace equation in spherical coordinates:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-l-1}]P_l(\cos\theta) \tag{35}$$

Suppose the potential is $V(\theta)$ on the surface of a sphere of radius a.

$$V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) \quad (\text{since } B_l = 0 \text{ as } \Phi \text{ must be finite at } r=0) \tag{36}$$

$$A_l = \frac{2l+1}{2a^l}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{37}$$

Example: Potential on a sphere of radius a:

$$V(\theta) = \begin{cases} +V, & 0 \leq \theta < \pi/2 \\ -V, & \pi/2 < \theta \leq \pi \end{cases} \tag{38}$$

Then the potential inside the sphere is

$$\Phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) - \dots\right] \tag{39}$$

On the symmetry axis, with $z = r$:

$$\Phi(z=r) = \sum_{l=0}^{\infty} A_l r^l, \quad \text{for } z > 0 \tag{40}$$

As for $z < 0$, each term must be multiplied by $(-1)^l$. We also have the form for the potential on the z-axis:

$$\Phi(z=r) = V\left[1 - \frac{r^2-a^2}{r\sqrt{r^2+a^2}}\right] \tag{41}$$

which can be expanded in powers of a/r.

$$\Phi(z=r) = \frac{V}{\sqrt{\pi}}\sum_{j=0}^{\infty}(-1)^{j+1}\frac{(2j-1)\Gamma(j-\frac{1}{2})}{j!}\left(\frac{a}{r}\right)^{2j+1} \tag{42}$$

Compared with the former expansion, we know that only odd l values enter. Thus, the solution is:

$$\Phi(r,\theta) = \frac{V}{\sqrt{\pi}}\sum_{j=0}^{\infty}(-1)^{j+1}\frac{(2j-1)\Gamma(j-\frac{1}{2})}{j!}\left(\frac{r}{a}\right)^{2j+1} P_{2j+1}(\cos\theta) \tag{43}$$

The expansion of the inverse distance is:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \tag{44}$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\vec{x}|$ and $|\vec{x}'|$, and γ is the angle between $\vec{x}$ and $\vec{x}'$.

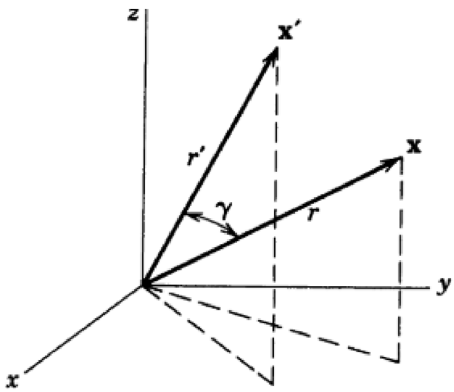

Figure 3.3

Figure 1:

Except at $\vec{x} = \vec{x}'$:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\gamma) \tag{45}$$

If $\vec{x}'$ is on the z-axis,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} \tag{46}$$

If $\vec{x}$ is on the z-axis,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l \tag{47}$$

1.4 Example: Potential due to a charged ring

Potential due to a total charge q uniformly distributed around a circular ring of radius a.

$$\Phi(z=r) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2+c^2-2rc\cos\alpha}} \tag{48}$$

where $c^2 = a^2+b^2$, $\alpha = \tan^{-1}(a/b)$. For $r > c$,

$$\Phi(z=r) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l(\cos\alpha) \tag{49}$$

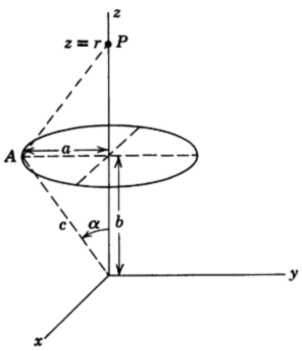

Figure 3.4 Ring of charge of radius a and total charge q located on the z axis with center at $z = b$.

Figure 2:

For $r < c$,

$$\Phi(z = r) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r^l}{c^{l+1}} P_l(\cos\alpha) \tag{50}$$

This leads to the general expression for the potential:

$$\Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta) \tag{51}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and c.

1.5 Behavior of Fields in a Conical Hole or Near a Sharp Point

For a range $0 \leq \theta \leq \beta$, $0 \leq \phi \leq 2\pi$.

- For $\beta < \pi/2 \rightarrow$ a deep conical hole bored in a conductor.
- For $\beta > \pi/2 \rightarrow$ surrounding a pointed conical conductor.

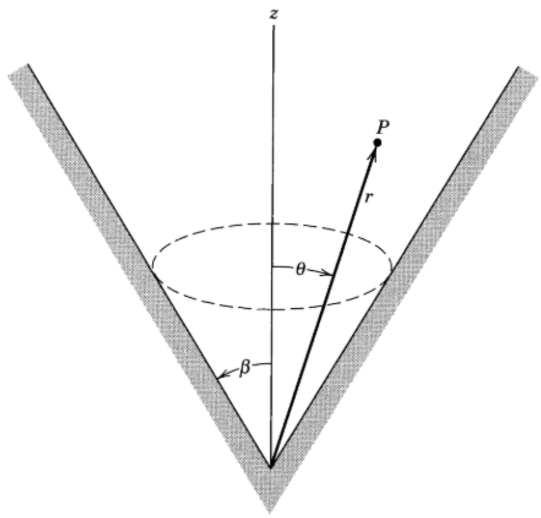

Figure 3.5

Figure 3:

We introduce $\xi = \cos\theta = \frac{1}{2}(1 - x)$. The Legendre equation becomes

$$\frac{d}{d\xi}\left[\xi(1 - \xi)\frac{dP}{d\xi}\right] + \nu(\nu + 1)P = 0 \tag{52}$$

With a power series solution $P(\xi) = \xi^\alpha \sum_{j=0}^{\infty} a_j \xi^j$.

$$\Rightarrow \frac{a_{j+1}}{a_j} = \frac{(j-\nu)(j+\nu+1)}{(j+1)^2} \quad \text{and} \quad \alpha = 0 \tag{53}$$

To normalize, we choose $a_0 = 1$ at $\xi = 0(\cos\theta = 1)$.

$$P_\nu(\xi) = 1 + \frac{(-\nu)(\nu+1)}{1! \cdot 1!}\xi + \frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{2! \cdot 2!}\xi^2 + \dots \tag{54}$$

For $\nu = l = 0, 1, 2, 3, \cdots \Rightarrow$ Legendre polynomials.

For ν not an integer, this is one example of a hypergeometric function ${}_2F_1(a, b, c; z)$ whose expansion is

$$ {}_2F_1(a, b, c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots \tag{55}$$

$$P_\nu(x) = {}_2F_1\left(-\nu, \nu+1, 1; \frac{1-x}{2}\right) \tag{56}$$

The basic solution is $Ar^\nu P_\nu(\cos\theta)$ for $\nu > 0$ and $P_\nu(\cos\beta) = 0$. The complete solution is

$$\Phi(r,\theta) = \sum_{k=0}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta) \quad \text{where } x = \cos\beta \text{ is the } k\text{th zero} \tag{57}$$

for $\nu = \nu_k$. We approximately write that

$$\Phi(r,\theta) \approx Ar^\nu P_\nu(\cos\theta) \tag{58}$$

where ν is the smallest root of $P_\nu(\cos\beta) = 0$. We obtain the electric field:

$$E_r = -\frac{\partial\Phi}{\partial r} \approx -\nu A r^{\nu-1} P_\nu(\cos\theta) \tag{59}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} \approx -Ar^{\nu-1}\sin\theta P_\nu'(\cos\theta) \tag{60}$$

The surface charge density is

$$\sigma(r) = -\frac{1}{4\pi}E_\theta|_{\theta=\beta} = -\frac{A}{4\pi}r^{\nu-1}\sin\beta P_\nu'(\cos\beta) \tag{61}$$

which is the surface-charge density on the conical conductor.

For an approximation for large ν and $\theta \ll 1$:

$$P_\nu(\cos\theta) \approx J_0((2\nu+1)\sin\frac{\theta}{2}) \tag{62}$$

2 Mathematical Methods: Spherical Harmonics

2.1 Associated Legendre Functions

The associated Legendre function is defined by:

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

By Rodrigues' formula for the Legendre polynomials $P_l(x)$:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

We can write:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

and for $m < 0$, with $m \to -m$:

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

The orthogonality relation for the associated Legendre functions is:

$$\int_{-1}^{1} P_l^m(x)P_k^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{lk}$$

2.2 Spherical Harmonics

The normalization condition is that the spherical harmonics $Y_{lm}(\theta,\phi)$ are defined as:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

and they satisfy the relation:

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi)$$

The orthonormality condition for spherical harmonics over the unit sphere is:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, Y_{l'm'}^*(\theta,\phi)Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

The completeness relation is:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

$$l=0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l=1 \quad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}}\cos\theta \end{cases}$$

$$l=2 \quad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}}\sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \end{cases}$$

$$l=3 \quad \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}}\sin^3\theta e^{3i\phi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\phi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}}\sin\theta\,(5\cos^2\theta - 1)e^{i\phi} \\ Y_{30} = \sqrt{\dfrac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \end{cases}$$

Figure 4:

2.2.1 Expansion of Functions

An arbitrary function $g(\theta, \phi)$ can be expanded in a series of spherical harmonics:

$$g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \phi)$$

where the coefficients A_{lm} are found by:

$$A_{lm} = \int d\Omega \, Y_{lm}^*(\theta, \phi) g(\theta, \phi)$$

with $d\Omega = \sin\theta d\theta d\phi$.

As a special case, for $m = 0$:

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

The general solution to Laplace's equation in spherical coordinates can be written as:

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta, \phi)$$

2.3 Addition Theorem for Spherical Harmonics

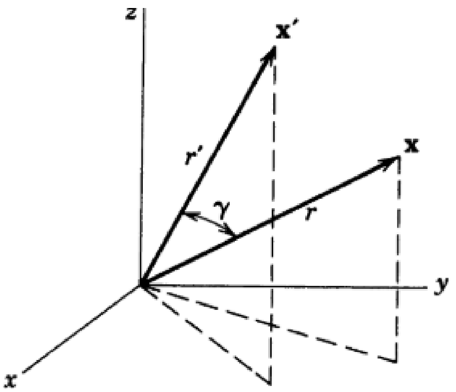

Figure 3.3

Figure 5:

We have an angle γ between two vectors $\vec{x}' = (r', \theta', \phi')$ and $\vec{x} = (r, \theta, \phi)$. The cosine of this angle is given by:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$$

The Legendre polynomial $P_l(\cos\gamma)$ satisfies the angular part of Laplace's equation:

$$\nabla^2 P_l(\cos\gamma) + \frac{l(l+1)}{r^2} P_l(\cos\gamma) = 0$$

We choose the coordinate system so that $\vec{x}'$ is on the z-axis, then $\theta' = 0$ and γ becomes the usual polar angle θ. We can expand $P_l(\cos\gamma)$ in a series of spherical harmonics in the (θ, ϕ) coordinates:

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_m(\theta', \phi') Y_{lm}(\theta, \phi)$$

where the coefficients A_m depend on the direction (θ', ϕ'). They can be found by projection:

$$A_m(\theta', \phi') = \int Y_{lm}^*(\theta, \phi) P_l(\cos\gamma) d\Omega$$

This integral is most easily evaluated by rotating the coordinate system so the z-axis points along the direction (θ', ϕ'). In this new frame, $\theta \to \gamma$, $P_l(\cos\gamma) \to P_l(\cos\theta_{new})$, and $Y^*_{lm}(\theta, \phi)$ becomes $Y^*_{lm}(\theta'_{new}, \phi'_{new})$. The integral simplifies greatly, yielding the result:

$$A_m(\theta', \phi') = \frac{4\pi}{2l+1} Y^*_{lm}(\theta', \phi')$$

Thus, we have the addition theorem in its symmetric form:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

Using the definition of Y_{lm}, this can be rewritten as:

$$P_l(\cos\gamma) = P_l(\cos\theta) P_l(\cos\theta') + 2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) P_l^m(\cos\theta') \cos[m(\phi - \phi')]$$

If the angle γ goes to zero, which means $\theta \to \theta'$ and $\phi \to \phi'$, we get $P_l(1) = 1$. This leads to Unsöld's theorem:

$$\sum_{m=-l}^{l} |Y_{lm}(\theta, \phi)|^2 = \frac{2l+1}{4\pi}$$

2.3.1 Expansion of the Green's function

We now obtain the expansion for the inverse distance between two points $\vec{x}$ and $\vec{x}'$:

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

where $r_<$ is the lesser of $|\vec{x}|$ and $|\vec{x}'|$, and $r_>$ is the greater.

3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

Derivation of Bessel's Equation

1. In the cylindrical coordinates (ρ, ϕ, z), the Laplace equation $\nabla^2\Phi = 0$ becomes:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

   We use separation of variables, assuming a solution of the form $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$. Substituting this into the equation and dividing by RQD yields:

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} + \frac{Z''}{Z} = 0$$

   This leads to a system of ordinary differential equations:

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \tag{63}$$

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0 \tag{64}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{65}$$

2. The solutions of (63) and (64) are:

$$Z(z) = e^{\pm ikz}$$

$$Q(\phi) = e^{\pm i\nu\phi}$$

   For Q to be single-valued, ν must be an integer.

3. By the change of variable $x = k\rho$, equation (65) becomes Bessel's equation:
$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0$$
We assume a series solution of the form $R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$. The indicial equation gives $\alpha = \pm\nu$. The recurrence relation for the coefficients is:
$$a_{2j} = \frac{-1}{4j(j+\alpha)} a_{2j-2} = \frac{(-1)^j}{2^{2j} j!(\alpha+1)(\alpha+2)\dots(\alpha+j)} a_0$$
If we choose $a_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}$, then the solution is the Bessel function of the first kind of order ν:
$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$
For $\alpha = -\nu$, the other solution is:
$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

Properties of Bessel Functions

- For $\nu = m$, an integer:
$$J_{-m}(x) = (-1)^m J_m(x)$$
In this case, $J_m(x)$ and $J_{-m}(x)$ are linearly dependent.
- If ν is not an integer, we have the Neumann function (or Bessel function of the second kind):
$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$
which is independent of $J_\nu(x)$. For integer orders m, $N_m(x) = \lim_{\nu\to m} N_\nu(x)$.
- **Hankel functions** (or Bessel functions of the third kind) are defined as:
$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$
- **Recurrence Relations:** The functions $J_\nu, N_\nu, H_\nu^{(1)}, H_\nu^{(2)}$ all satisfy the following relations (here denoted by $\Omega_\nu(x)$):
$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\Omega_\nu(x)$$
$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\frac{d}{dx}\Omega_\nu(x)$$

Asymptotic Forms

We only show the leading terms:

- For small arguments, $x \ll 1$:
$$J_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$
$$N_\nu(x) \to \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\dots\right], & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases}$$
- For large arguments, $x \gg 1$:
$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$
$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

Orthogonality and Fourier-Bessel Series

1. We'll be concerned with the roots of $J_\nu(x)$. Let $x_{\nu n}$ be the n-th root of $J_\nu(x)$, so $J_\nu(x_{\nu n}) = 0$ for $n = 1, 2, 3, \ldots$ The asymptotic formula for the roots is: $x_{\nu n} \approx n\pi + (\nu - \frac{1}{2})\frac{\pi}{2}$.

2. $J_\nu(x_{\nu n}\rho/a)$ satisfies the Sturm-Liouville equation:

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu}{d\rho}\right] + \left(\left(\frac{x_{\nu n}}{a}\right)^2 - \frac{\nu^2}{\rho^2}\right) J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) = 0$$

   This property leads to the orthogonality of Bessel functions. Consider two different roots $x_{\nu n}$ and $x_{\nu n'}$.

$$\int_0^a \rho J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\frac{x_{\nu n'}\rho}{a}\right)\right]d\rho + \int_0^a \rho\left(\left(\frac{x_{\nu n'}}{a}\right)^2 - \frac{\nu^2}{\rho^2}\right)\rho J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) J_\nu\left(\frac{x_{\nu n'}\rho}{a}\right)d\rho = 0$$

   After integration by parts and simplification, we get the orthogonality relation:

$$\left((x_{\nu n})^2 - (x_{\nu n'})^2\right)\int_0^a \rho J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) J_\nu\left(\frac{x_{\nu n'}\rho}{a}\right)d\rho = 0$$

   For $n \neq n'$, this implies:

$$\int_0^a \rho J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) J_\nu\left(\frac{x_{\nu n'}\rho}{a}\right)d\rho = \frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{nn'}$$

3. For an arbitrary function $f(\rho)$ on $0 \leq \rho \leq a$, we can write a **Fourier-Bessel series**:

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)$$

   where the coefficients are given by:

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})}\int_0^a \rho f(\rho) J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)d\rho$$

4. **Other Forms** of series expansions involving Bessel functions exist:
   - **Neumann series:** $\sum_{n=0}^{\infty} a_n J_n(z)$
   - **Kapteyn series:** $\sum_{n=0}^{\infty} a_n J_{\nu+n}((\nu+n)z)$
   - **Schlömilch series:** $\sum_{n=1}^{\infty} a_n J_0(nx)$

Modified Bessel Functions

1. We change k^2 to $-k^2$ in the radial equation (65).

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R = 0$$

   With the change of variable $x = k\rho$, this becomes:

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$$

   which is the **modified Bessel equation**.

2. The solutions are the modified Bessel functions:

$$I_\nu(x) = i^{-\nu}J_\nu(ix)$$
$$K_\nu(x) = \frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix)$$

   $I_\nu(x)$ is the modified Bessel function of the first kind, which is real and grows exponentially. $K_\nu(x)$ is the modified Bessel function of the second kind, which is real and decays exponentially.

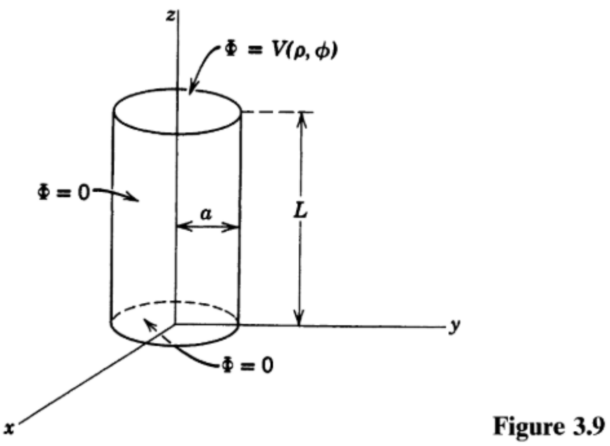

Figure 6:

3.8 Boundary-Value Problems in Cylindrical Coordinates

We make $\Phi = V(\rho, \phi)$ at $z = L$ and $\Phi = 0$ at $z = 0$. The solutions are:

$$\begin{aligned} Q(\phi) &= A\sin(m\phi) + B\cos(m\phi) \\ Z(z) &= \sinh(kz) \\ R(\rho) &= CJ_m(k\rho) + DN_m(k\rho) \end{aligned}$$

where $\nu = m$, an integer, and k is a constant. The potential is finite at $\rho = 0$, hence $D = 0$. The potential is zero at $\rho = a$, hence

$$J_m(k_n a) = 0$$

This implies $k_n = \frac{x_{mn}}{a}$, $(n = 1, 2, 3, \dots)$, where $J_m(x_{mn}) = 0$.

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)[A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)]$$

At $z = L$, $V(\rho, \phi) = \sum_{m,n}\sinh(k_{mn}L)J_m(k_{mn}\rho)[A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)]$. By Fourier series in ϕ and Fourier-Bessel series in ρ:

$$\begin{aligned} A_{mn} &= \frac{2\mathrm{cosech}(k_{mn}L)}{\pi a^2[J_{m+1}(k_{mn}a)]^2}\int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho, \phi) J_m(k_{mn}\rho)\sin(m\phi) \\ B_{mn} &= \frac{2\mathrm{cosech}(k_{mn}L)}{\pi a^2[J_{m+1}(k_{mn}a)]^2}\int_0^{2\pi} d\phi \int_0^a d\rho\, \rho V(\rho, \phi) J_m(k_{mn}\rho)\cos(m\phi) \end{aligned}$$

If the potential in charge-free space is finite for $z > 0$ and vanishes for $z \to \infty$, the general form must be (e^{-kz})

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\int_0^{\infty} dk\, e^{-kz} J_m(k\rho)[A_m(k)\sin(m\phi) + B_m(k)\cos(m\phi)]$$

and

$$V(\rho, \phi) = \sum_{m=0}^{\infty}\int_0^{\infty} dk\, J_m(k\rho)[A_m(k)\sin(m\phi) + B_m(k)\cos(m\phi)]$$

where $\frac{1}{\pi}\int_0^{2\pi} V(\rho, \phi)\begin{cases}\sin(m\phi) \\ \cos(m\phi)\end{cases} d\phi = \int_0^{\infty} J_m(k\rho)\begin{cases}A_m(k) \\ B_m(k)\end{cases} dk$. By $\int_0^{\infty} J_m(k\rho)J_m(k'\rho)\rho\, d\rho = \frac{1}{k}\delta(k -$

k'), we have

$$A_m(k) = \frac{k}{\pi}\int_0^{2\pi} d\phi \int_0^\infty d\rho\, \rho\, V(\rho,\phi) J_m(k\rho)\sin(m\phi)$$
$$B_m(k) = \frac{k}{\pi}\int_0^{2\pi} d\phi \int_0^\infty d\rho\, \rho\, V(\rho,\phi) J_m(k\rho)\cos(m\phi)$$

Also, $\int_0^\infty A(k)J_\nu(kx)dk$ where $\tilde{A}(k) = k\int_0^\infty xA(x)J_\nu(kx)dx$.

Spherical Bessel functions

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+\frac{1}{2}}(z)$$

The orthogonality becomes

$$\int_0^\infty r^2 j_l(kr)j_l(k'r)dr = \frac{\pi}{2k^2}\delta(k-k')$$

The completeness relation, with $r, k, k' > 0$

$$A(r) = \int_0^\infty \tilde{A}(k)j_l(kr)dk \quad \text{where} \quad \tilde{A}(k) = \frac{2k^2}{\pi}\int_0^\infty r^2 A(r)j_l(kr)dr$$

3.9 Expansion of Green Functions in Spherical Coordinates

① For the case of no boundary surfaces, except at infinity, we have the expansion of the Green function

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^\infty\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

We obtain:

$$G(\vec{x},\vec{x}') = 4\pi\sum_{l=0}^\infty\sum_{m=-l}^{l}\frac{1}{2l+1}\left[r_<^l - \frac{a^{2l+1}}{r_<^l}\right]\left[\frac{1}{r_>^{l+1}} - \frac{r_>}{b^{2l+1}}\right]Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

We exhibit the radial factors for $r < r'$ and $r > r'$.

$$\left[r^l - \frac{a^{2l+1}}{r^l}\right]\left[\frac{1}{(r')^{l+1}} - \frac{r'}{b^{2l+1}}\right] = \begin{cases} \frac{1}{r_>^{l+1}}\left[r_<^l - \frac{a^{2l+1}}{r_<^l}\right], & r < r' \\ \left[(r')^l - \frac{a^{2l+1}}{(r')^l}\right]\frac{1}{r^{l+1}}, & r > r' \end{cases}$$

② A Green function for a Dirichlet potential problem satisfies the equation:

$$\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \quad \text{and} \quad G(\vec{x},\vec{x}') = 0 \text{ on S}$$

We exploit the delta function:

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

By completeness relation

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^\infty\sum_{m=-l}^{l}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Then, $G(\vec{x},\vec{x}') = \sum_{l=0}^\infty\sum_{m=-l}^{l} A_{lm}(r|r',\theta',\phi') = g_l(r,r')Y_{lm}^*(\theta',\phi')$. Substitution leads to

$$A_{lm}(r|r',\theta',\phi') = g_l(r,r')Y_{lm}^*(\theta',\phi')$$

with

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_l(r,r')}{dr}\right) - \frac{l(l+1)}{r^2}g_l(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$$

$$\Longrightarrow \; g_l(r,r') = \begin{cases} Ar^l + Br^{-(l+1)}, & \text{for } r < r' \\ A'r^l + B'r^{-(l+1)}, & \text{for } r > r' \end{cases}$$

Since $g_l(r,r')$ vanishes for $r = a$ and $r = b$,

$$g_l(r,r') = A\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right), \quad r < r'$$

$$g_l(r,r') = B'\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right), \quad r > r'$$

By symmetry of $g_l(r,r')$ of r and r',

$$g_l(r,r') = Cr_<^l\left(1 - \frac{a^{2l+1}}{r_<^{2l+1}}\right)\frac{1}{r_>^{l+1}}\left(1 - \frac{r_>^{2l+1}}{b^{2l+1}}\right)$$

where $r_<(r_>)$ is the smaller (larger) of r and r'. To determine C, we multiply both sides of the differential equation of $g_l(r,r')$, and integrate over from $r = r' - \epsilon$ to $r = r' + \epsilon$, with ϵ very small.

$$\int_{r'-\epsilon}^{r'+\epsilon}\left[\frac{d}{dr}\left(r^2\frac{dg_l}{dr}\right) - l(l+1)g_l\right]dr = \int_{r'-\epsilon}^{r'+\epsilon} -4\pi\delta(r-r')dr$$

$$\left[r^2\frac{dg_l(r,r')}{dr}\right]_{r'-\epsilon}^{r'+\epsilon} = -4\pi$$

There is a discontinuity in slope at $r = r'$.

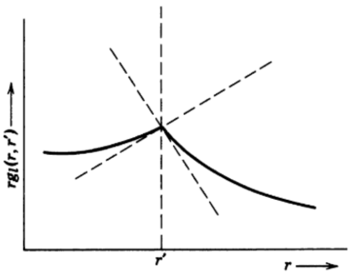

Figure 3.10 Discontinuity in slope of the radial Green function.

Figure 7:

For $r = r' + \epsilon, r_> = r, r_< = r'$, hence

$$\left\{\frac{d}{dr}[r^2 g_l(r,r')]\right\}_{r=r'+\epsilon} = C\left((r')^l - \frac{a^{2l+1}}{(r')^{l+1}}\right)\left[\frac{d}{dr}\left(r^2\frac{1}{r^{l+1}}\left(1 - \frac{r^{2l+1}}{b^{2l+1}}\right)\right)\right]_{r=r'}$$

$$= -C\left[1 - \left(\frac{a}{r'}\right)^{2l+1}\right][l + 1 + l(l+1)]$$

Similarly,

$$\left\{\frac{d}{dr}[rg_l(r,r')]\right\}_{r=r'-\epsilon} = C\left[1 - \left(\frac{r'}{b}\right)^{2l+1}\right][l + 1 + l(l+1)]$$

We thus find

$$C = \frac{4\pi}{(2l+1)[1-(a/b)^{2l+1}]}$$

③ Combining all these equations, the expansion of the Green function for a spherical shell bounded by $r = a$ and $r = b$ is

$$G(\vec{x}, \vec{x}') = 4\pi\sum_{l,m}\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)[1-(a/b)^{2l+1}]}\left(r_<^l - \frac{a^{2l+1}}{r_<^l}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>}{b^{2l+1}}\right)$$

3.10 Solution of Potential Problems with the Spherical Green Function Expansion

① The general solution to the Poisson equation with specified potential on the boundary

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}da'$$

We consider inside a sphere of radius b. With $a=0$ in the equation of $G(\vec{x},\vec{x}')$

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=b} = -\left.\frac{\partial G}{\partial r'}\right|_{r'=b} = -\frac{4\pi}{b^2}\sum_{l,m}\left(\frac{r}{b}\right)^l Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

The solution of the Laplace equation inside $r=b$ with $\Phi = V(\theta',\phi')$ on the surface

$$\Phi(\vec{x}) = \sum_{l,m}\left[\int V(\theta',\phi')Y^*_{lm}(\theta',\phi')d\Omega'\right]\left(\frac{r}{b}\right)^l Y_{lm}(\theta,\phi)$$

② We consider the linear superposition of a hollow grounded sphere of radius b with a concentric ring of charge of radius a and total Q.

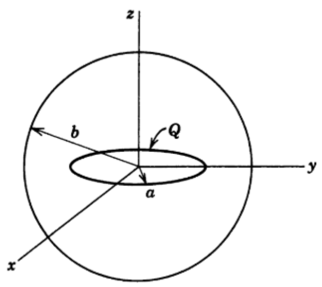

Figure 3.11 Ring of charge of radius a and total charge Q inside a grounded, conducting sphere of radius b.

Figure 8:

The charge density of the ring:

$$\rho(\vec{x}') = \frac{Q}{2\pi a^2}\delta(r'-a)\delta(\cos\theta')$$

Because of azimuthal symmetry, only terms in $m=0$ survive, and $a \to 0$, we find the $G(\vec{x},\vec{x}')$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x'$$

$$= \frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}P_l(0)r^l_<\left(\frac{1}{r^{l+1}_>} - \frac{r_>}{b^{2l+1}}\right)P_l(\cos\theta)$$

where $r_<(r_>)$ is the smaller (larger) of r and a. Using the fact that $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n\frac{(2n-1)!!}{(2n)!!}$:

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0 a}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n)!!}\left(\frac{r^{2n}_<}{r^{2n+1}_>} - \frac{r^{2n}_<r^{2n}_>}{b^{4n+1}}\right)P_{2n}(\cos\theta)$$

③ We consider another example, a hollow grounded sphere with a uniform line charge of total charge Q located on the z axis.

The volume-charge density:

$$\rho(\vec{x}') = \frac{Q}{2b}\frac{1}{2\pi r'\sin\theta'}[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)]$$

Thus we have

$$\Phi(\vec{x}) = \frac{Q}{8\pi\epsilon_0 b}\sum_{l=0}^{\infty}[P_l(1)+P_l(-1)]P_l(\cos\theta)\int_0^b \frac{1}{r^{l+1}_>}\left(r^l_< - \frac{r^{2l+1}_>}{b^{2l+1}}\right)dr'$$

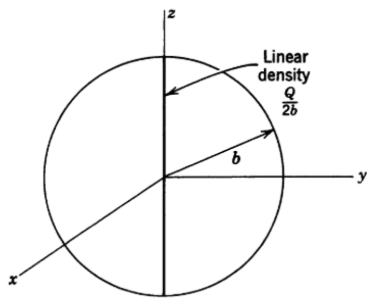

Figure 3.12 Uniform line charge of length $2b$ and total charge Q inside a grounded, conducting sphere of radius b.

Figure 9:

The integral can be broken up into $0 \le r' \le r$ and $r \le r' \le b$. We thus find:

$$\int_0^b = \frac{r^l}{r^{l+1}}\int_0^r dr' + r^l \int_r^b \frac{1}{(r')^{l+1}} dr' - \frac{r^l}{b^{2l+1}}\int_0^b (r')^l dr'$$
$$= \frac{2l+1}{l(l+1)}\left[1 - \left(\frac{r}{b}\right)^l\right]$$

For $l = 0$, $\int_0^b = \lim_{l\to 0} \frac{d/dl[1-(r/b)^l]}{d/dl[l(l+1)]} = \lim_{l\to 0} \frac{-(r/b)^l \ln(r/b)}{2l+1} = \ln(b/r)$. Using the fact that $P_l(1) = 1, P_l(-1) = (-1)^l$, we obtain

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0 b}\left\{\ln(b/r) + \sum_{j=1}^{\infty} \frac{4j+1}{2j(2j+1)}\left[1 - \left(\frac{r}{b}\right)^{2j}\right] P_{2j}(\cos\theta)\right\}$$

which diverges for $\cos\theta = \pm 1$ along the z axis. By differentiation of $\Phi(\vec{x})$, we obtain the surface-charge density on the grounded sphere

$$\sigma(\theta) = \epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=b} = \frac{Q}{4\pi b^2}\left[1 + \sum_{j=1}^{\infty} \frac{4j+1}{2j+1} P_{2j}(\cos\theta)\right]$$

3.11 Expansion of Green Functions in Cylindrical Coordinates

① The equation for the Green function

$$\nabla'^2 G(\vec{x}, \vec{x}') = -\frac{\delta(\rho - \rho')}{\rho'}\delta(\phi - \phi')\delta(z - z')$$

In terms of orthonormal functions

$$\delta(z - z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk\, \cos[k(z-z')]$$
$$\delta(\phi - \phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

In similar fashion

$$G(\vec{x}, \vec{x}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\phi-\phi')}\cos[k(z-z')] g_m(k, \rho, \rho')$$

The substitution leads to the equation for the radial Green function $g_m(k, \rho, \rho')$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2 + \frac{m^2}{\rho^2}\right) g_m = -\frac{4\pi}{\rho}\delta(\rho - \rho')$$

For $\rho \neq \rho'$, $\rightarrow$ modified Bessel functions, $I_m(k\rho)$ and $K_m(k\rho)$. Suppose some linear combination of I_m and K_m which satisfies the correct boundary conditions for $\rho < \rho'$ and that $\psi_2(k\rho)$ is another linearly independent combination for $\rho > \rho'$. By symmetry of Green function in ρ and ρ',

$$g_m(k, \rho, \rho') = \psi_1(k\rho_<)\psi_2(k\rho_>)$$

whose normalization is determined by the discontinuity in slope

$$\left.\frac{dg_m}{d\rho}\right|_+ - \left.\frac{dg_m}{d\rho}\right|_- = -\frac{4\pi}{\rho'}$$

where it means evaluated at $\rho = \rho' \pm \epsilon$.

where primes mean differentiation with respect to the argument, and $W[\psi_1, \psi_2]$ is the Wronskian.

3.1.2 Eigenfunction Expansions for Green Functions

We consider an elliptic differential equation

$$\nabla^2\Psi(\vec{x}) + [f(\vec{x}) + \lambda]\Psi(\vec{x}) = 0$$

where λ_n and $\Psi_n(\vec{x})$ are eigenvalues and eigenfunctions.

$$\nabla^2\Psi_n(\vec{x}) + [f(\vec{x}) + \lambda_n]\Psi_n(\vec{x}) = 0$$

Similarly, we have the orthogonality condition:

$$\int \Psi_m^*(\vec{x})\Psi_n(\vec{x})d^3x = \delta_{mn}$$

We now find the Green function for the equation:

$$\nabla_x^2 G(\vec{x}, \vec{x}') + [f(\vec{x}) + \lambda]G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$$

where we expand $G(\vec{x}, \vec{x}')$ in terms of the eigenfunctions Ψ_n:

$$G(\vec{x}, \vec{x}') = \sum_n a_n(\vec{x}')\Psi_n(\vec{x})$$

$$\implies \sum_n a_n(\vec{x}')(\lambda - \lambda_n)\Psi_n(\vec{x}) = -4\pi\delta(\vec{x} - \vec{x}')$$

From this, we find the coefficients $a_n(\vec{x}')$:

$$a_n(\vec{x}') = \frac{4\pi\Psi_n^*(\vec{x}')}{\lambda_n - \lambda}$$

Thus, the Green function is given by the expansion:

$$G(\vec{x}, \vec{x}') = 4\pi \sum_n \frac{\Psi_n^*(\vec{x}')\Psi_n(\vec{x})}{\lambda_n - \lambda}$$

Specially, we place $f(\vec{x}) = 0$, $\lambda = 0$.

$$(\nabla^2 + k^2)\Psi(\vec{x}) = 0$$

with the continuum of eigenvalues, k^2 and the eigenfunctions $\Psi_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{x}}$, which have the delta function normalization

$$\int \Psi_{\vec{k}}^*(\vec{x})\Psi_{\vec{k}'}(\vec{x})d^3x = \delta(\vec{k} - \vec{k}')$$

Then, the free space Green function has the form

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k^2}$$

We then consider the Green function for a Dirichlet problem inside a rectangular box defined by the six planes, $x = 0, y = 0, z = 0, x = a, y = b, z = c$. In terms of eigenfunctions of the wave equation:

$$(\nabla^2 + k_{lmn}^2)\Psi_{lmn}(x, y, z) = 0$$

where:

$$\Psi_{lmn}(x, y, z) = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right)$$

and:

$$k_{lmn}^2 = \pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

The expansion of the Green function:

$$G(\vec{x}, \vec{x}') = \frac{32\pi}{abc} \sum_{l,m,n=1}^{\infty} \frac{\sin(\frac{l\pi x}{a}) \sin(\frac{l\pi x'}{a}) \sin(\frac{m\pi y}{b}) \sin(\frac{m\pi y'}{b})}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}$$

As $(x, y) \to (\rho, \phi)$ and (z),

$$G(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \times \frac{\sinh(k_{lm}z_<) \sinh[k_{lm}(c - z_>)]}{k_{lm} \sinh(k_{lm}c)}$$

where $k_{lm}^2 = \pi^2(l^2/a^2 + m^2/b^2)$.

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi z}{c}) \sin(\frac{n\pi z'}{c})}{k_{lm}^2 + (\frac{n\pi}{c})^2} = \frac{c}{2} \frac{\sinh(k_{lm}z_<) \sinh[k_{lm}(c - z_>)]}{k_{lm} \sinh(k_{lm}c)}$$

The equation for $g_m(k, \rho, \rho')$ can be written in the Sturm-Liouville type.

$$\frac{d}{d\rho}\left[P(\rho)\frac{dy}{d\rho}\right] + g(\rho)y = 0$$

To normalize the Ψ_k, we have $W[\Psi_a(x), \Psi_b(x)] = -4\pi/k$. If there are no boundary surfaces, $g_m(k, \rho, \rho')$ must be finite at $\rho = 0$ and vanish at $\rho \to \infty$. Consequently, $\Psi_a(k\rho) = A \cdot I_m(k\rho)$ and $\Psi_b(k\rho) = K_m(k\rho)$ where A is determined by W. We find $W[I_m(x), K_m(x)] = -1/x$, so that $A = 4\pi$. Thus,

$$\begin{aligned}
\frac{1}{|\vec{x} - \vec{x}'|} &= \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} \cos[k(z - z')] I_m(k\rho_<) K_m(k\rho_>) \\
&= \frac{4}{\pi} \int_0^{\infty} dk\, \cos[k(z - z')] \left\{\frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) + \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] I_m(k\rho_<) K_m(k\rho_>)\right\}
\end{aligned}$$

Let $z' \to 0$, only the $m = 0$ term survives, we obtain

$$\frac{1}{\sqrt{\rho^2 + z^2}} = \frac{2}{\pi} \int_0^{\infty} \cos(kz) K_0(k\rho) dk$$

Replace ρ^2 by $R^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')$, we have

$$K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}) = I_0(k\rho_<) K_0(k\rho_>) + 2 \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] I_m(k\rho_<) K_m(k\rho_>)$$

We take the limit $k \to 0$,

$$\ln\left(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')\right) = 2\ln(\rho_>) + \sum_{m=1}^{\infty} \frac{2}{m} \left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi - \phi')]$$

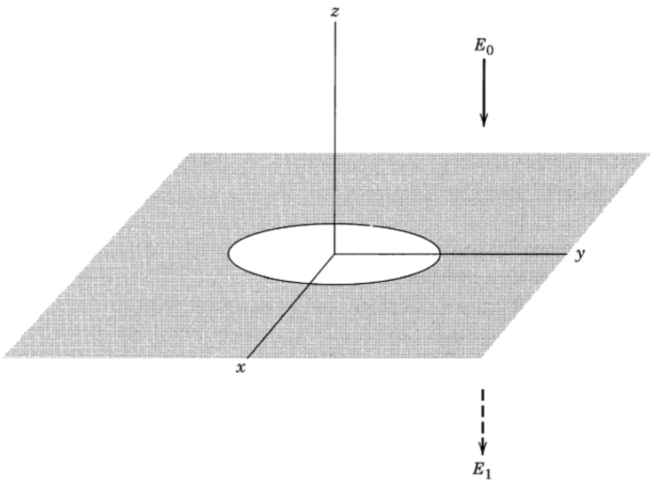

Figure 3.13

Figure 10:

3.1.3 Mixed Boundary Conditions; Conducting Plane with a Circular Hole

We consider the problem of an infinitely thin, grounded conducting plane with a circular hole of radius a cut in it and with the electric field far from the hole being normal to the plane.

We write the potential as $\Phi = \begin{cases} -E_0 z + \Phi^{(1)} \\ -E_1 z + \Phi^{(1)} \end{cases}$ since the electric field is far from the hole. The charge density is on the plane $z = 0$.

$$\Phi^{(1)}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma^{(1)}(x', y')dx'dy'}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}}$$

which is even in z, so that $E_x^{(1)}$ and $E_y^{(1)}$ are even in z and $E_z^{(1)}$ is odd. Since the total z component of electric field must be continuous across $z = 0$ in the hole, we have (for $\rho < a$)

$$-E_0 + E_z^{(1)}|_{z=0^+} = -E_1 + E_z^{(1)}|_{z=0^-}$$

and since $E_z^{(1)}|_{z=0^+} = -E_z^{(1)}|_{z=0^-} = \frac{1}{2}(E_0 - E_1)$, if (x, y) inside $(0 \leq \rho < a)$, we have the problem:

$$\left.\frac{\partial \Phi^{(1)}}{\partial z}\right|_{z=0^+} = -\frac{1}{2}(E_0 - E_1), \quad 0 \leq \rho < a$$

$$\Phi^{(1)}|_{z=0} = 0, \quad a \leq \rho < \infty$$

By azimuthal symmetry, in terms of cylindrical coordinates:

$$\Phi^{(1)}(\rho, z) = \int_0^\infty dk\, A(k) e^{-k|z|} J_0(k\rho)$$

We assume that $A(k)$ can be expanded around $k = 0$

$$A(k) = \sum_{l=0}^{\infty} \frac{k^l}{l!} \frac{d^l A}{dk^l}(0)$$

$$\Longrightarrow \Phi^{(1)}(\rho, z) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{d^l A}{dk^l}\right)_0 B_l(\rho, z)$$

where

$$
\begin{aligned}
B_l(\rho, z) &= \frac{1}{l!}\int_0^\infty dk\, k^l e^{-k|z|} J_0(k\rho) \\
&= \frac{1}{l!}\left(-\frac{\partial}{\partial |z|}\right)^l \int_0^\infty dk\, e^{-k|z|} J_0(k\rho) \\
&= \left(-\frac{\partial}{\partial |z|}\right)^l \left(\frac{1}{\sqrt{\rho^2 + z^2}}\right) \\
&= \frac{P_l(\cos\theta)}{r^{l+1}}
\end{aligned}
$$

where $\cos\theta = z/r$ and $r = \sqrt{\rho^2 + z^2}$. Thus,

$$
\Phi^{(1)} = \sum_{l=0}^{\infty} \frac{d^l A}{dk^l}(0) \frac{P_l(\cos\theta)}{r^{l+1}}
$$

where $A(0)$ is the total charge. For the mixed boundary value problem:

$$
\int_0^\infty dk\, kA(k) J_0(k\rho) = \frac{1}{2}(E_0 - E_1), \quad 0 \le \rho \le a
$$

$$
\int_0^\infty dk\, A(k) J_0(k\rho) = 0, \quad a < \rho < \infty
$$

We consider Weber's formulas:

$$
\int_0^\infty dy\, g(y) J_n(yx) = x^n, \quad 0 \le x < 1
$$

$$
\int_0^\infty dy\, g(y) J_n(yx) = 0, \quad 1 \le x < \infty
$$

$$
\implies g(y) = \frac{\Gamma(n+1)}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} j_n(y) = \sqrt{\frac{\pi}{2y}} J_{n+1/2}(y)
$$

We have $n = 0, x = \rho/a, y = ka$, thus

$$
A(k) = \frac{(E_0 - E_1)a^2}{\pi} j_1(ka) = \frac{(E_0 - E_1)a^2}{\pi}\left[\frac{\sin(ka)}{(ka)^2} - \frac{\cos(ka)}{ka}\right]
$$

which means:

$$
\Phi^{(1)} \to \frac{(E_0 - E_1)a^2}{3\pi} \frac{|z|}{r^3}
$$

which falls off with $1/r^2$, and has the effective electric dipole moment:

$$
p_z = \frac{4}{3}\pi\epsilon_0 (E_0 - E_1) a^3, \quad z \ge 0
$$

In the neighborhood of the opening

$$
\Phi^{(1)}(\rho, z) = \frac{(E_0 - E_1)a}{\pi}\int_0^\infty dk\, j_1(ka) e^{-k|z|} J_0(k\rho) = \frac{(E_0 - E_1)a}{\pi}\left[\sqrt{\frac{R - x}{2}} - \frac{|z|}{a}\tan^{-1}\left(\sqrt{\frac{R - x}{2}}\right)\right]
$$

where $x = \frac{1}{a^2}(\rho^2 + z^2 - a^2)$, $R = \sqrt{x^2 + 4z^2/a^2}$.

The added potential on the axis ($\rho = 0$):

$$
\Phi^{(1)}(0, z) = \frac{(E_0 - E_1)a}{\pi}\left[1 - \frac{|z|}{a}\tan^{-1}\left(\frac{a}{|z|}\right)\right]
$$

which reduces to $\frac{(E_0 - E_1)a^3}{3\pi} \frac{1}{|z|^2}$ for $|z| \gg a$ and $r \approx |z|$.

In the plane of opening ($z = 0$):

$$\Phi^{(1)}(\rho, 0) = \frac{(E_0 - E_1)}{\pi}\sqrt{a^2 - \rho^2}, \quad 0 \leq \rho < a$$

The tangential electric field in the opening is a radial field:

$$E'_{tan}(\rho, 0) = \frac{(E_0 - E_1)}{\pi}\frac{\rho}{\sqrt{a^2 - \rho^2}}$$

and

$$E_z(\rho, 0) = -\frac{1}{2}(E_0 + E_1)$$

Near the circular hole for the full potential, the equipotential contours when $E_1 = 0$.

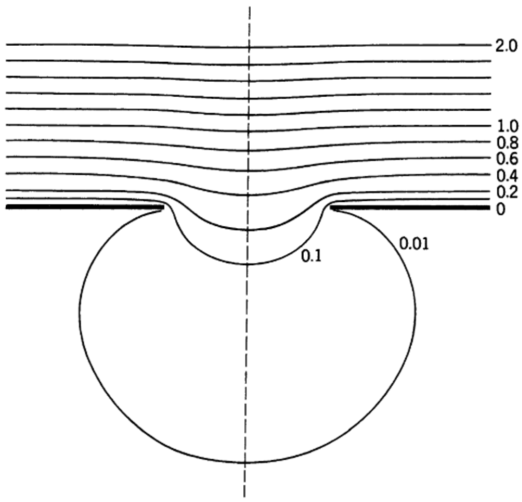

Figure 3.14 Equipotential contours near a circular hole in a conducting plane with a normal electric field E_0 far from the hole on one side and no field asymptotically on the other ($E_1 = 0$). The numbers are the values of the potential Φ in units of aE_0. The distribution is rotationally symmetric about the vertical dashed line through the center of the hole.

Figure 11: