数学物理方法笔记 (Fourier Analysis Notes)

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1 傅立叶级数 (Fourier Series)

1.1 一般理论 (General Theory)

给定一个周期为 $2L$ 的函数 $f(x)$，即 $f(x+2L)=f(x)$，其傅立叶级数展开为：

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}))$$

其中系数由以下积分给出：

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dt$$

对于周期为 2π 的函数（即 $L=\pi$）：

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

通过在 $[0,2\pi]$ 上积分可以推导出系数：

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))\right]dx = 2\pi a_0$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

利用三角函数的正交性:

$$\int_0^{2\pi} f(x)\cos(mx)dx = \sum_{n=1}^{\infty} a_n \int_0^{2\pi}\cos(nx)\cos(mx)dx = a_m\pi$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx$$

同理可得 b_n:

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx$$

狄利克雷定理 (Dirichlet's Theorem): 如果 $f(x)$ 在 $[-L, L]$ 上除有限个点外连续，且有有限个极值点。在 $(-L, L)$ 外为周期延拓，周期为 $2L$。则 $f(x)$ 的傅立叶级数收敛于：

$$\frac{f(x+0)+f(x-0)}{2}$$

1.2 正交函数系展开 (Expansion in Orthogonal Function Systems)

正弦级数 (Sine Series): 若函数 $\phi(x)$ 在 $(0, L)$ 上展开为正弦级数:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L$$

利用正交关系 $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}$，可得系数：

$$\int_0^L \phi(x)\sin\left(\frac{m\pi x}{L}\right)dx = C_m\frac{L}{2}$$

$$\Rightarrow C_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

余弦级数 (Cosine Series): 若函数 $\phi(x)$ 在 $(0, L)$ 上展开为余弦级数:

$$\phi(x) = D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{L}\right)$$

利用正交关系可得系数：

$$D_0 = \frac{1}{L}\int_0^L \phi(x)dx$$

$$D_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

1.3 傅立叶级数示例 (Fourier Series Examples)

例1: $f(x) = \frac{1}{2}(\pi - x)$ on $(0, 2\pi)$, with $f(x+2\pi) = f(x)$.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)dx = 0 \\
a_n &= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\cos(nx)dx = 0 \\
b_n &= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin(nx)dx \\
&= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos(nx)}{n}\right)\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi}(-1)\left(-\frac{\cos(nx)}{n}\right)dx \\
&= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] - 0 = \frac{1}{n}
\end{aligned}$$

所以:

$$f(x) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

例2: 将 $\phi(x) = \sin x$ 在 $[0, \pi]$ 上展开为余弦级数。这里 $L = \pi$。

$$
\begin{aligned}
D_0 &= \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi} \\
D_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos(nx)\, dx = \frac{1}{\pi}\int_0^\pi [\sin((1+n)x) + \sin((1-n)x)]\, dx \\
&= \frac{1}{\pi}\left[-\frac{\cos((1+n)x)}{1+n} - \frac{\cos((1-n)x)}{1-n}\right]_0^\pi \quad (n \neq 1) \\
&= \frac{1}{\pi}\left(\frac{1-\cos((n+1)\pi)}{n+1} + \frac{1-\cos((n-1)\pi)}{n-1}\right) \\
&= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1} + \frac{1-(-1)^{n-1}}{n-1}\right) = \frac{1+(-1)^n}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) \\
&= \frac{2(1+(-1)^n)}{\pi(n^2-1)}
\end{aligned}
$$

当 $n = 1$ 时, $D_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\, dx = 0$。
D_n 仅在 n 为偶数时非零。令 $n = 2k$:

$$D_{2k} = \frac{4}{\pi((2k)^2 - 1)}$$

所以:

$$\phi(x) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4}{\pi(1-(2k)^2)} \cos(2kx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2 - 1}$$

2 傅立叶积分 (Fourier Integral)

2.1 从级数到积分 (From Series to Integral)

考虑傅立叶级数，当 $L \to \infty$ 时，$\omega_n = \frac{n\pi}{L}$, $\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$。级数求和变为积分: $\sum_{n=1}^{\infty} \to \frac{L}{\pi}\sum \Delta\omega \to \frac{1}{\pi}\int_0^\infty d\omega$。

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)] d\omega$$

其中系数为:

$$
\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t) dt \\
B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t) dt
\end{aligned}
$$

2.2 傅立叶积分示例 (Fourier Integral Examples)

例3: $f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$ 函数为偶函数，所以 $B(\omega) = 0$。

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t) dt = \frac{1}{\pi}\int_{-1}^{1} \cos(\omega t) dt = \frac{1}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

所以:

$$f(x) = \int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos(\omega x) d\omega = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos(\omega x)}{\omega} d\omega$$

例4: $f(x) = \begin{cases} \cos x & |x| \leq \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$ 函数为偶函数，所以 $B(\omega) = 0$。

$$\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cos(\omega t) dt = \frac{2}{\pi}\int_{0}^{\pi/2} \cos t \cos(\omega t) dt \\
&= \frac{1}{\pi}\int_{0}^{\pi/2} [\cos((1+\omega)t) + \cos((1-\omega)t)] dt \\
&= \frac{1}{\pi}\left[\frac{\sin((1+\omega)t)}{1+\omega} + \frac{\sin((1-\omega)t)}{1-\omega}\right]_0^{\pi/2} \quad (\omega \neq 1) \\
&= \frac{1}{\pi}\left(\frac{\sin(\frac{\pi}{2}(1+\omega))}{1+\omega} + \frac{\sin(\frac{\pi}{2}(1-\omega))}{1-\omega}\right) \\
&= \frac{1}{\pi}\left(\frac{\cos(\frac{\pi\omega}{2})}{1+\omega} + \frac{\cos(\frac{\pi\omega}{2})}{1-\omega}\right) = \frac{2\cos(\frac{\pi\omega}{2})}{\pi(1-\omega^2)}
\end{aligned}$$

所以:

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos(\frac{\pi\omega}{2})}{1-\omega^2} \cos(\omega x) d\omega$$

3 复数形式的傅立叶变换 (Complex Form and Fourier Transform)

傅立叶积分可以写为：

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\cos(\omega(x-t)) dt \\
&= \frac{1}{2\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t)[e^{i\omega(x-t)} + e^{-i\omega(x-t)}] dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) e^{i\omega(x-t)} dt
\end{aligned}$$

这引出了傅立叶变换对：

$$\text{傅立叶变换 (Fourier Transform):} \quad F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\text{傅立叶逆变换 (Inverse F.T.):} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

3.1 傅立叶变换示例 (Fourier Transform Example)

例4 (续): $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-a}^{a} 1 \cdot e^{-i\omega x} dx \\
&= \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-a}^{a} = \frac{e^{-i\omega a} - e^{i\omega a}}{-i\omega} = \frac{2\sin(\omega a)}{\omega}
\end{aligned}$$

通过逆变换得到傅立叶积分表示：

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(\omega a)}{\omega} e^{i\omega x} d\omega \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega a)}{\omega} (\cos(\omega x) + i\sin(\omega x)) d\omega
\end{aligned}$$

由于 $\frac{\sin(\omega a)}{\omega}\sin(\omega x)$ 是奇函数，其积分为零。

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega a)\cos(\omega x)}{\omega} d\omega = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(\omega a)\cos(\omega x)}{\omega} d\omega$$

根据收敛定理，该积分等于：

$$\begin{cases} 1 & |x| < a \\ 1/2 & |x| = a \\ 0 & |x| > a \end{cases}$$

4 傅立叶变换性质 (Properties of Fourier Transform)

令 $F(\omega)$ 是 $f(x)$ 的傅立叶变换, $F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$.

1. **导数 (Differentiation):**

$$\mathcal{F}\left[\frac{df(x)}{dx}\right] = i\omega F(\omega)$$

推导:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{-i\omega x} dx &= \left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega)e^{-i\omega x}dx \\ &= 0 + i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = i\omega F(\omega)\end{aligned}$$

2. **乘以 x (Multiplication by x):**

$$\mathcal{F}[xf(x)] = i\frac{dF(\omega)}{d\omega}$$

推导:

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} -ixf(x)e^{-i\omega x}dx = -i\mathcal{F}[xf(x)]$$

推广可得:

$$\mathcal{F}[x^n f(x)] = i^n \frac{d^n F(\omega)}{d\omega^n}$$

3. **积分 (Integration):** 若 $g(x) = \int_{-\infty}^{x} f(\xi)d\xi$, 且 $f(x) = g'(x)$, 则

$$F(\omega) = \mathcal{F}[g'(x)] = i\omega G(\omega)$$

$$G(\omega) = \mathcal{F}\left[\int_{-\infty}^{x} f(\xi)d\xi\right] = \frac{1}{i\omega}F(\omega) \quad (\text{可能需要加上 } \pi F(0)\delta(\omega) \text{ 项})$$

4. **位移 (Shifting):**

$$\mathcal{F}[f(x+\xi)] = e^{i\omega\xi}F(\omega)$$

推导 (令 $y = x + \xi$):

$$\int_{-\infty}^{\infty} f(x+\xi)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(y)e^{-i\omega(y-\xi)}dy = e^{i\omega\xi}\int_{-\infty}^{\infty} f(y)e^{-i\omega y}dy = e^{i\omega\xi}F(\omega)$$

5. **卷积 (Convolution):** 卷积定义: $f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(\xi)f_2(x-\xi)d\xi$.

$$\mathcal{F}[f_1 * f_2] = F_1(\omega)F_2(\omega)$$

推导:

$$\begin{aligned}
\mathcal{F}[f_1 * f_2] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\xi) f_2(x-\xi) d\xi\right] e^{-i\omega x} dx \\
&= \int_{-\infty}^{\infty} f_1(\xi) \left[\int_{-\infty}^{\infty} f_2(x-\xi) e^{-i\omega x} dx\right] d\xi \\
(\text{令 } y = x-\xi) &= \int_{-\infty}^{\infty} f_1(\xi) \left[\int_{-\infty}^{\infty} f_2(y) e^{-i\omega(y+\xi)} dy\right] d\xi \\
&= \left(\int_{-\infty}^{\infty} f_1(\xi) e^{-i\omega\xi} d\xi\right) \left(\int_{-\infty}^{\infty} f_2(y) e^{-i\omega y} dy\right) = F_1(\omega) F_2(\omega)
\end{aligned}$$

5 狄拉克 δ 函数 (Dirac Delta Function)

5.1 定义与性质 (Definition and Properties)

δ 函数定义为满足以下条件的分布:

$$\delta(x-x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases} \quad \text{且} \quad \int_{-\infty}^{\infty} \delta(x-x_0) dx = 1$$

筛选性质 (Sifting Property):

$$\int_{-\infty}^{\infty} f(x) \delta(x-x_0) dx = f(x_0)$$

$\delta(x-x_1)$ 为偶函数, 即 $\delta(x) = \delta(-x)$。

卷积性质 (Convolution Properties):

$$\begin{aligned}
\delta(x) * f(x) &= \int_{-\infty}^{\infty} f(\xi) \delta(x-\xi) d\xi = f(x) \\
\delta(x-a) * f(x) &= f(x-a) \\
\delta(x-a) * \delta(x-b) &= \delta(x-(a+b))
\end{aligned}$$

傅立叶变换 (Fourier Transform):

$$\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-i\omega x} dx = e^{-i\omega x_0}$$

特别地, 当 $x_0 = 0$ 时, $\mathcal{F}[\delta(x)] = 1$。反变换给出 δ 函数的一个积分表示:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} d\omega$$

5.2 δ 函数的极限表示 (Limit Representations of δ-Function)

1. **Sinc 函数:**
$$\delta(x) = \lim_{B\to\infty} \frac{\sin(Bx)}{\pi x}$$
   这来自一个带宽为 $[-B, B]$ 的理想低通滤波器的冲激响应。
$$\frac{1}{2\pi} \int_{-B}^{B} e^{i\omega x} d\omega = \frac{\sin(Bx)}{\pi x}$$

2. **洛伦兹函数 (Lorentzian Function):**

$$\delta(x) = \lim_{b \to 0^+} \frac{1}{\pi} \frac{b}{x^2 + b^2}$$

3. **狄利克雷核 (Dirichlet Kernel, 在周期区间上):**

$$\sum_{k=-\infty}^{\infty} \delta(x - 2\pi k) = \frac{1}{2\pi} \lim_{m \to \infty} D_m(x) = \frac{1}{2\pi} \lim_{m \to \infty} \sum_{n=-m}^{m} e^{inx}$$

其中 $D_m(x) = 1 + 2\sum_{n=1}^{m} \cos(nx) = \frac{\sin((m+1/2)x)}{\sin(x/2)}$。

6 傅立叶级数的收敛与狄利克雷核

6.1 部分和 (Partial Sum)

傅立叶级数的部分和 $S_m(x)$ 可以表示为与狄利克雷核的卷积:

$$\begin{aligned}
S_m(x) &= \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx)) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt + \sum_{n=1}^{m} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(t-x))dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{m} \cos(n(t-x)) \right] dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(t-x)dt
\end{aligned}$$

其中 $D_m(u) = \sum_{n=-m}^{m} e^{inu} = \frac{\sin((m+1/2)u)}{\sin(u/2)}$ 是狄利克雷核。令 $u = t - x$, 并利用周期性:

$$S_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+u) D_m(u)du$$

由于 $D_m(u)$ 是偶函数,

$$S_m(x) = \frac{1}{2\pi} \int_{0}^{\pi} [f(x+u) + f(x-u)] D_m(u)du$$

6.2 收敛证明概要 (Sketch of Convergence Proof)

利用狄利克雷核的性质 $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(u)du = 1$, 可得

$$\frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi} \int_{0}^{\pi} [f(x+0) + f(x-0)] D_m(u)du$$

考虑差值:

$$\begin{aligned}
S_m(x) - \frac{f(x+0) + f(x-0)}{2} &= \frac{1}{2\pi} \int_{0}^{\pi} [f(x+u) - f(x+0) + f(x-u) - f(x-0)] D_m(u)du \\
&= \frac{1}{2\pi} \int_{0}^{\pi} \left[\frac{f(x+u) - f(x+0)}{u} + \frac{f(x-u) - f(x-0)}{u} \right] u \frac{\sin((m+1/2)u)}{\sin(u/2)} du
\end{aligned}$$

当 $f(x)$ 满足狄利克雷条件时, 方括号内的函数在 $[0, \pi]$ 上绝对可积。根据黎曼-勒贝格引理 (Riemann-Lebesgue Lemma), 当 $m \to \infty$ 时, 该积分趋于零。因此, $\lim_{m \to \infty} S_m(x) = \frac{f(x+0)+f(x-0)}{2}$。

7 更多傅立叶变换示例 (More Fourier Transform Examples)

7.1 高斯函数 (Gaussian Function)

例： 求 $f(x) = e^{-ax^2}$ (其中 $a > 0$) 的傅立叶变换。

$$\begin{aligned} G(\omega) = \mathcal{F}[e^{-ax^2}] &= \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} e^{-(ax^2+i\omega x)} dx \end{aligned}$$

为了求解这个积分，我们使用配方法：

$$ax^2 + i\omega x = a\left(x^2 + \frac{i\omega}{a}x\right) = a\left(x + \frac{i\omega}{2a}\right)^2 - a\left(\frac{i\omega}{2a}\right)^2 = a\left(x + \frac{i\omega}{2a}\right)^2 + \frac{\omega^2}{4a}$$

代入积分中可得：

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} \exp\left[-a\left(x + \frac{i\omega}{2a}\right)^2 - \frac{\omega^2}{4a}\right] dx \\ &= e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\omega}{2a}\right)^2} dx \end{aligned}$$

令 $y = \sqrt{a}(x + \frac{i\omega}{2a})$, $dy = \sqrt{a}dx$。这是一个在复平面上的积分，但可以证明其路径可以平移回实轴而不改变积分值。因此，积分结果等于标准高斯积分 $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$。

$$G(\omega) = e^{-\frac{\omega^2}{4a}} \cdot \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$

这是一个非常重要的结论：**高斯函数的傅立叶变换仍然是高斯函数**。

应用： 利用傅立叶逆变换，我们可以求解一个重要的积分。

$$f(x) = e^{-ax^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} e^{i\omega x} d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4a}} (\cos(\omega x) + i\sin(\omega x)) d\omega = 2\pi\sqrt{\frac{a}{\pi}} e^{-ax^2} = \sqrt{4\pi a} e^{-ax^2}$$

取等式两边的实部，并设 $a = 1/4$，则有：

$$\int_{-\infty}^{\infty} e^{-\omega^2} \cos(\omega x) d\omega = \sqrt{\pi} e^{-x^2/4}$$

7.2 基本函数 (Basic Functions)

例： $f(x) = \sin(kx)$

$$\begin{aligned} \mathcal{F}[\sin(kx)] &= \int_{-\infty}^{\infty} \frac{e^{ikx} - e^{-ikx}}{2i} e^{-i\omega x} dx \\ &= \frac{1}{2i} \left[\int_{-\infty}^{\infty} e^{-i(\omega-k)x} dx - \int_{-\infty}^{\infty} e^{-i(\omega+k)x} dx\right] \\ &= \frac{1}{2i} [2\pi\delta(\omega - k) - 2\pi\delta(\omega + k)] \\ &= i\pi[\delta(\omega + k) - \delta(\omega - k)] \end{aligned}$$

例：$f(x) = e^{-|x|}$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx = \int_{-\infty}^{0} e^{x} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x} dx \\
&= \int_{-\infty}^{0} e^{(1-i\omega)x} dx + \int_{0}^{\infty} e^{-(1+i\omega)x} dx \\
&= \left[\frac{e^{(1-i\omega)x}}{1-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(1+i\omega)x}}{-(1+i\omega)} \right]_{0}^{\infty} \\
&= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{1+i\omega+1-i\omega}{1+\omega^2} = \frac{2}{1+\omega^2}
\end{aligned}$$

通过傅立叶逆变换：

$$f(x) = e^{-|x|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{i\omega x} d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{1+\omega^2} d\omega$$

7.3 双曲正割与三角脉冲 (Hyperbolic Secant and Triangular Pulse)

例：$f(x) = \textbf{sech}(kx)$ 其傅立叶变换为：

$$F(\omega) = \mathcal{F}[\text{sech}(kx)] = \frac{\pi}{k} \text{sech}\left(\frac{\pi\omega}{2k}\right)$$

例：三角脉冲函数 $f(x) = \Delta(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

$$\begin{aligned}
F(\omega) &= \int_{-1}^{1} (1-|x|) e^{-i\omega x} dx = \int_{-1}^{0} (1+x) e^{-i\omega x} dx + \int_{0}^{1} (1-x) e^{-i\omega x} dx \\
&= \dots \quad \text{(通过两次分部积分)} \\
&= \frac{2(1-\cos\omega)}{\omega^2} = \frac{4\sin^2(\omega/2)}{\omega^2} = \left(\frac{\sin(\omega/2)}{\omega/2} \right)^2
\end{aligned}$$

8 赫维赛德阶跃函数的变换 (Transform of Heaviside Step Function)

赫维赛德阶跃函数 $u(t)$ 的傅立叶变换需要通过极限过程来定义。我们首先考虑一个单边指数衰减函数 $f(t) = e^{-Bt}u(t)$, 其中 $B > 0$。

$$\mathcal{F}[e^{-Bt}u(t)] = \int_{0}^{\infty} e^{-Bt} e^{-i\omega t} dt = \int_{0}^{\infty} e^{-(B+i\omega)t} dt = \frac{1}{B+i\omega}$$

现在我们令 $B \to 0^+$，来得到阶跃函数的变换：

$$\mathcal{F}[u(t)] = \lim_{B\to 0^+} \frac{1}{B+i\omega}$$

这个极限在分布意义下等于：

$$F(\omega) = \pi\delta(\omega) + \frac{1}{i\omega}$$

其中 $\frac{1}{i\omega}$ 项在积分时需要取柯西主值 (Cauchy Principal Value)。

逆变换验证：

$$\begin{aligned}
f(t) = \mathcal{F}^{-1}\left[\pi\delta(\omega) + \frac{1}{i\omega}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(\omega) + \frac{1}{i\omega}\right)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\delta(\omega)e^{i\omega t}d\omega + \frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{i\omega}d\omega \\
&= \frac{1}{2}\cdot 1 + \frac{1}{2\pi i}\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(\omega t) + i\sin(\omega t)}{\omega}d\omega \\
&= \frac{1}{2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\sin(\omega t)}{\omega}d\omega \quad (\cos\text{项为奇函数，主值积分为0}) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin(\omega t)}{\omega}d\omega
\end{aligned}$$

利用狄利克雷积分 $\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$，我们得到：

$$f(t) = \frac{1}{2} + \frac{1}{\pi}\cdot\frac{\pi}{2}\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

这正是赫维赛德阶跃函数 $u(t)$。

9 傅立叶变换的应用 (Applications of Fourier Transforms)

9.1 求解常微分方程 (Solving Ordinary Differential Equations)

傅立叶变换可以将微分方程转换为代数方程，从而简化求解过程。基本原理是利用其微分性质 $\mathcal{F}[\frac{d^n f(t)}{dt^n}] = (i\omega)^n F(\omega)$。

例：受驱阻尼谐振子 (Driven Damped Harmonic Oscillator) 考虑二阶线性常微分方程：

$$\frac{d^2x(t)}{dt^2} + 2\gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

对整个方程进行傅立叶变换，令 $X(\omega) = \mathcal{F}[x(t)]$ 及 $F(\omega) = \mathcal{F}[f(t)]$，可得：

$$(i\omega)^2 X(\omega) + 2\gamma(i\omega)X(\omega) + \omega_0^2 X(\omega) = F(\omega)$$

$$\Rightarrow (-\omega^2 + 2i\gamma\omega + \omega_0^2)X(\omega) = F(\omega)$$

解出频域中的响应 $X(\omega)$:

$$X(\omega) = \frac{F(\omega)}{-\omega^2 + 2i\gamma\omega + \omega_0^2}$$

时域中的解 $x(t)$ 可通过对 $X(\omega)$ 进行傅立叶逆变换得到：

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{i\omega t}d\omega$$

传递函数与冲激响应 (Transfer Function and Impulse Response) 若驱动力为狄拉克 δ 函数，即 $f(t) = \delta(t)$, 则 $F(\omega) = 1$。此时的频域响应称为系统的传递函数 $H(\omega)$:

$$H(\omega) = \frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2}$$

其傅立叶逆变换 $h(t) = \mathcal{F}^{-1}[H(\omega)]$ 是系统的冲激响应（或格林函数）。对于任意输入 $f(t)$，系统的输出可以通过卷积得到：

$$x(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

9.2 更多变换对示例 (Further Transform Pair Examples)

例：符号函数 (Sign Function) 符号函数定义为 $\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$。它与阶跃函数的关系是 $\text{sgn}(x) = 2u(x) - 1$。其傅立叶变换为：

$$\mathcal{F}[\text{sgn}(x)] = \frac{2}{i\omega}$$

反之，我们有：

$$\mathcal{F}\left[\frac{1}{x}\right] = \int_{-\infty}^{\infty} \frac{1}{x} e^{-i\omega x} dx = -i\pi \text{sgn}(\omega)$$

例：单边余弦函数 考虑 $f(x) = u(x)\cos(ax)$，其中 $u(x)$ 是赫维赛德阶跃函数。

$$\begin{aligned} F(\omega) &= \int_0^{\infty} \cos(ax) e^{-i\omega x} dx = \int_0^{\infty} \frac{e^{iax} + e^{-iax}}{2} e^{-i\omega x} dx \\ &= \frac{1}{2}\left[\int_0^{\infty} e^{-i(\omega-a)x} dx + \int_0^{\infty} e^{-i(\omega+a)x} dx\right] \end{aligned}$$

利用 $\int_0^{\infty} e^{-i\alpha x} dx = \pi\delta(\alpha) + \frac{1}{i\alpha}$ 的结果，可得：

$$F(\omega) = \frac{1}{2}\left[\pi\delta(\omega - a) + \frac{1}{i(\omega - a)} + \pi\delta(\omega + a) + \frac{1}{i(\omega + a)}\right]$$

$$= \frac{\pi}{2}[\delta(\omega - a) + \delta(\omega + a)] + \frac{i\omega}{a^2 - \omega^2}$$

10 在量子力学中的应用 (Application in Quantum Mechanics)

傅立叶变换是连接量子力学中位置表象和动量表象的桥梁。位置波函数 $\Psi(x)$ 和动量波函数 $\Phi(k)$ 通过傅立叶变换对联系在一起（常数因子取决于约定）：

$$\Phi(k) = \mathcal{F}[\Psi(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx} dx$$

$$\Psi(x) = \mathcal{F}^{-1}[\Phi(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k) e^{ikx} dk$$

例：δ 势阱中的束缚态 (Bound State in a Delta Potential Well) 定态薛定谔方程为 $H\Psi = E\Psi$，其中哈密顿算符 $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$。考虑一个吸引的 δ 势阱, $V(x) = -\alpha\delta(x)$ (其中 $\alpha > 0$)。薛定谔方程变为：

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \alpha\delta(x)\Psi(x) = E\Psi$$

对于束缚态，能量 $E < 0$。我们对整个方程进行傅立叶变换：

$$-\frac{\hbar^2}{2m}(-k^2)\Phi(k) - \alpha\mathcal{F}[\delta(x)\Psi(x)] = E\Phi(k)$$

其中 $\mathcal{F}[\delta(x)\Psi(x)] = \int \delta(x)\Psi(x)e^{-ikx}dx = \Psi(0)$。

$$\frac{\hbar^2 k^2}{2m}\Phi(k) - \alpha\Psi(0) = E\Phi(k)$$

整理可得动量波函数：

$$\left(\frac{\hbar^2 k^2}{2m} - E\right)\Phi(k) = \alpha\Psi(0) \quad \Rightarrow \quad \Phi(k) = \frac{\alpha\Psi(0)}{\frac{\hbar^2 k^2}{2m} - E}$$

令 $K^2 = -\frac{2mE}{\hbar^2}$（因为 $E < 0$, 所以 K 是实数），则 $E = -\frac{\hbar^2 K^2}{2m}$。

$$\Phi(k) = \frac{\alpha\Psi(0)}{\frac{\hbar^2}{2m}(k^2 + K^2)} = \frac{2m\alpha\Psi(0)/\hbar^2}{k^2 + K^2}$$

现在，我们利用 $\Psi(0)$ 和 $\Phi(k)$ 的关系来求解 K:

$$\Psi(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2m\alpha\Psi(0)/\hbar^2}{k^2 + K^2}dk$$

两边消去 $\Psi(0)$（假设它非零）：

$$1 = \frac{2m\alpha}{\sqrt{2\pi}\hbar^2}\int_{-\infty}^{\infty}\frac{dk}{k^2 + K^2} = \frac{2m\alpha}{\sqrt{2\pi}\hbar^2}\left[\frac{1}{K}\arctan\left(\frac{k}{K}\right)\right]_{-\infty}^{\infty} = \frac{2m\alpha}{\sqrt{2\pi}\hbar^2}\frac{\pi}{K}$$

$$\Rightarrow K = \frac{\sqrt{2\pi}m\alpha}{\hbar^2}$$

将 K 代入能量表达式，得到束缚态能量：

$$E = -\frac{\hbar^2 K^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{2\pi m^2\alpha^2}{\hbar^4}\right) = -\frac{\pi m\alpha^2}{\hbar^2}$$

(注：笔记中的推导似乎省略了 $\sqrt{2\pi}$ 因子，导致最终能量表达式略有不同。此处的推导基于标准的傅立叶变换定义。) 最后，通过对 $\Phi(k)$ 进行傅立叶逆变换，可以得到位置空间中的波函数，其形式为 $\Psi(x) \propto e^{-K|x|}$。

拉普拉斯变换 (Laplace Transform)

1. 定义

设 $f(t)$ 为 $t \geq 0$ 的函数，则 $f(t)$ 的拉普拉斯变换为：

$$F(p) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-pt}dt$$

其中 $p = \beta + i\omega$。通常要求函数 $f(t)$ 满足增长条件 $|f(t)| \leq Me^{\alpha t}$，且在 $\mathrm{Re}(p) > \alpha$ 时积分收敛。

2. 基本性质

1. **线性性质 (Linearity)**

$$\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(p) + bF_2(p)$$

2. **微分性质 (Differentiation)**

$$\mathcal{L}[f'(t)] = \int_0^{\infty}\frac{df(t)}{dt}e^{-pt}dt = [e^{-pt}f(t)]_0^{\infty} + p\int_0^{\infty}f(t)e^{-pt}dt = pF(p) - f(0)$$
$$\mathcal{L}[f''(t)] = p^2F(p) - pf(0) - f'(0)$$
$$\mathcal{L}[f^{(n)}(t)] = p^nF(p) - p^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

3. **p域微分 (Differentiation in p-domain)**

$$\frac{dF(p)}{dp} = \int_0^\infty (-tf(t))e^{-pt}dt = -\mathcal{L}[tf(t)]$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(p)}{dp^n}$$

4. **积分性质 (Integration)** 设 $g(t) = \int_0^t f(\tau)d\tau$，且 $g(0) = 0$。则 $g'(t) = f(t)$。由微分性质 $\mathcal{L}[g'(t)] = p\mathcal{L}[g(t)] - g(0)$，可得 $F(p) = p\mathcal{L}[g(t)]$。

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(p)}{p}$$

5. **p域积分 (Integration in p-domain)**

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_p^\infty F(s)ds$$

6. **位移性质 (Shifting Theorems)**
   - p域位移 (Frequency Shifting):

   $$\mathcal{L}[e^{at}f(t)] = \int_0^\infty f(t)e^{-(p-a)t}dt = F(p-a)$$

   - t域位移 (Time Shifting):

   $$\mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-pt}dt = e^{-pa}F(p)$$

   其中 $u(t-a)$ 是单位阶跃函数。

7. **卷积定理 (Convolution Theorem)**

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

$$\mathcal{L}[f_1(t) * f_2(t)] = F_1(p)F_2(p)$$

8. **标度变换 (Scaling Property)**

$$\mathcal{L}[f(at)] = \int_0^\infty f(at)e^{-pt}dt = \frac{1}{a}\int_0^\infty f(t')e^{-p(t'/a)}dt' = \frac{1}{a}F\left(\frac{p}{a}\right)$$

9. **周期函数 (Periodic Functions)** 若 $f(t)$ 周期为 T，即 $f(t+T) = f(t)$，则：

$$F(p) = \frac{\int_0^T f(t)e^{-pt}dt}{1-e^{-pT}}$$

3. 常用拉普拉斯变换表

$f(t)$	$F(p)$	$f(t)$	$F(p)$
1	$\frac{1}{p}$	e^{at}	$\frac{1}{p-a}$
t^n	$\frac{n!}{p^{n+1}}$	$t^{n-1}e^{at}$	$\frac{(n-1)!}{(p-a)^n}$
$\sin(kt)$	$\frac{k}{p^2+k^2}$	$\cos(kt)$	$\frac{p}{p^2+k^2}$
$\sinh(kt)$	$\frac{k}{p^2-k^2}$	$\cosh(kt)$	$\frac{p}{p^2-k^2}$
$t\sin(kt)$	$\frac{2pk}{(p^2+k^2)^2}$	$t\cos(kt)$	$\frac{p^2-k^2}{(p^2+k^2)^2}$
$t\sinh(kt)$	$\frac{2pk}{(p^2-k^2)^2}$	$t\cosh(kt)$	$\frac{p^2+k^2}{(p^2-k^2)^2}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}}$	$\delta(t)$	1

应用：求解微分和积分方程

例1：求解常微分方程

$y'' + 3y' + 2y = e^{-3t}$, $y(0) = y'(0) = 1$.

$$\mathcal{L}[y'' + 3y' + 2y] = \mathcal{L}[e^{-3t}]$$
$$[p^2Y(p) - py(0) - y'(0)] + 3[pY(p) - y(0)] + 2Y(p) = \frac{1}{p+3}$$
$$(p^2 + 3p + 2)Y(p) - p - 1 - 3 = \frac{1}{p+3}$$
$$(p+1)(p+2)Y(p) = p + 4 + \frac{1}{p+3} = \frac{p^2 + 7p + 13}{p+3}$$
$$Y(p) = \frac{p^2 + 7p + 13}{(p+1)(p+2)(p+3)} = \frac{7/2}{p+1} - \frac{3}{p+2} + \frac{1/2}{p+3}$$
$$y(t) = \frac{7}{2}e^{-t} - 3e^{-2t} + \frac{1}{2}e^{-3t}$$

例2：求解联立微分方程

$x'' - y = 1$, $y'' - x = t$, given $x(0) = 1, x'(0) = 0, y(0) = -1, y'(0) = 0$.

$$\begin{cases} p^2X(p) - px(0) - x'(0) - Y(p) = \frac{1}{p} \\ p^2Y(p) - py(0) - y'(0) - X(p) = \frac{1}{p^2} \end{cases}$$
$$\begin{cases} p^2X - p - Y = \frac{1}{p} \\ p^2Y + p - X = \frac{1}{p^2} \end{cases}$$

From the second eq: $X = p^2Y + p - \dfrac{1}{p^2}$. Substitute into the first eq:

$$p^2(p^2Y + p - \frac{1}{p^2}) - p - Y = \frac{1}{p} \implies (p^4 - 1)Y = -p^3 + p + 1 + \frac{1}{p}$$
$$(p^2 - 1)(p^2 + 1)Y = \frac{-p^4 + p^2 + p + 1}{p} \implies Y(p) = \frac{-p^4 + p^2 + p + 1}{p(p^2 - 1)(p^2 + 1)}$$
$$Y(p) = \frac{-(p^2 - 1)(p^2 + 1) + p + p^2}{p(p^2 - 1)(p^2 + 1)}$$
$$Y(p) = -\frac{1}{p} + \frac{p^2 + p}{p(p-1)(p+1)(p^2 + 1)} = -\frac{1}{p} + \frac{p+1}{(p-1)(p+1)(p^2 + 1)}$$
$$Y(p) = -\frac{1}{p} + \frac{1}{(p-1)(p^2 + 1)}$$
$$Y(p) = -\frac{1}{p} + \frac{1/2}{p-1} - \frac{1/2(p+1)}{p^2 + 1} = -\frac{1}{p} + \frac{1}{2}\frac{1}{p-1} - \frac{1}{2}\frac{p}{p^2 + 1} - \frac{1}{2}\frac{1}{p^2 + 1}$$
$$y(t) = -1 + \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

From $x = y'' - t$,

$$x(t) = \frac{d^2}{dt^2}(-1 + \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t) - t = (\frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t) - t$$
$$x(t) = \frac{1}{2}(e^t + \cos t + \sin t) - t$$

例3: 求解积分方程 (Volterra Type)

$y(t) = at - \int_0^t (t-\tau)y(\tau)d\tau$. This is $y(t) = at - (t * y(t))$.

$$Y(p) = \mathcal{L}[at] - \mathcal{L}[t * y(t)] = \frac{a}{p^2} - \mathcal{L}[t]\mathcal{L}[y(t)]$$

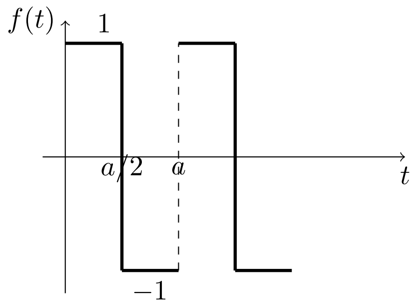
$$Y(p) = \frac{a}{p^2} - \frac{1}{p^2}Y(p)$$

$$Y(p)\left(1 + \frac{1}{p^2}\right) = \frac{a}{p^2} \implies Y(p)\left(\frac{p^2+1}{p^2}\right) = \frac{a}{p^2}$$

$$Y(p) = \frac{a}{p^2+1} \implies y(t) = a\sin t$$

例4: 周期函数的变换

求下图方波的拉普拉斯变换, $f(t+a) = f(t)$。



$$F(p) = \frac{\int_0^a f(t)e^{-pt}dt}{1-e^{-pa}}$$

$$\begin{aligned}
\int_0^a f(t)e^{-pt}dt &= \int_0^{a/2}(1)e^{-pt}dt + \int_{a/2}^a (-1)e^{-pt}dt \\
&= \left[-\frac{1}{p}e^{-pt}\right]_0^{a/2} - \left[-\frac{1}{p}e^{-pt}\right]_{a/2}^a \\
&= -\frac{1}{p}(e^{-pa/2} - 1) + \frac{1}{p}(e^{-pa} - e^{-pa/2}) = \frac{1}{p}(1 - 2e^{-pa/2} + e^{-pa}) = \frac{(1-e^{-pa/2})^2}{p}
\end{aligned}$$

$$\begin{aligned}
F(p) &= \frac{(1-e^{-pa/2})^2}{p(1-e^{-pa})} = \frac{(1-e^{-pa/2})^2}{p(1-e^{-pa/2})(1+e^{-pa/2})} = \frac{1-e^{-pa/2}}{p(1+e^{-pa/2})} \\
&= \frac{e^{pa/4} - e^{-pa/4}}{p(e^{pa/4} + e^{-pa/4})} = \frac{2\sinh(pa/4)}{p(2\cosh(pa/4))} = \frac{1}{p}\tanh\left(\frac{pa}{4}\right)
\end{aligned}$$

应用：求解偏微分方程

1. 弦振动方程 (Wave Equation)

方程为 $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$，其中 $a^2 = T/\rho$。设初始条件为 $u(x,0) = f(x)$, $u_t(x,0) = g(x)$。对时间 t 进行拉普拉斯变换：

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = a^2\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$p^2U(x,p) - pu(x,0) - u_t(x,0) = a^2\frac{d^2U(x,p)}{dx^2}$$

$$\frac{d^2U}{dx^2} - \frac{p^2}{a^2}U = -\frac{p}{a^2}f(x) - \frac{1}{a^2}g(x)$$

这是一个关于 x 的二阶常微分方程。求解 $U(x,p)$ 后再进行拉普拉斯逆变换得到 $u(x,t)$。
对于稳态振动解，可设 $u(x,t)=X(x)e^{i\omega t}$，代入原方程得到亥姆霍兹方程 (Helmholtz equation):

$$\frac{d^2X}{dx^2}+k^2X=0, \quad (k=\omega/a)$$

其通解为 $X(x)=Ae^{ikx}+Be^{-ikx}$，代表了沿 x 轴正负方向传播的波。

2. 输电线方程 (Telegrapher's Equation)

对于一段微元 Δx，电压和电流满足：

$$\frac{\partial V}{\partial x}=-RI-L\frac{\partial I}{\partial t}$$
$$\frac{\partial I}{\partial x}=-GV-C\frac{\partial V}{\partial t}$$

将两式联立消去 I，得到关于 V 的电报方程：

$$\frac{\partial^2 V}{\partial x^2}=LC\frac{\partial^2 V}{\partial t^2}+(RC+LG)\frac{\partial V}{\partial t}+GRV$$

无损耗情况: $R=0,G=0$。方程简化为波动方程 $\frac{\partial^2 V}{\partial x^2}=LC\frac{\partial^2 V}{\partial t^2}$。 **正弦稳态分析**: 设 $V(x,t)=V(x)e^{i\omega t}$，$I(x,t)=I(x)e^{i\omega t}$。

$$\frac{dV(x)}{dx}=-(R+i\omega L)I(x)=-ZI(x)$$
$$\frac{dI(x)}{dx}=-(G+i\omega C)V(x)=-YV(x)$$

其中 Z,Y 分别为串联阻抗和并联导纳。再次微分可得：

$$\frac{d^2V(x)}{dx^2}=ZY\cdot V(x)=\gamma^2V(x)$$

其中 $\gamma=\sqrt{ZY}=\sqrt{(R+i\omega L)(G+i\omega C)}$ 称为传播常数。 $\gamma=\alpha+i\beta$，α 是衰减常数，β 是相移常数。 **无失真条件**: 为了让信号在传播过程中波形不发生改变，要求相速度 $v_p=\omega/\beta$ 与频率无关。这发生在 $\frac{RC}{LC}=\frac{LG}{LC}$，即 $\frac{R}{L}=\frac{G}{C}$ (Heaviside condition)。

热传导方程

一维杆的热传导方程

考虑一维杆，长度为L，截面积为A。
物理量：

- $u(x,t)$: x点在t时刻的温度
- c: 比热容
- ρ: 密度
- Q: 热量

定律： 单位时间内截面热流量

$$Q = -kA\frac{\partial u}{\partial x}$$

其中k是热导率。

考虑$[x, x+\Delta x]$一小段，在Δt时间内热量变化：

$$\Delta Q = Q_1 - Q_2$$

$$Q_1 = -kA\frac{\partial u}{\partial x}\Big|_x \Delta t$$

$$Q_2 = -kA\frac{\partial u}{\partial x}\Big|_{x+\Delta x} \Delta t$$

$$\Delta Q = c\rho(A\Delta x)\Delta u = c\rho A\Delta x(u(t+\Delta t) - u(t))$$

（$P = m/V$, $\Delta m = \rho A\Delta x$, 质量守恒）

$$\Rightarrow kA\Delta t\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x\right) = c\rho A\Delta x\Delta u$$

$$\Rightarrow k\frac{\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x}{\Delta x} = c\rho\frac{\Delta u}{\Delta t}$$

令$\Delta x, \Delta t \to 0$:

$$k\frac{\partial^2 u}{\partial x^2} = c\rho\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \quad (a^2 = \frac{k}{c\rho})$$

热源情形： 若有热源$f(x,t)$

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

若热源由电流产生 $Q_{gen} = I^2R\Delta t = j^2\rho_e\delta\Delta x\Delta t$

$$Q_1 - Q_2 + Q_{gen} = \Delta Q$$

$$kA\Delta t\frac{\partial^2 u}{\partial x^2}\Delta x + j^2\rho_e\delta\Delta x\Delta t = c\rho\delta\Delta x\Delta u$$

$$\Rightarrow \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f$$

$$(c\rho\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + F)$$

例： 稳定状态 $\frac{\partial u}{\partial t} = 0$

$$a^2\frac{\partial^2 u}{\partial x^2} + f = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \quad (\text{若}f = 0)$$

$$u(x) = Ax + B$$

电磁波方程

麦克斯韦方程组 ($\rho=0,j=0$ 真空中)

$$\nabla\cdot E=0$$

$$\nabla\cdot B=0$$

$$\nabla\times E=-\frac{\partial B}{\partial t}$$

$$\nabla\times B=\mu_0\epsilon_0\frac{\partial E}{\partial t}$$

其中 $\epsilon_0\mu_0=1/c^2$。

推导波动方程： 利用恒等式 $\nabla\times(\nabla\times A)=\nabla(\nabla\cdot A)-\Delta A$

$$\nabla\times(\nabla\times E)=\nabla(\nabla\cdot E)-\Delta E=-\Delta E$$

$$\nabla\times(-\frac{\partial B}{\partial t})=-\frac{\partial}{\partial t}(\nabla\times B)=-\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow\Delta E=\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial t^2}=c^2\Delta E$$

同理对B可得

$$\frac{\partial^2 B}{\partial t^2}=c^2\Delta B$$

有源情况 ($\rho\neq 0,j\neq 0$)

$$\nabla\cdot E=\rho/\epsilon_0$$

$$\nabla\cdot B=0$$

$$\nabla\times E=-\frac{\partial B}{\partial t}$$

$$\nabla\times B=\mu_0 j+\mu_0\epsilon_0\frac{\partial E}{\partial t}$$

$$\nabla\times(\nabla\times E)=\nabla(\rho/\epsilon_0)-\Delta E=-\frac{\partial}{\partial t}(\mu_0 j+\mu_0\epsilon_0\frac{\partial E}{\partial t})$$

$$\Delta E-\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}=\nabla(\rho/\epsilon_0)+\mu_0\frac{\partial j}{\partial t}$$

泊松方程

当$\rho\neq 0,j=0$ (静电场)

$$\Delta E=\nabla(\rho/\epsilon_0)$$

引入电势ϕ，$E=-\nabla\phi$

$$\Delta(-\nabla\phi)=-\nabla(\Delta\phi)=\nabla(\rho/\epsilon_0)$$

$$\Delta\phi=-\rho/\epsilon_0$$

此为泊松方程。无源情形 ($\rho=0$)

$$\Delta\phi=0$$

此为拉普拉斯方程。

二阶线性偏微分方程的分类

考虑方程：

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

其中A,B,C,D,E,F,G是x,y的函数。令 $\Delta=B^2-AC$

- $\Delta>0$: 双曲型 (e.g. 波动方程)
- $\Delta=0$: 抛物型 (e.g. 热传导方程)
- $\Delta<0$: 椭圆型 (e.g. 拉普拉斯方程)

这与二次曲线的分类是类似的。

坐标变换 令 $\xi=\xi(x,y),\eta=\eta(x,y)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=\dots$$

代入原方程，得到新的方程：

$$a\frac{\partial^2 u}{\partial \xi^2}+2b\frac{\partial^2 u}{\partial \xi\partial \eta}+c\frac{\partial^2 u}{\partial \eta^2}+\dots=G$$

其中

$$a=A(\frac{\partial \xi}{\partial x})^2+2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+C(\frac{\partial \xi}{\partial y})^2$$

$$b=A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+B(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x})+C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}$$

$$c=A(\frac{\partial \eta}{\partial x})^2+2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+C(\frac{\partial \eta}{\partial y})^2$$

可以证明 $b^2-ac=(B^2-AC)(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x})^2$ 其中后面的行列式是坐标变换的雅可比行列式。

化为标准型

目标是选择ξ,η使得a,c中至少一个为0。令 $a=0$

$$A(\frac{\partial \xi}{\partial x})^2+2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+C(\frac{\partial \xi}{\partial y})^2=0$$

$$A\left(\frac{\partial\xi/\partial x}{\partial\xi/\partial y}\right)^2+2B\left(\frac{\partial\xi/\partial x}{\partial\xi/\partial y}\right)+C=0$$

根据隐函数定理，沿 $\xi(x,y)=const$ 曲线，有 $\frac{dy}{dx}=-\frac{\partial\xi/\partial x}{\partial\xi/\partial y}$

$$A\left(\frac{dy}{dx}\right)^2-2B\left(\frac{dy}{dx}\right)+C=0$$

解出 $\frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{2B\pm\sqrt{4B^2-4AC}}{2A}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

这就是特征方程。

(1) $\Delta = B^2 - AC > 0$ **(双曲型)** 有两个不同的实根 $\frac{dy}{dx} = \lambda_1, \frac{dy}{dx} = \lambda_2$。解这两个常微分方程，得到两个特征线族 $\phi(x,y) = c_1, \psi(x,y) = c_2$。令 $\xi = \phi(x,y), \eta = \psi(x,y)$。这样 $a = 0, c = 0$。方程化为

$$2b\frac{\partial^2 u}{\partial\xi\partial\eta} + \cdots = 0 \Rightarrow \frac{\partial^2 u}{\partial\xi\partial\eta} = \dots \quad \text{(标准型I)}$$

若再做变换 $\xi' = \xi + \eta, \eta' = \xi - \eta$，则

$$\frac{\partial^2 u}{\partial\xi'^2} - \frac{\partial^2 u}{\partial\eta'^2} = \dots \quad \text{(标准型II)}$$

(2) $\Delta = B^2 - AC = 0$ **(抛物型)** 只有一个实根 $\frac{dy}{dx} = \frac{B}{A}$。解得一个特征线族 $\phi(x,y) = c$。令 $\xi = \phi(x,y)$，则 $a = 0$。 η可以任取与ξ无关的函数，例如$\eta = x$。此时$b = 0$, $c \neq 0$。方程化为

$$c\frac{\partial^2 u}{\partial\eta^2} = \cdots \Rightarrow \frac{\partial^2 u}{\partial\eta^2} = \dots \quad \text{(标准型)}$$

(3) $\Delta = B^2 - AC < 0$ **(椭圆型)** 特征方程的根是共轭复数。

$$\frac{dy}{dx} = \frac{B \pm i\sqrt{AC - B^2}}{A}$$

解也是共轭的，$\phi(x,y) \pm i\psi(x,y) = const$。令 $\xi = \phi(x,y), \eta = \psi(x,y)$。可以证明 $a = c, b = 0$。方程化为

$$a\left(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\right) = \cdots \Rightarrow \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = \dots \quad \text{(标准型)}$$

再探标准型变换

(b) $\Delta = 0$: (应为 $u_{\eta\eta} = 0$) 取变换

$$\xi = x, \quad \eta = y - \frac{B}{A}x$$

则

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = -\frac{1}{A^2}(\dots)$$

(笔记此处似有误，应化为 $\frac{\partial^2 u}{\partial\eta^2} = \dots$)

(c) $\Delta < 0$: $u_{xx} + u_{yy} = 0$. 取变换

$$\xi = y - \frac{B}{A}x, \quad \eta = \frac{\sqrt{AC - B^2}}{A}x$$

$$\Rightarrow \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = \dots$$

总结特征线法： 从 $A(\frac{dy}{dx})^2 - 2B(\frac{dy}{dx}) + C = 0$ 出发

(a) $\Delta > 0$: 两条实特征线

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

解出 $\phi_1(x,y) = c_1, \phi_2(x,y) = c_2$。令 $\xi = \phi_1, \eta = \phi_2$。得标准型：

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = d\frac{\partial u}{\partial\xi} + e\frac{\partial u}{\partial\eta} + fu + g$$

(b) $\Delta = 0$: 一条实特征线

$$\frac{dy}{dx} = \frac{B}{A}$$

解出 $\phi(x, y) = c$。令 $\xi = \phi(x, y)$, η可任取（如$\eta = x$）。得标准型：

$$\frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu + g$$

(c) $\Delta < 0$: 无实特征线取 $\xi = y - \frac{B}{A}x, \eta = \frac{\sqrt{AC-B^2}}{A}x$。得标准型：

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu + g$$

例子

1. $u_{xx} - u_{tt} + au_t + bu_x = 0$ $A = 1, B = 0, C = -1$. $\Delta = 0 - (1)(-1) = 1 > 0$ (双曲型) 特征方程: $(\frac{dt}{dx})^2 - 1 = 0 \Rightarrow \frac{dt}{dx} = \pm 1$ 特征线: $t - x = c_1, t + x = c_2$ 令 $\xi = x - t, \eta = x + t$.
2. $u_{xx} - 2u_{xt} - u_t = 0$ $A = 1, B = -1, C = 0$. $\Delta = (-1)^2 - 0 = 1 > 0$ (双曲型) 特征方程: $(\frac{dt}{dx})^2 + 2(\frac{dt}{dx}) = 0 \Rightarrow \frac{dt}{dx}(\frac{dt}{dx} + 2) = 0$ $\frac{dt}{dx} = 0 \Rightarrow t = c_1$. $\frac{dt}{dx} = -2 \Rightarrow t + 2x = c_2$. 令 $\xi = t, \eta = t + 2x$.
3. $u_{xx} - 4u_{xy} + 3u_{yy} + 8u_y + x = 0$ $A = 1, B = -2, C = 3$. $\Delta = (-2)^2 - 1 \cdot 3 = 1 > 0$ (双曲型)
4. $yu_{xx} + xu_{yy} = 0$ $A = y, B = 0, C = x$. $\Delta = -xy$.
   - $xy > 0$ (I, III象限): 椭圆型
   - $xy < 0$ (II, IV象限): 双曲型
   - $x = 0$ 或 $y = 0$: 抛物型

定解问题

一维波动方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ u|_{x=0} = 0, u|_{x=L} = 0 & (\text{边界条件}) \\ u|_{t=0} = \phi(x) & (\text{初始位移}) \\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x) & (\text{初始速度}) \end{cases}$$

叠加原理： 对于线性齐次方程 $L(u) = 0$，若 u_1, u_2 是解，则 $c_1u_1 + c_2u_2$ 也是解。对于 $L(u) = F$ (非齐次方程)，其通解为 $u = u_p + u_h$，其中 u_p 是一个特解，u_h 是对应齐次方程的通解。此性质可用于分解问题。

一维热传导方程

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ u(0, t) = 0, u(L, t) = 0 & (t > 0) \\ u(x, 0) = f(x) & (0 \le x \le L) \end{cases}$$

这是一个定解问题。

例1：稳态解 如果边界条件不为零，如 $u(0,t)=T_1, u(L,t)=T_2$。稳态解 $u_E(x)$ 满足

$$\frac{d^2u_E}{dx^2}=0 \Rightarrow u_E(x)=c_1x+c_2$$

代入边界条件

$$u_E(0)=T_1 \Rightarrow c_2=T_1$$
$$u_E(L)=T_2 \Rightarrow c_1L+T_1=T_2 \Rightarrow c_1=\frac{T_2-T_1}{L}$$

所以

$$u_E(x)=\frac{T_2-T_1}{L}x+T_1$$

令 $u(x,t)=v(x,t)+u_E(x)$，则$v(x,t)$满足齐次边界条件。

分离变量法 (Separation of Variables Method)

弦振动 (String Vibration)

The governing partial differential equation (PDE):

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

Initial conditions:

$$u|_{t=0}=\phi(x)$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi(x)$$

Boundary conditions (fixed ends):

$$u|_{x=0}=0, \quad u|_{x=L}=0$$

Goal: Find the solution $u(x,t)$.

推导 (Derivation)

Assume the solution can be written as a product of functions of a single variable:

$$u(x,t)=X(x)T(t)$$

Substitute into the PDE:

$$X(x)T''(t)=a^2X''(x)T(t)$$

Rearrange the terms to separate variables:

$$\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}$$

Since the left side depends only on x and the right side only on t, both must be equal to a constant. Let's call this constant $-\lambda$.

$$\frac{d}{dx}\left[\frac{X''(x)}{X(x)}\right]=0$$

$$\frac{d}{dt}\left[\frac{T''(t)}{a^2T(t)}\right] = 0$$

This gives two ordinary differential equations (ODEs):

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + \lambda a^2 T(t) = 0$$

The boundary conditions for $u(x,t)$ translate to conditions for $X(x)$, since $T(t) \not\equiv 0$ for a non-trivial solution:

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$$

$$u(L,t) = X(L)T(t) = 0 \implies X(L) = 0$$

求解本征值问题 (Solving the Eigenvalue Problem for X(x))

We analyze the possible values of the separation constant λ.

Case 1: $\lambda = 0$ The equation for $X(x)$ is $X''(x) = 0$. The general solution is:

$$X(x) = Ax + B$$

Applying the boundary conditions:

$$X(0) = B = 0$$

$$X(L) = AL + B = 0 \implies A = 0$$

This gives $X(x) = 0$, which leads to the trivial solution $u(x,t) = 0$.

Case 2: $\lambda < 0$ Let $\lambda = -k^2$ where $k > 0$. The equation is $X''(x) - k^2X(x) = 0$. The general solution is:

$$X(x) = A\cosh(kx) + B\sinh(kx)$$

Applying the boundary conditions:

$$X(0) = A\cosh(0) + B\sinh(0) = A = 0$$

$$X(L) = B\sinh(kL) = 0$$

Since $k > 0$ and $L > 0$, $\sinh(kL) \neq 0$, so $B = 0$. This again leads to the trivial solution $X(x) = 0$.

Case 3: $\lambda > 0$ Let $\lambda = k^2$ where $k > 0$. The equation is $X''(x) + k^2X(x) = 0$. The general solution is:

$$X(x) = A\cos(kx) + B\sin(kx)$$

Applying the boundary conditions:

$$X(0) = A\cos(0) + B\sin(0) = A = 0$$

So,

$$X(x) = B\sin(kx)$$

$$X(L) = B\sin(kL) = 0$$

For a non-trivial solution, we must have $B \neq 0$, which implies:

$$\sin(kL) = 0$$

This means $kL = n\pi$ for $n = 1, 2, 3, \ldots$. The possible values for k are:

$$k_n = \frac{n\pi}{L}$$

These lead to the eigenvalues (本征值):

$$\lambda_n = k_n^2 = \left(\frac{n\pi}{L}\right)^2$$

The corresponding eigenfunctions (本征函数) are:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

求解 T(t) 并叠加 (Solving for T(t) and Superposition)

Now we solve for $T(t)$ using the found eigenvalues λ_n:

$$T_n''(t) + \lambda_n a^2 T_n(t) = 0$$

$$T_n''(t) + \left(\frac{n\pi a}{L}\right)^2 T_n(t) = 0$$

The general solution for $T_n(t)$ is:

$$T_n(t) = C_n \cos\left(\frac{n\pi at}{L}\right) + D_n \sin\left(\frac{n\pi at}{L}\right)$$

The solution for each mode n is $u_n(x,t) = X_n(x)T_n(t)$. We absorb the constant B_n into C_n and D_n.

$$u_n(x,t) = \left(C_n \cos\left(\frac{n\pi at}{L}\right) + D_n \sin\left(\frac{n\pi at}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

By the superposition principle (叠加原理), the general solution is the sum of all possible solutions:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi at}{L}\right) + D_n \sin\left(\frac{n\pi at}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

利用初始条件 (Using Initial Conditions)

We determine the coefficients C_n and D_n using the initial conditions. At $t = 0$:

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

This is a Fourier sine series for $\phi(x)$. The coefficients C_n are given by:

$$C_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Next, we find the derivative with respect to t:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-C_n \frac{n\pi a}{L} \sin\left(\frac{n\pi at}{L}\right) + D_n \frac{n\pi a}{L} \cos\left(\frac{n\pi at}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

At $t = 0$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = \sum_{n=1}^{\infty} D_n \frac{n\pi a}{L} \sin\left(\frac{n\pi x}{L}\right)$$

This is a Fourier sine series for $\psi(x)$. The coefficients are given by:

$$D_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$D_n = \frac{2}{n\pi a} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

分离变量法总结 (Summary of Separation of Variables)

1. **分离变量:** 设定 $u = TX$ (Let $u = TX$)
2. **定解:** 代入得 $X(x), T(t)$ 常微分方程 (Substitute to get ODEs for $X(x), T(t)$)
3. **边界条件:** 求解 $X(x)$, 边界条件 (齐次) $\Longrightarrow$ 本征值 λ_n 与本征函数 $X_n(x)$ (Solve for $X(x)$ using homogeneous boundary conditions to get eigenvalues λ_n and eigenfunctions $X_n(x)$)
4. **齐次:** 代入 $\lambda_n \to T_n(t)$ (Substitute λ_n to find $T_n(t)$)
5. **叠加原理:** $u(x,t) = \sum u_n(x,t)$ (Superposition principle)

基本解问题 (Examples of Fundamental Solutions)

例1 (Example 1: Plucked String)

Problem:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \phi(x) = \begin{cases} \frac{3}{2}x & 0 \le x \le 2/5 \\ 3(1-x) & 2/5 \le x \le 1 \end{cases} \quad \text{(with } L = 1\text{)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = 0$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

Since $\psi(x) = 0$, we have $D_n = 0$ for all n. We calculate C_n:

$$C_n = \frac{2}{1} \int_0^1 \phi(x) \sin(n\pi x) dx$$

$$C_n = 2 \left[\int_0^{2/5} \frac{3}{2} x \sin(n\pi x) dx + \int_{2/5}^1 3(1-x) \sin(n\pi x) dx \right]$$

After integration (result from notes):

$$C_n = \frac{9}{5n^2\pi^2} \sin\left(\frac{2n\pi}{5}\right)$$

The final solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{9}{5n^2\pi^2} \sin\left(\frac{2n\pi}{5}\right) \cos(n\pi at) \sin(n\pi x)$$

例2 (Example 2: Struck String)

Problem:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = \phi(x) = 0$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = \frac{K}{\rho}\delta(x-c) \quad \text{(impulse at } x=c\text{)}$$
$$u(0,t) = 0, \quad u(L,t) = 0$$

Since $\phi(x) = 0$, we have $C_n = 0$ for all n. We calculate D_n:

$$D_n = \frac{2}{n\pi a} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$D_n = \frac{2}{n\pi a} \int_0^L \frac{K}{\rho}\delta(x-c) \sin\left(\frac{n\pi x}{L}\right) dx$$

Using the sifting property of the Dirac delta function:

$$D_n = \frac{2K}{n\pi a\rho} \sin\left(\frac{n\pi c}{L}\right)$$

The final solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2K}{n\pi a\rho} \sin\left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi at}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

物理解释 (Physical Interpretation)

The solution for a single mode can be written in phase-amplitude form:

$$u_n(x,t) = N_n \sin(\omega_n t + \theta_n) \sin\left(\frac{n\pi x}{L}\right)$$

where the angular frequency is $\omega_n = \frac{n\pi a}{L}$. The amplitude N_n and phase θ_n are given by:

$$N_n = \sqrt{C_n^2 + D_n^2}$$

$$\tan\theta_n = \frac{C_n}{D_n}$$

An alternative form from the notes is:

$$u_n(x,t) = A_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$u_n(t) = B_n(x_0) \sin(\omega_n t + \theta_n)$$

驻波 (Standing Waves)

The solution $u_n(x,t)$ represents a standing wave.

- **节点 (Nodes):** Points that do not move. Occur when $\sin(\frac{n\pi x}{L}) = 0$.

$$\frac{n\pi x}{L} = m\pi, \quad m = 0, 1, \dots, n$$

$$x_m = \frac{m}{n}L$$

- **波腹 (Antinodes):** Points of maximum amplitude (x_0).

单模振动 (Single-mode Oscillation)

A single eigenfunction corresponds to a single mode of vibration.

$$E \sim \sin\left(\frac{n\pi x}{L}\right)$$

与量子力学类比 (Analogy to Quantum Mechanics)

The spatial part of the wave solution is analogous to the wave function for a particle in a 1D infinite potential well.

$$\Psi \sim \sin\left(\frac{n\pi x}{L}\right)$$

The time-dependent Schrödinger equation:

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U(x)\Psi$$

其他边界条件 (Other Boundary Conditions)

1. **两端固定 (Fixed-Fixed):** $u(0,t) = 0$, $u(L,t) = 0$.
2. **一端固定, 一端自由 (Fixed-Free):** $u(0,t) = 0$, $\frac{\partial u}{\partial x}\Big|_{x=L} = 0$.
3. **两端自由 (Free-Free):** $\frac{\partial u}{\partial x}\Big|_{x=0} = 0$, $\frac{\partial u}{\partial x}\Big|_{x=L} = 0$.
4. **辐射边界条件 (Radiation Boundary Condition):** $-k\frac{\partial u}{\partial x}\Big|_{x=L} = H(u(L,t) - u_0)$.

有阻尼波动方程与电报方程 (Damped Wave and Telegrapher's Equation)

例: 电报方程 (Example: Telegrapher's Equation)

The general form of the Telegrapher's equation is:

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t} + cu = 0$$

With initial conditions:

$$u|_{t=0} = \phi(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x)$$

And boundary conditions $(b, c > 0)$:

$$u|_{x=0} = 0, \quad u|_{x=L} = 0$$

Using separation of variables, $u(x,t) = X(x)T(t)$, we get:

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0, \quad X(L) = 0$$

and

$$T''(t) + 2bT'(t) + (\lambda a^2 + c)T(t) = 0$$

The solution for $X(x)$ is the same as for the standard wave equation:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

The solution for $T(t)$ is that of a damped harmonic oscillator. For the underdamped case, the solution has the form:

$$u(x,t) = e^{-bt} \sum_{n=1}^{\infty} (C_n \cos(q_n t) + D_n \sin(q_n t)) \sin\left(\frac{n\pi x}{L}\right)$$

where the new frequency q_n is:

$$q_n = \sqrt{\left|\left(\frac{n\pi a}{L}\right)^2 + c - b^2\right|}$$

The coefficients C_n, D_n are determined by the initial conditions.

例: 有阻尼波动方程 (Example: Damped Wave Equation)

This is a special case of the Telegrapher's equation where $c = 0$.

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial u}{\partial t} = 0$$

The equation for $T(t)$ becomes:

$$T_n''(t) + 2bT_n'(t) + \lambda_n a^2 T_n(t) = 0$$

The characteristic equation is $r^2 + 2br + (\frac{n\pi a}{L})^2 = 0$. The behavior depends on the discriminant. Let $q_n = \sqrt{|(\frac{n\pi a}{L})^2 - b^2|}$.

The general solution for $T_n(t)$ can be one of three cases for each mode n:

1. **Underdamped** ($\frac{n\pi a}{L} > b$):

$$T_n(t) = e^{-bt} \left(C_n \cos(q_n t) + D_n \sin(q_n t)\right)$$

2. **Critically Damped** ($\frac{n\pi a}{L} = b$):

$$T_n(t) = e^{-bt}(C_n + D_n t)$$

3. **Overdamped** ($\frac{n\pi a}{L} < b$):

$$T_n(t) = e^{-bt} \left(C_n \cosh(q_n t) + D_n \sinh(q_n t) \right)$$

Let's assume the underdamped case holds for all modes of interest ($\frac{bL}{\pi a} < 1$). The total solution is:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-bt} \left(C_n \cos(q_n t) + D_n \sin(q_n t) \right) \sin\left(\frac{n\pi x}{L}\right)$$

To find the coefficients from $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies C_n = \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\psi(x) = \sum_{n=1}^{\infty} (-bC_n + q_n D_n) \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies -bC_n + q_n D_n = \frac{2}{L} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\implies D_n = \frac{b}{q_n} C_n + \frac{2}{q_n L} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

If the initial velocity is zero, $\psi(x) = 0$, then $D_n = \frac{b}{q_n} C_n$.

热传导方程 (Heat Equation)

例: 傅里叶热棒 (Example: Fourier Heat Rod)

The problem describes the temperature $u(x,t)$ in a rod with insulated ends.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Initial condition:

$$u(t=0) = \phi(x)$$

Boundary conditions (insulated ends):

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Separating variables $u(x,t) = X(x)T(t)$ yields:

$$X''(x) + \lambda X(x) = 0, \quad \text{with } X'(0) = 0, X'(L) = 0$$

$$T'(t) + \lambda a^2 T(t) = 0$$

Solving the eigenvalue problem for $X(x)$:

- Case $\lambda = 0$: $X''(x) = 0 \implies X(x) = Ax + B$. $X'(0) = A = 0$. $X'(L) = A = 0$. So $X_0(x) = B_0$ (a constant) is an eigenfunction.
- Case $\lambda < 0$: Trivial solution $X(x) = 0$.
- Case $\lambda > 0$ ($\lambda = k^2$): $X(x) = A\cos(kx) + B\sin(kx)$. $X'(0) = Bk = 0 \implies B = 0$. $X'(L) = -Ak\sin(kL) = 0 \implies \sin(kL) = 0$. Thus $kL = n\pi$ for $n = 1, 2, 3, \ldots$.

The eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$ for $n = 0, 1, 2, \ldots$. The eigenfunctions are $X_n(x) = A_n \cos(\frac{n\pi x}{L})$. Solving for $T(t)$: For $n > 0$: $T_n'(t) + (\frac{n\pi a}{L})^2 T_n(t) = 0 \implies T_n(t) = C_n e^{-(\frac{n\pi a}{L})^2 t}$. For $n = 0$ ($\lambda_0 = 0$): $T_0'(t) = 0 \implies T_0(t) = C_0$. The general solution is by superposition:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \exp\left[-\left(\frac{n\pi a}{L}\right)^2 t\right] \cos\left(\frac{n\pi x}{L}\right)$$

Using the initial condition $u(x,0) = \phi(x)$:

$$\phi(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$$

This is a Fourier cosine series. The coefficients are:

$$C_0 = \frac{1}{L}\int_0^L \phi(x)dx$$

$$C_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

波动方程更多示例 (Further Examples for the Wave Equation)

例: 自由-固定端 (Example: Free-Fixed End)

The note appears to solve for a rod with a free end at $x = 0$ and a fixed end at $x = L$.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u|_{x=L} = 0$$

Separation of variables leads to $X''(x) + \lambda X(x) = 0$ with $X'(0) = 0, X(L) = 0$. Let $\lambda = k^2$. $X(x) = A\cos(kx) + B\sin(kx)$.

$$X'(0) = Bk = 0 \implies B = 0$$

$$X(L) = A\cos(kL) = 0 \implies kL = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

The eigenvalues and eigenfunctions are:

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$$

$$X_n(x) = A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

The general solution is:

$$u(x,t) = \sum_{n=0}^{\infty} \left(C_n \cos\left(\frac{(2n+1)\pi at}{2L}\right) + D_n \sin\left(\frac{(2n+1)\pi at}{2L}\right)\right) \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

Coefficients are found from initial conditions $\phi(x)$ and $\psi(x)$:

$$C_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

$$D_n = \frac{2}{L}\frac{2L}{(2n+1)\pi a}\int_0^L \psi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

Specific Case: If $u(x,0) = \cos(\frac{\pi x}{2L})$ and $u_t(x,0) = 0$. This corresponds to the $n = 0$ mode.

$$D_n = 0 \text{ for all } n$$

$$C_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

By orthogonality, this integral is non-zero only for $n = 0$.

$$C_0 = \frac{2}{L}\int_0^L \cos^2\left(\frac{\pi x}{2L}\right) dx = \frac{2}{L}\cdot\frac{L}{2} = 1$$

All other $C_n = 0$. The solution is:

$$u(x,t) = \cos\left(\frac{\pi at}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)$$

例: 固定-自由端 (Example: Fixed-Free End)

Another example shows fixed-free boundary conditions: $u(0,t) = 0, u_x(L,t) = 0$. Eigenfunctions: $\sin(\frac{(2n+1)\pi x}{2L})$. Initial conditions: $u(x,0) = E$ (a constant), $u_t(x,0) = 0$. Then $D_n = 0$ for all n.

$$C_n = \frac{2}{L}\int_0^L E\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx$$

$$C_n = \frac{2E}{L}\left[-\frac{2L}{(2n+1)\pi}\cos\left(\frac{(2n+1)\pi x}{2L}\right)\right]_0^L$$

$$C_n = -\frac{4E}{(2n+1)\pi}(\cos(\frac{(2n+1)\pi}{2}) - \cos(0)) = \frac{4E}{(2n+1)\pi}$$

The solution is:

$$u(x,t) = \frac{4E}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos\left(\frac{(2n+1)\pi at}{2L}\right)\sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

总结 (Summary)

$$X''(x) + \lambda X(x) = 0$$

$$u|_{x=0} = u|_{x=L} = 0 \quad \Rightarrow \quad \lambda_n = (\frac{n\pi}{L})^2, \quad X_n(x) = B_n\sin(\frac{n\pi x}{L})$$

$$\frac{\partial u}{\partial x}|_{x=0} = \frac{\partial u}{\partial x}|_{x=L} = 0 \quad \Rightarrow \quad \lambda_n = (\frac{n\pi}{L})^2, \quad X_n(x) = A_n\cos(\frac{n\pi x}{L})$$

$$u|_{x=0} = \frac{\partial u}{\partial x}|_{x=L} = 0 \quad \Rightarrow \quad \lambda_n = (\frac{(2n+1)\pi}{2L})^2, \quad X_n(x) = A_n\cos(\frac{(2n+1)\pi x}{2L})$$

二维波动方程 (2D Wave Equation)

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$\begin{cases} u|_{t=0} = \phi(x,y) \\ u_t|_{t=0} = \psi(x,y) \\ u|_{x=0} = u|_{x=a} = 0 \quad 0 \le y \le b \\ u|_{y=0} = u|_{y=b} = 0 \quad 0 \le x \le a \end{cases}$$

令 (Let)

$$u(x,y,t) = V(x,y)T(t)$$

$$\Rightarrow \frac{T''}{c^2T} = \frac{1}{V}(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}) = -\lambda$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \lambda V = 0$$

$$T'' + \lambda c^2 T = 0$$

再令 (Let again)

$$V(x,y) = X(x)Y(y)$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y'' + \lambda Y}{Y} = -\mu$$

$$\Rightarrow \begin{cases} X'' + \mu X = 0 \\ X(0) = X(a) = 0 \end{cases}$$

$$\begin{cases} Y'' + \nu Y = 0, \quad \nu = \lambda - \mu \\ Y(0) = Y(b) = 0 \end{cases}$$

解得 (Solution is)

$$\mu_m = (\frac{m\pi}{a})^2, \quad X_m(x) = \sin(\frac{m\pi x}{a})$$
$$\nu_n = (\frac{n\pi}{b})^2, \quad Y_n(y) = \sin(\frac{n\pi y}{b})$$

到 (Thus)

$$\lambda_{mn} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$
$$V_{mn}(x, y) = \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

代入 (Substitute into T)

$$T_{mn}(t) = C_{mn}\cos(\omega_{mn}t) + D_{mn}\sin(\omega_{mn}t)$$
$$\omega_{mn} = c\sqrt{\lambda_{mn}} = c\pi\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

基频 (Fundamental frequency)

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

叠加 (Superposition)

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(C_{mn}\cos(\omega_{mn}t) + D_{mn}\sin(\omega_{mn}t))\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$
$$\phi(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$
$$\psi(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \omega_{mn}D_{mn}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

正交性 (Orthogonality)

$$\int_0^a\int_0^b V_{mn}(x, y)V_{m'n'}(x, y)dxdy = (\int_0^a \sin(\frac{m\pi x}{a})\sin(\frac{m'\pi x}{a})dx)(\int_0^b \sin(\frac{n\pi y}{b})\sin(\frac{n'\pi y}{b})dy)$$
$$= \frac{ab}{4}\delta_{mm'}\delta_{nn'}$$

得 (We get)

$$C_{mn} = \frac{4}{ab}\int_0^a\int_0^b \phi(x, y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dxdy$$
$$D_{mn} = \frac{4}{ab\omega_{mn}}\int_0^a\int_0^b \psi(x, y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dxdy$$

特征函数集 (Set of eigenfunctions)

$$\{\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})\}$$

例 (Example)

$$a = b = 1, \quad c = \frac{1}{\pi}$$
$$\phi = x(1-x)y(1-y)$$

$$\psi = 0 \quad \Rightarrow \quad D_{mn} = 0$$

$$C_{mn} = 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin(m\pi x)\sin(n\pi y)dxdy$$

$$= 4[\int_0^1 x(1-x)\sin(m\pi x)dx][\int_0^1 y(1-y)\sin(n\pi y)dy]$$

$$\int_0^1 x(1-x)\sin(m\pi x)dx = \frac{2(1-(-1)^m)}{m^3\pi^3}$$

$$C_{mn} = 4\frac{2(1-(-1)^m)}{m^3\pi^3}\frac{2(1-(-1)^n)}{n^3\pi^3} = \frac{16(1-(-1)^m)(1-(-1)^n)}{m^3n^3\pi^6}$$

解 (Solution)

$$u = \sum_{m,n=1,3,5,\dots}^{\infty} \frac{64}{\pi^6 m^3 n^3}\cos(\sqrt{m^2+n^2}t)\sin(m\pi x)\sin(n\pi y)$$

二维热传导方程 (2D Heat Equation)

$$\frac{\partial u}{\partial t} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$u|_{t=0} = \phi(x,y)$$

$$u|_{x=0} = u|_{x=a} = 0, \quad u|_{y=0} = u|_{y=b} = 0$$

令 (Let)

$$u = V(x,y)T(t)$$

$$\Rightarrow \frac{1}{c^2T}\frac{\partial T}{\partial t} = \frac{1}{V}(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}) = -\lambda$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \lambda V = 0$$

$$T' + c^2\lambda T = 0$$

$$\Rightarrow \lambda_{mn} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$

$$V_{mn}(x,y) = \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

$$T_{mn}(t) = e^{-\omega_{mn}t}$$

$$\omega_{mn} = c^2\lambda_{mn} = c^2\pi^2((\frac{m}{a})^2 + (\frac{n}{b})^2)$$

解 (Solution)

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-\omega_{mn}t}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

$$C_{mn} = \frac{4}{ab}\int_0^a\int_0^b \phi(x,y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dxdy$$

例1 (Example 1)

$$a = b = 1, c = 1$$

$$\phi = x(1-x)y(1-y)$$

解 (Solution)

$$u(x,y,t) = \sum_{m,n=1,3,5,\dots}^{\infty} \frac{64}{m^3n^3\pi^6}e^{-[(m\pi)^2+(n\pi)^2]t}\sin(m\pi x)\sin(n\pi y)$$

中心温度 (Temperature at the center)

$$u(\frac{1}{2},\frac{1}{2},t)=\sum_{m,n=1,3,5,\ldots}^{\infty}\frac{64}{m^3n^3\pi^6}e^{-[(m\pi)^2+(n\pi)^2]t}\sin(\frac{m\pi}{2})\sin(\frac{n\pi}{2})$$

例2 (Example 2)

$$a=b=1,c=1$$

$$\phi=\sin(\pi x)\sin(\pi y)$$

解 (Solution)

$$C_{mn}=4\int_0^1\int_0^1\sin(\pi x)\sin(\pi y)\sin(m\pi x)\sin(n\pi y)dxdy$$
$$=\delta_{m1}\delta_{n1}$$

$$u=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\delta_{m1}\delta_{n1}e^{-\omega_{mn}t}\sin(m\pi x)\sin(n\pi y)$$

$$u=e^{-2\pi^2t}\sin(\pi x)\sin(\pi y)$$

$$u(\frac{1}{2},\frac{1}{2},t)=e^{-2\pi^2t}$$

一维热传导方程相关 (Related 1D Heat Equation Concepts) (a) 稳态温度 (Steady-state temperature) ($t\to\infty$)

$$u(x,\infty)=C_0=\frac{1}{L}\int_0^L\phi(x)dx$$

$$u(x,0)=\phi(x)\quad u(x,\infty)=\text{常数(constant)}$$

平均温度 (Average temperature)

$$U(t)=\frac{1}{L}\int_0^Lu(x,t)dx=C_0$$

(b) 若 (If) $\phi(x)=x$:

$$C_0=\frac{L}{2}$$

$$C_n=\frac{2}{L}\int_0^Lx\cos(\frac{n\pi x}{L})dx=\frac{2L}{n^2\pi^2}[(-1)^n-1]$$

$$u(x,t)=\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}[(-1)^n-1]e^{-[(n\pi/L)c]^2t}\cos(\frac{n\pi x}{L})$$

(c) 若 (If) $\phi(x)=1+\cos(\frac{2\pi x}{L})$:

$$C_0=1$$

$$C_n=\frac{2}{L}\int_0^L\cos(\frac{2\pi x}{L})\cos(\frac{n\pi x}{L})dx=\delta_{2n}$$

$$u(x,t)=1+e^{-[(2\pi c/L)]^2t}\cos(\frac{2\pi x}{L})$$

特征函数正交性 (Orthogonality of Eigenfunctions)

设 $X_n(x)$ 和 $X_m(x)$ 是以下特征值问题的特征函数，其中 $x \in [0, L]$：

$$X''(x) + \lambda X(x) = 0$$

所以，我们有：

$$X_n''(x) + \lambda_n X_n(x) = 0$$

$$X_m''(x) + \lambda_m X_m(x) = 0$$

从微分方程的恒等式出发：

$$\frac{d}{dx}(X_m' X_n - X_n' X_m) = X_m'' X_n - X_n'' X_m$$

将特征值方程代入上式：

$$X_m'' X_n - X_n'' X_m = (-\lambda_m X_m) X_n - (-\lambda_n X_n) X_m = (\lambda_n - \lambda_m) X_n X_m$$

两边从 0 到 L 积分：

$$\int_0^L (\lambda_n - \lambda_m) X_n X_m dx = \int_0^L \frac{d}{dx}(X_m' X_n - X_n' X_m) dx$$

$$(\lambda_n - \lambda_m) \int_0^L X_n X_m dx = [X_m' X_n - X_n' X_m]_0^L$$

如果边界项 $Q = [X_m' X_n - X_n' X_m]_0^L = 0$，并且特征值不同 $(\lambda_n \neq \lambda_m)$，那么特征函数是正交的：

$$\int_0^L X_n(x) X_m(x) dx = 0 \quad (n \neq m)$$

热传导问题 1

考虑以下热传导方程、初始条件和边界条件：

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = \phi(x)$$

$$\frac{\partial u}{\partial x}(0, t) = 0$$

$$\frac{\partial u}{\partial x}(L, t) + hu(L, t) = 0$$

使用分离变量法 $u(x, t) = X(x)T(t)$，我们得到两个常微分方程：

$$X''(x) + \lambda X(x) = 0, \quad \text{with } X'(0) = 0, X'(L) + hX(L) = 0$$

$$T'(t) + \lambda a^2 T(t) = 0$$

求解特征值问题

对于 $X(x)$ 的方程:

- **情况 1:** $\lambda = 0$
  $X''(x) = 0 \Rightarrow X(x) = Ax + B$。
  $X'(0) = 0 \Rightarrow A = 0$。
  $X'(L) + hX(L) = 0 \Rightarrow 0 + hB = 0$。如果 $h \neq 0$, 则 $B = 0$ (平凡解)。如果 $h = 0$, $\lambda = 0$ 是一个特征值。

- **情况 2:** $\lambda > 0$
  设 $\lambda = \mu^2$ $(\mu > 0)$。通解为:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

应用边界条件:

$$X'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$$

$$X'(0) = B\mu = 0 \Rightarrow B = 0$$

所以 $X(x) = A\cos(\mu x)$。应用第二个边界条件:

$$X'(L) + hX(L) = -A\mu\sin(\mu L) + hA\cos(\mu L) = 0$$

假设 $A \neq 0$，我们得到特征方程:

$$\cot(\mu L) = \frac{\mu}{h}$$

令 $\alpha = \mu L$，则方程变为 $\cot(\alpha) = \frac{\alpha}{hL}$。此方程的正根 α_n (通过图解法求得) 给出特征值 $\lambda_n = \mu_n^2 = (\frac{\alpha_n}{L})^2$。

对应的特征函数为:

$$X_n(x) = \cos(\mu_n x)$$

通解和系数

$T(t)$ 的解为 $T_n(t) = C_n e^{-\lambda_n a^2 t} = C_n e^{-\mu_n^2 a^2 t}$。总解是这些解的叠加:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t}\cos(\mu_n x)$$

应用初始条件 $u(x,0) = \phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \cos(\mu_n x)$$

为了求系数 C_n，我们利用特征函数的正交性。将两边乘以 $\cos(\mu_m x)$ 并从 0 到 L 积分:

$$\int_0^L \phi(x)\cos(\mu_m x)dx = \sum_{n=1}^{\infty} C_n \int_0^L \cos(\mu_n x)\cos(\mu_m x)dx$$

正交积分的计算如下:

$$\int_0^L \cos(\mu_n x)\cos(\mu_m x)dx = \begin{cases} 0 & m \neq n \\ \int_0^L \cos^2(\mu_n x)dx & m = n \end{cases}$$

当 $m = n$ 时：

$$\int_0^L \cos^2(\mu_n x)dx = \int_0^L \frac{1+\cos(2\mu_n x)}{2}dx = \left[\frac{x}{2} + \frac{\sin(2\mu_n x)}{4\mu_n}\right]_0^L = \frac{L}{2} + \frac{\sin(2\mu_n L)}{4\mu_n}$$

因此，系数 C_n 为：

$$C_n = \frac{\int_0^L \phi(x)\cos(\mu_n x)dx}{\frac{L}{2} + \frac{\sin(2\mu_n L)}{4\mu_n}} = \frac{2}{L(1+\frac{\sin(2\mu_n L)}{2\mu_n L})}\int_0^L \phi(x)\cos(\mu_n x)dx$$

总热量

系统中的总热量 $U(t)$ 是 $u(x,t)$ 在空间域上的积分：

$$U(t) = \int_0^L u(x,t)dx = \int_0^L \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t}\cos(\mu_n x)dx$$

$$U(t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t}\int_0^L \cos(\mu_n x)dx$$

$$\int_0^L \cos(\mu_n x)dx = \left[\frac{\sin(\mu_n x)}{\mu_n}\right]_0^L = \frac{\sin(\mu_n L)}{\mu_n}$$

所以：

$$U(t) = \sum_{n=1}^{\infty} C_n \left(\frac{\sin(\mu_n L)}{\mu_n}\right) e^{-\mu_n^2 a^2 t}$$

热传导问题 2

考虑具有不同边界条件的热传导问题：

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \phi(x)$$

$$u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) + hu(L,t) = 0$$

分离变量得到与之前相同的方程，但边界条件不同：

$$X''(x) + \lambda X(x) = 0, \quad \text{with } X(0) = 0, X'(L) + hX(L) = 0$$

$$T'(t) + \lambda a^2 T(t) = 0$$

求解特征值问题

对于 $X(x)$ 的方程:

- **情况 1:** $\lambda = 0$
  $X(x) = Ax+B$。$X(0) = 0 \Rightarrow B = 0$。$X'(L)+hX(L) = 0 \Rightarrow A+h(AL) = 0 \Rightarrow A(1+hL) = 0$。通常 $1 + hL \neq 0$, 所以 $A = 0$ (平凡解)。

- **情况 2:** $\lambda > 0$
  设 $\lambda = \mu^2$ ($\mu > 0$)。通解为：

  $$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

  应用边界条件：

  $$X(0) = A = 0$$

  所以 $X(x) = B\sin(\mu x)$。应用第二个边界条件：

  $$X'(L) + hX(L) = B\mu\cos(\mu L) + hB\sin(\mu L) = 0$$

  假设 $B \neq 0$，我们得到特征方程：

  $$\tan(\mu L) = -\frac{\mu}{h}$$

  令 $\alpha = \mu L$，则方程变为 $\tan(\alpha) = -\frac{\alpha}{hL}$。此方程的正根 α_n (通过图解法求得) 给出特征值 $\lambda_n = \mu_n^2 = (\frac{\alpha_n}{L})^2$。

  对应的特征函数为：

  $$X_n(x) = \sin(\mu_n x)$$

通解和系数

$T(t)$ 的解为 $T_n(t) = C'_n e^{-\mu_n^2 a^2 t}$。总解是这些解的叠加（令 $C_n = B_n C'_n$）：

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t} \sin(\mu_n x)$$

应用初始条件 $u(x,0) = \phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin(\mu_n x)$$

为了求系数 C_n，我们利用正交性。对于这组边界条件，我们首先验证边界项 Q 为零：

$$Q = [X'_m X_n - X'_n X_m]_0^L$$

在 $x = L$ 处: $X'_m = -hX_m$ and $X'_n = -hX_n$。

$$X'_m(L)X_n(L) - X'_n(L)X_m(L) = (-hX_m(L))X_n(L) - (-hX_n(L))X_m(L) = 0$$

在 $x = 0$ 处: $X_n(0) = 0$ and $X_m(0) = 0$, 所以项为零。因此 $Q = 0$，特征函数是正交的。正交积分的计算如下：

$$\int_0^L \sin(\mu_n x)\sin(\mu_m x)dx = \begin{cases} 0 & m \neq n \\ \int_0^L \sin^2(\mu_n x)dx & m = n \end{cases}$$

当 $m = n$ 时：

$$\int_0^L \sin^2(\mu_n x)dx = \int_0^L \frac{1-\cos(2\mu_n x)}{2}dx = \left[\frac{x}{2} - \frac{\sin(2\mu_n x)}{4\mu_n}\right]_0^L = \frac{L}{2} - \frac{\sin(2\mu_n L)}{4\mu_n}$$

因此，系数 C_n 为：

$$C_n = \frac{\int_0^L \phi(x)\sin(\mu_n x)dx}{\frac{L}{2} - \frac{\sin(2\mu_n L)}{4\mu_n}}$$

利用 $\tan(\mu_n L) = -\mu_n/h$, 可以进一步化简分母。$\sin(2\mu_n L) = 2\sin(\mu_n L)\cos(\mu_n L)$。

总热量

系统中的总热量 $U(t)$:

$$U(t) = \int_0^L u(x,t)dx = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t} \int_0^L \sin(\mu_n x)dx$$

$$\int_0^L \sin(\mu_n x)dx = \left[-\frac{\cos(\mu_n x)}{\mu_n}\right]_0^L = \frac{1-\cos(\mu_n L)}{\mu_n}$$

所以：

$$U(t) = \sum_{n=1}^{\infty} C_n \left(\frac{1-\cos(\mu_n L)}{\mu_n}\right) e^{-\mu_n^2 a^2 t}$$