III The Central Force Problem

3.1 Reduction to the equivalent one-body problem

Consider a monogenic system of two mass points m_1 and m_2.

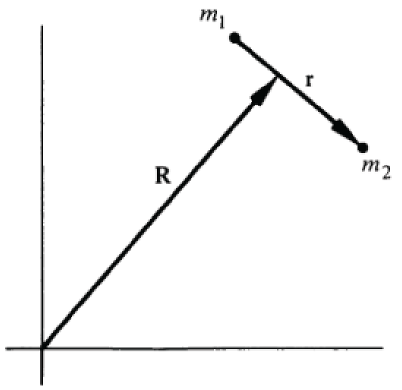

FIGURE 3.1 Coordinates for the two-body problem.

Figure 1:

Forces are all due to an interaction potential U depending on $\vec{r} = \vec{r}_1 - \vec{r}_2$, which is the vector between the two particles.

We choose $\vec{R}$ as the radius vector of the center of mass. The system has 6 degrees of freedom.

The Lagrangian is $L = T(\vec{R}, \vec{r}) - U(\vec{r}_1 - \vec{r}_2)$.

Where $T = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + T'$ with $T' = \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2$ which is the kinetic energy of motion about the center of mass.

Here $\vec{r}_1'$ and $\vec{r}_2'$ are the radii vectors of two particles relative to the center of mass and are related to $\vec{r}$ by

$$\vec{r}_1' = -\frac{m_2}{m_1 + m_2}\vec{r} \quad , \quad \vec{r}_2' = \frac{m_1}{m_1 + m_2}\vec{r}$$

Anyway, we obtain

$$L = \frac{m_1 + m_2}{2}\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\vec{r}}|^2 - U(\vec{r}_1, \vec{r}_2, \dots)$$

We define the reduced mass as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

3.2 The equations of motion and first integrals

(1) We now restrict ourselves to conservative central forces and problems that are spherically symmetric.

The total angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ is conserved, thus $\vec{r}$ is always perpendicular to the fixed position of $\vec{L}$ in space.

We now express in plane polar coordinates; the Lagrangian

$$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

where θ is a cyclic coordinate, whose corresponding canonical momentum is the angular momentum of the system

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\Rightarrow \dot{p}_\theta = \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

with the immediate integral

$$mr^2\dot{\theta} = l \quad \text{as a constant magnitude of the angular momentum.}$$

Also, we obtain $\frac{d}{dt}(\frac{1}{2}r^2\dot{\theta}) = 0$

$$(\text{Tip:}\frac{dp_\theta}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} = 0 \text{ for a cyclic } \theta)$$

where $\frac{1}{2}r^2\dot{\theta}$ is the areal velocity.

The differential area swept out in time dt being

$$dA = \frac{1}{2}r(rd\theta)$$

Hence, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$

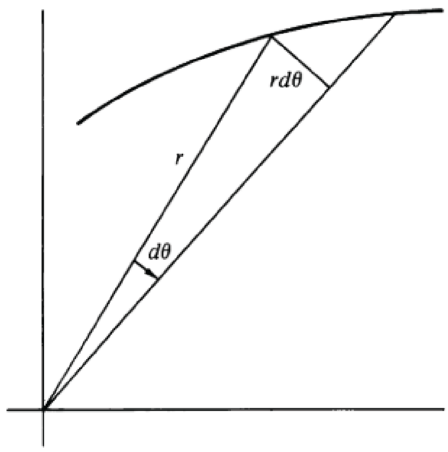

FIGURE 3.2 The area swept out by the radius vector in a time dt.

Figure 2:

Here we have the proof of the Kepler's second law of planetary motion.

(2) For the coordinate r, the Lagrange equation is

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

where we designate the force along $\vec{r}$ as $-\frac{\partial V}{\partial r}$. So $m\ddot{r} - mr\dot{\theta}^2 = f(r)$.

Using $mr^2\dot{\theta} = l$ to eliminate $\dot{\theta}$, we have

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r)$$

We consider the conservation of energy

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{l^2}{2mr^2}\right)$$

and $m\ddot{r} = \frac{d}{dt}(\frac{1}{2}m\dot{r}^2)$.

Using $\frac{d}{dt}g(r) = \frac{dg}{dr}\dot{r}$, we have:

$$\frac{d}{dt}(\frac{1}{2}m\dot{r}^2) = -\frac{d}{dr}\left(V + \frac{l^2}{2mr^2}\right)\dot{r}$$

or

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V\right) = 0$$

which is equivalent to $E =$ constant.

(Tip: $\frac{l^2}{2mr^2} = \frac{1}{2m}m^2r^4\frac{\dot{\theta}^2}{r^2} = \frac{mr^2\dot{\theta}^2}{2}$)

Solving for $\dot{r}$, we have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)} \quad \text{or} \quad dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

At time $t = 0$, let r have the initial value r_0.

$$\Rightarrow t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

3.3 The equivalent one-dimensional problem and classification of orbits

(1) The magnitude v follows the conservation of energy

$$E = \frac{1}{2}mv^2 + V(r)$$

$$\Rightarrow v = \sqrt{\frac{2}{m}(E - V(r))}$$

And the particle is subject to a force

$$f' = f + \frac{l^2}{mr^3}$$

where the second term is the centrifugal force. The particle also has the energy which is fictitious

$$V' = V + \frac{l^2}{2mr^2}$$

Note that:

$$f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{l^2}{mr^3}$$

Thus the energy conservation theorem

$$E = V' + \frac{1}{2}m\dot{r}^2$$

(2) We consider an example of an attractive inverse-square law of force.

$$f = -\frac{k}{r^2} \Rightarrow V = -\frac{k}{r}$$

$$V' = -\frac{k}{r} + \frac{l^2}{2mr^2}$$

(3) Also, for $mr^2\dot{\theta} = l$.

$$\Rightarrow d\theta = \frac{ldt}{mr^2}$$

$$\Rightarrow \theta = l\int \frac{dt}{mr^2(t)} + \theta_0$$

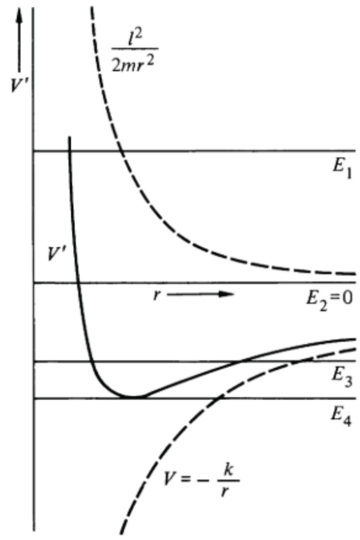

FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

Figure 3:

3.4 The Virial Theorem

(1) Consider a system of mass points with position vectors $\vec{r}_i$ and applied forces $\vec{F}_i$. The equations of motion: $\dot{\vec{p}_i} = \vec{F}_i$. The quantity:

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dG}{dt} = \sum_i \dot{\vec{p}_i} \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \dot{\vec{r}_i}$$

where the second term:

$$\sum_i \vec{p}_i \cdot \dot{\vec{r}_i} = \sum_i m_i \dot{\vec{r}_i} \cdot \dot{\vec{r}_i} = \sum_i m_i v_i^2 = 2T$$

the first term

$$\sum_i \dot{\vec{p}_i} \cdot \vec{r}_i = \sum_i \vec{F}_i \cdot \vec{r}_i$$

Thus:

$$\frac{d}{dt} \sum_i \vec{p}_i \cdot \vec{r}_i = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i$$

The time average over τ:

$$\frac{1}{\tau} \int_0^\tau \frac{dG}{dt} dt = \frac{d\bar{G}}{dt} = 2\bar{T} + \sum_i \overline{\vec{F}_i \cdot \vec{r}_i} = \frac{1}{\tau}[G(\tau) - G(0)]$$

Whether the motion is periodic, for τ sufficiently long, $G(\tau) - G(0) = 0$.

$$\Rightarrow \bar{T} = -\frac{1}{2} \sum_i \overline{\vec{F}_i \cdot \vec{r}_i} \text{which is the virial theorem}$$

(2) We consider a gas consisting of N atoms confined within a container of volume V. By the equipartition theorem of kinetic energy, the average kinetic energy is given by $\frac{3}{2}k_B T$, where k_B is the Boltzmann constant. Thus:

$$\bar{T} = \frac{3}{2} N k_B T$$

On the other hand, $\vec{F}_i$ include both the forces of interaction and the forces of constraint on the system. As for perfect gas, we only consider the forces due to the collisions with the walls.

$$d\vec{F}_i = -p\hat{n} dA$$

where P is the pressure. or

$$\frac{1}{2}\sum_i \overline{\vec{F}_i \cdot \vec{r}_i} = -\frac{1}{2}\int p\hat{n}\cdot\vec{r}dA$$

By Gauss's theorem

$$\int \vec{r}\cdot\hat{n}dA = \int \nabla\cdot\vec{r}dV = 3V$$

Hence the virial theorem for perfect gas is

$$\frac{3}{2}Nk_BT = \frac{3}{2}PV\text{: which is the ideal gas law}$$

(3) For conservative forces

$$\bar{T} = \frac{1}{2}\sum_i \overline{\nabla V\cdot\vec{r}_i}$$

For a single particle

$$\bar{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r}r}$$

If $V = ar^{n+1}$

$$\Rightarrow \frac{\partial V}{\partial r}r = (n+1)V$$

Then, we obtain

$$\bar{T} = \frac{n+1}{2}\bar{V}$$

One special example is when n=-2

$$\bar{T} = -\frac{1}{2}\bar{V}$$

3.5 The Differential Equation for the Orbit, and Integrable Power-Law Potentials.

(1) As stated before: $ldt = mr^2d\theta$.

$$\Rightarrow \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

which can be used to convert the equation of motion into a differential equation for the orbit.

$$\Rightarrow \frac{l}{r}\frac{d}{d\theta}(\frac{l}{mr^2}\frac{dr}{d\theta}) - \frac{l^2}{mr^3} = f(r)$$

substitute $u = \frac{1}{r}$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}V(\frac{1}{u})$$

(Tip: $\frac{d}{du} = \frac{1}{u^2}\frac{d}{dr}$) The equation is such that the orbit is symmetric about two adjacent turning points. Further: $u = u(\theta)$, $(\frac{du}{d\theta})_0 = 0$ for $\theta = 0$ will be unaffected whether we use θ or $-\theta$. $\Rightarrow$ The orbit is invariant under reflection about the apsidal vectors.

(2)

$$d\theta = \frac{ldr}{mr^2\sqrt{\frac{2}{m}(E-V) - \frac{l^2}{2mr^2}}}$$

$$\theta = \int_r^0 \frac{ldr}{r^2\sqrt{2mE - 2mV - \frac{l^2}{r^2}}} + \theta_0$$

$u = \frac{1}{r}$

$$\theta = \theta_0 - \int_{u_0}^{u}\frac{ldu}{\sqrt{2mE - 2mV - l^2u^2}}$$

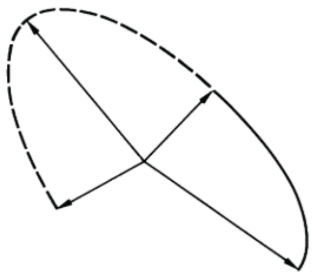

FIGURE 3.12 Extension of the orbit by reflection of a portion about the apsidal vectors.

Figure 4:

Assume that $V = \alpha r^{n+1}$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2m\alpha}{l^2} u^{-(n+1)} - u^2}}$$

i) Specially, for n=1 (linear force),

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2m\alpha}{l^2} u^{-2} - u^2}}$$

Substitute $u^2 = x$, $du = \frac{dx}{2\sqrt{x}}$

$$\theta = \theta_0 - \frac{1}{2} \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2mE}{l^2} x - \frac{2m\alpha}{l^2} - x^2}}$$

The same for $n = 1, -2, -3$. 2) For $n = 5, 3, 0, -4, -5, -7$ we have the elliptic functions.

3.6 Conditions for Closed Orbits (Bertrand's Theorem)

For closed orbits, the equivalent potential $V'(r)$ must have a minimum or maximum at the orbit radius r_0, with the total energy being $E = V'(r_0)$. For a circular orbit, this requires that the force is attractive and satisfies:

$$f(r_0) = -\frac{L^2}{mr_0^3}$$

The total energy for this orbit is:

$$E = V(r_0) + \frac{L^2}{2mr_0^2} \quad \text{(which ensures that } \dot{r} = 0)$$

To make the orbit **stable**, the second derivative of $V'(r)$ at r_0 must be positive (i.e., the potential is concave up).

$$\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0} = -\left.\frac{\partial f}{\partial r}\right|_{r=r_0} + \frac{3L^2}{mr_0^4} > 0$$

This leads to the stability conditions:

$$\left.\frac{\partial f}{\partial r}\right|_{r=r_0} < -\frac{3f(r_0)}{r_0}$$

$$\text{or} \quad \left.\frac{d\ln f}{d\ln r}\right|_{r=r_0} > -3$$

If we consider a power-law force $f = -kr^n$ (with $k > 0$), the stability condition becomes:

$$-knr^{n-1} < 3kr^{n-1} \quad \Rightarrow \quad n > -3$$

This means that any power-law attractive potential that varies more slowly than $1/r^2$ can produce stable circular orbits.

For an orbit to be closed, small perturbations must also result in a closed path. This restricts the possible force laws. For a perturbed orbit, the condition for closure is related to β^2, where:

$$\beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\bigg|_{r=r_0}$$

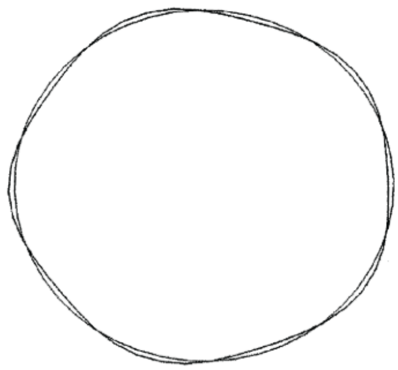

FIGURE 3.13 Orbit for motion in a central force deviating slightly from a circular orbit for $\beta = 5$.

Figure 5:

Bertrand's Theorem shows that only two force laws produce closed orbits for all bounded energies:

- $\beta^2 = 1 \Rightarrow$ The **inverse-square law** ($f \propto 1/r^2$)
- $\beta^2 = 4 \Rightarrow$ **Hooke's law** ($f \propto r$)

3.7 The Kepler Problem: Inverse-Square Law of Force

Derivation of the Orbit Equation

For an inverse-square law of force:

$$f = -\frac{k}{r^2}, \quad V = -\frac{k}{r}$$

The integral for the orbit's angle θ, with $u = 1/r$, is:

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{L^2} + \frac{2mk}{L^2}u - u^2}}$$

Using the standard integral form $\int \frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos\left(-\frac{\beta+2\gamma x}{\sqrt{\beta^2-4\alpha\gamma}}\right)$, we identify:

$$\alpha = \frac{2mE}{L^2}, \quad \beta = \frac{2mk}{L^2}, \quad \gamma = -1$$

Solving the integral and rearranging gives the equation of the orbit:

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{mk^2}}\cos(\theta - \theta')\right)$$

Classification of Orbits

The general equation for a conic section is $\frac{1}{r} = C[1 + e\cos(\theta - \theta')]$, where e is the **eccentricity**. By comparing this to the orbit equation, we find the eccentricity is:

$$e = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

The shape of the orbit is determined by the energy E and eccentricity e:

- $E > 0 \implies e > 1$: **Hyperbola**
- $E = 0 \implies e = 1$: **Parabola**
- $E < 0 \implies e < 1$: **Ellipse**
- $E = -\frac{mk^2}{2L^2} \implies e = 0$: **Circle**

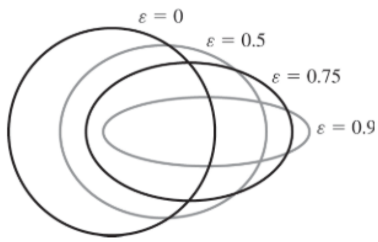

FIGURE 3.14 Ellipses with the same major axes and eccentricities from 0.0 to 0.9.

Figure 6:

Properties of Orbits

Virial Theorem: For an inverse-square force, the virial theorem relates the total energy E to the potential energy V for a stable orbit. For a circular orbit of radius r_0:

$$E = \frac{V}{2} \quad \Rightarrow \quad E = -\frac{k}{2r_0}$$

Circular Motion: The condition for a circular orbit is found by equating the central force to the required centripetal force, which gives the radius:

$$\frac{k}{r_0^2} = \frac{L^2}{mr_0^3} \quad \Rightarrow \quad r_0 = \frac{L^2}{mk}$$

This leads to the specific energy required for a circular orbit:

$$E = -\frac{mk^2}{2L^2}$$

Apsidal Distances: The minimum and maximum radial distances (r_1 and r_2) are the points where the radial velocity is zero. They are the roots of the radial energy equation:

$$E - \frac{L^2}{2mr^2} + \frac{k}{r} = 0$$

Elliptical Orbits: The semi-major axis a and eccentricity e can be expressed as:

$$a = \frac{k}{2|E|} \quad \text{and} \quad e = \sqrt{1 - \frac{L^2}{mka}}$$

The orbit equation in terms of a and e is:

$$r = \frac{a(1-e^2)}{1+e\cos(\theta)}$$

Velocity Vector: The velocity vector $\vec{v}_{||}$ of the particle along its elliptical path has radial and tangential components:

$$\vec{v}_{||} = v_r\hat{r} + v_\theta\hat{\theta}$$

where $v_r = |\dot{r}| = \frac{p_r}{m}$ and $v_\theta = r\dot{\theta} = \frac{L}{mr}$.

3.8 The Motion in Time in the Kepler Problem

1. For an inverse-square force, the time taken to travel from the perihelion distance r_0 to a distance r is given by the integral:

$$t = \sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{r\,dr}{\sqrt{Er^2 + kr - \frac{L^2}{2m}}}$$

   In terms of the parameters a, e, k:

$$t = \sqrt{\frac{m}{2k}}\int_{r_0}^{r} \frac{r\,dr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-e^2)}{2}}}$$

   where r_0 is the perihelion distance.

   Similarly, using the conservation of angular momentum, $dt = \frac{mr^2}{L}d\theta$, which leads to:

$$t = \frac{L^3}{mk^2}\int_0^{\theta} \frac{d\theta}{[1+e\cos\theta]^2}$$

2. We now consider the parabolic motion ($e = 1$). By using the identity $1+\cos\theta = 2\cos^2\frac{\theta}{2}$ and setting θ' to zero, the time integral becomes:

$$t = \frac{L^3}{4mk^2}\int_0^{\theta} \sec^4\frac{\theta}{2}\,d\theta$$

   By a change of variable to $x = \tan\frac{\theta}{2}$, leading to $dx = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$ and $1+x^2 = \sec^2\frac{\theta}{2}$:

$$t = \frac{L^3}{2mk^2}\int_0^{\tan\frac{\theta}{2}} (1+x^2)\,dx = \frac{L^3}{2mk^2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

   In this equation, $-\pi < \theta < \pi$. For $t \to \infty$, the particle approaches infinity at $\theta = \pm\pi$, and $t = 0$ corresponds to $\theta = 0$, where the particle is at perihelion.

3. We introduce an eccentric anomaly ψ, defined by:

$$r = a(1 - e\cos\psi), \quad 0 < \psi < 2\pi$$

   In terms of ψ, the time integral is:

$$t = \sqrt{\frac{ma^2}{k}}\int_0^{\psi} (1 - e\cos\psi)\,d\psi$$

   If this is carried out over the full range of ψ of 2π, the period is $T = 2\pi a\sqrt{\frac{ma}{k}}$.

   In another fashion, from the conservation of angular momentum, using areal velocity:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m}$$

Integrating over a full period gives $\int_0^T dA = A = \frac{LT}{2m}$, where $A = \pi ab$ is the area of the ellipse. The semi-minor axis is $b = a\sqrt{1-e^2} = a\sqrt{\frac{L^2}{mak}}$. The period is therefore:

$$T = \frac{2m}{L}\pi a^2 \sqrt{\frac{L^2}{mak}} = 2\pi a^{3/2}\sqrt{\frac{m}{k}}$$

The square of the period is proportional to the cube of the major axis $\implies$ Kepler's Third Law.

4. For a two-body system, we have the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$. Thus, the gravitational attractive force is $F = -G\frac{m_1 m_2}{r^2}$ where $k = Gm_1m_2$ is a constant. Under these conditions, the period is:

$$T = \frac{2\pi a^{3/2}}{\sqrt{G(m_1+m_2)}} \approx \frac{2\pi a^{3/2}}{\sqrt{Gm_s}}$$

where we neglect the mass of the planet.

3.9 The Laplace-Runge-Lenz Vector

1. By Newton's second law: $\dot{\vec{p}} = f(r)\frac{\vec{r}}{r}$.

$$\dot{\vec{p}} \times \vec{L} = \frac{m}{r} f(r)[\vec{r} \times (\vec{r} \times \dot{\vec{r}})] = \frac{m}{r} f(r)[(\vec{r}\cdot\dot{\vec{r}})\vec{r} - r^2\dot{\vec{r}}]$$

Note that $r\dot{r} = \vec{r}\cdot\dot{\vec{r}}$ and $\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\dot{r}}{r^2}$.

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = -mf(r)r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

For an inverse-square force law, $f = -k/r^2$, $V = -k/r$.

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = \frac{d}{dt}\left(mk\frac{\vec{r}}{r}\right)$$

We define a conserved vector, which is called the Runge-Lenz vector:

$$\vec{A} = \vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}$$

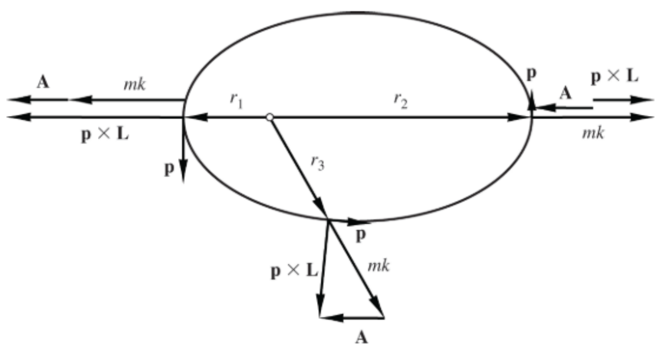

FIGURE 3.18 The vectors **p**, **L**, and **A** at three positions in a Keplerian orbit. At perihelion (extreme left) $|\mathbf{p}\times\mathbf{L}| = mk(1+e)$ and at aphelion (extreme right) $|\mathbf{p}\times\mathbf{L}| = mk(1-e)$. The vector **A** always points in the same direction with a magnitude mke.

Figure 7:

We can easily see that $\vec{A}\cdot\vec{L} = 0$ since $\vec{L}$ is perpendicular to $\vec{p}\times\vec{L}$ and $\vec{p}\times\vec{L}$ is perpendicular to $\vec{L}$, and $\vec{A}$ belongs to the plane of the orbit.

Now consider the dot product of $\vec{A}$ with $\vec{r}$:

$$\vec{A}\cdot\vec{r} = Ar\cos\theta = \vec{r}\cdot(\vec{p}\times\vec{L}) - mkr$$

Using the vector identity $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$, we have $\vec{r}\cdot(\vec{p}\times\vec{L}) = (\vec{r}\times\vec{p})\cdot\vec{L} = \vec{L}\cdot\vec{L} = L^2$. So, $Ar\cos\theta = L^2 - mkr$. Rearranging this gives the orbit equation:

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 + \frac{A}{mk}\cos\theta\right)$$

Compared with the standard conic section equation $\frac{1}{r} = C(1+\epsilon\cos(\theta-\theta'))$ and $\epsilon = \sqrt{1+\frac{2EL^2}{mk^2}}$, we have:

$$A = mk\epsilon \quad \text{or} \quad A^2 = m^2k^2\epsilon^2 = m^2k^2 + 2mEL^2$$

3.10 Scattering in a Central Force Field

1. The incident beam is characterized by specifying its intensity/flux density I, which is the number of particles crossing a unit area normal to the beam in unit time. The cross-section for scattering in a given direction is defined by:

$$\sigma(\Omega)d\Omega = \frac{\text{number of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity}}$$

By symmetry, $d\Omega = 2\pi\sin\Theta d\Theta$, where Θ is the angle between the scattered and incident directions.

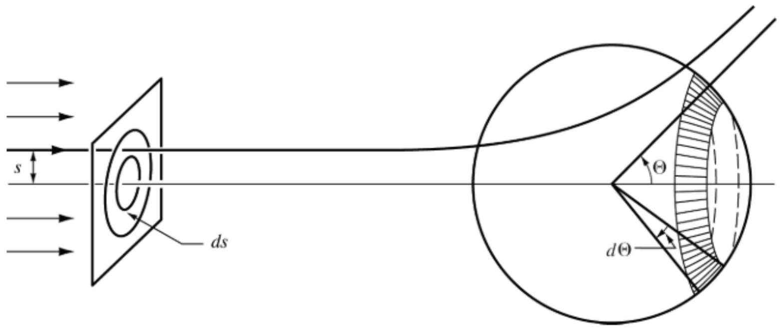

FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

Figure 8:

We have $L = mvs = s\sqrt{2mE}$, where s is the perpendicular distance between the center of force and the incident velocity.

Between s and $s+ds$, the number of particles scattered into $d\Omega$ is equal to the number of incident particles lying between s and $s+ds$.

$$2\pi I|s\,ds| = 2\pi\sigma(\Theta)I\sin\Theta|d\Theta|$$

We consider a function $s = s(\Theta, E)$.

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \quad \text{and} \quad \Theta = |\pi - 2\Psi|$$

where Ψ is the angle between the direction of the incoming asymptote and the periapsis direction. By setting $r = r_m$ when $\theta = \Psi$ and $\theta = \pi$ when $r\to\infty$:

$$\Psi = \int_{r_m}^{\infty}\frac{dr}{r^2\sqrt{\frac{2mE}{L^2}+\frac{2mV(r)}{L^2}-\frac{1}{r^2}}} \quad \text{and} \quad L = s\sqrt{2mE}$$

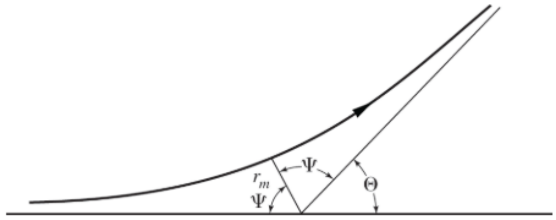

FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

Figure 9:

This implies:

$$\Psi(\Theta) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r^2\sqrt{1 - \frac{V(r)}{E} - \frac{s^2}{r^2}}}$$

Changing r to $u = s/r$:

$$\Psi(s) = \pi - 2\int_{0}^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(s/u)}{E} - u^2}}$$

2. The scattering force field is produced by $-Ze$ acting on the incident particles charged $-Z'e$. Thus, the force is $f = \frac{ZZ'e^2}{r^2}$. The orbit is a hyperbola with the eccentricity:

$$\epsilon = \sqrt{1 + \frac{2EL^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2Es}{ZZ'e^2}\right)^2}$$

We choose θ' to be π and periapsis corresponds to $\theta = 0$ and the orbit equation becomes:

$$\frac{1}{r} = \frac{mZZ'e^2}{L^2}(E\cos\theta - 1) \quad \text{(Mistake in notes, should be } \epsilon\cos\theta - 1\text{)}$$

Let's correct it to: $\frac{1}{r} = \frac{mZZ'e^2}{L^2}(\epsilon\cos\theta - 1)$. As $r \to \infty$, $\epsilon\cos\Psi - 1 = 0$. We have $\cos\Psi = \frac{1}{\epsilon}$. Hence $\cot^2\Psi = \epsilon^2 - 1$.

$$\cot^2\frac{\Theta}{2} = \cot^2\Psi = \epsilon^2 - 1 = \left(\frac{2Es}{ZZ'e^2}\right)^2 \quad \text{and} \quad \cot\frac{\Theta}{2} = \frac{2Es}{ZZ'e^2}$$

We thus obtain $s = \frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}$ and

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \csc^4\frac{\Theta}{2}$$

which gives the Rutherford formula for the scattering of α particles by atomic nuclei.

3. The total scattering cross-section σ_T is defined by:

$$\sigma_T = \int \sigma(\Omega)d\Omega = 2\pi\int_0^{\pi} \sigma(\Theta)\sin\Theta\,d\Theta$$

(etc. page 110)