XI Special Theory of Relativity

11.1 Einstein's Two Postulates

We consider two reference frames k and k' with (x, y, z, t) and (x', y', z', t').

By Galilean relativity, we have

$$\vec{x}' = \vec{x} - \vec{v}t$$
$$t' = t$$

For example, we consider a group of particles interacting via two-body central potentials. The equation of motion of the i-th particle in k'

$$m_i \frac{d\vec{v}_i}{dt} = -\nabla'_i \sum_j V_{ij}(|\vec{x}'_i - \vec{x}'_j|) = -\nabla_i \sum_j V_{ij}(|\vec{x}_i - \vec{x}_j|)$$

We assume that a field $\psi(x', t')$ satisfies

$$\left[\sum_i \left(\frac{\partial^2}{\partial {x'_i}^2}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi = 0$$

in the frame k'.

In terms of the coordinates in the frame k, it becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\vec{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}(\vec{v}\cdot\nabla)(\vec{v}\cdot\nabla)\right)\psi = 0$$

$\Longrightarrow$ The wave equation is not invariant under Galilean transformations. $\Longrightarrow$ Another preferred reference frame where the luminiferous either was at rest. $\Longrightarrow$ Lorentz transformation

$$L = L_0\sqrt{1 - v^2/c^2}$$

Einstein's special theory of relativity is based on two postulates.

1. **Postulate of relativity**
   There exists a triply infinite set of equivalent Euclidean reference frames moving with constant velocities in rectilinear paths relative to one another in which all physical phenomena occur in an identical manner. $\Longrightarrow$ inertial reference.

2. **Postulate of the constancy of the speed of light**
   The speed of light is infinite and independent of the motion of its source.

11.2 Some Experiments

A: Ether - Drift based on the Mössbauer effect

For a plane electromagnetic wave in vacuum, its phase as observed in the inertial frames k and k' are connected by Galilean transformation.

$$\phi = \omega\left(t - \frac{\vec{n}\cdot\vec{x}}{c}\right) = \omega'\left(t' - \frac{\vec{n}'\cdot\vec{x}'}{c'}\right)$$

$$= \omega'\left[t'\left(1 - \frac{\vec{n}'\cdot\vec{v}}{c'}\right) - \frac{\vec{n}'\cdot\vec{x}'}{c'}\right] \quad \text{for all } t' \text{ and } \vec{x}'$$

We find

$$\vec{n} = \vec{n}'$$
$$\frac{\omega}{c} = \frac{\omega'}{c'}(1 - \frac{\vec{n}'\cdot\vec{v}}{c'})$$
$$c' = c - \vec{n}\cdot\vec{v}$$

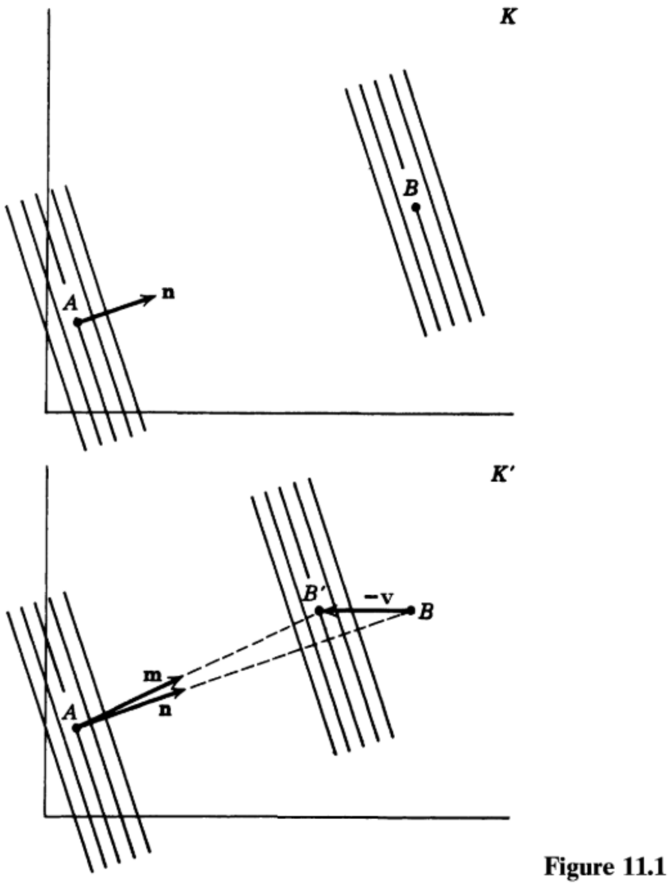

Figure 1:

In k', the direction of motion of the wave packet is the direction of energy flow, along a unit vector

$$\vec{m} = \frac{c\vec{n}' - \vec{v}}{|c\vec{n}' - \vec{v}|} \approx (1 - \frac{\vec{n} \cdot \vec{v}}{c})\vec{m} + \frac{\vec{v}}{c}$$

where $\vec{v}_0$ is the velocity of the laboratory relative to the either rest frame.

We now consider a phase wave whose frequency is ω in the ether rest from, ω_0 in the laboratory and ω_1 in an inertial frame k, moving with $\vec{v} = \vec{v}_1 - \vec{v}_0$ relative to the either rest from. Thus

$$\omega_1 = \omega \left(1 - \frac{\vec{n} \cdot \vec{v}_1}{c}\right)$$
$$\omega_0 = \omega \left(1 - \frac{\vec{n} \cdot \vec{v}_0}{c}\right)$$

If ω_1 is expressed in terms of ω_0 and eliminate $\vec{n}$

$$\omega_1 \approx \omega_0 \left[1 - \frac{\vec{u}_1 \cdot (\vec{m} + \vec{v}_0')}{c}\right]$$

where $\vec{u}_1$ is the velocity of k relative to the laboratory, $\vec{m}$ is the direction of energy propagation in the laboratory, ω_0 is the frequency of the wave in the laboratory, $\vec{v}_0'$ is velocity of the laboratory relative to the ether.

Consider two Mössbauer systems, with $\vec{u}_1$ and $\vec{u}_2$. The difference is:

$$\frac{\omega_1 - \omega_2}{\omega_0} = \frac{1}{c}(\vec{u}_2 - \vec{u}_1) \cdot (\vec{m} + \frac{\vec{v}_0}{c})$$

[If the emitter and absorber are located on the opposite ends of a rod of length $2R$ that is rotated about its center with angular velocity Ω

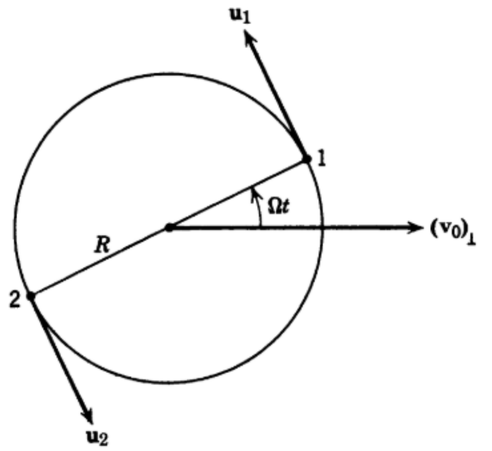

Figure 11.2

Figure 2:

Then $(\vec{u}_2 - \vec{u}_1) \cdot \vec{m} = 0$.

$$\frac{\omega_1 - \omega_2}{\omega_0} = \frac{2\Omega R}{c^2} \sin \Omega t |(\vec{v}_0)_\perp|$$

where $|(\vec{v}_0)_\perp|$ is the component of $\vec{v}_0$ perpendicular to the axis of rotation.

B: Speed of Light from a Moving Source

C: Frequency Dependence of the Speed of Light in Vacuum

11.3 Lorentz Transformations and Basic Kinematic Results of Special Relativity

A: Simple Lorentz Transformations of Coordinates

We consider k and k' with a relative $\vec{v}$ and the coordinates (t, x, y, z) and (t', x', y', z'). Let the origins of the coordinates in k and k' be coincident at $t = t' = 0$.

The wave front reaches a point (x, y, z) in k at t.

$$c^2t^2 - (x^2 + y^2 + z^2) = 0$$

Similarly in k',

$$c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0$$

We assume that the space-time is homogeneous and isotropic, the equations are related by

$$c^2t^2 - (x^2 + y^2 + z^2) = \lambda^2[c^2t'^2 - (x'^2 + y'^2 + z'^2)]$$

where $\lambda = \lambda(|\vec{v}|)$ is a possible change of scale. We can see that $\lambda(v) = 1$ and the lorentz transformation between k' and k.

$$\begin{aligned} x'_0 &= \gamma(x_0 - \beta x_1) \\ x'_1 &= \gamma(x_1 - \beta x_0) \\ x'_2 &= x_2 \\ x'_3 &= x_3 \end{aligned}$$

where we introduce $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$. and $\vec{\beta} = \frac{\vec{v}}{c}$, $\beta = |\vec{\beta}|$, $\gamma = (1 - \beta^2)^{-1/2} \implies$

$$\begin{aligned} x_0 &= \gamma(x'_0 + \beta x'_1) \\ x_1 &= \gamma(x'_1 + \beta x'_0) \\ x_2 &= x'_2 \\ x_3 &= x'_3 \end{aligned}$$

Lorentz Transformation for an Arbitrary Direction

If $\vec{v}$ is in an arbitrary direction, we generalize that

$$\begin{cases} x_0' = \gamma(x_0 - \vec{\beta} \cdot \vec{x}) \\ \vec{x}' = \vec{x} + (\frac{\gamma-1}{\beta^2})(\vec{\beta} \cdot \vec{x})\vec{\beta} - \gamma\vec{\beta}x_0 \end{cases}$$

Since $0 \le \beta \le 1$, $1 \le \gamma \le \infty$, we replace with

$$\begin{cases} \beta = \tanh\zeta \\ \gamma = \cosh\zeta \\ \gamma\beta = \sinh\zeta \end{cases} \quad \text{where } \zeta \text{ is the boost parameter}$$

$$\Longrightarrow \; \begin{aligned} x_0' &= x_0\cosh\zeta - x_i\sinh\zeta \\ x_i' &= -x_0\sinh\zeta + x_i\cosh\zeta \end{aligned}$$

B: 4-vectors

For (A_0, A_1, A_2, A_3), the Lorentz transformation is

$$\begin{cases} A_0' = \gamma(A_0 - \vec{\beta} \cdot \vec{A}) \\ A_\parallel' = \gamma(A_\parallel - \beta A_0) \\ A_\perp' = A_\perp \end{cases}$$

where the relative velocity $\vec{v} = c\vec{\beta}$.

The invariance is

$$A_0^2 - |\vec{A}|^2 = A_0'^2 - |\vec{A}'|^2$$

We can also prove that for $\begin{cases} (A_0, A_1, A_2, A_3) \\ (B_0, B_1, B_2, B_3) \end{cases}$

$$A_0B_0 - \vec{A} \cdot \vec{B} = A_0'B_0' - \vec{A}' \cdot \vec{B}'$$

C: Light Cone, Proper Time and Time Dilation

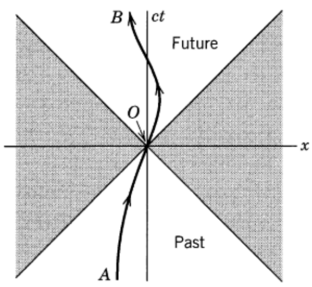

Figure 11.3 World line of a system and the light cone. The unshaded interior of the cone represents the past and the future, while the shaded region outside the cone is called "elsewhere." A point inside (outside) the light cone is said to have a timelike (spacelike) separation from the origin.

Figure 3:

(only in one dimension) We divide the space-time into three regions by a "cone" (light cone) whose surface is specified by $x^2 + y^2 + z^2 = c^2t^2$. As time goes on, it would trace out a path called world line. When $t > 0$, the path of the system lies inside the upper half-cone $\Longrightarrow$ future. Similarly, the lower half-cone $\Longrightarrow$ past. And the shaded region outside the light cone is the elsewhere.

We consider two events with the interval S_{12}

$$P_1(t_1, \vec{x}_1), \quad P_2(t_2, \vec{x}_2)$$

$$S_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$$

There are three possibilities:

$$\begin{cases} 1) \quad S_{12}^2 > 0 & \text{: time like separation} \\ 2) \quad S_{12}^2 < 0 & \text{: spacelike separation} \\ 3) \quad S_{12}^2 = 0 & \text{: lightlike separation} \end{cases}$$

1. We can find a Lorentz transformation such that $\vec{x}_1' = \vec{x}_2'$, then $S_{12}^2 = c^2(t_1' - t_2')^2 > 0$.
2. We can find an inertial frame k'' where $t_1'' = t_2''$, then $S_{12}^2 = -|\vec{x}_1'' - \vec{x}_2''|^2 < 0$.
3. We can only connect two events by light signals.

We consider a system of a particle moving with an instantaneous velocity $\vec{u}(t)$ relative to k. In a time interval dt, the position change is $d\vec{x} = \vec{u}dt$. Thus,

$$ds^2 = c^2dt^2 - |d\vec{x}|^2 = c^2dt^2(1-\beta^2) \quad \text{where } \beta = u/c$$

In k', the space-time increments are $dt' = d\tau$, $d\vec{x}' = 0$ since the system is at rest. Thus the invariant interval is $ds = cd\tau$ and $d\tau$ is a Lorentz invariant quantity that takes the form

$$d\tau = dt\sqrt{1-\beta^2(t)} = \frac{dt}{\gamma(t)}$$

The time τ is the proper time, and $\tau_2 - \tau_1$ is given by

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} \frac{dt}{\gamma(t)} = \int_{t_1}^{t_2} \sqrt{1-\beta^2(t)}dt$$

$$\Longrightarrow \text{ Time dilation}$$

D: Relativistic Doppler Shift

We consider a plane wave with frequency ω and wave vector $\vec{k}$ in k and ω', $\vec{k}'$ in k'. The phase of the wave is an invariant.

$$\phi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'$$

For light waves $|\vec{k}| = k_0$, $|\vec{k}'| = k_0'$.

$$\begin{cases} k_0' = \gamma(k_0 - \beta k_x) \\ k_x' = \gamma(k_x - \beta k_0) \\ k_y' = k_y \\ k_z' = k_z \end{cases} \quad \Longrightarrow \quad \begin{cases} \omega' = \gamma\omega(1-\beta\cos\theta) \\ \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \end{cases}$$

11.4 Addition of Velocities: 4-Vector

We consider a moving point P whose velocity $\vec{u}'$ has coordinates (u_x', u_y', u_z') in k', which is moving with $\vec{v} = c\vec{\beta}$ in the positive x_1 direction with respect to k.

$$\begin{aligned} dx_0 &= \gamma_v(dx_0' + \beta dx_1') \\ dx_1 &= \gamma_v(dx_1' + \beta dx_0') \\ dx_2 &= dx_2' \\ dx_3 &= dx_3' \end{aligned}$$

The velocity components in each frame are $u_i' = c\, dx_i'/dx_0'$ and $u_i = c\, dx_i/dx_0$.

$$\Longrightarrow \begin{cases} u_\parallel = \frac{u_\parallel' + v}{1 + \frac{u_\parallel' v}{c^2}} \\ u_\perp = \frac{u_\perp'}{\gamma_v(1 + \frac{u_\parallel' v}{c^2})} \end{cases}$$

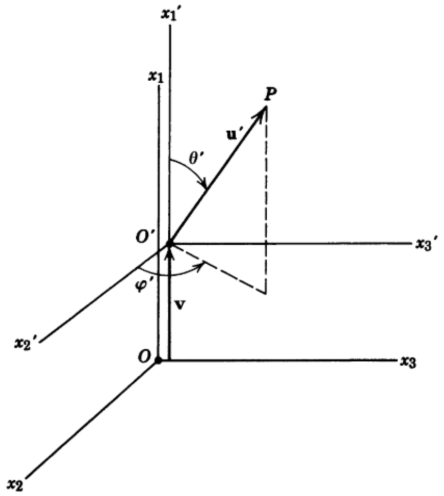

Figure 11.4 Addition of velocities.

Figure 4:

Since $u_2/u_3 = u_2'/u_3'$, the azimuthal angles in the two frames are equal.

$$\tan\theta = \frac{u'\sin\theta'}{\gamma_v(u'\cos\theta' + v)}$$

$$u = \sqrt{u_{\parallel}^2 + u_{\perp}^2} = \frac{\sqrt{(u'\cos\theta' + v)^2 + (u'\sin\theta'/\gamma_v)^2}}{1 + \frac{u'v}{c^2}\cos\theta'}$$

As $u, v \ll c$, we obtain the Galilean transformation.

Addition of Velocities: 4-Vector (Continued)

But if $\vec{u}'$ or $\vec{u}$ is close to c, we have the case of parallel velocities addition law

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

And if $u' = c$, we have $u = c$ which is the one example of Einstein's second postulate.

In terms of 4-vector

The expression $(1 + \vec{u} \cdot \vec{v}/c^2)$ can be expressed through

$$\gamma_u = \gamma_v \gamma_{u'}(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2})$$

And thus we have

$$\begin{cases} \gamma_u u_{\parallel} = \gamma_v(\gamma_{u'} u_{\parallel}' + \gamma_{u'} v) \\ \gamma_u u_{\perp} = \gamma_{u'} u_{\perp}' \end{cases}$$

Hence, the four quantities $(\gamma_u c, \gamma_u \vec{u})$ form a 4-vector. Lorentz transformations $\implies$ time-space components of the 4-velocity $(U_0, \vec{U})$ where

$$\begin{cases} U_0 = \frac{dx_0}{d\tau} = \frac{dx_0}{dt}\frac{dt}{d\tau} = \gamma_u c \\ \vec{U} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \gamma_u \vec{u} \end{cases}$$

11.5 Relativistic Momentum and Energy of a Particle

For a particle with speed small compared to c

We have

$$\begin{cases} \vec{p} = m\vec{u} \\ E = E(0) + \frac{1}{2}mu^2 \end{cases}$$

where $E(0)$ is the rest energy. The only possible generalization consistent with the first postulate are

$$\begin{cases} \vec{p} = M(u)\vec{u} \\ E = \mathcal{E}(u) \end{cases} \quad \text{where } M \text{ and } \mathcal{E} \text{ are functions of } |\vec{u}|.$$

We see that

$$\begin{cases} M(0) = m \\ (-\frac{\partial \mathcal{E}}{\partial u^2}|_{u=0}) = \frac{m}{2} \end{cases}$$

Collision Analysis

We now consider the collision in k and k' connected by Lorentz transformation. The two identical particles have initial velocities $\vec{u}_a = \vec{v}$, $\vec{u}_b = -\vec{v}$ along the z axis. After collision the final velocities are $\vec{u}_c, \vec{u}_d$. (c.f. Fig 5, page 534)

In k', the conservation are

$$\begin{cases} \vec{p'_a} + \vec{p'_b} = \vec{p'_c} + \vec{p'_d} \\ E'_a + E'_b = E'_c + E'_d \end{cases}$$

or

$$\begin{cases} M(v')\vec{v'} - M(v')\vec{v'} = M(v'')\vec{v''} + M(v'')(-\vec{v''}) \\ \mathcal{E}(v') + \mathcal{E}(v') = \mathcal{E}(v'') + \mathcal{E}(v'') \end{cases}$$

where we require $\mathcal{E}(v') = \mathcal{E}(v'')$. We have, then $v' = v'' = v$.

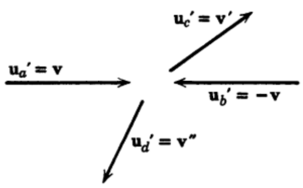

Figure 11.5 Initial and final velocity vectors in the frame K' for the collision of two identical particles.

Figure 5:

We now consider the collision in k, moving with $-\vec{v}$ in the z direction with respect to k'. We see that b is at rest in k while a is incident along the z axis with a velocity

$$u_a = \frac{2v}{1 + v^2/c^2} = \frac{2c\beta}{1 + \beta^2}$$

The components in k are

$$(u_c)_x = \frac{c\beta \sin\theta'}{\gamma(1 + \beta^2 \cos\theta')} \qquad (u_c)_z = \frac{c\beta(1 + \cos\theta')}{1 + \beta^2 \cos\theta'}$$

$$(u_d)_x = -\frac{c\beta \sin\theta'}{\gamma(1 - \beta^2 \cos\theta')} \qquad (u_d)_z = \frac{c\beta(1 - \cos\theta')}{1 - \beta^2 \cos\theta'}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The equations of conservation in k are

$$M(u_a)\vec{u}_a + M(u_b)\vec{u}_b = M(u_c)\vec{u}_c + M(u_d)\vec{u}_d$$

$$\mathcal{E}(u_a) + \mathcal{E}(u_b) = \mathcal{E}(u_c) + \mathcal{E}(u_d)$$

The x-component is $((u_b)_x = (u_d)_x = 0)$

$$0 = M(u_c)\frac{c\beta\sin\theta'}{\gamma(1+\beta^2\cos\theta')} - M(u_d)\frac{c\beta\sin\theta'}{\gamma(1-\beta^2\cos\theta')}$$

$$\implies M(u_c)(1-\beta^2\cos\theta') = M(u_d)(1+\beta^2\cos\theta') \quad \text{for all } \theta'$$

Specifically for $\theta' = 0$ where $u_c = u_a$, $u_d = 0$.

$$M(u_a)\frac{1-\beta^2}{1+\beta^2} = M(0)\frac{1}{\sqrt{1-u_a^2/c^2}} = \gamma_a$$

with $M(0) = m$, we have $M(u_a) = \gamma_a m$.

$$\implies \vec{p} = \gamma m\vec{u} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

Energy Derivation

For small θ', we have

$$\mathcal{E}(u_a) + \mathcal{E}(0) = \mathcal{E}(u_c) + \mathcal{E}(u_d)$$

where u_c and u_d are functions of θ'. Correct to order θ'^2 inclusive,

$$u_c^2 = u_a^2 - \frac{\eta}{\gamma_a^2} + O(\theta'^4)$$
$$u_d^2 = \eta + O(\theta'^4)$$

where γ_a is given by $\frac{1+\beta^2}{1-\beta^2}$ and $\eta = c^2\beta^2\theta'^2/(1-\beta^2)$ is a convenient expansion parameter.

We now expand in Taylor series

$$\mathcal{E}(u_a) + \mathcal{E}(0) = \mathcal{E}(u_c) + \mathcal{E}(u_d) = \mathcal{E}(u_a) + \eta\cdot(\frac{d\mathcal{E}(u_c)}{du_c^2}|_{u_c=u_a}\frac{\partial u_c^2}{\partial\eta}|_{\eta=0} + \frac{d\mathcal{E}(u_d)}{du_d^2}|_{u_d=0}) + \dots$$

The first-order terms yield

$$0 = -\frac{1}{\gamma_a^2}\frac{d\mathcal{E}(u_a)}{du_a^2} + [\frac{d\mathcal{E}(u_d)}{du_d^2}]_{u_d=0}$$

We thus find:

$$\frac{d\mathcal{E}(u_a)}{du_a^2} = \frac{1}{2}m\gamma_a^3 = \frac{m}{2(1-u_a^2/c^2)^{3/2}}$$

Integration yields

$$\mathcal{E}(u) = \frac{mc^2}{\sqrt{1-u^2/c^2}} + [\mathcal{E}(0) - mc^2]$$

And the kinetic energy is given by

$$T(u) = \mathcal{E}(u) - \mathcal{E}(0) = mc^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right]$$

Part 1

We further consider the decay of a neutral K-meson into two photons, $k^0 \to \gamma\gamma$, where $\mathcal{E}_k(0)$ is the rest energy.

For another decay mode of a neutral K-meson into two pions, the kinetic energy of each pion in the K-meson's rest frame must be:

$$T_\pi = \frac{1}{2}\mathcal{E}_k(0) - \mathcal{E}_\pi(0)$$

where $\mathcal{E}(0) = mc^2$.

The total energy of a particle with mass m and velocity $\vec{u}$ is given by:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

The energy and momentum conservation for an arbitrary collision are:

$$\sum_{\text{initial}} (P_0)_a - \sum_{\text{final}} (P_0)_b = \Delta_0$$

$$\sum_{\text{initial}} \vec{P}_a - \sum_{\text{final}} \vec{P}_b = \vec{\Delta}$$

where $(\Delta_0, \vec{\Delta})$ is a 4-vector with $\Delta = 0$. And

$$c\Delta_0 = \sum_{\text{final}} [\mathcal{E}_b(0) - m_b c^2] - \sum_{\text{initial}} [\mathcal{E}_a(0) - m_a c^2]$$

If $\Delta = 0$ in all inertial frames, it's necessary that $\Delta_0 = 0$ for each particle.

Part 2

The velocity in terms of momentum $\vec{p}$ and energy E is:

$$\vec{u} = \frac{c^2\vec{p}}{E}$$

The invariant length of $(P_0 = E/c, \vec{P})$ is:

$$P_0^2 - \vec{p}\cdot\vec{p} = (mc)^2$$

And the energy in terms of momentum is:

$$E = \sqrt{c^2p^2 + m^2c^4}$$

We consider a particle with momentum $\vec{p}$ in frame K with transverse momentum $\vec{p}_\perp$ and a z component $p_\parallel$. There is a unique Lorentz transformation in the z direction to a frame K'. In K', the momentum and energy are:

$$p'_\perp = p_\perp, \quad E' = \Omega = \sqrt{p_\perp^2 + m^2c^2}$$

Let the rapidity parameter from K to K' be ξ. In the frame K, the momentum components are:

$$p_\parallel = \Omega \sinh\xi, \quad \frac{E}{c} = \Omega \cosh\xi$$

with $\Omega = \sqrt{p_\perp^2 + m^2c^2}$, and Ω is the transverse mass. If the particle is at rest in K', that is $p'_\parallel = 0$, then:

$$p = mc\sinh\xi, \quad E = mc^2\cosh\xi$$

Part 3

11.6 Mathematical Properties of the Space-Time of Special Theory

The Lorentz transformation of the four-dimensional coordinates $(x_0, \vec{x})$ follow from the invariance of

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

The group of all transformations that leave s^2 invariant is the homogeneous Lorentz group. The group of transformations that leave invariant $s^2(x,y) = (x_0-y_0)^2 - (x_1-y_1)^2 - (x_2-y_2)^2 - (x_3-y_3)^2$ is the inhomogeneous Lorentz group / Poincaré group.

In a non-Euclidean vector space, the space-time continuum is defined in terms of x^0, x^1, x^2, x^3 and the transformation yields x'^0, x'^1, x'^2, x'^3.

$$x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3) \quad (\alpha = 0,1,2,3)$$

For the contravariant vector A^α with A^0, A^1, A^2, A^3:

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \quad (\text{sum on } \beta = 0,1,2,3)$$

$$A'^0 = \frac{\partial x'^0}{\partial x^0} A^0 + \frac{\partial x'^0}{\partial x^1} A^1 + \dots$$

For the covariant vector B_α:

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \quad (\text{sum on } \beta)$$

Part 4

For a tensor of rank two, it consists of 16 quantities.

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \quad \text{(sum on } \gamma, \delta)$$

$$G'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G^{\gamma}_{\delta} \quad \text{(sum on } \delta, \gamma)$$

$$H'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} H_{\gamma\delta} \quad \text{(sum on } \gamma, \delta)$$

We have some properties:

$$B \cdot A = B_{\alpha} A^{\alpha}$$

$$B' \cdot A' = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} = \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B_{\beta} A^{\beta} = B \cdot A$$

In differential form, the infinitesimal interval is:

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$(ds)^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

where $g_{\alpha\beta}$ is the metric tensor, and $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$.

By the Kronecker delta in four dimensions, $\delta^{\alpha}_{\beta} = g^{\alpha\gamma} g_{\gamma\beta}$. We can obtain the covariant coordinate 4-vector x_{α} in terms of the contravariant x^{β} and $g_{\alpha\beta}$, that is:

$$x_{\alpha} = g_{\alpha\beta} x^{\beta}$$

and its inverse

$$g^{\alpha\beta} x_{\beta} = x^{\alpha}$$

11.7 Matrix Representation of Lorentz Transformations, Infinitesimal Generators

We define the coordinates x^{α} as a vector

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

and the Matrix A scalar products of 4-vectors as

$$(a, b) = \tilde{a} b$$

with the

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{and} \quad g^2 = I$$

we see that

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

we also read that

$$a \cdot b = (a, gb) = (\tilde{a} g) b = \tilde{a} g b$$

We now seek a group of linear transformations on the coordinates

$$x' = Ax$$

where A is a 4x4 matrix such that (x, gx) is invariant

$$\tilde{x}' g x' = \tilde{x} g x$$

By substitution, we have

$$\tilde{x}\tilde{A}gAx = \tilde{x}gx \quad \text{for all } x.$$

$$\Rightarrow \tilde{A}gA = g$$

Taking the determinant:

$$\det\left(\tilde{A}gA\right) = \det(g)(\det A)^2 = \det g = -1 \neq 0$$

$$\Rightarrow \det A = \pm 1$$

$\det A = -1 :$ improper Lorentz transformations (sufficient but not necessary)
$\det A = +1 :$ proper Lorentz transformations

We make the ansatz $A = e^L$ where L is 4x4. The determinant is $\det A = \det\left(e^L\right) = e^{\mathrm{Tr} L}$. If A is traceless, $\det A = +1 \Rightarrow$ proper Lorentz transformation. Thus $\tilde{A}gA = g \Rightarrow g\tilde{A}gA = I$ since $\tilde{A} = A^{-1}$ with $A = e^L$ and $g^2 = I$. We have: $\tilde{A} = e^{\tilde{L}}, g\tilde{A}g = ge^{\tilde{L}}g = e^{g\tilde{L}g}$, $A^{-1} = e^{-L}$

$$\Rightarrow g\tilde{L}g = -L \quad \text{or} \quad \tilde{gL} = -gL$$

From these properties, we have the general form

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

The set of fundamental matrices defined by

$$S_1 = \begin{pmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \quad S_2 = \begin{pmatrix} & & & \\ & & & 1 \\ & & & \\ & -1 & & \end{pmatrix}, \quad S_3 = \begin{pmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{pmatrix}$$

$$K_1 = \begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & \\ & & & \end{pmatrix}, \quad K_2 = \begin{pmatrix} & & -1 & \\ & & & \\ -1 & & & \\ & & & \end{pmatrix}, \quad K_3 = \begin{pmatrix} & & & -1 \\ & & & \\ & & & \\ -1 & & & \end{pmatrix}$$

S_i generate rotations in three dimensions K_i produce boost

The square of those are

$$S_1^2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad K_1^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$S_2^2 = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}, \quad K_2^2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

$$S_3^2 = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}, \quad K_3^2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$

we can also prove that:

$$(\epsilon \cdot S)^2 = -\epsilon^2 \cdot S$$

$$(\epsilon' \cdot K)^2 = \epsilon' \cdot K$$

where ϵ, ϵ' are any real unit 3-vectors.

The general result for L can be shown as

$$L = -w \cdot S - \zeta \cdot K$$

where w and ζ are constant 3-vectors

$$A = e^{-w \cdot S - \zeta \cdot K}$$

We consider a situation in which $w = 0, \zeta = \zeta \hat{k}_1$. The $L = -\zeta K_1$ and $K_1^2 = K_1$ and S_i^2, K_j^2

$$A = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If $\zeta = 0$ and $w = w\hat{E}_3$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(w) & \sin(w) & 0 \\ 0 & -\sin(w) & \cos(w) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For a boost in an arbitrary direction:

$$A = e^{-\vec{\zeta} \cdot \vec{K}}$$

where $\vec{\zeta} = \hat{\beta} \tanh^{-1} \beta$, where $\hat{\beta}$ is a unit vector in the direction of the relative velocity of the two inertial frames.

The pure boost is

$$A_{boost}(\beta) = e^{-\vec{\zeta} \cdot \vec{K}} \tanh^{-1} \beta$$

$$= \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1 + (\gamma - 1)\frac{\beta_1^2}{\beta^2} & (\gamma - 1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma - 1)\frac{\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & (\gamma - 1)\frac{\beta_2\beta_1}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_2^2}{\beta^2} & (\gamma - 1)\frac{\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & (\gamma - 1)\frac{\beta_3\beta_1}{\beta^2} & (\gamma - 1)\frac{\beta_3\beta_2}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

The six matrices S_i, K_j are a representation of the infinitesimal generators of the Lorentz group. They satisfy the commutation relations

$$[S_i, S_j] = \epsilon_{ijk} S_k$$

$$[S_i, K_j] = \epsilon_{ijk} K_k \quad \text{where } [A, B] = AB - BA$$

$$[K_i, K_j] = \epsilon_{ijk} S_k$$

$$\Rightarrow F_{\alpha\beta} = g_{\alpha\delta} F^{\delta}_{\beta}$$

$$G_{\dots\alpha..} = g_{\alpha\beta} G^{\dots\beta}_{\dots}$$

$$A^0, A^1, A^2, A^3 \quad \rightarrow \quad A_0 = A^0, A_1 = -A^1, A_2 = -A^2, A_3 = -A^3$$

We write this concisely:

$$A^{\alpha} = (A^0, \vec{A}), \quad A_{\alpha} = (A^0, -\vec{A})$$

where $\vec{A}$ is a 3-vector with $A^1 A^2 A^3$ and

$$B \cdot A = B_{\alpha} A^{\alpha} = B^0 A^0 - \vec{B} \cdot \vec{A}$$

We have the operator

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}}$$

we employ the notation

$$\partial^{\alpha} = \frac{\partial}{\partial x_{\alpha}} = (\frac{\partial}{\partial x_0}, -\nabla)$$

$$\partial_{\alpha} = \frac{\partial}{\partial x^{\alpha}} = (\frac{\partial}{\partial x^0}, \nabla)$$

$$\partial_{\alpha}\partial^{\alpha} = \frac{\partial}{\partial x^0}\frac{\partial}{\partial x_0} + \nabla \cdot (-\nabla) \text{ is the invariant}$$

The four-dimensional Laplacian operator is defined to be the invariant contraction

$$\square = \partial_{\alpha}\partial^{\alpha} = \frac{\partial^2}{\partial x_0^2} - \nabla^2$$

which is the operator of the wave equation in vacuum.

11.8 Thomas Precession

By Uhlenbeck-Goudsmit hypothesis

The electron possesses a spin angular momentum $\vec{s}$ and a magnetic moment $\vec{\mu}$ related by

$$\vec{\mu} = \frac{ge}{2mc}\vec{s} \quad \text{where } g \approx 2$$

We suppose the electron moves with $\vec{v}$ in fields $\vec{E}$ and $\vec{B}$. The equation of motion in its rest frame is:

$$\left(\frac{d\vec{s}}{dt}\right)_{\text{rest frame}} = \vec{\mu} \times \vec{B}'$$

where $\vec{B}' \approx \vec{B} - \frac{\vec{v}}{c} \times \vec{E}$. Then,

$$\left(\frac{d\vec{s}}{dt}\right)_{\text{rest frame}} = \vec{\mu} \times \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)$$

which is equivalent to an energy of interaction of the electron spin

$$U' = -\vec{\mu} \cdot \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)$$

We view $e\vec{E} = -\vec{\nabla} V = -\frac{\vec{r}}{r}\frac{dV}{dr}$, then

$$U' = -\frac{ge}{2mc}\vec{s} \cdot \vec{B} + \frac{g}{2m^2c^2}(\vec{s} \cdot \vec{L})\frac{1}{r}\frac{dV}{dr}$$

where $\vec{L} = m(\vec{r} \times \vec{v})$. This gives the anomalous Zeeman effect.

Coordinate System Rotation

If the coordinate system rotates, any vector $\vec{G}$ is given by

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{non-rot}} = \left(\frac{d\vec{G}}{dt}\right)_{\text{rest frame}} + \vec{w}_T \times \vec{G}$$

where $\vec{w}_T$ is the angular velocity of rotation. Thus:

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{non-rot}} = \vec{s} \times \left(\frac{ge\vec{B}}{2mc} - \vec{w}_T\right)$$

The corresponding energy of interaction is

$$U = U' + \vec{s} \cdot \vec{w}_T$$

where U' is the electromagnetic spin interaction.

We consider an electron moving with $\vec{v}(t)$ with respect to a laboratory inertial frame. Let the velocity of the rest frame with respect to the laboratory at laboratory time t be $\vec{v}(t) = c\vec{\beta}$ and at $t + \delta t$ be $\vec{v}(t + \delta t) = c(\vec{\beta} + \delta\vec{\beta})$. The connection is

$$\begin{cases} \vec{x}' = A\text{boost}(\vec{\beta})\vec{x} \\ \vec{x}'' = A\text{boost}(\vec{\beta} + \delta\vec{\beta})\vec{x} \end{cases}$$

$$\implies \vec{x}'' = A\text{boost}(\vec{\beta} + \delta\vec{\beta})A^{-1}\text{boost}(\vec{\beta})\vec{x}' = AT\vec{x}'$$

where $AT = A\text{boost}(\vec{\beta} + \delta\vec{\beta})A^{-1}\text{boost}(\vec{\beta}) = A\text{boost}(\vec{\beta} + \delta\vec{\beta})A\text{boost}(-\vec{\beta})$.

We choose a proper laboratory frame where $\vec{\beta}$ at t is parallel to the 1-axis and the $\delta\vec{\beta}$ lies in the 1-2 plane.

Hence,

$$A\text{boost}(-\vec{\beta}) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

since $-\beta_1 = \beta$, $\beta_2 = \beta_3 = 0$. Similarly,

$$A\text{boost}(\vec{\beta} + \delta\vec{\beta}) = \begin{pmatrix} \gamma^3\beta\delta\beta_1 & -(\gamma\beta + \gamma^3\delta\beta_1) & -\gamma\delta\beta_2 & 0 \\ -(\gamma\beta + \gamma^3\delta\beta_1) & \gamma + \gamma^3\beta\delta\beta_1 & (\frac{\gamma-1}{\beta^2})\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & -(\frac{\gamma-1}{\beta^2})\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus:

$$AT = \begin{pmatrix} 1 & -\gamma^2\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_1 & 1 & (\frac{\gamma-1}{\beta^2})\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & -(\frac{\gamma-1}{\beta^2})\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= I - \frac{\gamma^2}{\gamma + 1}(\vec{\beta} \times \delta\vec{\beta}) \cdot \vec{S} - (\gamma^2\beta_1\delta\beta_1 + \gamma\delta\beta_2)\vec{K}$$

where $\delta\vec{\beta}_1$ and $\delta\vec{\beta}_2$ are components of $\delta\vec{\beta}$.

To first order in $\delta\vec{\beta}$,

$$AT = A\text{boost}(\Delta\vec{\beta})R(\delta\vec{\Omega}) \approx A\text{boost}(\delta\vec{\beta}')$$

where $\{A\text{boost}(\Delta\vec{\beta}') = I - \Delta\vec{\beta}' \cdot \vec{K}\}$, $\{R(\Delta\vec{\Omega}) = I - \Delta\vec{\Omega} \cdot \vec{S}\}$ with

$$\Delta\vec{\beta}' = \gamma^2\delta\vec{\beta}_1 + \delta\vec{\beta}_2$$

$$\Delta\vec{\Omega} = (\frac{\gamma - 1}{\beta^2})\vec{\beta} \times \delta\vec{\beta} = \frac{\gamma^2}{\gamma + 1}\vec{\beta} \times \delta\vec{\beta}$$

That is: a pure Lorentz boost to the frame with $c(\vec{\beta} + \delta\vec{\beta})$ is a boost to/from a frame with $c\vec{\beta}$, followed by an infinitesimal transformation consisting of a boost with $c\Delta\vec{\beta}$ and a rotation $\Delta\vec{\Omega}$.

Rest Frame Coordinates

We now consider the rest-frame coordinates at $t + \delta t$ that are given from those at t by the Boost($\Delta\vec{\beta}$)

$$\vec{x}''' = A\text{boost}(\Delta\vec{\beta})\vec{x}' = R(-\Delta\vec{\Omega})A\text{boost}(\vec{\beta} + \delta\vec{\beta})\vec{x}$$

If a vector $\vec{G}$ has a time rate of change $\frac{d\vec{G}}{dt}$ in the rest frame, the precession of the rest-frame axes with respect to the laboratory makes the vector have a total time rate of change with respect to the laboratory axes of:

$$\vec{\omega}_T = \lim_{\delta t \to 0} \frac{\Delta\vec{\Omega}}{\delta t} = \frac{\gamma^2}{\gamma + 1} \frac{\vec{a} \times \vec{v}}{c^2}$$

where $\vec{a}$ is the acceleration in the laboratory frame. To be more precise

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{rest frame}} = +(\gamma - 1)\left(\frac{d\vec{G}}{dt}\right)_{\text{rest frame}}$$

For an electron in the atom

$$\vec{\omega}_T = -\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L} = -\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}(\vec{r} \times \vec{p})$$

The Thomas precession reduces the spin-orbit coupling, yielding

$$U = \frac{ge}{2mc}\vec{s} \cdot \vec{B} + (\frac{g-1}{2m^2c^2})\vec{s} \cdot \vec{L}\frac{1}{r}\frac{dV}{dr}$$

With $g = 2$, the spin-orbit interaction is reduced by $\frac{1}{2}$.

Continuation of Previous Notes

Rotating Coordinate Systems and Composition of Boosts

If the coordinate system rotates, any vector $\vec{G}$ is given by

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{non-rot}} = \left(\frac{d\vec{G}}{dt}\right)_{\text{rest frame}} + \vec{w}_T \times \vec{G}$$

where $\vec{w}_T$ is the angular velocity of rotation. Thus,

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{non-rot}} = S \times \left(\frac{geB}{2mc} - \vec{w}_T\right)$$

The corresponding energy of interaction

$$U = U' + S \cdot \vec{w}_T$$

where U' is the electromagnetic spin interaction.

We consider an electron moving with $\vec{v}(t)$ with respect to a laboratory inertial frame. Let the velocity of the rest frame with respect to the laboratory at laboratory time $t : \vec{v}(t) = c\vec{\beta}$ and at $t+\delta t : \vec{v}(t+\delta t) = c(\vec{\beta} + \delta\vec{\beta})$. The connection is:

$$\begin{aligned} x' &= A_{\text{boost}}(\vec{\beta})x \\ x'' &= A_{\text{boost}}(\vec{\beta} + \delta\vec{\beta})x \\ \Rightarrow x'' &= A'x' = A'A^{-1}x \end{aligned}$$

where $AT = A_{\text{boost}}(\vec{\beta} + \delta\vec{\beta})A^{-1}_{\text{boost}}(\vec{\beta}) = A_{\text{boost}}(\vec{\beta} + \delta\vec{\beta})A_{\text{boost}}(-\vec{\beta})$.

We choose a proper laboratory frame where $\vec{\beta}$ at t is parallel to the 1-axis and the $\delta\vec{\beta}$ lies in the 1-2 plane. Hence

$$A_{\text{boost}}(-\vec{\beta}) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

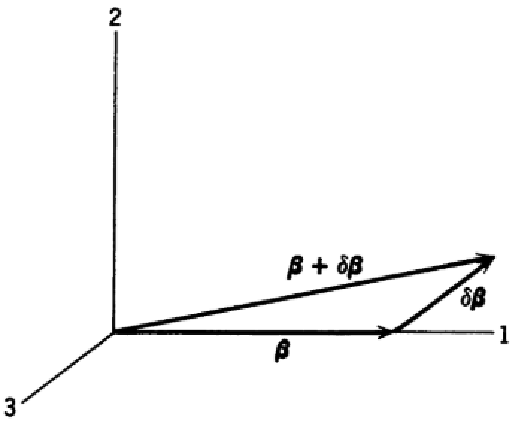

Figure 11.7

Figure 6:

since $-\beta_1 = \beta$, $\beta_2 = \beta_3 = 0$.

Similarly,

$$A_{\text{boost}}(\vec{\beta} + \delta\vec{\beta}) = \begin{pmatrix} \gamma + \gamma^3\beta\delta\beta_1 & -(\gamma\beta + \gamma^3\delta\beta_1) & -(\gamma + \gamma^3\beta^2)\delta\beta_2 & 0 \\ -(\gamma\beta + \gamma^3\delta\beta_1) & \gamma + \gamma^3\beta^2\delta\beta_1 & (\gamma - 1)\frac{\delta\beta_2}{\beta} & 0 \\ -\gamma\delta\beta_2 & -\gamma\beta\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\begin{aligned} AT &= \begin{pmatrix} 1 & -\gamma^2\delta\beta_1 & -(\gamma\beta)\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_1 & 1 & (\frac{\gamma-1}{\beta})\delta\beta_2 & 0 \\ -\gamma\delta\beta_2 & -\frac{\gamma(\gamma-1)}{\beta}\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= I - (\frac{\gamma}{\beta})(\vec{\beta} \times \delta\vec{\beta}) \cdot \vec{S} - (\gamma^2\beta\delta\beta_1 + \gamma\delta\beta_2)\vec{K} \end{aligned}$$

where $\delta\beta_1$ and $\delta\beta_2$ are components of $\delta\vec{\beta}$.

To first order in $\delta\vec{\beta}$

$$AT = A_{\text{boost}}(\Delta\vec{\beta}')R(\delta\vec{\Omega}) \approx A_{\text{boost}}(\Delta\vec{\beta}')$$

where $\{A_{\text{boost}}(\Delta\vec{\beta}') = I - \Delta\vec{\beta}' \cdot \vec{K}\}$. $\{R(\Delta\vec{\Omega}) = I - \Delta\vec{\Omega} \cdot \vec{S}\}$ with $\Delta\vec{\beta}' = \gamma^2\delta\beta_1\hat{\imath} + \gamma\delta\beta_2\hat{\jmath}$.

$$\Delta\vec{\Omega} = (\frac{\gamma - 1}{\beta^2})\vec{\beta} \times \delta\vec{\beta} = \frac{\gamma^2}{c^2}\vec{v} \times \delta\vec{v}$$

That is: a pure Lorentz boost to the frame with $c(\vec{\beta} + \delta\vec{\beta})$ is a boost to/from with $c\vec{\beta}$, followed by an infinitesimal transformation consisting of a boost with $c\Delta\vec{\beta}$ and a rotation $\Delta\vec{\Omega}$.

11.9 Invariance of Electric Charge; Covariance of Electrodynamics

Lorentz Force

Consider the Lorentz force for a particle

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$$

We know $\vec{p}$ transforms as the space part of the 4-vector of energy and momentum

$$P^\alpha = (P_0, \vec{P}) = m(U_0, \vec{U})$$

where $P_0 = E/c$ and U^α is the 4-velocity. We use proper time τ:

$$\frac{d\vec{P}}{d\tau} = \frac{q}{c}(U_0\vec{E} + \vec{U} \times \vec{B})$$

where the left-hand side is the space part. The corresponding time component

$$\frac{dP_0}{d\tau} = \frac{q}{c}\vec{U} \cdot \vec{E}$$

Maxwell Equations

We then consider the Maxwell equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

We postulate that ρ and $\vec{J}$ form a 4-vector $\vec{J}$

$$J^\alpha = (c\rho, \vec{J})$$

The continuity equation becomes

$$\partial_\alpha J^\alpha = 0$$

The 4-dimensional volume element $d^4x = dx^0dx^1dx^2dx^3$ is a Lorentz invariant

$$d^4x' = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x'^0, x'^1, x'^2, x'^3)} d^4x = \det A d^4x = d^4x$$

In the Lorentz family of gauges the wave equations for $\vec{A}'$ and Φ:

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \dfrac{4\pi}{c}\vec{J} \\ \dfrac{1}{c^2}\dfrac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho \end{cases}$$

with Lorentz condition

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

The Lorentz covariance requires that Φ and $\vec{A}$ form a 4-vector potential

$$A^\alpha = (\Phi, \vec{A})$$

$$\Rightarrow \Box A^\alpha = \frac{4\pi}{c} J^\alpha$$

$$\partial_\alpha A^\alpha = 0$$

We express $\vec{E}, \vec{B}$ in terms of $\Phi, \vec{A}$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \Phi$$

$$\vec{B} = \nabla \times \vec{A}$$

The x-components

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

These equations indicate that $\vec{E}$ and $\vec{B}$ form a second-rank, antisymmetric field-strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

In matrix form, the field-strength tensor $F^{\alpha\beta}$ is:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The covariant tensor $F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$ is:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

We define the dual field-strength tensor by

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita pseudotensor.

Maxwell's Equations in Covariant Form

The two inhomogeneous equations

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

can be written compactly as

$$\Rightarrow \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

Similarly, the two homogeneous equations

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

become

$$\Rightarrow \partial_\alpha \mathcal{F}^{\alpha\beta} = 0$$

which is equivalent to the cyclic identity

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0$$

Lorentz Force in Covariant Form

We can write the Lorentz force equation as

$$\frac{dP^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta$$

Its components are:

$$\frac{d\vec{P}}{dt} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$$

$$\frac{dP_0}{d\tau} = \frac{q}{c}\vec{U} \cdot \vec{E}$$

As we denote for $(\vec{E}, \vec{B}) \to (\vec{D}, \vec{H})$, the covariant form is (we denote for $(\vec{E}, \vec{B})$ as $F^{\alpha\beta}$, and for $(\vec{D}, \vec{H})$ as $G^{\alpha\beta}$)

$$\begin{cases} \partial_\alpha G^{\alpha\beta} = \dfrac{4\pi}{c}j^\beta \\ \partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \end{cases}$$

11.10 Transformations of Electromagnetic Fields

We express the second-rank tensor in k' in terms of the values in k.

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}F^{\gamma\delta}$$

$$F' = AFA^T$$

where A is the Lorentz transformation.

For example: we consider a frame k' changed to another frame k with $\vec{\beta}$ along the x_1 axis.

$$\begin{aligned} E'_1 &= E_1, & B'_1 &= B_1 \\ E'_2 &= \gamma(E_2 - \beta B_3), & B'_2 &= \gamma(B_2 + \beta E_3) \\ E'_3 &= \gamma(E_3 + \beta B_2), & B'_3 &= \gamma(B_3 - \beta E_2) \end{aligned}$$

By putting $\vec{E} \to -\vec{B}$, we get the transformation

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma + 1}\vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma + 1}\vec{\beta}(\vec{\beta} \cdot \vec{B})$$

$\Rightarrow \vec{E}', \vec{B}'$ are interrelated.

We consider one example that there exists no magnetic field in k' and one or more point charges at rest in k'. In the k we have

$$\vec{B} = \vec{\beta} \times \vec{E}$$

As an important example: we now consider the fields in k when a point charge q moves by in a straight-line path with $\vec{v}$. The charge is at rest in k'. We suppose the charge moves in the positive x_1.

The observer is at P. In the k', P has coordinates $x'_1 = -vt'$, $x'_2 = b$, $x'_3 = 0$, and a distance $r' = \sqrt{(vt')^2 + b^2}$ away from q. With the transformation $t' = \gamma[t - (v/c^2)x_1] = \gamma t$ since $x_1 = 0$ for P in k.

In the rest frame k' the fields at P are

$$E'_1 = \frac{-qvt'}{(b^2 + v^2t'^2)^{3/2}}$$

$$E'_2 = \frac{qb}{(b^2 + v^2t'^2)^{3/2}}$$

$$E'_3 = 0$$

$$B'_1 = 0, \quad B'_2 = 0, \quad B'_3 = 0$$

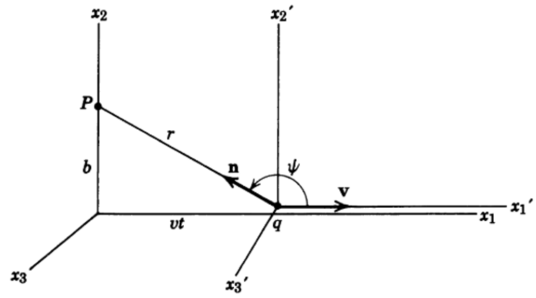

Figure 11.8 Particle of charge q moving at constant velocity $\mathbf{v}$ passes an observation point P at impact parameter b.

Figure 7:

Substituting $t' = \gamma t$:

$$E_1' = \frac{-q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$
$$E_2' = \frac{qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

The inverse in the k:

$$E_1 = E_1' = \frac{-q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$
$$E_2 = \gamma E_2' = \frac{\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$
$$B_3 = \gamma\beta E_2' = \beta E_2$$

with the other components vanishing.

As $\epsilon \to 0$, we see that the magnetic field becomes almost equal to E_2. When $\gamma \approx 1$,

$$B' \approx \frac{q}{c}\frac{\vec{v} \times \vec{r}}{r^3}$$

which is the Ampere-Biot-Savart expression.

11.11 Relativistic Equation of Motion for Spin in Uniform or Slowly Varying External Fields

The equation of motion for the spin in the rest frame

$$\frac{d\vec{s}}{dt} = \frac{ge}{2mc}\vec{s} \times \vec{B}$$

A: Covariant Equation of Motion

We define an axial vector S^α, then the time-component in the rest frame k' is

$$S'^0 = \gamma(S^0 - \vec{\beta} \cdot \vec{s}) = 0$$

where U^α is the 4-velocity. By the covariant constraint,

$$U_\alpha S^\alpha = 0$$

In an inertial frame where the particle has $\vec{\beta}$, the time component of S is

$$S_0 = \vec{\beta} \cdot \vec{s}$$

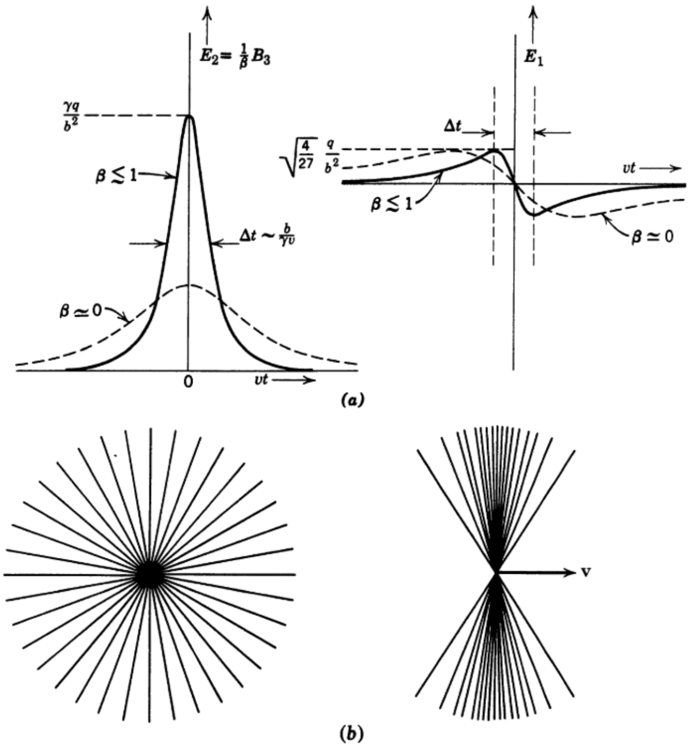

Figure 11.9 Fields of a uniformly moving charged particle. (*a*) Fields at the observation point *P* in Fig. 11.8 as a function of time. (*b*) Lines of electric force for a particle at rest and in motion ($\gamma = 3$). The field lines emanate from the *present* position of the charge.

sion of the lines of force in the transverse direction can be viewed as a consequence of the FitzGerald–Lorentz contraction.

Figure 8:

$$
\begin{aligned}
\vec{s}' &= \vec{s} - \frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{s})\vec{\beta} \\
\vec{s} &= \vec{s}' + \frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{s}')\vec{\beta} \\
S_0 &= \gamma\vec{\beta}\cdot\vec{s}'
\end{aligned}
$$

A. Construction of 4-vectors and Equation of Motion

We construct the 4-vectors

$$F^{\alpha\beta}S_\beta, \quad (S_\lambda F^{\lambda\mu}U_\mu)U^\alpha, \quad \left(S_\beta\frac{dU^\beta}{d\tau}\right)U^\alpha$$

The equation of motion

$$\frac{dS^\alpha}{d\tau} = A_1F^{\alpha\beta}S_\beta + A_2(S_\lambda F^{\lambda\mu}U_\mu)U^\alpha + A_3\left(S_\beta\frac{dU^\beta}{d\tau}\right)U^\alpha$$

The constraint equation must hold at all times which requires

$$\frac{d}{d\tau}(U_\alpha S^\alpha) = S^\alpha\frac{dU_\alpha}{d\tau} + U_\alpha\frac{dS^\alpha}{d\tau} = 0$$

Hence,

$$(A_1 - A_2)U_\alpha F^{\alpha\beta}S_\beta + (A_3+1)S_\beta\frac{dU^\beta}{d\tau} = 0$$

If non-electromagnetic or field gradient forces are allowed, it's necessary $A_1 = A_2$, $A_3 = -1$. Reduction to the rest frame gives $A_1 = \frac{ge}{2mc}$. Thus,

$$\frac{dS^\alpha}{d\tau} = \frac{ge}{2mc}\left[F^{\alpha\beta}S_\beta + \frac{1}{c^2}U^\alpha(S_\lambda F^{\lambda\mu}U_\mu)\right] - \frac{1}{c^2}U^\alpha\left(S_\beta\frac{dU^\beta}{d\tau}\right)$$

If the electromagnetic fields are uniform, or if gradient force terms like $\nabla(\vec{s}\cdot\vec{B})$ can be neglected, and there are no other appreciable forces on the particle, the translational motion is described by

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$$

Then we have the BMT equation

$$\frac{dS^\alpha}{d\tau} = \frac{e}{mc}\left[\frac{g}{2}F^{\alpha\beta}S_\beta + \frac{1}{c^2}(\frac{g}{2}-1)U^\alpha(S_\lambda F^{\lambda\mu}U_\mu)\right]$$

B. Connection to the Thomas Precession

We consider the rest-frame spin $\vec{s}$

$$S_0 = \frac{\gamma}{c}\vec{v}\cdot\vec{s}$$

and $\vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}\frac{(\vec{\beta}\cdot\vec{s})\vec{\beta}}{c^2}$ We find:

$$\begin{cases} \frac{d\vec{S}}{dt} = \vec{F} + \gamma^2\vec{v}(\vec{S}\cdot\frac{d\vec{\beta}}{dt}) \\ \frac{dS_0}{dt} = \vec{F}_0 + \gamma^2(\vec{S}\cdot\frac{d\vec{\beta}}{dt}) \end{cases}$$

can be combined to give

$$\frac{d\vec{s}}{dt} = \vec{F} - \frac{\gamma\vec{\beta}}{\gamma+1}\vec{F}_0 + \frac{\gamma^2}{\gamma+1}\left[\vec{s}\cdot\vec{\beta}\times\frac{d\vec{\beta}}{dt}\right]$$

By $F_0 = \vec{\beta}\cdot\vec{F}$ and definition of $\vec{\omega}_T$

$$\frac{d\vec{s}}{dt} = \frac{1}{\gamma}\vec{F} + \vec{\omega}_T\times\vec{s}$$

For motion in electromagnetic fields where $\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$ holds,

$$\frac{d\vec{p}}{dt} = \frac{e}{mc}\left[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})\right]$$

By the transformation properties

$$\vec{F} = \frac{ge}{2mc^2}\left[\vec{s}\times\left(\vec{B} - \frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{B})\vec{\beta} - \vec{\beta}\times\vec{E}\right)\right]$$

$$\Rightarrow \frac{d\vec{s}}{dt} = \frac{e}{mc}\vec{s}\times\left[\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{B} - \left(\frac{g}{2}-1\right)\frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\vec{\beta}\times\vec{E}\right]$$

Thomas's equation

C. Rate of Change of Longitudinal Polarization

We consider the rate of change of the component of $\vec{s}$ parallel to the velocity or net helicity of the particle. The longitudinal polarization is $\hat{\beta}\cdot\vec{S}$. Thus:

$$\frac{d}{dt}(\hat{\beta}\cdot\vec{S}) = \frac{d\hat{\beta}}{dt}\cdot\vec{S} + \hat{\beta}\cdot\frac{d\vec{S}}{dt}$$

By $\frac{d\hat{\beta}}{dt}$ and $\frac{d\vec{S}}{dt}$ we have

$$\frac{d}{dt}(\hat{\beta}\cdot\vec{S}) = -\frac{e}{mc}\vec{s}_\perp\cdot\left[-\left(\frac{g}{2}-1\right)\hat{\beta}\times\vec{B} + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\vec{E}_\perp\right]$$

where $\vec{s}_\perp$ is the component of $\vec{s}$ perpendicular to the velocity.

11.12 Note on Notation and Units in Relativistic Kinematics

We replace

$$\begin{pmatrix} S_0 \\ c\vec{p} \\ E \\ mc^2 \\ \frac{\vec{v}}{c} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} P_0 \\ \vec{P} \\ E \\ m \\ \vec{v} \end{pmatrix}$$

We can also write that

$$a \cdot b = a_\alpha b^\alpha = a_0 b_0 - \vec{a} \cdot \vec{b}$$

Lorentz Transformations

Derivation from Postulates

We consider one-dimensional relative movement for two frames K and K'. So, $y' = y$, $z' = z$. Frame K' is moving with velocity V with respect to K. We introduce γ so that

$$x' = \gamma(x - vt) \tag{1}$$

By the principle of relativity, the inverse transformation must have the same form, with $v \to -v$:

$$x = \gamma(x' + vt') \tag{2}$$

At $t = t' = 0$, the origins of K and K' are coincided.

We imagine a light pulse propagating along the positive x-axis. After some time t, the light pulse reaches at $x = ct$ in frame K. While in K', it reaches at $x' = ct'$. We consider the postulate that the speed of light is invariant in all inertial frames.

$$\begin{aligned} x' = ct' &= \gamma(x - vt) = \gamma(ct - vt) \\ x = ct &= \gamma(x' + vt') = \gamma(ct' + vt') \end{aligned}$$

Multiply them:

$$\begin{aligned} c^2tt' &= \gamma^2(ct - vt)(ct' + vt') \\ c^2tt' &= \gamma^2(c^2tt' + cvtt' - vctt' - v^2tt') \\ c^2tt' &= \gamma^2 tt'(c^2 - v^2) \\ c^2 &= \gamma^2(c^2 - v^2) \\ \gamma^2 &= \frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2} \end{aligned}$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3}$$

By substituting $x' = \gamma(x - vt)$ into $x = \gamma(x' + vt')$, we can solve for t'.

$$\begin{aligned} x &= \gamma(\gamma(x - vt) + vt') \\ \frac{x}{\gamma} &= \gamma(x - vt) + vt' \\ vt' &= \frac{x}{\gamma} - \gamma x + \gamma vt \\ t' &= \frac{x}{v\gamma} - \frac{\gamma x}{v} + \gamma t \\ t' &= \gamma t + \frac{x}{v}(\frac{1}{\gamma} - \gamma) = \gamma t + \frac{x}{v}\left(\frac{1 - \gamma^2}{\gamma}\right) \\ t' &= \gamma t + \frac{x}{v\gamma}\left(1 - \frac{1}{1 - v^2/c^2}\right) = \gamma t + \frac{x}{v\gamma}\left(\frac{-v^2/c^2}{1 - v^2/c^2}\right) \\ t' &= \gamma t + \frac{x}{v\gamma}(-\frac{v^2}{c^2}\gamma^2) = \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$

The complete Lorentz transformations are:

$$\begin{cases} t' = \gamma\left(t - \frac{vx}{c^2}\right) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4}$$

4-Vectors

A 4-vector is a quantity $(A_0, A_1, A_2, A_3) \to (A_0, \vec{A})$. We may view it as a complex number with one real part and three imaginary parts:

$$q = w + i\pi + jy + zk \quad \text{where } i^2 = j^2 = k^2 = ijk = -1$$

The Lorentz transformation of 4-vectors: For $A = (A_0, \vec{A}) \to (A_0', \vec{A}')$

$$\begin{cases} A_0' = \gamma(A_0 - \vec{\beta} \cdot \vec{A}) \\ A_{\parallel}' = \gamma(A_{\parallel} - \beta A_0) \\ A_{\perp}' = A_{\perp} \end{cases} \quad \text{where } \vec{\beta} = \frac{\vec{v}}{c}, A_0 = ct, A_0' = ct' \tag{5}$$

The scalar product is invariant:

$$A' \cdot B' = A_0' B_0' - \vec{A}' \cdot \vec{B}' = A_0 B_0 - \vec{A} \cdot \vec{B} \tag{6}$$

and

$$A_0'^2 - |\vec{A}'|^2 = A_0^2 - |\vec{A}|^2 \tag{7}$$

Relativistic Kinematics

4-Velocity and Time Dilation

The 4-velocity is defined as:

$$U = \frac{dx}{d\tau} = \left(\frac{d(ct)}{d\tau}, \frac{d\vec{x}}{d\tau}\right) \tag{8}$$

where τ is the proper time. The 4-velocity is invariant under Lorentz transformation.

We first derive time dilation. From the Lorentz transformation for time:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

For a clock at rest in the moving frame K', its position is $x = vt$.

$$\Rightarrow t' = \gamma\left(t - \frac{v(vt)}{c^2}\right) = \gamma t\left(1 - \frac{v^2}{c^2}\right) = \gamma t \frac{1}{\gamma^2} = \frac{t}{\gamma}$$

We see that t' is the proper time and t is the laboratory time. Thus, the proper time interval is $d\tau = \frac{1}{\gamma} dt$.

The components of the 4-velocity are:

$$U^0 = \frac{d(ct)}{d\tau} = c\frac{dt}{d\tau} = c\gamma$$

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \vec{u}\gamma$$

So, the 4-velocity is $U = (U^0, \vec{U}) = (\gamma c, \gamma\vec{u})$.

Velocity Addition

With the Lorentz transformation differentials:

$$\begin{aligned}
dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) \\
dx &= \gamma(dx' + vdt') \\
dy &= dy' \\
dz &= dz'
\end{aligned}$$

$$\Rightarrow u_x = \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + \frac{v}{c^2}dx')} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{v}{c^2}dx')} = \frac{\frac{dy'}{dt'}}{\gamma(1 + \frac{v}{c^2}\frac{dx'}{dt'})} = \frac{u'_y}{\gamma(1 + \frac{vu'_x}{c^2})}$$

The velocity addition formulas are:

$$\Rightarrow \begin{cases} u_\parallel = \dfrac{u'_\parallel + v}{1 + \frac{vu'_\parallel}{c^2}} \\ u_\perp = \dfrac{u'_\perp}{\gamma\left(1 + \frac{vu'_\parallel}{c^2}\right)} \end{cases} \tag{9}$$

Especially, we see that if $u'_x = c$, then $u_x = \frac{c+v}{1+vc/c^2} = \frac{c+v}{(c+v)/c} = c$.

Relativistic Dynamics

Relativistic Doppler Effect

We then consider the Doppler effect. The observer is in frame S and the light source is in frame S', which is moving with $\vec{v}$ with respect to S. We use the 4-wavevector $(k_0, \vec{k})$ and $(k'_0, \vec{k}')$. The transformation for the time-like component is:

$$k'_0 = \gamma(k_0 - \vec{\beta}\cdot\vec{k})$$

The timelike component is $k_0 = \omega/c$. For a photon, $|\vec{k}| = \omega/c$.

$$\Rightarrow \frac{\omega_s}{c} = \gamma\left(\frac{\omega_o}{c} - \vec{\beta}\cdot\vec{k}\right) = \gamma\frac{\omega_o}{c}(1 - \beta\cos\theta)$$

where $|\vec{k}| = \omega_o/c$. Thus:

$$\omega_o = \frac{\omega_s}{\gamma(1 - \beta\cos\theta)} \tag{10}$$

where ω_o is the observed frequency and ω_s is the fixed (source) frequency. This implies, using $f = \omega/(2\pi)$ and $\gamma = 1/\sqrt{1-\beta^2}$:

$$f_o = f_s\frac{\sqrt{1-\beta^2}}{1 - \beta\cos\theta} \tag{11}$$

4-Momentum and Mass-Energy Equivalence

We further consider the 4-momentum.

$$P = m_0 U = m_0(\gamma c, \gamma\vec{u}) = (\gamma m_0 c, \gamma m_0 \vec{u}) \tag{12}$$

We first prove the mass-energy equation. Let K be the kinetic energy.

$$\begin{aligned}
dK &= \frac{d\vec{p}}{dt}\cdot d\vec{x} = \vec{u}\cdot d\vec{p} \\
&= \vec{u}\cdot d(\gamma m_0\vec{u}) \\
&= m_0(\vec{u}\gamma d\vec{u} + \vec{u}\cdot\vec{u}d\gamma) \\
&= m_0 u\gamma du\left(\frac{\gamma^2 u}{c^2} + 1\right) = m_0 u\gamma^3 du
\end{aligned}$$

And, since $d\gamma = d(1-u^2/c^2)^{-1/2} = \frac{\gamma^3 u}{c^2}du$:

$$d(m_0c^2\gamma) = m_0c^2 d\gamma = m_0c^2\frac{\gamma^3 u}{c^2}du = m_0 u\gamma^3 du = dK$$

Therefore:

$$K = \int_0^K dK' = \int_{u=0}^{u} d(m_0c^2\gamma) = \gamma m_0c^2 - \gamma(u=0)m_0c^2 = \gamma m_0c^2 - m_0c^2$$

$$\Rightarrow \gamma m_0c^2 = K + m_0c^2 \tag{13}$$

which is the total energy E. And we shall define m_0c^2 as the static energy. Hence, the 4-momentum can be written as:

$$P = \left(\frac{E}{c}, \gamma m_0\vec{u}\right) = \left(\frac{E}{c}, \vec{p}\right) \tag{14}$$

We have already seen that in all inertial frames, the scalar product $P \cdot P$ is an invariant:

$$P \cdot P = (P^0)^2 - |\vec{p}|^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2$$

We consider the special case in which $\vec{p} = 0$ (the rest frame). In this frame, $E = m_0c^2$, so that

$$\left(\frac{E}{c}\right)^2 - |\vec{p}|^2 = (m_0c)^2$$

Since this quantity is an invariant, it holds in all frames.

$$E^2 = (pc)^2 + (m_0c^2)^2 \tag{15}$$

(5)

We know that, $\vec{P} = m\vec{u}$

$$E = E(0) + \frac{1}{2}mu^2$$

We assume... under $M(\vec{u})\ldots E \to E(u)$ where $M(0) = m$, $\frac{\partial^2 E(0)}{\partial u_i \partial u_j} = m\delta_{ij}$

(6)

(The tensor part has been discussed in other notes)

For space-time, in all inertial frames the space-time interval is invariant.

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

since

$$(c\Delta t)^2 - (\Delta\vec{x})^2 = (c\Delta t')^2 - (\Delta\vec{x}')^2$$

$$\Rightarrow (ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta \quad \text{metric tensor}$$

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\begin{cases} \frac{\partial}{\partial x^\alpha} = (\frac{\partial}{\partial x^0}, \nabla) \\ \\ \frac{\partial}{\partial x_\alpha} = (\frac{\partial}{\partial x^0}, -\nabla) \end{cases}$$

$$\Delta^\beta = g^{\beta\alpha}\Delta_\alpha$$

$$\partial'_\lambda = g_{\lambda\alpha}\partial^\alpha = \cdots = (-)\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x_1}$$

(7) In matrix notation

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} (ds)^2 &= (ct)^2 - x^2 - y^2 - z^2 \\ &= x^T g x \\ (ds')^2 &= x'^T g x' \end{aligned}$$

We have the transformation $x' = Ax$ and $x'^T = x^T A^T$

$$\Rightarrow x^T g x = x'^T g x' = x^T A^T g A x \quad \text{for all } x$$

$$\Rightarrow A^T g A = g$$

Take the norm: $\det A = \pm 1 \Rightarrow A^T = A^{-1}$ We define $A = e^L$ where L, a 4x4 matrix, is the generator of A. We have: $\tilde{A} g A = g$.

$$\tilde{A} g A = g \cdot g = I$$

$$g \tilde{A} g A A^{-1} = A^{-1}$$

$$A = e^L$$

$$A^{-1} = e^{-L}$$

For infinitesimal transformation: $e^L \approx I + L$

$$\Rightarrow g \tilde{L} g = -L$$

Decomposition

For the elements of L, $L_{\mu\nu}$, with $g = \text{diag}(1, -1, -1, -1)$ we find

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

decomposed

$$S_1 = \begin{pmatrix} 0 & & & \\ & 0 & 0 & 0 \\ & 0 & 0 & -1 \\ & 0 & 1 & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & & & \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & -1 & 0 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & & & \\ & 0 & -1 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix} \quad K_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ & & & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$

$$S_1^2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad S_2^2 = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

$$S_3^2 = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \quad K_1^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$K_2^2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \quad K_3^2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$

where S_1, S_2, S_3 represent the rotations along x,y,z axes while K_1, K_2, K_3 represent the boost along x,y,z axes.

Lorentz Transformation Expression

$$\Rightarrow \begin{cases} L = -w \cdot S - \zeta \cdot K \\ \\ A = e^{-w \cdot S - \zeta \cdot K} \end{cases}$$

For example: only boost along x-axis: $w = 0$,

$$A = e^{-\zeta K_1}$$

$$L = -\zeta K_1$$

By Taylor series

$$A = e^{-\zeta K_1} = I + (-\zeta K_1) + \dots$$

with $K_1^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$, $K_1^3 = K_1,\ K_1^4 = K_1^2, \dots$

$$A = I + \left\{ S_1^2 \frac{\zeta^2}{2!} + S_1^4 \frac{\zeta^4}{4!} + \dots \right\} - \left(S_1 K_1 + \frac{S_1^3}{3!} K_1^3 + \dots \right)$$

And we note that

$$\begin{cases} \cosh\zeta = 1 + \frac{\zeta^2}{2!} + \dots \\ \\ \sinh\zeta = \zeta + \frac{\zeta^3}{3!} + \dots \end{cases}$$

$$\Rightarrow A = I + (\cosh\zeta - 1) K_1^2 - \sinh(\zeta) K_1$$

Similarly, for only rotation along z-axis: $\zeta = 0$,

$$A = e^{-wS_3}$$

$$L = -wS_3$$

$$A = \begin{pmatrix} 1 & & & \\ & \cos w & -\sin w & \\ & \sin w & \cos w & \\ & & & 1 \end{pmatrix}$$

Commutation Relations

By definitions of K_i, S_j, we note that

$$\begin{aligned} [S_i, S_j] &= i\epsilon_{ijk} S_k \\ [S_i, K_j] &= i\epsilon_{ijk} K_k \quad \text{where } [A, B] = AB - BA \\ [K_i, K_j] &= -i\epsilon_{ijk} S_k \end{aligned}$$

Two boosts in two directions produces an additional rotation $\Rightarrow$ Thomas Precession.

1 The interaction energy of the electron spin

The interaction energy is given by

$$U' = -\vec{\mu}\cdot\left(\vec{B} - \frac{\vec{v}\times\vec{E}}{c}\right)$$

where $\vec{\mu} = \frac{ge}{2mc}\vec{S}$, $\vec{L} = \vec{r}\times m\vec{v}$, and $\vec{E} = -\frac{\vec{r}}{er}\frac{dV}{dr}$.

$$= U' = \frac{ge}{2mc}\vec{S}\cdot\vec{B} + \frac{g}{2m^2c^2}(\vec{S}\cdot\vec{L})\frac{1}{r}\frac{dV}{dr}$$

For a spin electron in $\vec{B}'$, $\vec{\mu}$ will be acted by a torque, leading to Larmor precession.

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{Larmor}} = \vec{\mu}\times\vec{B}' \quad \text{where } \vec{B}' = \vec{B} - \frac{\vec{v}\times\vec{E}}{c}$$

$$U' = -\vec{\mu}\cdot\vec{B}' = \frac{g}{2m^2c^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV}{dr}$$

where $g = 2 \Rightarrow U'$ is twice the experiment. The equation $\frac{d\vec{S}}{dt} = \vec{\mu}\times\vec{B}'$ is only valid in an inertial frame, but the spin electron's rest frame is rotating.

$$\Rightarrow \left(\frac{d\vec{G}}{dt}\right)_{\text{nonrot}} = \left(\frac{d\vec{G}}{dt}\right)_{\text{rest frame}} + \vec{\omega}_T\times\vec{G}$$

where $\vec{\omega}_T$ is the Thomas precession frequency. The Larmor precession is given by

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{Larmor}} = \vec{\omega}_L\times\vec{S} = \vec{\mu}\times\vec{B}'$$

$$\begin{aligned}\vec{\omega}_T &= \frac{e}{2mc}(\vec{v}\times\vec{E}) = -\frac{g}{2mc^2}\vec{S}\times(\vec{v}\times\vec{E})\\ &= \frac{g}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}(\vec{L}\cdot\vec{S})\end{aligned}$$

2 Thomas Precession from consecutive boosts

We now consider two boosts: $x' = A_{\text{boost}}(\vec{\beta})x$ and $x'' = A_{\text{boost}}(\vec{\beta}+\delta\vec{\beta})x'$. Then $x'' = A_T x'$, where

$$A_T = A_{\text{boost}}(\vec{\beta}+\delta\vec{\beta})A^{-1}_{\text{boost}}(\vec{\beta}) = A_{\text{boost}}(\vec{\beta}+\delta\vec{\beta})A_{\text{boost}}(-\vec{\beta})$$

By the expression of $A_{\text{boost}}(\vec{\beta})$, we have

$$A_{\text{boost}}(-\vec{\beta}) = \begin{pmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Since we only consider a boost $\vec{\beta}$ in the 1-axis, and we consider $\vec{\beta}+\delta\vec{\beta}$ with $\delta\beta_2, \delta\beta_3 \neq 0$. And we only consider $\vec{B}+\delta\vec{B}$ in the same plane with $\vec{B}$ ($B\perp\delta B$).

$$\begin{aligned}A_{\text{boost}}(\delta\vec{\beta}+\vec{\beta}) &= \begin{pmatrix}\gamma(\gamma+\delta\gamma\beta_1) & -(\gamma\beta_1+\gamma\delta\beta_1) & -\gamma\delta\beta_2 & 0\\ -(\gamma\beta_1+\gamma\delta\beta_1) & \gamma^2\delta\beta_1 & (\frac{\gamma^2}{\gamma+1})\delta\beta_1 & 0\\ -\gamma\delta\beta_2 & (\frac{\gamma^2}{\gamma+1})\delta\beta_2 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\\ = \gamma A_T &= \begin{pmatrix}1 & -\gamma^2\delta\beta_1 & -\gamma\delta\beta_2 & 0\\ \gamma^2\delta\beta_1 & 1 & (\frac{\gamma^2}{\gamma+1})\delta\beta_2 & 0\\ \gamma\delta\beta_2 & -(\frac{\gamma^2}{\gamma+1})\delta\beta_2 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\\ &= I - \left(\tfrac{\gamma^2}{\gamma+1}(\vec{\beta}\times\delta\vec{\beta})\cdot\vec{S} - (\gamma^2\delta\beta_1+\gamma\delta\beta_1)\vec{K}\right)\\ &= A_{\text{boost}}(\delta\vec{\beta})R(\delta\vec{\Omega}) = R(\delta\vec{\Omega})A_{\text{boost}}(\delta\vec{\beta})\end{aligned}$$

where $R(\delta\vec{\Omega}) = I - \delta\vec{\Omega} \cdot \vec{S}$ and $A_{\text{boost}}(\delta\vec{\beta}) = I - \delta\vec{\beta} \cdot \vec{K}$. with $\delta\vec{\beta}' = \gamma^2 \delta\beta_{||} + \gamma\delta\beta_{\perp}$ (velocity). and $\delta\vec{\Omega} = (\frac{\gamma^2}{\gamma+1})(\vec{\beta} \times \delta\vec{\beta}) - \frac{\gamma-1}{v^2}\vec{\beta} \times \delta\vec{v}$ (angle).

Relativistic addition of velocities:

$$\vec{v}_f = \frac{\vec{u} + \vec{w}_{||} + \sqrt{1 - u^2/c^2}\vec{w}_{\perp}}{1 + \vec{u} \cdot \vec{w}/c^2}$$

Let $\vec{u} = c\vec{\beta}$, $\vec{w} = c\delta\vec{\beta}$. $w_{||} = \delta\beta_{||}$, $w_{\perp} = \delta\beta_{\perp}$. $\Rightarrow v_{f||} \approx \gamma^2\delta\beta_{||}$, $v_{f\perp} \approx \gamma\delta\beta_{\perp}$.

If the boost is arbitrary with no rotation, $A = e^{-\vec{\zeta}\cdot\vec{K}}$. We define a matrix M by $K_i K_j K_k$

$$M = \begin{pmatrix} 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{pmatrix}, \quad \text{where } n_1^2 + n_2^2 + n_3^2 = 1$$

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2^2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3^2 \end{pmatrix}$$

$$M^3 = M$$

Let $\vec{\zeta} = \hat{\beta} \tanh^{-1} \beta$.

$$A = e^{-\vec{\zeta}\cdot\vec{M}} = I - \zeta M + \frac{\zeta^2}{2!} M^2 - \frac{\zeta^3}{3!} M^3 + \dots$$

By $\sinh \zeta = \zeta + \frac{\zeta^3}{3!} + \dots$ and $\cosh \zeta = 1 + \frac{\zeta^2}{2!} + \dots$

$$A = I - (\sinh \zeta) M + (\cosh \zeta - 1) M^2$$

and $\sinh \zeta = \sinh\big(\tanh^{-1} \beta\big) = \gamma\beta$ and $\cosh \zeta = \gamma$.

$$\Rightarrow A = I - (\gamma\beta) M + (\gamma - 1) M^2$$

where $n_i = \frac{\beta_i}{\beta}$.

3 Derivation of Thomas Precession Frequency

$$\vec{\omega}_T = -\frac{1}{c^2} \frac{\gamma^2}{\gamma + 1} (\vec{v} \times \vec{a}) = -\frac{\gamma - 1}{v^2} \vec{v} \times \vec{a}$$

where $\vec{a}$ is the laboratory acceleration, which is produced by Coulomb force ($e\vec{E}$). Thus,

$$\vec{\omega}_T = \frac{\gamma^2}{(\gamma + 1)c^2} \frac{1}{m\gamma} \frac{1}{r} \frac{dV}{dr} (\vec{v} \times \vec{r}) = -\frac{\gamma^2}{(\gamma + 1)c^2 m\gamma} \frac{1}{r} \frac{dV}{dr} \vec{L}$$

Since $\vec{L} = m(\vec{r} \times \gamma\vec{v}) = \gamma m(\vec{r} \times \vec{v})$. Hence,

$$\vec{\omega}_T = -\frac{\gamma^2}{(\gamma + 1)c^2 m^2 \gamma} \frac{1}{r} \frac{dV}{dr} \vec{L} = -\frac{\gamma}{(\gamma + 1)m^2 c^2} \frac{1}{r} \frac{dV}{dr} \vec{L}$$

The contribution of Thomas precession to the energy is

$$U_T = \vec{s} \cdot \vec{\omega}_T \approx -\frac{1}{2m^2 c^2} \frac{1}{r} \frac{dV}{dr} \vec{s} \cdot \vec{L} = -\frac{1}{2} U_{\text{total}}$$

4 Covariance of Electrodynamics

Lorentz force: $\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$. By 4-momentum: $p^\alpha = (p_0, \vec{p}) = m(U_0, \vec{U})$. Using proper time τ such that $\gamma d\tau = dt$.

$$\frac{dp^i}{d\tau} = \frac{q}{c}(U_0 E^i + (\vec{U} \times \vec{B})^i) \quad \frac{dp_0}{d\tau} = \frac{q}{c} \vec{U} \cdot \vec{E}$$

$$\frac{dp_0}{dt} = \frac{1}{\gamma} \frac{dp_0}{d\tau} = \frac{q}{\gamma c} \vec{U} \cdot \vec{E} = \frac{q}{c} \vec{v} \cdot \vec{E}$$

$\vec{v} = \frac{\vec{U}}{U_0}$. $\gamma = U_0$.

$$\frac{dp_0}{dE} = \frac{q}{c} \vec{U} \cdot \vec{E}$$

.

5 Relativistic Electrodynamics Formalism

By continuous equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$
we construct $j^\alpha = (c\rho, \vec{j})$. Then, $\partial_\alpha j^\alpha = 0$ where $\partial_\alpha = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$.

The four-dimensional volume element $d^4x = dx^0 d^3x$ is a Lorentz invariant.
$$d^4x' = \left| \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} \right| d^4x = d^4x$$
$\det A = 1$.

By Maxwell equations
$$\Rightarrow \begin{cases} \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{j} \\ \\ \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 4\pi\rho \end{cases}$$
With Lorentz condition, we make the transformation
$$\begin{cases} \vec{A}' = \vec{A} + \nabla\chi \\ \phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t} \end{cases} \qquad \text{to such that} \qquad \begin{cases} \vec{E}' = \vec{E} \\ \vec{B}' = \vec{B} \end{cases}$$
where χ is scalar.

The D'Alembert operator
$$\Box = \partial_\mu \partial^\mu = g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$
which is a Lorentz invariant.

The Lorentz condition:
$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

6 The Electromagnetic Field Tensor

With the condition we thus transform the equation into: $\partial_\beta \partial^\beta A^\alpha = 0$ [where $A^\alpha = (\phi, \vec{A})$]
$$\Leftarrow \left(\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} \right) + \nabla(\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}) = \frac{4\pi}{c}\vec{j} \quad \text{coupled term source}$$
And
$$\begin{cases} \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$
The components:
$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$
where
$$\begin{cases} A^1 = A_x \\ A^2 = A_y \\ A^3 = A_z \\ A^0 = \phi \end{cases}$$
We now construct the tensor of field. We find $E_x, E_y, E_z, B_x, B_y, B_z$ can all be written in the form like $-(\partial^0 A^1 - \partial^1 A^0)$, $-(\partial^2 A^3 - \partial^3 A^2)$.
$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad \text{(antisymmetric)}$$

$$\begin{cases} F^{01} = \partial^0 A^1 - \partial^1 A^0 = -E_x \\ F^{02} = -E_y \\ F^{03} = -E_z \end{cases}$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -(\frac{\partial}{\partial x}(A_y) - (-\frac{\partial}{\partial y} A_x)) = -(\nabla \times \vec{A})_z = -B_z$$

$$F^{23} = -B_x$$

$$F^{31} = -B_y$$

By antisymmetry

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} \quad \text{where } g_{\alpha\beta} = (1, -1, -1, -1)$$

($F^{\alpha\beta} = +F_{\alpha\beta}$, $F^{0i} = -F_{0i}$)

The dual tensor

$$\mathcal{G}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

where $\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \\ 0 & \text{any two equal} \end{cases}$

By Maxwell equations

$$\nabla \cdot \vec{E} = 4\pi\rho, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}, \quad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

In terms of $F^{\alpha\beta}, \mathcal{G}^{\alpha\beta}$:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta \quad \text{(summation on } \alpha\text{)}$$

$$\partial_\alpha \mathcal{G}^{\alpha\beta} = 0$$

By Bianchi identity

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

7 Fields in Media and Transformations

In fields with medium we transform

$$F^{\alpha\beta} = (\vec{E}, \vec{B}) \Rightarrow G^{\alpha\beta} = (\vec{D}, \vec{H})$$

Thus: $\partial_\alpha G^{\alpha\beta} = 0$, $\partial_\alpha F^{\alpha\beta} = 0$.

In conductors:

How source produces fields $\begin{cases} \nabla \cdot \vec{E} = \rho/\epsilon_0 \\ \nabla \times \vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} \end{cases} \rightarrow \partial_\beta F^{\alpha\beta} = \frac{4\pi}{c} j^\beta$ ($\downarrow$ with media $E \& B \rightarrow G$)

The structure of fields themselves $\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{cases} \rightarrow \partial_\beta \mathcal{G}^{\alpha\beta} = 0$

(10) Transformation of Electromagnetic Fields

We know that

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}F^{\gamma\delta}$$

$$\Rightarrow F' = AF\tilde{A} \quad \text{A is the Lorentz transformation}$$

Consider one example that K' is moving at v along positive x with respect to K.

$$\begin{aligned} x' &= \gamma(x - vt) & x^\alpha &= (ct, x, y, z) \\ y' &= y, z' = z & x'^\alpha &= (ct', x', y', z') \\ t' &= \gamma(t - vx/c^2) & \text{and } x'^\alpha &= \Lambda^\alpha_\beta x^\beta \end{aligned} \quad \Rightarrow A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} E'_1 = E_1, & E'_2 = \gamma(E_2 - \beta B_3), & B'_2 = \gamma(B_2 + \beta E_3) \\ B'_1 = B_1, & E'_3 = \gamma(E_3 + \beta B_2), & B'_3 = \gamma(B_3 - \beta E_2) \end{matrix}$$

Example: Field of a Moving Charge

As an example, consider the fields produced by a moving charge with q, v. In K' where q is at rest

$$\vec{B}' = 0, \quad \vec{E}' = \frac{q\vec{r'}}{r'^3}$$

$$r' = \sqrt{(x'_1)^2 + (x'_2)^2 + (x'_3)^2} = \sqrt{\gamma(1 - v/c^2)t^2 + \dots}$$

since $\vec{r'} = (\gamma vt, b, 0)$.

From $K' \to K$

$$\begin{cases} E_1 = E'_1 \\ E_2 = \gamma(E'_2 + vB'_3) = \gamma E'_2 \\ E_3 = \gamma(E'_3 - vB'_2) = 0 \\ B_1 = B'_1 = 0 \\ B_2 = \gamma(B'_2 - \frac{v}{c^2}E'_3) = 0 \\ B_3 = \gamma(B'_3 + \frac{v}{c^2}E'_2) = \gamma\beta E'_2 \end{cases} \qquad \begin{aligned} E_2 &= \gamma(E'_2 + (-B'_3)) = \gamma E'_2 \\ B_2 &= \gamma(B'_2 - (-E'_3)) = 0 \\ B_3 &= \gamma(B'_3 + (-E'_2)) = \gamma\beta E'_2 \end{aligned}$$

This shows that magnetic fields are actually a relativistic effect. The electric field is amplified by γ times in the direction perpendicular to $\vec{v}$.