Hamilton-Jacobi Theory and Action-Angle Variables

0.1 10.1 The Hamilton-Jacobi Equation for Hamilton's Principal Function

We ensure the new variables to be constant in time by requiring the transformed Hamiltonian K to be zero. The equations of motion are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0$$
$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0$$

And K, P_i are generated by

$$K = H + \frac{\partial F}{\partial t}$$

Only to zero if: $H(q,p,t) + \frac{\partial F}{\partial t} = 0$.

For convenience, we take F as a function of q_i (the new momenta) and time: $F_2(q, P, t)$. By the transformation $p_i = \frac{\partial F_2}{\partial q_i}$, we obtain:

$$H\left(q_1, \ldots, q_n; \frac{\partial F_2}{\partial q_1}, \ldots, \frac{\partial F_2}{\partial q_n}; t\right) + \frac{\partial F_2}{\partial t} = 0$$

This is the Hamilton-Jacobi equation.

We denote F_2 by S which is the Hamilton's principal function

$$F_2 = S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_{n+1}; t)$$

where α_i are $n+1$ independent constants.

We note that if S is a solution to the H-J equation, so is $S + \alpha$ (where α is a constant). Hence the complete solution to the equation can be written as

$$S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n, t)$$

We can take the n constants to be the new momenta $P_i = \alpha_i$. And the n transformation equations

$$p_i = \frac{\partial S(q, P, t)}{\partial q_i}$$
$$Q_i = \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}$$

We may also represent q_j in terms of α, β, t

$$q_j = q_j(\alpha, \beta, t)$$

so as for P_i: $P_i = P_i(\alpha, \beta, t)$. This gives solutions for Hamilton's equations of motion.

The total time derivative of S is

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \quad \text{since } P_i's \text{ are constants}$$

$$\frac{dS}{dt} = p_i \dot{q}_i - H = L$$

So that

$$S = \int L dt + \text{constant}$$

which is another expression of the Hamilton's principle.

When Hamiltonian doesn't depend explicitly upon time

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t$$

where W is Hamilton's characteristic function such that:

$$\frac{dW}{dt} = \frac{\partial W}{\partial q_i}\dot{q}_i$$

We have : $p_i = \frac{\partial W}{\partial q_i}$. Hence,

$$\frac{dW}{dt} = p_i\dot{q}_i$$

This implies

$$W = \int p_i\dot{q}_i dt = \int p_i dq_i$$

0.2 10.2 The Harmonic Oscillator Problem as an Example

1. For a one-dimensional harmonic oscillator, the Hamiltonian is

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) = E$$

where $k = \sqrt{\frac{k}{m}}$ is the force constant.

With $p = \frac{\partial S}{\partial q}$, we have the requirement the new Hamiltonian must vanish

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

with the solution of the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t$$

where α is constant. We obtain:

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

By the relation $\frac{\partial S}{\partial t} + H = 0$ which reduces to $H = \alpha$. We thus have

$$W = \sqrt{2m\alpha}\int dq\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}$$

This implies

$$S = \sqrt{2m\alpha}\int dq\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} - \alpha t$$

With the transformation equation

$$\beta' = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t$$

we can integrate to give

$$t + \beta' = \frac{1}{\omega}\arcsin\left(q\sqrt{\frac{m\omega^2}{2\alpha}}\right)$$

which yields:

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \beta)$$

and

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}$$
$$= \sqrt{2m\alpha(1 - \sin^2(\omega t + \beta))}$$
$$= \sqrt{2m\alpha}\cos(\omega t + \beta)$$

With the initial condition q_0, p_0 at $t = 0$

$$2m\alpha = p_0^2 + m^2\omega^2 q_0^2$$
$$\tan\beta = \frac{m\omega q_0}{p_0}$$

2. By substitution of q into Hamilton's principal function

$$S = 2\alpha \int \cos^2(\omega t + \beta) dt - \alpha t = 2\alpha \int \left(\frac{\cos(2(\omega t + \beta)) + 1}{2} \right) dt$$

Now the Lagrangian is

$$L = \frac{1}{2m}(p^2 - m^2\omega^2 q^2)$$
$$= \alpha(\cos^2(\omega t + \beta) - \sin^2(\omega t + \beta))$$
$$= \alpha\left(\cos^2(\omega t + \beta) - (1 - \cos^2(\omega t + \beta))\right)$$
$$= \alpha(2\cos^2(\omega t + \beta) - 1)$$

3. We then consider the two-dimensional anisotropic harmonic oscillator

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2)$$

where $\omega_i = \sqrt{\frac{k_i}{m}}$, $i = x, y$.

We separate the principal function into two characteristic functions

$$S(x, y, \alpha, t) = W_x(x, \alpha) + W_y(y, \alpha) - \alpha t$$

The H-J equation becomes

$$\frac{1}{2m}\left[\left(\frac{\partial W_x}{\partial x}\right)^2 + m^2\omega_x^2 x^2 + \left(\frac{\partial W_y}{\partial y}\right)^2 + m^2\omega_y^2 y^2\right] = \alpha$$

The y-part:

$$\frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 + \frac{1}{2}m\omega_y^2 y^2 = \alpha_y$$

Which yields

$$\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2 + \frac{1}{2}m\omega_x^2 x^2 = \alpha_x = \alpha - \alpha_y$$

This implies

$$x = \sqrt{\frac{2\alpha_x}{m\omega_x^2}}\sin(\omega_x t + \beta_x)$$
$$y = \sqrt{\frac{2\alpha_y}{m\omega_y^2}}\sin(\omega_y t + \beta_y)$$
$$p_x = \sqrt{2m\alpha_x}\cos(\omega_x t + \beta_x)$$
$$p_y = \sqrt{2m\alpha_y}\cos(\omega_y t + \beta_y)$$

The total energy $E = \alpha_x + \alpha_y = \alpha$.

0.3 The Linear Case: $\beta = 0$

$$\Rightarrow y = \frac{\sqrt{4\alpha}}{m\omega^2}\sin\omega t, \quad p_y = \sqrt{4m\alpha}\cos\omega t$$

$$\theta = \frac{\pi}{4}, \quad p_\theta = 0$$

The other case: $\beta = \frac{\pi}{2}$

$$\Rightarrow r = r_0 = \sqrt{\frac{2\alpha}{m\omega^2}}, \quad p_r = 0$$

$$\theta = \omega t, \quad p_\theta = mr_0^2\omega$$

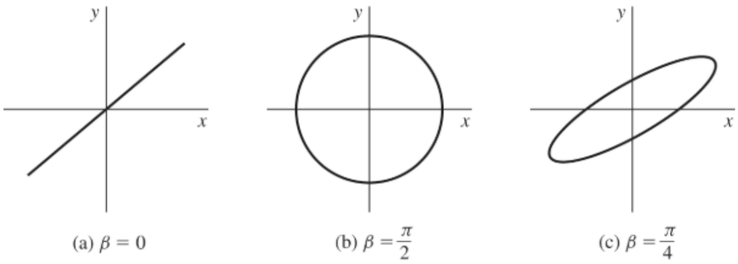

FIGURE 10.1 The two limiting cases (a) and (b) for the harmonic oscillator and an intermediate example (c).

Figure 1:

0.4 10.3 The Hamilton-Jacobi Equation for Hamilton's Characteristic Function

1. If H doesn't involve time explicitly then the restricted H-J equation

$$H(q_i, \frac{\partial W}{\partial q_i}) = \alpha_1$$

If the generating function for a canonical transformation is $W(q, P)$, then the equations of transformation are

$$p_i = \frac{\partial W}{\partial q_i}, \quad Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i}$$

Thus

$$H(q_i, p_i) = \alpha_1$$

which is the new canonical momentum. It also becomes:

$$H(q_i, \frac{\partial W}{\partial q_i}) = \alpha_1$$

Since W doesn't involve time, it follows that $K = \alpha_1$.

2. We again consider the 2-dimensional harmonic oscillator which is isotropic so that $k_x = k_y = k$ and $\omega_x = \omega_y = \omega$. And use polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$p_x = m\dot{x}, \quad p_y = m\dot{y}, \quad p_\theta = mr^2\dot{\theta}$$

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + m^2\omega^2r^2)$$

which is cyclic in θ.

The principal function

$$S(r, \theta, \alpha, \alpha_\theta) = W_r(\alpha, r) + W_\theta(\theta, \alpha_\theta) - \alpha t$$

where

$$W_r(r,\alpha) + \theta\alpha_\theta - \alpha t$$

a cyclic coordinate q_i always has the characteristic function component $W_{q_i} = q_i\alpha_i$.

The canonical momentum p_θ with θ is given by

$$p_\theta = \frac{\partial S}{\partial \theta} = \alpha_\theta$$

By substitution we find

$$\frac{1}{2m}\left((\frac{\partial W_r}{\partial r})^2 + \frac{\alpha_\theta^2}{r^2} \right) + \frac{1}{2}m\omega^2 r^2 = \alpha$$

We write the Cartesian coordinate solution

$$x = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \epsilon), \quad p_x = \sqrt{2m\alpha}\cos(\omega t + \epsilon)$$

$$y = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\omega t, \quad p_y = \sqrt{2m\alpha}\cos\omega t$$

And polar counterparts

$$r = \sqrt{\frac{2\alpha}{m\omega^2}}\sqrt{\sin^2\omega t + \sin^2(\omega t + \epsilon)}, \quad p_r = m\dot{r}$$

$$\theta = \tan^{-1}\left[\frac{\sin\omega t}{\sin(\omega t + \epsilon)}\right], \quad p_\theta = mr^2\dot{\theta}$$

3. The momenta conjugate to the cyclic coordinates are all constant:

$$p_i = \frac{\partial k}{\partial \dot{q}_i} = 0, \quad P_i = \alpha_i$$

$$\Rightarrow \dot{Q}_i = \frac{\partial k}{\partial \alpha_i} = 1, \quad i \neq 1$$
$$= 0, \quad i \neq 1$$

with the solutions

$$Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1}$$

$$Q_i = \beta_i = \frac{\partial W}{\partial \alpha_i}, \quad i \neq 1$$

We denote the transformed momenta as γ_i

$$\dot{Q}_i = \frac{\partial k}{\partial \gamma_i} = v_i$$

where v_i are function of γ_i

$$\Rightarrow Q_i = v_i t + \beta_i$$

4. In the conditions following, we can solve by either Hamilton's principal or characteristic function. When the Hamiltonian is any function of t,

$$H(q,p,t)$$

is conserved

$$H(q,p) = \text{constant}$$

We seek canonical transformation such that

(a) all Q_i, P_i are constants of motion
(b) all P_i are constants

We demand that the new Hamiltonian

(a) shall vanish

$$K = 0$$

(b) shall be cyclic in all the coordinates

$$K = H(P_i) = \alpha_1$$

5. Then the new equations of motion are

(a)

$$\dot{Q}_i = \frac{\partial k}{\partial P_i} = 0, \quad \dot{P}_i = -\frac{\partial k}{\partial Q_i} = 0$$

(b)

$$\dot{Q}_i = \frac{\partial k}{\partial P_i} = v_i, \quad \dot{P}_i = -\frac{\partial k}{\partial Q_i} = 0$$

with the solutions

(a)

$$Q_i = \beta_i, \quad P_i = \gamma_i^2$$

(b)

$$Q_i = v_i t + \beta_i, \quad P_i = \gamma_i$$

The generating functions are

(a) Principal Function $S(q, P, t)$

(b) Characteristic Function $W(q, P)$

Satisfying the H-J equations:

(a)

$$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$$

(b)

$$H(q, \frac{\partial W}{\partial q}) - \alpha_1 = 0$$

A complete solution contains

(a) n nontrivial constants of integration $\alpha, \ldots, \alpha_n$

(b) $n-1$ nontrivial constants of integration with $\alpha_1, \ldots, \alpha_n$

The new constant momenta $P_i = \gamma_i$ can be chosen as any independent functions of n constants of integration.

(a)

$$P_i = \gamma_i(\alpha_1, \ldots, \alpha_n)$$

(b)

$$P_i = \gamma_i(\alpha_1, \ldots, \alpha_n)$$

So the complete solutions to H-J equation can be considered as functions of new momenta.

(a)

$$S = S(q_i, \gamma_i, t)$$

(b)

$$W = (q_i, \gamma_i)$$

6. In particular, we choose one-half of the transformations equations as γ_i, one-half of the

(a)

$$P_i = \frac{\partial S}{\partial q_i}$$

(b)

$$P_i = \frac{\partial W}{\partial q_i}$$

The other half

(a)

$$Q_i = \frac{\partial S}{\partial \gamma_i} = \beta_i$$

(b)

$$Q_i = \frac{\partial W}{\partial \gamma_i} = v_i(\gamma)t + \beta_i$$

The generating functions are related by

$$S(q, P, t) = W(q, P) - \alpha_1 t$$

10.4 Separation of Variables in the H-J Equation

1. We take q_i as the separable coordinates. Thus we can use Hamilton's principal function of the form

$$S = \sum_i S_i(q_i; \alpha_1, \ldots, \alpha_n; t)$$

to split Hamilton-Jacobi equation into n equations

$$H_i\left(q_i, \frac{\partial S_i}{\partial q_i}; \alpha_1, \ldots, \alpha_n; t\right) + \frac{\partial S_i}{\partial t} = 0$$

If the Hamiltonian doesn't depend upon time for each S_i we have

$$S_i(q_i; \alpha_1, \ldots, \alpha_n; t) = W_i(q_i; \alpha_1, \ldots, \alpha_n) - \alpha_i t$$

which provide n restricted H-J equations

$$H_i\left(q_i, \frac{\partial W_i}{\partial q_i}; \alpha_1, \ldots, \alpha_n\right) = \alpha_i$$

where α_i are separation constants.

10.5 Ignorable Coordinates and the Kepler Problem

1. We suppose the cyclic coordinate is q_1 and the conjugate momentum p_1 is a constant, say γ. The H-J equation for W is

$$H\left(q_2, \ldots, q_n; \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1$$

We try a separated solution of the form

$$W = W_1(q_1, \alpha) + W'(q_2, \ldots, q_n, \alpha)$$

where W_1 is the solution to

$$p_1 = \gamma = \frac{\partial W_1}{\partial q_1}$$

and W' is the only function in the H-J equation. The solution is $W_1 = \gamma q_1$. Thus, $W = W' + \gamma q_1$.

2. We let s of the coordinates except be non-cyclic. The Hamiltonian is of the form:

$$H(q_1, \ldots, q_s, \frac{\partial S}{\partial q_{s+1}}, \ldots, \frac{\partial S}{\partial q_n})$$

The characteristic function

$$W(q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{s} W_i(q_i; \alpha_1, \ldots, \alpha_n) + \sum_{j=s+1}^{n} q_j \alpha_j$$

There are s H-J equations

$$H(q_i; \frac{\partial W_i}{\partial q_i}; \alpha_1, \ldots, \alpha_n) = \alpha_i$$

3. In general, q_j is separable if the conjugate momentum p_j can be segregated in the Hamiltonian in to some function $f(q_j, p_j)$. We seek a trial solution

$$W = W_j(q_j, \alpha) + W'(q_i, \alpha)$$

where q_i represents all q's except q_j. then the H-J equation is

$$H\left(q_i, \frac{\partial W'}{\partial q_i}, f\left(q_j, \frac{\partial W_j}{\partial q_j}\right)\right) = \alpha_1$$

We can solve for f:

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = g\left(q_i, \frac{\partial W'}{\partial q_i}, \alpha_1\right)$$

This can hold only if

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = \alpha_j = g\left(q_i, \frac{\partial W'}{\partial q_i}\right)$$

where α_j is independent of all q's.

4. The Staeckel conditions for the separation of the H-J equations.

1. The Hamiltonian is conserved.
2. The Lagrangian is no more than a quadratic function of the generalized velocity.

$$H = \frac{1}{2}(\vec{p} - \vec{a})T^{-1}(\vec{p} - \vec{a}) + V(q)$$

3. $\vec{a}$ is elements a_i are functions of the corresponding coordinate only. $a_i = a_i(q_i)$.
4. $V(q) = \sum_i \frac{V_i(q_i)}{\Phi_{i1}}$
5. $\Phi_{ij}^{-1} = \frac{1}{T_{ii}}$ (no summation at i) where

$$\left(\frac{\partial W_i}{\partial q_i} - a_i\right)^2 = 2\delta_{ik}\Phi_{ij}\gamma_j$$

with γ a constant unspecified vector.

For a particle in an external force field. the matrix T is diagonal, the elements of T^{-1}

$$\Phi_{ij}^{-1} = \frac{1}{T_{ii}} = \frac{1}{m} \quad \text{(no summation)}$$

If the Staeckel conditions are satisfied, then Hamilton's characteristic function is completely separable.

$$W(q) = \sum_i W_i(q_i)$$

with W_i satisfying

$$\left(\frac{\partial W_i}{\partial q_i} - a_i\right)^2 = -2V_i(q_i) + 2\Phi_{ij}\gamma_j$$

where γ_j are constants of integration.

5. We first consider the central force problem in terms of the polar coordinates (r, ψ) in the plane of the orbit.

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\psi^2}{r^2}) + V(r)$$

which is cyclic in ψ. Hamilton's characteristic function

$$W = W(r) + \alpha_\psi \psi$$

where α_ψ is the constant angular momentum p_ψ conjugate to ψ. The H-J equation becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\psi^2}{r^2} + 2mV(r) = 2m\alpha_1$$

where α_1 is the total energy. Then we obtain the solution

$$\frac{dW_r}{dr} = \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\psi^2}{r^2}}$$

$$\Rightarrow W = \int dr \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\psi^2}{r^2}} + \alpha_\psi \psi$$

10.6 Action-Angle Variables in Systems of one Degree of Freedom

1. We now consider a conserved system with one degree of freedom so that

   $$H(q, p) = \alpha_1$$

   Solving for the momentum, we have that

   $$p = p(q, \alpha_1)$$

   Two types of periodic motion may be distinguished

   (a) p, q are both functions of time that are periodic with the same frequency.
   $\Rightarrow$ libration (a)

   (b) p is some periodic function of q with period q_0.
   $\Rightarrow$ rotation (b)

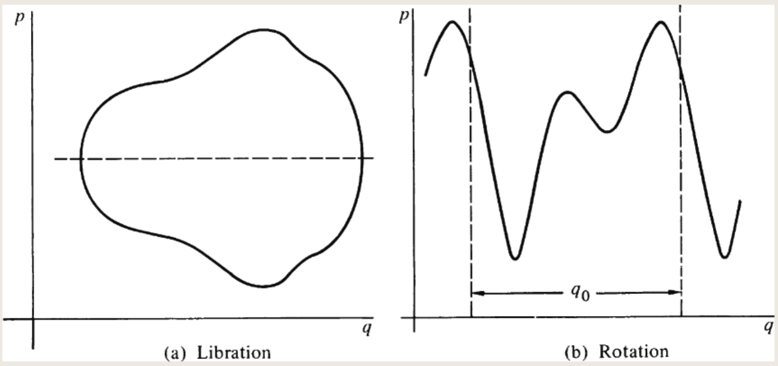

FIGURE 10.2 Orbit of the system point in phase space for periodic motion of one-

Figure 2:

We may consider the simple pendulum where q is the angle of deflection θ. The equation with the length l and the potential energy taken as 0 at the point of suspension gives the total energy

$$E = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta \quad \text{or} \quad p_\theta = \sqrt{2m^2l^2(E + mgl\cos\theta)}$$

In matrix notation, we consider a diagonal T

$$T_{ii} = m, \quad T_{\theta\theta} = ml^2, \quad T_{\phi\phi} = mr^2\sin^2\theta$$

$$\Rightarrow V(q) = V_r(r) + \frac{V_\theta(\theta)}{r^2} + \frac{V_\phi(\phi)}{r^2\sin^2\theta}$$

If $E < mgl$, then the physical motion of the system can only occur for $|\theta| < \arccos\left(-\frac{E}{mgl}\right) = \theta_0$.
$\Rightarrow$ libration

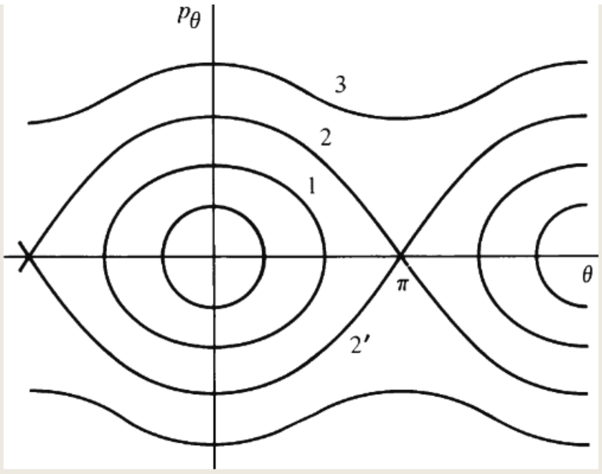

FIGURE 10.3 Phase space orbits for the simple pendulum.

Figure 3:

If $E > mgl$, all θ correspond to physical motion and θ can increase without limit.
$\Rightarrow$ rotation

2. For both types of motion, we introduce a new variable J to replace α_1 as the transformed constant momentum. The action variable:

$$J = \oint p\,dq$$

By $p = p(q, \alpha_1)$ we note that J is always some function of α_1.

$$\alpha_1 = H = H(J)$$

Hamilton's characteristic function can be written

$$W = W(q, J)$$

And the generalized coordinate conjugate to J known as the angle variable is defined by

$$w = \frac{\partial W}{\partial J}$$

The equation of motion for w is

$$\dot{w} = \frac{\partial H(J)}{\partial J} = \nu(J)$$

where ν is a constant function of J only. The solution is

$$w = \nu t + \epsilon$$

3. Consider the change in w as q goes through a complete cycle of libration or rotation.

$$\Delta w = \oint \frac{\partial w}{\partial q}\, dq$$

$$= \oint \frac{\partial^2 W}{\partial q \partial J}\, dq$$

Since J is a constant,

$$\Delta w = \frac{d}{dJ} \oint p\, dq = \frac{dJ}{dJ} = 1$$

Which means that w changes by unity as q goes through a complete period. If τ is the period of q, then

$$\Delta w = 1 = \nu\tau$$

where $\nu = \frac{1}{\tau}$ is the reciprocal of the period which is the frequency associated with q.

4. We consider the linear harmonic oscillator

$$J = \oint p\, dq = \oint \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq$$

where α is the constant total energy and $\omega = \sqrt{\frac{k}{m}}$. By substitution $q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin\theta$, we obtain

$$J = \frac{2\alpha}{\omega} \int_0^{2\pi} \cos^2\theta\, d\theta = \frac{2\pi\alpha}{\omega}$$

or

$$\alpha = H = \frac{J\omega}{2\pi}$$

The frequency is

$$\frac{\partial H}{\partial J} = \nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

By letting $2\pi w = \omega t + \epsilon$,

$$q = \sqrt{\frac{J}{\pi m \omega}} \sin(2\pi w)$$

$$p = \sqrt{\frac{m\omega J}{\pi}} \cos(2\pi w)$$

Further Examination of the Central Force Problem

The transformation equations are

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \int \frac{m\,dr}{\sqrt{2m(\alpha_1 - V) - \frac{\alpha_\theta^2}{r^2}}}$$

$$\beta_2 = \frac{\partial W}{\partial \alpha_\theta} = -\int \frac{\alpha_\theta\,dr}{r^2\sqrt{2m(\alpha_1 - V) - \frac{\alpha_\theta^2}{r^2}}} + \psi$$

By $u = \frac{1}{r}$, we can reduce to

$$\psi = \beta_2 - \int \frac{du}{\sqrt{\frac{2m(\alpha_1 - V)}{l^2} - u^2}}$$

6. We further examine the same central force problem in spherical polar coordinates.

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(r)$$

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$$

where ϕ is cyclic $\Rightarrow W_\phi = \alpha_\phi \phi$. The H-J equation

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta}\right] + 2mV(r) = 2mE$$

We note that

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2$$

$$\Rightarrow \left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m(E - V(r))$$

Note that $\alpha_\phi = p_\phi = \frac{\partial W_\phi}{\partial \phi}$. And $p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = \alpha_\theta^2$. So that

$$H = \frac{1}{2m}\left(p_r^2 + \frac{\alpha_\theta^2}{r^2}\right) + V(r)$$

We see that $\alpha_\theta = p_\theta = L$, $\alpha_1 = E$.

10.7 Action-Angle Variables for Completely Seperable Systems

(No summation)

Complete separability

Complete separability means:

$$p_i = \frac{\partial W_i(q_i; \alpha_1, \ldots, \alpha_n)}{\partial q_i}$$
$$= p_i(q_i; \alpha_1, \ldots, \alpha_n)$$

The action variables J_i are defined by

$$J_i = \oint p_i dq_i$$

For all types of rotations, the period is 2π, then $J_i = 2\pi p_i$ for all cyclic variables whose conjugate momentums are constants.

It can also be written as

$$J = \oint \frac{\partial W_i(q_i; \alpha_1, \ldots, \alpha_n)}{\partial q_i} dq_i$$

Expressing α_i's in terms of J_i's

$$W = W(q_1, \ldots, q_n; J_1, \ldots, J_n) = \sum_i W_j(q_j; J_1, \ldots, J_n)$$

while the Hamiltonian

$$H = \alpha_1 = H(J_1, \ldots, J_n)$$

In the system of one degree of freedom, we define conjugate angle variables w_i by

$$w_i = \frac{\partial W}{\partial J_i} = \sum_j \frac{\partial W_j(q_j; J_1, \ldots, J_n)}{\partial J_i}$$

or

$$w_i = w_i(q_1, \ldots, q_n; J_1, \ldots, J_n)$$

and the equations of motion:

$$\dot{w}_i = \frac{\partial H(J_1, \ldots, J_n)}{\partial J_i} = \nu_i(J_1, \ldots, J_n)$$

where ν_i's are constants.

$$\Rightarrow w_i = \nu_i t + \delta_i$$

Infinitesimal change

We consider an infinitesimal change denoted by δ.

$$\delta W_i = \sum_j \frac{\partial W_i}{\partial J_j} dJ_j = \sum_j \int \frac{\partial p_i(q_i, J)}{\partial J_j} dq_i dJ_j$$

$$= \frac{\partial}{\partial J_j} \oint p_i(q_i, J) dJ_j$$

The total change in w_i is therefore

$$\Delta w_i = \frac{\partial}{\partial J_i} \oint p_j(q_j, J) dJ_j$$

Since the J_i's are all independent constants,

$$\Delta w_i = m_i$$

or $\Delta \vec{w} = \vec{m}$.

We assume all separable motions are libration. We define $\vec{J}$ as a function of $\vec{w}$ so that $\Delta \vec{J} = 0$ corresponds to $\Delta \vec{w} = \vec{m}$. Since the number of cycles in the chosen motion of q_j are arbitrary, m_j can be taken as zero except for $m = 1$. And all components of q remain unchanged or return to their original values. $\Rightarrow$ some periodic functions of w_i whose periods are unity.

Fourier Expansion

By Fourier expansion:

$$q_k = \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} a^{(k)}_{j_1, \ldots, j_n} e^{2\pi i(j_1 w_1 + \cdots + j_n w_n)}$$

(Libration) where we can view j's as a vector in the same n-dimensional space with $\vec{w}$.

$$\Rightarrow q_k = \sum_j a^{(k)}_j e^{2\pi i \vec{j} \cdot \vec{w}} \quad \text{(Libration)}$$

If we write: $\vec{w} = \vec{\nu} t + \vec{\delta}$

$$q_k(t) = \sum_j a^{(k)}_j e^{2\pi i \vec{j} \cdot (\vec{\nu} t + \vec{\delta})} \quad \text{(Libration)}$$

where

$$a_j^{(k)} = \int_0^1 \cdots \int_0^1 q_k(\vec{w}) e^{-2\pi i \vec{j}\cdot\vec{w}} (d\vec{w})$$

We note that w_k increased by unity during the cycle so $q_k - w_k q_k$ returns to its initial value.

$$q_k - w_k q_k = \sum_j a_j^{(k)} e^{2\pi i \vec{j}\cdot\vec{w}} \quad \text{(Rotation)}$$

or

$$q_k = Q_k(v_k t + B_k) + \sum_j a_j^{(k)} e^{2\pi i \vec{j}\cdot(\vec{\nu}t+\vec{\delta})}$$

Thus, for multiply periodic functions of the w's

$$f(q,p) = \sum_j b_j e^{2\pi i \vec{j}\cdot\vec{w}} = \sum_j b_j e^{2\pi i \vec{j}\cdot(\vec{\nu}t+\vec{\delta})}$$

2D Anisotropic Harmonic Oscillator

We consider a 2-dimensional anisotropic harmonic oscillator. In Cartesian coordinates, the Hamiltonian is

$$H = \frac{1}{2m}[(p_x^2 + p_y^2 + m^2(\nu_x^2 x^2 + \nu_y^2 y^2)) + (p_x'^2 + p_y'^2 + m^2(\nu_x'^2 x'^2 + \nu_y'^2 y'^2))]$$

Suppose the coordinates are rotated by α about the z-axis.

$$x' = \frac{1}{\sqrt{2}}[x_0 \cos\alpha\pi(\nu_x t + \beta_x) + y_0 \cos\alpha\pi(\nu_y t + \beta_y)]$$
$$y' = \frac{1}{\sqrt{2}}[y_0 \cos\alpha\pi(\nu_y t + \beta_y) - x_0 \cos\alpha\pi(\nu_x t + \beta_x)]$$

If ν_x/ν_y is a rational number, these two expressions will be commensurate, corresponding to closed Lissajous figures.

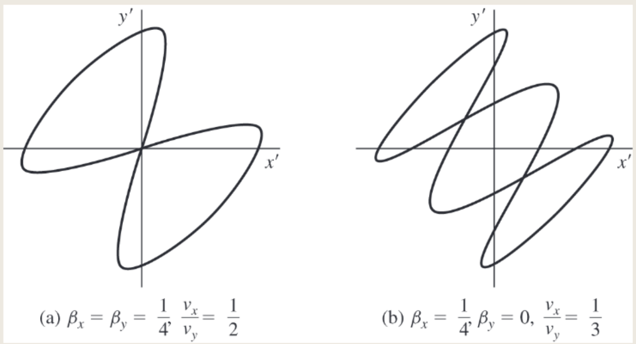

FIGURE 10.4 Lissajous figures for Eq. (10.118). (a) $\beta_x = \beta_y = \frac{1}{4}, \frac{v_x}{v_y} = \frac{1}{2}$ (b) $\beta_x = \frac{1}{4}$, $\beta_y = 0, \frac{v_x}{v_y} = \frac{1}{3}$.

Figure 4:

As the time interval T containing m complete cycles of q_k plus a fraction of a cycle increases indefinitely.

$$\lim_{T\to\infty} \frac{m}{T} = \nu_k$$

As q_k goes through a complete cycle, when w_k changes by a unity, the characteristic function increases by J_k.

$$W' = W - \sum_k w_k J_k$$

remains unchanged when w_k is increased by unity where $w_k = \frac{\partial W}{\partial J_k}$.

Degeneracy Conditions

We now consider m degeneracy conditions

$$\sum_{k=1}^{n} j_{ki}\nu_i = 0, \quad k = 1, \ldots, m.$$

Considering $(w, J) \to (w', J')$, the generating function is

$$F_2 = \sum_{i=1}^{m}\sum_{k=1}^{n} J_i' j_{ki} w_k + \sum_{k=m+1}^{n} J_k' w_k$$

$$w_k' = \sum_{i=1}^{m} J_i' j_{ki}, \quad k = 1, \ldots, m$$

$$w_k' = w_k, \quad k = m+1, \ldots, n$$

The new frequencies

$$\nu_i' = \dot{w}_k' = \sum_{j=1}^{n} j_{ki}\nu_j = 0, \quad k = 1, \ldots, m$$

$$\nu_k' = \nu_k, \quad k = m+1, \ldots, n$$

where w_k' are the angle variables. And the corresponding constant action variables

$$J_i = \sum_{k=1}^{m} j_{ik} J_k' + \sum_{k=m+1}^{n} J_k' \delta_{ki}$$

Since $\nu_i' = \frac{\partial H}{\partial J_i'}$, the Hamiltonian must be independent of J_i'. As for $(w, p) \to (w', J')$, we have

$$w_i' = \frac{\partial W}{\partial J_i'}$$