Canonical Transformations

Contents

1 The Equations of Canonical Transformation

1.1 Motivation

We consider the situation where the Hamiltonian is a constant of the motion and all coordinates q_i are cyclic. The conjugate momenta are constants:

$$P_i = \alpha_i$$

and the Hamiltonian is a function of these constants:

$$H = H(\alpha_1, \ldots, \alpha_n)$$

Then Hamilton's equations for the coordinates q_i are:

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i$$

where ω_i are functions of the constants α_j. The solutions are straightforwardly found by integration:

$$q_i = \omega_i t + \beta_i$$

where β_i are constants of integration. The goal of canonical transformations is to find a set of new coordinates (Q_i, P_i) from the old ones (q_i, p_i) such that the new Hamiltonian has this simple, cyclic form.

1.2 The Fundamental Principle

The transformation from the old canonical variables (q, p) to the new ones (Q, P) is called a canonical transformation if there exists a new Hamiltonian $K(Q, P, t)$ such that the motion of the system is governed by Hamilton's equations in the new variables:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Both sets of variables must satisfy their respective versions of Hamilton's principle:

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H(q, p, t) \right) dt = 0$$

$$\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K(Q, P, t) \right) dt = 0$$

Since the variations at the endpoints are zero, the two integrands can differ at most by the total time derivative of an arbitrary function F, called the **generating function**.

$$\lambda \left(\sum_i p_i \dot{q}_i - H \right) = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$$

where F is any function of the phase space coordinates with continuous second derivatives, and λ is a scale factor independent of the canonical coordinates and time.

1.3 Scale Transformations

This formulation allows for scale transformations. For instance, suppose we transform (q_i, p_i) to (Q_i, P_i) by:

$$Q_i = \mu q_i, \quad P_i = \nu p_i$$

The transformed Hamiltonian is $K(Q, P) = \mu\nu H(q, p)$, and the integrands are related by:

$$\mu\nu(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K'$$

Here, the scale factor is $\lambda = \mu\nu$.

- If $\lambda = 1$, the transformation is called a **canonical transformation**.
- If $\lambda \neq 1$, it is an **extended canonical transformation**.

For the remainder of this text, we will consider $\lambda = 1$, so the fundamental equation is:

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$$

2 Generating Functions

The generating function F can be a function of different combinations of old and new variables. We explore two common types.

2.1 Type 1: $F = F_1(q, Q, t)$

We first define F to be a function of the old coordinates q, the new coordinates Q, and time t. The total time derivative of F_1 is:

$$\frac{dF_1}{dt} = \frac{\partial F_1}{\partial t} + \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i$$

Substituting this into our fundamental equation:

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i$$

Rearranging the terms, we get:

$$\sum_i \left(p_i - \frac{\partial F_1}{\partial q_i} \right) \dot{q}_i - \sum_i \left(P_i + \frac{\partial F_1}{\partial Q_i} \right) \dot{Q}_i - \left(H - K + \frac{\partial F_1}{\partial t} \right) = 0$$

Since the velocities $\dot{q}_i$ and $\dot{Q}_i$ are independent, their coefficients must vanish separately. This gives us the transformation equations:

$$p_i = \frac{\partial F_1(q, Q, t)}{\partial q_i}$$

$$P_i = -\frac{\partial F_1(q, Q, t)}{\partial Q_i}$$

And the relation between the Hamiltonians:

$$K = H + \frac{\partial F_1}{\partial t}$$

2.2 Type 2: $F = F_2(q, P, t)$

We can change the independent variables of the generating function using a Legendre transformation. Let's define a new generating function F_2 such that $F = F_2(q, P, t) - \sum_i Q_i P_i$.

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Generating Function	Generating Function Derivatives		Trivial Special Case		
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i,$	$Q_i = p_i,$	$P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i,$	$Q_i = q_i,$	$P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i,$	$Q_i = -q_i,$	$P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i,$	$Q_i = p_i,$	$P_i = -q_i$

Figure 1:

This leads to the following set of transformation equations:

$$p_i = \frac{\partial F_2(q, P, t)}{\partial q_i}$$

$$Q_i = \frac{\partial F_2(q, P, t)}{\partial P_i}$$

And the Hamiltonians are related by:

$$K = H + \frac{\partial F_2}{\partial t}$$

3 Examples of Canonical Transformations

Let's consider some examples using the F_2 generating function.

3.1 Identity Transformation

Consider a simple generating function $F_2 = \sum_i q_i P_i$. The transformation equations are:

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i$$

Since F_2 is not an explicit function of time, $\frac{\partial F_2}{\partial t} = 0$, so $K = H$. This generating function simply returns the original coordinates and momenta, thus generating the **identity transformation**.

3.2 Point Transformations

More generally, consider $F_2 = \sum_i f_i(q_1, \ldots, q_n, t) P_i$. The new coordinates are:

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, \ldots, q_n, t)$$

This represents a general point transformation where the new coordinates are functions of the old coordinates.

3.3 Linear Transformations

We consider an even more general form:

$$F_2 = \sum_i f_i(q, t) P_i + g(q, t)$$

The transformation equations for the momenta are (summing over index i):

$$p_j = \frac{\partial F_2}{\partial q_j} = \sum_i \frac{\partial f_i}{\partial q_j} P_i + \frac{\partial g}{\partial q_j}$$

In matrix notation, this becomes:

$$\mathbf{p} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right)^T \mathbf{P} + \frac{\partial g}{\partial \mathbf{q}}$$

In two dimensions, this is written as:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_2}{\partial q_1} \\ \frac{\partial f_1}{\partial q_2} & \frac{\partial f_2}{\partial q_2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{bmatrix}$$

We can solve for the new momenta $\mathbf{P}$:

$$\mathbf{P} = \left[\left(\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right)^T\right]^{-1} \left(\mathbf{p} - \frac{\partial g}{\partial \mathbf{q}}\right)$$

In two dimensions:

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_2}{\partial q_1} \\ \frac{\partial f_1}{\partial q_2} & \frac{\partial f_2}{\partial q_2} \end{bmatrix}^{-1} \left(\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{bmatrix} \right)$$

9.3 The Harmonic Oscillator

We consider a canonical transformation to solve the problem of the simple oscillator in one dimension. The force constant is k. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}$$

With $k/m = \omega^2$ we can write

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

We find a canonical transformation

$$p = f(P)\cos Q$$

$$q = \frac{f(P)}{m\omega}\sin Q$$

so

$$H = \frac{f^2(P)}{2m}(\cos^2 Q + \sin^2 Q) = \frac{f^2(P)}{2m}$$

so that Q is cyclic.

We use $F_1 = \frac{m\omega q^2}{2}\cot Q$. Then

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

We then thus have

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q$$

$$p = \sqrt{2Pm\omega}\cos Q$$

We see that

$$f(P) = \sqrt{2m\omega P}$$

It follows that

$$H = \omega P$$

and

$$P = \frac{E}{\omega}$$

The equation of motion for Q reduces to

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$

with the solution

$$Q = \omega t + \alpha$$

where α is a constant of integration.

$$\Rightarrow \begin{cases} q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha) \\ p = \sqrt{2mE}\cos(\omega t + \alpha) \end{cases}$$

It is an ellipse with the semi-major axes

$$a = \sqrt{\frac{2E}{m\omega^2}}, \quad b = \sqrt{2mE}$$

And the area

$$A = \pi ab = \frac{2\pi E}{\omega}.$$

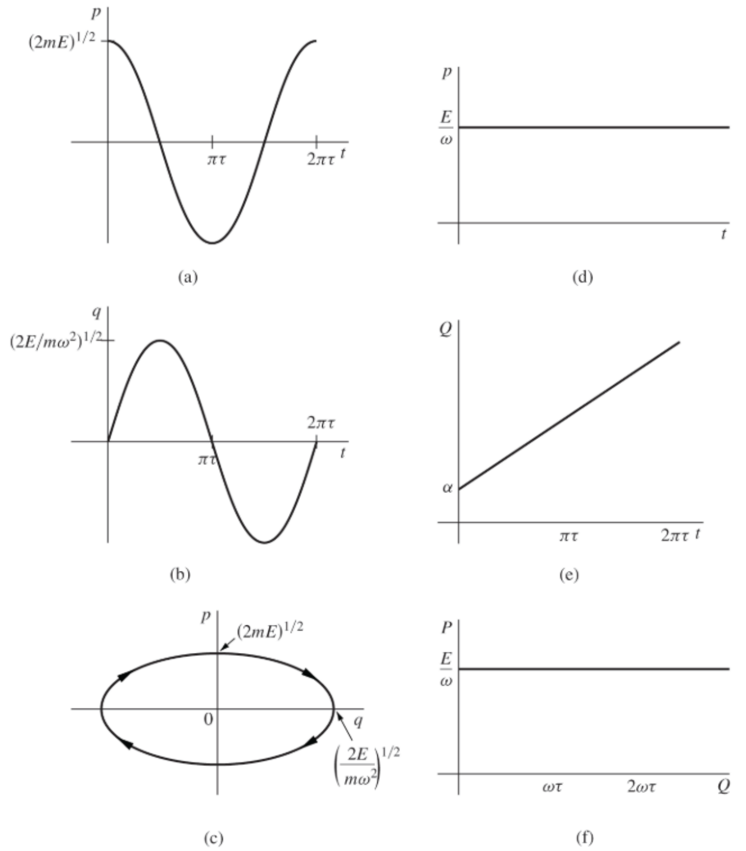

FIGURE 9.1 The harmonic oscillator in two canonical coordinate systems. Drawings (a)–(c) show the q, p system and (d)–(f) show the P, Q system.

Figure 2:

Generating Function of Type 4

We now consider $F_4 = q_k Q_k$, which leads to

$$p_i = \frac{\partial F_4}{\partial q_i} = Q_i$$

$$P_i = -\frac{\partial F_4}{\partial Q_i} = -q_i$$

For a system of two degrees of freedom the transformation are

$$Q_1 = q_2, \quad P_1 = p_1$$

$$Q_2 = p_2, \quad P_2 = -q_2$$

which is generated by the function:

$$F = q_2 p_1 + q_2 Q_2.$$

9.4 The Symplectic Approach to Canonical Transformation

We consider the transformation without time

$$Q_i = Q_i(q, p)$$

$$P_i = P_i(q, p)$$

Then

$$\begin{aligned}\dot{Q}_i &= \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j \\ &= \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}\end{aligned}$$

The inverse of the transformation

$$q_j = q_j(Q, P)$$

$$p_j = p_j(Q, P)$$

enables us to consider $H(Q, P, t)$ as a function of Q and P.

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i}$$

When

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad ①$$

$$\left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

the transformation is canonical. There is

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}$$

Also we can obtain

$$\left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P} \qquad ②$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P}$$

① & ② are the direct conditions.

In matrix notation:

$$\dot{\eta} = J\frac{\partial H}{\partial \eta}$$

where J is the antisymmetric matrix and η is a column matrix with $2n$ elements, q_i, p_i.
For restricted canonical transformation

$$\zeta = S(\eta)$$

The time derivative

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial \eta_j}\dot{\eta}_j, \quad i, j = 1, \ldots, 2n$$

In matrix notation

$$\dot{\zeta} = M\dot{\eta}$$

where M is the Jacobian matrix with $M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}$. Then by $\dot{\eta} = J\frac{\partial H}{\partial \eta}$, we can have

$$\frac{\partial H}{\partial \zeta_i} = \frac{\partial H}{\partial \eta_j}\frac{\partial \eta_j}{\partial \zeta_i} \quad \text{or} \quad \frac{\partial H}{\partial \eta} = \tilde{M}\frac{\partial H}{\partial \zeta}$$

Combined these equations:

$$\dot{\zeta} = MJ\tilde{M}\frac{\partial H}{\partial \zeta}$$

If $MJ\tilde{M} = \lambda J$, we can say the transformation is canonical.

2.

We consider the generating function $F = F_2(q, P, t) + \delta F(q, P, t)$. The column vectors are

$$\eta = \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{and} \quad \dot{\eta} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

The transformation S is $\eta' = M\eta$.

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & \delta t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

Hamilton's equations for $\dot{\eta} = S\frac{\partial H}{\partial \eta}$ are:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

where $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$.

3.

We consider a canonical transformation of the form $S = S(q, p, t)$. If the transformation is canonical, $y \to Y(y, t)$ and $y(t_0) \to Y(t)$. The so is $y \to Y(y, \epsilon)$ and $S(q, \epsilon) \to S(\epsilon)$.

We introduce the infinitesimal canonical transformation (I.C.T) where we only retain the first-order terms. The transformation equations are

$$Q_i = q_i + \delta q_i$$
$$P_i = p_i + \delta p_i$$

$\implies S = \delta S$

A proper generating function would be $F_2 = q_i P_i + \epsilon G(q, P, t)$, when ϵ is some infinitesimal parameter and G is any differentiable function of its $2n + 1$ arguments. The transformation equations for the momenta

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i + \epsilon\frac{\partial G}{\partial q_i}$$

or

$$\delta p_i = P_i - p_i = -\epsilon \frac{\partial G}{\partial q_i}$$

Similarly we also have:

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i + \epsilon \frac{\partial G}{\partial P_i} \quad \text{or} \quad \delta q_i = \epsilon \frac{\partial G}{\partial P_i}$$

As an example from t_0 to $t_0 + dt$:

$$\delta q = \dot{q} dt \implies \dot{q} = \frac{\partial G}{\partial p}$$

The Jacobian for an infinitesimal transformation:

$$M = \frac{\partial(Q,P)}{\partial(q,p)} \text{ or } M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$

Whose elements are

$$M = I + \epsilon J \text{ or } M = \begin{pmatrix} 1 + \epsilon \frac{\partial^2 G}{\partial q \partial p} & \epsilon \frac{\partial^2 G}{\partial p^2} \\ -\epsilon \frac{\partial^2 G}{\partial q^2} & 1 - \epsilon \frac{\partial^2 G}{\partial p \partial q} \end{pmatrix}$$

The transpose of M is M^T and the symplectic condition becomes

$$M^T J M = J$$

$$(I + \epsilon J^T) J (I + \epsilon J) \approx J + \epsilon (J^T J + JJ) = J$$

Thus, the symplectic condition would be

$$J^T J + JJ = 0$$

This means that the symplectic condition holds for any I.C.T.

9.5 Poisson Brackets And Other Canonical Invariants

Definition of Poisson Bracket

We define the Poisson bracket of u, v with respect to the canonical variables (q, p) as

$$[u, v]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

through which we can see that q is coupled with p and p with $-q$.

In matrix form:

$$[u, v] = \left(\frac{\partial u}{\partial \xi} \right)^T J \left(\frac{\partial v}{\partial \xi} \right)$$

By definition, we see that

$$[q_j, q_k] = 0, \quad [p_j, p_k] = 0, \quad [q_j, p_k] = \delta_{jk}$$

We introduce a square matrix Poisson bracket whose elements are $[\xi_j, \xi_k]$.

$$\implies [J]_{jk} = [\xi_j, \xi_k]$$

$$\implies [\xi, \xi] = J$$

If $q \to Q$ is canonical we have

$$[Q, P] = \frac{\partial(Q, P)}{\partial(q, p)} = MJM^T$$

Hence $[Q, P] = J$. We see that the fundamental Poisson brackets are invariant under canonical transformation. Since $[Q, P] = J$.

Lagrange Brackets and Properties

We now consider two functions u, v with respect to the j set of coordinates. We have the relation:

$$\tilde{M} = \frac{\partial \xi}{\partial \zeta} \quad \text{and} \quad M = \frac{\partial \zeta}{\partial \xi}$$

Hence $[u, v]_\xi = \left(\frac{\partial u}{\partial \zeta}\right)^T \tilde{M} J \tilde{M}^T \left(\frac{\partial v}{\partial \zeta}\right)$. If the transformation is canonical, $[u, v]_\zeta = \frac{\partial u}{\partial \zeta_r} J_{rs} \frac{\partial v}{\partial \zeta_s}$. Then all Poisson brackets are canonical invariants.

We have some properties:

1. $[u, u] = 0$ (antisymmetry)
2. $[u, c] = 0$
3. $[u + v, w] = [u, w] + [v, w]$ (linearity)
4. $[uv, w] = u[v, w] + [u, w]v$
5. $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ (Jacobi identity)

Proof: We denote partial derivatives of u, v, w by corresponding canonical variable. (Functions with continuous second derivatives)

$$\sum_k \left(\frac{\partial u}{\partial \xi_k} \frac{\partial [v, w]}{\partial \eta_k} - \frac{\partial [v, w]}{\partial \xi_k} \frac{\partial u}{\partial \eta_k} \right)$$

Then $[u, v] = u_j J_{jk} v_k$, where J_{ij} is the element. Thus: $[[u, v], w] = (u_{ij} J_{jk} v_k + u_j J_{jk} v_{ki}) J_{il} w_l$. Since $J_{ji} + J_{ij} = 0$ and $w_{kl} = w_{lk}$, we thus prove the Jacobi identity.

Further Properties and Transformations

The Lagrange bracket of u and v with respect to the $(\mathbf{q}, \mathbf{p})$ variables is defined as

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right)$$

or

$$\{u, v\} = \left(\frac{\partial \xi}{\partial u} \right)^T J^{-1} \left(\frac{\partial \xi}{\partial v} \right)$$

And the fundamental Lagrange brackets

$$\{q_j, q_k\} = 0, \quad \{p_j, p_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk}$$

or $\{\xi_i, \xi_j\} = J$. Then these have the character: $\{u, v\}[u, v] = 1$.

We consider the transformations from one phase space with coordinates ξ_j to another with ζ_j in $2n$ dimensional. The volume element:

$$dV = dq_1 dq_2 \dots dq_n dp_1 \dots dp_n$$

$$dV_\zeta = dQ_1 \dots dQ_n dP_1 \dots dP_n$$

They are connected by the value of Jacobian determinant $|M|$.

$$dV_\zeta = |M| dV_\xi$$

For example, for $s = 1$: $(\mathbf{q}, \mathbf{p}) \to (\mathbf{Q}, \mathbf{P})$.

$$dQdP = \left| \frac{\partial(Q, P)}{\partial(q, p)} \right| dqdp = |[Q, P]| dqdp$$

The volume of an arbitrary region in phase space is a canonical invariant.

$$J_V = \int \cdots \int dV_\xi$$

By symplectic condition $MJM^T = J$, we obtain $|M| = \pm 1$.

For systems with many degrees of freedom, we suppose the first of the equations of transformation $Q = Q(q, p)$, $P = P(q, p)$ is invertible, say $q = q(Q, P)$. Substitution in the second equation yields, say $p = p(Q, P)$. We expect the transformation is generated by F_1.

$$p = \frac{\partial F_1(q, Q)}{\partial q}, \quad P = -\frac{\partial F_1(q, Q)}{\partial Q}$$

Then $\frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q}$. Conversely, if this equation is valid, there must exist F_1 such that p and P are given. If $Q = Q(q, p)$, $\phi = \phi(q, p)$, the condition for the transformation to be canonical is

$$[Q, \phi]_{q,p} = 1$$

In the same spirit, we write that $p = p(q, Q)$ and since $P = P(q, Q)$,

$$[q, p]_{Q,P} = \left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} \right)_{q,Q} = 1$$

Hence $\left.\frac{\partial p}{\partial Q}\right|_q = -\left(\frac{\partial q}{\partial Q}\right)_p^{-1} = -\left.\frac{\partial P}{\partial q}\right|_Q$.

$$-\frac{\partial p}{\partial Q} = \frac{\partial P}{\partial q}$$

9.6 Equations of Motion. Infinitesimal Canonical Transformations, and Conservation Theorems in the Poisson Bracket Formulation.

1

For the function $u(q, p, t)$

$$\frac{du}{dt} = \sum_i \left(\frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i \right) + \frac{\partial u}{\partial t}$$

$$= \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial u}{\partial t}$$

$$= [u, H] + \frac{\partial u}{\partial t}$$

or

$$\frac{du}{dt} = \sum_j \frac{\partial u}{\partial \eta_j}\dot{\eta}_j + \frac{\partial u}{\partial t} = [u, H] + \frac{\partial u}{\partial t}$$

And we obtain the Hamilton's equations

$$\dot{q}_i = [q_i, H], \quad \dot{p}_i = [p_i, H]$$

or

$$\dot{\eta}_j = [\eta_j, H]$$

If u is a constant of the motion

$$[H, u] = -\frac{\partial u}{\partial t}$$

By Jacobian Identity

$$[H, [u, v]] = 0 \quad \text{if u,v are constants of motion}$$

That is $[u, v]$ is always a constant in time.

2

We consider an I.C.T. (Infinitesimal Canonical Transformation)

$$\zeta = \eta + \delta\eta$$

In terms of the generator G

$$\delta\eta = \epsilon\frac{\partial G}{\partial \eta}$$

By definition: $[y, u] = \frac{\partial u}{\partial y}$. If we set $u = G$, we see the I.C.T can be written as

$$\delta y = \epsilon[y, G]$$

We let $\epsilon = dt$.

$$\delta y = dt[y, H] = \dot{y}dt = dy$$

We can say that the Hamiltonian is the generator of the system motion with time.

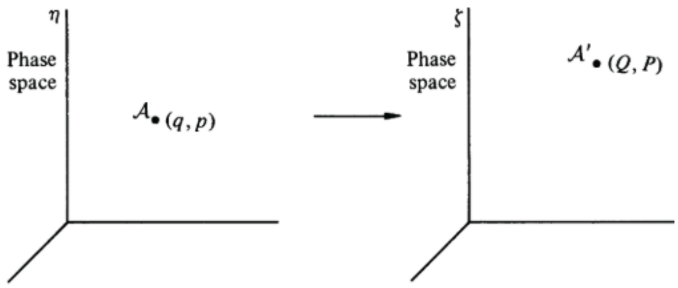

FIGURE 9.2 The passive view of a canonical transformation.

Figure 3:

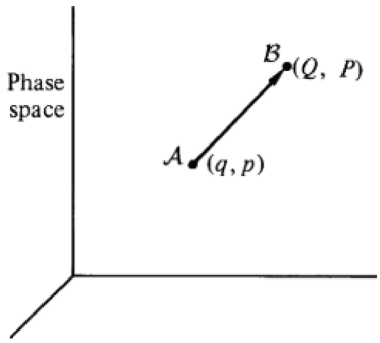

FIGURE 9.3 The active view of a canonical transformation.

Figure 4:

3

We denote a change in the value of a function about an active I.C.T.

$$\delta u = u(B) - u(A)$$

where A and B will be infinitesimally close. Or:

$$\delta u = u(\eta + \delta\eta) - u(\eta)$$

Expanding in a Taylor series and retaining the first order

$$\delta u = \frac{\partial u}{\partial y}\delta y = \epsilon \frac{\partial u}{\partial y}\frac{\partial G}{\partial y} = \epsilon[u, G]$$

$$\delta u = \epsilon[u, G]$$

4

When the canonical transformation depends upon the time

$$H \to K$$

The difference is:

$$\delta H = H(B) - K(A)$$

The function itself doesn't change under the transformation

$$u(A') = u(A)$$

K and H are related by

$$K = H + \frac{\partial f}{\partial t}$$

Let the generator be G, as a function of time

$$K(A') = H(A') + \epsilon \frac{\partial G}{\partial t} = H(A) + \epsilon [H, G] + \epsilon \frac{\partial G}{\partial t}$$

$$\delta H = H(B) - H(A) - \epsilon \frac{\partial G}{\partial t} = \epsilon [H, G] - \epsilon \frac{\partial G}{\partial t}$$

with G as u.

$$\delta H = -\epsilon \frac{dG}{dt}$$

It indicates that the constants of the motion are the generating functions of those I.C.T.s that leave the Hamiltonian invariant.

5

Consider a transformation generated by the generalized momentum conjugate to q_i:

$$G(q, P) = p_i$$

The ICT is:

$$\delta q_j = \epsilon \delta_{ij}, \quad \delta p_i = 0$$

That is if a coordinate is cyclic, its conjugate momentum is a constant of motion. If the generating function of an I.C.T. is

$$G_1 = (v_j \eta_j)_L = v_j p_j$$

The equations of transformation are

$$\delta \eta_k = \epsilon [\eta_k, G_1] = \epsilon \frac{\partial G_1}{\partial \eta_k} = \epsilon J_{ks} \frac{\partial G_1}{\partial \eta_s} = \epsilon J_{ks} v_s = \epsilon S_{kl} v_l$$

6

We consider an infinitesimal rotation along the z-axis. The changes in the particle coordinates

$$\delta x = -y d\theta, \quad \delta y = x d\theta, \quad \delta z = 0$$

The momenta conjugate to the coordinates

$$\delta p_x = -p_y d\theta, \quad \delta p_y = p_x d\theta, \quad \delta p_z = 0$$

The corresponding generating function

$$G = x p_y - y p_x$$

with $d\theta$ as ϵ. We note that:

$$\delta x = \epsilon \frac{\partial G}{\partial p_x} = -y d\theta, \quad \delta p_x = -\epsilon \frac{\partial G}{\partial x} = -p_y d\theta$$

$$\delta y = \epsilon \frac{\partial G}{\partial p_y} = x d\theta, \quad \delta p_y = -\epsilon \frac{\partial G}{\partial y} = p_x d\theta$$

It has the physical significance that G is the z-component of the total angular momentum.

$$G = L_z = (\vec{r} \times \vec{p})_z = \vec{L} \cdot \vec{n}$$

where $\vec{n}$ is the unit vector along the rotation.

7

We consider a continuous function along the trajectory (with the initial $u_0 = u(0)$). The infinitesimal change of u on the trajectory

$$\delta u = d\theta [u, G]$$

$$\frac{du}{d\alpha} = [u, G]$$

By the Taylor series

$$u(\alpha) = u_0 + \alpha \frac{du}{d\alpha}|_0 + \dots$$

we have

$$\frac{du}{d\alpha}|_0 = [u, G]_0 \quad \text{and} \quad \frac{d\eta}{d\alpha} = [\eta, G]$$

9.7 The Angular Momentum Poisson Bracket Relations

We consider a vector function $\vec{F}$ of the system configuration. The change in F under an infinitesimal canonical transformation generated by G is given by

$$\delta F_i = d\alpha [F_i, G]$$

With $G = \vec{L} \cdot \hat{n}$, the generator of rotations, we have

$$\delta \vec{F} = d\theta [\vec{F}, \vec{L} \cdot \hat{n}]$$

The change in $\vec{F}$ under an infinitesimal rotation about an axis $\hat{n}$ is

$$d\vec{F} = d\alpha (\hat{n} \times \vec{F})$$

For a system $\vec{F}$, the change under an I.C.T. generated by $\vec{L} \cdot \hat{n}$ is

$$\delta \vec{F} = d\theta [\vec{F}, \vec{L} \cdot \hat{n}] = d\alpha (\hat{n} \times \vec{F})$$

$$\Rightarrow [\vec{F}, \vec{L} \cdot \hat{n}] = \hat{n} \times \vec{F}$$

In Cartesian coordinates, if the $\hat{n}$ is along the z-axis, so that $L_z = xp_y - yp_x$:

$$\begin{cases} [p_x, xp_y - yp_x] = -p_y \\ [p_y, xp_y - yp_x] = p_x \\ [p_z, xp_y - yp_x] = 0 \end{cases}$$

For an arbitrary $\hat{n}$, the k-th component is

$$[F_i, L_j] = \epsilon_{ijk} F_k$$

In particular, if we let $\vec{F} = \vec{L}$, we get the angular momentum algebra:

$$\Rightarrow [L_i, L_j] = \epsilon_{ijk} L_k$$

or, for components in cyclic order (l, m, n):

$$[L_l, L_m] = L_n$$

For the product of two system vectors $\vec{F}$, $\vec{G}$:

$$[\vec{F} \cdot \vec{G}, \vec{L} \cdot \hat{n}] = \vec{F} \cdot [\vec{G}, \vec{L} \cdot \hat{n}] + \vec{G} \cdot [\vec{F}, \vec{L} \cdot \hat{n}]$$

$$= \vec{F} \cdot (\hat{n} \times \vec{G}) + \vec{G} \cdot (\hat{n} \times \vec{F})$$

Using the scalar triple product identity $\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$:

$$= \vec{F} \cdot (\hat{n} \times \vec{G}) - \vec{F} \cdot (\hat{n} \times \vec{G}) = 0$$

Taking $\vec{F} = \vec{G} = \vec{L}$:

$$[\vec{L} \cdot \vec{L}, \hat{n} \cdot \vec{L}] = [L^2, \vec{L} \cdot \hat{n}] = 0$$

From the general relation for a vector, we also have for momentum $\vec{p}$:

$$[p_i, \vec{L} \cdot \hat{n}] = (\hat{n} \times \vec{p})_i$$

$$[p_i, L_j] = \epsilon_{ijk} p_k$$

If p_z, L_x, L_y are all constants of motion (i.e., their Poisson brackets with the Hamiltonian are zero), then

$$[p_z, L_x] = p_y$$

$$[p_z, L_y] = -p_x$$

This implies that p_x and p_y are also conserved, and therefore $\vec{L}$ and $\vec{p}$ are conserved.

If p_x, p_y, L_z are constants of motion:

$$[p_x, p_y] = 0$$

$$[p_x, L_z] = p_y$$

$$[p_y, L_z] = -p_x$$

9.9 Liouville's Theorem

The total time derivation of D can be written as

$$\frac{dD}{dt} = [D, H] + \frac{\partial D}{\partial t}$$

We then consider an infinitesimal volume in phase space that has a motion (L.Fig 9, page 420).

Since the number of systems in the infinitesimal region dN and the volume dV are constants, we have

$$0 = \frac{dN}{dV} \text{ must be constant, that is}$$

$$\frac{dD}{dt} = 0 \quad \text{or} \quad \frac{\partial D}{\partial t} = -[D, H]$$

Plus by continuous equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

where $\vec{v} = (\dot{q}, \dot{p})$ is the velocity of phase space.

$$\nabla \cdot (\rho \vec{v}) = \sum_{i=1}^{N} \left(\frac{\partial(\rho \dot{q}_i)}{\partial q_i} + \frac{\partial(\rho \dot{p}_i)}{\partial p_i} \right)$$

With Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The continuity equation becomes

$$\Rightarrow \frac{\partial \rho}{\partial t} + [\rho, H] = 0$$