数学物理方法笔记 (Fourier Analysis Notes)

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1 傅立叶级数 (Fourier Series)

1.1 一般理论 (General Theory)

给定一个周期为 $2L$ 的函数 $f(x)$，即 $f(x+2L)=f(x)$，其傅立叶级数展开为：

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{L})+b_n\sin(\frac{n\pi x}{L}))$$

其中系数由以下积分给出：

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(t)dt$$
$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos(\frac{n\pi t}{L})dt$$
$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin(\frac{n\pi t}{L})dt$$

对于周期为 2π 的函数（即 $L=\pi$）：

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$$

通过在 $[0,2\pi]$ 上积分可以推导出系数：

$$\int_0^{2\pi}f(x)dx=\int_0^{2\pi}\left[a_0+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))\right]dx=2\pi a_0$$

$$\Rightarrow a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx$$

利用三角函数的正交性:

$$\int_0^{2\pi}f(x)\cos(mx)dx=\sum_{n=1}^{\infty}a_n\int_0^{2\pi}\cos(nx)\cos(mx)dx=a_m\pi$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)dx$$

同理可得 b_n:

$$\Rightarrow b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx$$

狄利克雷定理 (Dirichlet's Theorem): 如果 $f(x)$ 在 $[-L, L]$ 上除有限个点外连续，且有有限个极值点。在 $(-L, L)$ 外为周期延拓，周期为 $2L$。则 $f(x)$ 的傅立叶级数收敛于：

$$\frac{f(x+0)+f(x-0)}{2}$$

1.2 正交函数系展开 (Expansion in Orthogonal Function Systems)

正弦级数 (Sine Series): 若函数 $\phi(x)$ 在 $(0, L)$ 上展开为正弦级数:

$$\phi(x)=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right), \quad 0<x<L$$

利用正交关系 $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{L}{2}\delta_{mn}$，可得系数：

$$\int_0^L \phi(x)\sin\left(\frac{m\pi x}{L}\right)dx=C_m\frac{L}{2}$$

$$\Rightarrow C_n=\frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

余弦级数 (Cosine Series): 若函数 $\phi(x)$ 在 $(0, L)$ 上展开为余弦级数:

$$\phi(x)=D_0+\sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{L}\right)$$

利用正交关系可得系数：

$$D_0=\frac{1}{L}\int_0^L \phi(x)dx$$

$$D_n=\frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

1.3 傅立叶级数示例 (Fourier Series Examples)

例1: $f(x)=\frac{1}{2}(\pi-x)$ on $(0, 2\pi)$, with $f(x+2\pi)=f(x)$.

$$\begin{aligned}
a_0&=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)dx=0\\
a_n&=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\cos(nx)dx=0\\
b_n&=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\sin(nx)dx\\
&=\frac{1}{2\pi}\left[(\pi-x)\left(-\frac{\cos(nx)}{n}\right)\right]_0^{2\pi}-\frac{1}{2\pi}\int_0^{2\pi}(-1)\left(-\frac{\cos(nx)}{n}\right)dx\\
&=\frac{1}{2\pi}\left[\frac{\pi}{n}+\frac{\pi}{n}\right]-0=\frac{1}{n}
\end{aligned}$$

所以:

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

例2: 将 $\phi(x) = \sin x$ 在 $[0, \pi]$ 上展开为余弦级数。这里 $L = \pi$。

$$
\begin{aligned}
D_0 &= \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi} \\
D_n &= \frac{2}{\pi}\int_0^\pi \sin x\cos(nx)\,dx = \frac{1}{\pi}\int_0^\pi [\sin((1+n)x) + \sin((1-n)x)]\,dx \\
&= \frac{1}{\pi}\left[-\frac{\cos((1+n)x)}{1+n} - \frac{\cos((1-n)x)}{1-n}\right]_0^\pi \quad (n \neq 1) \\
&= \frac{1}{\pi}\left(\frac{1-\cos((n+1)\pi)}{n+1} + \frac{1-\cos((n-1)\pi)}{n-1}\right) \\
&= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1} + \frac{1-(-1)^{n-1}}{n-1}\right) = \frac{1+(-1)^n}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) \\
&= \frac{2(1+(-1)^n)}{\pi(n^2-1)}
\end{aligned}
$$

当 $n = 1$ 时, $D_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = 0$。
D_n 仅在 n 为偶数时非零。令 $n = 2k$:

$$
D_{2k} = \frac{4}{\pi((2k)^2-1)}
$$

所以:

$$
\phi(x) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4}{\pi(1-(2k)^2)}\cos(2kx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}
$$

2 傅立叶积分 (Fourier Integral)

2.1 从级数到积分 (From Series to Integral)

考虑傅立叶级数，当 $L \to \infty$ 时，$\omega_n = \frac{n\pi}{L}$, $\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$。级数求和变为积分: $\sum_{n=1}^{\infty} \to \frac{L}{\pi}\sum \Delta\omega \to \frac{1}{\pi}\int_0^\infty d\omega$。

$$
f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega
$$

其中系数为:

$$
\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt \\
B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt
\end{aligned}
$$

2.2 傅立叶积分示例 (Fourier Integral Examples)

例3: $f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$ 函数为偶函数，所以 $B(\omega) = 0$。

$$
A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt = \frac{1}{\pi}\int_{-1}^{1}\cos(\omega t)dt = \frac{1}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}
$$

所以:

$$
f(x) = \int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos(\omega x)d\omega = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos(\omega x)}{\omega}d\omega
$$

例4: $f(x) = \begin{cases} \cos x & |x| \le \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$ 函数为偶函数，所以 $B(\omega) = 0$。

$$\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cos(\omega t) dt = \frac{2}{\pi}\int_0^{\pi/2} \cos t \cos(\omega t) dt \\
&= \frac{1}{\pi}\int_0^{\pi/2} [\cos((1+\omega)t) + \cos((1-\omega)t)] dt \\
&= \frac{1}{\pi}\left[\frac{\sin((1+\omega)t)}{1+\omega} + \frac{\sin((1-\omega)t)}{1-\omega}\right]_0^{\pi/2} \quad (\omega \ne 1) \\
&= \frac{1}{\pi}\left(\frac{\sin(\frac{\pi}{2}(1+\omega))}{1+\omega} + \frac{\sin(\frac{\pi}{2}(1-\omega))}{1-\omega}\right) \\
&= \frac{1}{\pi}\left(\frac{\cos(\frac{\pi\omega}{2})}{1+\omega} + \frac{\cos(\frac{\pi\omega}{2})}{1-\omega}\right) = \frac{2\cos(\frac{\pi\omega}{2})}{\pi(1-\omega^2)}
\end{aligned}$$

所以:

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\frac{\pi\omega}{2})}{1-\omega^2}\cos(\omega x) d\omega$$

3 复数形式的傅立叶变换 (Complex Form and Fourier Transform)

傅立叶积分可以写为：

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty f(t)\cos(\omega(x-t)) dt \\
&= \frac{1}{2\pi}\int_0^\infty d\omega \int_{-\infty}^\infty f(t)[e^{i\omega(x-t)} + e^{-i\omega(x-t)}] dt \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty d\omega \int_{-\infty}^\infty f(t) e^{i\omega(x-t)} dt
\end{aligned}$$

这引出了傅立叶变换对：

傅立叶变换 (Fourier Transform): $F(\omega) = \int_{-\infty}^\infty f(x) e^{-i\omega x} dx$

傅立叶逆变换 (Inverse F.T.): $f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega) e^{i\omega x} d\omega$

3.1 傅立叶变换示例 (Fourier Transform Example)

例4 (续): $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^\infty f(x) e^{-i\omega x} dx = \int_{-a}^a 1 \cdot e^{-i\omega x} dx \\
&= \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-a}^a = \frac{e^{-i\omega a} - e^{i\omega a}}{-i\omega} = \frac{2\sin(\omega a)}{\omega}
\end{aligned}$$

通过逆变换得到傅立叶积分表示：

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{2\sin(\omega a)}{\omega} e^{i\omega x} d\omega \\
&= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin(\omega a)}{\omega}(\cos(\omega x) + i\sin(\omega x)) d\omega
\end{aligned}$$

由于 $\frac{\sin(\omega a)}{\omega}\sin(\omega x)$ 是奇函数，其积分为零。

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega a)\cos(\omega x)}{\omega}d\omega = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(\omega a)\cos(\omega x)}{\omega}d\omega$$

根据收敛定理，该积分等于：

$$\begin{cases} 1 & |x| < a \\ 1/2 & |x| = a \\ 0 & |x| > a \end{cases}$$

4 傅立叶变换性质 (Properties of Fourier Transform)

令 $F(\omega)$ 是 $f(x)$ 的傅立叶变换, $F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$.

1. **导数 (Differentiation):**

$$\mathcal{F}\left[\frac{df(x)}{dx}\right] = i\omega F(\omega)$$

推导:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{df(x)}{dx}e^{-i\omega x}dx &= \left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega)e^{-i\omega x}dx \\ &= 0 + i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = i\omega F(\omega)\end{aligned}$$

2. **乘以 x (Multiplication by x):**

$$\mathcal{F}[xf(x)] = i\frac{dF(\omega)}{d\omega}$$

推导:

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} -ixf(x)e^{-i\omega x}dx = -i\mathcal{F}[xf(x)]$$

推广可得:

$$\mathcal{F}[x^n f(x)] = i^n\frac{d^nF(\omega)}{d\omega^n}$$

3. **积分 (Integration):** 若 $g(x) = \int_{-\infty}^{x} f(\xi)d\xi$, 且 $f(x) = g'(x)$, 则

$$F(\omega) = \mathcal{F}[g'(x)] = i\omega G(\omega)$$

$$G(\omega) = \mathcal{F}\left[\int_{-\infty}^{x} f(\xi)d\xi\right] = \frac{1}{i\omega}F(\omega) \quad \text{(可能需要加上 } \pi F(0)\delta(\omega) \text{ 项)}$$

4. **位移 (Shifting):**

$$\mathcal{F}[f(x+\xi)] = e^{i\omega\xi}F(\omega)$$

推导 (令 $y = x + \xi$):

$$\int_{-\infty}^{\infty} f(x+\xi)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(y)e^{-i\omega(y-\xi)}dy = e^{i\omega\xi}\int_{-\infty}^{\infty} f(y)e^{-i\omega y}dy = e^{i\omega\xi}F(\omega)$$

5. **卷积 (Convolution):** 卷积定义: $f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(\xi)f_2(x-\xi)d\xi$.

$$\mathcal{F}[f_1 * f_2] = F_1(\omega)F_2(\omega)$$

推导:

$$\begin{aligned}
\mathcal{F}[f_1 * f_2] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\xi) f_2(x-\xi) d\xi\right] e^{-i\omega x} dx \\
&= \int_{-\infty}^{\infty} f_1(\xi) \left[\int_{-\infty}^{\infty} f_2(x-\xi) e^{-i\omega x} dx\right] d\xi \\
(\text{令 } y = x-\xi) &= \int_{-\infty}^{\infty} f_1(\xi) \left[\int_{-\infty}^{\infty} f_2(y) e^{-i\omega(y+\xi)} dy\right] d\xi \\
&= \left(\int_{-\infty}^{\infty} f_1(\xi) e^{-i\omega\xi} d\xi\right) \left(\int_{-\infty}^{\infty} f_2(y) e^{-i\omega y} dy\right) = F_1(\omega) F_2(\omega)
\end{aligned}$$

5 狄拉克 δ 函数 (Dirac Delta Function)

5.1 定义与性质 (Definition and Properties)

δ 函数定义为满足以下条件的分布:

$$\delta(x-x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases} \quad \text{且} \quad \int_{-\infty}^{\infty} \delta(x-x_0) dx = 1$$

筛选性质 (Sifting Property):

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0) dx = f(x_0)$$

$\delta(x-x_1)$ 为偶函数, 即 $\delta(x) = \delta(-x)$。

卷积性质 (Convolution Properties):

$$\begin{aligned}
\delta(x) * f(x) &= \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi) d\xi = f(x) \\
\delta(x-a) * f(x) &= f(x-a) \\
\delta(x-a) * \delta(x-b) &= \delta(x-(a+b))
\end{aligned}$$

傅立叶变换 (Fourier Transform):

$$\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-i\omega x} dx = e^{-i\omega x_0}$$

特别地, 当 $x_0 = 0$ 时, $\mathcal{F}[\delta(x)] = 1$。反变换给出 δ 函数的一个积分表示:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} d\omega$$

5.2 δ 函数的极限表示 (Limit Representations of δ-Function)

1. **Sinc 函数:**

$$\delta(x) = \lim_{B\to\infty} \frac{\sin(Bx)}{\pi x}$$

这来自一个带宽为 $[-B, B]$ 的理想低通滤波器的冲激响应。

$$\frac{1}{2\pi} \int_{-B}^{B} e^{i\omega x} d\omega = \frac{\sin(Bx)}{\pi x}$$

2. **洛伦兹函数 (Lorentzian Function):**

$$\delta(x) = \lim_{b \to 0^+} \frac{1}{\pi} \frac{b}{x^2 + b^2}$$

3. **狄利克雷核 (Dirichlet Kernel, 在周期区间上):**

$$\sum_{k=-\infty}^{\infty} \delta(x - 2\pi k) = \frac{1}{2\pi} \lim_{m \to \infty} D_m(x) = \frac{1}{2\pi} \lim_{m \to \infty} \sum_{n=-m}^{m} e^{inx}$$

其中 $D_m(x) = 1 + 2\sum_{n=1}^{m} \cos(nx) = \frac{\sin((m+1/2)x)}{\sin(x/2)}$。

6 傅立叶级数的收敛与狄利克雷核

6.1 部分和 (Partial Sum)

傅立叶级数的部分和 $S_m(x)$ 可以表示为与狄利克雷核的卷积:

$$\begin{aligned}
S_m(x) &= \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx)) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{n=1}^{m} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(t-x)) dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{m} \cos(n(t-x)) \right] dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(t-x) dt
\end{aligned}$$

其中 $D_m(u) = \sum_{n=-m}^{m} e^{inu} = \frac{\sin((m+1/2)u)}{\sin(u/2)}$ 是狄利克雷核。令 $u = t - x$, 并利用周期性:

$$S_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+u) D_m(u) du$$

由于 $D_m(u)$ 是偶函数,

$$S_m(x) = \frac{1}{2\pi} \int_{0}^{\pi} [f(x+u) + f(x-u)] D_m(u) du$$

6.2 收敛证明概要 (Sketch of Convergence Proof)

利用狄利克雷核的性质 $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(u) du = 1$, 可得

$$\frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi} \int_{0}^{\pi} [f(x+0) + f(x-0)] D_m(u) du$$

考虑差值:

$$\begin{aligned}
S_m(x) - \frac{f(x+0) + f(x-0)}{2} &= \frac{1}{2\pi} \int_{0}^{\pi} [f(x+u) - f(x+0) + f(x-u) - f(x-0)] D_m(u) du \\
&= \frac{1}{2\pi} \int_{0}^{\pi} \left[\frac{f(x+u) - f(x+0)}{u} + \frac{f(x-u) - f(x-0)}{u} \right] u \frac{\sin((m+1/2)u)}{\sin(u/2)} du
\end{aligned}$$

当 $f(x)$ 满足狄利克雷条件时, 方括号内的函数在 $[0, \pi]$ 上绝对可积。根据黎曼-勒贝格引理 (Riemann-Lebesgue Lemma), 当 $m \to \infty$ 时, 该积分趋于零。因此, $\lim_{m \to \infty} S_m(x) = \frac{f(x+0)+f(x-0)}{2}$。

7 更多傅立叶变换示例 (More Fourier Transform Examples)

7.1 高斯函数 (Gaussian Function)

例： 求 $f(x) = e^{-ax^2}$ (其中 $a > 0$) 的傅立叶变换。

$$\begin{aligned} G(\omega) &= \mathcal{F}[e^{-ax^2}] = \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} e^{-(ax^2+i\omega x)} dx \end{aligned}$$

为了求解这个积分，我们使用配方法：

$$ax^2 + i\omega x = a\left(x^2 + \frac{i\omega}{a}x\right) = a\left(x + \frac{i\omega}{2a}\right)^2 - a\left(\frac{i\omega}{2a}\right)^2 = a\left(x + \frac{i\omega}{2a}\right)^2 + \frac{\omega^2}{4a}$$

代入积分中可得：

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} \exp\left[-a\left(x + \frac{i\omega}{2a}\right)^2 - \frac{\omega^2}{4a}\right] dx \\ &= e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\omega}{2a}\right)^2} dx \end{aligned}$$

令 $y = \sqrt{a}(x + \frac{i\omega}{2a})$, $dy = \sqrt{a}dx$。这是一个在复平面上的积分，但可以证明其路径可以平移回实轴而不改变积分值。因此，积分结果等于标准高斯积分 $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$。

$$G(\omega) = e^{-\frac{\omega^2}{4a}} \cdot \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$

这是一个非常重要的结论：**高斯函数的傅立叶变换仍然是高斯函数**。

应用： 利用傅立叶逆变换，我们可以求解一个重要的积分。

$$f(x) = e^{-ax^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} e^{i\omega x} d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4a}} (\cos(\omega x) + i\sin(\omega x)) d\omega = 2\pi\sqrt{\frac{a}{\pi}} e^{-ax^2} = \sqrt{4\pi a} e^{-ax^2}$$

取等式两边的实部，并设 $a = 1/4$，则有：

$$\int_{-\infty}^{\infty} e^{-\omega^2} \cos(\omega x) d\omega = \sqrt{\pi} e^{-x^2/4}$$

7.2 基本函数 (Basic Functions)

例： $f(x) = \sin(kx)$

$$\begin{aligned} \mathcal{F}[\sin(kx)] &= \int_{-\infty}^{\infty} \frac{e^{ikx} - e^{-ikx}}{2i} e^{-i\omega x} dx \\ &= \frac{1}{2i}\left[\int_{-\infty}^{\infty} e^{-i(\omega-k)x} dx - \int_{-\infty}^{\infty} e^{-i(\omega+k)x} dx\right] \\ &= \frac{1}{2i}[2\pi\delta(\omega - k) - 2\pi\delta(\omega + k)] \\ &= i\pi[\delta(\omega + k) - \delta(\omega - k)] \end{aligned}$$

例：$f(x) = e^{-|x|}$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx = \int_{-\infty}^{0} e^{x} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x} dx \\
&= \int_{-\infty}^{0} e^{(1-i\omega)x} dx + \int_{0}^{\infty} e^{-(1+i\omega)x} dx \\
&= \left[\frac{e^{(1-i\omega)x}}{1-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\right]_{0}^{\infty} \\
&= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{1+i\omega+1-i\omega}{1+\omega^2} = \frac{2}{1+\omega^2}
\end{aligned}$$

通过傅立叶逆变换：

$$f(x) = e^{-|x|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{i\omega x} d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{1+\omega^2} d\omega$$

7.3 双曲正割与三角脉冲 (Hyperbolic Secant and Triangular Pulse)

例：$f(x) = \mathbf{sech}(kx)$ 其傅立叶变换为：

$$F(\omega) = \mathcal{F}[\operatorname{sech}(kx)] = \frac{\pi}{k} \operatorname{sech}\left(\frac{\pi\omega}{2k}\right)$$

例：三角脉冲函数 $f(x) = \Delta(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

$$\begin{aligned}
F(\omega) &= \int_{-1}^{1} (1-|x|) e^{-i\omega x} dx = \int_{-1}^{0} (1+x) e^{-i\omega x} dx + \int_{0}^{1} (1-x) e^{-i\omega x} dx \\
&= \dots \quad (\text{通过两次分部积分}) \\
&= \frac{2(1-\cos\omega)}{\omega^2} = \frac{4\sin^2(\omega/2)}{\omega^2} = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2
\end{aligned}$$

8 赫维赛德阶跃函数的变换 (Transform of Heaviside Step Function)

赫维赛德阶跃函数 $u(t)$ 的傅立叶变换需要通过极限过程来定义。我们首先考虑一个单边指数衰减函数 $f(t) = e^{-Bt}u(t)$, 其中 $B > 0$。

$$\mathcal{F}[e^{-Bt}u(t)] = \int_{0}^{\infty} e^{-Bt} e^{-i\omega t} dt = \int_{0}^{\infty} e^{-(B+i\omega)t} dt = \frac{1}{B+i\omega}$$

现在我们令 $B \to 0^+$，来得到阶跃函数的变换：

$$\mathcal{F}[u(t)] = \lim_{B\to 0^+} \frac{1}{B+i\omega}$$

这个极限在分布意义下等于：

$$F(\omega) = \pi\delta(\omega) + \frac{1}{i\omega}$$

其中 $\frac{1}{i\omega}$ 项在积分时需要取柯西主值 (Cauchy Principal Value)。

逆变换验证：

$$
\begin{aligned}
f(t) = \mathcal{F}^{-1}\left[\pi\delta(\omega) + \frac{1}{i\omega}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(\omega) + \frac{1}{i\omega}\right)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\delta(\omega)e^{i\omega t}d\omega + \frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{i\omega}d\omega \\
&= \frac{1}{2}\cdot 1 + \frac{1}{2\pi i}\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(\omega t) + i\sin(\omega t)}{\omega}d\omega \\
&= \frac{1}{2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\sin(\omega t)}{\omega}d\omega \quad (\cos\text{项为奇函数，主值积分为}0) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin(\omega t)}{\omega}d\omega
\end{aligned}
$$

利用狄利克雷积分 $\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$，我们得到：

$$
f(t) = \frac{1}{2} + \frac{1}{\pi}\cdot\frac{\pi}{2}\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}
$$

这正是赫维赛德阶跃函数 $u(t)$。

9 傅立叶变换的应用 (Applications of Fourier Transforms)

9.1 求解常微分方程 (Solving Ordinary Differential Equations)

傅立叶变换可以将微分方程转换为代数方程，从而简化求解过程。基本原理是利用其微分性质 $\mathcal{F}[\frac{d^n f(t)}{dt^n}] = (i\omega)^n F(\omega)$。

例：受驱阻尼谐振子 (Driven Damped Harmonic Oscillator) 考虑二阶线性常微分方程：

$$
\frac{d^2x(t)}{dt^2} + 2\gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)
$$

对整个方程进行傅立叶变换，令 $X(\omega) = \mathcal{F}[x(t)]$ 及 $F(\omega) = \mathcal{F}[f(t)]$，可得：

$$
(i\omega)^2 X(\omega) + 2\gamma(i\omega)X(\omega) + \omega_0^2 X(\omega) = F(\omega)
$$

$$
\Rightarrow (-\omega^2 + 2i\gamma\omega + \omega_0^2)X(\omega) = F(\omega)
$$

解出频域中的响应 $X(\omega)$:

$$
X(\omega) = \frac{F(\omega)}{-\omega^2 + 2i\gamma\omega + \omega_0^2}
$$

时域中的解 $x(t)$ 可通过对 $X(\omega)$ 进行傅立叶逆变换得到：

$$
x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega
$$

传递函数与冲激响应 (Transfer Function and Impulse Response) 若驱动力为狄拉克 δ 函数，即 $f(t) = \delta(t)$, 则 $F(\omega) = 1$。此时的频域响应称为系统的传递函数 $H(\omega)$:

$$
H(\omega) = \frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2}
$$

其傅立叶逆变换 $h(t) = \mathcal{F}^{-1}[H(\omega)]$ 是系统的冲激响应（或格林函数）。对于任意输入 $f(t)$，系统的输出可以通过卷积得到：

$$x(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

9.2 更多变换对示例 (Further Transform Pair Examples)

例：符号函数 (Sign Function) 符号函数定义为 $\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$。它与阶跃函数的关系是 $\mathrm{sgn}(x) = 2u(x) - 1$。其傅立叶变换为：

$$\mathcal{F}[\mathrm{sgn}(x)] = \frac{2}{i\omega}$$

反之，我们有：

$$\mathcal{F}\left[\frac{1}{x}\right] = \int_{-\infty}^{\infty} \frac{1}{x} e^{-i\omega x} dx = -i\pi \mathrm{sgn}(\omega)$$

例：单边余弦函数 考虑 $f(x) = u(x)\cos(ax)$，其中 $u(x)$ 是赫维赛德阶跃函数。

$$\begin{aligned} F(\omega) &= \int_0^{\infty} \cos(ax) e^{-i\omega x} dx = \int_0^{\infty} \frac{e^{iax} + e^{-iax}}{2} e^{-i\omega x} dx \\ &= \frac{1}{2}\left[\int_0^{\infty} e^{-i(\omega-a)x} dx + \int_0^{\infty} e^{-i(\omega+a)x} dx\right] \end{aligned}$$

利用 $\int_0^{\infty} e^{-i\alpha x} dx = \pi\delta(\alpha) + \frac{1}{i\alpha}$ 的结果，可得：

$$F(\omega) = \frac{1}{2}\left[\pi\delta(\omega-a) + \frac{1}{i(\omega-a)} + \pi\delta(\omega+a) + \frac{1}{i(\omega+a)}\right]$$

$$= \frac{\pi}{2}[\delta(\omega-a) + \delta(\omega+a)] + \frac{i\omega}{a^2 - \omega^2}$$

10 在量子力学中的应用 (Application in Quantum Mechanics)

傅立叶变换是连接量子力学中位置表象和动量表象的桥梁。位置波函数 $\Psi(x)$ 和动量波函数 $\Phi(k)$ 通过傅立叶变换对联系在一起（常数因子取决于约定）：

$$\Phi(k) = \mathcal{F}[\Psi(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx} dx$$

$$\Psi(x) = \mathcal{F}^{-1}[\Phi(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k) e^{ikx} dk$$

例：δ 势阱中的束缚态 (Bound State in a Delta Potential Well) 定态薛定谔方程为 $H\Psi = E\Psi$，其中哈密顿算符 $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$。考虑一个吸引的 δ 势阱, $V(x) = -\alpha\delta(x)$ (其中 $\alpha > 0$)。薛定谔方程变为：

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \alpha\delta(x)\Psi(x) = E\Psi$$

对于束缚态，能量 $E < 0$。我们对整个方程进行傅立叶变换：

$$-\frac{\hbar^2}{2m}(-k^2)\Phi(k) - \alpha\mathcal{F}[\delta(x)\Psi(x)] = E\Phi(k)$$

其中 $\mathcal{F}[\delta(x)\Psi(x)] = \int \delta(x)\Psi(x)e^{-ikx}dx = \Psi(0)$。

$$\frac{\hbar^2 k^2}{2m}\Phi(k) - \alpha\Psi(0) = E\Phi(k)$$

整理可得动量波函数：

$$\left(\frac{\hbar^2 k^2}{2m} - E\right)\Phi(k) = \alpha\Psi(0) \quad \Rightarrow \quad \Phi(k) = \frac{\alpha\Psi(0)}{\frac{\hbar^2 k^2}{2m} - E}$$

令 $K^2 = -\frac{2mE}{\hbar^2}$（因为 $E < 0$, 所以 K 是实数），则 $E = -\frac{\hbar^2 K^2}{2m}$。

$$\Phi(k) = \frac{\alpha\Psi(0)}{\frac{\hbar^2}{2m}(k^2 + K^2)} = \frac{2m\alpha\Psi(0)/\hbar^2}{k^2 + K^2}$$

现在，我们利用 $\Psi(0)$ 和 $\Phi(k)$ 的关系来求解 K:

$$\Psi(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2m\alpha\Psi(0)/\hbar^2}{k^2 + K^2}dk$$

两边消去 $\Psi(0)$（假设它非零）：

$$1 = \frac{2m\alpha}{\sqrt{2\pi}\hbar^2}\int_{-\infty}^{\infty}\frac{dk}{k^2 + K^2} = \frac{2m\alpha}{\sqrt{2\pi}\hbar^2}\left[\frac{1}{K}\arctan\left(\frac{k}{K}\right)\right]_{-\infty}^{\infty} = \frac{2m\alpha}{\sqrt{2\pi}\hbar^2}\frac{\pi}{K}$$

$$\Rightarrow K = \frac{\sqrt{2\pi}m\alpha}{\hbar^2}$$

将 K 代入能量表达式，得到束缚态能量：

$$E = -\frac{\hbar^2 K^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{2\pi m^2\alpha^2}{\hbar^4}\right) = -\frac{\pi m\alpha^2}{\hbar^2}$$

(注：笔记中的推导似乎省略了 $\sqrt{2\pi}$ 因子，导致最终能量表达式略有不同。此处的推导基于标准的傅立叶变换定义。) 最后，通过对 $\Phi(k)$ 进行傅立叶逆变换，可以得到位置空间中的波函数，其形式为 $\Psi(x) \propto e^{-K|x|}$。

拉普拉斯变换 (Laplace Transform)

1. 定义

设 $f(t)$ 为 $t \geq 0$ 的函数，则 $f(t)$ 的拉普拉斯变换为：

$$F(p) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-pt}dt$$

其中 $p = \beta + i\omega$。通常要求函数 $f(t)$ 满足增长条件 $|f(t)| \leq Me^{\alpha t}$，且在 $\mathrm{Re}(p) > \alpha$ 时积分收敛。

2. 基本性质

1. **线性性质 (Linearity)**

$$\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(p) + bF_2(p)$$

2. **微分性质 (Differentiation)**

$$\mathcal{L}[f'(t)] = \int_0^{\infty}\frac{df(t)}{dt}e^{-pt}dt = [e^{-pt}f(t)]_0^{\infty} + p\int_0^{\infty}f(t)e^{-pt}dt = pF(p) - f(0)$$
$$\mathcal{L}[f''(t)] = p^2F(p) - pf(0) - f'(0)$$
$$\mathcal{L}[f^{(n)}(t)] = p^nF(p) - p^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

3. **p域微分 (Differentiation in p-domain)**

$$\frac{dF(p)}{dp} = \int_0^\infty (-tf(t))e^{-pt}dt = -\mathcal{L}[tf(t)]$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(p)}{dp^n}$$

4. **积分性质 (Integration)** 设 $g(t) = \int_0^t f(\tau)d\tau$，且 $g(0) = 0$。则 $g'(t) = f(t)$。由微分性质 $\mathcal{L}[g'(t)] = p\mathcal{L}[g(t)] - g(0)$，可得 $F(p) = p\mathcal{L}[g(t)]$。

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(p)}{p}$$

5. **p域积分 (Integration in p-domain)**

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_p^\infty F(s)ds$$

6. **位移性质 (Shifting Theorems)**

   - p域位移 (Frequency Shifting):

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty f(t)e^{-(p-a)t}dt = F(p-a)$$

   - t域位移 (Time Shifting):

$$\mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-pt}dt = e^{-pa}F(p)$$

   其中 $u(t-a)$ 是单位阶跃函数。

7. **卷积定理 (Convolution Theorem)**

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

$$\mathcal{L}[f_1(t) * f_2(t)] = F_1(p)F_2(p)$$

8. **标度变换 (Scaling Property)**

$$\mathcal{L}[f(at)] = \int_0^\infty f(at)e^{-pt}dt = \frac{1}{a}\int_0^\infty f(t')e^{-p(t'/a)}dt' = \frac{1}{a}F\left(\frac{p}{a}\right)$$

9. **周期函数 (Periodic Functions)** 若 $f(t)$ 周期为 T，即 $f(t+T) = f(t)$，则：

$$F(p) = \frac{\int_0^T f(t)e^{-pt}dt}{1-e^{-pT}}$$

3. 常用拉普拉斯变换表

$f(t)$	$F(p)$	$f(t)$	$F(p)$
1	$\frac{1}{p}$	e^{at}	$\frac{1}{p-a}$
t^n	$\frac{n!}{p^{n+1}}$	$t^{n-1}e^{at}$	$\frac{(n-1)!}{(p-a)^n}$
$\sin(kt)$	$\frac{k}{p^2+k^2}$	$\cos(kt)$	$\frac{p}{p^2+k^2}$
$\sinh(kt)$	$\frac{k}{p^2-k^2}$	$\cosh(kt)$	$\frac{p}{p^2-k^2}$
$t\sin(kt)$	$\frac{2pk}{(p^2+k^2)^2}$	$t\cos(kt)$	$\frac{p^2-k^2}{(p^2+k^2)^2}$
$t\sinh(kt)$	$\frac{2pk}{(p^2-k^2)^2}$	$t\cosh(kt)$	$\frac{p^2+k^2}{(p^2-k^2)^2}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}}$	$\delta(t)$	1

应用：求解微分和积分方程

例1：求解常微分方程

$y'' + 3y' + 2y = e^{-3t}$, $y(0) = y'(0) = 1$.

$$\mathcal{L}[y'' + 3y' + 2y] = \mathcal{L}[e^{-3t}]$$
$$[p^2Y(p) - py(0) - y'(0)] + 3[pY(p) - y(0)] + 2Y(p) = \frac{1}{p+3}$$
$$(p^2 + 3p + 2)Y(p) - p - 1 - 3 = \frac{1}{p+3}$$
$$(p+1)(p+2)Y(p) = p + 4 + \frac{1}{p+3} = \frac{p^2+7p+13}{p+3}$$
$$Y(p) = \frac{p^2+7p+13}{(p+1)(p+2)(p+3)} = \frac{7/2}{p+1} - \frac{3}{p+2} + \frac{1/2}{p+3}$$
$$y(t) = \frac{7}{2}e^{-t} - 3e^{-2t} + \frac{1}{2}e^{-3t}$$

例2：求解联立微分方程

$x'' - y = 1$, $y'' - x = t$, given $x(0) = 1, x'(0) = 0, y(0) = -1, y'(0) = 0$.

$$\begin{cases} p^2X(p) - px(0) - x'(0) - Y(p) = \frac{1}{p} \\ p^2Y(p) - py(0) - y'(0) - X(p) = \frac{1}{p^2} \end{cases}$$
$$\begin{cases} p^2X - p - Y = \frac{1}{p} \\ p^2Y + p - X = \frac{1}{p^2} \end{cases}$$

From the second eq: $X = p^2Y + p - \dfrac{1}{p^2}$. Substitute into the first eq:

$$p^2(p^2Y + p - \frac{1}{p^2}) - p - Y = \frac{1}{p} \implies (p^4 - 1)Y = -p^3 + p + 1 + \frac{1}{p}$$
$$(p^2-1)(p^2+1)Y = \frac{-p^4+p^2+p+1}{p} \implies Y(p) = \frac{-p^4+p^2+p+1}{p(p^2-1)(p^2+1)}$$
$$Y(p) = \frac{-(p^2-1)(p^2+1)+p+p^2}{p(p^2-1)(p^2+1)}$$
$$Y(p) = -\frac{1}{p} + \frac{p^2+p}{p(p-1)(p+1)(p^2+1)} = -\frac{1}{p} + \frac{p+1}{(p-1)(p+1)(p^2+1)}$$
$$Y(p) = -\frac{1}{p} + \frac{1}{(p-1)(p^2+1)}$$
$$Y(p) = -\frac{1}{p} + \frac{1/2}{p-1} - \frac{1/2(p+1)}{p^2+1} = -\frac{1}{p} + \frac{1}{2}\frac{1}{p-1} - \frac{1}{2}\frac{p}{p^2+1} - \frac{1}{2}\frac{1}{p^2+1}$$
$$y(t) = -1 + \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

From $x = y'' - t$,

$$x(t) = \frac{d^2}{dt^2}(-1 + \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t) - t = (\frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t) - t$$
$$x(t) = \frac{1}{2}(e^t + \cos t + \sin t) - t$$

例3: 求解积分方程 (Volterra Type)

$y(t) = at - \int_0^t (t-\tau)y(\tau)d\tau$. This is $y(t) = at - (t * y(t))$.

$$Y(p) = \mathcal{L}[at] - \mathcal{L}[t * y(t)] = \frac{a}{p^2} - \mathcal{L}[t]\mathcal{L}[y(t)]$$

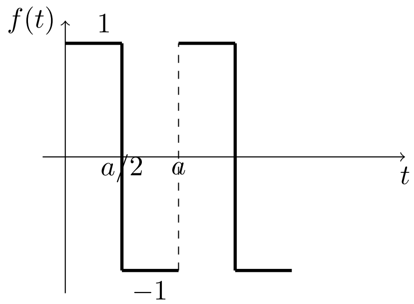
$$Y(p) = \frac{a}{p^2} - \frac{1}{p^2}Y(p)$$

$$Y(p)\left(1+\frac{1}{p^2}\right) = \frac{a}{p^2} \implies Y(p)\left(\frac{p^2+1}{p^2}\right) = \frac{a}{p^2}$$

$$Y(p) = \frac{a}{p^2+1} \implies y(t) = a\sin t$$

例4: 周期函数的变换

求下图方波的拉普拉斯变换, $f(t+a) = f(t)$。



$$F(p) = \frac{\int_0^a f(t)e^{-pt}dt}{1-e^{-pa}}$$

$$\int_0^a f(t)e^{-pt}dt = \int_0^{a/2}(1)e^{-pt}dt + \int_{a/2}^a (-1)e^{-pt}dt$$

$$= \left[-\frac{1}{p}e^{-pt}\right]_0^{a/2} - \left[-\frac{1}{p}e^{-pt}\right]_{a/2}^a$$

$$= -\frac{1}{p}(e^{-pa/2}-1) + \frac{1}{p}(e^{-pa} - e^{-pa/2}) = \frac{1}{p}(1-2e^{-pa/2}+e^{-pa}) = \frac{(1-e^{-pa/2})^2}{p}$$

$$F(p) = \frac{(1-e^{-pa/2})^2}{p(1-e^{-pa})} = \frac{(1-e^{-pa/2})^2}{p(1-e^{-pa/2})(1+e^{-pa/2})} = \frac{1-e^{-pa/2}}{p(1+e^{-pa/2})}$$

$$= \frac{e^{pa/4}-e^{-pa/4}}{p(e^{pa/4}+e^{-pa/4})} = \frac{2\sinh(pa/4)}{p(2\cosh(pa/4))} = \frac{1}{p}\tanh\left(\frac{pa}{4}\right)$$

应用：求解偏微分方程

1. 弦振动方程 (Wave Equation)

方程为 $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$，其中 $a^2 = T/\rho$。设初始条件为 $u(x,0) = f(x)$, $u_t(x,0) = g(x)$。对时间 t 进行拉普拉斯变换：

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = a^2\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$p^2U(x,p) - pu(x,0) - u_t(x,0) = a^2\frac{d^2U(x,p)}{dx^2}$$

$$\frac{d^2U}{dx^2} - \frac{p^2}{a^2}U = -\frac{p}{a^2}f(x) - \frac{1}{a^2}g(x)$$

这是一个关于 x 的二阶常微分方程。求解 $U(x,p)$ 后再进行拉普拉斯逆变换得到 $u(x,t)$。
对于稳态振动解，可设 $u(x,t)=X(x)e^{i\omega t}$，代入原方程得到亥姆霍兹方程 (Helmholtz equation):

$$\frac{d^2X}{dx^2}+k^2X=0, \quad (k=\omega/a)$$

其通解为 $X(x)=Ae^{ikx}+Be^{-ikx}$，代表了沿 x 轴正负方向传播的波。

2. 输电线方程 (Telegrapher's Equation)

对于一段微元 Δx，电压和电流满足：

$$\frac{\partial V}{\partial x}=-RI-L\frac{\partial I}{\partial t}$$
$$\frac{\partial I}{\partial x}=-GV-C\frac{\partial V}{\partial t}$$

将两式联立消去 I，得到关于 V 的电报方程：

$$\frac{\partial^2 V}{\partial x^2}=LC\frac{\partial^2 V}{\partial t^2}+(RC+LG)\frac{\partial V}{\partial t}+GRV$$

无损耗情况: $R=0, G=0$。方程简化为波动方程 $\frac{\partial^2 V}{\partial x^2}=LC\frac{\partial^2 V}{\partial t^2}$。 **正弦稳态分析**: 设 $V(x,t)=V(x)e^{i\omega t}$，$I(x,t)=I(x)e^{i\omega t}$。

$$\frac{dV(x)}{dx}=-(R+i\omega L)I(x)=-ZI(x)$$
$$\frac{dI(x)}{dx}=-(G+i\omega C)V(x)=-YV(x)$$

其中 Z,Y 分别为串联阻抗和并联导纳。再次微分可得：

$$\frac{d^2V(x)}{dx^2}=ZY\cdot V(x)=\gamma^2V(x)$$

其中 $\gamma=\sqrt{ZY}=\sqrt{(R+i\omega L)(G+i\omega C)}$ 称为传播常数。 $\gamma=\alpha+i\beta$，α 是衰减常数，β 是相移常数。 **无失真条件**: 为了让信号在传播过程中波形不发生改变，要求相速度 $v_p=\omega/\beta$ 与频率无关。这发生在 $\frac{RC}{LC}=\frac{LG}{LC}$，即 $\frac{R}{L}=\frac{G}{C}$ (Heaviside condition)。

热传导方程

一维杆的热传导方程

考虑一维杆，长度为L，截面积为A。
物理量：

- $u(x,t)$: x点在t时刻的温度
- c: 比热容
- ρ: 密度
- Q: 热量

定律： 单位时间内截面热流量

$$Q = -kA\frac{\partial u}{\partial x}$$

其中k是热导率。

考虑$[x, x+\Delta x]$一小段，在Δt时间内热量变化：

$$\Delta Q = Q_1 - Q_2$$

$$Q_1 = -kA\frac{\partial u}{\partial x}\Big|_x \Delta t$$

$$Q_2 = -kA\frac{\partial u}{\partial x}\Big|_{x+\Delta x} \Delta t$$

$$\Delta Q = c\rho(A\Delta x)\Delta u = c\rho A\Delta x(u(t+\Delta t) - u(t))$$

（$P = m/V$, $\Delta m = \rho A\Delta x$, 质量守恒）

$$\Rightarrow kA\Delta t\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x\right) = c\rho A\Delta x\Delta u$$

$$\Rightarrow k\frac{\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x}{\Delta x} = c\rho\frac{\Delta u}{\Delta t}$$

令$\Delta x, \Delta t \to 0$:

$$k\frac{\partial^2 u}{\partial x^2} = c\rho\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \quad (a^2 = \frac{k}{c\rho})$$

热源情形： 若有热源$f(x,t)$

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

若热源由电流产生 $Q_{gen} = I^2R\Delta t = j^2\rho_e\delta\Delta x\Delta t$

$$Q_1 - Q_2 + Q_{gen} = \Delta Q$$

$$kA\Delta t\frac{\partial^2 u}{\partial x^2}\Delta x + j^2\rho_e\delta\Delta x\Delta t = c\rho\delta\Delta x\Delta u$$

$$\Rightarrow \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f$$

$$(c\rho\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + F)$$

例： 稳定状态 $\frac{\partial u}{\partial t} = 0$

$$a^2\frac{\partial^2 u}{\partial x^2} + f = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \quad (\text{若}f = 0)$$

$$u(x) = Ax + B$$

电磁波方程

麦克斯韦方程组 ($\rho = 0, j = 0$ 真空中)

$$\nabla \cdot E = 0$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

其中 $\epsilon_0 \mu_0 = 1/c^2$。

推导波动方程： 利用恒等式 $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A$

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \Delta E = -\Delta E$$

$$\nabla \times (-\frac{\partial B}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow \Delta E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial t^2} = c^2 \Delta E$$

同理对B可得

$$\frac{\partial^2 B}{\partial t^2} = c^2 \Delta B$$

有源情况 ($\rho \neq 0, j \neq 0$)

$$\nabla \cdot E = \rho/\epsilon_0$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 j + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times (\nabla \times E) = \nabla(\rho/\epsilon_0) - \Delta E = -\frac{\partial}{\partial t}(\mu_0 j + \mu_0 \epsilon_0 \frac{\partial E}{\partial t})$$

$$\Delta E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \nabla(\rho/\epsilon_0) + \mu_0 \frac{\partial j}{\partial t}$$

泊松方程

当$\rho \neq 0, j = 0$ (静电场)

$$\Delta E = \nabla(\rho/\epsilon_0)$$

引入电势ϕ，$E = -\nabla \phi$

$$\Delta(-\nabla \phi) = -\nabla(\Delta \phi) = \nabla(\rho/\epsilon_0)$$

$$\Delta \phi = -\rho/\epsilon_0$$

此为泊松方程。无源情形 ($\rho = 0$)

$$\Delta \phi = 0$$

此为拉普拉斯方程。

二阶线性偏微分方程的分类

考虑方程：

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

其中A,B,C,D,E,F,G是x,y的函数。令 $\Delta=B^2-AC$

- $\Delta>0$: 双曲型 (e.g. 波动方程)
- $\Delta=0$: 抛物型 (e.g. 热传导方程)
- $\Delta<0$: 椭圆型 (e.g. 拉普拉斯方程)

这与二次曲线的分类是类似的。

坐标变换 令 $\xi=\xi(x,y),\eta=\eta(x,y)$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=\dots$$

代入原方程，得到新的方程：

$$a\frac{\partial^2 u}{\partial \xi^2}+2b\frac{\partial^2 u}{\partial \xi\partial \eta}+c\frac{\partial^2 u}{\partial \eta^2}+\cdots=G$$

其中

$$a=A(\frac{\partial \xi}{\partial x})^2+2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+C(\frac{\partial \xi}{\partial y})^2$$

$$b=A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+B(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x})+C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}$$

$$c=A(\frac{\partial \eta}{\partial x})^2+2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+C(\frac{\partial \eta}{\partial y})^2$$

可以证明 $b^2-ac=(B^2-AC)(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x})^2$ 其中后面的行列式是坐标变换的雅可比行列式。

化为标准型

目标是选择ξ,η使得a,c中至少一个为0。令 $a=0$

$$A(\frac{\partial \xi}{\partial x})^2+2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+C(\frac{\partial \xi}{\partial y})^2=0$$

$$A\left(\frac{\partial \xi/\partial x}{\partial \xi/\partial y}\right)^2+2B\left(\frac{\partial \xi/\partial x}{\partial \xi/\partial y}\right)+C=0$$

根据隐函数定理，沿 $\xi(x,y)=const$ 曲线，有 $\frac{dy}{dx}=-\frac{\partial\xi/\partial x}{\partial\xi/\partial y}$

$$A\left(\frac{dy}{dx}\right)^2-2B\left(\frac{dy}{dx}\right)+C=0$$

解出 $\frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{2B\pm\sqrt{4B^2-4AC}}{2A}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

这就是特征方程。

(1) $\Delta = B^2 - AC > 0$ (双曲型) 有两个不同的实根 $\frac{dy}{dx} = \lambda_1, \frac{dy}{dx} = \lambda_2$。解这两个常微分方程，得到两个特征线族 $\phi(x,y) = c_1, \psi(x,y) = c_2$。令 $\xi = \phi(x,y), \eta = \psi(x,y)$。这样 $a = 0, c = 0$。方程化为

$$2b\frac{\partial^2 u}{\partial\xi\partial\eta} + \dots = 0 \Rightarrow \frac{\partial^2 u}{\partial\xi\partial\eta} = \dots \quad \text{(标准型I)}$$

若再做变换 $\xi' = \xi + \eta, \eta' = \xi - \eta$，则

$$\frac{\partial^2 u}{\partial\xi'^2} - \frac{\partial^2 u}{\partial\eta'^2} = \dots \quad \text{(标准型II)}$$

(2) $\Delta = B^2 - AC = 0$ (抛物型) 只有一个实根 $\frac{dy}{dx} = \frac{B}{A}$。解得一个特征线族 $\phi(x,y) = c$。令 $\xi = \phi(x,y)$，则 $a = 0$。 η可以任取与ξ无关的函数，例如$\eta = x$。此时$b = 0$, $c \neq 0$。方程化为

$$c\frac{\partial^2 u}{\partial\eta^2} = \dots \Rightarrow \frac{\partial^2 u}{\partial\eta^2} = \dots \quad \text{(标准型)}$$

(3) $\Delta = B^2 - AC < 0$ (椭圆型) 特征方程的根是共轭复数。

$$\frac{dy}{dx} = \frac{B \pm i\sqrt{AC - B^2}}{A}$$

解也是共轭的，$\phi(x,y) \pm i\psi(x,y) = const$。令 $\xi = \phi(x,y), \eta = \psi(x,y)$。可以证明 $a = c, b = 0$。方程化为

$$a\left(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\right) = \dots \Rightarrow \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = \dots \quad \text{(标准型)}$$

再探标准型变换

(b) $\Delta = 0$: (应为 $u_{\eta\eta} = 0$) 取变换

$$\xi = x, \quad \eta = y - \frac{B}{A}x$$

则

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = -\frac{1}{A^2}(\dots)$$

(笔记此处似有误，应化为 $\frac{\partial^2 u}{\partial\eta^2} = \dots$)

(c) $\Delta < 0$: $u_{xx} + u_{yy} = 0$. 取变换

$$\xi = y - \frac{B}{A}x, \quad \eta = \frac{\sqrt{AC - B^2}}{A}x$$

$$\Rightarrow \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = \dots$$

总结特征线法： 从 $A(\frac{dy}{dx})^2 - 2B(\frac{dy}{dx}) + C = 0$ 出发

(a) $\Delta > 0$: 两条实特征线

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

解出 $\phi_1(x,y) = c_1, \phi_2(x,y) = c_2$。令 $\xi = \phi_1, \eta = \phi_2$。得标准型：

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = d\frac{\partial u}{\partial\xi} + e\frac{\partial u}{\partial\eta} + fu + g$$

(b) $\Delta = 0$: 一条实特征线

$$\frac{dy}{dx} = \frac{B}{A}$$

解出 $\phi(x,y)=c$。令 $\xi=\phi(x,y)$, η可任取（如$\eta=x$）。得标准型：

$$\frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu + g$$

(c) $\Delta < 0$: 无实特征线取 $\xi = y - \frac{B}{A}x, \eta = \frac{\sqrt{AC-B^2}}{A}x$。得标准型：

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu + g$$

例子

1. $u_{xx} - u_{tt} + au_t + bu_x = 0$ $A=1, B=0, C=-1$. $\Delta = 0-(1)(-1) = 1 > 0$ (双曲型) 特征方程: $(\frac{dt}{dx})^2 - 1 = 0 \Rightarrow \frac{dt}{dx} = \pm 1$ 特征线: $t-x=c_1, t+x=c_2$ 令 $\xi = x-t, \eta = x+t$.

2. $u_{xx} - 2u_{xt} - u_t = 0$ $A=1, B=-1, C=0$. $\Delta = (-1)^2 - 0 = 1 > 0$ (双曲型) 特征方程: $(\frac{dt}{dx})^2 + 2(\frac{dt}{dx}) = 0 \Rightarrow \frac{dt}{dx}(\frac{dt}{dx}+2) = 0$ $\frac{dt}{dx} = 0 \Rightarrow t = c_1$. $\frac{dt}{dx} = -2 \Rightarrow t+2x = c_2$. 令 $\xi = t, \eta = t+2x$.

3. $u_{xx} - 4u_{xy} + 3u_{yy} + 8u_y + x = 0$ $A=1, B=-2, C=3$. $\Delta = (-2)^2 - 1\cdot 3 = 1 > 0$ (双曲型)

4. $yu_{xx} + xu_{yy} = 0$ $A=y, B=0, C=x$. $\Delta = -xy$.

   - $xy > 0$ (I, III象限): 椭圆型
   - $xy < 0$ (II, IV象限): 双曲型
   - $x=0$ 或 $y=0$: 抛物型

定解问题

一维波动方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} & 0<x<L, t>0 \\ u|_{x=0} = 0, u|_{x=L} = 0 & \text{(边界条件)} \\ u|_{t=0} = \phi(x) & \text{(初始位移)} \\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x) & \text{(初始速度)} \end{cases}$$

叠加原理： 对于线性齐次方程 $L(u)=0$，若 u_1, u_2 是解，则 $c_1u_1 + c_2u_2$ 也是解。对于 $L(u) = F$ (非齐次方程)，其通解为 $u = u_p + u_h$，其中 u_p 是一个特解，u_h 是对应齐次方程的通解。此性质可用于分解问题。

一维热传导方程

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & 0<x<L, t>0 \\ u(0,t) = 0, u(L,t) = 0 & (t>0) \\ u(x,0) = f(x) & (0 \le x \le L) \end{cases}$$

这是一个定解问题。

例1：稳态解 如果边界条件不为零，如 $u(0,t)=T_1, u(L,t)=T_2$。稳态解 $u_E(x)$ 满足

$$\frac{d^2u_E}{dx^2}=0 \Rightarrow u_E(x)=c_1x+c_2$$

代入边界条件

$$u_E(0)=T_1 \Rightarrow c_2=T_1$$

$$u_E(L)=T_2 \Rightarrow c_1L+T_1=T_2 \Rightarrow c_1=\frac{T_2-T_1}{L}$$

所以

$$u_E(x)=\frac{T_2-T_1}{L}x+T_1$$

令 $u(x,t)=v(x,t)+u_E(x)$，则$v(x,t)$满足齐次边界条件。

分离变量法 (Separation of Variables Method)

弦振动 (String Vibration)

The governing partial differential equation (PDE):

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$$

Initial conditions:

$$u|_{t=0}=\phi(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi(x)$$

Boundary conditions (fixed ends):

$$u|_{x=0}=0, \quad u|_{x=L}=0$$

Goal: Find the solution $u(x,t)$.

推导 (Derivation)

Assume the solution can be written as a product of functions of a single variable:

$$u(x,t)=X(x)T(t)$$

Substitute into the PDE:

$$X(x)T''(t)=a^2X''(x)T(t)$$

Rearrange the terms to separate variables:

$$\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}$$

Since the left side depends only on x and the right side only on t, both must be equal to a constant. Let's call this constant $-\lambda$.

$$\frac{d}{dx}\left[\frac{X''(x)}{X(x)}\right]=0$$

$$\frac{d}{dt}\left[\frac{T''(t)}{a^2T(t)}\right] = 0$$

This gives two ordinary differential equations (ODEs):

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + \lambda a^2 T(t) = 0$$

The boundary conditions for $u(x,t)$ translate to conditions for $X(x)$, since $T(t) \not\equiv 0$ for a non-trivial solution:

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$$

$$u(L,t) = X(L)T(t) = 0 \implies X(L) = 0$$

求解本征值问题 (Solving the Eigenvalue Problem for X(x))

We analyze the possible values of the separation constant λ.

Case 1: $\lambda = 0$ The equation for $X(x)$ is $X''(x) = 0$. The general solution is:

$$X(x) = Ax + B$$

Applying the boundary conditions:

$$X(0) = B = 0$$

$$X(L) = AL + B = 0 \implies A = 0$$

This gives $X(x) = 0$, which leads to the trivial solution $u(x,t) = 0$.

Case 2: $\lambda < 0$ Let $\lambda = -k^2$ where $k > 0$. The equation is $X''(x) - k^2X(x) = 0$. The general solution is:

$$X(x) = A\cosh(kx) + B\sinh(kx)$$

Applying the boundary conditions:

$$X(0) = A\cosh(0) + B\sinh(0) = A = 0$$

$$X(L) = B\sinh(kL) = 0$$

Since $k > 0$ and $L > 0$, $\sinh(kL) \neq 0$, so $B = 0$. This again leads to the trivial solution $X(x) = 0$.

Case 3: $\lambda > 0$ Let $\lambda = k^2$ where $k > 0$. The equation is $X''(x) + k^2X(x) = 0$. The general solution is:

$$X(x) = A\cos(kx) + B\sin(kx)$$

Applying the boundary conditions:

$$X(0) = A\cos(0) + B\sin(0) = A = 0$$

So,

$$X(x) = B\sin(kx)$$

$$X(L) = B\sin(kL) = 0$$

For a non-trivial solution, we must have $B \neq 0$, which implies:

$$\sin(kL) = 0$$

This means $kL = n\pi$ for $n = 1, 2, 3, \ldots$. The possible values for k are:

$$k_n = \frac{n\pi}{L}$$

These lead to the eigenvalues (本征值):

$$\lambda_n = k_n^2 = \left(\frac{n\pi}{L}\right)^2$$

The corresponding eigenfunctions (本征函数) are:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

求解 T(t) 并叠加 (Solving for T(t) and Superposition)

Now we solve for $T(t)$ using the found eigenvalues λ_n:

$$T_n''(t) + \lambda_n a^2 T_n(t) = 0$$

$$T_n''(t) + \left(\frac{n\pi a}{L}\right)^2 T_n(t) = 0$$

The general solution for $T_n(t)$ is:

$$T_n(t) = C_n \cos\left(\frac{n\pi a t}{L}\right) + D_n \sin\left(\frac{n\pi a t}{L}\right)$$

The solution for each mode n is $u_n(x,t) = X_n(x)T_n(t)$. We absorb the constant B_n into C_n and D_n.

$$u_n(x,t) = \left(C_n \cos\left(\frac{n\pi a t}{L}\right) + D_n \sin\left(\frac{n\pi a t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

By the superposition principle (叠加原理), the general solution is the sum of all possible solutions:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi a t}{L}\right) + D_n \sin\left(\frac{n\pi a t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

利用初始条件 (Using Initial Conditions)

We determine the coefficients C_n and D_n using the initial conditions. At $t = 0$:

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

This is a Fourier sine series for $\phi(x)$. The coefficients C_n are given by:

$$C_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Next, we find the derivative with respect to t:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-C_n \frac{n\pi a}{L} \sin\left(\frac{n\pi a t}{L}\right) + D_n \frac{n\pi a}{L} \cos\left(\frac{n\pi a t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

At $t = 0$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = \sum_{n=1}^{\infty} D_n \frac{n\pi a}{L} \sin\left(\frac{n\pi x}{L}\right)$$

This is a Fourier sine series for $\psi(x)$. The coefficients are given by:

$$D_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$D_n = \frac{2}{n\pi a} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

分离变量法总结 (Summary of Separation of Variables)

1. **分离变量:** 设定 $u = TX$ (Let $u = TX$)
2. **定解:** 代入得 $X(x), T(t)$ 常微分方程 (Substitute to get ODEs for $X(x), T(t)$)
3. **边界条件:** 求解 $X(x)$, 边界条件 (齐次) $\implies$ 本征值 λ_n 与本征函数 $X_n(x)$ (Solve for $X(x)$ using homogeneous boundary conditions to get eigenvalues λ_n and eigenfunctions $X_n(x)$)
4. **齐次:** 代入 $\lambda_n \to T_n(t)$ (Substitute λ_n to find $T_n(t)$)
5. **叠加原理:** $u(x,t) = \sum u_n(x,t)$ (Superposition principle)

基本解问题 (Examples of Fundamental Solutions)

例1 (Example 1: Plucked String)

Problem:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \phi(x) = \begin{cases} \frac{3}{2}x & 0 \le x \le 2/5 \\ 3(1-x) & 2/5 \le x \le 1 \end{cases} \quad \text{(with } L = 1\text{)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = 0$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

Since $\psi(x) = 0$, we have $D_n = 0$ for all n. We calculate C_n:

$$C_n = \frac{2}{1} \int_0^1 \phi(x) \sin(n\pi x) dx$$

$$C_n = 2 \left[\int_0^{2/5} \frac{3}{2} x \sin(n\pi x) dx + \int_{2/5}^1 3(1-x) \sin(n\pi x) dx \right]$$

After integration (result from notes):

$$C_n = \frac{9}{5n^2\pi^2} \sin\left(\frac{2n\pi}{5}\right)$$

The final solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{9}{5n^2\pi^2} \sin\left(\frac{2n\pi}{5}\right) \cos(n\pi at) \sin(n\pi x)$$

例2 (Example 2: Struck String)

Problem:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = \phi(x) = 0$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = \frac{K}{\rho}\delta(x-c) \quad \text{(impulse at } x = c\text{)}$$
$$u(0,t) = 0, \quad u(L,t) = 0$$

Since $\phi(x) = 0$, we have $C_n = 0$ for all n. We calculate D_n:

$$D_n = \frac{2}{n\pi a} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$D_n = \frac{2}{n\pi a} \int_0^L \frac{K}{\rho}\delta(x-c) \sin\left(\frac{n\pi x}{L}\right) dx$$

Using the sifting property of the Dirac delta function:

$$D_n = \frac{2K}{n\pi a\rho} \sin\left(\frac{n\pi c}{L}\right)$$

The final solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2K}{n\pi a\rho} \sin\left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi at}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

物理解释 (Physical Interpretation)

The solution for a single mode can be written in phase-amplitude form:

$$u_n(x,t) = N_n \sin(\omega_n t + \theta_n) \sin\left(\frac{n\pi x}{L}\right)$$

where the angular frequency is $\omega_n = \frac{n\pi a}{L}$. The amplitude N_n and phase θ_n are given by:

$$N_n = \sqrt{C_n^2 + D_n^2}$$

$$\tan\theta_n = \frac{C_n}{D_n}$$

An alternative form from the notes is:

$$u_n(x,t) = A_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$u_n(t) = B_n(x_0) \sin(\omega_n t + \theta_n)$$

驻波 (Standing Waves)

The solution $u_n(x,t)$ represents a standing wave.

- **节点 (Nodes):** Points that do not move. Occur when $\sin(\frac{n\pi x}{L}) = 0$.
$$\frac{n\pi x}{L} = m\pi, \quad m = 0, 1, \ldots, n$$
$$x_m = \frac{m}{n} L$$
- **波腹 (Antinodes):** Points of maximum amplitude (x_0).

单模振动 (Single-mode Oscillation)

A single eigenfunction corresponds to a single mode of vibration.
$$E \sim \sin\left(\frac{n\pi x}{L}\right)$$

与量子力学类比 (Analogy to Quantum Mechanics)

The spatial part of the wave solution is analogous to the wave function for a particle in a 1D infinite potential well.
$$\Psi \sim \sin\left(\frac{n\pi x}{L}\right)$$
The time-dependent Schrödinger equation:
$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U(x)\Psi$$

其他边界条件 (Other Boundary Conditions)

1. **两端固定 (Fixed-Fixed):** $u(0,t) = 0$, $u(L,t) = 0$.
2. **一端固定, 一端自由 (Fixed-Free):** $u(0,t) = 0$, $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$.
3. **两端自由 (Free-Free):** $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$, $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$.
4. **辐射边界条件 (Radiation Boundary Condition):** $-k\left.\frac{\partial u}{\partial x}\right|_{x=L} = H(u(L,t) - u_0)$.

有阻尼波动方程与电报方程 (Damped Wave and Telegrapher's Equation)

例: 电报方程 (Example: Telegrapher's Equation)

The general form of the Telegrapher's equation is:
$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t} + cu = 0$$
With initial conditions:
$$u|_{t=0} = \phi(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x)$$

And boundary conditions ($b, c > 0$):

$$u|_{x=0} = 0, \quad u|_{x=L} = 0$$

Using separation of variables, $u(x,t) = X(x)T(t)$, we get:

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0, \quad X(L) = 0$$

and

$$T''(t) + 2bT'(t) + (\lambda a^2 + c)T(t) = 0$$

The solution for $X(x)$ is the same as for the standard wave equation:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

The solution for $T(t)$ is that of a damped harmonic oscillator. For the underdamped case, the solution has the form:

$$u(x,t) = e^{-bt} \sum_{n=1}^{\infty} (C_n \cos(q_n t) + D_n \sin(q_n t)) \sin\left(\frac{n\pi x}{L}\right)$$

where the new frequency q_n is:

$$q_n = \sqrt{\left|\left(\frac{n\pi a}{L}\right)^2 + c - b^2\right|}$$

The coefficients C_n, D_n are determined by the initial conditions.

例: 有阻尼波动方程 (Example: Damped Wave Equation)

This is a special case of the Telegrapher's equation where $c = 0$.

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t} = 0$$

The equation for $T(t)$ becomes:

$$T_n''(t) + 2bT_n'(t) + \lambda_n a^2 T_n(t) = 0$$

The characteristic equation is $r^2 + 2br + (\frac{n\pi a}{L})^2 = 0$. The behavior depends on the discriminant. Let $q_n = \sqrt{|(\frac{n\pi a}{L})^2 - b^2|}$.

The general solution for $T_n(t)$ can be one of three cases for each mode n:

1. **Underdamped** ($\frac{n\pi a}{L} > b$):

$$T_n(t) = e^{-bt} \left(C_n \cos(q_n t) + D_n \sin(q_n t)\right)$$

2. **Critically Damped** ($\frac{n\pi a}{L} = b$):

$$T_n(t) = e^{-bt}(C_n + D_n t)$$

3. **Overdamped** ($\frac{n\pi a}{L} < b$):

$$T_n(t) = e^{-bt}\left(C_n \cosh(q_n t) + D_n \sinh(q_n t)\right)$$

Let's assume the underdamped case holds for all modes of interest ($\frac{bL}{\pi a} < 1$). The total solution is:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-bt}\left(C_n \cos(q_n t) + D_n \sin(q_n t)\right) \sin\left(\frac{n\pi x}{L}\right)$$

To find the coefficients from $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies C_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\psi(x) = \sum_{n=1}^{\infty}(-bC_n + q_n D_n) \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies -bC_n + q_n D_n = \frac{2}{L}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\implies D_n = \frac{b}{q_n}C_n + \frac{2}{q_n L}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

If the initial velocity is zero, $\psi(x) = 0$, then $D_n = \frac{b}{q_n}C_n$.

热传导方程 (Heat Equation)

例: 傅里叶热棒 (Example: Fourier Heat Rod)

The problem describes the temperature $u(x,t)$ in a rod with insulated ends.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Initial condition:

$$u(t=0) = \phi(x)$$

Boundary conditions (insulated ends):

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Separating variables $u(x,t) = X(x)T(t)$ yields:

$$X''(x) + \lambda X(x) = 0, \quad \text{with } X'(0) = 0, X'(L) = 0$$

$$T'(t) + \lambda a^2 T(t) = 0$$

Solving the eigenvalue problem for $X(x)$:

- Case $\lambda = 0$: $X''(x) = 0 \implies X(x) = Ax + B$. $X'(0) = A = 0$. $X'(L) = A = 0$. So $X_0(x) = B_0$ (a constant) is an eigenfunction.
- Case $\lambda < 0$: Trivial solution $X(x) = 0$.
- Case $\lambda > 0$ ($\lambda = k^2$): $X(x) = A\cos(kx) + B\sin(kx)$. $X'(0) = Bk = 0 \implies B = 0$. $X'(L) = -Ak\sin(kL) = 0 \implies \sin(kL) = 0$. Thus $kL = n\pi$ for $n = 1, 2, 3, \dots$.

The eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$ for $n = 0, 1, 2, \dots$. The eigenfunctions are $X_n(x) = A_n \cos(\frac{n\pi x}{L})$. Solving for $T(t)$: For $n > 0$: $T_n'(t) + (\frac{n\pi a}{L})^2 T_n(t) = 0 \implies T_n(t) = C_n e^{-(\frac{n\pi a}{L})^2 t}$. For $n = 0$ ($\lambda_0 = 0$): $T_0'(t) = 0 \implies T_0(t) = C_0$. The general solution is by superposition:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \exp\left[-\left(\frac{n\pi a}{L}\right)^2 t\right] \cos\left(\frac{n\pi x}{L}\right)$$

Using the initial condition $u(x,0) = \phi(x)$:

$$\phi(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$$

This is a Fourier cosine series. The coefficients are:

$$C_0 = \frac{1}{L}\int_0^L \phi(x)dx$$

$$C_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

波动方程更多示例 (Further Examples for the Wave Equation)

例: 自由-固定端 (Example: Free-Fixed End)

The note appears to solve for a rod with a free end at $x = 0$ and a fixed end at $x = L$.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u|_{x=L} = 0$$

Separation of variables leads to $X''(x) + \lambda X(x) = 0$ with $X'(0) = 0, X(L) = 0$. Let $\lambda = k^2$. $X(x) = A\cos(kx) + B\sin(kx)$.

$$X'(0) = Bk = 0 \implies B = 0$$

$$X(L) = A\cos(kL) = 0 \implies kL = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

The eigenvalues and eigenfunctions are:

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$$

$$X_n(x) = A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

The general solution is:

$$u(x,t) = \sum_{n=0}^{\infty} \left(C_n \cos\left(\frac{(2n+1)\pi at}{2L}\right) + D_n \sin\left(\frac{(2n+1)\pi at}{2L}\right)\right) \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

Coefficients are found from initial conditions $\phi(x)$ and $\psi(x)$:

$$C_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

$$D_n = \frac{2}{L}\frac{2L}{(2n+1)\pi a}\int_0^L \psi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

Specific Case: If $u(x,0) = \cos(\frac{\pi x}{2L})$ and $u_t(x,0) = 0$. This corresponds to the $n = 0$ mode.

$$D_n = 0 \text{ for all } n$$

$$C_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

By orthogonality, this integral is non-zero only for $n = 0$.

$$C_0 = \frac{2}{L}\int_0^L \cos^2\left(\frac{\pi x}{2L}\right) dx = \frac{2}{L}\cdot\frac{L}{2} = 1$$

All other $C_n = 0$. The solution is:

$$u(x,t) = \cos\left(\frac{\pi at}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)$$

例: 固定-自由端 (Example: Fixed-Free End)

Another example shows fixed-free boundary conditions: $u(0,t)=0, u_x(L,t)=0$. Eigenfunctions: $\sin(\frac{(2n+1)\pi x}{2L})$. Initial conditions: $u(x,0)=E$ (a constant), $u_t(x,0)=0$. Then $D_n=0$ for all n.

$$C_n = \frac{2}{L}\int_0^L E\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx$$

$$C_n = \frac{2E}{L}\left[-\frac{2L}{(2n+1)\pi}\cos\left(\frac{(2n+1)\pi x}{2L}\right)\right]_0^L$$

$$C_n = -\frac{4E}{(2n+1)\pi}(\cos(\frac{(2n+1)\pi}{2})-\cos(0)) = \frac{4E}{(2n+1)\pi}$$

The solution is:

$$u(x,t) = \frac{4E}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos\left(\frac{(2n+1)\pi at}{2L}\right)\sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

总结 (Summary)

$$X''(x)+\lambda X(x)=0$$

$$u|_{x=0}=u|_{x=L}=0 \quad\Rightarrow\quad \lambda_n=(\frac{n\pi}{L})^2,\quad X_n(x)=B_n\sin(\frac{n\pi x}{L})$$

$$\frac{\partial u}{\partial x}|_{x=0}=\frac{\partial u}{\partial x}|_{x=L}=0 \quad\Rightarrow\quad \lambda_n=(\frac{n\pi}{L})^2,\quad X_n(x)=A_n\cos(\frac{n\pi x}{L})$$

$$u|_{x=0}=\frac{\partial u}{\partial x}|_{x=L}=0 \quad\Rightarrow\quad \lambda_n=(\frac{(2n+1)\pi}{2L})^2,\quad X_n(x)=A_n\cos(\frac{(2n+1)\pi x}{2L})$$

二维波动方程 (2D Wave Equation)

$$\frac{\partial^2 u}{\partial t^2}=c^2(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$$

$$\begin{cases} u|_{t=0}=\phi(x,y) \\ u_t|_{t=0}=\psi(x,y) \\ u|_{x=0}=u|_{x=a}=0 \quad 0\le y\le b \\ u|_{y=0}=u|_{y=b}=0 \quad 0\le x\le a \end{cases}$$

令 (Let)

$$u(x,y,t)=V(x,y)T(t)$$

$$\Rightarrow \frac{T''}{c^2T}=\frac{1}{V}(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2})=-\lambda$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\lambda V=0$$

$$T''+\lambda c^2T=0$$

再令 (Let again)

$$V(x,y)=X(x)Y(y)$$

$$\Rightarrow \frac{X''}{X}=-\frac{Y''+\lambda Y}{Y}=-\mu$$

$$\Rightarrow \begin{cases} X''+\mu X=0 \\ X(0)=X(a)=0 \end{cases}$$

$$\begin{cases} Y'' + \nu Y = 0, & \nu = \lambda - \mu \\ Y(0) = Y(b) = 0 \end{cases}$$

解得 (Solution is)

$$\mu_m = (\frac{m\pi}{a})^2, \quad X_m(x) = \sin(\frac{m\pi x}{a})$$

$$\nu_n = (\frac{n\pi}{b})^2, \quad Y_n(y) = \sin(\frac{n\pi y}{b})$$

到 (Thus)

$$\lambda_{mn} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$

$$V_{mn}(x,y) = \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

代入 (Substitute into T)

$$T_{mn}(t) = C_{mn}\cos(\omega_{mn}t) + D_{mn}\sin(\omega_{mn}t)$$

$$\omega_{mn} = c\sqrt{\lambda_{mn}} = c\pi\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

基频 (Fundamental frequency)

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

叠加 (Superposition)

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(C_{mn}\cos(\omega_{mn}t) + D_{mn}\sin(\omega_{mn}t))\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

$$\phi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

$$\psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \omega_{mn}D_{mn}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

正交性 (Orthogonality)

$$\int_0^a\int_0^b V_{mn}(x,y)V_{m'n'}(x,y)dxdy = (\int_0^a \sin(\frac{m\pi x}{a})\sin(\frac{m'\pi x}{a})dx)(\int_0^b \sin(\frac{n\pi y}{b})\sin(\frac{n'\pi y}{b})dy)$$

$$= \frac{ab}{4}\delta_{mm'}\delta_{nn'}$$

得 (We get)

$$C_{mn} = \frac{4}{ab}\int_0^a\int_0^b \phi(x,y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dxdy$$

$$D_{mn} = \frac{4}{ab\omega_{mn}}\int_0^a\int_0^b \psi(x,y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dxdy$$

特征函数集 (Set of eigenfunctions)

$$\{\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})\}$$

例 (Example)

$$a = b = 1, \quad c = \frac{1}{\pi}$$

$$\phi = x(1-x)y(1-y)$$

$$\psi = 0 \quad \Rightarrow \quad D_{mn} = 0$$

$$C_{mn} = 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin(m\pi x)\sin(n\pi y)dxdy$$

$$= 4[\int_0^1 x(1-x)\sin(m\pi x)dx][\int_0^1 y(1-y)\sin(n\pi y)dy]$$

$$\int_0^1 x(1-x)\sin(m\pi x)dx = \frac{2(1-(-1)^m)}{m^3\pi^3}$$

$$C_{mn} = 4\frac{2(1-(-1)^m)}{m^3\pi^3}\frac{2(1-(-1)^n)}{n^3\pi^3} = \frac{16(1-(-1)^m)(1-(-1)^n)}{m^3n^3\pi^6}$$

解 (Solution)

$$u = \sum_{m,n=1,3,5,\ldots}^{\infty} \frac{64}{\pi^6 m^3 n^3}\cos(\sqrt{m^2+n^2}t)\sin(m\pi x)\sin(n\pi y)$$

二维热传导方程 (2D Heat Equation)

$$\frac{\partial u}{\partial t} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$u|_{t=0} = \phi(x,y)$$

$$u|_{x=0} = u|_{x=a} = 0, \quad u|_{y=0} = u|_{y=b} = 0$$

令 (Let)

$$u = V(x,y)T(t)$$

$$\Rightarrow \frac{1}{c^2T}\frac{\partial T}{\partial t} = \frac{1}{V}(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}) = -\lambda$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \lambda V = 0$$

$$T' + c^2\lambda T = 0$$

$$\Rightarrow \lambda_{mn} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$

$$V_{mn}(x,y) = \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

$$T_{mn}(t) = e^{-\omega_{mn}t}$$

$$\omega_{mn} = c^2\lambda_{mn} = c^2\pi^2((\frac{m}{a})^2 + (\frac{n}{b})^2)$$

解 (Solution)

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-\omega_{mn}t}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

$$C_{mn} = \frac{4}{ab}\int_0^a\int_0^b \phi(x,y)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})dxdy$$

例1 (Example 1)

$$a = b = 1, c = 1$$

$$\phi = x(1-x)y(1-y)$$

解 (Solution)

$$u(x,y,t) = \sum_{m,n=1,3,5,\ldots}^{\infty} \frac{64}{m^3n^3\pi^6}e^{-[(m\pi)^2+(n\pi)^2]t}\sin(m\pi x)\sin(n\pi y)$$

中心温度 (Temperature at the center)

$$u(\frac{1}{2},\frac{1}{2},t)=\sum_{m,n=1,3,5,\ldots}^{\infty}\frac{64}{m^3n^3\pi^6}e^{-[(m\pi)^2+(n\pi)^2]t}\sin(\frac{m\pi}{2})\sin(\frac{n\pi}{2})$$

例2 (Example 2)

$$a=b=1,c=1$$

$$\phi=\sin(\pi x)\sin(\pi y)$$

解 (Solution)

$$C_{mn}=4\int_0^1\int_0^1\sin(\pi x)\sin(\pi y)\sin(m\pi x)\sin(n\pi y)dxdy$$

$$=\delta_{m1}\delta_{n1}$$

$$u=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\delta_{m1}\delta_{n1}e^{-\omega_{mn}t}\sin(m\pi x)\sin(n\pi y)$$

$$u=e^{-2\pi^2t}\sin(\pi x)\sin(\pi y)$$

$$u(\frac{1}{2},\frac{1}{2},t)=e^{-2\pi^2t}$$

一维热传导方程相关 (Related 1D Heat Equation Concepts) (a) 稳态温度 (Steady-state temperature) ($t\to\infty$)

$$u(x,\infty)=C_0=\frac{1}{L}\int_0^L\phi(x)dx$$

$$u(x,0)=\phi(x)\quad u(x,\infty)=\text{常数(constant)}$$

平均温度 (Average temperature)

$$U(t)=\frac{1}{L}\int_0^L u(x,t)dx=C_0$$

(b) 若 (If) $\phi(x)=x$:

$$C_0=\frac{L}{2}$$

$$C_n=\frac{2}{L}\int_0^L x\cos(\frac{n\pi x}{L})dx=\frac{2L}{n^2\pi^2}[(-1)^n-1]$$

$$u(x,t)=\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}[(-1)^n-1]e^{-[(n\pi/L)c]^2t}\cos(\frac{n\pi x}{L})$$

(c) 若 (If) $\phi(x)=1+\cos(\frac{2\pi x}{L})$:

$$C_0=1$$

$$C_n=\frac{2}{L}\int_0^L\cos(\frac{2\pi x}{L})\cos(\frac{n\pi x}{L})dx=\delta_{2n}$$

$$u(x,t)=1+e^{-[(2\pi c/L)]^2t}\cos(\frac{2\pi x}{L})$$

特征函数正交性 (Orthogonality of Eigenfunctions)

设 $X_n(x)$ 和 $X_m(x)$ 是以下特征值问题的特征函数，其中 $x \in [0, L]$:

$$X''(x) + \lambda X(x) = 0$$

所以，我们有:

$$X_n''(x) + \lambda_n X_n(x) = 0$$

$$X_m''(x) + \lambda_m X_m(x) = 0$$

从微分方程的恒等式出发:

$$\frac{d}{dx}(X_m' X_n - X_n' X_m) = X_m'' X_n - X_n'' X_m$$

将特征值方程代入上式:

$$X_m'' X_n - X_n'' X_m = (-\lambda_m X_m) X_n - (-\lambda_n X_n) X_m = (\lambda_n - \lambda_m) X_n X_m$$

两边从 0 到 L 积分:

$$\int_0^L (\lambda_n - \lambda_m) X_n X_m dx = \int_0^L \frac{d}{dx}(X_m' X_n - X_n' X_m) dx$$

$$(\lambda_n - \lambda_m) \int_0^L X_n X_m dx = [X_m' X_n - X_n' X_m]_0^L$$

如果边界项 $Q = [X_m' X_n - X_n' X_m]_0^L = 0$，并且特征值不同 $(\lambda_n \neq \lambda_m)$，那么特征函数是正交的:

$$\int_0^L X_n(x) X_m(x) dx = 0 \quad (n \neq m)$$

热传导问题 1

考虑以下热传导方程、初始条件和边界条件:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = \phi(x)$$

$$\frac{\partial u}{\partial x}(0, t) = 0$$

$$\frac{\partial u}{\partial x}(L, t) + hu(L, t) = 0$$

使用分离变量法 $u(x, t) = X(x)T(t)$，我们得到两个常微分方程:

$$X''(x) + \lambda X(x) = 0, \quad \text{with } X'(0) = 0, X'(L) + hX(L) = 0$$

$$T'(t) + \lambda a^2 T(t) = 0$$

求解特征值问题

对于 $X(x)$ 的方程:

- **情况 1:** $\lambda = 0$
  $X''(x) = 0 \Rightarrow X(x) = Ax + B$。
  $X'(0) = 0 \Rightarrow A = 0$。
  $X'(L) + hX(L) = 0 \Rightarrow 0 + hB = 0$。如果 $h \neq 0$, 则 $B = 0$ (平凡解)。如果 $h = 0$, $\lambda = 0$ 是一个特征值。

- **情况 2:** $\lambda > 0$
  设 $\lambda = \mu^2$ $(\mu > 0)$。通解为:

  $$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

  应用边界条件:

  $$X'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$$

  $$X'(0) = B\mu = 0 \Rightarrow B = 0$$

  所以 $X(x) = A\cos(\mu x)$。应用第二个边界条件:

  $$X'(L) + hX(L) = -A\mu\sin(\mu L) + hA\cos(\mu L) = 0$$

  假设 $A \neq 0$,我们得到特征方程:

  $$\cot(\mu L) = \frac{\mu}{h}$$

  令 $\alpha = \mu L$,则方程变为 $\cot(\alpha) = \frac{\alpha}{hL}$。此方程的正根 α_n (通过图解法求得) 给出特征值 $\lambda_n = \mu_n^2 = (\frac{\alpha_n}{L})^2$。
  对应的特征函数为:

  $$X_n(x) = \cos(\mu_n x)$$

通解和系数

$T(t)$ 的解为 $T_n(t) = C_n e^{-\lambda_n a^2 t} = C_n e^{-\mu_n^2 a^2 t}$。总解是这些解的叠加:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t}\cos(\mu_n x)$$

应用初始条件 $u(x,0) = \phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n\cos(\mu_n x)$$

为了求系数 C_n,我们利用特征函数的正交性。将两边乘以 $\cos(\mu_m x)$ 并从 0 到 L 积分:

$$\int_0^L \phi(x)\cos(\mu_m x)dx = \sum_{n=1}^{\infty} C_n \int_0^L \cos(\mu_n x)\cos(\mu_m x)dx$$

正交积分的计算如下:

$$\int_0^L \cos(\mu_n x)\cos(\mu_m x)dx = \begin{cases} 0 & m \neq n \\ \int_0^L \cos^2(\mu_n x)dx & m = n \end{cases}$$

当 $m = n$ 时：

$$\int_0^L \cos^2(\mu_n x)dx = \int_0^L \frac{1+\cos(2\mu_n x)}{2}dx = \left[\frac{x}{2} + \frac{\sin(2\mu_n x)}{4\mu_n}\right]_0^L = \frac{L}{2} + \frac{\sin(2\mu_n L)}{4\mu_n}$$

因此，系数 C_n 为：

$$C_n = \frac{\int_0^L \phi(x)\cos(\mu_n x)dx}{\frac{L}{2} + \frac{\sin(2\mu_n L)}{4\mu_n}} = \frac{2}{L(1+\frac{\sin(2\mu_n L)}{2\mu_n L})}\int_0^L \phi(x)\cos(\mu_n x)dx$$

总热量

系统中的总热量 $U(t)$ 是 $u(x,t)$ 在空间域上的积分：

$$U(t) = \int_0^L u(x,t)dx = \int_0^L \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t}\cos(\mu_n x)dx$$

$$U(t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t}\int_0^L \cos(\mu_n x)dx$$

$$\int_0^L \cos(\mu_n x)dx = \left[\frac{\sin(\mu_n x)}{\mu_n}\right]_0^L = \frac{\sin(\mu_n L)}{\mu_n}$$

所以：

$$U(t) = \sum_{n=1}^{\infty} C_n \left(\frac{\sin(\mu_n L)}{\mu_n}\right) e^{-\mu_n^2 a^2 t}$$

热传导问题 2

考虑具有不同边界条件的热传导问题：

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \phi(x)$$

$$u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) + hu(L,t) = 0$$

分离变量得到与之前相同的方程，但边界条件不同：

$$X''(x) + \lambda X(x) = 0, \quad \text{with } X(0) = 0, X'(L) + hX(L) = 0$$

$$T'(t) + \lambda a^2 T(t) = 0$$

求解特征值问题

对于 $X(x)$ 的方程:

- **情况 1:** $\lambda = 0$
  $X(x) = Ax+B$。$X(0) = 0 \Rightarrow B = 0$。$X'(L)+hX(L) = 0 \Rightarrow A+h(AL) = 0 \Rightarrow A(1+hL) = 0$。通常 $1 + hL \neq 0$, 所以 $A = 0$ (平凡解)。

- **情况 2:** $\lambda > 0$
  设 $\lambda = \mu^2$ ($\mu > 0$)。通解为：

  $$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

  应用边界条件：

  $$X(0) = A = 0$$

  所以 $X(x) = B\sin(\mu x)$。应用第二个边界条件：

  $$X'(L) + hX(L) = B\mu\cos(\mu L) + hB\sin(\mu L) = 0$$

  假设 $B \neq 0$，我们得到特征方程：

  $$\tan(\mu L) = -\frac{\mu}{h}$$

  令 $\alpha = \mu L$，则方程变为 $\tan(\alpha) = -\frac{\alpha}{hL}$。此方程的正根 α_n (通过图解法求得) 给出特征值 $\lambda_n = \mu_n^2 = (\frac{\alpha_n}{L})^2$。

  对应的特征函数为：

  $$X_n(x) = \sin(\mu_n x)$$

通解和系数

$T(t)$ 的解为 $T_n(t) = C'_n e^{-\mu_n^2 a^2 t}$。总解是这些解的叠加（令 $C_n = B_n C'_n$）：

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t} \sin(\mu_n x)$$

应用初始条件 $u(x,0) = \phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin(\mu_n x)$$

为了求系数 C_n，我们利用正交性。对于这组边界条件，我们首先验证边界项 Q 为零：

$$Q = [X'_m X_n - X'_n X_m]_0^L$$

在 $x = L$ 处: $X'_m = -hX_m$ and $X'_n = -hX_n$。

$$X'_m(L)X_n(L) - X'_n(L)X_m(L) = (-hX_m(L))X_n(L) - (-hX_n(L))X_m(L) = 0$$

在 $x = 0$ 处: $X_n(0) = 0$ and $X_m(0) = 0$, 所以项为零。因此 $Q = 0$，特征函数是正交的。正交积分的计算如下：

$$\int_0^L \sin(\mu_n x)\sin(\mu_m x)dx = \begin{cases} 0 & m \neq n \\ \int_0^L \sin^2(\mu_n x)dx & m = n \end{cases}$$

当 $m = n$ 时：

$$\int_0^L \sin^2(\mu_n x)dx = \int_0^L \frac{1-\cos(2\mu_n x)}{2}dx = \left[\frac{x}{2} - \frac{\sin(2\mu_n x)}{4\mu_n}\right]_0^L = \frac{L}{2} - \frac{\sin(2\mu_n L)}{4\mu_n}$$

因此，系数 C_n 为：

$$C_n = \frac{\int_0^L \phi(x)\sin(\mu_n x)dx}{\frac{L}{2} - \frac{\sin(2\mu_n L)}{4\mu_n}}$$

利用 $\tan(\mu_n L) = -\mu_n/h$, 可以进一步化简分母。$\sin(2\mu_n L) = 2\sin(\mu_n L)\cos(\mu_n L)$。

总热量

系统中的总热量 $U(t)$:

$$U(t) = \int_0^L u(x,t)dx = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2 a^2 t} \int_0^L \sin(\mu_n x)dx$$

$$\int_0^L \sin(\mu_n x)dx = \left[-\frac{\cos(\mu_n x)}{\mu_n}\right]_0^L = \frac{1-\cos(\mu_n L)}{\mu_n}$$

所以：

$$U(t) = \sum_{n=1}^{\infty} C_n \left(\frac{1-\cos(\mu_n L)}{\mu_n}\right) e^{-\mu_n^2 a^2 t}$$

11 问题描述 (Problem Statement)

我们求解一个一维热传导方程，带有Robin边界条件。

$$
\begin{aligned}
\text{PDE: } & \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } x \in (0, L), t > 0 \\
\text{BCs: } & u(0, t) = 0 \\
& \frac{\partial u}{\partial x}(L, t) - hu(L, t) = 0 \\
\text{IC: } & u(x, 0) = \phi(x)
\end{aligned}
$$

这里的 h 是一个常数。当 $h > 0$ 时，此边界条件描述了在杆的末端 $x = L$ 处有热量散失到周围介质中。

12 分离变量法 (Method of Separation of Variables)

我们假设解的形式为 $u(x, t) = X(x)T(t)$。将其代入偏微分方程得到：

$$X(x)T'(t) = a^2 X''(x)T(t)$$

分离变量后，我们得到：

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

这里 λ 是一个常数。这引出了两个常微分方程：

$$X''(x) + \lambda X(x) = 0 \tag{1}$$

$$T'(t) + \lambda a^2 T(t) = 0 \tag{2}$$

相应的边界条件变为：

$$
\begin{aligned}
X(0) &= 0 \\
X'(L) - hX(L) &= 0
\end{aligned}
$$

13 特征值问题 (The Eigenvalue Problem)

我们现在求解 $X(x)$ 的方程，需要根据 λ 的符号分情况讨论。

13.1 情况 1: $\lambda = 0$

$X''(x) = 0$ 的通解是 $X(x) = Ax + B$。

- 从 $X(0) = 0$ 可得 $B = 0$。
- 于是 $X(x) = Ax$，$X'(x) = A$。代入第二个边界条件得到 $A - h(AL) = A(1 - hL) = 0$。
- 如果 $hL \neq 1$，则 $A = 0$。这导致 $X(x) = 0$，是一个平凡解。
- 如果 $hL = 1$，任何 A 都是解，但这种情况通常单独处理，这里我们假设 $hL \neq 1$。

因此，$\lambda = 0$ 不是一个特征值。

13.2 情况 2: $\lambda = k^2 > 0$ (衰减模式)

$X''(x) + k^2 X(x) = 0$ 的通解是 $X(x) = A\cos(kx) + B\sin(kx)$。

- 从 $X(0) = 0$ 可得 $A = 0$。
- 于是 $X(x) = B\sin(kx)$，$X'(x) = Bk\cos(kx)$。代入第二个边界条件得到：

$$Bk\cos(kL) - hB\sin(kL) = 0$$

- 为得到非平凡解 ($B \neq 0$)，我们必须有 $k\cos(kL) - h\sin(kL) = 0$，即：

$$\tan(kL) = \frac{k}{h}$$

该超越方程的正根 k_n ($n = 1, 2, 3, \dots$) 确定了正特征值 $\lambda_n = k_n^2$。对应的特征函数是 $X_n(x) = \sin(k_n x)$。

13.3 情况 3: $\lambda = -\mu^2 < 0$ (增长模式)

$X''(x) - \mu^2 X(x) = 0$ 的通解是 $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$。

- 从 $X(0) = 0$ 可得 $A = 0$。
- 于是 $X(x) = B\sinh(\mu x)$，$X'(x) = B\mu\cosh(\mu x)$。代入第二个边界条件得到：

$$B\mu\cosh(\mu L) - hB\sinh(\mu L) = 0$$

- 为得到非平凡解 ($B \neq 0$)，我们必须有 $\mu\cosh(\mu L) - h\sinh(\mu L) = 0$，即：

$$\tanh(\mu L) = \frac{\mu}{h} = \frac{\mu L}{hL}$$

这个方程的解的存在性取决于 hL 的值。通过图形分析，仅当直线 $y = (\frac{1}{hL})x$ 的斜率小于双曲正切函数 $y = \tanh(x)$ 在原点的斜率（即 1）时，才存在正实数解 μ_0。

$$\frac{1}{hL} < 1 \implies hL > 1$$

如果 $hL > 1$，则存在一个正解 μ_0，对应一个负特征值 $\lambda_0 = -\mu_0^2$ 和特征函数 $X_0(x) = \sinh(\mu_0 x)$。这个模式会随时间指数增长，因为它对应的时间解为 $T_0(t) = C_0 e^{\mu_0^2 a^2 t}$。

14 通解与系数确定

通过叠加原理，通解是所有可能解的线性组合。

- **如果** $hL \leq 1$: 只存在衰减模式。

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k_n^2 a^2 t} \sin(k_n x)$$

- **如果** $hL > 1$: 存在一个增长模式和无穷多个衰减模式。

$$u(x,t) = C_0 e^{\mu_0^2 a^2 t} \sinh(\mu_0 x) + \sum_{n=1}^{\infty} C_n e^{-k_n^2 a^2 t} \sin(k_n x)$$

系数 C_0 和 C_n 由初始条件 $u(x,0)=\phi(x)$ 和特征函数的正交性确定。

$$\phi(x) = C_0 \sinh(\mu_0 x) + \sum_{n=1}^{\infty} C_n \sin(k_n x)$$

系数公式为：

$$C_0 = \frac{\int_0^L \phi(x)\sinh(\mu_0 x)dx}{\int_0^L \sinh^2(\mu_0 x)dx}$$
$$C_n = \frac{\int_0^L \phi(x)\sin(k_n x)dx}{\int_0^L \sin^2(k_n x)dx}$$

其中分母中的归一化积分为：

$$\int_0^L \sinh^2(\mu_0 x)dx = \frac{\sinh(2\mu_0 L)}{4\mu_0} - \frac{L}{2}$$
$$\int_0^L \sin^2(k_n x)dx = \frac{L}{2} - \frac{\sin(2k_n L)}{4k_n}$$

15 笔记中的例子 (Examples from the Notes)

15.1 例 1: 存在增长模式

这个例子探讨了存在增长解的情况。

- **参数:** $a^2=1, L=1, h=3.66$。
- **初始条件:** $\phi(x)=\sin(\pi x)$。
- **增长模式检验:** $hL = 3.66\times 1 = 3.66 > 1$，所以存在一个增长模式，由 $\tanh(\mu_0)=\mu_0/3.66$ 确定 μ_0。
- **衰减模式:** 由 $\tan(k_n)=k_n/3.66$ 确定 k_n。
- **解的形式:**

$$u(x,t) = C_0 e^{\mu_0^2 t}\sinh(\mu_0 x) + \sum_{n=1}^{\infty} C_n e^{-k_n^2 t}\sin(k_n x)$$

- **系数 C_0 的计算:**

$$C_0 = \frac{\int_0^1 \sin(\pi x)\sinh(\mu_0 x)dx}{\int_0^1 \sinh^2(\mu_0 x)dx}$$

  分子积分可得：

$$\int_0^1 \sin(\pi x)\sinh(\mu_0 x)dx = \frac{\pi(1+\cosh(\mu_0))}{\mu_0^2+\pi^2}$$

  因此，

$$C_0 = \frac{\frac{\pi(1+\cosh(\mu_0))}{\mu_0^2+\pi^2}}{\frac{\sinh(2\mu_0)}{4\mu_0} - \frac{1}{2}}$$

15.2 例 2: 仅有衰减模式

这个例子展示了只有衰减解的情况，并使用了不同的边界条件 $u_x(L,t)+hu(L,t)=0$。

- **边界条件:** $u(0,t)=0, u_x(1,t)+\frac{1}{2}u(1,t)=0$。
- **参数:** $L=1, h=1/2$。
- **初始条件:** $\phi(x)=x(1-x)$。
- **增长模式检验:** 对于边界条件 $u_x+hu=0$，特征方程为 $\tanh(\mu L)=-\mu/h$。对于正的 μ, h, L，此方程无正解。因此没有增长模式。
- **衰减模式:** 特征方程为 $\tan(kL)=-k/h$，即 $\tan(k)=-2k$。
- **解的形式:**
$$u(x,t)=\sum_{n=1}^{\infty} C_n e^{-k_n^2 a^2 t}\sin(k_n x)$$
- **系数 C_n 的计算:**
$$C_n=\frac{\int_0^1 x(1-x)\sin(k_n x)dx}{\int_0^1 \sin^2(k_n x)dx}$$
分子积分可得:
$$\int_0^1 (x-x^2)\sin(k_n x)dx=\frac{2k_n-2\sin(k_n)}{k_n^3}$$
因此,
$$C_n=\frac{\frac{2(k_n-\sin k_n)}{k_n^3}}{\frac{1}{2}-\frac{\sin(2k_n)}{4k_n}}$$

Solving the Heat Equation

Consider the heat equation
$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2},\quad 0<x<L,\quad t>0$$
with boundary conditions
$$u(0,t)=0,\quad u(L,t)=0$$
and initial condition
$$u(x,0)=f(x).$$

Step 1: Separation of Variables

We assume a solution of the form $u(x,t)=X(x)T(t)$. Substituting into the PDE:
$$X(x)T'(t)=\alpha^2 X''(x)T(t)$$
Dividing by $\alpha^2 X(x)T(t)$, we separate variables:
$$\frac{T'(t)}{\alpha^2 T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$
where $-\lambda$ is the separation constant. This yields two ordinary differential equations (ODEs):
$$T'(t)=-\lambda\alpha^2 T(t)$$
$$X''(x)+\lambda X(x)=0$$

Step 2: Solving for $X(x)$ using Boundary Conditions

We analyze the eigenvalue problem $X''(x) + \lambda X(x) = 0$ with $X(0) = 0$ and $X(L) = 0$.

Case 1: $\lambda > 0$

Let $\lambda = k^2$ where $k > 0$. The characteristic equation is $r^2 + k^2 = 0$, so $r = \pm ik$. The general solution for $X(x)$ is

$$X(x) = A\cos(kx) + B\sin(kx).$$

Applying the boundary condition $X(0) = 0$:

$$X(0) = A\cos(0) + B\sin(0) = A = 0.$$

So, $X(x) = B\sin(kx)$. Applying the boundary condition $X(L) = 0$:

$$X(L) = B\sin(kL) = 0.$$

For a non-trivial solution (i.e., $B \neq 0$), we must have $\sin(kL) = 0$. This implies $kL = n\pi$ for $n = 1, 2, 3, \ldots$. Thus, $k_n = \frac{n\pi}{L}$. The eigenvalues are $\lambda_n = k_n^2 = \left(\frac{n\pi}{L}\right)^2$. The corresponding eigenfunctions are $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ (we absorb the constant B into the constant for the full solution later).

Case 2: $\lambda = 0$

If $\lambda = 0$, then $X''(x) = 0$. Integrating twice, $X(x) = Ax + B$. Applying $X(0) = 0 \implies B = 0$. So $X(x) = Ax$. Applying $X(L) = 0 \implies AL = 0 \implies A = 0$. Thus, only the trivial solution $X(x) = 0$ exists for $\lambda = 0$.

Case 3: $\lambda < 0$

Let $\lambda = -k^2$ where $k > 0$. The characteristic equation is $r^2 - k^2 = 0$, so $r = \pm k$. The general solution for $X(x)$ is

$$X(x) = Ae^{kx} + Be^{-kx}.$$

Applying $X(0) = 0 \implies A + B = 0 \implies B = -A$. So, $X(x) = A(e^{kx} - e^{-kx}) = 2A\sinh(kx)$. Applying $X(L) = 0 \implies 2A\sinh(kL) = 0$. Since $k > 0$ and $L > 0$, $\sinh(kL) \neq 0$. Thus, $A = 0$, which leads to $X(x) = 0$. Only the trivial solution exists for $\lambda < 0$.

Therefore, the only valid eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with eigenfunctions $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

Step 3: Solving for $T(t)$

For each λ_n, the ODE for $T(t)$ is

$$T_n'(t) = -\lambda_n \alpha^2 T_n(t).$$

Integrating, we get

$$T_n(t) = C_n e^{-\lambda_n \alpha^2 t} = C_n e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t}.$$

Step 4: Forming the General Solution

By the principle of superposition, the general solution is a sum of all possible solutions:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t}$$

where B_n is a new constant that combines B and C_n.

Step 5: Applying the Initial Condition

At $t = 0$, we have $u(x,0) = f(x)$. Substituting into the general solution:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

This is a Fourier sine series for $f(x)$. The coefficients B_n are given by the orthogonality of sine functions:

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Problem 1: 1D Heat Equation with Robin Boundary Conditions

This problem considers the one-dimensional heat equation with a Neumann boundary condition at one end and a Robin condition at the other.

Problem Statement

The governing Partial Differential Equation (PDE) is:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

with Boundary Conditions (BCs):

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) + hu(L,t) = 0 \quad (h > 0)$$

and an Initial Condition (IC):

$$u(x,0) = f(x)$$

Separation of Variables

Let $u(x,t) = X(x)T(t)$. Substituting into the PDE gives:

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

This leads to two ordinary differential equations:

$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + \lambda\alpha^2 T(t) = 0$$

The boundary conditions for $X(x)$ are $X'(0) = 0$ and $X'(L) + hX(L) = 0$.

Eigenvalue Problem (Sturm-Liouville)

We analyze the possible values for the eigenvalue λ.

Case 1: $\lambda = \mu^2 > 0$ The solution for $X(x)$ is $X(x) = A\cos(\mu x) + B\sin(\mu x)$.

- From $X'(0) = 0$: $X'(x) = -\mu A\sin(\mu x) + \mu B\cos(\mu x) \implies \mu B = 0 \implies B = 0$. So, $X(x) = A\cos(\mu x)$.
- From $X'(L) + hX(L) = 0$: $-\mu A\sin(\mu L) + hA\cos(\mu L) = 0$. Assuming $A \neq 0$, we get the eigenvalue equation for μ_n:
$$\tan(\mu L) = \frac{h}{\mu}$$

Let the positive roots be μ_n for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are $X_n(x) = \cos(\mu_n x)$.

Case 2: $\lambda = 0$ The solution is $X(x) = Ax + B$.

- From $X'(0) = 0 \implies A = 0$.
- From $X'(L) + hX(L) = 0 \implies 0 + hB = 0$. Since $h > 0$, we must have $B = 0$.

Thus, $\lambda = 0$ is not an eigenvalue.

Case 3: $\lambda = -\mu^2 < 0$ The solution for $X(x)$ is $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$.

- From $X'(0) = 0$: $X'(x) = \mu A\sinh(\mu x) + \mu B\cosh(\mu x) \implies \mu B = 0 \implies B = 0$. So, $X(x) = A\cosh(\mu x)$.
- From $X'(L) + hX(L) = 0$: $\mu A\sinh(\mu L) + hA\cosh(\mu L) = 0$. Assuming $A \neq 0$, we get the equation for μ:
$$\tanh(\mu L) = -\frac{h}{\mu}$$

For $h > 0$ and $L > 0$, a graphical analysis shows there is one positive root, which we denote μ_0. This gives one negative eigenvalue $\lambda_0 = -\mu_0^2$. The corresponding eigenfunction is $X_0(x) = \cosh(\mu_0 x)$.

General Solution

The time-dependent solutions are $T_n(t) = e^{-\alpha^2\mu_n^2 t}$ for $n \geq 1$ and $T_0(t) = e^{\alpha^2\mu_0^2 t}$. The general solution for $u(x,t)$ is a superposition of all product solutions:

$$u(x,t) = c_0\cosh(\mu_0 x)e^{\alpha^2\mu_0^2 t} + \sum_{n=1}^{\infty} c_n\cos(\mu_n x)e^{-\alpha^2\mu_n^2 t}$$

The coefficients c_n are determined by the initial condition $u(x,0) = f(x)$.

$$f(x) = c_0\cosh(\mu_0 x) + \sum_{n=1}^{\infty} c_n\cos(\mu_n x)$$

Using the orthogonality of the eigenfunctions:

$$c_n = \frac{\int_0^L f(x)X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

The normalization integrals are:

$$\int_0^L \cosh^2(\mu_0 x)\,dx = \frac{L}{2} + \frac{\sinh(2\mu_0 L)}{4\mu_0} = \frac{L}{2} - \frac{h\cosh^2(\mu_0 L)}{2\mu_0^2}$$
$$\int_0^L \cos^2(\mu_n x)\,dx = \frac{L}{2} + \frac{\sin(2\mu_n L)}{4\mu_n} = \frac{L}{2} + \frac{h\cos^2(\mu_n L)}{2\mu_n^2}$$

Problem 2: Laplace Equation on a Rectangle

This problem solves the Laplace equation $\nabla^2 u = 0$ inside a rectangular domain with specified values on the boundary (Dirichlet problem).

Problem Statement

The governing PDE is:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{for } 0 < x < a,\ 0 < y < b$$

with general Dirichlet Boundary Conditions:

$$\begin{aligned} u(x,0) &= f_1(x), & u(x,b) &= f_2(x) \\ u(0,y) &= g_1(y), & u(a,y) &= g_2(y) \end{aligned}$$

By the principle of superposition, the problem can be split into four simpler problems. The notes focus on the case where $g_1(y) = g_2(y) = 0$.

Solution for Homogeneous Vertical Boundaries

We solve for $u(0,y) = 0$ and $u(a,y) = 0$. The solution $u(x,y)$ can be further split into $u(x,y) = u_1(x,y) + u_2(x,y)$, where:

- u_1 solves the problem with $u_1(x,0) = f_1(x)$ and $u_1(x,b) = 0$.
- u_2 solves the problem with $u_2(x,0) = 0$ and $u_2(x,b) = f_2(x)$.

Separation of Variables Let $u(x,y) = X(x)Y(y)$. This leads to:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

For the boundary conditions $X(0) = 0$ and $X(a) = 0$, we get the eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \ldots$$

The corresponding equation for $Y(y)$ is $Y'' - \lambda_n Y = 0$, with solution:

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right)$$

Solution for $u_1(x,y)$

- BCs: $u_1(x,0) = f_1(x)$, $u_1(x,b) = 0$.
- The condition $u_1(x,b) = 0$ requires $Y_n(b) = 0$ for each n. It is more convenient to write the solution for $Y_n(y)$ in a basis that satisfies this condition automatically:

  $$Y_n(y) = C_n \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

  This form satisfies $Y_n(b) = 0$.

- The solution for u_1 is a superposition:

$$u_1(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

- Applying the final BC, $u_1(x, 0) = f_1(x)$:

$$f_1(x) = \sum_{n=1}^{\infty} \left[C_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right)$$

- This is a Fourier sine series for $f_1(x)$. The coefficient is found by:

$$C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$C_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Solution for $u_2(x, y)$

- BCs: $u_2(x, 0) = 0$, $u_2(x, b) = f_2(x)$.
- The condition $u_2(x, 0) = 0$ requires $Y_n(0) = 0$. The standard hyperbolic sine term works:

$$Y_n(y) = D_n \sinh\left(\frac{n\pi y}{a}\right)$$

- The solution for u_2 is a superposition:

$$u_2(x, y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

- Applying the final BC, $u_2(x, b) = f_2(x)$:

$$f_2(x) = \sum_{n=1}^{\infty} \left[D_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right)$$

- The coefficient is found by:

$$D_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

The complete solution for homogeneous vertical boundaries is $u(x, y) = u_1(x, y) + u_2(x, y)$.

Partial Differential Equations Notes

1D Heat Equation

The one-dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Let's consider the problem:

$$u_t = u_{xx} - \gamma u, \quad (a = 1, b = -\gamma)$$

Given by $u_t = u_{xx} - 3u$, which is $a = 1, \gamma = 3$. Let $u(x,t) = e^{-\gamma t}v(x,t)$. Then $v_t = a^2 v_{xx}$. The solution is given by separation of variables, assuming $u(x,t) = X(x)T(t)$.

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$$

This leads to two ordinary differential equations:

$$X'' + \lambda X = 0$$

$$T' + a^2 \lambda T = 0$$

For the boundary conditions $u(0,t) = u(L,t) = 0$, we have $X(0) = X(L) = 0$. This gives non-trivial solutions for $X(x)$ only if $\lambda > 0$. Let $\lambda = k^2$.

$$X(x) = A\cos(kx) + B\sin(kx)$$

$X(0) = 0 \implies A = 0$. $X(L) = 0 \implies B\sin(kL) = 0 \implies kL = n\pi \implies k = \frac{n\pi}{L}$ for $n = 1, 2, 3, \ldots$ So, the eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$ and eigenfunctions are $X_n(x) = \sin(\frac{n\pi x}{L})$. The solution for $T(t)$ is $T_n(t) = C_n e^{-a^2\lambda_n t} = C_n e^{-a^2(n\pi/L)^2 t}$. The general solution is a superposition:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-a^2(n\pi/L)^2 t} \sin(\frac{n\pi x}{L})$$

The coefficients C_n are determined by the initial condition $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L})$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

Wave Equation

Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), u_t(x,0) = g(x)$$

Boundary conditions: $u(0,t) = u(\pi,t) = 0$. Separation of variables $u(x,t) = X(x)T(t)$:

$$\frac{T''}{4T} = \frac{X''}{X} = -\lambda$$

$X'' + \lambda X = 0$, with $X(0) = X(\pi) = 0$. This yields $\lambda_n = n^2$ and $X_n(x) = \sin(nx)$ for $n = 1, 2, \ldots$. $T'' + 4n^2 T = 0$, so $T_n(t) = A_n\cos(2nt) + B_n\sin(2nt)$. The general solution is:

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(2nt) + B_n\sin(2nt))\sin(nx)$$

Initial conditions give:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\sin(nx) \implies A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} 2nB_n\sin(nx) \implies 2nB_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)dx$$

Fourier Transform Method

Consider the heat equation on an infinite domain:

$$u_t = ku_{xx}, \quad -\infty < x < \infty, t > 0$$

$$u(x, 0) = f(x)$$

Let $\mathcal{F}[u(x,t)](\omega) = \hat{u}(\omega, t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$. The transformed equation is:

$$\frac{\partial \hat{u}}{\partial t} = k(i\omega)^2\hat{u} = -k\omega^2\hat{u}$$

The solution is $\hat{u}(\omega, t) = Ce^{-k\omega^2 t}$. From the initial condition, $\hat{u}(\omega, 0) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$. So, $C = \hat{f}(\omega)$.

$$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-k\omega^2 t}$$

The solution $u(x,t)$ is the inverse Fourier transform:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-k\omega^2 t}e^{i\omega x}d\omega$$

Using the convolution theorem, $\mathcal{F}^{-1}[e^{-k\omega^2 t}] = \sqrt{\frac{\pi}{kt}}e^{-x^2/(4kt)}$.

$$u(x,t) = f(x) * \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)} = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/(4kt)}dy$$

Example 1: Solve $u_t = u_{xx}$ with $u(x,0) = f(x) = e^{-|x|}$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-|x|}e^{-i\omega x}dx = \frac{2}{1+\omega^2}$$

$$\hat{u}(\omega, t) = \frac{2}{1+\omega^2}e^{-\omega^2 t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\omega^2}e^{-\omega^2 t}e^{i\omega x}d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{1+\omega^2}e^{-\omega^2 t}d\omega$$

Fourier Sine and Cosine Transforms

For problems on a semi-infinite interval $[0, \infty)$. **Example 2:** Solve $u_t = u_{xx}$ for $x > 0, t > 0$, with $u(0,t) = 0$ and $u(x,0) = f(x)$. Use Fourier Sine Transform:

$$\mathcal{F}_s[u(x,t)] = U_s(\omega, t) = \int_0^{\infty} u(x,t)\sin(\omega x)dx$$

$$\frac{dU_s}{dt} = \int_0^{\infty} u_t \sin(\omega x)dx = \int_0^{\infty} u_{xx}\sin(\omega x)dx$$
$$= [u_x \sin(\omega x) - \omega u\cos(\omega x)]_0^{\infty} - \omega^2 U_s(\omega, t)$$

Assuming $u, u_x \to 0$ as $x \to \infty$, and using $u(0,t) = 0$:

$$\frac{dU_s}{dt} = -\omega^2 U_s(\omega, t)$$

Solution: $U_s(\omega, t) = U_s(\omega, 0)e^{-\omega^2 t}$, where $U_s(\omega, 0) = \int_0^{\infty} f(x)\sin(\omega x)dx$.

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} U_s(\omega, t)\sin(\omega x)d\omega$$

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} f(y)\sin(\omega y)dy\right)e^{-\omega^2 t}\sin(\omega x)d\omega$$

Laplace's Equation in a Disk

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Boundary condition: $u(a,\theta) = f(\theta)$. Using separation of variables, $u(r,\theta) = R(r)\Theta(\theta)$.

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$\Theta'' + \lambda\Theta = 0$ with periodic boundary conditions $\Theta(\theta) = \Theta(\theta + 2\pi)$. This gives $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$. $\Theta_0(\theta) = A_0$ $\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$ for $n \geq 1$. The radial equation is $r^2R'' + rR' - n^2R = 0$. The solutions are $R_0(r) = C_0 + D_0 \ln r$ and $R_n(r) = C_n r^n + D_n r^{-n}$ for $n \geq 1$. For the solution to be bounded at $r = 0$, we must have $D_0 = 0$ and $D_n = 0$. The general solution is:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos(n\theta) + B_n \sin(n\theta))$$

Using the boundary condition $u(a,\theta) = f(\theta)$:

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n(A_n \cos(n\theta) + B_n \sin(n\theta))$$

These are the Fourier series coefficients for $f(\theta)$:

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$a^n A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta$$

$$a^n B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta$$

Nonhomogeneous Equations

Consider the nonhomogeneous heat equation:

$$u_t = a^2 u_{xx} + F(x,t)$$

with homogeneous boundary conditions, e.g., $u(0,t) = u(L,t) = 0$. And initial condition $u(x,0) = f(x)$.

Method 1: Eigenfunction Expansion Expand the solution in terms of the eigenfunctions of the corresponding homogeneous problem. The eigenfunctions for the operator $\frac{d^2}{dx^2}$ with $X(0) = X(L) = 0$ are $\sin(\frac{n\pi x}{L})$. Let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi x}{L})$$

Expand the source term $F(x,t)$ as well:

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t)\sin(\frac{n\pi x}{L})$$

where $F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin(\frac{n\pi x}{L})dx$. Substitute these into the PDE:

$$\sum_{n=1}^{\infty} u_n'(t)\sin(\frac{n\pi x}{L}) = a^2\sum_{n=1}^{\infty} u_n(t)(-\frac{n^2\pi^2}{L^2})\sin(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} F_n(t)\sin(\frac{n\pi x}{L})$$

This yields an ODE for each coefficient $u_n(t)$:

$$u_n'(t) + a^2(\frac{n\pi}{L})^2 u_n(t) = F_n(t)$$

This is a first-order linear ODE. Its solution is:

$$u_n(t) = e^{-\lambda_n a^2 t}\left(u_n(0) + \int_0^t F_n(\tau)e^{\lambda_n a^2 \tau}d\tau\right)$$

where $\lambda_n = (\frac{n\pi}{L})^2$. The initial coefficients $u_n(0)$ are found from the initial condition $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} u_n(0)\sin(\frac{n\pi x}{L}) \implies u_n(0) = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

1. One-Dimensional Heat Equation with Robin Boundary Conditions

This section details the solution to the heat equation in one dimension with a Robin boundary condition at one end.

Problem Statement

The problem is defined by the following partial differential equation and boundary/initial conditions:

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & 0 < x < l, \quad t > 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(l,t) + hu(l,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Here, k is the thermal diffusivity and h is a positive constant. (Note: The handwritten notes assume $k = 1$).

Method: Separation of Variables

We assume a solution of the form $u(x,t) = X(x)T(t)$. Substituting this into the PDE gives:

$$X(x)T'(t) = kX''(x)T(t)$$

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

where $-\lambda$ is the separation constant. This leads to two ordinary differential equations:

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + k\lambda T(t) = 0$$

The boundary conditions for $X(x)$ become:

$$X'(0) = 0 \quad \text{and} \quad X'(l) + hX(l) = 0$$

Solving the Eigenvalue Problem

We analyze the problem for $X(x)$ based on the sign of λ.

- **Case 1:** $\lambda = 0$
  $X''(x) = 0 \implies X(x) = Ax + B$.
  $X'(0) = A = 0$. So, $X(x) = B$.
  $X'(l) + hX(l) = 0 + hB = 0$. Since $h > 0$, we must have $B = 0$. This gives only the trivial solution. (Note: If $h = 0$, $\lambda_0 = 0$ is an eigenvalue with eigenfunction $X_0(x) = 1$).

- **Case 2:** $\lambda < 0$
  Let $\lambda = -\mu^2$ where $\mu > 0$. The equation is $X''(x) - \mu^2 X(x) = 0$. The general solution is $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$. $X'(0) = B\mu = 0 \implies B = 0$. So, $X(x) = A\cosh(\mu x)$. $X'(l) + hX(l) = A\mu\sinh(\mu l) + hA\cosh(\mu l) = 0$. Since $A \neq 0$, $\mu > 0$, $l > 0$, and $h > 0$, the term $\mu\tanh(\mu l) = -h$ has no solution because $\tanh(\mu l) > 0$. This case also leads to the trivial solution.

- **Case 3:** $\lambda > 0$
  Let $\lambda = \mu^2$ where $\mu > 0$. The equation is $X''(x) + \mu^2 X(x) = 0$. The general solution is $X(x) = A\cos(\mu x) + B\sin(\mu x)$. $X'(0) = -A\mu\sin(0) + B\mu\cos(0) = B\mu = 0 \implies B = 0$. So, $X(x) = A\cos(\mu x)$. The second boundary condition gives:

  $$-A\mu\sin(\mu l) + hA\cos(\mu l) = 0$$

  For a non-trivial solution ($A \neq 0$), we must have:

  $$\mu\tan(\mu l) = h$$

  This is the characteristic equation for the eigenvalues μ_n. It can be solved graphically by finding the intersections of $y = \tan(\mu l)$ and $y = h/\mu$. Let the positive roots be $\mu_1, \mu_2, \ldots$. The corresponding eigenvalues are $\lambda_n = \mu_n^2$ and eigenfunctions are $X_n(x) = \cos(\mu_n x)$.

General Solution

The solution for $T(t)$ is $T_n(t) = C_n e^{-k\lambda_n t} = C_n e^{-k\mu_n^2 t}$. The general solution for $u(x,t)$ is a superposition of all product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} C_n e^{-k\mu_n^2 t}\cos(\mu_n x)$$

The coefficients C_n are determined by the initial condition $u(x,0) = \phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \cos(\mu_n x)$$

The eigenfunctions $\cos(\mu_n x)$ are orthogonal. The coefficients are given by:

$$C_n = \frac{\int_0^l \phi(x)\cos(\mu_n x)\,dx}{\int_0^l \cos^2(\mu_n x)\,dx}$$

The norm squared in the denominator can be calculated as:

$$\int_0^l \cos^2(\mu_n x)\,dx = \frac{l(h^2 + \mu_n^2) + h}{2(h^2 + \mu_n^2)}$$

2. Two-Dimensional Laplace's Equation on a Rectangle

This section addresses the steady-state heat distribution (Laplace's equation) in a rectangular domain with specified boundary temperatures.

Problem Statement and Superposition

The governing PDE is Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

With the general non-homogeneous boundary conditions:

$$\begin{cases} u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = g_1(y) \\ u(a,y) = g_2(y) \end{cases}$$

The problem is linear, so we can use the principle of superposition. The solution $u(x,y)$ is the sum of four solutions, $u = u_1 + u_2 + u_3 + u_4$, where each sub-problem has only one non-homogeneous boundary condition.

Solution for Each Boundary Condition

We solve for each case using separation of variables. Let $u(x,y) = X(x)Y(y)$. This leads to $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$.

Case 1: Bottom Boundary $u(x,0) = f_1(x)$ (u_1)

The problem is $u_1(x,0) = f_1(x)$ with other boundaries being zero.

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Case 2: Top Boundary $u(x,b) = f_2(x)$ (u_2)

The problem is $u_2(x,b) = f_2(x)$ with other boundaries being zero.

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

$$B_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{n\pi x}{a}\right) dx = \frac{2}{a} \operatorname{csch}\left(\frac{n\pi b}{a}\right) \int_0^a f_2(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Case 3: Left Boundary $u(0, y) = g_1(y)$ **(u_3)**

The problem is $u_3(0, y) = g_1(y)$ with other boundaries being zero.

$$u_3(x, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$C_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_1(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

Case 4: Right Boundary $u(a, y) = g_2(y)$ **(u_4)**

The problem is $u_4(a, y) = g_2(y)$ with other boundaries being zero.

$$u_4(x, y) = \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$D_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_2(y) \sin\left(\frac{n\pi y}{b}\right) dy = \frac{2}{b} \operatorname{csch}\left(\frac{n\pi a}{b}\right) \int_0^b g_2(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

Example: Constant Top Boundary Temperature

Consider the case where $f_2(x) = f_0$ (constant), and all other boundary conditions are zero ($f_1 = g_1 = g_2 = 0$). The solution is simply $u(x, y) = u_2(x, y)$. We calculate the coefficients B_n:

$$\begin{aligned}
B_n &= \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_0 \sin\left(\frac{n\pi x}{a}\right) dx \\
&= \frac{2f_0}{a \sinh\left(\frac{n\pi b}{a}\right)} \left[-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right]_0^a \\
&= \frac{2f_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} (-\cos(n\pi) + \cos(0)) \\
&= \frac{2f_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} (1 - (-1)^n)
\end{aligned}$$

The coefficient is non-zero only for odd values of n:

$$B_n = \begin{cases} \frac{4f_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Letting $n = 2k - 1$ for $k = 1, 2, 3, \ldots$, the final solution is:

$$u(x, y) = \sum_{k=1}^{\infty} \frac{4f_0}{(2k-1)\pi \sinh\left(\frac{(2k-1)\pi b}{a}\right)} \sin\left(\frac{(2k-1)\pi x}{a}\right) \sinh\left(\frac{(2k-1)\pi y}{a}\right)$$

16 Problems on Infinite and Semi-Infinite Domains (Fourier Transform Methods)

16.1 Heat Equation on a Semi-Infinite Rod (Homogeneous Dirichlet BC)

Problem: Solve the heat equation on a semi-infinite rod with the end held at zero temperature.

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad x > 0, t > 0 \\ u(x, 0) = f(x) \\ u(0, t) = 0 \end{cases}$$

We use the Fourier Sine Transform. The solution is given by:

$$u(x,t) = \int_0^\infty B(\omega) e^{-k\omega^2 t} \sin(\omega x)\, d\omega$$

where the coefficient $B(\omega)$ is determined by the initial condition:

$$u(x,0) = f(x) = \int_0^\infty B(\omega) \sin(\omega x)\, d\omega$$

Thus, $B(\omega)$ is the Fourier Sine Transform of $f(x)$:

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin(\omega x)\, dx$$

16.2 Heat Equation on an Infinite Rod

Problem: Solve the heat equation on an infinite rod.

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) \end{cases}$$

We use the Fourier Transform. The solution is given by:

$$u(x,t) = \int_{-\infty}^\infty C(\omega) e^{-k\omega^2 t} e^{i\omega x}\, d\omega$$

An alternative real form is:

$$u(x,t) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)] e^{-k\omega^2 t}\, d\omega$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(x) \cos(\omega x)\, dx$$
$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(x) \sin(\omega x)\, dx$$

Example 1: Let $k = 1$ and $f(x) = \frac{1}{x^2+1}$. The Fourier transform of $f(x)$ is $\mathcal{F}[f(x)](\omega) = \pi e^{-|\omega|}$. The solution becomes:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty \pi e^{-|\omega|} e^{-\omega^2 t} e^{i\omega x}\, d\omega$$

Note: The notes might be slightly simplified. A full derivation involves the convolution theorem. The solution is the convolution of the initial condition with the heat kernel.

16.3 Heat Equation on a Semi-Infinite Rod (Homogeneous Neumann BC)

Problem: Solve the heat equation on a semi-infinite rod with an insulated end.

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, & x > 0, t > 0 \\ u(x,0) = f(x) \\ \frac{\partial u}{\partial x}(0,t) = 0 \end{cases}$$

We use the Fourier Cosine Transform. The solution is given by:

$$u(x,t) = \int_0^\infty A(\omega) e^{-k\omega^2 t} \cos(\omega x)\, d\omega$$

where $A(\omega)$ is the Fourier Cosine Transform of $f(x)$:

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(x)\cos(\omega x)\, dx$$

Example 2: Let $k = 1$ and $f(x) = e^{-x}$.

$$A(\omega) = \frac{2}{\pi} \int_0^\infty e^{-x}\cos(\omega x)\, dx = \frac{2}{\pi}\frac{1}{1+\omega^2}$$

The solution is:

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\cos(\omega x)}{1+\omega^2} e^{-\omega^2 t}\, d\omega$$

16.4 Laplace's Equation on the Upper Half-Plane

Problem: Solve Laplace's equation in the upper half-plane $y > 0$.

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & -\infty < x < \infty, y > 0 \\ u(x,0) = f(x) \end{cases}$$

Using Fourier Transform with respect to x, let $U(\omega, y) = \mathcal{F}[u(x,y)]$. The PDE becomes:

$$-\omega^2 U(\omega, y) + \frac{d^2U}{dy^2} = 0$$

The solution is $U(\omega, y) = C_1(\omega)e^{\omega y} + C_2(\omega)e^{-\omega y}$. For the solution to be bounded as $y \to \infty$, we require $U(\omega, y) = C(\omega)e^{-|\omega|y}$. From the boundary condition, $U(\omega, 0) = \mathcal{F}[f(x)](\omega) = C(\omega)$. So, $U(\omega, y) = \mathcal{F}[f(x)](\omega)e^{-|\omega|y}$. Taking the inverse Fourier transform and using the convolution theorem, we get the Poisson Integral Formula for the upper half-plane:

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^\infty \frac{f(\xi)}{(x-\xi)^2 + y^2}\, d\xi$$

17 Problems on Finite Domains (Separation of Variables)

17.1 Heat Equation with Homogeneous Neumann Boundary Conditions

Problem:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Using separation of variables, $u(x,t) = X(x)T(t)$, we find the eigenfunctions are cosines. The solution is a superposition:

$$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^\infty C_n \cos\left(\frac{n\pi x}{L}\right) e^{-(n\pi a/L)^2 t}$$

Using the initial condition $u(x,0) = f(x)$:

$$f(x) = \frac{C_0}{2} + \sum_{n=1}^\infty C_n \cos\left(\frac{n\pi x}{L}\right)$$

The coefficients are given by the Fourier cosine series formulas:

$$C_n = \frac{2}{L} \int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx, \quad n = 0, 1, 2, \ldots$$

17.2 Heat Equation with Mixed Boundary Conditions

Problem:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Separation of variables leads to eigenfunctions of the form $\cos(\lambda x)$ where $\cos(\lambda L) = 0$. This implies $\lambda_n L = \frac{(2n+1)\pi}{2}$, so $\lambda_n = \frac{(2n+1)\pi}{2L}$ for $n = 0, 1, 2, \ldots$ The solution is a superposition:

$$u(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n+1)\pi x}{2L}\right) \exp\left\{-\left[\frac{(2n+1)\pi a}{2L}\right]^2 t\right\}$$

The coefficients are found from the initial condition:

$$f(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

18 Laplace's Equation in Polar Coordinates

Problem: Solve Laplace's equation inside a disk of radius a.

$$\begin{cases} \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \le r < a \\ u(a,\theta) = f(\theta) \end{cases}$$

Let $u(r,\theta) = R(r)\Theta(\theta)$. This separates the PDE into two ODEs:

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

Periodicity of $\Theta(\theta)$ requires $\lambda = n^2$ for $n = 0, 1, 2, \ldots$. The solutions for Θ are $1, \cos(n\theta), \sin(n\theta)$. The radial equation is a Cauchy-Euler equation with solutions r^n and r^{-n} (or $\ln r$ for $n = 0$). For the problem inside the disk, we need the solution to be bounded at $r = 0$, so we discard r^{-n} and $\ln r$. The general solution is a superposition:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta))$$

Applying the boundary condition at $r = a$:

$$f(\theta) = u(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n a^n \cos(n\theta) + b_n a^n \sin(n\theta))$$

This is the Fourier series for $f(\theta)$. The coefficients are:

$$a_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta \implies a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta$$

$$b_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta \implies b_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta$$

Example: Solve $\nabla^2 u = 0$ in the unit disk ($a = 1$) with $u(1, \theta) = x + y = \cos\theta + \sin\theta$. The boundary condition is $f(\theta) = \cos\theta + \sin\theta$. Comparing this with the general series for $f(\theta)$:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

We can see by inspection that $a_1 = 1$, $b_1 = 1$, and all other coefficients are zero. The solution is:

$$u(r, \theta) = a_1 r^1 \cos(1 \cdot \theta) + b_1 r^1 \sin(1 \cdot \theta) = r\cos\theta + r\sin\theta$$

In Cartesian coordinates, since $x = r\cos\theta$ and $y = r\sin\theta$, the solution is:

$$u(x, y) = x + y$$

19 Non-homogeneous Problems (Eigenfunction Expansion)

Problem: Solve the non-homogeneous heat equation with homogeneous boundary conditions.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \quad 0 < x < L, t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

The eigenfunctions for the associated homogeneous problem are $\sin\left(\frac{n\pi x}{L}\right)$. We seek a solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

We also expand the source term $F(x,t)$ and the initial condition $f(x)$ in terms of these eigenfunctions:

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi x}{L}\right), \quad \text{where } F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where } f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Substitute the series for $u(x,t)$ and $F(x,t)$ into the PDE. Equating coefficients of $\sin\left(\frac{n\pi x}{L}\right)$, we obtain an ODE for each $g_n(t)$:

$$g_n'(t) = -a^2 \left(\frac{n\pi}{L}\right)^2 g_n(t) + F_n(t)$$

Rewriting, we get a first-order linear ODE:

$$g_n'(t) + \left(\frac{n\pi a}{L}\right)^2 g_n(t) = F_n(t)$$

The initial condition is $g_n(0) = f_n$. The solution to this ODE is:

$$g_n(t) = f_n e^{-(n\pi a/L)^2 t} + \int_0^t e^{-(n\pi a/L)^2 (t-\tau)} F_n(\tau)\, d\tau$$

The final solution is obtained by substituting $g_n(t)$ back into the series for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} \left[f_n e^{-(n\pi a/L)^2 t} + \int_0^t e^{-(n\pi a/L)^2 (t-\tau)} F_n(\tau)\, d\tau \right] \sin\left(\frac{n\pi x}{L}\right)$$

Problem 1: Wave Equation with Homogeneous Dirichlet BC

The problem is to solve the one-dimensional wave equation with a source term $f(x,t)$, subject to homogeneous Dirichlet boundary conditions and given initial conditions.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

with boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0$$

and initial conditions:

$$u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi(x)$$

Let the solution be expressed as a sine series:

$$u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin \frac{n\pi x}{L}$$

The source term is also expanded in a sine series:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{L}$$

where

$$f_n(t) = \frac{2}{L} \int_0^L f(x,t) \sin \frac{n\pi x}{L} dx$$

Substituting the series for $u(x,t)$ into the PDE, we get an ODE for $g_n(t)$:

$$g_n''(t) + \left(\frac{n\pi a}{L}\right)^2 g_n(t) = f_n(t)$$

The initial conditions for $g_n(t)$ are derived from the initial conditions for $u(x,t)$:

$$g_n(0) = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx = c_n$$

$$g_n'(0) = \frac{2}{L} \int_0^L \psi(x) \sin \frac{n\pi x}{L} dx = d_n$$

The solution to the ODE for $g_n(t)$ is given by:

$$g_n(t) = c_n \cos \frac{n\pi at}{L} + \frac{d_n L}{n\pi a} \sin \frac{n\pi at}{L} + \frac{L}{n\pi a} \int_0^t f_n(\tau) \sin \left[\frac{n\pi a}{L}(t-\tau)\right] d\tau$$

The final solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(c_n \cos \frac{n\pi at}{L} + \frac{d_n L}{n\pi a} \sin \frac{n\pi at}{L} + \frac{L}{n\pi a} \int_0^t f_n(\tau) \sin \left[\frac{n\pi a}{L}(t-\tau)\right] d\tau \right) \sin \frac{n\pi x}{L}$$

Let's consider the specific case where $f(x,t) = \sin(\frac{\pi x}{L}) \sin(\omega t)$. Then $f_n(t) = 0$ for $n \neq 1$ and $f_1(t) = \sin(\omega t)$. The solution for $g_1(t)$ is:

$$g_1(t) = c_1 \cos \frac{\pi at}{L} + \frac{d_1 L}{\pi a} \sin \frac{\pi at}{L} + \frac{L}{\pi a} \int_0^t \sin(\omega\tau) \sin \left[\frac{\pi a}{L}(t-\tau)\right] d\tau$$

Assuming $c_1 = 0$ and $d_1 = 0$, and let $\omega_1 = \frac{\pi a}{L}$.

$$g_1(t) = \frac{L}{\pi a} \int_0^t \sin(\omega\tau) \sin[\omega_1(t-\tau)] d\tau$$

Using the product-to-sum formula $2\sin A \sin B = \cos(A-B) - \cos(A+B)$:

$$g_1(t) = \frac{L}{2\pi a}\int_0^t \left(\cos((\omega-\omega_1)\tau + \omega_1 t) - \cos((\omega+\omega_1)\tau - \omega_1 t)\right) d\tau$$

If $\omega \neq \omega_1$:

$$g_1(t) = \frac{L}{2\pi a}\left[\frac{\sin(\omega t) - \sin(\omega_1 t)}{\omega - \omega_1} - \frac{\sin(\omega t) + \sin(\omega_1 t)}{\omega + \omega_1}\right]$$

If $\omega = \omega_1$:

$$g_1(t) = \frac{L}{2\pi a}\left(t\cos(\omega_1 t) - \frac{\sin(\omega_1 t)}{2\omega_1}\right)$$

This demonstrates the phenomenon of resonance.

Problem 2: Heat Equation with Non-Homogeneous Dirichlet BC

This problem appears to deal with the heat equation with time-independent boundary conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions:

$$u(0,t) = T_1, \quad u(L,t) = T_2$$

and initial condition:

$$u(x,0) = f(x)$$

The solution is decomposed into a steady-state solution $u_s(x)$ and a transient solution $v(x,t)$.

$$u(x,t) = u_s(x) + v(x,t)$$

The steady-state solution satisfies $\frac{d^2 u_s}{dx^2} = 0$, which gives $u_s(x) = Ax + B$. Applying the boundary conditions: $u_s(0) = B = T_1$ and $u_s(L) = AL + T_1 = T_2$, so $A = \frac{T_2 - T_1}{L}$.

$$u_s(x) = \frac{T_2 - T_1}{L}x + T_1$$

The transient solution $v(x,t)$ satisfies the homogeneous heat equation:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

with homogeneous boundary conditions:

$$v(0,t) = u(0,t) - u_s(0) = T_1 - T_1 = 0$$

$$v(L,t) = u(L,t) - u_s(L) = T_2 - T_2 = 0$$

and initial condition:

$$v(x,0) = u(x,0) - u_s(x) = f(x) - \left(\frac{T_2 - T_1}{L}x + T_1\right)$$

The solution for $v(x,t)$ is a standard sine series solution:

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-k(n\pi/L)^2 t}\sin\frac{n\pi x}{L}$$

where

$$c_n = \frac{2}{L}\int_0^L v(x,0)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L [f(x) - u_s(x)]\sin\frac{n\pi x}{L}dx$$

The final solution is $u(x,t) = u_s(x) + v(x,t)$.

Problem 3: Wave Equation with Homogeneous Neumann BC

Here we solve the wave equation with a source, but with homogeneous Neumann boundary conditions.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

with boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

and initial conditions:

$$u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi(x)$$

The solution is expanded in a cosine series:

$$u(x,t) = g_0(t) + \sum_{n=1}^{\infty} g_n(t) \cos \frac{n\pi x}{L}$$

The source term is also expanded in a cosine series:

$$f(x,t) = f_0(t) + \sum_{n=1}^{\infty} f_n(t) \cos \frac{n\pi x}{L}$$

where

$$f_0(t) = \frac{1}{L} \int_0^L f(x,t)dx, \quad f_n(t) = \frac{2}{L} \int_0^L f(x,t) \cos \frac{n\pi x}{L} dx$$

Substituting into the PDE gives ODEs for the coefficients $g_n(t)$. For $n = 0$:

$$g_0''(t) = f_0(t)$$

For $n \geq 1$:

$$g_n''(t) + \left(\frac{n\pi a}{L}\right)^2 g_n(t) = f_n(t)$$

The initial conditions are:

$$g_0(0) = \frac{1}{L} \int_0^L \phi(x)dx = c_0, \quad g_0'(0) = \frac{1}{L} \int_0^L \psi(x)dx = d_0$$

$$g_n(0) = \frac{2}{L} \int_0^L \phi(x) \cos \frac{n\pi x}{L} dx = c_n, \quad g_n'(0) = \frac{2}{L} \int_0^L \psi(x) \cos \frac{n\pi x}{L} dx = d_n$$

Solving for $g_n(t)$:

$$g_0(t) = c_0 + d_0 t + \int_0^t \int_0^\tau f_0(s)ds d\tau$$

$$g_n(t) = c_n \cos \frac{n\pi at}{L} + \frac{d_n L}{n\pi a} \sin \frac{n\pi at}{L} + \frac{L}{n\pi a} \int_0^t f_n(\tau) \sin \left[\frac{n\pi a}{L}(t-\tau)\right] d\tau$$

The final solution is $u(x,t) = g_0(t) + \sum_{n=1}^{\infty} g_n(t) \cos \frac{n\pi x}{L}$.

Problem 4: Heat Equation with a Source and Dirichlet BC

The problem is the heat equation with a source term and zero boundary conditions.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

with boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0$$

and initial condition:

$$u(x,0) = f(x)$$

Let's assume the solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{L}$$

Expand the source term $Q(x,t)$ and the initial condition $f(x)$ in sine series:

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}, \quad q_n(t) = \frac{2}{L} \int_0^L Q(x,t) \sin \frac{n\pi x}{L} dx$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}, \quad c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Substituting into the PDE, we get an ODE for $u_n(t)$:

$$u_n'(t) + k \left(\frac{n\pi}{L}\right)^2 u_n(t) = q_n(t)$$

The initial condition is $u_n(0) = c_n$. The solution to this first-order linear ODE is:

$$u_n(t) = c_n e^{-k(n\pi/L)^2 t} + \int_0^t e^{-k(n\pi/L)^2 (t-\tau)} q_n(\tau) d\tau$$

The full solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(c_n e^{-k(n\pi/L)^2 t} + \int_0^t e^{-k(n\pi/L)^2 (t-\tau)} q_n(\tau) d\tau \right) \sin \frac{n\pi x}{L}$$

20 求解拉普拉斯方程 (Solving Laplace's Equation)

20.1 矩形域上的泊松方程 (Poisson's Equation on a Rectangle)

考虑在矩形域 $D = \{(x,y) | 0 < x < a, 0 < y < b\}$ 上的泊松方程，边界条件为零（齐次狄利克雷边界条件）。

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

边界条件 (BC):

$$u(x,0) = u(x,b) = 0$$

$$u(0,y) = u(a,y) = 0$$

我们使用特征函数展开法。对应于零边界条件的拉普拉斯算子的特征函数为:

$$v_{mn}(x,y) = \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

其对应的特征值为:

$$\nabla^2 v_{mn} = -\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \right] v_{mn} = -\lambda_{mn} v_{mn}$$

假设解的形式为:

$$u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

代入泊松方程中：

$$\nabla^2 u(x,y)=\sum_{m,n} A_{mn}\nabla^2 v_{mn}=\sum_{m,n} -A_{mn}\lambda_{mn}v_{mn}=f(x,y)$$

利用正交性求解系数 A_{mn}。将上式两边同乘以 v_{pq} 并在区域 D上积分：

$$\int_0^b\int_0^a\left(\sum_{m,n} -A_{mn}\lambda_{mn}v_{mn}\right)v_{pq}\,dx\,dy=\int_0^b\int_0^a f(x,y)v_{pq}\,dx\,dy$$

由于特征函数的正交性，只有当 $m=p$ 且 $n=q$ 时左侧积分不为零：

$$-A_{mn}\lambda_{mn}\int_0^b\int_0^a \sin^2\frac{n\pi x}{a}\sin^2\frac{m\pi y}{b}\,dx\,dy=\int_0^b\int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$$

计算范数平方积分：

$$\int_0^b\int_0^a \sin^2\frac{n\pi x}{a}\sin^2\frac{m\pi y}{b}\,dx\,dy=\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)=\frac{ab}{4}$$

因此，系数 A_{mn} 为：

$$A_{mn}=-\frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$$

其中 $\lambda_{mn}=\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2$。

20.2 示例：常数源项的泊松方程

问题：求解 $\nabla^2 u=T_0$ (常数)，其中 $a=b=L$，边界条件为 $u=0$。

$$f(x,y)=T_0$$

$$\lambda_{mn}=\frac{\pi^2(m^2+n^2)}{L^2}$$

计算积分：

$$\begin{aligned}
B_{mn}&=\int_0^L\int_0^L T_0\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L}\,dx\,dy\\
&=T_0\left[\int_0^L\sin\frac{m\pi x}{L}\,dx\right]\left[\int_0^L\sin\frac{n\pi y}{L}\,dy\right]\\
&=T_0\left[-\frac{L}{m\pi}\cos\frac{m\pi x}{L}\right]_0^L\left[-\frac{L}{n\pi}\cos\frac{n\pi y}{L}\right]_0^L\\
&=T_0\left[\frac{L}{m\pi}(1-(-1)^m)\right]\left[\frac{L}{n\pi}(1-(-1)^n)\right]
\end{aligned}$$

当 m,n 均为奇数时，该积分不为零：

$$B_{mn}=T_0\left(\frac{2L}{m\pi}\right)\left(\frac{2L}{n\pi}\right)=\frac{4L^2T_0}{mn\pi^2}\quad(m,n\text{ are odd})$$

系数 A_{mn}：

$$A_{mn}=-\frac{4}{L^2\lambda_{mn}}B_{mn}=-\frac{4}{L^2\frac{\pi^2(m^2+n^2)}{L^2}}\frac{4L^2T_0}{mn\pi^2}=-\frac{16L^2T_0}{mn\pi^4(m^2+n^2)}$$

最终解为 (注意手稿中似乎遗漏了符号和一些因子，此处为修正后的推导结果)：

$$u(x,y)=-\frac{16T_0L^2}{\pi^4}\sum_{m,n\text{ odd}}\frac{\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L}}{mn(m^2+n^2)}$$

21 求解热传导方程 (Solving the Heat Equation)

通用形式的非齐次热传导方程：

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + F(x,t)$$

21.1 齐次狄利克雷边界条件

问题陈述：

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \quad 0 < x < L, t > 0$$

BC: $u(0,t) = 0, \quad u(L,t) = 0$ IC: $u(x,0) = f(x)$

使用特征函数展开法，设解为：

$$u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin \frac{n\pi x}{L}$$

将源项 $F(x,t)$ 和初值 $f(x)$ 也按特征函数展开：

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi x}{L}, \quad F_n(t) = \frac{2}{L} \int_0^L F(x,t) \sin \frac{n\pi x}{L}\, dx$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}, \quad c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

代入 PDE 中，记 $\mu_n^2 = \alpha^2(\frac{n\pi}{L})^2$：

$$\sum_{n=1}^{\infty} g_n'(t) \sin \frac{n\pi x}{L} = \alpha^2 \sum_{n=1}^{\infty} g_n(t) \left(-\left(\frac{n\pi}{L}\right)^2\right) \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi x}{L}$$

比较系数得到关于 $g_n(t)$ 的常微分方程 (ODE)：

$$g_n'(t) + \mu_n^2 g_n(t) = F_n(t)$$

由初始条件 $u(x,0) = f(x)$，可知 $g_n(0) = c_n$。求解此一阶线性 ODE：

$$g_n(t) = c_n e^{-\mu_n^2 t} + \int_0^t F_n(\tau) e^{-\mu_n^2 (t-\tau)}\, d\tau$$

最终解为：

$$u(x,t) = \sum_{n=1}^{\infty} \left[c_n e^{-\mu_n^2 t} + \int_0^t F_n(\tau) e^{-\mu_n^2 (t-\tau)}\, d\tau \right] \sin \frac{n\pi x}{L}$$

21.2 齐次诺依曼边界条件

问题陈述：

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \quad 0 < x < L, t > 0$$

BC: $u_x(0,t) = 0, \quad u_x(L,t) = 0$ IC: $u(x,0) = f(x)$

特征函数为 $\cos \frac{n\pi x}{L}$。设解为：

$$u(x,t) = \frac{1}{2} g_0(t) + \sum_{n=1}^{\infty} g_n(t) \cos \frac{n\pi x}{L}$$

展开 $F(x,t)$ 和 $f(x)$：

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\cos\frac{n\pi x}{L}\,dx$$

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

代入 PDE 得到 ODE (记 $\mu_n^2 = c^2(\frac{n\pi}{L})^2$)：

$$g_n'(t) + \mu_n^2 g_n(t) = F_n(t) \quad (n \geq 1)$$

$$\frac{1}{2}g_0'(t) = \frac{1}{2}F_0(t) \implies g_0'(t) = F_0(t)$$

初值为 $g_n(0) = c_n$。 ODE 的解为：

$$g_0(t) = c_0 + \int_0^t F_0(\tau)\,d\tau$$

$$g_n(t) = c_n e^{-\mu_n^2 t} + \int_0^t F_n(\tau)e^{-\mu_n^2(t-\tau)}\,d\tau \quad (n \geq 1)$$

21.2.1 非齐次诺依曼边界条件

当边界条件为 $u_x(0,t) = g_1(t), u_x(L,t) = g_2(t)$ 时，我们通过对 $u(x,t)$ 的余弦变换来求解。

$$g_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\frac{n\pi x}{L}\,dx$$

$$g_n'(t) = \frac{2}{L}\int_0^L u_t\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L (c^2 u_{xx} + F(x,t))\cos\frac{n\pi x}{L}\,dx$$

对 u_{xx} 项分部积分两次：

$$\begin{aligned}
\int_0^L u_{xx}\cos\frac{n\pi x}{L}\,dx &= \left[u_x\cos\frac{n\pi x}{L}\right]_0^L + \frac{n\pi}{L}\int_0^L u_x\sin\frac{n\pi x}{L}\,dx \\
&= u_x(L,t)\cos(n\pi) - u_x(0,t) - \left(\frac{n\pi}{L}\right)^2\int_0^L u\cos\frac{n\pi x}{L}\,dx \\
&= (-1)^n g_2(t) - g_1(t) - \left(\frac{n\pi}{L}\right)^2\frac{L}{2}g_n(t)
\end{aligned}$$

代入 $g_n'(t)$ 的表达式中，得到 ODE：

$$g_n'(t) + c^2\left(\frac{n\pi}{L}\right)^2 g_n(t) = F_n(t) + \frac{2c^2}{L}((-1)^n g_2(t) - g_1(t))$$

21.3 混合边界条件

问题陈述：

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + F(x,t), \quad 0 < x < L, t > 0$$

BC: $u(0,t) = 0, \quad u_x(L,t) = 0$ IC: $u(x,0) = f(x)$

特征函数为 $\sin\frac{(2n+1)\pi x}{2L}$。设解为：

$$u(x,t) = \sum_{n=0}^{\infty} g_n(t)\sin\frac{(2n+1)\pi x}{2L}$$

展开 $F(x,t)$ 和 $f(x)$：

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{(2n+1)\pi x}{2L}\,dx$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n+1)\pi x}{2L}\,dx$$

代入 PDE 得到 ODE (记 $\lambda_n^2 = a^2(\frac{(2n+1)\pi}{2L})^2$)：

$$g_n'(t) + \lambda_n^2 g_n(t) = F_n(t)$$

初值为 $g_n(0) = c_n$。ODE 的解为：

$$g_n(t) = c_n e^{-\lambda_n^2 t} + \int_0^t F_n(\tau)e^{-\lambda_n^2(t-\tau)}\,d\tau$$

最终解为：

$$u(x,t) = \sum_{n=0}^{\infty}\left[c_n e^{-\lambda_n^2 t} + \int_0^t F_n(\tau)e^{-\lambda_n^2(t-\tau)}\,d\tau\right]\sin\frac{(2n+1)\pi x}{2L}$$

问题一：二维热传导方程

$$\begin{aligned}
\text{PDE: } & \nabla^2 u = u_t \\
\text{BCs: } & u(x,0,t) = 0, \quad u(x,b,t) = 0 \\
& u(0,y,t) = 0, \quad u(a,y,t) = 0 \\
\text{IC: } & u(x,y,0) = f(x,y)
\end{aligned}$$

解的形式为：

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} A_{km}e^{-\lambda_{km}^2 t}\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

其中

$$\lambda_{km}^2 = \frac{k^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

$$A_{km} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dxdy$$

问题二：圆盘上的拉普拉斯方程

求解内部问题（区域为圆盘，在 $r=0$ 处有界）。

$$\begin{aligned}
\text{PDE: } & \nabla^2 u = 0 \\
\text{BC: } & u(1,\theta) = -2\cos\theta
\end{aligned}$$

在极坐标下，拉普拉斯方程为：

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$$

采用分离变量法，通解形式为：

$$u(r,\theta) = A_0(r) + \sum_{n=1}^{\infty}\left(A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta)\right)$$

代入 PDE，得到关于径向函数 $A_n(r)$ 和 $B_n(r)$ 的常微分方程（欧拉方程）：

$$r^2R''(r) + rR'(r) - n^2R(r) = 0$$

情况分析:

- 对于 $n = 0$:
$$A_0''(r) + \frac{1}{r}A_0'(r) = 0 \implies A_0(r) = c_0 \ln r + d_0$$
因为解在 $r = 0$ 处必须有界，所以 $c_0 = 0$。

- 对于 $n \geq 1$:
$$A_n(r) = c_n r^n + d_n r^{-n}, \quad B_n(r) = e_n r^n + f_n r^{-n}$$
因为解在 $r = 0$ 处必须有界，所以 $d_n = 0$ 和 $f_n = 0$。

因此，内部问题的解具有以下形式：

$$u(r, \theta) = d_0 + \sum_{n=1}^{\infty} (c_n r^n \cos(n\theta) + e_n r^n \sin(n\theta))$$

应用边界条件 $u(1, \theta) = -2\cos\theta$:

$$u(1, \theta) = d_0 + \sum_{n=1}^{\infty} (c_n \cos(n\theta) + e_n \sin(n\theta)) = -2\cos\theta$$

通过傅里叶级数系数匹配，我们得到：

$$d_0 = 0, \quad e_n = 0 \text{ for all } n$$

$$c_n = 0 \text{ for } n \neq 1, \quad c_1 = -2$$

将系数代回，得到最终解：

$$u(r, \theta) = -2r\cos\theta$$

主题：非齐次偏微分方程

方法一：处理非齐次边界条件

核心思想: 叠加原理。将解 $u(x, t)$ 分解为一个稳态解（或能处理边界条件的函数）$v(x)$ 和一个瞬态解 $w(x, t)$。

$$u(x, t) = v(x) + w(x, t)$$

目标是选择合适的 $v(x)$ 使得 $w(x, t)$ 满足齐次边界条件。

示例:

PDE: $u_t = a^2 u_{xx}$

BCs: $u(0, t) = T_1, \quad u(L, t) = T_2$

IC: $u(x, 0) = f(x)$

令 $v(x)$ 为满足边界条件的稳态解：

$$v''(x) = 0 \implies v(x) = Ax + B$$

应用边界条件 $v(0) = T_1, v(L) = T_2$ 得:

$$v(x) = \frac{T_2 - T_1}{L}x + T_1$$

现在，我们求解 $w(x,t) = u(x,t) - v(x)$ 的问题:

$$\begin{aligned}
\text{PDE: } & w_t = u_t = a^2 u_{xx} = a^2(w_{xx} + v_{xx}) = a^2 w_{xx} \quad (\text{因为 } v_{xx} = 0) \\
\text{BCs: } & w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0 \\
& w(L,t) = u(L,t) - v(L) = T_2 - T_2 = 0 \\
\text{IC: } & w(x,0) = u(x,0) - v(x) = f(x) - \left(\frac{T_2 - T_1}{L}x + T_1\right)
\end{aligned}$$

这样，$w(x,t)$ 的问题就转化为了一个具有齐次边界条件的标准热传导方程问题，可以用分离变量法求解。

$$w(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n\pi a/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

其中

$$C_n = \frac{2}{L}\int_0^L [f(x) - v(x)] \sin\left(\frac{n\pi x}{L}\right) dx$$

最终解为 $u(x,t) = v(x) + w(x,t)$。

方法二：处理非齐次源项（特征函数展开法）

核心思想: 将解和源项按照给定边界条件对应的特征函数展开。

示例:

$$\begin{aligned}
\text{PDE: } & u_t = a^2 u_{xx} + F(x,t) \\
\text{BCs: } & u(0,t) = 0, \quad u(L,t) = 0 \\
\text{IC: } & u(x,0) = f(x)
\end{aligned}$$

对于给定的齐次 Dirichlet 边界条件，特征函数是 $\sin(n\pi x/L)$。我们将 $u(x,t)$ 和 $F(x,t)$ 进行傅里叶正弦级数展开：

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi x}{L}\right), \quad \text{其中 } F_n(t) = \frac{2}{L}\int_0^L F(x,t) \sin\left(\frac{n\pi x}{L}\right) dx$$

将级数代入原 PDE，可得关于时间系数 $u_n(t)$ 的一阶线性常微分方程：

$$u_n'(t) + a^2 \left(\frac{n\pi}{L}\right)^2 u_n(t) = F_n(t)$$

初始条件由 $f(x)$ 决定：

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\implies u_n(0) = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

求解该常微分方程即可得到 $u_n(t)$，进而得到最终解 $u(x,t)$。

22 Sturm-Liouville 理论与正交性

本节内容对应于Sturm-Liouville型方程，并证明其特征值的性质以及特征函数的正交性。

22.1 Sturm-Liouville 方程与算符

给定一个二阶线性常微分方程：

$$\frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y + \lambda p(x)y = 0, \quad a \le x \le L$$

其中 $p(x) > 0$。边界条件为：

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(L) + \beta_2 y'(L) = 0 \end{cases}$$

定义Sturm-Liouville算符 L 为：

$$L[y] = \frac{d}{dx}\left(k(x)y'\right) - q(x)y$$

于是，原方程可以写成算符形式的本征值问题：

$$L[y] = -\lambda p(x)y$$

22.2 特征函数正交性证明

设 $y_m(x)$ 和 $y_n(x)$ 是对应于不同特征值 λ_m 和 λ_n 的特征函数，其中 $\lambda_m \neq \lambda_n$。

$$\begin{cases} L[y_n] = -\lambda_n p y_n & \cdots(1) \\ L[y_m] = -\lambda_m p y_m & \cdots(2) \end{cases}$$

用 y_m 乘以(1)式，用 y_n 乘以(2)式，然后相减可得：

$$y_m L[y_n] - y_n L[y_m] = (\lambda_m - \lambda_n)p(x)y_m y_n$$

对上式在区间 $[a, L]$ 上积分：

$$\int_a^L (y_m L[y_n] - y_n L[y_m])dx = (\lambda_m - \lambda_n)\int_a^L p(x)y_m y_n dx$$

考察积分的左侧（拉格朗日恒等式）：

$$\begin{aligned} \int_a^L (y_m L[y_n] - y_n L[y_m])dx &= \int_a^L \left[y_m\left((ky_n')' - qy_n\right) - y_n\left((ky_m')' - qy_m\right)\right]dx \\ &= \int_a^L \left[y_m(ky_n')' - y_n(ky_m')'\right]dx \\ &= \left[y_m(ky_n') - y_n(ky_m')\right]_a^L \\ &= k(L)[y_m(L)y_n'(L) - y_n(L)y_m'(L)] - k(a)[y_m(a)y_n'(a) - y_n(a)y_m'(a)] \end{aligned}$$

由于 y_m 和 y_n 都满足边界条件，可以证明对于所有齐次边界条件，上式结果为0。因此：

$$(\lambda_m - \lambda_n)\int_a^L p(x)y_m(x)y_n(x)dx = 0$$

因为 $\lambda_m \neq \lambda_n$，所以我们得到正交关系：

$$\int_a^L p(x)y_m(x)y_n(x)dx = 0 \quad (m \neq n)$$

22.3 瑞利商 (Rayleigh Quotient)

利用瑞利商可以证明在特定条件下特征值为正。

$$
\begin{aligned}
Q &= \frac{-\int_a^L yL[y]dx}{\int_a^L py^2 dx} \\
&= \frac{-\int_a^L y[(ky')' - qy]dx}{\int_a^L py^2 dx} \\
&= \frac{-[y(ky')]_a^L + \int_a^L (ky'^2 + qy^2)dx}{\int_a^L py^2 dx}
\end{aligned}
$$

在齐次边界条件下，$[y(ky')]_a^L = 0$。如果 $k(x) > 0$ 且 $q(x) \geq 0$，则积分项 $\int_a^L (ky'^2 + qy^2)dx \geq 0$。由于 $p(x) > 0$, $\int_a^L py^2 dx > 0$。因此 $\lambda = Q \geq 0$。

23 常微分方程边值问题：梁方程

笔记中给出了一个四阶常微分方程的例子，这在弹性力学中描述了梁的振动。

23.1 两端固支梁 (Clamped-Clamped Beam)

考虑一个两端固定的均匀梁，其振动方程和边界条件为：

$$
\frac{d^4y}{dx^4} - \lambda y = 0
$$

$$
y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0
$$

求解过程:

1. 假设存在非零解，$\lambda > 0$。令 $\lambda = \beta^4$ $(\beta > 0)$。
2. 特征方程为 $r^4 - \beta^4 = 0$，其根为 $r = \pm\beta, \pm i\beta$。
3. 通解为 $y(x) = C_1 \cosh(\beta x) + C_2 \sinh(\beta x) + C_3 \cos(\beta x) + C_4 \sin(\beta x)$。
4. 应用边界条件 $y(0) = 0$ 和 $y'(0) = 0$：

$$
\begin{aligned}
y(0) = C_1 + C_3 = 0 &\implies C_3 = -C_1 \\
y'(0) = \beta C_2 + \beta C_4 = 0 &\implies C_4 = -C_2
\end{aligned}
$$

5. 将此代入通解，得到满足左端边界条件的解：

$$
y(x) = C_1(\cosh(\beta x) - \cos(\beta x)) + C_2(\sinh(\beta x) - \sin(\beta x))
$$

6. 应用右端的边界条件 $y(L) = 0$ 和 $y'(L) = 0$：

$$
\begin{aligned}
C_1(\cosh(\beta L) - \cos(\beta L)) + C_2(\sinh(\beta L) - \sin(\beta L)) = 0 \\
C_1\beta(\sinh(\beta L) + \sin(\beta L)) + C_2\beta(\cosh(\beta L) - \cos(\beta L)) = 0
\end{aligned}
$$

7. 为了使 C_1, C_2 有非零解，其系数矩阵的行列式必须为零：

$$
\begin{vmatrix} \cosh(\beta L) - \cos(\beta L) & \sinh(\beta L) - \sin(\beta L) \\ \sinh(\beta L) + \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \end{vmatrix} = 0
$$

8. 展开行列式：

$$
\begin{aligned}
&(\cosh(\beta L)-\cos(\beta L))^2-(\sinh(\beta L)-\sin(\beta L))(\sinh(\beta L)+\sin(\beta L))=0\\
\Longrightarrow\ &(\cosh^2(\beta L)-2\cosh(\beta L)\cos(\beta L)+\cos^2(\beta L))-(\sinh^2(\beta L)-\sin^2(\beta L))=0\\
\Longrightarrow\ &(\cosh^2(\beta L)-\sinh^2(\beta L))+(\cos^2(\beta L)+\sin^2(\beta L))-2\cosh(\beta L)\cos(\beta L)=0\\
\Longrightarrow\ &1+1-2\cosh(\beta L)\cos(\beta L)=0
\end{aligned}
$$

9. 最终得到特征方程：

$$\cosh(\beta L)\cos(\beta L)=1$$

23.2 其他边界条件

笔记中还提到了另一组边界条件（可能是固支-简支梁）的特征方程：

$$\tan(\beta L)=\tanh(\beta L)$$

其特征值由该方程的根 β_n 决定：$\lambda_n=(\beta_n/L)^4$。

24 偏微分方程求解实例

24.1 波动方程：自由端边界条件

$$\begin{cases}u_{tt}=a^2u_{xx}\\ u_x(0,t)=0,\quad u_x(L,t)=0\quad(\text{自由端})\\ u(x,0)=f(x)\\ u_t(x,0)=g(x)\end{cases}$$ **求解（分离变量法）：**

1. 令 $u(x,t)=X(x)T(t)$，得到两个常微分方程：

$$X''(x)+\lambda X(x)=0,\quad X'(0)=0,X'(L)=0$$

$$T''(t)+\lambda a^2T(t)=0$$

2. 求解 $X(x)$ 的本征值问题，得到特征值和特征函数：

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\quad X_n(x)=\cos\left(\frac{n\pi x}{L}\right),\quad n=0,1,2,\dots$$

3. 对每一个 λ_n，求解 $T(t)$ 方程：

$$T_n(t)=A_n\cos\left(\frac{n\pi at}{L}\right)+B_n\sin\left(\frac{n\pi at}{L}\right)$$

4. 叠加得到通解：

$$u(x,t)=\frac{A_0+B_0t}{2}+\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi at}{L}\right)+B_n\sin\left(\frac{n\pi at}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

(注意：笔记中 $n=0$ 项的 B_0t 形式更通用，但对于 $g(x)$ 的傅里叶系数 B_0 来说，通常为 $T_0(t)=A_0+B_0t$)

5. 根据初始条件确定系数：

$$u(x,0)=f(x) \implies A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n=0,1,2,\dots)$$

$$u_t(x,0)=g(x) \implies B_n\frac{n\pi a}{L}=\frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n=1,2,\dots)$$

$$B_0=\frac{1}{L}\int_0^L g(x)dx$$

24.2 热传导方程：有热量损失

$$\begin{cases} u_t=a^2u_{xx}-bu & (b>0) \\ u_x(0,t)=0, \quad u_x(L,t)=0 & (\text{绝热边界}) \\ u(x,0)=f(x) \end{cases}$$ **求解（辅助变量法）：**

1. 做替换，令 $u(x,t)=v(x,t)e^{-bt}$。

2. 代入原方程，可以消去 $-bu$ 项，得到关于 $v(x,t)$ 的标准热传导方程：

$$v_t=a^2v_{xx}$$

3. 变换后的边界条件和初始条件为：

$$\begin{aligned} v_x(0,t)&=u_x(0,t)e^{bt}=0 \\ v_x(L,t)&=u_x(L,t)e^{bt}=0 \\ v(x,0)&=u(x,0)e^0=f(x) \end{aligned}$$

4. 这是一个具有绝热边界的标准热方程，其解为：

$$v(x,t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_ne^{-\left(\frac{n\pi a}{L}\right)^2t}\cos\left(\frac{n\pi x}{L}\right)$$

其中系数由初始条件 $v(x,0)=f(x)$ 决定：

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

5. 将 $v(x,t)$ 的解代回，得到原问题的最终解：

$$u(x,t)=e^{-bt}\left[\frac{A_0}{2}+\sum_{n=1}^{\infty}A_ne^{-\left(\frac{n\pi a}{L}\right)^2t}\cos\left(\frac{n\pi x}{L}\right)\right]$$

示例： 当初始条件为 $f(x)=x$ 时，计算傅里叶系数：

$$\begin{aligned} A_0&=\frac{2}{L}\int_0^L x\,dx=\frac{2}{L}\left[\frac{x^2}{2}\right]_0^L=L \\ A_n&=\frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx \\ &=\frac{2}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\bigg|_0^L-\int_0^L\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx\right] \\ &=\frac{2}{L}\left[0-\left(-\frac{L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right)_0^L\right] \\ &=\frac{2L}{(n\pi)^2}[\cos(n\pi)-1]=\frac{2L}{(n\pi)^2}[(-1)^n-1] \end{aligned}$$

所以，当 n 为偶数时 $A_n = 0$ $(n > 0)$，当 n 为奇数时 $A_n = -\frac{4L}{(n\pi)^2}$。

24.3 非齐次波动方程

$$\begin{cases} u_{tt} = a^2 u_{xx} + F(x,t) \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

求解（特征函数展开法）：

1. 对应的齐次问题的特征函数为 $\sin\left(\frac{n\pi x}{L}\right)$。
2. 将解和非齐次项 $F(x,t)$ 按特征函数展开：
$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$$
$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad \text{其中 } F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$$
3. 代入原PDE，得到关于 $u_n(t)$ 的常微分方程：
$$u_n''(t) + \left(\frac{n\pi a}{L}\right)^2 u_n(t) = F_n(t)$$
4. 初始条件也需要展开：
$$u_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$u_n'(0) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
5. 这个二阶非齐次常微分方程的解（通过常数变易法）为：
$$u_n(t) = u_n(0)\cos\left(\frac{n\pi at}{L}\right) + \frac{u_n'(0)L}{n\pi a}\sin\left(\frac{n\pi at}{L}\right) + \frac{L}{n\pi a}\int_0^t F_n(\tau)\sin\left(\frac{n\pi a(t-\tau)}{L}\right)d\tau$$
6. 将每个 $u_n(t)$ 的解代回级数，即可得到原问题的最终解。

一、Sturm-Liouville 理论

1. Sturm-Liouville 型方程与算子

考虑一个正则的 Sturm-Liouville 问题，其核心是二阶齐次线性微分方程：

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda\rho(x)y = 0, \quad x \in (a,b)$$

其中，我们要求 $p(x) > 0$ 和权函数 $\rho(x) > 0$。

为了方便讨论，定义 Sturm-Liouville 算子 L 如下：

$$L[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y$$

于是，Sturm-Liouville 方程可以简洁地表示为：

$$L[y] + \lambda\rho(x)y = 0$$

2. Lagrange 恒等式与算子的自伴随性

Lagrange 恒等式是证明 Sturm-Liouville 算子性质的关键。考虑积分 $\int_a^b (y_2 L[y_1] - y_1 L[y_2])dx$。通过两次分部积分，可以得到：

$$
\begin{aligned}
\int_a^b (y_2 L[y_1] - y_1 L[y_2])dx &= \int_a^b \left[y_2 \left((py_1')' - qy_1\right) - y_1 \left((py_2')' - qy_2\right)\right] dx \\
&= \int_a^b \left[y_2 (py_1')' - y_1 (py_2')'\right] dx \\
&= \left[y_2 (py_1') - y_1 (py_2')\right]_a^b \\
&= p(x) \left[y_2(x) y_1'(x) - y_1(x) y_2'(x)\right]_a^b
\end{aligned}
$$

如果边界条件使得上式右端为零，即 $p(x)[y_2 y_1' - y_1 y_2']_a^b = 0$，则称算子 L 在该边界条件下是自伴随的。此时，我们有：

$$
\int_a^b y_2 L[y_1] dx = \int_a^b y_1 L[y_2] dx
$$

这表明算子 L 是一个 Hermitian 算子。

3. 特征值与特征函数的正交性

设 $y_m(x)$ 和 $y_n(x)$ 是对应于不同特征值 $\lambda_m \neq \lambda_n$ 的特征函数，它们满足：

$$
\begin{cases}
L[y_m] + \lambda_m \rho y_m = 0 \\
L[y_n] + \lambda_n \rho y_n = 0
\end{cases}
$$

在自伴随的边界条件下，利用算子的自伴随性：

$$
\int_a^b (y_n L[y_m] - y_m L[y_n]) dx = 0
$$

将 $L[y_m] = -\lambda_m \rho y_m$ 和 $L[y_n] = -\lambda_n \rho y_n$ 代入上式：

$$
\int_a^b (y_n(-\lambda_m \rho y_m) - y_m(-\lambda_n \rho y_n)) dx = 0
$$

$$
(\lambda_n - \lambda_m) \int_a^b \rho(x) y_m(x) y_n(x) dx = 0
$$

由于我们假设 $\lambda_m \neq \lambda_n$，因此必须有：

$$
\int_a^b \rho(x) y_m(x) y_n(x) dx = 0, \quad (m \neq n)
$$

这即是 Sturm-Liouville 问题特征函数系 $\{y_n(x)\}$ 在权函数 $\rho(x)$ 下的正交性关系。

4. 基于特征函数的级数展开

对于定义区间 $[a, b]$ 内满足相应齐次边界条件的任意函数 $f(x)$，可以将其展开为特征函数的无穷级数：

$$
f(x) = \sum_{n=1}^{\infty} C_n y_n(x)
$$

利用特征函数的正交性，可以确定展开系数 C_n。将上式两边同乘以 $\rho(x)y_m(x)$ 并积分：

$$\int_a^b f(x)y_m(x)\rho(x)dx = \sum_{n=1}^{\infty} C_n \int_a^b y_n(x)y_m(x)\rho(x)dx$$

根据正交性，右边的积分仅在 $n = m$ 时不为零，因此：

$$C_m = \frac{\int_a^b f(x)y_m(x)\rho(x)dx}{\int_a^b y_m^2(x)\rho(x)dx}$$

二、高阶边值问题求解示例

问题描述

求解如下四阶微分方程的边值问题，这通常出现在梁的振动或屈曲分析中：

$$\begin{cases} \frac{d^4y}{dx^4} + \lambda \frac{d^2y}{dx^2} = 0 \\ y(0) = 0, \quad y'(0) = 0 \\ y(L) = 0, \quad y'(L) = 0 \end{cases}$$

求解过程

其特征方程为 $r^4 + \lambda r^2 = 0$, 即 $r^2(r^2 + \lambda) = 0$。

情况 1: $\lambda > 0$

设 $\lambda = \beta^2$ $(\beta > 0)$。特征根为 $r = 0$ (二重根) 和 $r = \pm i\beta$。通解为：

$$y(x) = A + Bx + C\cos(\beta x) + D\sin(\beta x)$$

应用边界条件： $y(0) = A + C = 0 \implies A = -C$ $y'(0) = B + \beta D = 0 \implies B = -\beta D$

将 A, B 代回，并应用在 $x = L$ 处的边界条件：

$$y(L) = -C + (-\beta D)L + C\cos(\beta L) + D\sin(\beta L) = 0$$
$$y'(L) = -\beta D - \beta C\sin(\beta L) + \beta D\cos(\beta L) = 0$$

整理成关于 C 和 D 的线性方程组：

$$\begin{cases} C(\cos(\beta L) - 1) + D(\sin(\beta L) - \beta L) = 0 \\ C(-\sin(\beta L)) + D(\cos(\beta L) - 1) = 0 \end{cases}$$

为了得到非平凡解，系数行列式必须为零：

$$\begin{vmatrix} \cos(\beta L) - 1 & \sin(\beta L) - \beta L \\ -\sin(\beta L) & \cos(\beta L) - 1 \end{vmatrix} = 0$$

展开行列式得到特征方程：

$$(\cos(\beta L) - 1)^2 + \sin(\beta L)(\sin(\beta L) - \beta L) = 0$$

$$\cos^2(\beta L) - 2\cos(\beta L) + 1 + \sin^2(\beta L) - \beta L\sin(\beta L) = 0$$

$$2 - 2\cos(\beta L) - \beta L \sin(\beta L) = 0$$

利用三角半角公式 $1 - \cos(x) = 2\sin^2(x/2)$ 和 $\sin(x) = 2\sin(x/2)\cos(x/2)$:

$$2\left(2\sin^2\left(\frac{\beta L}{2}\right)\right) - \beta L\left(2\sin\left(\frac{\beta L}{2}\right)\cos\left(\frac{\beta L}{2}\right)\right) = 0$$

$$4\sin\left(\frac{\beta L}{2}\right)\left[\sin\left(\frac{\beta L}{2}\right) - \frac{\beta L}{2}\cos\left(\frac{\beta L}{2}\right)\right] = 0$$

若 $\sin(\frac{\beta L}{2}) \neq 0$, 则必须满足：

$$\tan\left(\frac{\beta L}{2}\right) = \frac{\beta L}{2}$$

令 $\mu_n = \frac{\beta_n L}{2}$, 特征值 $\lambda_n = \beta_n^2 = (\frac{2\mu_n}{L})^2$，其中 μ_n 是超越方程 $\tan\mu = \mu$ 的正根。

情况 2: $\lambda = 0$ **和** $\lambda < 0$

对 $\lambda = 0$ (四重根 $r = 0$) 和 $\lambda < 0$ (二重根 $r = 0$ 和一对实根 $r = \pm\mu$) 的情况，代入边界条件均只能得到 $A = B = C = D = 0$ 的平凡解。因此不存在零特征值或负特征值。

三、非齐次偏微分方程求解

1. 非齐次边界条件的处理：辅助函数法

考虑一般的非齐次热传导方程：

$$\begin{cases} u_t = a^2 u_{xx} + F(x,t) \\ u(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

为将非齐次边界条件齐次化，引入辅助函数 $w(x,t)$，作变量代换 $u(x,t) = v(x,t) + w(x,t)$。选取 $w(x,t)$ 来满足原问题的边界条件。一个简单的选择是沿 x 的线性函数：

$$w(x,t) = A(t) + \frac{B(t) - A(t)}{L}x$$

将 $u = v + w$ 代入原方程组，可以得到一个关于 $v(x,t)$ 的、具有齐次边界条件的新问题：

- **PDE:** $v_t = a^2 v_{xx} + \left[F(x,t) - \frac{\partial w}{\partial t}\right]$
- **边界条件:** $v(0,t) = 0, \quad v(L,t) = 0$
- **初始条件:** $v(x,0) = f(x) - w(x,0)$

这个新的 v 问题就可以通过标准的特征函数展开法进行求解。

2. 特征函数展开法求解（混合边界条件示例）

考虑如下带有源项和混合边界条件的问题：

$$\begin{cases} u_t = a^2 u_{xx} + F(x,t) \\ u_x(0,t) = 0, \quad u(L,t) = 0 \quad \text{(一端绝热，一端零温)} \\ u(x,0) = f(x) \end{cases}$$

该边值问题对应的特征函数族为 $X_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right)$, for $n = 0, 1, 2, \ldots$ 我们将解 $u(x,t)$，源项 $F(x,t)$ 和初值 $f(x)$ 均按此特征函数系展开：

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

通过代入原方程并利用正交性，可以求得完整的解。其最终形式（通过Duhamel原理）为：

$$u(x,t) = \sum_{n=0}^{\infty} \left[C_n e^{-\lambda_n a^2 t} + \int_0^t e^{-\lambda_n a^2 (t-\tau)} F_n(\tau) d\tau\right] \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

其中，特征值 $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$。展开系数 C_n 和 $F_n(t)$ 由初始条件和源项决定：

$$C_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

$$F_n(t) = \frac{2}{L} \int_0^L F(x,t) \cos\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

将 C_n 和 $F_n(t)$ 的积分表达式代回，可以得到解的完整积分形式：

$$\begin{aligned} u(x,t) = &\sum_{n=0}^{\infty} \left\{\frac{2}{L} \int_0^L f(\xi) \cos\frac{(2n+1)\pi\xi}{2L} d\xi\right\} e^{-(\frac{(2n+1)\pi a}{2L})^2 t} \cos\frac{(2n+1)\pi x}{2L} \\ &+ \sum_{n=0}^{\infty} \left\{\frac{2}{L} \int_0^t e^{-(\frac{(2n+1)\pi a}{2L})^2 (t-\tau)} \left(\int_0^L F(\xi,\tau) \cos\frac{(2n+1)\pi\xi}{2L} d\xi\right) d\tau\right\} \cos\frac{(2n+1)\pi x}{2L} \end{aligned}$$

25 一维波动方程

25.1 驻波法 (分离变量法)

给定一个一维波动方程：

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

边界条件 (B.C.) 为：

$$u|_{x=0} = 0$$

$$u|_{x=L} = 0$$

初始条件为：

$$u|_{t=0} = f(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

使用分离变量法，设解的形式为 $u(x,t) = X(x)T(t)$。代入原方程得到：

$$X(x)T''(t) = a^2 X''(x)T(t)$$

分离变量后可得：

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

由此我们得到两个常微分方程 (ODEs):

$$X'' + \lambda^2 X = 0$$
$$T'' + a^2\lambda^2 T = 0$$

结合边界条件 $X(0) = 0$ 和 $X(L) = 0$，解得本征值和本征函数：

$$\lambda_m = \frac{m\pi}{L}, \quad X_m(x) = \sin\left(\frac{m\pi x}{L}\right), \quad (m = 1, 2, \dots)$$

对于时间部分的方程，其通解为：

$$T_m(t) = A_m \cos\left(\frac{m\pi a}{L}t\right) + B'_m \sin\left(\frac{m\pi a}{L}t\right)$$

因此，波动方程的通解为所有驻波解的叠加：

$$u(x,t) = \sum_{m=1}^{\infty} u_m(x,t) = \sum_{m=1}^{\infty} \left(A_m \cos\left(\frac{m\pi a}{L}t\right) + B_m \sin\left(\frac{m\pi a}{L}t\right)\right) \sin\left(\frac{m\pi x}{L}\right)$$

其中系数 A_m 和 B_m (这里 $B_m = B'_m$) 由初始条件确定：

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right) = f(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty} B_m \left(\frac{m\pi a}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = g(x)$$

利用傅里叶级数展开，我们可以得到系数的表达式：

$$A_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$B_m = \frac{2}{m\pi a}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

25.2 行波法 (d'Alembert 解法)

对于无界空间中的波动方程，我们采用行波法求解。

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

引入新的坐标变量：

$$\xi = x + at$$
$$\eta = x - at$$

通过链式法则计算 u 对 t 和 x 的偏导数：

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = a\frac{\partial u}{\partial \xi} - a\frac{\partial u}{\partial \eta} \\
\frac{\partial^2 u}{\partial t^2} &= a\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial t}\right) - a\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t}\right) \\
&= a^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x} \\
&= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}
\end{aligned}
$$

将以上结果代入原波动方程 $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$ 中，得到：

$$
-4a^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \implies \frac{\partial^2 u}{\partial \xi \partial \eta} = 0
$$

该方程的通解为：

$$
u(\xi, \eta) = \phi(\xi) + \psi(\eta)
$$

换回原变量 x, t，我们得到波动方程的通解，即 d'Alembert 解：

$$
u(x, t) = \phi(x + at) + \psi(x - at)
$$

现在利用初始条件求解 ϕ 和 ψ：

$$
\begin{aligned}
&u(x, 0) = \phi(x) + \psi(x) = f(x) \\
&\left.\frac{\partial u}{\partial t}\right|_{t=0} = a\phi'(x) - a\psi'(x) = g(x)
\end{aligned}
$$

对 (25.2) 式两边积分，得到：

$$
\phi(x) - \psi(x) = \frac{1}{a}\int_0^x g(s)ds + C
$$

联立 (25.2) 和上式，可以解出 $\phi(x)$ 和 $\psi(x)$:

$$
\begin{aligned}
\phi(x) &= \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(s)ds + \frac{C}{2} \\
\psi(x) &= \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(s)ds - \frac{C}{2}
\end{aligned}
$$

代入通解 $u(x, t) = \phi(x + at) + \psi(x - at)$ 中，最终得到 d'Alembert 公式：

$$
u(x, t) = \frac{1}{2}\left[f(x + at) + f(x - at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds
$$

26 算符理论

26.1 厄米算符与实本征值

一个算符 $\hat{F}$ 如果是厄米算符，则它满足以下关系：

$$
\int \psi^*(\hat{F}\phi)dx = \int (\hat{F}\psi)^*\phi dx
$$

现在证明厄米算符的本征值是实数。考虑本征方程 $\hat{F}\psi = \lambda\psi$。将 $\phi = \psi$ 代入厄米算符的定义式中：

$$
\int \psi^*(\hat{F}\psi)dx = \int (\hat{F}\psi)^*\psi dx
$$

将本征方程代入上式：

$$\int \psi^*(\lambda\psi)dx = \int (\lambda\psi)^*\psi dx$$

$$\lambda \int \psi^*\psi dx = \lambda^* \int \psi^*\psi dx$$

$$(\lambda - \lambda^*) \int |\psi|^2 dx = 0$$

因为 ψ 是本征函数，$\int |\psi|^2 dx \neq 0$，所以必须有：

$$\lambda - \lambda^* = 0 \implies \lambda = \lambda^*$$

这证明了厄米算符的本征值 λ 必为实数。

26.2 本征函数的正交性

证明：属于不同本征值的本征函数是正交的。设有两个本征方程：

$$\hat{F}\psi_m = \lambda_m\psi_m$$
$$\hat{F}\psi_n = \lambda_n\psi_n$$

考虑积分 $\int \psi_n^*(\hat{F}\psi_m)dx$。一方面：

$$\int \psi_n^*(\hat{F}\psi_m)dx = \int \psi_n^*\lambda_m\psi_m dx = \lambda_m \int \psi_n^*\psi_m dx$$

另一方面，利用 $\hat{F}$ 的厄米性：

$$\int \psi_n^*(\hat{F}\psi_m)dx = \int (\hat{F}\psi_n)^*\psi_m dx$$

将第二个本征方程代入：

$$\int (\hat{F}\psi_n)^*\psi_m dx = \int (\lambda_n\psi_n)^*\psi_m dx = \lambda_n^* \int \psi_n^*\psi_m dx$$

因为我们已经证明厄米算符的本征值是实数，所以 $\lambda_n^* = \lambda_n$。因此，我们得到：

$$\lambda_m \int \psi_n^*\psi_m dx = \lambda_n \int \psi_n^*\psi_m dx$$

$$(\lambda_m - \lambda_n) \int \psi_n^*\psi_m dx = 0$$

如果本征值不同，即 $\lambda_m \neq \lambda_n$，则必然有：

$$\int \psi_n^*\psi_m dx = 0 \quad (m \neq n)$$

这证明了属于不同本征值的本征函数是相互正交的。若 $m = n$，则积分 $\int \psi_m^*\psi_m dx = \int |\psi_m|^2 dx$ 是一个非零常数。如果 $\int |\psi_m|^2 dx = 1$，则称本征函数是归一化的。正交归一关系可以写为：

$$\int \psi_m^*\psi_n dx = \delta_{mn}$$

27 本征函数展开

任何一个行为良好的函数 $f(x)$ 都可以用一个完备的正交函数系 $\{\psi_n(x)\}$ 来展开。

$$f(x) = \sum_n C_n \psi_n(x)$$

为了确定展开系数 C_n，我们将上式两边同乘以 $\psi_m^*(x)$，然后在整个定义域上积分：

$$\int \psi_m^*(x) f(x) dx = \int \psi_m^*(x) \left(\sum_n C_n \psi_n(x) \right) dx$$

交换积分和求和的顺序：

$$\int \psi_m^*(x) f(x) dx = \sum_n C_n \int \psi_m^*(x) \psi_n(x) dx$$

利用本征函数的正交归一性 $\int \psi_m^*(x)\psi_n(x)dx = \delta_{mn}$：

$$\int \psi_m^*(x) f(x) dx = \sum_n C_n \delta_{mn}$$

由于克罗内克 δ_{mn} 的性质，求和号右边只有在 $n = m$ 的项不为零，因此：

$$\int \psi_m^*(x) f(x) dx = C_m$$

所以，展开系数 C_n 的表达式为：

$$C_n = \int \psi_n^*(x) f(x) dx$$

（注：这是在 $\{\psi_n(x)\}$ 是一个标准正交基的前提下。如果基底仅是正交的，则 $C_n = \frac{\int \psi_n^*(x) f(x) dx}{\int |\psi_n(x)|^2 dx}$）。

28 Sturm-Liouville 理论

28.1 Sturm-Liouville 型方程

Sturm-Liouville (S-L) 型方程的一般形式为：

$$\frac{d}{dx}\left[p(x) \frac{dy}{dx} \right] - q(x) y + \lambda \rho(x) y = 0, \quad a \le x \le L$$

其中 $p(x) > 0, \rho(x) > 0$。该方程通常伴随有齐次边界条件，例如：

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(L) + \beta_2 y'(L) = 0 \end{cases}$$

使得方程有非零解的 λ 称为特征值（或本征值），对应的非零解 $y(x)$ 称为特征函数（或本征函数）。

我们可以定义 Sturm-Liouville 算子 L 为：

$$L[y] = \frac{d}{dx}\left(p(x) \frac{dy}{dx} \right) - q(x) y$$

于是 S-L 方程可以写为 $L[y] + \lambda \rho(x) y = 0$。

28.2 特征函数的正交性

S-L 问题的特征函数系是加权正交的。设 $\lambda_m \neq \lambda_n$ 是两个不同的特征值，它们对应的特征函数分别为 $y_m(x)$ 和 $y_n(x)$。它们满足方程：

$$\frac{d}{dx}\left(p\frac{dy_m}{dx}\right) - qy_m + \lambda_m \rho y_m = 0 \quad \cdots(1)$$
$$\frac{d}{dx}\left(p\frac{dy_n}{dx}\right) - qy_n + \lambda_n \rho y_n = 0 \quad \cdots(2)$$

将 (1) 式乘以 y_n，(2) 式乘以 y_m，然后相减得到：

$$y_n\frac{d}{dx}\left(p\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(p\frac{dy_n}{dx}\right) + (\lambda_m - \lambda_n)\rho y_m y_n = 0$$

注意到左边前两项可以写作一个全导数：

$$\frac{d}{dx}\left[p\left(y_n\frac{dy_m}{dx} - y_m\frac{dy_n}{dx}\right)\right] = y_n\frac{d}{dx}\left(p\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(p\frac{dy_n}{dx}\right)$$

所以

$$(\lambda_m - \lambda_n)\rho y_m y_n = -\frac{d}{dx}\left[p\left(y_n y_m' - y_m y_n'\right)\right]$$

对上式从 a 到 L 积分：

$$(\lambda_m - \lambda_n)\int_a^L \rho(x)y_m(x)y_n(x)dx = -\left[p(x)(y_n y_m' - y_m y_n')\right]_a^L$$

由于 y_m 和 y_n 满足齐次边界条件，可以证明右边的边界项为零。因此，当 $\lambda_m \neq \lambda_n$ 时，我们得到正交关系：

$$\int_a^L \rho(x)y_m(x)y_n(x)dx = 0$$

这表明特征函数系 $\{y_n(x)\}$ 关于权函数 $\rho(x)$ 在区间 $[a, L]$ 上是正交的。

28.3 特征函数展开

任意满足相同边界条件的函数 $f(x)$ 可以展开为 S-L 特征函数的级数：

$$f(x) = \sum_{n=1}^{\infty} C_n y_n(x)$$

为了求系数 C_n，我们将上式两边同乘以 $\rho(x)y_m(x)$，并在 $[a, L]$ 上积分：

$$\int_a^L f(x)y_m(x)\rho(x)dx = \int_a^L \left(\sum_{n=1}^{\infty} C_n y_n(x)\right) y_m(x)\rho(x)dx$$

$$\int_a^L f(x)y_m(x)\rho(x)dx = \sum_{n=1}^{\infty} C_n \int_a^L y_n(x)y_m(x)\rho(x)dx$$

利用正交性，右边的积分项仅在 $n = m$ 时不为零：

$$\int_a^L f(x)y_m(x)\rho(x)dx = C_m \int_a^L y_m^2(x)\rho(x)dx$$

因此，系数 C_m 为：

$$C_m = \frac{\int_a^L f(x)y_m(x)\rho(x)dx}{\int_a^L y_m^2(x)\rho(x)dx} = \frac{\langle f, y_m\rangle}{\|y_m\|^2}$$

29 本征值问题求解实例

29.1 例1：一个四阶常微分方程边值问题

求解方程和边界条件：

$$\begin{cases} \frac{d^4y}{dx^4} + \lambda \frac{d^2y}{dx^2} = 0, \quad 0 < x < L \\ y(0) = 0, \quad y'(0) = 0 \\ y(L) = 0, \quad y'(L) = 0 \end{cases}$$

该问题的特征方程为 $r^4 + \lambda r^2 = 0$, 即 $r^2(r^2 + \lambda) = 0$.

29.1.1 情况 1: $\lambda > 0$

令 $\lambda = \beta^2$ $(\beta > 0)$。特征根为 $r = 0, 0, \pm i\beta$. 通解为:

$$y(x) = C_1 + C_2 x + C_3 \cos(\beta x) + C_4 \sin(\beta x)$$

其导数为:

$$y'(x) = C_2 - C_3 \beta \sin(\beta x) + C_4 \beta \cos(\beta x)$$

代入边界条件得到线性方程组：

1. $y(0) = 0 \implies C_1 + C_3 = 0$
2. $y'(0) = 0 \implies C_2 + C_4 \beta = 0$
3. $y(L) = 0 \implies C_1 + C_2 L + C_3 \cos(\beta L) + C_4 \sin(\beta L) = 0$
4. $y'(L) = 0 \implies C_2 - C_3 \beta \sin(\beta L) + C_4 \beta \cos(\beta L) = 0$

由 (1) 和 (2) 得 $C_3 = -C_1$ 和 $C_2 = -C_4\beta$。代入 (3) 和 (4):

$$C_1(1 - \cos(\beta L)) + C_4(\sin(\beta L) - \beta L) = 0$$

$$-C_4\beta + C_1 \beta \sin(\beta L) + C_4 \beta \cos(\beta L) = 0 \implies C_1 \sin(\beta L) + C_4(\cos(\beta L) - 1) = 0$$

为了使 C_1, C_4 有非零解，系数行列式必须为零：

$$\begin{vmatrix} 1 - \cos(\beta L) & \sin(\beta L) - \beta L \\ \sin(\beta L) & \cos(\beta L) - 1 \end{vmatrix} = 0$$

展开行列式：

$$-(1 - \cos(\beta L))^2 - \sin(\beta L)(\sin(\beta L) - \beta L) = 0$$

$$-(1 - 2\cos(\beta L) + \cos^2(\beta L)) - \sin^2(\beta L) + \beta L \sin(\beta L) = 0$$

$$-1 + 2\cos(\beta L) - (\cos^2(\beta L) + \sin^2(\beta L)) + \beta L \sin(\beta L) = 0$$

$$-2 + 2\cos(\beta L) + \beta L \sin(\beta L) = 0$$

利用三角恒等变换 $1 - \cos(\theta) = 2\sin^2(\frac{\theta}{2})$ 和 $\sin(\theta) = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$:

$$\beta L \left(2 \sin \frac{\beta L}{2} \cos \frac{\beta L}{2} \right) = 2 \left(2 \sin^2 \frac{\beta L}{2} \right)$$

若 $\sin(\frac{\beta L}{2})\neq 0$, 则两边可约去 $2\sin(\frac{\beta L}{2})$:

$$\beta L\cos\frac{\beta L}{2}=2\sin\frac{\beta L}{2}$$

得到超越方程:

$$\tan\left(\frac{\beta L}{2}\right)=\frac{\beta L}{2}$$

令 $\mu_n=\frac{\beta_n L}{2}$，则 $\tan(\mu_n)=\mu_n$。该方程有无穷多正根 μ_n。特征值为 $\lambda_n=\beta_n^2=\left(\frac{2\mu_n}{L}\right)^2$。

29.1.2 情况 2: $\lambda=0$

方程为 $y''''(x)=0$，通解为 $y(x)=Ax^3+Bx^2+Cx+D$。代入边界条件，易得 $A=B=C=D=0$，只有零解。

29.1.3 情况 3: $\lambda<0$

令 $\lambda=-\beta^2$ $(\beta>0)$。特征根为 $r=0,0,\pm\beta$。通解为 $y(x)=C_1+C_2x+C_3\cosh(\beta x)+C_4\sinh(\beta x)$。代入边界条件同样会得到只有零解。

29.2 例2：一个二阶常微分方程边值问题

求解方程和边界条件：

$$\begin{cases}X''(x)+\lambda X(x)=0, & 0<x<L\\ X(0)=0, \quad X'(L)=0\end{cases}$$

29.2.1 情况 1: $\lambda>0$

令 $\lambda=k^2$ $(k>0)$。通解为

$$X(x)=A\cos(kx)+B\sin(kx)$$

应用边界条件：

1. $X(0)=0\implies A=0$.
2. $X'(x)=Bk\cos(kx)$.
3. $X'(L)=0\implies Bk\cos(kL)=0$.

为得到非零解，必须有 $B\neq 0$ 和 $k\neq 0$，因此

$$\cos(kL)=0$$

这给出 $kL=\frac{(2n+1)\pi}{2}$ for $n=0,1,2,\ldots$. 特征值为：

$$\lambda_n=k_n^2=\left(\frac{(2n+1)\pi}{2L}\right)^2,\quad n=0,1,2,\ldots$$

对应的特征函数为：

$$X_n(x)=\sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

29.2.2 情况 2: $\lambda = 0$

方程为 $X''(x) = 0$, 通解 $X(x) = Ax + B$. $X(0) = 0 \implies B = 0$. $X'(L) = 0 \implies A = 0$. 只有零解。

29.2.3 情况 3: $\lambda < 0$

令 $\lambda = -k^2$ $(k > 0)$。通解为 $X(x) = A\cosh(kx) + B\sinh(kx)$. $X(0) = 0 \implies A = 0$. $X'(L) = 0 \implies Bk\cosh(kL) = 0$. 因为 $k > 0, L > 0, \cosh(kL) \geq 1$, 所以 $B = 0$. 只有零解。

30 求解非齐次偏微分方程

30.1 方法一：特征函数展开法 (处理源项)

考虑如下带源项的非齐次热传导方程：

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \\ \text{齐次边界条件 (e.g., } u(0,t) = 0, u(L,t) = 0) \\ u(x,0) = \phi(x) \end{cases}$$

1. **求解对应的本征值问题**: 求解与边界条件相关的 S-L 问题 $X'' + \lambda X = 0$ 来找到特征函数系 $\{X_n(x)\}$ 和特征值 $\{\lambda_n\}$。 2. **展开**: 将解 $u(x,t)$ 和源项 $f(x,t)$ 按特征函数展开：

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x), \quad \text{其中 } f_n(t) = \frac{\int_0^L f(x,t) X_n(x) dx}{\int_0^L X_n^2(x) dx}$$

3. **求解常微分方程**: 将展开式代入原 PDE，利用 $X_n''(x) = -\lambda_n X_n(x)$, 得到关于 $u_n(t)$ 的常微分方程组：

$$\sum_{n=1}^{\infty} \left[u_n'(t) X_n(x) + a^2 \lambda_n u_n(t) X_n(x) \right] = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

根据正交性，每个系数必须相等：

$$u_n'(t) + a^2 \lambda_n u_n(t) = f_n(t)$$

4. **确定初始条件**: 展开初始条件 $u(x,0) = \phi(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) X_n(x) = \phi(x) \implies u_n(0) = \frac{\int_0^L \phi(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}$$

5. **求解**: 求解上述一阶线性 ODE，得到 $u_n(t)$，再带回 $u(x,t)$ 的级数表达式中得到最终解。其通解为：

$$u_n(t) = u_n(0) e^{-a^2 \lambda_n t} + \int_0^t e^{-a^2 \lambda_n (t-\tau)} f_n(\tau) d\tau$$

30.2 方法二：辅助函数法 (处理非齐次边界条件)

考虑如下带有非齐次边界条件的热传导方程：

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = g_1(t), \quad u(L,t) = g_2(t) \\ u(x,0) = \phi(x) \end{cases}$$

核心思想是将解拆分为两部分 $u(x,t) = v(x,t) + w(x,t)$。

1. **构造辅助函数** $w(x,t)$: 选择一个简单的函数 $w(x,t)$ 来满足非齐次的边界条件。一个常见的选择是构造一个关于 x 的线性函数：

$$w(x,t) = A(t)x + B(t)$$

代入边界条件：

$$w(0,t) = B(t) = g_1(t)$$

$$w(L,t) = A(t)L + g_1(t) = g_2(t) \implies A(t) = \frac{g_2(t) - g_1(t)}{L}$$

所以

$$w(x,t) = \left(\frac{g_2(t) - g_1(t)}{L} \right) x + g_1(t)$$

2. **导出新问题**: 令 $v(x,t) = u(x,t) - w(x,t)$。将 $u = v + w$ 代入原方程组，得到关于 $v(x,t)$ 的新问题。

- **PDE**: $\frac{\partial(v+w)}{\partial t} = a^2 \frac{\partial^2(v+w)}{\partial x^2}$. 因为 w 对 x 是线性的, $\frac{\partial^2 w}{\partial x^2} = 0$.

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial w}{\partial t}$$

  这是一个带源项 $F(x,t) = -\frac{\partial w}{\partial t}$ 的非齐次方程。

- **边界条件**:

$$v(0,t) = u(0,t) - w(0,t) = g_1(t) - g_1(t) = 0$$

$$v(L,t) = u(L,t) - w(L,t) = g_2(t) - g_2(t) = 0$$

  $v(x,t)$ 的边界条件是齐次的。

- **初始条件**:

$$v(x,0) = u(x,0) - w(x,0) = \phi(x) - w(x,0)$$

3. **求解**: 新的关于 $v(x,t)$ 的问题是一个具有齐次边界条件和源项的问题，可以用上一节的特征函数展开法求解。 4. **合成解**: 求出 $v(x,t)$ 后，最终解为 $u(x,t) = v(x,t) + w(x,t)$。

31 一维弦的横振动 (分离变量法)

给定一维波动方程描述弦的振动：

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

其中 $a^2 = T/\rho$。

我们采用分离变量法，设解的形式为 $u(x,t) = X(x)T(t)$。代入原方程得到：

$$X(x)T''(t) = a^2 X''(x)T(t)$$

分离变量后可得：

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \quad (\text{常数})$$

这导出两个常微分方程：

$$X''(x) + \lambda^2 X(x) = 0$$
$$T''(t) + a^2\lambda^2 T(t) = 0$$

考虑固定端点的边界条件 (B.C.):

$$u(0,t) = 0, \quad u(L,t) = 0$$

这意味着对于所有 t，$X(0) = 0$ 且 $X(L) = 0$。

解空间方程 $X''(x) + \lambda^2 X(x) = 0$，其通解为 $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$。

- 由 $X(0) = 0$ 得 $A = 0$。
- 由 $X(L) = 0$ 得 $B\sin(\lambda L) = 0$。为得到非平凡解，$B \neq 0$，因此 $\sin(\lambda L) = 0$。

这给出了本征值：

$$\lambda L = n\pi \implies \lambda_n = \frac{n\pi}{L}, \quad (n = 1, 2, 3, \dots)$$

对应的本征函数为：

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

对于每一个 λ_n，时间方程 $T_n''(t) + (a\lambda_n)^2 T_n(t) = 0$ 的解为：

$$T_n(t) = A_n \cos\left(\frac{an\pi t}{L}\right) + B_n \sin\left(\frac{an\pi t}{L}\right)$$

其中 A_n 和 B_n 是待定系数。

根据叠加原理，波动方程的通解是所有可能解的线性组合：

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{an\pi t}{L}\right) + B_n \sin\left(\frac{an\pi t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

系数 A_n 和 B_n 由初始条件确定。设初始条件 (I.C.) 为：

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

可得：

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{an\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

利用傅里叶级数的正交性，可以解出系数：

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$
$$B_n\left(\frac{an\pi}{L}\right) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

32　算符的厄米性与本征函数

32.1　厄米算符 (Hermitian Operator)

一个算符 F 被称为厄米算符，如果它满足以下关系：

$$\int \psi^*(F\phi)\,dx = \int (F\psi)^*\phi\,dx$$

性质1：厄米算符的本征值是实数

证明：设 F 是厄米算符，其本征方程为 $F\psi = \lambda\psi$。

$$\int \psi^*(F\psi)\,dx = \int \psi^*(\lambda\psi)\,dx = \lambda \int |\psi|^2\,dx$$
$$\int (F\psi)^*\psi\,dx = \int (\lambda\psi)^*\psi\,dx = \lambda^* \int |\psi|^2\,dx$$

根据厄米算符的定义，左边的两式相等，因此：

$$\lambda \int |\psi|^2\,dx = \lambda^* \int |\psi|^2\,dx$$

由于 $\int |\psi|^2\,dx \neq 0$，我们必然得到 $\lambda = \lambda^*$，这证明了本征值 λ 是实数。

性质2：厄米算符的期望值是实数

证明：算符 F 的期望值定义为 $\langle F\rangle = \int \psi^* F\psi\,dx$。其复共轭为：

$$\langle F\rangle^* = \left(\int \psi^* F\psi\,dx\right)^* = \int (\psi^* F\psi)^*\,dx = \int \psi (F\psi)^*\,dx$$

根据厄米算符的定义 $\int (F\psi)^*\phi\,dx = \int \psi^*(F\phi)\,dx$，令 $\phi = \psi$，则有：

$$\int (F\psi)^*\psi\,dx = \int \psi^* F\psi\,dx = \langle F\rangle$$

因此，我们得到 $\langle F\rangle^* = \langle F\rangle$，证明了期望值是实数。

32.2　本征函数的正交性与完备性

证明：不同本征值对应的本征函数正交

设有两个本征方程：

$$F\psi_n = \lambda_n\psi_n$$
$$F\psi_m = \lambda_m\psi_m$$

考虑积分 $\int \psi_m^*(F\psi_n)\,dx - \int (F\psi_m)^*\psi_n\,dx$。根据厄米定义，此值为0。

$$\begin{aligned}
0 &= \int \psi_m^*(F\psi_n)\,dx - \int (F\psi_m)^*\psi_n\,dx \\
&= \int \psi_m^*(\lambda_n\psi_n)\,dx - \int (\lambda_m\psi_m)^*\psi_n\,dx \\
&= \lambda_n \int \psi_m^*\psi_n\,dx - \lambda_m^* \int \psi_m^*\psi_n\,dx
\end{aligned}$$

因为厄米算符的本征值为实数，$\lambda_m^* = \lambda_m$，所以：

$$(\lambda_n - \lambda_m) \int \psi_m^* \psi_n \, dx = 0$$

如果本征值不同，即 $\lambda_n \neq \lambda_m$，则必须有：

$$\int \psi_m^* \psi_n \, dx = 0$$

这证明了对应不同本征值的本征函数是正交的。通常我们可以将其归一化，使得 $\int \psi_m^* \psi_n \, dx = \delta_{mn}$。

完备性与展开系数

厄米算符的本征函数集构成一个完备集。任何一个行为良好的函数 $f(x)$ 都可以用这个本征函数集展开：

$$f(x) = \sum_n C_n \psi_n(x)$$

为了求展开系数 C_m，我们将上式两边同乘 $\psi_m^*(x)$，然后在整个空间积分：

$$\begin{aligned}
\int \psi_m^*(x) f(x) \, dx &= \int \psi_m^*(x) \left(\sum_n C_n \psi_n(x) \right) dx \\
&= \sum_n C_n \int \psi_m^*(x) \psi_n(x) \, dx \\
&= \sum_n C_n \delta_{mn} \\
&= C_m
\end{aligned}$$

因此，展开系数为：

$$C_n = \int \psi_n^*(x) f(x) \, dx$$

33 波动方程的达朗贝尔 (d'Alembert) 解法

我们再次考虑一维波动方程 $u_{tt} - a^2 u_{xx} = 0$。引入特征坐标：

$$\xi = x + at, \quad \eta = x - at$$

利用链式法则计算偏导数：

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} = a \frac{\partial}{\partial \xi} - a \frac{\partial}{\partial \eta}
\end{aligned}$$

计算二阶偏导：

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) = \frac{\partial^2}{\partial \xi^2} + 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \\
\frac{\partial^2}{\partial t^2} &= a \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) a \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) = a^2 \left(\frac{\partial^2}{\partial \xi^2} - 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right)
\end{aligned}$$

将二阶偏导代入波动方程：

$$\begin{aligned}u_{tt}-a^2u_{xx}&=a^2(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta})-a^2(u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta})\\&=-4a^2u_{\xi\eta}=0\end{aligned}$$

这就得到了简化的方程 $u_{\xi\eta}=0$。对 η 积分得到 $u_\xi=g'(\xi)$（g'是任意函数），再对 ξ 积分得到 $u(\xi,\eta)=g(\xi)+f(\eta)$。将原坐标代回，得到波动方程的通解：

$$u(x,t)=g(x+at)+f(x-at)$$

对于具有初始条件 $u(x,0)=\phi(x)$ 和 $u_t(x,0)=\psi(x)$ 的初值问题，我们可以确定函数 f 和 g 的形式。

$$\begin{aligned}u(x,0)&=g(x)+f(x)=\phi(x) &(1)\\u_t(x,t)&=ag'(x+at)-af'(x-at)\\u_t(x,0)&=ag'(x)-af'(x)=\psi(x) &(2)\end{aligned}$$

对(2)式积分得到 $g(x)-f(x)=\frac{1}{a}\int_{x_0}^{x}\psi(s)ds+K$。(1)和此式联立求解 f,g:

$$\begin{aligned}g(z)&=\frac{1}{2}\phi(z)+\frac{1}{2a}\int_{x_0}^{z}\psi(s)ds+\frac{K}{2}\\f(z)&=\frac{1}{2}\phi(z)-\frac{1}{2a}\int_{x_0}^{z}\psi(s)ds-\frac{K}{2}\end{aligned}$$

代入通解 $u(x,t)=g(x+at)+f(x-at)$，得到达朗贝尔公式：

$$u(x,t)=\frac{\phi(x+at)+\phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$$

33.1 应用实例

例1

求解 $u_{tt}=u_{xx}$ $(a=1)$ 满足初始条件 $u(x,0)=x^3$ 和 $u_t(x,0)=\cos x$。这里 $\phi(x)=x^3$，$\psi(x)=\cos x$。

$$\begin{aligned}u(x,t)&=\frac{(x+t)^3+(x-t)^3}{2}+\frac{1}{2}\int_{x-t}^{x+t}\cos(s)ds\\&=\frac{(x^3+3x^2t+3xt^2+t^3)+(x^3-3x^2t+3xt^2-t^3)}{2}+\frac{1}{2}[\sin(s)]_{x-t}^{x+t}\\&=\frac{2x^3+6xt^2}{2}+\frac{1}{2}(\sin(x+t)-\sin(x-t))\\&=x^3+3xt^2+\cos(x)\sin(t)\end{aligned}$$

注：使用了和差化积公式 $\sin(A)-\sin(B)=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$。

例2

求解 $u_{tt}-u_{xx}=0$ ($a=1$) 满足初始条件 $u(x,0)=e^x$ 和 $u_t(x,0)=\sin x$。这里 $\phi(x)=e^x$，$\psi(x)=\sin x$。

$$\begin{aligned}
u(x,t) &= \frac{e^{x+t}+e^{x-t}}{2}+\frac{1}{2}\int_{x-t}^{x+t}\sin(s)ds \\
&= \frac{e^xe^t+e^xe^{-t}}{2}+\frac{1}{2}[-\cos(s)]_{x-t}^{x+t} \\
&= e^x\left(\frac{e^t+e^{-t}}{2}\right)-\frac{1}{2}(\cos(x+t)-\cos(x-t)) \\
&= e^x\cosh(t)-\frac{1}{2}(-2\sin(x)\sin(t)) \\
&= e^x\cosh(t)+\sin(x)\sin(t)
\end{aligned}$$

注：使用了和差化积公式 $\cos(A)-\cos(B)=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$。