Assignment 2.

7.(i) Given $a < 0$, $-a > 0$.

$\therefore$ by Archimedean property $\exists$ $n_0 \in \mathbb{N}$

st $n_0(-a) > 1$ $\Rightarrow$ $-1 > an$

$\Rightarrow$ $-\frac{1}{n_0} > a$

Also $\exists$ $n_1 \in \mathbb{N}$ st $n_1 > -a$

$\Rightarrow$ $a > -n_1$

Let $N = \max\{n_1, n_0\}$.

Then $N(-a) > 1$, $N > -a$

implies $-N < a < -\frac{1}{N}$.

(ii) Let $S = \{-n : n \in \mathbb{N}\}$.

Claim : S is not bdd below.

On the contrary, suppose S is bdd below

i.e $\exists$ $n_0 \in \mathbb{R}$ st $n_0 < -n$ $\forall$ $n \in \mathbb{N}$.

Hence by completeness axiom $\exists$ $m \in \mathbb{R}$

st $m = \inf\{-n : n \in \mathbb{N}\}$.

given $\epsilon > 0$, $m - \epsilon < m < -n$ $\forall n \in \mathbb{N}$

$\therefore$ $m - \epsilon < 0$ $\Rightarrow$ $\epsilon - m > 0$

and by Archimedean property $\exists$

$K \in \mathbb{N}$ st $K > \epsilon - m$

$\Rightarrow$ $m - \epsilon > -K$

i.e $\exists$ $k \in \mathbb{N}$ st $-k < m - t < m$

which contradicts that m is the infimum of the set $\{-n : n \in \mathbb{N}\}$.

Hence the set $\{-n : n \in \mathbb{N}\}$ cannot be bdd below.

iii. Given $y > 0$, by Archimedean property $\exists$ $n \in \mathbb{N}$ st $ny > 1$ — ⊛

If we prove that $2^n > n$ $\forall$ $n \in \mathbb{N}$ then using the fact that $y > 0$ and ⊛ we would have

$$2^n y > ny > 1.$$

$$\therefore \quad y > \frac{1}{2^n}.$$

By applying induction on n, we show that $2^n > n$ $\forall$ $n \in \mathbb{N}$.

For $n = 1$, $2 > 1$ is clearly true.

Assume $2^n > n$ $\forall$ $n \leq k$.

$$\therefore \quad 2^{k+1} = 2 \cdot 2^k > 2 \cdot k > k + 1 \;\; \forall \; k \geq 1.$$

Hence we see that given $y > 0$, $\exists$ $n \in \mathbb{N}$ st $y > \frac{1}{2^n}$.

7(iv). If $a, b \in \mathbb{R}$ are such that

$a \le b + \frac{1}{n} \quad \forall\, n \in \mathbb{N}$ then $a \le b$.

We first show that the given condition implies that $\forall\, \epsilon > 0$, $a \le b + \epsilon$.

Notice given $\epsilon > 0$, by Archimedean property $\exists\, n_\epsilon \in \mathbb{N}$ st $n_\epsilon \epsilon > 1$. This implies that $\frac{1}{n_\epsilon} < \epsilon$.

$\because$ by given condition, $a \le b + \frac{1}{n}$ $\forall n \in \mathbb{N}$,

$\therefore$ we see that for $\epsilon > 0$,

$$a \le b + \frac{1}{n_\epsilon} \le b + \epsilon.$$

Hence $a \le b + \epsilon$ $\forall\, \epsilon > 0$.

If $a \nleq b$, then by order axiom $a > b$, implying $a - b > 0$.

$\therefore$ putting $\epsilon = \frac{a-b}{2}$ we get

$$a \le b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a$$

which is weird.

Hence $a \le b$.

8. Given $x \in \mathbb{R}$, prove $\exists\,!\; n \in \mathbb{Z}$ such that $n-1 \le x < n$.

For $x \in \mathbb{R}$, $|x| \geq 0$. By Archimedean property $\exists\, n_0 \in \mathbb{N}$ st $n_0 > |x|$. Then by problem 1(i),

$$-n_0 < x < n_0.$$

Let $S_{n_0} = \{ j \in \mathbb{Z} : -n_0 \leq j \leq n_0 \}$

Clearly S_{n_0} is a finite set.

Let $S_{n_0}^x = \{ j \in S_{n_0} : j > x \}$

$\because n_0 \in S_{n_0}^x$ $S_n^0 \neq \phi$. Further $S_{n_0}^x$ is bdd below by x $\therefore$ $\inf S_{n_0}^x$ exists.

However as $S_{n_0}^x$ is finite by problem 3(ii) $\inf S_{n_0}^x \in S_{n_0}^x$ i.e $\exists$ $m \in S_{n_0}^x$ such that $\inf S_{n_0}^x = m$.

By problem 3(i), $\inf S_{n_0}^k$ is unique

$\therefore$ such an integer m is unique.

Since $x < m$ and $k < x$

$\therefore$ $k \leq m-1$, This implies $m-1 \in S_k$ but since $m-1 < m$ and $m = \inf S_k^x$ $\therefore$ $m-1 \notin S_k^x$.

Hence $m-1 \leq x$.

$\therefore$ From the uniqueness of $\inf S_k^x$ it follows that given $x \in \mathbb{R}$, $\exists !\, m \in \mathbb{N}$ such that $m-1 \leq x < m$.

9. Let $S \subseteq \mathbb{R}$, $S \neq \phi$.

Let $u \in \mathbb{R}$ be such that $u - \frac{1}{n}$ is not an upper bound of S $\forall$ $n \in \mathbb{N}$;

and $u + \frac{1}{n}$ is an upper bound of S $\forall n \in \mathbb{N}$.

Claim: $u = \sup S$.

Since $u + \frac{1}{n}$ is an upper bound of S $\forall\, n \in \mathbb{N}$, $\therefore$ by completeness axiom $\sup S$ exists and

$$\sup S \leq u + \frac{1}{n} \quad \forall\, n \in \mathbb{N}.$$

For $\epsilon > 0$, by Archimedean property $\exists$ $n_\epsilon \in \mathbb{N}$ s.t. $n_\epsilon \epsilon > 1$

$$\therefore u + \epsilon > \frac{1}{n_\epsilon} + u > \sup S$$

$$\Rightarrow \sup S < u + \epsilon \quad \forall\, \epsilon > 0. \text{ --- } (*)$$

Hence $\sup S \leq u$. — (i)

(otherwise if $u < \sup S$ then putting $\epsilon = \frac{\sup S - u}{2}$ we would get from $(*)$,

$$\sup S < u + \frac{\sup S - u}{2} = \frac{u + \sup S}{2} < \sup S$$

which is weird.)

On the other hand $\because$ $u - \frac{1}{n}$ is not an upper bd for S,

$$\therefore u - \frac{1}{n} < \sup S \leq u \quad \forall\, n \in \mathbb{N}.$$

$$\therefore u - \sup S < \frac{1}{n} \quad \forall\, n \in \mathbb{N}. \text{ — (ii)}$$

Using same argument as above we see that condition (ii) implies that $\forall\, \epsilon > 0$,

$u < \sup S + \epsilon$ and hence $u \le \sup S$. — (iii)

Using (i) and (iii) we can now conclude that $u \le \sup S \le u$

$\therefore \sup S = u$.

Conversely, if $u = \sup S$, then $\forall\ \epsilon > 0$

$u + \epsilon > u$ is an upper bound for S.

In particular $\because \frac{1}{n} > 0\ \forall\ n \in \mathbb{N}$,

$u + \frac{1}{n}$ is an upper bound for S.

By defn of supremum of a set S,

given $\epsilon > 0\ \exists\ s_\epsilon \in S$ st

$u - \epsilon < s_\epsilon \le \sup S$.

$\therefore$ for $n \in \mathbb{N}$, $\frac{1}{n} > 0$, $\exists\ s_n \in S$ st

$u - \frac{1}{n} < s_n \le \sup S$ implying

that $u - \frac{1}{n}$ is not an upper bound of $S\ \forall\ n \in \mathbb{N}$.

10. (i) Given $a < b$, by denseness of rationals in $\mathbb{R}$ $\exists\ x_1 \in \mathbb{Q}$ st

$a < x_1 < b$.

Let $S_{a,b} = \{x \in \mathbb{Q} : a < x < b\}$.

By Denseness of $\mathbb{Q}$ in $\mathbb{R}$, $S_{a,b}$ is a non-empty subset of $\mathbb{R}$. Since $S_{a,b}$ is bdd above, $\sup S_{a,b}$ exists.

If $S_{a,b}$ is finite then by problem 3 (ii) $\sup S_{a,b} \in S_{a,b}$ i.e $\exists$ $s \in \mathbb{Q}$ st

$a < s < b$ and $\forall$ $x \in \mathbb{Q}$ st $a < x < b$,

$x \leq s$.

But $s \in \mathbb{R}$ and $s < b$, $\therefore$ by denseness of $\mathbb{Q}$ in $\mathbb{R}$ $\exists$ $x' \in \mathbb{Q}$ st $s < x' < b$.

This would mean $a < s < x' < b$ and $x' \in \mathbb{Q}$, implying $x' \in S_{a,b}$ and $x' > \sup S_{a,b}$ which contradicts that $s = \sup S_{a,b}$.

Hence our assumption that $S_{a,b}$ is finite is wrong.

(ii). We know for $a, b \in \mathbb{R}$ st $a < b$

$a + \sqrt{2}$, $b + \sqrt{2} \in \mathbb{R}$ and

$$a + \sqrt{2} < b + \sqrt{2}$$

By denseness of rationals in $\mathbb{R}$, $\exists$ $r \in \mathbb{Q}$ st

$$a + \sqrt{2} < r < b + \sqrt{2}$$

$$\Rightarrow \quad a < r - \sqrt{2} < b$$

Claim: if $r \in \mathbb{Q}$ then $r - \sqrt{2}$ is an irrational number.

On the contrary assume

$x = r - \sqrt{2}$ is rational

then $r - x = \sqrt{2}$ would be rational

which is not true.

Hence we see that given $a, b \in \mathbb{R}$

$\exists$ $x \in \mathbb{I}$ st $a < x < b$.

(iii). If $a, b \in \mathbb{R}$ are such that $a < b$,

$b - a > 0$. $\therefore$ given any real

number $u > 0$, by Archimedean

property $\exists$ $n \in \mathbb{N}$ st $n(b-a) > u$.

Now using the same proof as the

denseness of rationals in $\mathbb{R}$ it can be

shown that $\exists$ $r \in \mathbb{Q}$ st

$$a < ru < b.$$