MTH101: Symmetry
Problem Set 4

Disclaimer: These are informally written solutions, meant to explain the problem to you. If I leave an argument incomplete, it means that you are supposed to complete it. Do not treat these as "model solutions".

(For Hints to Problem 1-6, see Lecture 26.)

Problem 7. A square matrix is said to be symmetric if it is equal to its transpose. Show that the set of all symmetric square matrices of size n is a subspace of $M_{n\times n}(F)$.

Solution. A matrix $A = (a_{ij})_{i,i}$ is symmetric if and only if $a_{ij} = a_{ji}$. We need to check that these matrices form a subspace. See Lecture 22, Slide 14 for the criterion for a set to be a subspace. We need to just check that these matrices are closed under addition and scaler multiplication.

Let $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ be symmetric matrices. So, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. Let $A + B = (c_{ij})_{i,j}$. Then $c_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = c_{ji}$. Thus, $A + B$ is symmetric.

Similarly, you can check that a scaler multiple of a symmetric matrix is symmetric. □

Problem 8. Show that the space of symmetric 2×2 matrices is 3-dimensional.

Solution. A 2×2 symmetric matrix is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. We can write this as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the space of symmetric 2×2 matrices is spanned by the three matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have to check if these are linearly independent.

Suppose we have a linear relation

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The LHS is equal to $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. So, we have the equation

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Comparing entries, we see that $a = b = c = 0$. Thus, the three matrices are linearly independent. Thus, the space is 3-dimensional. □

Problem 9. Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation. Let $\mathbf{w} \in W$. Show that the set

$$T^{-1}(\mathbf{w}) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{w}\}$$

is a subspace of V if and only if $\mathbf{w} = \mathbf{0}$.

Solution. First, suppose that $\mathbf{w} = 0$. We want to show that $T^{-1}(\mathbf{0})$ is a subspace of V. But this is the kernel of T and we already know that this is a subspace. (You should see if you can write the proof for this explicitly. Use the criterion in Lecture 22, slide 14.)

Conversely, suppose $\mathbf{w}$ is such that $T^{-1}(\mathbf{w})$ is a subspace. Let $\mathbf{x}$ and $\mathbf{y}$ be in $T^{-1}(\mathbf{w})$. Then $T(\mathbf{x}+\mathbf{y}) = \mathbf{w}+\mathbf{w} = 2\mathbf{w}$. As $\mathbf{w} \neq 0$, $2\mathbf{w} \neq \mathbf{w}$. Thus, $T(\mathbf{x}+\mathbf{y}) \neq T(\mathbf{x}) + T(\mathbf{y})$. Thus, $T^{-1}(\mathbf{w})$ is not a subspace. $\square$

Problem 10. Let V be the set of all polynomials of degree $\leqslant 3$. Show that V is a 4-dimensional vector space.

Solution. First, you have to check that V is a vector space, i.e. it is a subspace of the vector space of all polynomials. For this, you may use the criterion in Lecture 22, slide 14.

A polynomial in V is of the form $a_0 + a_1X + a_2X^2 + a_3X^3$. Thus, it is a linear combination of the polynomials 1, X, X^2, X^3. You can prove that these are linearly independent by using an argument similar to the one in the solution of Problem 8. $\square$

Problem 11. Let $T : \mathbb{R}^4 \to \mathbb{R}$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \end{bmatrix}.$$

Find a basis for $ker(T)$. Prove that it is a basis for this space.

Proof. See the example in Lecture 24. $\square$

Problem 12. Let V be a finite dimensional vector space over $\mathbb{R}$. Show that any injective linear transformation $T : V \to V$ is also surjective. (Hint: Pick a basis of V and look at its image under T.)

Solution. Let $v_1, \dots, v_n$ be a basis for V. The argument in Problem 4 shows that $T(v_1), \dots, T(v_n)$ are linearly independent. As V is n-dimensional, these elements form a basis of V. Thus, V is in the span of these elements. Thus, $V \subset Im(T)$, which shows that V is surjective. $\square$

Problem 13. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad \text{and} \qquad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Compute $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Solution. Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the form $x_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (You will have to solve a system of linear equations for this.)

Then,

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

$\square$

Problem 14. In the vector space $\mathbb{R}^3$, consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}$$

Is $\mathbf{v}_4$ in $span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

Solution. Solve the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{v}_4$. (You can solve this with row reduction.)

If there exists a solution, $\mathbf{v}_4$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. If there is no solution, it does not lie in the span. $\square$

Problem 15. Let V be a 3-dimensional vector space. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of V. Prove that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is also a basis of V.

Solution. You have to prove that there is no non-trivial solution to the equation

$$x_1(\mathbf{v}_1 + \mathbf{v}_2) + x_2(\mathbf{v}_2 + \mathbf{v}_3) + x_3(\mathbf{v}_3 + \mathbf{v}_1) = \mathbf{0}.$$

This can be re-written as

$$(x_1 + x_3)\mathbf{v}_1 + (x_1 + x_2)\mathbf{v}_2 + (x_2 + x_3)\mathbf{v}_3 = \mathbf{0}.$$

As $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, this gives us the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

Solve this system of equations and verify that it has only the trivial solution. $\square$