Lecture 26

Problem set 4 — Hints and solutions

1) In $\mathbb{R}^2$, $\mathcal{B} = [v_1, v_2]$ — ordered basis

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Find matrix representations for $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution: We want to find x_1, x_2 such that

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & -1 & -1 \end{array}\right]$$ — solve using row reduction or Cramer's rule

$$M_{\mathcal{B}}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Same for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\rightsquigarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Suppose you were asked to calculate the matrix repn for $\left[\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \mathcal{A}$.

$$M_{\mathcal{B}}(\mathcal{A}) = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

2) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x) = Ax$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -2 \end{bmatrix}$$

$\mathcal{B} = [v_1, v_2]$, $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathcal{C} = [w_1, w_2, w_3]$, $w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $w_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Calculate $M_{\mathcal{C}}^{\mathcal{B}}(T)$

<u>Solution</u> $\mathcal{E} = [e_1, e_2]$ be the standard basis for $\mathbb{R}^2$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathcal{F} = [f_1, f_2, f_3]$ be the standard basis for $\mathbb{R}^3$, $f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, etc.

$$A = M_{\mathcal{F}}^{\mathcal{E}}(T)$$

$$M_{\mathcal{C}}^{\mathcal{B}}(T) = \underline{M_{\mathcal{C}}(\mathcal{F})} \cdot M_{\mathcal{F}}^{\mathcal{E}}(T) \cdot \underline{M_{\mathcal{E}}(\mathcal{B})}$$

$M_{\mathcal{E}}(\mathcal{B}) = ?$

$$M_{\mathcal{E}}(\mathcal{B}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$M_{\mathcal{F}}(\mathcal{C}) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$M_{\mathcal{C}}(\mathcal{F}) = M_{\mathcal{F}}(\mathcal{C})^{-1}$ — calculate with row reduction

3) $S = \{v_1, v_2, v_3\} \subseteq \mathbb{R}^4$.

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}$$

Is this linearly indep.?

Solution: Find all x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve this linear system using row reduction. If you find a non-zero solution, then they are linearly dependent.

Other way: Consider the matrix $\begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & 1 \\ 4 & 2 & 4 \\ 5 & 3 & 3 \end{bmatrix}$

If this has a 3x3 minor with non-zero determinant, they are linearly independent.

4) V, W — vector spaces

$\{v_1, \ldots v_m\}$ — linearly indep set in V.

$T: V \to W$ is a one-to-one linear trans.

Show $\{T(v_1), \ldots, T(v_m)\}$ is linearly indep.

Proof: Want to show that if
$x_1 T(v_1) + x_2 T(v_2) + \cdots + x_m T(v_m) = 0$
then $x_i = 0 \quad \forall i.$

As T is linear,
$x_1 T(v_1) + x_2 T(v_2) + \cdots + x_m T(v_m)$
$= T(x_1 v_1 + x_2 v_2 + \cdots + x_m v_m)$
As this is equal to 0,
$T(x_1 v_1 + \cdots + x_m v_m) = 0 = T(0)$
$\Rightarrow x_1 v_1 + \cdots + x_m v_m = 0$ (as T is a one-to-one)
$\Rightarrow x_i = 0 \quad \forall i$ (as S is linearly independent)

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5) V, W – finite dim vector spaces.
$T: V \to W$ is an onto linear trans.
Show $\dim(V) \geq \dim(W)$
Proof: $\text{Rank}(T) + \dim(\ker(T)) = \dim(V)$
$\text{Rank}(T) = \dim(\text{Im}(T))$
As T is onto, $\text{Im}(T) = W$.
$\text{Rank}(T) = \dim(W)$.
So $\dim(W) + \underbrace{\dim(\ker(T))}_{\geq 0} = \dim(V)$

So as $\dim(\ker(T)) \geq 0$, $\dim(V) \geq \dim(W)$

Proof 2: Let $B = \{v_1, \ldots, v_n\}$ be a basis of V.
Then and v in V is of the form
$v = x_1 v_1 + x_2 v_2 \cdots + x_n v_n.$

Let $w \in W$.
As T is onto, $\exists$ $v \in V$ such that $T(v) = w$.
$v = x_1 v_1 + \cdots + x_n v_n$ for some x_i
So $w = T(v) = T(x_1 v_1 + x_2 v_2 + \cdots x_n v_n)$
$= x_1 T(v_1) + x_2 T(v_2) + \cdots + x_n T(v_n)$
So $w \in \text{Span}(T(v_1), \cdots, T(v_n))$

So $\{T(v_1), \cdots, T(v_n)\}$ is a spanning set of W.
So, it contains a basis of W.
So $\dim(W) \leq n = \dim(V)$

6) Show that $M_{m \times n}(\mathbb{R})$ is mn-dimensional.

Proof: $m=1$, $n=2$ $[*, *]$
Basis $[1, 0]$, $[0, 1]$

$m=2$, $n=2$ $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$

Basis $-$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows it is a spanning set.
If $x_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + x_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Then $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies x_1 = 0,\ x_2 = 0,\ x_3 = 0$
$x_4 = 0$.

For general m, n.

Let E_{ij} be the matrix having 1 in (i,j) position. and 0 elsewhere)

Then, if $A = (a_{ij})_{ij}$

$A = \sum_{i,j} a_{ij} \cdot E_{ij}$ — so this is a spanning set.

If $\sum_{i,j} x_{ij} E_{ij} = 0$,

Then we see that $\sum_{i,j} x_{ij} E_{ij}$ has x_{ij} as its (i,j) entry. So $x_{ij} = 0 \ \forall \ i,j$.

So $\{E_{ij}\}_{i,j}$ is a linearly indep set.

So, it is a basis.