Lecture 22

We defined a vector space to be a set V with two operations:

$$+ : V \times V \longrightarrow V \quad \text{and}$$

$$\cdot : \mathbb{R} \times V \longrightarrow V$$

$(V, +)$ is an abelian group.

$\cdot$ is required to satisfy the following properties:

(1) $1 \cdot v = v \quad \forall \quad v \in V$

(2) $(\alpha\beta)v = \alpha \cdot (\beta v)$ for $\alpha, \beta \in \mathbb{R}$ and $v \in V$

(3) $(\alpha+\beta)v = \alpha v + \beta v$ for $\alpha, \beta \in \mathbb{R}$, $v \in V$

(4) $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$ for $\alpha \in \mathbb{R}$, $v_1, v_2 \in V$.

We now look at some basic properties.

Basic properties

(1) $0 \cdot v = \bar{0} \quad \forall \; v \in \mathbb{R}$

the 0 in $\mathbb{R}$

identity for + in V

Proof: $0 \cdot v = (0+0)v = 0 \cdot v + 0 \cdot v$

So $0 \cdot v = \bar{0}$. //

(2) $-v = (-1) \cdot v \quad \forall \; v \in \mathbb{R}$.

<u>Proof</u>: $v + (-1) \cdot v = 1 \cdot v + (-1) \cdot v$

$= (1 + (-1)) \cdot v = 0 \cdot v = \bar{0}$

Thus $(-1) \cdot v = -v$. //

Examples of vector spaces

(1) Let S be any set. Let F be the set of functions from S to $\mathbb{R}$.

For functions $f, g \in F$, we define the function $f+g$ by the formula $(f+g)(s) = f(s) + g(s) \;\; \forall s \in S$

F is an abelian group under $+$.

For $\alpha \in \mathbb{R}$ and $f \in F$, we define the function αf by $(\alpha f)(s) = \alpha \cdot f(s)$.
F is a vector space with these operations.

(2) Let $\mathbb{R}[X]$ denotc the set of polynomials in the variable X, with coefficients in $\mathbb{R}$.
With the usual additron operation

on polynomials, $\mathbb{R}[X]$ is an abelian group.

If $\alpha \in \mathbb{R}$ and p is the polynomial $a_0 + a_1 X + \cdots + a_n X^n$, we define αp to be $\alpha a_0 + (\alpha a_1) X + \cdots + (\alpha a_n) X^n$.

This gives $\mathbb{R}[X]$ the structure of a vector space.

(3) Let m and n be positive integers.

Suppose we have a system of m equations in n variables as follows:

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= 0.
\end{aligned}
$$

Notice that all constant terms are zero.

Such a system is called a homogeneous system.

Recall that we write this system as the matrix equation $AX = 0$.

A solution is an element $v \in \mathbb{R}^n$ such that $Av = 0$.

If v_1 and v_2 are solutions, then $A(v_1 + v_2) = Av_1 + Av_2 = 0$.

Thus $v_1 + v_2$ is a solution as well.

Similarly, if $\alpha \in \mathbb{R}$ and v is a solution, then $A(\alpha v) = 0$.

[If $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\alpha v = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$.

As $a_{11}x_1 + \cdots + a_{1n}x_n = 0$

we have $a_{11}(\alpha x_1) + \cdots + a_{1n}(\alpha x_n) = 0$]

It is easy to check from this that the set of solutions is a vector space in which addition and scalar multiplication are given by the usual operations on $\mathbb{R}^n$. Thus, it is an example of a "subspace" of $\mathbb{R}^n$.

Subspaces

Let V be a vector space. A subspace W of V is a subset $W \subset V$ such that it is a vector space under the operations of addition and scalar multiplication on V. Thus, to begin with, it should be closed under these operations

Thus, we must have at least the following two conditions:

(1) If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$.

(2) If $w \in W$ and $\alpha \in \mathbb{R}$, then $\alpha w \in V$.

Apart from this, other conditions (e.g. existence of inverse) must also be checked.

However, these follow immediately.

__Existence of $\bar{0}$__: Suppose W is non-empty. Take any $w \in W$. As $0 \in \mathbb{R}$, $0 \cdot w \in W$, i.e. $\bar{0} \in W$.

__Existence of inverse__: If $w \in W$, $(-1) \cdot w \in W$. But $(-1) \cdot w = -w$. So $-w \in W$.

Other properties (associativity of $+$, properties of scalar mult., etc.) hold as they hold on the whole of V.

Thus, we see that:

<u>Lemma</u>: Let V be a vector space. A non-empty subset $W \subset V$ is a subspace if the following two conditions hold:

(i) $w_1, w_2 \in W \implies w_1 + w_2 \in W$.

(ii) $\alpha \in \mathbb{R}$ and $w \in W \implies \alpha w \in W$. //

This criterion can be expressed concisely as follows:

<u>Lemma</u> V – a vector space. $W \subset V$. Then, W is a subspace if and only if for any $w_1, w_2 \in W$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, we have $\alpha_1 w_1 + \alpha_2 w_2 \in W$. //

⌈If we take $\alpha_1 = \alpha_2 = 1$, we get (i) from previous lemma. If we take $\alpha_1 = \alpha$, $\alpha_2 = 0$, we get (ii). ⌋

Examples:

(1) Let V be a vector space.
Then, V is a subspace of V.
Also, $\{\vec{0}\}$ is a subspace of V.
(This is the "zero subspace.")

(2) As we saw, the solutions of a homogeneous system in n variables is a subspace of $\mathbb{R}^n$,

We can adapt many results and ideas from group theory to get analogous results about subspaces.

(1) The intersection of any family of subspaces of V is a subspace of V.

(2) Let V be a vector space and $S \subset V$ be a subset.

The intersection of all subspaces of V containing S is a subspace of V (by (1)).
This is called the <u>span</u> of S and is written as $\mathrm{Span}(S)$.
(This is analogous to the notion of the subgroup generated by a set.)

We have another description for Span(S):

$$\mathrm{Span}(S) = \left\{ a_1 v_1 + \cdots + a_n v_n \;\middle|\; \begin{array}{l} v_1, v_2, \ldots, v_n \in S \\ a_1, a_2, \ldots, a_n \in \mathbb{R} \end{array} \right\}$$

(Exercise: Prove this. You can compare with the analogous statement on groups from Lecture 10, slides 8–17.).

The elements of Span(S) are called the <u>linear (or $\mathbb{R}$-linear) combinations of S.</u>

Note that S may be an infinite set, but a linear combination only involves finitely many terms.

Example V - a vector space.

$v \in V$. Let $S = \{v\}$.

Then, $\mathrm{Span}(S) = \{\alpha v \mid \alpha \in \mathbb{R}\}$.

For example, if $V = \mathbb{R}^2$ or $\mathbb{R}^3$, and if $v \neq \overline{0}$, $\mathrm{Span}(S)$ is the line joining $\overline{0}$ and $\overline{v}$.

If $v = \overline{0}$, $\mathrm{Span}(S) = \{0\}$.

Example

Let $V = \mathbb{R}^3$ and let $v_1, v_2 \in \mathbb{R}$.

$\mathrm{Span}(\{v_1, v_2\}) = \{a_1 v_1 + a_2 v_2 \mid a_1, a_2 \in \mathbb{R}\}$.

What does this look like?

If $v_1 = v_2 = \overline{0}$, $\mathrm{Span}(\{v_1, v_2\}) = \{0\}$.

If $v_1 \neq 0$, and $v_2 = \alpha v_1$ for some $\alpha \in \mathbb{R}$, then $a_1 v_1 + a_2 v_2$ is just $(a_1 + a_2 \alpha) v_1 \in \mathrm{Span}(\{v_1\})$.

Thus, in this case, Span$(\{v_1, v_2\})$ is the line through 0 and v_1.

Now, finally, assume $v_1 \neq 0$, $v_2 \neq 0$ and $v_2 \neq \alpha v_1$ for any α.

(So, v_2 is not on the line containing $\bar{0}$ and v_1.)

Then, Span$(\{v_1, v_2\})$ can be shown to be the plane containing $\bar{0}$, v_1 and v_2.

__Example__ Let $V = \mathbb{R}^n$.

Let $S = \{e_1, \cdots e_n\}$.

Then, $\mathrm{Span}(S) = \mathbb{R}^n$.

Indeed, if $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then

$$v = a_1 e_1 + \cdots + a_n e_n \in \mathrm{Span}(S).$$

Definition V - a vector space.
A subset $S \subset V$ is a **spanning set** of V if $V = \text{Span}(S)$.

Note that if $S_1 \subset S_2$, then $\text{Span}(S_1) \subset \text{Span}(S_2)$.
So, if S is a spanning set of V, so is an subset of V which contains S.

Geometrical intuition for Span

For any $v \in V$, αv is another vector having the same direction as v, but possibly a different length.

$\alpha_1 v_1 + \alpha_2 v_2$ is a vector (i.e. position in space) reached by moving some distance along the direction of v_1 and then some distance in the direction of v_2.

Similarly,

Span(S) = collection of all locations in space that can be reached through movements in directions provided by elements of S.

Finite dimensionality

It seems reasonable, now, to say that V is finite-dimensional if V has a finite spanning set.

But what is dimension?

(next time).