MTH101: Symmetry
Problem Set 4

Problem 1. In the vector space $\mathbb{R}^2$, let $\mathcal{B}$ be the ordered basis $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$ where

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}3\\-1\end{bmatrix}$$

Find the matrix representations (as 2×1 matrices) for

$$\mathbf{w}_1 = \begin{bmatrix}2\\-1\end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

with respect to $\mathcal{B}$.

Problem 2. Consider the linear transformation from $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix}1 & -1\\2 & 0\\3 & -2\end{bmatrix}.$$

Compute the matrix representation for this linear transformation with respect to the ordered bases $\mathcal{B} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{bmatrix}$ of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

and

$$\mathbf{w}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \mathbf{w}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Problem 3. Consider the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^4$ (column matrices) where

$$\mathbf{v}_1 = \begin{bmatrix}2\\3\\4\\5\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}1\\-1\\2\\3\end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix}2\\1\\4\\3\end{bmatrix}.$$

Is this set linearly independent or not? Prove your claim.

Problem 4. Let V and W be vector spaces. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a linearly independent set in V. Let $T : V \to W$ be a *one-to-one* linear transformation. Prove that the set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is linearly independent.

Problem 5. Let V and W be finite dimensional vector spaces. Let $T : V \to W$ be an *onto* linear transformation. Show that $\dim(V) \geqslant \dim(W)$.

Problem 6. Let m and n be positive integers. Show that the space $M_{m\times n}(F)$ of $m \times n$ matrices is mn-dimensional. (Hint: Try to guess a basis to begin with.)

Problem 7. A square matrix is said to be symmetric if it is equal to its transpose. Show that the set of all symmetric square matrices of size n is a subspace of $M_{n\times n}(F)$.

Problem 8. Show that the space of symmetric 2×2 matrices is 3-dimensional.

Problem 9. Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation. Let $\mathbf{w} \in W$. Show that the set

$$T^{-1}(\mathbf{w}) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{w}\}$$

is a subspace of V if and only if $\mathbf{w} = \mathbf{0}$.

Problem 10. Let V be the set of all polynomials of degree $\leqslant 3$. Show that V is a 4-dimensional vector space.

Problem 11. Let $T : \mathbb{R}^4 \to \mathbb{R}$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \end{bmatrix}.$$

Find a basis for $ker(T)$. Prove that it is a basis for this space.

Problem 12. Let V be a finite dimensional vector space over $\mathbb{R}$. Show that any injective linear transformation $T : V \to V$ is also surjective. (Hint: Pick a basis of V and look at its image under T.)

Problem 13. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad \text{and} \qquad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Compute $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Problem 14. In the vector space $\mathbb{R}^3$, consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}$$

Is $\mathbf{v}_4$ in $span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

Problem 15. Let V be a 3-dimensional vector space. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of V. Prove that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is also a basis of V.