Lecture 25

Let V be a vector space.

An $m \times n$ matrix with entries in V is rectangular arrangement, with m rows and n columns, of mn elements of V.

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ \vdots & & & \\ v_{m1} & \cdots & & v_{mn} \end{bmatrix}$$

$v_{ij} \in V$ for all i, j

The set of such matrices is denoted by $M_{m\times n}(V)$.
This is actually a vector space, with addition and scaler multiplication being defined component-wise.
We will mostly be concerned with the case $m=1$ (row matrices).

Convention A 1×1 matrix with entries in V is of the form $[v]$, where $v \in V$.
In this case, we will identify it with the vector v and not write the brackets.

Matrices with vector entries will generally be written with "curly" letters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, etc.

Matrix multiplication

Given an element of $M_{m \times n}(V)$, it can be multiplied on the left or right by matrices with entries from $\mathbb{R}$ (with appropriate sizes).

e.g.

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} v_{11} + 4v_{12} & 2v_{11} + 5v_{12} & 3v_{11} + 6v_{12} \\ v_{21} + 4v_{22} & 2v_{21} + 5v_{22} & 3v_{21} + 6v_{22} \end{bmatrix}$$

Notice that we use scalar multiplication to multiply a real number with a vector. This operation is associative and distributes over addition in the appropriate sense.

e.g. $(A \cdot B) \cdot \mathcal{C} = A \cdot (B \cdot \mathcal{C})$

A, B - entries from $\mathbb{R}$.

$\mathcal{C}$ - entries from V.

An ordered basis as a row matrix

Given a basis of V with a chosen ordering of its elements, we will view it as a row matrix.

So, an <u>ordered basis</u> $\mathcal{B}$ of an n-dim vector space V is a $1 \times n$ matrix of the form $[v_1, \dots v_n]$ where $\{v_1, \dots, v_n\}$ is a basis.

Given an ordered basis $\mathcal{B}$, to any $v \in V$, we associate a column matrix in $M_{n\times 1}(\mathbb{R}) = \mathbb{R}^n$.

We write v as a linear combination of the basis elements.

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

The numbers $a_1, a_2, \ldots, a_n$ are uniquely determined.

Then the column matrix associated to v is $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$.

We call this <u>the matrix representation of v with respect to $\mathcal{B}$</u> and is denoted by $M_{\mathcal{B}}(v)$.

So, we have a function

$$\varphi_{\mathcal{B}} : V \to \mathbb{R}^n, \quad \varphi_{\mathcal{B}}(v) = M_{\mathcal{B}}(v).$$

We saw (in Lecture 24) that this is a linear isomorphism with inverse given by $\psi_B : \mathbb{R}^n \longrightarrow V$,

$$\psi_B\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 v_1 + \cdots + a_n v_n.$$

Observe that $[a_1 v_1 + \cdots a_n v_n]$

$$= [v_1, \cdots\cdots, v_n] \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$$

So, for any vector v, we see that

$$[v] = \mathcal{B} \cdot M_{\mathcal{B}}(v).$$

By our convention, we write $[v]$ as just v and so we have the formula

$$\boxed{v = \mathcal{B} \cdot M_{\mathcal{B}}(v)}$$

__Remark__ The fact that $\psi_{\mathcal{B}}$ is a 1-1 correspondence tells us that if for $x, y \in \mathbb{R}^n$, we have $\mathcal{B} \cdot x = \mathcal{B} \cdot y$, then $x = y$.
So $\mathcal{B}$ can be "cancelled" from both sides.
Note that this is only true when $\mathcal{B}$ is an ordered basis.

Example Suppose $V = \mathbb{R}^n$.

Let $\mathcal{B}$ be the standard basis

$[e_1, e_2, \cdots, e_n]$.

Then if $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, we see that

$v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$.

So $M_{\mathcal{B}}(v) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, which is v itself!

Example The vectors $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

are linearly independent and so form a basis of $\mathbb{R}^2$. Let $\mathcal{B} = [v_1, v_2]$.

Let $v = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. What is $M_{\mathcal{B}}(v)$?

We want to find x_1, x_2 such that

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This is a system of linear equations represented by

$$\left[\begin{array}{cc|c} 2 & 4 & 4 \\ 3 & 1 & 5 \end{array}\right].$$

We can solve this by row reduction or Cramer's rule and find that the only solution is $x_1 = 8/5$, $x_2 = 1/5$

So $M_{\mathcal{B}}(v) = \begin{bmatrix} 8/5 \\ 1/5 \end{bmatrix}$.

Matrix representation for row of vectors

Given a row matrix of vectors, say $\mathcal{A} = [w_1, \cdots, w_m]$ and an ordered basis $\mathcal{B} = [v_1, \cdots, v_n]$, we define the matrix representation of $\mathcal{A}$ with respect to $\mathcal{B}$, denoted by $M_{\mathcal{B}}(\mathcal{A})$ to be the $n \times m$ matrix whose i-th column is $M_{\mathcal{B}}(w_i)$.

Thus, we get a function

$$M_{1\times m}(V) \longrightarrow M_{n\times m}(\mathbb{R})$$

$$\mathcal{A} \longmapsto M_{\mathcal{B}}(\mathcal{A}).$$

So $M_{\mathcal{B}}(\mathcal{A}) = [M_{\mathcal{B}}(w_1) \;\; M_{\mathcal{B}}(w_2) \;\; \cdots \;\; M_{\mathcal{B}}(w_m)]$

These are <u>columns</u>, not single entries

Fact: $\boxed{\mathcal{A} = \mathcal{B} \cdot M_{\mathcal{B}}(\mathcal{A})}$ (Try to prove this!)

Example Let $V = \mathbb{R}^n$.

Let $\mathcal{B} = [e_1, \cdots e_n]$.

Then, for any matrix $\mathcal{A} = [v_1, \cdots v_m]$, the matrix $M_{\mathcal{B}}(\mathcal{A})$ is the $n \times m$ matrix, whose i^{th} column is $M_{\mathcal{B}}(v_i)$. But we saw earlier that when $\mathcal{B}$ is the standard basis, $M_{\mathcal{B}}(v_i)$ is v_i itself.

So, $M_{\mathcal{B}}(\mathcal{A})$ has v_i as its i^{th} column.

So, if $V = \mathbb{R}^2$, $\mathcal{B} = [e_1, e_2]$ and $\mathcal{A} = \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right]$, then

$$M_{\mathcal{B}}(\mathcal{A}) = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix}.$$

Example $V = \mathbb{R}^2$. $\mathcal{B} = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$

Suppose $\mathcal{A} = \left[\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right]$

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = (-2) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-3) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -1 \end{bmatrix} = 11 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-6) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 11 \\ -6 \end{bmatrix}$$

So $M_{\mathcal{B}}(\mathcal{A}) = \begin{bmatrix} -2 & -3 & 11 \\ 2 & 2 & -6 \end{bmatrix}$

$$\underset{\overset{\shortparallel}{\mathcal{B}}}{\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]} \cdot \underset{\overset{\shortparallel}{M_{\mathcal{B}}(\mathcal{A})}}{\begin{bmatrix} -2 & -3 & 11 \\ 2 & 2 & -6 \end{bmatrix}}$$

$$= \left[\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right] = \mathcal{A}.$$

Theorem Let V be a vector space. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered bases of V. Then,

$$\boxed{M_{\mathcal{B}_2}(v) = M_{\mathcal{B}_2}(\mathcal{B}_1) \cdot M_{\mathcal{B}_1}(v).}$$

Proof $\mathcal{B}_2 \cdot M_{\mathcal{B}_2}(v) = v.$

$$\mathcal{B}_2 \cdot \left(M_{\mathcal{B}_2}(\mathcal{B}_1) \cdot M_{\mathcal{B}_1}(v)\right) = \left(\mathcal{B}_2 \cdot M_{\mathcal{B}_2}(\mathcal{B}_1)\right) \cdot M_{\mathcal{B}_1}(v)$$

$$= \mathcal{B}_1 \cdot M_{\mathcal{B}_1}(v) = v.$$

Thus,

$$\mathcal{B}_2 \cdot M_{\mathcal{B}_2}(v) = \mathcal{B}_2 \cdot M_{\mathcal{B}_2}(\mathcal{B}_1) \cdot M_{\mathcal{B}_1}(v).$$

As $\mathcal{B}_2$ is an <u>ordered basis</u>, we know that it can be "cancelled" on the left.

So, $M_{\mathcal{B}_2}(v) = M_{\mathcal{B}_2}(\mathcal{B}_1) \cdot M_{\mathcal{B}_1}(v)$. //

Theorem V - fin. dim vector space. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered bases. Then the matrix $M_{\mathcal{B}_2}(\mathcal{B}_1)$ is invertible, and its inverse is $M_{\mathcal{B}_1}(\mathcal{B}_2)$.

Proof: $\mathcal{B}_1 = \mathcal{B}_2 \cdot M_{\mathcal{B}_2}(\mathcal{B}_1)$

$$= \mathcal{B}_1 \cdot M_{\mathcal{B}_1}(\mathcal{B}_2) \cdot M_{\mathcal{B}_2}(\mathcal{B}_1).$$

So

$$\mathcal{B}_1 \cdot I_n = \mathcal{B}_1 \cdot \left(M_{\mathcal{B}_1}(\mathcal{B}_2) \cdot M_{\mathcal{B}_2}(\mathcal{B}_1) \right)$$

$(n = \dim V)$.

Thus (since $\mathcal{B}_1$ can be "cancelled"),

$$I_n = M_{\mathcal{B}_1}(\mathcal{B}_2) \cdot M_{\mathcal{B}_2}(\mathcal{B}_1).$$

This proves the theorem. //

<u>Remark</u> Conversely an invertible matrix A is of the form $M_{\mathcal{B}_2}(\mathcal{B}_1)$ for some choice of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Indeed, take $\mathcal{B}_2$ = standard basis of $\mathbb{R}^n$

$= [e_1 \cdots \cdots e_n]$

As A is invertible, the columns of A are linearly independent, and

so they form a basis of $\mathbb{R}^n$.
Suppose the columns of A are $v_1, \cdots, v_n$ (in that order).
Take $\mathcal{B}_1 = [v_1, v_2, \cdots, v_n]$
Then $M_{\mathcal{B}_2}(\mathcal{B}_1) = A$.

Linear transformations

Recall that if $A \in M_{m\times n}(\mathbb{R})$, then $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $T(x) = Ax$ is a linear transformation.

The i^{th} column of A is $Ae_i = T(e_i)$.

Conversely, if $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is any linear transformation, let A be the $m\times n$ matrix whose i^{th} column is $T(e_i)$.

Then $T(e_i) = Ae_i$ for all i.

So, if $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$,

then
$$T(x) = x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n)$$
$$= A(x_1e_1) + A(x_2e_2) + \cdots + A(x_ne_n)$$
$$= A(x_1e_1 + \cdots + x_ne_n) = Ax.$$

Thus we have a 1-1 correspondence

$$\left\{\begin{array}{c}\text{linear transformations}\\ T: \mathbb{R}^n \longrightarrow \mathbb{R}^m\end{array}\right\} \longleftrightarrow M_{m\times n}(\mathbb{R}).$$

Give two <u>abstract</u> finite dim. vector spaces V and W, and a linear transformation $T: V \to W$, we cannot immediately associate a matrix to it. We need to choose ordered bases for V and W. Let $\mathcal{B}$ be an ordered basis for V and $\mathcal{C}$ be an ordered basis for W.

Let $\mathcal{B} = [v_1, \dots, v_n]$

and $\mathcal{C} = [w_1, \dots, w_m]$

These bases give isomorphisms

$V \longrightarrow \mathbb{R}^n$ and $W \longrightarrow \mathbb{R}^m$.

Then $T : V \longrightarrow W$ corresponds to a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^m$.

We want to calculate the matrix of this transformation.

Let us denote this matrix by A.

We have the following diagram

$$\begin{tikzcd}
 & v \arrow[r, mapsto] & M_{\mathcal{B}}(v) & \\
v \arrow[d, mapsto] & V \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[d] & x \arrow[d, mapsto] \\
T(v) & W \arrow[r] & \mathbb{R}^m & Ax \\
 & w \arrow[r, mapsto] & M_{\mathcal{C}}(w) &
\end{tikzcd}$$

Let us see what happens to $v \in V$.

$$
\begin{tikzcd}
v \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & M_{\mathcal{B}}(v) \arrow[ddd, mapsto] \\
& V \arrow[r] \arrow[dd] & \mathbb{R}^n \arrow[dd] & \\
& & & \\
& W \arrow[r] & \mathbb{R}^m & A \cdot M_{\mathcal{B}}(v). \\
T(v) \arrow[rrr, mapsto] & & & M_{\mathcal{C}}(T(v))
\end{tikzcd}
$$

So, $A \cdot M_{\mathcal{B}}(v) = M_{\mathcal{C}}(T(v))$.

Thus A is a matrix such that $M_{\mathcal{E}}(T(v)) = A \cdot M_{\mathcal{B}}(v)$ for all $v \in V$.

Let $e_1, \dots, e_n$ be the standard basis vectors in $\mathbb{R}^n$.

Then $M_{\mathcal{B}}(v_i) = e_i$ for all i, $1 \leq i \leq n$.

So $M_{\mathcal{E}}(T(v_i)) = Ae_i$, which is the i-th column of A.

Thus A is the $m \times n$ matrix whose i-th column is $M_{\mathcal{C}}(T(v_i))$.

We will denote this matrix by $M^{\mathcal{B}}_{\mathcal{C}}(T)$. (This is the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{C}$.)

Note that

$$T(v) = \mathcal{C} \cdot M_{\mathcal{C}}(T(v)).$$

As $M_{\mathcal{C}}(T(v)) = A \cdot M_{\mathcal{B}}(v)$

$$= M_{\mathcal{C}}^{\mathcal{B}}(T) \cdot M_{\mathcal{B}}(v),$$

we get

$$\boxed{T(v) = \mathcal{C} \cdot M_{\mathcal{C}}^{\mathcal{B}}(T) \cdot M_{\mathcal{B}}(v)}$$

Changing bases

Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are ordered bases on V.

Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are ordered bases on W.

What is the relation between $M^{\mathcal{B}_1}_{\mathcal{C}_1}(T)$ and $M^{\mathcal{B}_2}_{\mathcal{C}_2}(T)$?

$$T(v) = \mathcal{C}_1 \cdot M_{\mathcal{C}_1}^{\mathcal{B}_1}(T) \cdot M_{\mathcal{B}_1}(v)$$

We know that $\mathcal{C}_1 = \mathcal{C}_2 \cdot M_{\mathcal{C}_2}(\mathcal{C}_1)$

and $M_{\mathcal{B}_1}(v) = M_{\mathcal{B}_1}(\mathcal{B}_2) \cdot M_{\mathcal{B}_2}(v)$.

So $T(v) = \mathcal{C}_2 \cdot M_{\mathcal{C}_2}(\mathcal{C}_1) \cdot M_{\mathcal{C}_1}^{\mathcal{B}_1} \cdot M_{\mathcal{B}_1}(\mathcal{B}_2) \cdot M_{\mathcal{B}_2}(v)$

Also $T(v) = \mathcal{C}_2 \cdot M_{\mathcal{C}_2}^{\mathcal{B}_2}(T) \cdot M_{\mathcal{B}_2}(v)$.

So,

$$e_2 \cdot M_{e_2}^{B_2}(T) \cdot M_{B_2}(v) = e_2 \cdot M_{e_2}(e_1) \cdot M_{e_1}^{B_1}(T) \cdot M_{B_1}(B_2) \cdot M_{B_2}(v).$$

Cancelling e_2, we get

$$M_{e_2}^{B_2}(T) \cdot M_{B_2}(v) = M_{e_2}(e_1) \cdot M_{e_1}^{B_1}(T) \cdot M_{B_1}(B_2) \cdot M_{B_2}(v)$$

for all $v \in V$.

Choose v such that $M_{B_2}(v) = e_i$

Then we see that

$$M_{\mathcal{E}_2}^{\mathcal{B}_2}(T) \cdot e_i = M_{\mathcal{E}_2}(\mathcal{E}_1) \cdot M_{\mathcal{E}_1}^{\mathcal{B}_1}(T) \cdot M_{\mathcal{B}_1}(\mathcal{B}_2) \cdot e_i$$

Thus, the matrices $M_{\mathcal{E}_2}^{\mathcal{B}_2}(T)$ and $M_{\mathcal{E}_2}(\mathcal{E}_1) \cdot M_{\mathcal{E}_1}^{\mathcal{B}_1}(T) \cdot M_{\mathcal{B}_1}(\mathcal{B}_2)$ have the same i-th column for any i.

So

$$\boxed{M_{\mathcal{E}_2}^{\mathcal{B}_2}(T) = M_{\mathcal{E}_2}(\mathcal{E}_1) \cdot M_{\mathcal{E}_1}^{\mathcal{B}_1}(T) \cdot M_{\mathcal{B}_1}(\mathcal{B}_2)}$$