Lecture 18

Let m and n be positive integers.

The set of $m \times n$ matrices (with entries from $\mathbb{R}$) is denoted by $M_{m\times n}(\mathbb{R})$.

Notation:

If A is an $m \times n$ matrix, the (i,j)-entry of which is a_{ij}, we write $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, or just $A = (a_{ij})_{i,j}$ if m and n are known from the context.

Let $0_{m \times n}$ denote the $m \times n$ matrix in which every entry is equal to 0. Again, if m and n are clear from the context, we simply write 0 instead of $0_{m \times n}$.

Addition of matrices

Let $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ be elements of $M_{m \times n}(\mathbb{R})$.

We define $A + B$ to be the element $C = (c_{ij})_{i,j} \in M_{m \times n}(\mathbb{R})$ where $c_{ij} = a_{ij} + b_{ij}$.

Example

$$\begin{bmatrix} 1 & 0 & 8 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -2 & 3 \\ 7 & 2 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 6 & 5 \\ 9 & 6 & 2 & 12 \end{bmatrix}$$

$M_{m \times n}(\mathbb{R})$ forms an abelian group under this binary operation.

$0_{m \times n}$ is the identity

If $A = (a_{ij})_{i,j}$, then we define $-A$ to be $(-a_{ij})_{i,j}$. Clearly $-A$ is the inverse of A.

Matrix multiplication

Let $A = (a_{ij})_{i,j}$ be an $m \times n$ matrix.

Let $B = (b_{ij})_{i,j}$ be an $n \times p$ matrix.

Then we define the <u>product</u> AB to be the matrix $C = (c_{ij})_{i,j} \in M_{m \times p}(\mathbb{R})$ such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

In other words,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots$$
$$\cdots + a_{in}b_{nj}.$$

$[a_{i1}\ a_{i2} \cdots\ a_{in}]$ — i^{th} row of A

$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{jn} \end{bmatrix}$ - j^{th} column of B

Multiply the i^{th} terms in each for $1 \leq i < n$ and add.

Example

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 4 \\ 1 & 3 & 0 & 1 \\ 5 & 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Let us compute this

$$2 \cdot 1 + 3 \cdot 0 + 4 \cdot 4 = 18.$$

Example

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot(-1) + 1\cdot 7 & 2\cdot 5 + 1\cdot 6 \\ 4\cdot(-1) + 3\cdot 7 & 4\cdot 5 + 3\cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 16 \\ 17 & 38 \end{bmatrix}$$

Note

The product AB is defined only if the number of colums of A is equal to the number of rows of B.
So, even if AB is defined, BA may not be defined.

Furthermore, even if AB and BA are both defined, they may not be equal.

In fact, they may not even have the same shape.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \quad \text{but}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Even if AB and BA have the same shape, they may not be equal.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$

Identity matrices

For any integer n, let I_n denote the $n \times n$ matrix having 1's on the "diagonal" (top left to bottom right) and 0's elsewhere.

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Properties of multiplication

(1) Let A be an $m \times n$ matrix. Then,

$I_m \times A = A$ and

$A \times I_n = A$.

(2) $A \in M_{m \times n}(\mathbb{R})$, $B = M_{n \times p}(\mathbb{R})$ and $C \in M_{m \times p}(\mathbb{R})$. Then

$(AB)C = A(BC)$ (associativity)

(3) $A \in M_{m \times n}(\mathbb{R})$ and
$B, C \in M_{n \times p}(\mathbb{R})$.
Then
$A(B+C) = AB + AC$

(4) $A, B \in M_{m \times n}(\mathbb{R})$ and
$C \in M_{n \times p}(\mathbb{R})$.
Then, $(A+B) \cdot C = AC + BC$.

Systems of linear equations

Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

be a system of linear equations.

Let A be the $m \times n$ matrix $(a_{ij})_{i,j}$.

$$X = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

Then the above system can be written as $AX = B$.

Row operations through matrix multiplication

A — some $m \times n$ matrix.

B — matrix obtained from A by performing some row operation.

E — $m \times m$ matrix obtained by performing <u>same</u> operation on I_m.

Fact:

Then $EA = B$.

So row operations are the same as multiplying by some appropriate matrix on the <u>left</u>.

Example

$$A = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 0 & 2 & 8 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

Perform $R_2 + 2R_3$.

$$B = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 7 & 4 & 4 & 10 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

E is obtained by performing $R_2 + 2R_3$ on I_3

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 0 & 2 & 8 \\ 3 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 8 & 4 & 4 & 10 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

$$= B.$$

Exercise Try out some more examples and see if you can understand why this happens.