# Chapter 3

# Bondi's k-Calculus Approach to Special Relativity



## 3.1 Introduction

Bondi [4] has developed a unique highly intuitive approach to special relativity that displays some of the essential characteristics with spacetime diagrams.

> Sir Hermann Bondi (1919–2005) was a distinguished mathematician and cosmologist. With T. Gold and F. Hoyle, he developed the Steady State theory of the universe, widely believed but eventually discarded upon the discovery of the cosmic microwave background radiation. As well as his development of the k-calculus approach to special relativity, Bondi wrote several important papers on a variety of subjects in general relativity.

It makes extensive use of the relativistic Doppler factor $k$ that relates inertial reference frames in relative motion and the fundamental new aspect of relativity vis-a-vis Newtonian physics, the invariance of the speed of light. We will see how clearly this approach resolves the so-called twin paradox, a key stumbling block for many who are first introduced to relativity. The word "calculus" to describe the Bondi method is misleading as his method relies upon elementary

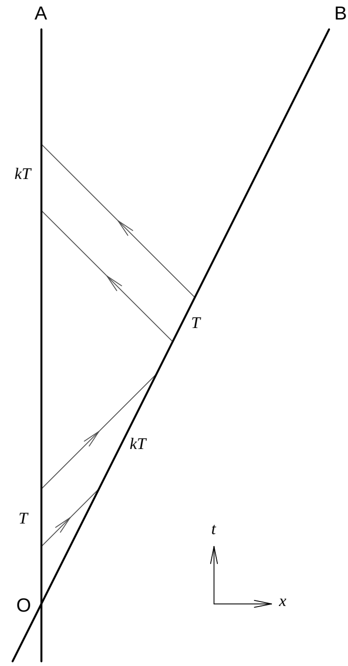


Figure 3.1: Light shone from Ava to Beta and vice-versa with intervals displaying their complete equivalence.



algebra and could be taught to junior high school students.

We consider two inertial observer twins Ava (A)[a] and Beta (B)[b] in relative motion. In Figure 3.1, the horizontal axis is $x$ and the vertical axis is $t$. The diagram is displayed from the perspective of Ava who is shown to be at rest in $(x, t)$ and Beta is seen to move in the positive $x$ direction.

The spacetime diagrams that follow have the closest connections to simple plane geometry from the vantage point of the rest observer.

Some time after Ava meets Beta (the point at which their lines cross in the diagram and at which point they both set their clocks to 0), Ava shines light toward Beta for a period $T$ by her (Ava's) clock. Beta receives the light for a proportional amount of time $kT$

[a]Ava–Greek–"An eagle".
[b]Beta–Czech–"Dedicated to God".

where the $k$ constant symbol is the Doppler factor connecting the observers. The lines with arrows indicate light rays.

Later, Beta sends light toward Ava for the same period $T$ and since Ava and Beta are totally equivalent physically, Ava observes the light for the same period as Beta had observed the earlier light from Ava, namely $kT$. Note that the geometrical appearance aspects such as the slopes of outgoing light rays, are necessarily biased in favor of the rest observer. Only relative to the rest observer are the light ray slopes always at 45 deg.



## 3.2 Velocity–Doppler factor connection

Ava wishes to determine Beta's speed relative to herself. For this, a different process is required, as shown in Figure 3.2. Immediately upon meeting Beta at O, Ava shines light to Beta for a period $T$ and as in Figure 3.1, Beta receives the light for a period $kT$. Ava and Beta agree that as long as Beta receives light from Ava, she (namely Beta) sends light back to Ava.

Since this emission period from Beta is $kT$, the reception of Beta's light by Ava is the Doppler factor $k$ times the emission period, i.e. $k(kT) = k^2T$.

From here it is a simple procedure to deduce the relative velocity between Beta and Ava. All that is required is to determine the distance between Ava and Beta at a convenient point and how much time has elapsed since their meeting at O to achieve that separation. The point P that is chosen is where Beta has received the last photon from Ava and has sent her last photon back to Ava. Ava reckons that she has sent her last photon reaching P at time $T$ and has received the last photon back from Beta at time $k^2T$. Thus, the time for the photon to reach P and return is $k^2T - T$. Multiplying this by $c$, the photon speed, we get twice the OP separation. Therefore the OP separation is $D = c(k^2T - T)/2$. Ava also reckons that the last photon reached P at the half-way point in time from the emission at time $T$ and the reception at time $k^2T$, namely at time $T^* = T + (k^2T - T)/2 = (k^2 + 1)T/2$. Finally, the relative velocity $v$ is $D/T^*$ or

$$v = c\frac{k^2 - 1}{k^2 + 1}. \tag{3.1}$$

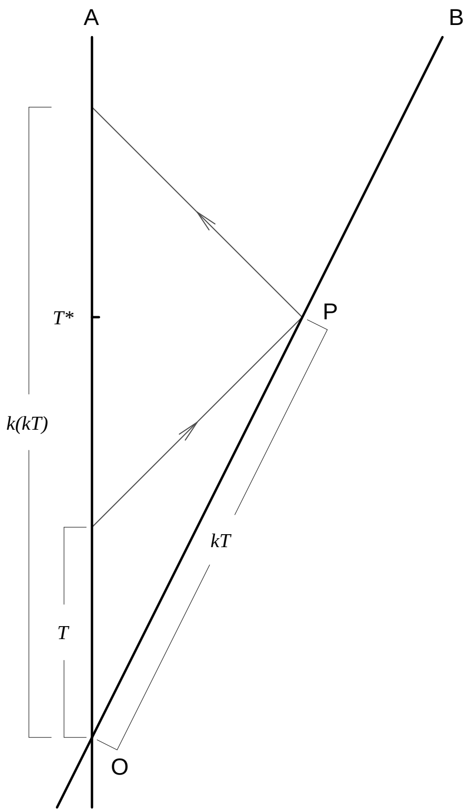


Figure 3.2: Determination of the speed of Beta relative to Ava.

Solving for $k$ yields the relativistic Doppler factor $k$ in terms of relative velocity[c]

$$k = \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{3.2}$$

The following properties are noted:

1) $k = 1$ for $v = 0$ which is logical since there is no Doppler shift when there is no relative velocity.

2) $k > 1$ if $v > 0$ which is logical since a relative recession entails an increase in period, decrease in frequency, increase in wavelength, or red-shift.

3) Similarly, $k < 1$ for $v < 0$. In this case there is a relative approach and hence a blue shift.

4) If $v \longrightarrow -v$ then $k \longrightarrow 1/k$.



## 3.3 Composition law for velocities and Doppler factors

We now determine the relativistic composition law for velocities. Consider a third observer Cayla (C),[d] introduced as in Figure 3.3. Let Ava emit light to Beta for a period $T$. Beta receives the light for a proportional time period $k_{\mathrm{AB}}T$ where $k_{\mathrm{AB}}$ is the Doppler factor between Ava and Beta. For as long as Beta receives the light, she transmits to Cayla who receives it for a period $k_{\mathrm{BC}}(k_{\mathrm{AB}}T)$ where $k_{\mathrm{BC}}$ is the Doppler factor between Beta and Cayla. At this point we invoke the key feature of special relativity, the invariance of the speed of light: we can view the direct transmission of light from Ava to Cayla for a period $T$ being received by Cayla for a time $k_{\mathrm{AC}}$ with the same photon lines that were already used for the previous two-step process. This is because the light speed does not get boosted in the two-step process as compared to the direct transmission. As a result, we can equate the reception times by Cayla of the one-step and two-step process. Canceling the common factor $T$, we have

$$k_{\mathrm{AC}} = k_{\mathrm{AB}}k_{\mathrm{BC}}. \tag{3.3}$$

[c]The positive root solution is chosen to maintain the same direction of flow of time for the two observers.

[d]Cayla–“pure”.

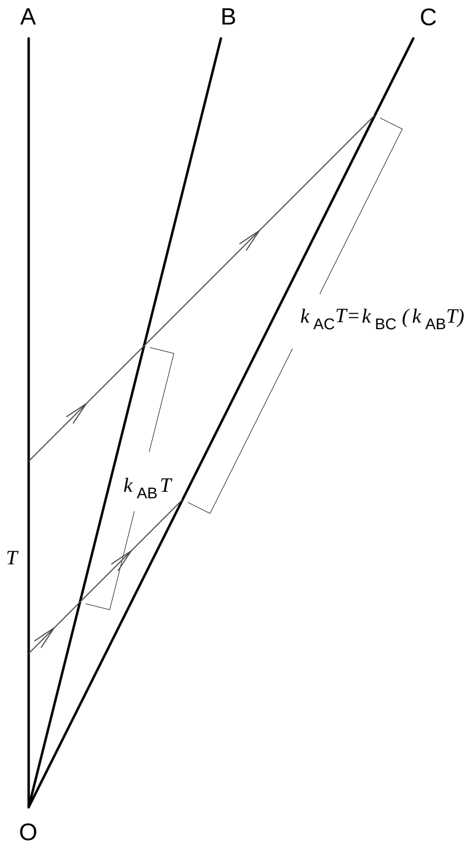


Figure 3.3: Determination of the compounding velocities in relativity.

Similarly, if we were to introduce a fourth observer Della (D)[e], we would have

$$k_{\mathrm{AD}} = k_{\mathrm{AB}} k_{\mathrm{BC}} k_{\mathrm{CD}} \tag{3.4}$$

and so on for an arbitrary sequence of observers. Let us now work in units where $c = 1$ (we will restore the symbol later). Returning to the three observers in Figure 3.3, from (3.1) and (3.3) we have

$$v_{\mathrm{AC}} = \frac{{k_{\mathrm{AC}}}^2 - 1}{{k_{\mathrm{AC}}}^2 + 1} = \frac{{k_{\mathrm{AB}}}^2 {k_{\mathrm{BC}}}^2 - 1}{{k_{\mathrm{AB}}}^2 {k_{\mathrm{BC}}}^2 + 1}. \tag{3.5}$$

Now using (3.2) repeatedly in (3.5) for the $k$ factors in terms of relative velocities, after simplification we find (with the explicit $c$ now restored)

$$v_{\mathrm{AC}} = \frac{v_{\mathrm{AB}} + v_{\mathrm{BC}}}{1 + v_{\mathrm{AB}} v_{\mathrm{BC}}/c^2}. \tag{3.6}$$

This is the familiar relativistic law for the composition of velocities. If we let $c$ approach infinity, we retrieve the usual Newtonian velocity composition law

$$v_{\mathrm{AC}} = v_{\mathrm{AB}} + v_{\mathrm{BC}}. \tag{3.7}$$

If the relative velocities of the observers are small compared to c, the effect of using (3.6) instead of (3.7) is small. However, for "relativistic" velocities, the effect is dramatic. For example, with $v_{\mathrm{AB}} = v_{\mathrm{BC}} = 3c/4$, the correct composition law (3.6) yields $v_{\mathrm{AC}} = 24c/25$, a velocity less than c as must be the case and considerably different from $1.5c$ that would result from (3.7).

Of particular interest is the elegant simplicity of the composition law for Doppler factors as a simple multiplicative sequence in (3.4) in contrast to the awkward composition law for velocities in relativity (3.6). The simplicity of the former is understandable as it reflects the invariance of the speed of light and it is the Doppler factor that characterizes the connection between observers vis-a-vis light propagation. Note, however, that the Newtonian composition law, (3.7) (and its familiar extension to more observers, even non-collinearly) for velocities does have the simplicity that the Doppler factor composition displays. This is a reflection of the absoluteness of time in Newtonian physics. The Newtonian composition proceeds pictorially

[e] Della-English-"A woman from the island of Delos".

with vectors pasted end-to-end, having each observer's clock in step with all the rest.

In the next section, we display the non-absoluteness of time as an algebraic formula.

## 3.4 Derivation of the Lorentz transformation

The Lorentz transformation is the recipe for the labeling of event coordinates by one inertial observer relative to another. To derive this recipe, we return to our two twin observers Ava and Beta and consider an event E which Ava labels $(t, x)$ and Beta labels $(t^*, x^*)$. Ava and Beta synchronize their clocks when they meet at O and Ava decides to send a photon so that it arrives in coincidence with the event E. At that point, it is reflected back to Ava. Since for Ava, the event is at position $x$, the photon must have been sent from Ava at time $t - x/c$ to arrive at time $t$ at E. It will arrive back to Ava a time $x/c$ later, i.e. at time $t + x/c$.

Since the speed of the photon is also $c$ for Beta, the photon intersects with Beta at time $t^* - x^*/c$ for the inbound path and at time $t^* + x^*/c$ for the outbound path. From our earlier discussion regarding the relationship between time intervals, we see from Figure 3.4 that

$$t^* - x^*/c = k(t - x/c) \tag{3.8}$$

where $k$ is the Doppler factor between Ava and Beta.

Similarly we can focus on an emission time interval of $t^* + x^*/c$ from Beta to Ava with a reception time interval $t + x/c$ by Ava's reckoning. Therefore we have

$$t + x/c = k(t^* + x^*/c) \tag{3.9}$$

Eliminating $x^*$ between (3.8) and (3.9), we find

$$2t^* = t(k + 1/k) - (x/c)(k - 1/k) \tag{3.10}$$

$$2x^* = x(k + 1/k) - (ct)(k - 1/k) \tag{3.11}$$

Substituting the expression for $k$ from (3.2) into (3.10) and (3.11), we find as in (2.5), the important Lorentz transformation

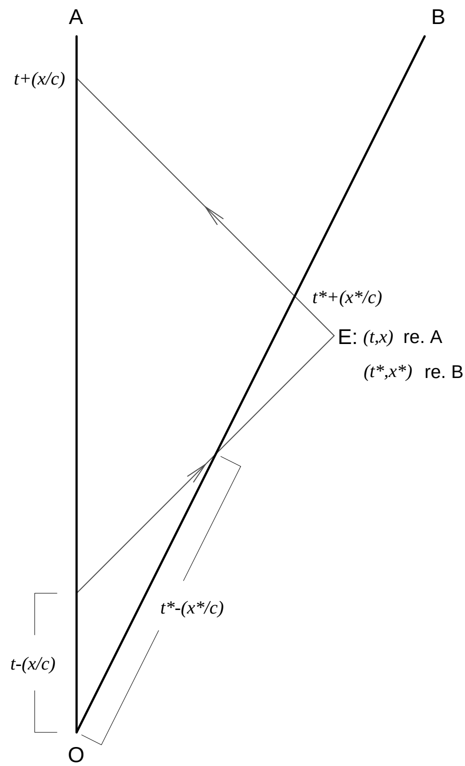




Figure 3.4: Derivation of the Lorentz transformation by the Bondi method.

$$x^* = \gamma(x - vt), ct^* = \gamma(ct - vx/c) \tag{3.12}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

In the second of (3.12), we see the complexity of the relationship between times read in the two frames in relative motion.

## 3.5 The twin or clock paradox

Because of the nature of time in the theory of relativity, it turns out that we can describe a very interesting scenario involving our pair of twins, Ava (A) and Beta (B). Ava stays home while Beta, the adventurer, takes off on a long journey at high speed, turns around eventually from homesickness and heads back to reunite with her twin sister Ava. If the conditions of time interval and speed are sufficient, Beta could return aged by let us say one year while Ava is long gone, unavailable for the planned reunion. Instead, Beta finds that she is meeting Ava's great-great-grandchildren. The supposed paradox consists of considering Beta to have been at "rest" while Ava is to have made the long journey and returned. Then it might at first glance appear from "relativity" that Ava should have aged only one year while Beta's space ship should have the future descendents of Beta emerging for the reunion. Logically it cannot be both. Which is correct?

The paradox is resolved using the k-calculus with some additional logical arguments. There is an essential asymmetry between Ava and Beta in this exercise in that while Ava follows an inertial spacetime trajectory, Beta undergoes a period of travel where she undergoes deceleration followed by acceleration. Acceleration and deceleration in her spaceship are translated into sensations that Beta experiences in her spaceship such as the variations in pressure against her seat. Such physical manifestations are not felt by Ava. Thus Beta's journey, unlike that of Ava, cannot be wholly one of following an inertial spacetime trajectory. This is best illustrated by bringing in our third observer Cayla (C) as shown in Figure 3.5.

At a velocity $v$, Beta leaves Ava at O where they synchronize their clocks. Immediately upon separation, Beta sends light back to Ava for a time $T$ by Beta's reckoning and Ava receives this emission for a time $kT$. At the time $T$ by Beta's clock, she meets Cayla who is

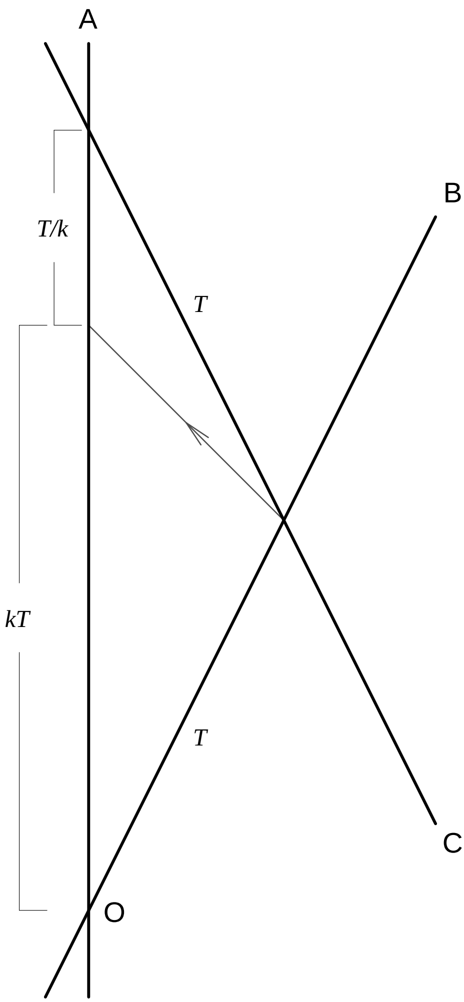


Figure 3.5: Comparison of times with three inertial observers.

traveling towards Ava with velocity $-v$. Beta and Cayla synchronize their clocks to the time $T$ when they cross at which point Cayla begins to beam light to Ava until Cayla meets Ava.

The mathematics is very simple. By symmetry, Cayla beams her light to Ava for the same period $T$ as Beta had beamed to Ava until she met Cayla. Thus when Cayla meets Ava, Cayla notes that her clock reads $T + T = 2T$ o'clock. We recall that when $v$ goes to $-v$, $k$ goes to $1/k$. Therefore the period of light reception by Ava of Cayla's transmission to her is $(1/k)T$. From Figure 3.5, we see that Ava has witnessed a period of time $kT + (1/k)T = (k + 1/k)T$ from the time she said farewell to Beta until she met Cayla. Since $k + 1/k$ is greater than 2 unless $k = 1$ (in which case there would not have been any relative motion), we see that Ava concludes that more time has elapsed for her than has elapsed for the combined journeys of Beta and Cayla between the three meetings.

Rather than introduce the third person Cayla, we could have considered Beta to have undergone a short deceleration period just before the point of Beta meeting Cayla followed by a short acceleration period. In this manner, her spacetime journey relative to Ava closely approximates the three-person plot of Figure 3.5. In this case, we have simulated the picture described earlier of Beta making a return trip and meeting Ava's great-great-grandchildren upon returning home. It is the deceleration/acceleration phase of Beta that is not present in the entirely inertial spacetime trajectory of Ava that makes Ava and Beta physically non-equivalent.

Some have argued that the periods of deceleration and acceleration will always compensate to remove the time difference and make Beta return to Ava at the same time. This is an untenable argument. Consider a sequence of journeys by Beta of *different* durations with the same velocities and with the identical reversals at the turnaround points as shown in Figure 3.6. Since the spacetime itself does not evolve in time, the physical effects that accrue at each one of the turnarounds must be identical because the turnarounds were identical. However, the return journeys of different durations require *different* amounts of compensation at the turnarounds to have the twins always unite with the same clock readings. Therefore the assumption that the non-inertial periods of travel will compensate and remove the time difference, is faulty.

It is also to be emphasized that while it is the period of acceleration by Beta that breaks the otherwise physically equivalent inertial observer symmetry of the journeys of Ava and Beta, it is the lengths in time of Beta's segments before and after the acceleration period that determine the extent to which their clock readings differ at the time of reunion.

Even more significantly, experiments with atomic clocks taken on return flights have displayed the effect. One might have objected that since Ava is always an inertial observer and Beta is an inertial observer for all but the very small spacetime trajectory segment near the turnaround point, it would seem that they should read essentially the same time upon reunion, i.e. that the spacetime segments are all straight lines apart from a very tiny segment. Bondi provides a very astute analogy as a counter to this argument. He considers journeys of Ava and Beta in the $x - y$ plane as shown in Figure 3.7, a plot in two *spatial* dimensions as in a conventional map. Ava's journey is a straight line and Beta's journey is almost a straight line apart from the kink that changes Beta's direction at the extreme point. It is that kink at R that breaks the symmetry and renders Beta's spatial distance covered longer than that covered by Ava. *The essential point is that in relativity, time is a route-dependent quantity just as distance is a route-dependent quantity.* It is also to be noted that it is the lengths of the segments before and after the kink that determine just how much longer Beta's journey will be than that of Ava. The existence of the kink makes the journeys non-equivalent but it does not determine how non-equivalent they are in distance covered. This extends the analogy with the twins regarding the degree of time difference between the clocks of the twins.

Another point to note is that while the greater *distance* covered is pictorially greater for Beta than for Ava in the space-space diagram of Figure 3.7 (the shortest distance between two points is a straight line), the shorter *time* for Beta is pictorially longer than that for Ava in the space-time diagram of Figure 3.6. This should come as no surprise. Time is not just another dimension like $x, y$ or $z$. Space and time are different concepts.

Space and time are unified in Einstein's special relativity but they are not equivalent. They have a reciprocal connection, identified from the outset by the differing signs in the spacetime interval. We have

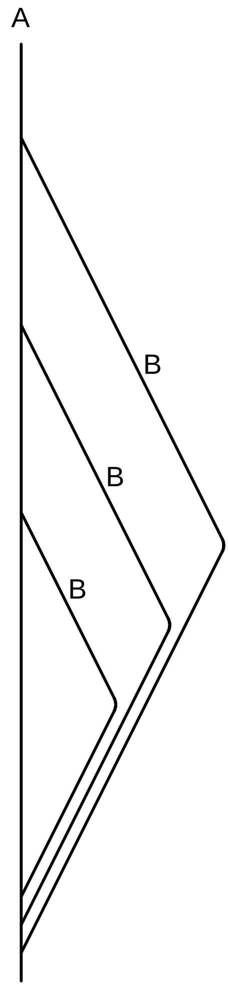


Figure 3.6: Comparison of times for two observers with journeys of different durations.

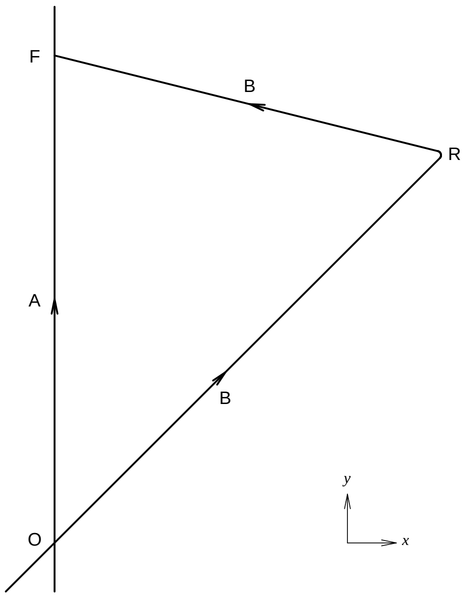


Figure 3.7: Comparison of distance covered for two observers over different routes.

outlined the essentials of special relativity, Einstein's theory of space and time in the absence of gravity. The incorporation of gravity into the relativistic framework is our primary focus in the chapters to follow.