Lecture 17

Example

$$
\begin{aligned}
-x + y + 2z + 3w &= -6 \\
-5x - 2y + 11w &= -27 \\
-2x + z + 5w &= -11
\end{aligned}
$$

Multiply eqn 1 by (-1).

Eqn(2) $\rightsquigarrow$ Eqn(2) + 5 Eqn(1)

Eqn(3) $\rightsquigarrow$ Eqn(3) + 2 Eqn(1).

$$\begin{array}{rrrrl} X & -\,Y & -\,2Z & -\,3W & = 6 \\ & -\,7Y & -\,10Z & -\,4W & = 3 \\ & -\,2Y & -\,3Z & -\,W & = 1 \end{array}$$

$$\text{Eqn }(2) \rightsquigarrow -\frac{1}{7}\,\text{Eqn }(2)$$

$$\text{Eqn }(3) \rightsquigarrow \text{Eqn }(3) + 2\,\text{Eqn }(2)$$

$$
\begin{aligned}
x \quad - y \quad - 2z \quad - 3w &= 6 \\
y \quad + \tfrac{10}{7} z \quad + \tfrac{4}{7} w &= -3/7 \\
\boxed{-\tfrac{1}{7} z + \tfrac{1}{7} w = \tfrac{1}{7}}
\end{aligned}
$$

Solve last eqn for z.

w can take any value, say $t \in \mathbb{R}$.

So $z = t - 1$

Substituting $W = t$ and $Z = t - 1$ in eqn 2, we get $Y = -2t + 1$.

Substituting in eqn 1, we get $X = 3t + 5$.

So, the solution set is

$$\{(3t+5, -2t+1, t-1, t) \mid t \in \mathbb{R}\}.$$

Notice that we do not need to focus on the variables, only on the coefficients and their positions. So, we could have written the system as

$$\left[\begin{array}{cccc|c} -1 & 1 & 2 & 3 & -6 \\ -5 & -2 & 0 & 11 & -27 \\ -2 & 0 & 1 & 5 & -11 \end{array}\right]$$

Matrices

Let m and n be integers. An $m \times n$ matrix is a collection of mn numbers arranged in a rectangular array with m rows and n columns.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The number in the i-th row and j-th column is called the (i,j)-entry of the matrix.

An augmented matrix is a matrix in which the last column is separated from the rest using a separator.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Row operations

Operations on equations in a system $\rightsquigarrow$ operations on rows in a matrix.

(1) Replace Row i with Row i + $a\times$ Row j

$(i \neq j)$

where $a \in \mathbb{R}$; written as $R_i + aR_j$

(2) Interchange rows i and j : $R_i \longleftrightarrow R_j$

(3) Multiply row i by a, $a \neq 0$:

written as aR_i

<u>Row reduced echelon matrix.</u>

A matrix is in row reduced echelon form if the following conditions hold:

(1) Leftmost non-zero entry in each row is a 1. This is called a <u>pivot</u>.

(2) If a column contains a pivot, all other entries in that column are 0.

(3) If $i < j$, and if both row i and row j have pivots, the pivot in row j is to the right of the pivot in row i.

(4) All zero rows (i.e. rows which contain only zeros) occur at the bottom of the matrix.

Examples

$$\begin{bmatrix} \boxed{1} & 4 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix} \qquad \begin{bmatrix} 0 & \boxed{1} & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 8 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Non-examples</u>

$$\begin{bmatrix} \boxed{1} & 5 & 2 & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & \boxed{1} & 0 & 4 \end{bmatrix}$$

Row reduction algorithm

Want to create a row reduced echelon matrix from given $\underline{m \times n}$ matrix using the three kinds of row operations listed earlier.

The algorithm has n steps. Step i checks if a pivot can be created in column i. If this is possible, we create the pivot and then reduce all other entries in the column to 0.

Step 1

- Start scanning column 1 from the top and try to find a non-zero entry.
- If there is no non-zero entry, go on to Step 2.

- If the first non-zero entry is in row i and $i > 1$, perform $R_i \longleftrightarrow R_j$
- Divide R_1 by the $(1,1)$-entry so that this entry becomes 1.
- Kill all other entries in column 1. If (j,i)-entry is a_{ji}, perform $R_j - a_{ji} R_1$ for all $j > 1$.

Step k for $k \geq 2$.

- Let i be the first row from the top which does not have a pivot.

  Start scanning column k from row i.
- Find first non-zero entry.

- If no such entry, go to Step $k+1$ (or if $k=n$, just STOP.)
- If there is a non-zero entry move the row where you found this entry to position i (if necessary).

Divide Row i by the (i,k)-entry.

- Kill all other entries in column k. So, if (j,k)-entry is a_{jk}, perform $R_j + (-a_{jk})R_i$ for all $j \neq i$.
- If $k < n$, go to Step $k+1$.
  If $k = n$, STOP.

Example

$$\begin{bmatrix} 0 & 1 & 4 & 2 \\ 3 & 2 & 1 & 0 \\ 4 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} \boxed{1} & 2/3 & 1/3 & 0 \\ 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 - 4R_1} \begin{bmatrix} \boxed{1} & 2/3 & 1/3 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & -5/3 & -4/3 & 3 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} \boxed{1} & 2/3 & 1/3 & 0 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & -5/3 & -4/3 & 3 \end{bmatrix} \xrightarrow{R_1 - 2/3 R_2} \begin{bmatrix} \boxed{1} & 0 & -7/3 & -4/3 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & -5/3 & -4/3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 + \frac{5}{3} R_2} \begin{bmatrix} \boxed{1} & 0 & -7/3 & -4/3 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 16/3 & 19/3 \end{bmatrix}$$

Step 3

$$\begin{bmatrix} \boxed{1} & 0 & -7/3 & -4/3 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 16/3 & 19/3 \end{bmatrix} \xrightarrow{\frac{3}{16} R_3} \begin{bmatrix} \boxed{1} & 0 & -7/3 & -4/3 \\ 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & \boxed{1} & 19/16 \end{bmatrix}$$

STOP

Example

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & \boxed{1} & 2 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3-3R_1} \begin{bmatrix} 0 & \boxed{1} & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & -5 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow{R_3 + 5R_2} \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Solving systems of linear equations

We want to solve the system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots &+ a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots &+ a_{2n}x_n = b_2 \\
\vdots \qquad\qquad \ddots \qquad\qquad & \qquad\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots &+ a_{mn}x_n = b_m
\end{aligned}
$$

We represent it by the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Perform row reduction on the left block. However, simultaneously perform the same operations on the right block as well. After getting a row reduced echelon matrix in the left block, check which columns have pivots.

Each column corresponds to a variable.

Suppose columns $i_1, i_2, \cdots i_r$ do not have pivots.

Then $x_{i_1}, x_{i_2}, \cdots x_{i_r}$ are the "free variables" — they can take any value.

Set $x_{i_1} = t_1$, $x_{i_2} = t_2$, $\cdots$ $x_{i_r} = t_r$ where $t_1, t_2, \cdots t_r$ denote <u>arbitrary</u> elements of $\mathbb{R}$.

Solve for the remaining variables using the row reduced system of equations.

Example

Consider the system

$$3x_1 - 2x_2 + 4x_3 + 7x_4 = 11$$
$$x_1 + 5x_2 - x_3 + 6x_4 = 4$$
$$-x_1 + 3x_2 + 3x_3 + 2x_4 = -1$$

Write the corresponding matrix.

$$\left[\begin{array}{cccc|c} 3 & -2 & 4 & 7 & 11 \\ 1 & 5 & -1 & 6 & 4 \\ -1 & 3 & 3 & 2 & -1 \end{array}\right]$$

$\Downarrow$ (see pg 24 of 2019 notes)

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 11/5 & 16/5 \\ 0 & \boxed{1} & 0 & 13/15 & 23/90 \\ 0 & 0 & \boxed{1} & 8/15 & 43/90 \end{array}\right]$$

Column 4 has no pivots.

So set $x_4 = t$ and solve

using the reduced system:

$$
\begin{aligned}
x_1 + 0 + 0 + \tfrac{11}{5}x_4 &= 16/5 \\
0 + x_2 + 0 + \tfrac{13}{15}x_4 &= 23/90 \\
0 + 0 + x_3 + \tfrac{8}{15}x_4 &= 43/90
\end{aligned}
$$

This gives

$$x_1 = \frac{16}{5} - \frac{11}{5}t$$

$$x_2 = \frac{23}{90} - \frac{13}{15}t$$

$$x_3 = \frac{43}{90} - \frac{8}{15}t$$

So, the solution set is

$$\left\{\left(\frac{16}{5} - \frac{11t}{5},\ \frac{23}{90} - \frac{13t}{15},\ \frac{43}{90} - \frac{8t}{15},\ t\right) \middle|\ t \in \mathbb{R}\right\}$$