Tutorial 1 solutions

1) ℓ – line in the plane

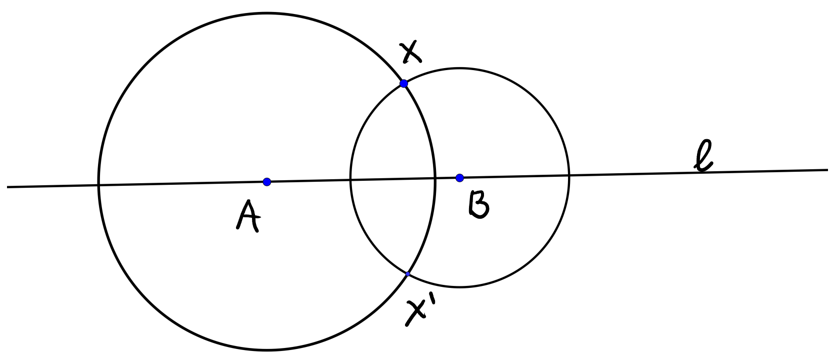
(a) Find the group of isometries σ of the plane such that $\sigma(P) = P$ for all P in ℓ.

(b) Find the group of isometries σ of the plane such that $\sigma(P)$ is in ℓ for all P in ℓ.

Solution : (a) Let G be the required group and let σ be in G. Then, for a point X in the plane, what can $\sigma(x)$ be?

If X is on ℓ, we know that $\sigma(x) = X$.

But what if X is not on ℓ?



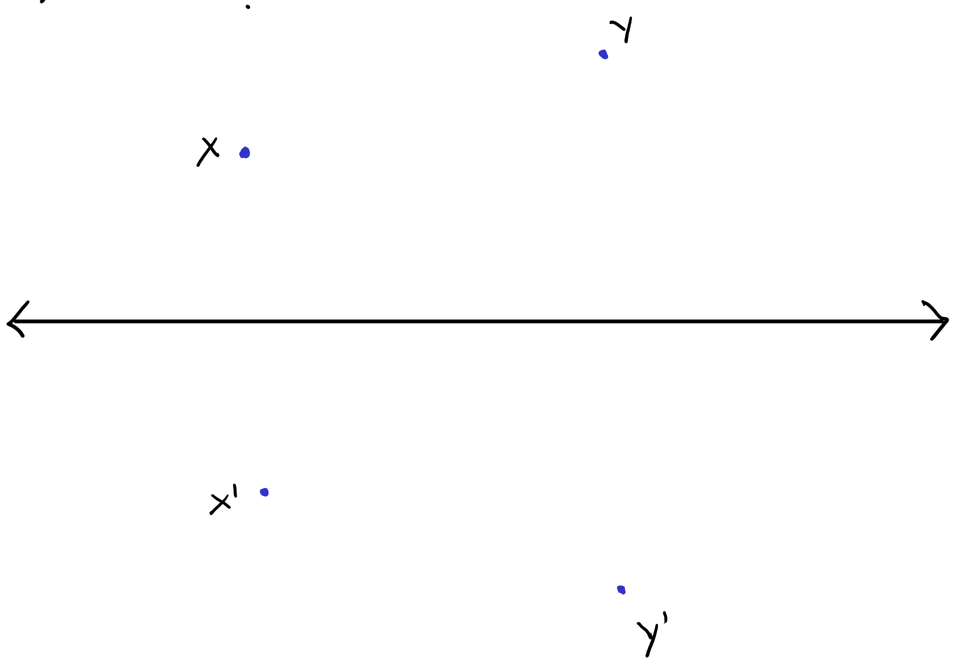
Let A, B be two distinct points on ℓ

$\text{dist}(A, x) = \text{dist}(\sigma(A), \sigma(x)) = \text{dist}(A, \sigma(x))$.

So, $\sigma(X)$ must lie on a circle with centre A and radius equal to $\text{dist}(A, X)$

Similarly, $\sigma(x)$ must lie on the circle with centre B and radius $\text{dist}(BX)$. So, $\sigma(x)$ can only be X or X'.

So, σ maps every point to itself, or to its reflection in ℓ.
Let X be a point that is not on ℓ.
Suppose σ maps the point X to its reflection X'.
Then, for a point Y, is it possible that $\sigma(Y) = Y$?



Suppose this is so.
dist $(X Y) =$ dist $(\sigma(X), \sigma(Y)) =$ dist (X', Y)
But, we also have dist $(X, Y) =$ dist (X', Y')

So, X' is equidistant from Y and Y'.
If Y is not on ℓ, $Y \neq Y'$ and the perpendicular bisector of YY' is ℓ.
So X' is on ℓ — contra.

So, it cannot happen that $\sigma(Y) = Y$ unless Y is on ℓ. So $\sigma(Y)$ is the reflection of Y in ℓ for every point Y.

Let τ denote the reflection in ℓ.

Then $G = \{1, \tau\}$ and the multiplication table is

	1	τ
1	1	τ
τ	τ	1

(b) Let G denote the required group.
We can readily list some elements of G.
- reflection in ℓ, denoted by τ_ℓ
- reflection in any line m which is perpendicular to ℓ. We denote this by τ_m.
- translation along ℓ through distance d. We fix a direction on ℓ and any translation in that direction will be considered positive. Thus, for every real number d, we have a translation σ_d.
- rotation around a point P of ℓ through π radians. We call this ρ_P.

We want to know if there are any other isometries. Also, we want to know the composition rules.

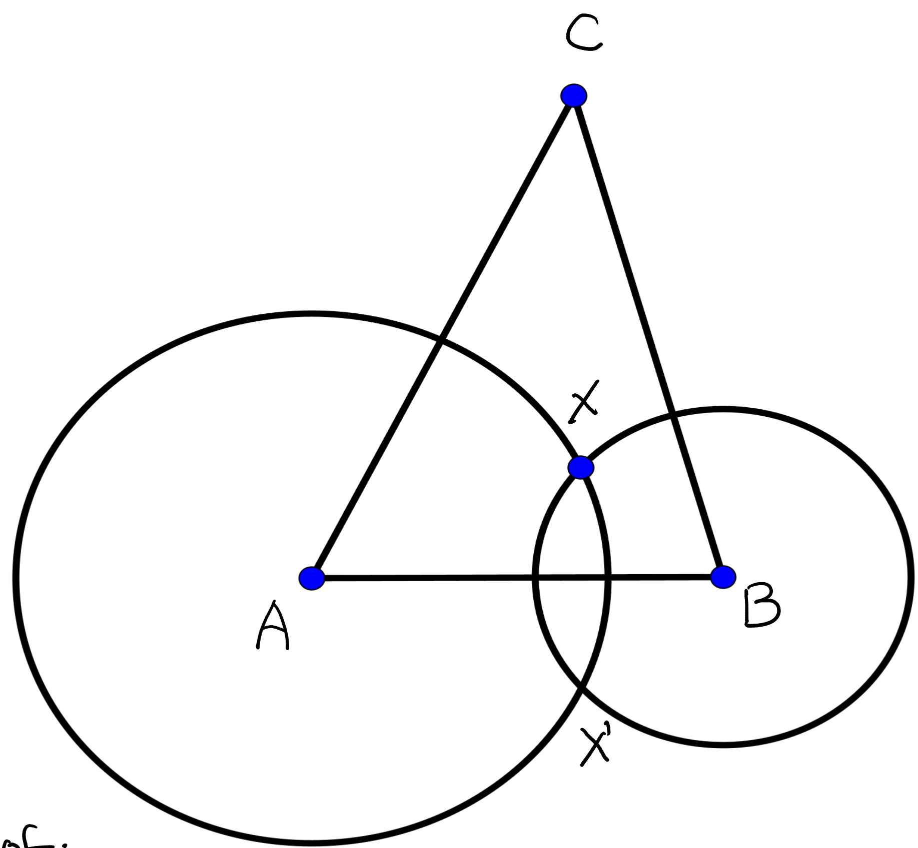
First, we observe a general principle:

(*) $\boxed{\text{Any isometry of the plane is completely determined by the images of three non-collinear points.}}$

This means, if σ and τ are two isometries and A, B, C are <u>non-collinear</u> points such that $\sigma(A) = \tau(A)$, $\sigma(B) = \tau(B)$ and $\sigma(C) = \tau(C)$, then $\sigma(P) = \tau(P)$ for all points P.

What is the proof of this claim?
This follows from another basic observation:

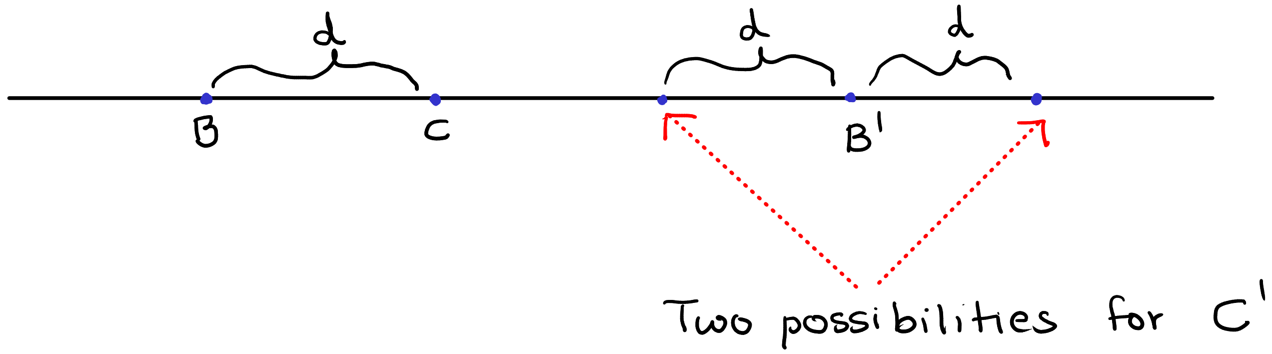
(∗∗) Let A, B, C be three points in the plane. Suppose X and Y are points such that $dist(AX) = dist(AY)$, $dist(BX) = dist(BY)$ and $dist(CX) = dist(CY)$. Then $X = Y$.



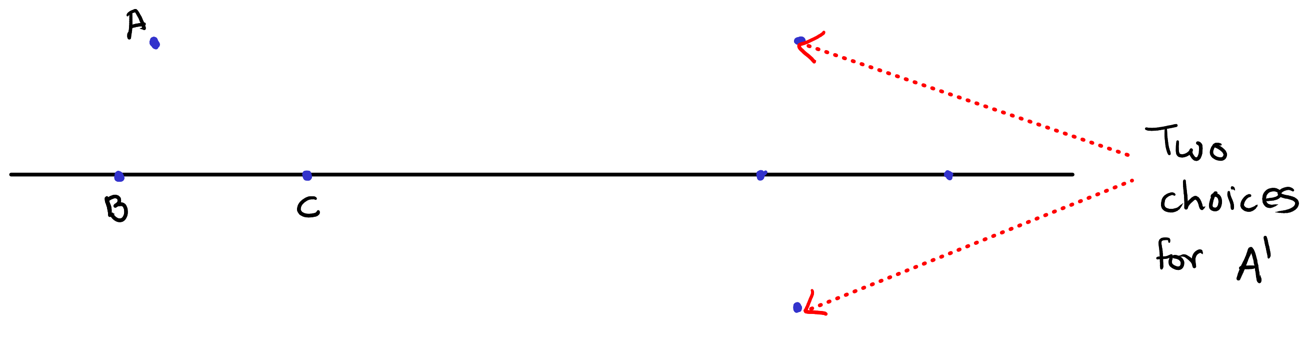
<u>Proof:</u>
$dist(AX) = dist(AY)$ and $dist(B,X) = dist(B,Y)$ forces $X = Y$ <u>OR</u> $X = X'$. As C does not lie on AB, $dist(C,X) \neq dist(C,X')$. So $Y \neq X' \Rightarrow Y = X$ //

Returning to the problem:
To identify an isometry σ in G, we need to understand what it does to any three non-collinear points of our choice.

So, we choose three points A, B, C such that B and C lie on ℓ, but A does not. We want to choose A', B' and C' so that $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Suppose B' is chosen. Then, we have two possibilities for C'.



Once C' is fixed to be one of these two choices, then A' can be chosen in 2 ways. For example:

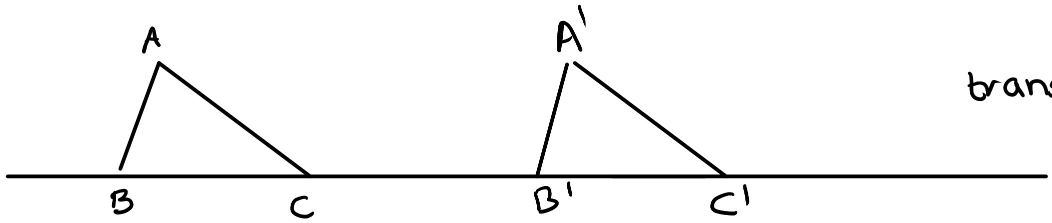


So, once B' is chosen, we have 2 choices for C'. After C' is chosen, we have two choices for A'.

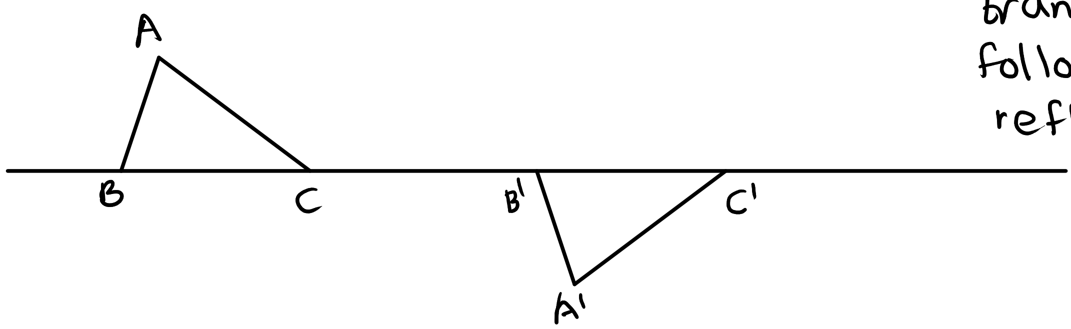
Question Once we choose three points A', B', C' so that the triangles ABC and $A'B'C'$ are congruent, does there always exist some isometry σ such that $\sigma(A) = A'$, $\sigma(B) = B'$ and $\sigma(C) = C'$.

Answer Yes. (Exercise in basic geometry.)

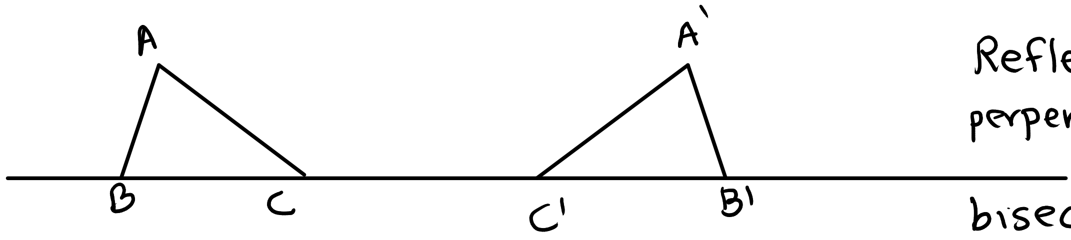
We will not prove this here, but in our problem, the isometries are easy to guess



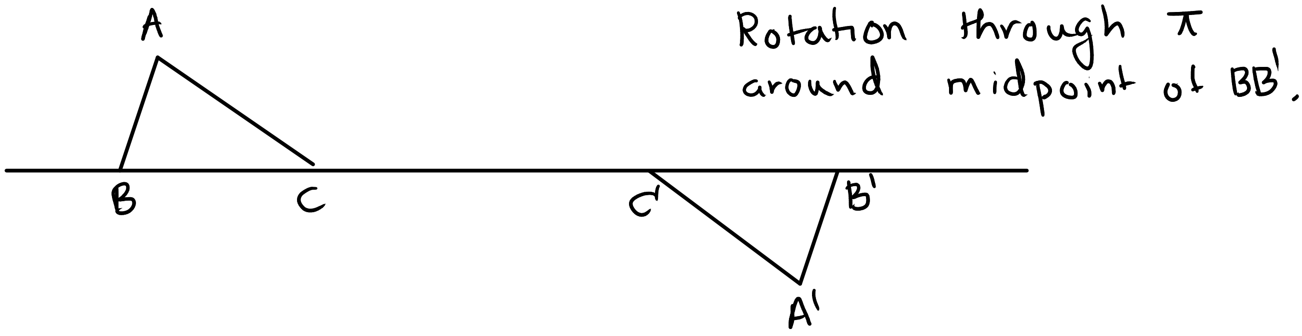
translation.



translation followed by reflection in ℓ



Reflection in perpendicular bisector of BB'



Thus a complete list of isometries is as follows:

(1) translations along ℓ. (σ_d, d in $\mathbb{R}$)

(2) translation along ℓ followed by reflection in ℓ ($\tau_\ell \circ \sigma_d$) (If $d=0$, this is just τ_ℓ)

(3) Reflection in any line perpendicular to ℓ (τ_v, v perpendicular to ℓ)

(4) Rotation through π around a point on ℓ. (ρ_P — P is any point on ℓ)

To be able to do computations, you need to work out various compositions. This can be done by explicitly computing various compositions.

Here are some relations. (Ex. Check this.)

(1) $\sigma_{d_1} \circ \sigma_{d_2} = \sigma_{d_2} \circ \sigma_{d_1} = \sigma_{d_1+d_2}$ for any real numbers d_1, d_2

(2) $\sigma_d \circ \tau_\ell = \tau_\ell \circ \sigma_d$ for any real number d.

(3) Let v_1 and v_2 be two lines that are perpendicular to ℓ.

Then if P_1 is the point of intersection of v_1 and ℓ, P_2 is the point of intersection of v_2 and ℓ, then

$$\tau_{v_2} \circ \tau_{v_1} = \sigma_{2d}$$

where d = Signed distance from P_1 to P_2 (so, this is dist(P_1, P_2) with a positive or negative sign depending on whether the direction $\overrightarrow{P_1 P_2}$ is positive or negative.)

(4) $\sigma_d \circ \tau_v = \tau_w$ where w is the line $\tau_{d/2}(v)$.

(5) If v is any line perpendicular to ℓ, then $\tau_v \circ \tau_\ell = \tau_\ell \circ \tau_v = \rho_Q$

where Q = point of intersection of v and ℓ.

Using these five relations, any composition can be calculated easily.

2) We have seen that the composition of two rotations around a point O is again a rotation. Suppose σ_1 is a rotation around a point P_1 through θ_1 radians and σ_2 is a rotation around a point P_2 through θ_2 radians. What can you say about the isometry $\sigma_2 \circ \sigma_1$?

Solution 1 :

Recall : Rotations and translations preserve orientation. Reflections reverse orientation.

Composition of two rotations preserves orientation. So, we guess that the composition $\sigma_2 \circ \sigma_1$ may be a rotation or a translation.

(Fact : It can be proved that any orientation preserving isometry is a rotation or a translation. We will not prove this here.)

Given any line ℓ, if ℓ' is its image under rotation through θ radians, the angle between ℓ and ℓ' is θ radians. (We think of the angle between parallel lines as being 0 or π radians.)

On the other hand, if ℓ' is the image of ℓ under a translation, then the angle between ℓ and ℓ' is zero.

So, if $\theta_1 + \theta_2 \neq \pi$ or 2π, then $\sigma_2 \circ \sigma_1$ cannot be a translation.

So, for now, let us assume that $\theta_1 + \theta_2 \neq \pi$ or 2π and explore our guess that $\sigma_2 \circ \sigma_1$ might be a rotation.

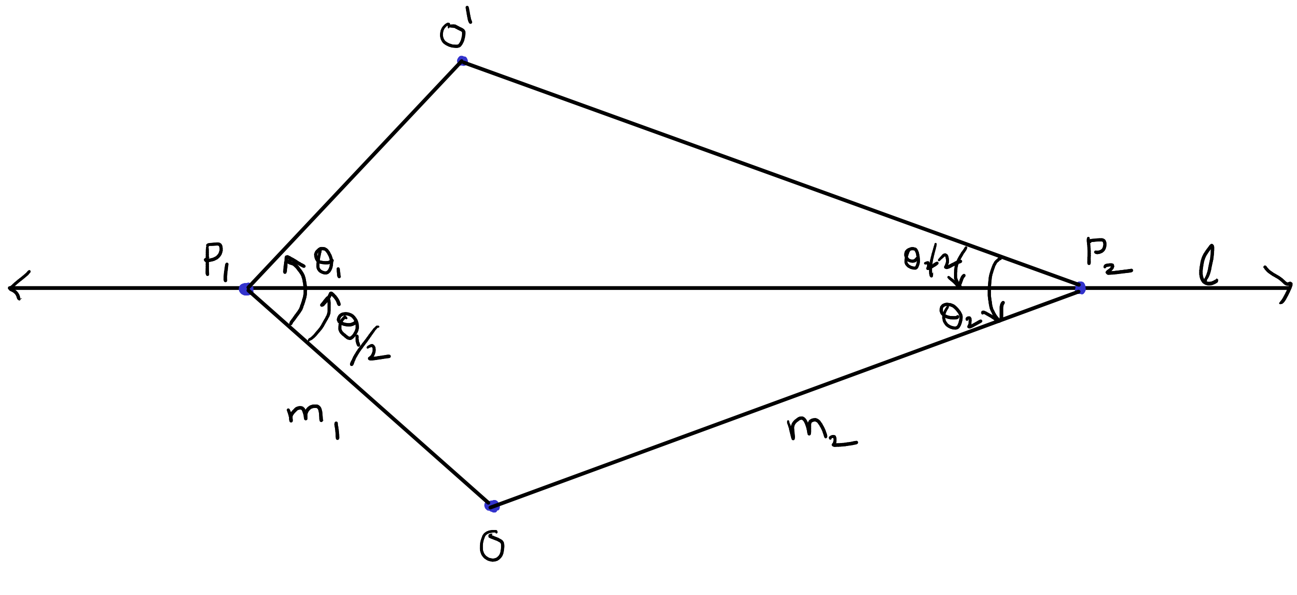
The centre of a rotation is invariant under the rotation. (In other words, it is not moved by the rotation.)

So, we will try to find if there exists a point O such that $\sigma_2 \circ \sigma_1 (O) = O$.

If such a point O exists, let us denote $\sigma_1(O)$ by O'.

Then $\sigma_1(O) = O'$ and $\sigma_2(O') = O$.
So $\text{dist}(O, P_1) = \text{dist}(O', P_1)$ and $\text{dist}(O, P_2) = \text{dist}(O', P_2)$

So P_1 and P_2 lie on the perpendicular bisector of OO'. So O' is the reflection of O in P_1P_2.



Construct a point O such that $\angle OP_1P_2 = \theta_1/2$ and $\angle P_1P_2O = \theta_2/2$.

(Note that angles are measured anticlockwise. So, $\angle P_2P_1O = -\angle OP_1P_2$, etc.)

The point O can be constructed if and only if $\theta_1 \neq -\theta_2$. If $\theta_1 = -\theta_2$, the two lines being constructed become parallel. So, for now we assume only that $\theta_1 \neq -\theta_2$.

If $\theta_1 \neq -\theta_2$, we construct the line m_1 by rotating the line $l = P_1P_2$ around P_1 through $-\theta_1/2$.

We construct m_2 by rotating $l = P_1P_2$ around P_2 through $\theta_2/2$.

O is then obtained as the intersection of m_1 and m_2.

We claim that $\sigma_2 \circ \sigma_1$ is the rotation around O through $\theta_1 + \theta_2$.

First observe that for any point $X \neq O$,

$$\begin{aligned} \text{dist}(O, X) &= \text{dist}(\sigma_1(O), \sigma_1(X)) \\ &= \text{dist}(O', \sigma_1(X)) \\ &= \text{dist}(\sigma_2(O'), \sigma_2 \circ \sigma_1(X)) \\ &= \text{dist}(O, \sigma_2 \circ \sigma_1(X)) \end{aligned}$$

So, if $X' = \sigma_2 \circ \sigma_1(X)$, then O is equidistant from X and X'.

It remains to be seen that $\angle XOX' = \theta_1 + \theta_2$.

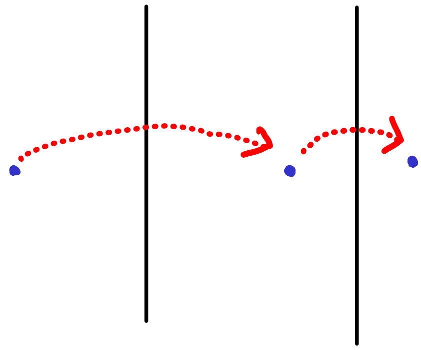
This can be established by "angle chasing". (In other words, draw the picture and calculate angles.) — Omitted.

Once this is established, we see that $\sigma_2 \circ \sigma_1$ is a rotation around O through $\theta_1 + \theta_2$.

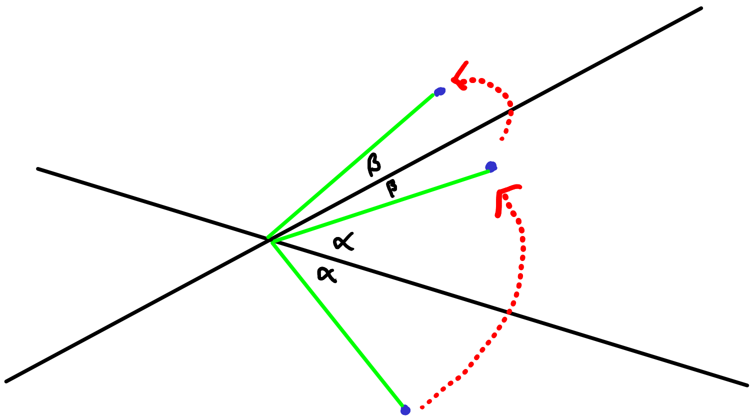
If $\theta_1 = -\theta_2$, the point O does not exist as the lines m_1 and m_2 are parallel. In this case, it is easy to check that $\sigma_2 \circ \sigma_1$ is a translation. (Also see solution 2 below.)

Solution 2: First, we will note a basic fact about composition of reflections.

Let l_1 and l_2 be two lines in the plane and let τ_{l_1} and τ_{l_2} be reflections in these lines.
Then, what is $\tau_{l_2} \circ \tau_{l_1}$?



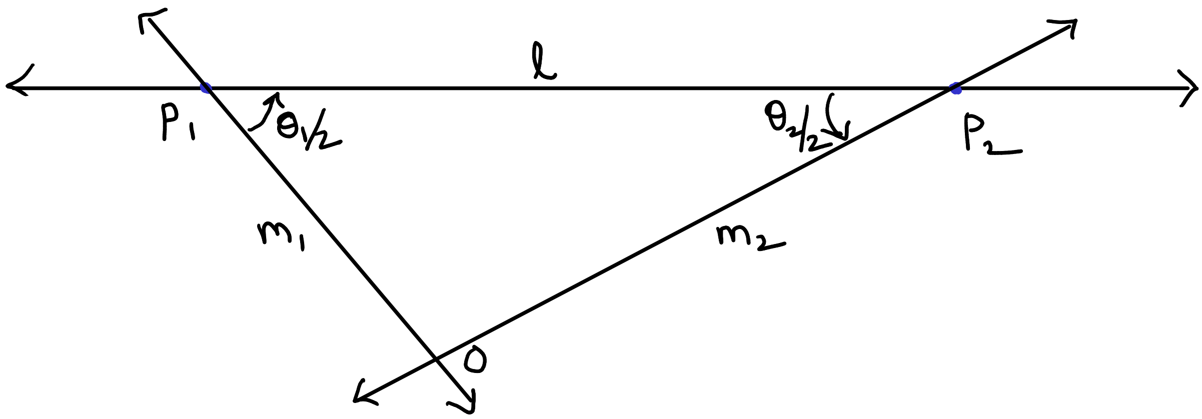
If l_1 and l_2 are parallel, then one can easily check that $\tau_{l_2} \cdot \tau_{l_1}$ is a translation in the direction perpendicular to l_1 and l_2 through a distance of $2d$ where d = distance between l_1 and l_2



If l_1 and l_2 intersect in a point O, then $\tau_{l_2} \circ \tau_{l_1}$ is a rotation around O through 2θ where θ is the angle through which l_1 must be rotated to coincided with l_2 (measured in the anticlockwise sense).

Let us now use this in our problem.
The idea is to write both σ_1 and σ_2 as products of reflections in appropriately chosen lines.

Let us go back to the earlier picture.



Let m_1 be obtained by rotating l around P_1 through $-\theta_1/2$.

Let m_2 be obtained by rotating l around P_2 through $\theta_2/2$.

Then $\sigma_1 = \tau_l \circ \tau_{m_1}$, $\sigma_2 = \tau_{m_2} \circ \tau_l$

So, $\sigma_2 \circ \sigma_1 = \tau_{m_2} \circ \tau_l \circ \tau_l \circ \tau_{m_1}$

$$= \tau_{m_2} \circ \tau_{m_1}$$

Suppose m_1, m_2 intersect in a point O. In that case, $T_{m_2} \circ T_{m_1}$ is the rotation around O through

If m_1 and m_2 are parallel, (i.e. $\theta_1 = -\theta_2$) then $T_{m_2} \circ T_{m_1}$ is a translation.

3) Describe the group of isometries of a line.

<u>Solution</u>:

Observe that any isometry of a line is completely determined by the images of two points.
(Proof omitted. Compare with the analogous result for the plane in the solution of 1(b).)

Let σ be any isometry of the line.
Let P, Q be two points. Let $P' = \sigma(P)$, $Q' = \sigma(Q)$.

Once P' is chosen, we have exactly two choices for Q' since $\text{dist}(P', Q') = \text{dist}(P, Q)$.

<u>Case 1</u>

This is a translation.

Case 2

This is a reflection in a line perpendicular to ℓ.

So, we have two families of isometries:

(1) σ_d = translation through d units
(fix a direction on ℓ as "positive".
Translation by a negative distance means translation in the opposite direction.)
d — any real number

(2) τ_v — reflection in any line v perpendicular to ℓ.

Fix a line v_0. Let v be any other line. If distance from v_0 to v is d (signed distance!), then we can check that

$$\boxed{\tau_v = \sigma_d \circ \tau \circ \sigma_{-d}.}$$

Also, we can check that for any real number x, $\boxed{\tau \sigma_x = \sigma_{-x} \tau.}$

So, $\boxed{\tau_v = \sigma_d \,\tau\, \sigma_{-d} = \sigma_{2d}\,\tau.}$

Thus, our list can simply be written as:

(1) Translations: σ_x for x in $\mathbb{R}$.

(2) Reflections: $\sigma_x \tau$ for all x in $\mathbb{R}$.

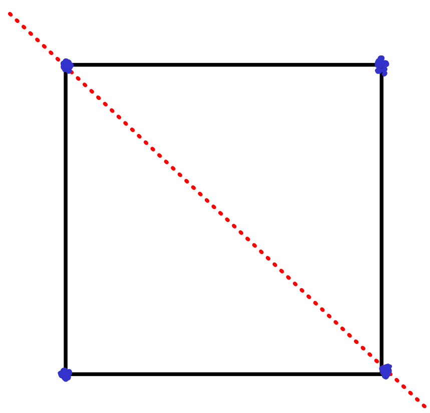
(Note the τ is a reflection in the fixed line v_0.)

Composition can be easily calculated using the relation $\tau\sigma_x = \sigma_{-x}\tau$.

(This should remind you of the dihedral group.)

4) σ – rotation through $\pi/2$

τ – reflection in the red diagonal



All symmetries: $\{1, \rho, \rho^2, \rho^3, \tau, \rho\tau, \rho^2\tau, \rho^3\tau\}$
We know that $\rho^4 = 1$, $\tau^2 = 1$, $\tau\rho = \rho^{-1}\tau$.

	1	ρ	ρ^2	ρ^3	τ	$\rho\tau$	$\rho^2\tau$	$\rho^3\tau$
1	1	ρ	ρ^2	ρ^3	τ	$\rho\tau$	$\rho^2\tau$	$\rho^3\tau$
ρ	ρ	ρ^2	ρ^3	1	$\rho^3\tau$	τ	$\rho\tau$	$\rho^2\tau$
ρ^2	ρ^2	ρ^3	1	ρ	$\rho^2\tau$	$\rho^3\tau$	τ	$\rho\tau$
ρ^3	ρ^3	1	ρ	ρ^2	$\rho\tau$	$\rho^2\tau$	$\rho^3\tau$	τ
τ	τ	$\rho\tau$	$\rho^2\tau$	$\rho^3\tau$	1	ρ	ρ^2	ρ^3
$\rho\tau$	$\rho\tau$	$\rho^2\tau$	$\rho^3\tau$	τ	ρ^3	1	ρ	ρ^2
$\rho^2\tau$	$\rho^2\tau$	$\rho^3\tau$	τ	$\rho\tau$	ρ^2	ρ^3	1	ρ
$\rho^3\tau$	$\rho^3\tau$	τ	$\rho\tau$	$\rho^2\tau$	ρ	ρ^2	ρ^3	1