MTH101: Symmetry

Mid-semester examination: 23/12/2022

Time: 9:30 a.m. to 11:30 a.m.

Total points: 25

Note: Calculators or electronic devices of any kind are not permitted.

Problem 1. (1 point each) Write the statements of the following theorems:

(a) Lagrange's theorem

(b) First Isomorphism Theorem of group theory.

Problem 2. (1 point each) Define the following terms:

(a) Group homomorphism

(a) Normal subgroup

Problem 3. (2 points each)

(a) What is the order of $\langle \overline{6} \rangle$ in $\mathbb{Z}/111\mathbb{Z}$?

(b) What is the order of the group $U(36)$?

(c) Use the Euclidean algorithm to find the greatest common divisor of 10532 and 2324.

Problem 4. (3 points) In the group S_8, consider the elements $\sigma = (3, 2, 4)(5, 1, 8, 7)$ and $\tau = (2, 1, 7)(5, 4, 63)$. Compute the order of the element $\tau\sigma\tau^{-1}$.

Problem 5. (3 points) Let $\mathbb{Q}$ denote the group of rational numbers under addition and let $\mathbb{Z}$ denote the group of integers under addition. Let G denote the quotient group $\mathbb{Q}/\mathbb{Z}$. Prove that every element of G has finite order.

Problem 6. (3 points) Let G be an abelian group. Let S be the set of all elements of G having finite order. Prove that S is a subgroup of G.

Problem 7. (3 points) Let $\phi : S_6 \to S_6$ be the function defined by $\phi(x) = x^2$. Is ϕ a group homomorphism? Justify your answer. (Do not just answer "yes" or "no". You must prove your claim.)

Problem 8. (3 points) Prove that S_{10} does not have any element of order 35. (Hint: If such an element existed, what would its cycle decomposition look like?)
