Lecture 23

V - vector space, $S \subset V$.

Then, any element of V is a linear combination of finitely many elements of S.

We can write a general element of S as $\sum_{w \in S} a_w \cdot w$ where it is to be understood that only finitely many of the a_w are non-zero.

When does the span grow?

Let V be a vector space and let $S \subset V$.

Let $v \in V$. We know that

$$\mathrm{Span}(S) \subseteq \mathrm{Span}(S \cup \{v\})$$

When is $\mathrm{Span}(S \cup \{v\})$ strictly bigger than $\mathrm{Span}(S)$?

Clearly, $v \in \mathrm{Span}(S \cup \{v\})$. So, if $v \notin \mathrm{Span}(S)$, then $\mathrm{Span}(S \cup \{v\}) \neq \mathrm{Span}(S)$.

Conversely, suppose $\mathrm{Span}(S \cup \{v\})$ is strictly bigger than $\mathrm{Span}(S)$.

Then we claim that $v \notin \mathrm{Span}(S)$.

Indeed, suppose $v \in \mathrm{Span}(S)$.

So, $v = \sum_{w \in S} a_w \cdot w$, $\quad a_w \in \mathbb{R} \quad \forall\, w$.

Any element $x \in \mathrm{Span}(S \cup \{v\})$ is of the form $x = b_v \cdot v + \sum_{w \in S} b_w \cdot w$.

So $x = \sum_{w \in S} (b_v a_w + b_w) \cdot w$.

Thus $x \in \mathrm{Span}(S)$.

Thus $\mathrm{Span}(S \cup \{v\}) \subset \mathrm{Span}(S)$

$\longleftarrow$ contra.

So $v \notin \mathrm{Span}\, S$.

So, we have proved:

Theorem $\mathrm{Span}(S \cup \{v\}) \neq \mathrm{Span}(S)$

$$\iff v \notin \mathrm{Span}(S)$$

Linear independence

Let $S \subset V$.

Suppose $v \in S$ is such that $v \in \operatorname{Span}(S \setminus \{v\})$.

Then $\operatorname{Span}(S \setminus \{v\}) = \operatorname{Span}(S)$.

Thus, v can be removed from S without altering the span.

On the other hand, if $v \notin \operatorname{Span}(S \setminus \{v\})$, then removing v shrinks the span.

<u>Defn</u>: A set $S \subset V$ is said to be <u>linearly independent</u> if for any $v \in S$, we have $v \notin \mathrm{Span}(S \setminus \{v\})$.

This means that no element can be written as a linear combination of other elements.
No element can be thrown out without altering the span.

Characterization of linear independence

Let V be a vector space.
Let $S \subset V$. Then, the following three conditions are equivalent.

(1) S is linearly independent.

(2) Any element of $\mathrm{Span}(S)$ can be <u>uniquely</u> written in the form

$$\sum_{v \in S} a_v \cdot v$$

(3) If $\sum_{v \in S} a_v \cdot v = \bar{0}$, then we must have $a_v = 0 \ \forall \ v$.

(In other words, there is no collection of real numbers $\{a_v\}_{v \in S}$ such that at least one of them is non-zero and $\sum a_v \cdot v = 0$.)

Proof of (1) $\Rightarrow$ (3):

We assume S is linearly independent.

Suppose $\sum_{v \in S} a_v \cdot v = 0$ for some choice of real numbers a_v.

We want to prove that $a_v = 0$ for all v.

Suppose, for some $v_0 \in S$, $a_{v_0} \neq 0$.

Then $a_{v_0} \cdot v_0 + \sum_{v \neq v_0} a_v \cdot v = 0$.

So, $v_0 = \sum_{v \neq v_0} \left(\frac{-a_v}{a_{v_0}} \right) \cdot v$.

Thus $v_0 \in \text{Span}(S \setminus \{v_0\})$

— contra.

Thus $a_v = 0 \quad \forall \; v \in S$.

Thus (3) is true.

This proves (1) $\Longrightarrow$ (3).

Proof of (3) $\Longrightarrow$ (1):

We assume (3) is true.

Suppose S is not linearly independent.

Thus, for some $v \in S$, we have $v \in \operatorname{Span}(S \setminus \{v\})$.

So $v = \sum_{w \neq v} a_w \cdot w$.

So, $-v + \sum_{w \neq v} a_w \cdot w = \overline{0}$.

In this expression, the coefficient of v is -1, which is non-zero.

This contradicts our assumption that (3) is true.

Thus S must be linearly independent.

Thus (3) $\Longrightarrow$ (1).

Proof of (3) $\Longrightarrow$ (2):

Assume (3) is true.

If (2) is not true, some $v \in \text{Span}(S)$ has two <u>distinct</u> expressions

$$v = \sum_{w \in S} a_w \cdot w \quad \text{and} \quad v = \sum_{w \in S} b_w \cdot w.$$

So $\overline{0} = \sum_{w \in S} (a_w - b_w) \cdot w.$

As the two expressions are distinct, $\exists$ some $w_0 \in S$ such that $a_{w_0} \neq b_{w_0}$.

So $a_{w_0} - b_{w_0} \neq 0$.

As $\sum_{w \in S} (a_w - b_w) \cdot w = 0$, this shows that (3) does not hold. — contra.

Thus (2) must be true.

Thus (3) $\Longrightarrow$ (2).

Proof of (2) $\Longrightarrow$ (3).

Assume (2). Suppose (3) is not true.

So there exists an equality

$\sum_{w \in S} a_w \cdot w = \overline{0}$ where not all a_w are equal to 0.

But, we also have $\sum_{w \in S} 0 \cdot w = \overline{0}$.

This violates (2) (for $v = \overline{0}$) — contra.

So, we have proved

$$(1) \iff (3) \iff (2).$$

Thus the conditions are equivalent.

An expression of the form $\sum_{v \in S} a_v \cdot v$ is called a <u>linear relation</u> between the elements of S.

If all $a_r = 0$, then we say that it is the trivial relation. If not, we call it a <u>non-trivial linear relation</u> between the elements of S.

Thus a set is linearly independent if and only if there is no non-trivial linear relation between its elements.

If a set is not linearly independent, we say that it is <u>linearly dependent</u>.

<u>Examples</u>:

(1) In $\mathbb{R}^n$, $\{e_1, \dots, e_n\}$ is a linearly independent set.

(2) $\{\bar{0}\}$ is linearly dependent.

(3) ϕ is linearly independent.

Finite dimensional spaces

Recall that a space is finite dimensional if it has a finite spanning set.

Suppose V is a finite dimensional vector space and $S = \{v_1, \dots, v_n\}$ is a spanning set.

If S is not linearly independent, there is some v_i such that

$v_i \in \text{Span}(S \setminus \{v_i\})$.

So, $\text{Span}(S \setminus \{v_i\}) = \text{Span}(S) = V$.

Thus $S \setminus \{v_i\}$ is a <u>smaller</u> spanning set.

Again, if $S \setminus \{v_i\}$ is not linearly independent, we can remove another element to create an even smaller spanning set.

This can continue at most n times.

So, at some point, we must end up with a spanning set which is linearly independent.

Defn: V - a vector space.

A **basis** of V is a spanning set which is linearly independent.

We have proved that every finite dimensional vector space has a basis. In fact, any finite spanning set of a finite dimensional vector space has a subset which is a basis.

Fact:

For any vector space V, any spanning set of V contains a basis.

In particular, (taking the spanning set to be V itself) we see that any vector space has a basis.

(We will not prove this.)

Let V be a finite dimensional vector space.

Suppose $\mathcal{B} = \{v_1, v_2, \cdots v_n\}$ is a basis. Then, any $v \in V$ can be <u>uniquely</u> written in the form $v = a_1 v_1 + a_2 v_2 + \cdots a_n v_n$.

We fix the order $v_1, \cdots, v_n$ on the basis $\mathcal{B}$.

We define a function $\varphi_B : V \longrightarrow \mathbb{R}^n$ by $\varphi_B(v) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

if $v = a_1 v_1 + \cdots + a_n v_n$.

As any such expression for v is unique, φ_B is well-defined.

It is easy to see that φ_B is a <u>linear transformation</u>.

φ_B is a 1-1 function:

If $\varphi_B(v) = \varphi_B(w) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then

$$v = a_1 v_1 + a_2 v_2 + \cdots a_n v_n = w.$$

φ_B is an onto function

Let $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$. Let $v = a_1 v_1 + \cdots + a_n v_n$

Then, $\varphi_B(v) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Thus, an <u>ordered basis</u> (i.e. a basis with a given order) $\mathcal{B} = (v_1, \cdots, v_n)$ defines a linear transformation $\varphi_{\mathcal{B}} : V \to \mathbb{R}^n$.

$\varphi_{\mathcal{B}}$ is 1-1 and onto.

$\varphi_{\mathcal{B}}^{-1} : \mathbb{R}^n \to V$ is given by

$$\varphi_{\mathcal{B}}^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 v_1 + \cdots + a_n v_n.$$

<u>Definition</u> A linear transformation $T: V \to W$ is said to be an <u>isomorphism of vector spaces</u> or a <u>linear isomorphism</u> if T is a 1-1 correspondence.

<u>Exercise</u>: If $T: V \to W$ is a linear isomorphism, prove that T^{-1} is a linear isomorphism.

<u>Exercise</u> Let $T: V \to W$ be a linear isomorphism.

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a subset of V.

(1) Prove that $\mathcal{B}$ is a spanning set of V if and only if $T(\mathcal{B})$ is a spanning set of W.

(2) Prove that $\mathcal{B}$ is linearly independent if and only if $T(\mathcal{B})$ is linearly independent.

Theorem Let $\{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ be a spanning set. Then $m \geq n$.

Proof: Let $v_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$. Let $A = (a_{ij})_{i,j}$

So A is an $n \times m$ matrix.

Any element $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ of $\mathbb{R}^n$ can be

expressed as $x_1 v_1 + x_2 v_2 + \cdots + x_m v_m$.

So the matrix equation $AX = B$ always has a solution.

Let C be the row-reduced echelon matrix associated to A.

So $C = EA$ for some invertible matrix E.

Choose $B = E^{-1} e_n$. $\left(e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$

Suppose $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ is a solution of

$AX = B$.

Thus $A \cdot x = E^{-1} e_n$.

So $EAx = EE^{-1}e_n$.

$\Rightarrow \quad Cx = e_n$.

If $m < n$, C has a zero row at the bottom.

So the product Cx must have a zero row at the bottom.

— contradiction.

So $m \geqslant n$. //

Theorem Let $\{v_1, \dots, v_m\}$ be a linearly independent set in $\mathbb{R}^n$. Then, $n \geq m$.

Proof: As in the proof of the previous theorem, let $v_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$

and let $A = (a_{ij})_{i,j}$

Let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be in $\mathrm{Span}(\{v_1 \dots v_m\})$.

Then, as $\{v_1 \dots v_n\}$ is linearly independent, the equation $x_1 v_1 + \dots + x_n v_n = B$ has a unique solution.

So the matrix equation $AX = B$ has a unique solution. Suppose $m<n$. But we know that if $m > n$, there are infinitely many solutions if there exists a solution at all. — contra. //

Thus any basis of $\mathbb{R}^n$ has exactly n elements.

Let V be a finite dimensional vector space with an ordered basis $\mathcal{B} = (v_1, \ldots v_n)$.
This gives us an isomorphism $\varphi_{\mathcal{B}} : V \to \mathbb{R}^n$. $\varphi_{\mathcal{B}}$ maps any basis of V to a basis of $\mathbb{R}^n$.

Thus, any basis of V has n elements.

We have proved:

<u>Theorem</u> Let V be a finite dimensional vector space. Then, any basis of V has the same number of elements.

Definition

Let V be a finite dimensional vector space. The <u>dimension of V</u>, written as $\dim(V)$, is the number of elements in a basis of V.