Lecture 14

Let G be a group. We defined a normal subgroup of G to be a subgroup H such that $gHg^{-1} \subseteq H$ for any $g \in G$.

We proved that if H is a normal subgroup of G, then G/H is a group.

The binary operation is defined to be $(g_1H)\cdot(g_2H) = g_1g_2H$.
We also proved that if H is a normal subgroup of G, any left coset is also a right coset.

In fact, the converse is true.

<u>Proposition 1</u> Suppose H is a subgroup of G such that every left coset of H is also a right coset. Then H is a normal subgroup of G.

Proof: Let $g \in G$. Then the left coset gH is also a right coset, i.e. it is of the form Hg' for some $g' \in G$.

However, as $g \in gH = Hg'$, we see that $Hg' \cap Hg \neq \emptyset$ and so $Hg' = Hg$. So $gH = Hg$.

So, for any $h \in H$, gh is equal to $h'g$ for some $h' \in H$.

So $ghg^{-1} \in H$.

As this is true for any $h \in H$, we see that H is normal. //

<u>Remark:</u> Gallian's book defines normal subgroups using this property.

Notation

Let G and H be groups.

$H \leq G$ means "H is a subgroup of G"

$H < G$ means "H is a proper subgroup of G"

$H \lhd G$ means "H is a normal subgroup of G".

Quotient homomorphism

Let $H \trianglelefteq G$. Consider the function $\varphi : G \to G/H$ defined by $\varphi(g) = gH$.

Then $\varphi(g_1 g_2) = g_1 g_2 H$

$$= (g_1 H) \cdot g_2 H = \varphi(g_1) \cdot \varphi(g_2)$$

So φ is a group homomorphism.

This homomorphism is called the <u>quotient homomorphism</u> from G to G/H.

Observe that $\ker(\varphi) = H$.

So, every normal subgroup is the kernel of some homomorphism.

Recall

We have already seen that the kernel of a group homomorphism $\varphi : G \longrightarrow H$ is a normal subgroup of G.

Now we see that any normal occurs as the kernel of some homomorphism.

More about homomorphisms

Let $\varphi : G \to H$ be a group homomorphism and let $K = \ker(\varphi)$.

When is $\varphi(x) = \varphi(y)$ for $x, y \in G$?

$$\varphi(x) = \varphi(y) \iff \varphi(x^{-1}y) = 1_H$$

$$\iff x^{-1}y \in K$$

$$\iff y \in xK \iff xK = yK.$$

Conclusions

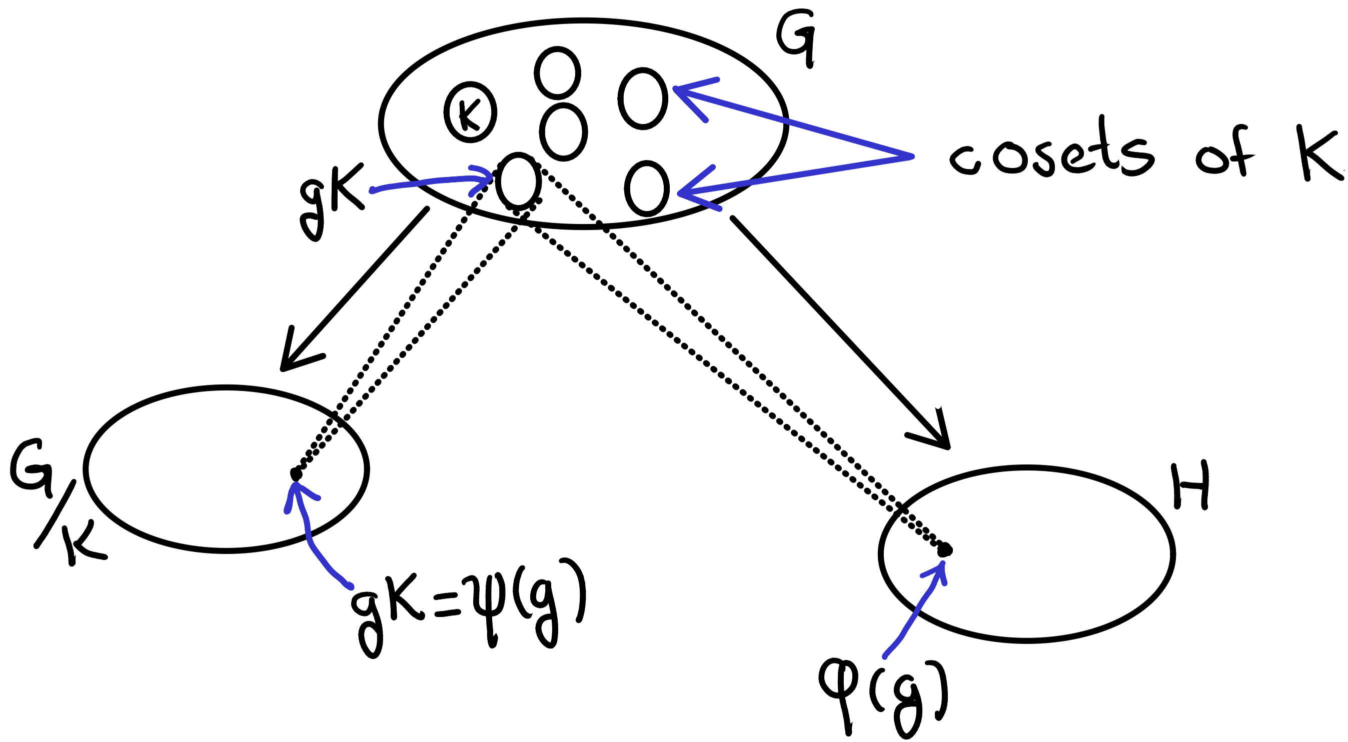
1) Let $\psi : G \longrightarrow G/K$ be the quotient homomorphism.

We have observed that

$$\varphi(x) = \varphi(y) \iff \psi(x) = \psi(y).$$

2) φ is a one-to-one function

$$\iff K = \{1\}.$$


G
K
gK
cosets of K
G/K
gK=ψ(g)
H
φ(g)

We define a function $\tilde{\varphi} : G/K \to H$ by the formula $\tilde{\varphi}(gK) = \varphi(g)$

This is <u>well-defined</u> because if $g_1 K = g_2 K$, then $\varphi(g_1) = \varphi(g_2)$.

Observe that $\tilde{\varphi} \circ \psi = \varphi$.

$$\begin{tikzcd} & G \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ G/K \arrow[rr, "\tilde{\varphi}"'] & & H \end{tikzcd}$$

Also, if $\tilde{\varphi}(g_1K) = \tilde{\varphi}(g_2K)$

then $\varphi(g_1) = \varphi(g_2) \Longrightarrow g_1K = g_2K$.

$\boxed{\text{So, } \tilde{\varphi} \text{ is a one-to-one function..}}$

Let $g_1, g_2 \in G$.

Then, $\tilde{\varphi}(g_1 g_2 K) = \varphi(g_1 g_2)$

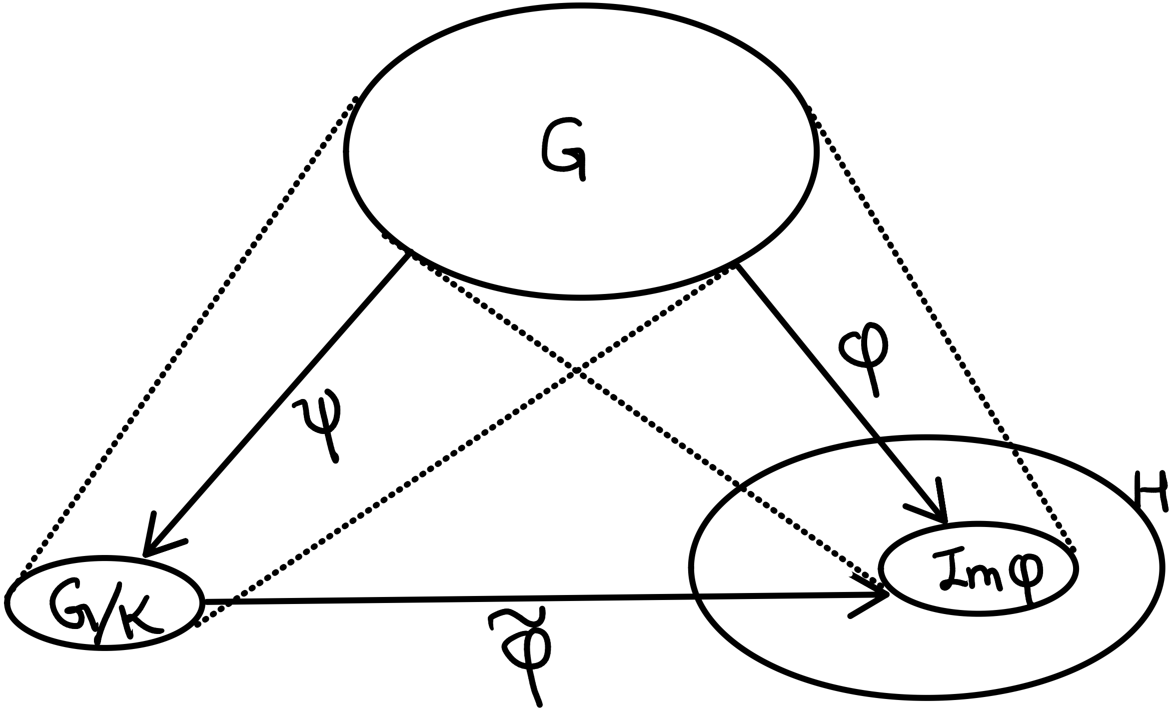
$$= \varphi(g_1) \cdot \varphi(g_2)$$

$$= \tilde{\varphi}(g_1 K) \cdot \tilde{\varphi}(g_2 K).$$

$\boxed{\text{Thus, } \tilde{\varphi} \text{ is a group homomorphism.}}$

$\tilde{\varphi}$ is one-to-one, but it may not be a group isomorphism since it may not be onto.

However, if we consider $\tilde{\varphi}$ as a function from G/K to $\mathrm{Im}(\varphi)$, it becomes onto.


G
ψ
φ
H
G/K
Im φ
φ̃

Thus, $\tilde{\varphi}$ gives a 1-1 correspondence from G/K to $\mathrm{Im}(\varphi)$.

Let $g_1, g_2 \in G$.

$$\tilde{\varphi}(g_1K \cdot g_2K) = \tilde{\varphi}(g_1g_2K) = \varphi(g_1g_2)$$
$$= \varphi(g_1)\cdot\varphi(g_2)$$
$$= \tilde{\varphi}(g_1K)\cdot\tilde{\varphi}(g_2K).$$

Thus $\tilde{\varphi}$ is a group isomorphism.

<u>Theorem</u> Let $\varphi : G \to H$ be a group homomorphism and let $K = \ker(\varphi)$. Then the function $gK \longmapsto \varphi(g)$ is an isomorphism from G/K to $\operatorname{Im}(\varphi)$.

This is called the <u>First Isomorphism theorem</u> of group theory.

<u>Easy consequence</u>

$$|G| = |\ker(\varphi)| \cdot |\operatorname{Im}(\varphi)|$$

<u>Proof</u>: $|\operatorname{Im}\varphi| = |G/K| = |G|/|K|$. //

Subgroups of the quotient

Let G be a group and let $K \triangleleft G$. Let $\psi : G \to G/K$ be the quotient homomorphism.

We want to understand the relationship between subgroups of G and subgroups of G/K.

We know that if H is a subgroup of G, $\psi(H)$ is a subgroup of G/K.

$$\left\{\begin{array}{c}\text{subgroups of} \\ G\end{array}\right\} \xrightarrow{\text{Image}} \left\{\begin{array}{c}\text{Subgroups of} \\ G/K\end{array}\right\}$$

Exercise Let $f: A \to B$ be a group homomorphism. Let $C \leq B$.

(1) Prove that $f^{-1}(C)$ is a subgroup of A.

(2) Prove that if $C \lhd B$, then $f^{-1}(C) \lhd A$.

Thus, we see that if $L \leq G/K$, then $\psi^{-1}(L)$ is a subgroup of G.

$$\left\{ \begin{array}{c} \text{Subgroups of} \\ G \end{array} \right\} \underset{\text{Preimage}}{\overset{\text{Image}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Subgroups of} \\ G/K \end{array} \right\}$$

What is the relationship between these maps?

If $L \leq G/K$, then

$\psi^{-1}(L) = \{g \mid g \in G, \psi(g) \in L\}$

So if $g \in \psi^{-1}(L)$, $\psi(g) \in L$.

Thus $\psi(\psi^{-1}(L)) \subseteq L$.

In fact, if $x \in L$, there always exists some $g \in G$ such that $\psi(g) = x$ as ψ is <u>onto</u>.

Thus, if $x \in L$, $x \in \psi(\psi^{-1}(L))$.

Thus, $L \subseteq \psi(\psi^{-1}(L))$.

So $L = \psi(\psi^{-1}(L))$.

$$\left\{ \text{Subgroups of } G \right\} \underset{\text{Preimage}}{\overset{\text{Image}}{\rightleftarrows}} \left\{ \text{Subgroups of } G/K \right\}$$

This composition is identity.

However, if $H \leq G$, it is <u>not</u> true that $\psi^{-1}(\psi(H)) = H$.

Indeed, take $H = \{1_G\}$.

Then, $\psi(H) = \{1_H\}$ and $\psi^{-1}(\psi(H)) = K$.

What is $\psi^{-1}(\psi(H))$ for a general H?

__Notation__: Let A be a group and let $B \leqslant A$. For any subset $S \subseteq A$, we define SB to be the set

$$\{s \cdot b \mid s \in S,\ b \in B\} = \bigcup_{s \in S} sB$$

Now, suppose C is another subgroup of A. Then, the set CB <u>may not</u> be a subgroup of A.

<u>Example</u> In D_3, consider the subgroups $B = \{1, \tau\}$ and $C = \{1, \rho\tau\}$. Then $BC = \{1, \rho\tau, \tau, \rho^2\}$ is not a subgroup.

However, we have the following:

<u>Lemma:</u> Let A be a group and let $B \trianglelefteq A$. Then for any subgroup $C \leq A$, the set CB is a subgroup of A.

Proof: Exercise.

Returning to our original question...

Let $H \leq G$. What is $\psi^{-1}(\psi(H))$?

Let $x \in \psi^{-1}(\psi(H))$.

Then $\psi(x) \in \psi(H)$. Thus,

$\exists\, h \in H$ such that $\psi(x) = \psi(h)$.

$\psi(h^{-1}x) = 1_{G/K}$. So $h^{-1}x \in \ker\psi = K$.

Thus, $x = h(h^{-1}x) \in HK$.

Thus, $\psi^{-1}(\psi(H)) \subseteq HK$.

It is easy to see that

$HK \subseteq \psi^{-1}(\psi(H))$.

$$\boxed{\text{So } \psi^{-1}(\psi(H)) = HK.}$$

Exercise: Show that $HK = \langle H \cup K \rangle$

(Recall: For any $S \subseteq G$, $\langle S \rangle$ is the subgroup generated by S.)

So the function

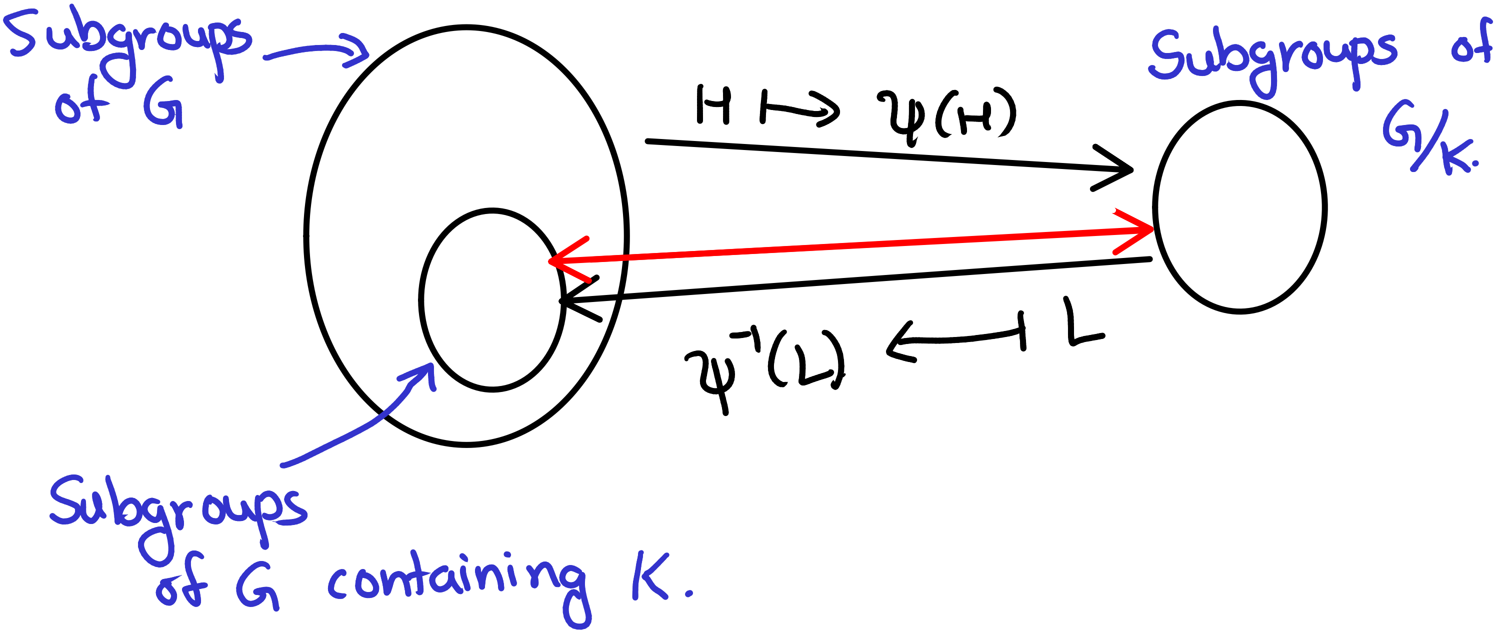
$$\left\{ \begin{array}{c} \text{Subgroups of} \\ G \end{array} \right\} \longleftarrow \left\{ \begin{array}{c} \text{Subgroups of} \\ G/K \end{array} \right\}$$

$$\psi^{-1}(L) \longleftarrow\!\shortmid\; L$$

is not onto since we may have $H \neq HK$.

<u>Exercise</u>. Let $H \leq G$ and $K \trianglelefteq G$. Show that $HK = H \iff K \subseteq H$.

So we have the following situation.



Consider the restrictions of ψ to the subgroups H and HK of G.

$$\ker(\psi|_H) = \{x \mid x \in H, \psi(x) = 1\} = H \cap K.$$

$$\ker(\psi|_{HK}) = \{x \mid x \in HK, \psi(x) = 1\} = K.$$

By the First Isomorphism theorem,

$$H/H\cap K \simeq \psi(H) \quad \text{and}$$

$$HK/K \simeq \psi(H).$$

So $\boxed{H/H\cap K \simeq HK/K}$

(Second Isomorphism Theorem).

Exercise: Let G be a group. Let H, K be <u>normal</u> subgroups of G such that $K \subseteq H$. Then $H/K \trianglelefteq G/K$ and

$$\boxed{(G/K)\big/(H/K) \cong G/H.}$$

(Third Isomorphism theorem).