MTH101: Symmetry

Quiz 1

December 15, 2022

Duration: 40 minutes Total points: 10

Problem 1. (3 points) Let S_7 denote the group of permutations of the set $\{1, 2, \ldots, 7\}$. In this group, consider the elements $\sigma = (1, 4, 3, 5)(2, 7)$ and $\tau = (1, 5, 2, 4)(6, 3, 7)$. Compute the orders of the elements $\sigma\tau$ and $\tau\sigma$.

Solution. By computing values of $\sigma\tau$ and $\tau\sigma$ on each element, we see that $\sigma\tau = (2, 3)(5, 7, 6)$ and $\tau\sigma = (2, 6, 3)(4, 7)$.

The order of a permutation is equal to the lcm of the lengths of the cycles appearing in its cycle decomposition. Thus, $ord(\sigma\tau) = 6$ and $ord(\tau\sigma) = 6$. □

Problem 2. (2 points) Find integers x, y such that $37x + 128y = 2$.

Proof. First we compute the gcd of 37 and 128 using the Euclidean algorithm.

$$\begin{aligned}
128 &= 37 \times 3 + 17\\
37 &= 17 \times 2 + 3\\
17 &= 3 \times 5 + 2\\
3 &= 2 \times 1 + 1\\
2 &= 1 \times 2 + 0
\end{aligned}$$

Thus, $gcd(128, 37) = 1$. We first express 1 in the form $37m + 128n$ where m and n are integers.

$$\begin{aligned}
17 &= 37 \times (-3) + 128 \times 1\\
3 &= 37 - 17 \times 2\\
&= 37 - (37 \times (-3) + 128 \times (-1)) \times 2\\
&= 37 \times (7) + 128 \times (-2)\\
2 &= 17 - 3 \times 5\\
&= (37 \times (-3) + 128 \times 1) - (37 \times 7 + 128 \times (-2)) \times 5\\
&= 37 \times (-38) + 128 \times 11\\
1 &= 3 - 2 \times 1\\
&= 37 \times 7 + 128 \times (-2) - 37 \times (-38) - 128 \times 11\\
&= 37 \times (45) + 128 \times (-13)
\end{aligned}$$

So we have the equation $1 = 37 \times 45 + 128 \times (-13)$. We can multiply this by 2 to get $2 = 37 \times 90 + 128 \times (-26)$. Thus, we have the solution $(x, y) = (90, -26)$.

This is the standard way to solve such equations. To obtain a solution of $ax + by = m$, first check if $gcd(a, b)$ divides m. If not, there is no integer solution. If $gcd(a, b)|m$, then first obtain a solution to $ax + by = gcd(a, b)$ using the Euclidean algorithm and then multiply it by $m/gcd(a, b)$.

However, in this particular problem, we can also notice that we got lucky and encountered at a much earlier stage of our calculations. We have obtained the equation $2 = 37 \times (-38) + 128 \times 11$. We could have just stopped when we reached this expression. But this was just a matter of luck. The standard method is the one explained above. □

Problem 3. Let H denote the subgroup $\langle \overline{35} \rangle$ of $\mathbb{Z}/55\mathbb{Z}$.

(a) (1 points) Compute $|H|$.

(b) (1 points) List all generators of H.

Solution. The order of H is equal to the order of the element $\overline{35}$. Since the order of $\overline{1}$ is 55, the order of $\overline{35} = 35 \times \overline{1}$ is equal to $\frac{55}{gcd(55,35)} = 11$. Thus $|H| = 11$.

The generators of H are of the form $x \cdot \overline{35}$ where x can be an integer such that $0 \leqslant x < ord(\overline{35})$ such that $gcd(x, ord(\overline{35})) = 1$. (This is an acceptable answer.)

Thus, the generators of H are $\overline{35}$, $2 \cdot \overline{35}$, $3 \cdot \overline{35}$, $4 \cdot \overline{35}$, $5 \cdot \overline{35}$, $6 \cdot \overline{35}$, $7 \cdot \overline{35}$, $8 \cdot \overline{35}$, $9 \cdot \overline{35}$ and $10 \cdot \overline{35}$.

It would be also acceptable to say that the subgroup $\langle \overline{35} \rangle$ is actually generated by $gcd(35, 55) \cdot \overline{1} = \overline{5}$ and so the generators are of the form $x \cdot \overline{5}$ where $0 \leqslant x \leqslant 11$ and $gcd(x, 11) = 1$. $\square$

Problem 4. (3 points) Let D_6 denote the group of isometries of the regular hexagon. Let $\rho \in D_6$ be the rotation through $\pi/3$ radians. Determine whether the subgroup $\langle \rho^2 \rangle$ is a normal subgroup of D_6.

Solution. The group $\langle \rho^2 \rangle$ consists of the powers of ρ^2. Thus, a general element is of the form ρ^{2n} for some integer n. (In fact, we can see that this is just the group $\{\rho^0, \rho^2, \rho^4\}$.) This group is normal if and only if $ghg^{-1} \in \langle \rho^2 \rangle$ where g is any element of D_6 and h is any element of $\langle \rho^2 \rangle$.

Any element of D_6 is either of the form ρ^i where $0 \leqslant i < 6$, or it is of the form $\tau\rho^i$ where $0 \leqslant i < 6$.

If g is of the form ρ^i, for $0 \leqslant i < 6$, then we observe that $\rho^i \rho^{2n} \rho^{-i} = \rho^{2n}$ which is in $\langle \rho^2 \rangle$. . If g is of the form $\tau\rho^i$ for $0 \leqslant i < 6$, then

$$\begin{aligned}
(\tau\rho^i)\rho^{2n}(\tau\rho^i)^{-1} &= \tau\rho^i\rho^{2n}\rho^{-i}\tau^{-1} \\
&= \tau\rho^{2n}\tau \\
&= \tau \cdot \tau \cdot \rho^{-2n} \\
&= \rho^{-2n}.
\end{aligned}$$

As ρ^{-2n} is also an element of $\langle \rho^2 \rangle$, we see that $\langle \rho^2 \rangle$ is a normal subgroup. $\square$