Lecture 17

Multiply eqn 1 by
$$(-1)$$
.
Eqn(2) \longrightarrow Eqn(2)+ 5 Eqn(1)

Eqn(3) $\sim > Eqn(3) + 2 Eqn(1)$.

2Z

$$+ 11 W = -27$$

 $+ 5W = -11$

$$+ 3W = -6$$

 $+ 11W = -2$

$$-2Y - 3Z - W = 1$$

Eqn(3) ~>> Eqn(3) + 2 Eqn(2)

Eqn (2) ~~> - + Eqn (2)

Solve last eqn for Z.

W can take any value, say telk.

So Z = t - 1

Substituting W=t and Z=t-1 in eqn 2, we get Y=-2t+1. Substituting in eqn 1, we get

Substituting in eqn 1, we get x = 3t + 5.

So, the solution set is

 $\{(3t+5, -2t+1, t-1, t) \mid t \in \mathbb{R}\}.$

Notice that we do not need to focus on the variables, only on the coefficients and their positions. So, we could have written the system

30, we could have written the as
$$\begin{bmatrix} -1 & 1 & 2 & 3 & | -6 \\ -5 & -2 & 0 & 11 & | -27 \\ | -2 & 0 & 1 & 5 & | -11 \end{bmatrix}$$

Matrices

Let m and n be integers.

An mxn matrix is a collection of mn numbers arranged in a

rectangular array with m rows and n columns.

 $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ The number in the i-th row and j-th column is called the

(i,j) - entry of the matrix.

separated from the rest using a separator. $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_{1} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{mn} \end{bmatrix}$

An augmented matrix is a matrix

in which the last column is

Row operations

Operations on equations in a system

my operations on rows in a matrix.

(1) Replace Row i with Row i + ax Rowj

where a elk: written as Ritakj

(2) Interchange rows i and j: Ri Ri (3) Multiply row i by a, a = 0: written as a Ri

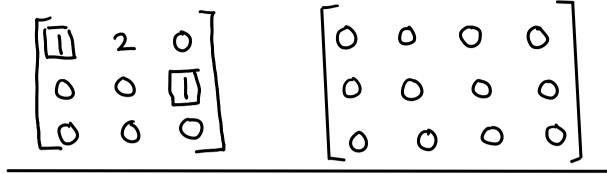
Row reduced echelon matrix. A matrix is in row reduced

echelon form if the following conditions hold:

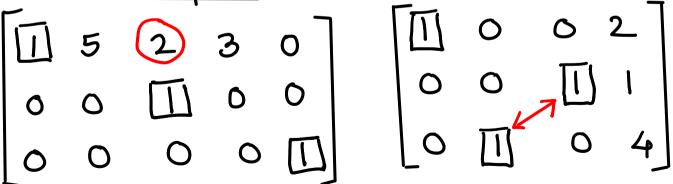
(1) Leftmost non-zero entry in each row is a 1. This is called a pivot.

2) If a column contains a pivot, all other entries in that column are 0. (3) If i<j, and if both row i and row j have pivots, the pivot in rowj is to the right of the pivot in row i.

(4) All zero rows (i.e. rows which contain only zeros) occur at the bottom of the matrix.



Non-examples



Row reduction algorithm Want to create a row reduced

echelon matrix from given

mxn matrix using the three kinds of row operations

listed earlier.

The algorithm has n steps. Step i checks if a pivot can be created in column i. If this is possible, we create the pivot and then reduce all other entries in the column to 0.

Step 1 -Start scanning column 1 from non-zero entry.

the top and try to find a - If there is no non-zero

entry, go on to Step 2.

is in row i and i>1, perform Ri > Rj - Divide R, by the (1,1)-entry so that this entry becomes 1. - Kill all other entries in column 1 If (j,i)-entry is aji, perform Rj-aji, for all j>1.

- If the first non-zero entry

Stepk for k > 2.

-Let i be the first row from the top which does not have a pivot.

Start scanning column k from row i.

- Find first non-zero entry.

-If no such entry, go to Step k+1 (or if k = n, just STOP.). - If there is a non-zero entry move the row where you found this entry to position i

(if necessary).

Divide Row i by the (i,k)-entry.

- Kill all other entries in column

k. So, if (j,k)-entry is ajk,

porform R: + (-a,)R: for all

per form $R_j + (-a_{jk})R_i$ for all $j \neq i$.

- If k<n, go to Step k+1.

If k=n, STOP.

Example

$$\begin{bmatrix}
0 & 1 & 4 & 2 \\
3 & 2 & 1 & 0
\end{bmatrix}
\xrightarrow{R_1 \to R_2}
\begin{bmatrix}
3 & 2 & 1 & 0 \\
0 & 1 & 4 & 2 \\
4 & 1 & 0 & 3
\end{bmatrix}$$

$$\frac{1}{3}R_{1} = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_{3}-4R_{1}} \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

Step 3

[] 0 -7/3 -4/3 0 [] 4 2 0 0 1/4 1/3	3 R3	[[] 0 -% -	-4/3
0 11 4 2	16	0 1 4 2	2
001818		[O O [] 9	16

STOP

Example

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & \boxed{1} & 2 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

Solving systems of linear equations

We want to solve the system

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1$$

 $a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2$

We represent it by the augmented motrix

a,,	Q ₁₂ · · · · ·	a _{in} a _{nn}	b, b ₂
			•
•			
ani	$a_{m2} \cdots$	amn	pm

Perform row reduction on the left block. However, simultaneously perform the same operations on the right block as well. After getting a row reduced echelon matrix in the left block, check which columns have pivots.

Each column corresponds to a variable.

Suppose columns i, iz, ... ir do not have pivots.

Then X_{i_1} , X_{i_2} , ... X_{i_r} are the "free variables" — they can take any value.

Set $X_{i_1} = t_1$, $X_{i_2} = t_2$, $X_{i_r} = t_r$ where $t_1, t_2, \dots t_r$ denote arbitrary elements of IR

arbitrary elements of IR.

Solve for the remaining variables using the row reduced system

using the row reduced system of equations.

Example

 $3X_1 - 2X_2 + 4X_3 + 7X_4 = 11$

 $X_1 + 5X_2 - X_3 + 6X_4 = 4$

 $-X_1 + 3X_2 + 3X_3 + 2X_4 = -1$

Write the corresponding matrix.

$$\begin{bmatrix} 3 & -2 & 4 & 7 & | 1 \\ 1 & 5 & -1 & 6 & | 4 \\ -1 & 3 & 3 & 2 & | -1 \end{bmatrix}$$

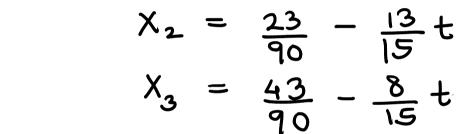
$$\begin{cases} \text{(see pg 24 of 2019 notes)} \\ \text{(o)} & \text{(o)} & \text{(i)} & \text{(i)} & \text{(o)} & \text{$$

Column 4 has no pivots. So set $X_4 = t$ and solve using the reduced system: $X_1 + 0 + 0 + \frac{11}{5}X_4 = \frac{16}{5}$ $0 + X_2 + 0 + \frac{13}{15}X_4 = \frac{23}{90}$ $0 + 0 + X_3 + \frac{8}{15}X_4 = \frac{43}{90}$

This gives
$$X_{1} = \frac{16}{5} - \frac{11}{5}t$$

$$X_{2} = \frac{23}{5} - \frac{13}{5}t$$

$$X_{2} = \frac{23}{90} - \frac{13}{15} + \frac{13}{15$$



So, the solution set is

 $2\frac{16}{5} - \frac{111}{5}, \frac{23}{90} - \frac{13t}{15}, \frac{43}{90} - \frac{8t}{15}, t + telk$