Lecture 1 : Introduction to symmetry

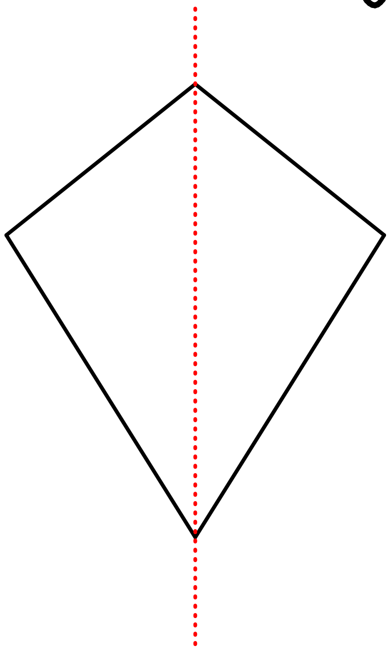
Question: What is symmetry?

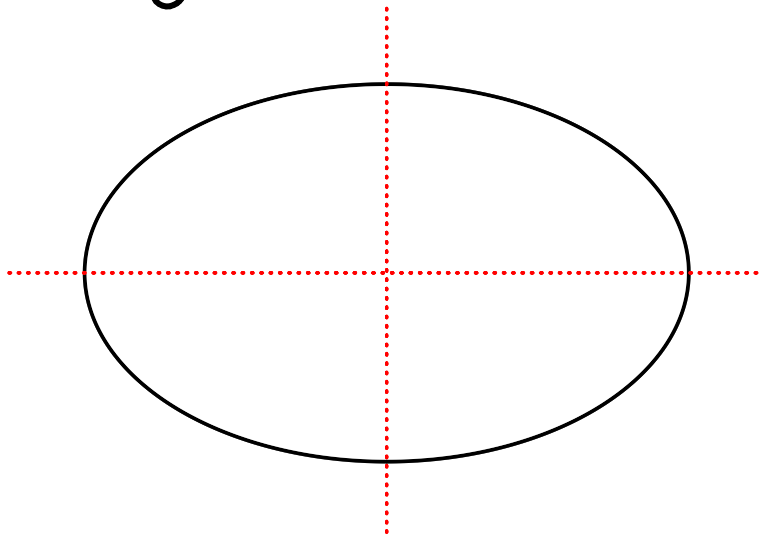
We will focus on geometrical symmetry to begin with.

Three kinds of geometrical symmetry:

- Reflective symmetry.
- Rotational symmetry
- Translational symmetry.

Reflective symmetry





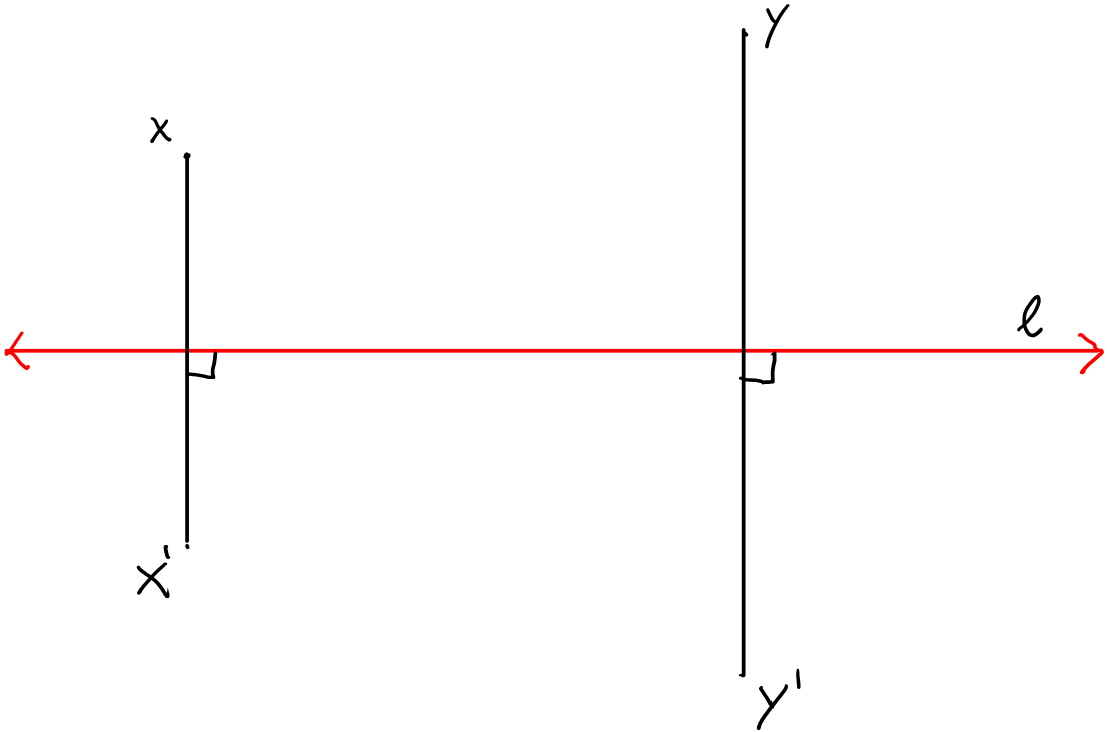
Reflection

$\mathcal{P}$ - Euclidean plane

ℓ - line in the plane

X - point in $\mathcal{P}$

To reflect X in ℓ, draw perpendicular from X to ℓ and extend to X' so that ℓ bisects XX'.



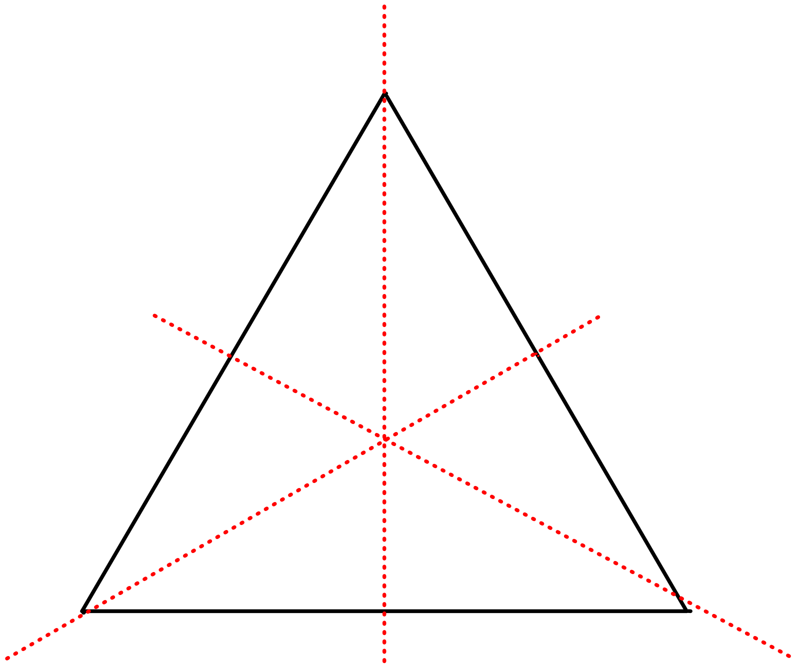
X
Y
ℓ
X'
Y'

Reflective symmetry

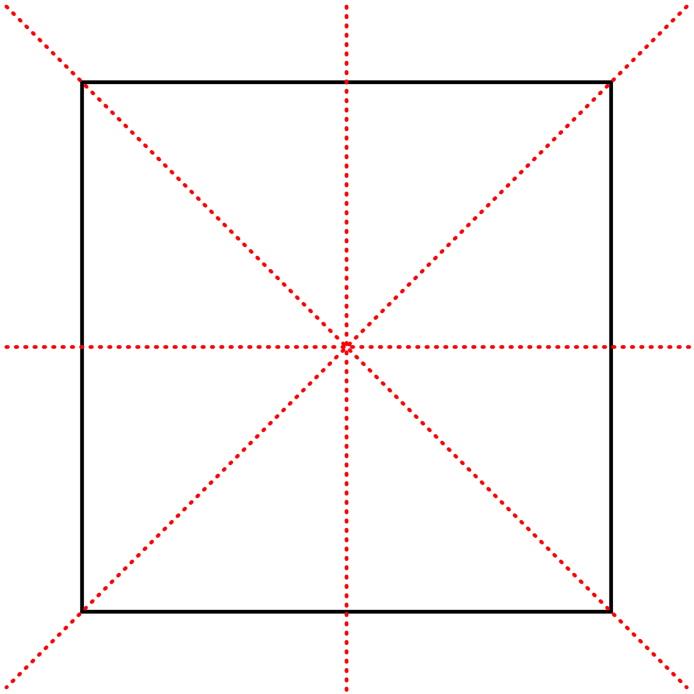
Given a subset A of $\mathcal{P}$, we say A has reflective symmetry if there is a line ℓ such that the reflection of A in ℓ coincides with A.

Example Equilateral triangle

3 reflective symmetries

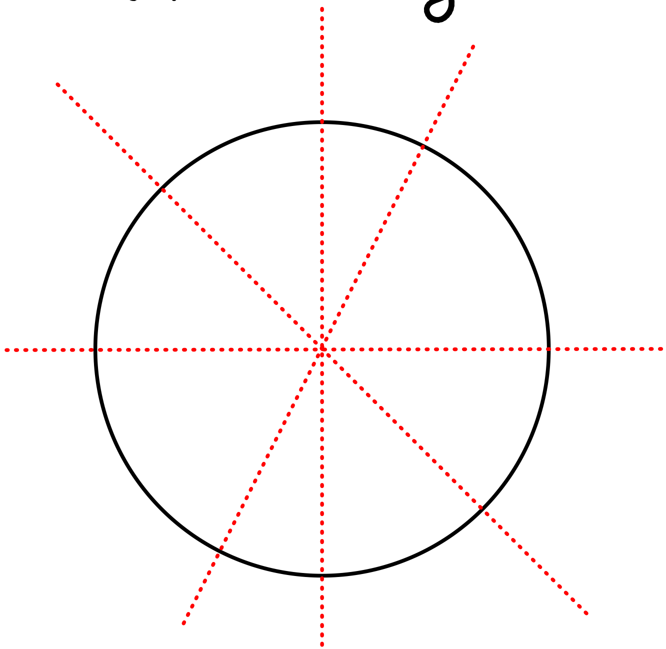


Square 4 reflective symmetries



Circle

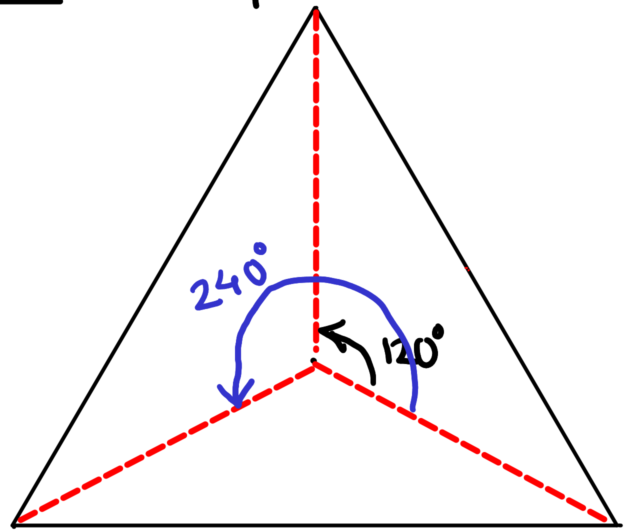
Infinitely many reflective symmetries



Rotational symmetries

We say that a subset A of the plane has rotational symmetry if there is a point O and an angle θ such that the image of A, under rotation around O through θ, coincides with A.

Example Equilateral triangle



3 rotational symmetries.

Angles – 0°, 120°, 240°

Translational symmetry

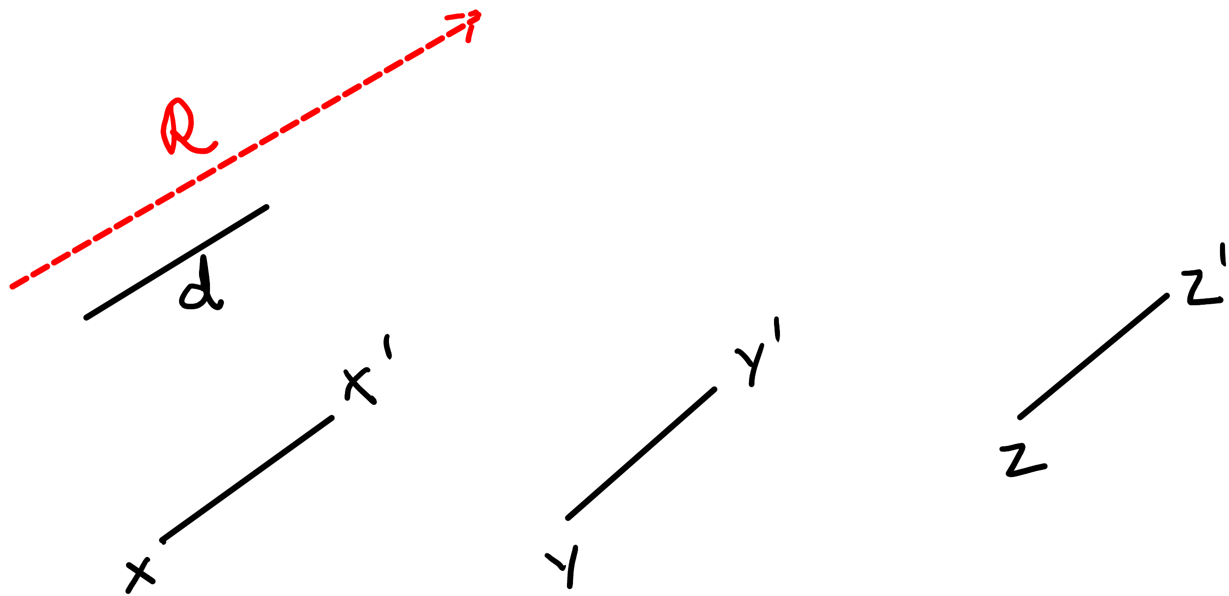
Given a ray R (i.e. a direction) in the plane and a distance d.

X – a point.

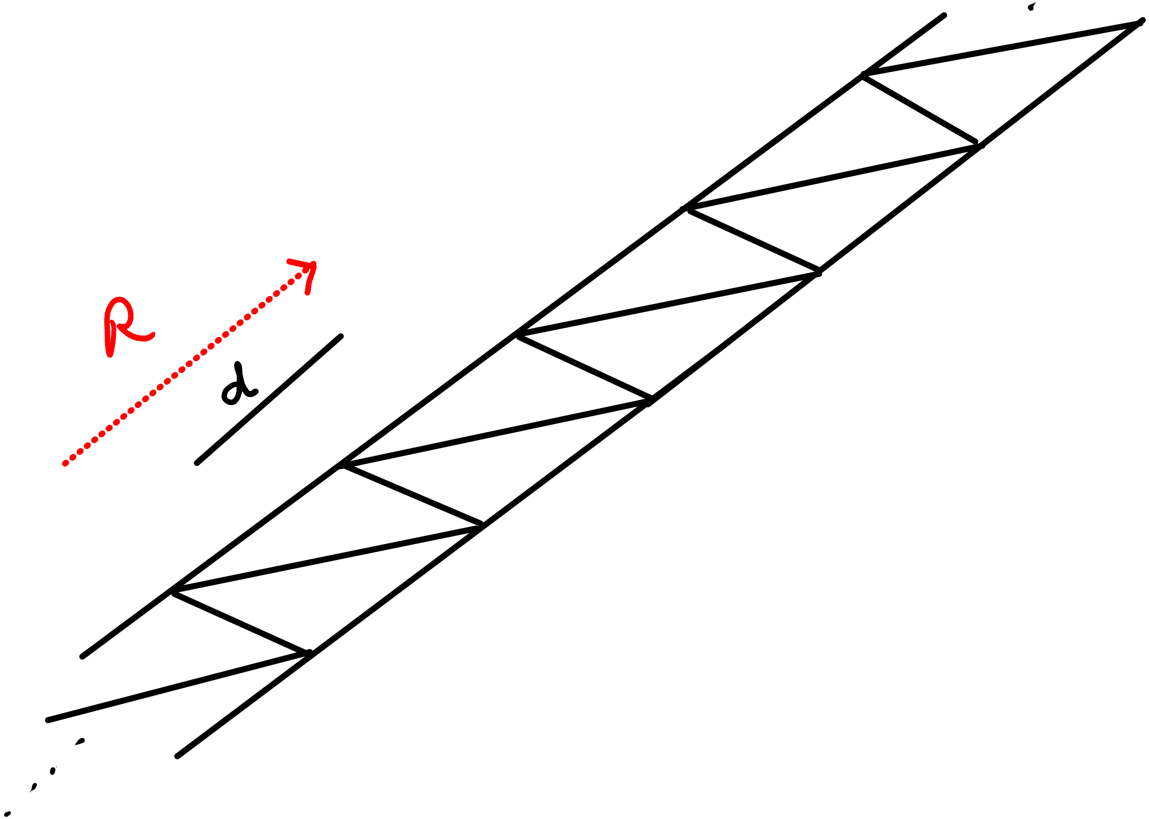
X' – unique point so that $\overrightarrow{XX'}$ points in the same direction as R and $\text{dist}(X X') = d$.

Then X' is the image of X under translation through d along R.

Example



Example



Why care about these operations?

They preserve "shape" and "size".

This means they preserve the distance between any two points.

Isometries

Let A be a subset of $\mathbb{P}$.
An isometry of A is a function $f: A \longrightarrow A$ such that:

(1) f is a 1-1 correspondence

(2) $\text{dist}(x, y) = \text{dist}(f(x), f(y))$ for any x, y in A.

Remark

The notion of isometry makes sense for any set on which we have a notion of distance.

Fact

Any isometry of a subset of the plane is a reflection, rotation or translation.
(Try to prove this!)
So, in this context, symmetry and isometry mean the same.

Properties of isometries

1) Let f, g be isometries of A.

So f, g are 1-1 correspondences

So $g \circ f$ is a 1-1 correspondence.

If x, y are in A,

$$\begin{aligned} \text{dist}(x, y) &= \text{dist}(f(x), f(y)) \\ &= \text{dist}(g(f(x)), g(f(y))). \end{aligned}$$

So $\underline{g \circ f \text{ is an isometry.}}$

2) If f is an isometry,

f is a 1-1 correspondence

$\Rightarrow f^{-1}$ exists and is als a

1-1 correspondence.

$$\operatorname{dist}(f^{-1}(x), f^{-1}(y)) = \operatorname{dist}(f(f^{-1}(x)), f(f^{-1}(y)))$$

$$= \operatorname{dist}(x, y)$$

So f^{-1} is an isometry.

3) Suppose f, g, h are isometries of A.

Then $\underline{f \circ (g \circ h) = (f \circ g) \circ h}$.

This is because for any x in A, $(f \circ (g \circ h))(x)$ and $((f \circ g) \circ h)(x)$ are both equal to $f(g(h(x)))$.

Remark

The last property is true for **all functions**, not just isometries.

We say that **composition of functions is associative.**

Symmetry in geometry

We saw that in the context of geometry, a symmetry is a 1-1 correspondence which preserve the notion of distance.

Symmetry in other contexts

In general, given a set A with some extra structure, a symmetry is a 1-1 correspondence $A \longrightarrow A$ which "preserves that structure" in some sense.

Permutations

Given a set A, a **permutation** of A is a 1-1 correspondence from A to itself.

The set of all permutations of A is denoted by $\text{Perm}(A)$.

Properties of permutations

1) The identity function

$id_A : A \longrightarrow A$ defined by

$$id_A(x) = x$$

is an element of Perm (A)

For any f in Perm (A)

$$f \circ id_A = id_A \circ f = f.$$

2) If f, g are in Perm(A), the composition $g \circ f$ is also in Perm(A)

Recall that $g \circ f(x) = g(f(x))$

(So, the function on the right acts first.)

3) If f is in Perm(A), f^{-1} is also in Perm (A)

4) If f, g, h are in Perm(A)
$(f \circ g) \circ h = f \circ (g \circ h)$.

Example

In a class, some chairs are reserved for boys, others are reserved for girls.

No. of chairs = no. of students

Assume everyone is seated to begin with.

$A =$ set of students

Now suppose all students want to shift positions.

Each student x must choose a student $f(x)$ whose chair he/she will take.

(1) f is in $\mathrm{Perm}(A)$

(2) f preserves gender.

This is an example of a non-geometric "structure" on A. In this context, we may study the set of symmetries of the given seating arrangement. Let G denote this set.

Observe :

1) If f, g are in G, so is $g \circ f$
2) identity function is in G.
3) If f is in G, so is f^{-1}
4) Composition is associative.

Where are we going?

The four properties listed above, define a group. Groups are algebraic objects designed to study symmetries.