MTH101: Symmetry
Problem Set 2

Problem 1. Find the order of the subgroup $\langle \overline{20}, \overline{24}, \overline{30} \rangle$ of $\mathbb{Z}/40\mathbb{Z}$.

Problem 2. Prove that $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/20\mathbb{Z}$ is not a cyclic group.

Problem 3. Find an element of order 24 in S_{10}.

Problem 4. Let G be a group. For any element $g \in G$, define a function $\lambda_g : G \to G$ by $\lambda_g(x) = gx$.

(a) Prove that λ_g is a permutation of G for any $g \in G$.

(b) Prove that the function $g \mapsto \lambda_g$ is a group homomorphism from G to $Perm(G)$.

Problem 5. Let G be a finite group and let H be a subgroup such that $|G| = 2|H|$. Prove that G is normal. (Hint: Try to prove that every left coset is a right coset.)

Problem 6. Let G be a finite group. Let $\phi : G \to \mathbb{Z}$ be a group homomorphism. Prove that $\phi(g) = 0$ for any $g \in G$.

Problem 7. Let $f : A \to B$ be a group homomorphism. Let C be a subgroup of B.

(a) Prove that $f^{-1}(C)$ is a subgroup of A.

(b) Prove that if C is a normal subgroup of B then $f^{-1}(C)$ is a normal subgroup of A.

Problem 8. Let G be a group and, let $H \leqslant G$ and let $K \lhd G$.

(a) Prove that
$$HK := \{hk | h \in H, k \in K\}.$$
is a subgroup of G.

(b) Prove that $HK = \langle H \cup K \rangle$. (Thus, HK is the subgroup of G generated by H and K.)

(c) Prove that $HK = H$ if and only if $K \subset H$.

Problem 9. Let G be a group and let H and K be normal subgroups of G such that $K \subset H$.

(a) Prove that H/K is a normal subgroup of G/K.

(b) There exists an isomorphism $G/K)/(H/K) \cong G/H$. (Third Isomorphism Theorem)