Lecture 19

We saw that if A is an $m \times n$ matrix and if B is obtained from A by performing a row operation, then $B = EA$ where E is the matrix obtained by performing the same row operation on I_m.

Recall that here the row operations being referred to are of three types:

(1) $R_i + aR_j$, $a \in \mathbb{R}$. $1 \leq i, j \leq m$

(2) $R_i \longleftrightarrow R_j$, $1 \leq i, j \leq m$

(3) aR_i , $a \in \mathbb{R}$, $a \neq 0$, $1 \leq i \leq m$.

These are called <u>elementary row operations</u>.

If σ denotes any such operation, let E_σ denote the matrix obtained by performing σ on I_m. Then, to perform σ on an $m \times n$ matrix A (for any n) is the same as multiplying A on the left by E_σ.

These are called <u>elementary matrices</u>.

If B is obtained from A by performing a series of operations $\sigma_1, \sigma_2, \cdots, \sigma_r$, then we see that

$$B = E_{\sigma_r} \cdot E_{\sigma_{r-1}} \cdot \cdots E_{\sigma_1} \cdot A$$

Thus $B = EA$ where E is a product of elementary matrices. E is obtained from I_m by performing $\sigma_1, \sigma_2, \cdots, \sigma_r$.

Invertible matrices

Let m be any positive integer. Then, matrix multiplication gives a binary operation on $M_{m \times m}(\mathbb{R})$ (square matrices of size m).

However $M_{m \times m}(\mathbb{R})$ is not a group under this operation as every matrix may not have a multiplicative inverse.

Left/right inverses

Let A be an $m \times m$ matrix.
An $m \times m$ matrix B is said to be a <u>left inverse</u> of A if $BA = I_m$
Similarly, B is a <u>right inverse</u> of A if $AB = I_m$.
We say that A is invertible if A has both left and right inverses

Lemma Suppose A is an $m \times m$ matrix which has a left inverse B_1 and a right inverse B_2. Then $B_1 = B_2$

Proof: We know $B_1 A = I_m$ and $AB_2 = I_m$

So, $$B_1 = B_1 I_m = B_1 (A B_2)$$
$$= (B_1 A) B_2 = I_m B_2$$
$$= B_2 \quad //$$

Question

Suppose we only know that $A \in M_{n\times n}(\mathbb{R})$ has a left inverse. Can we conclude that it also has a right inverse?

Elementary matrices are invertible

Indeed we saw that if σ is an elementary row operation,

$$E_\sigma \cdot E_{(\sigma^{-1})} = I_m \quad \text{and} \quad E_{(\sigma^{-1})} \cdot E_\sigma = I_m$$

Also a product of elementary matrices is invertible.

$$(E_{\sigma_1} \cdot E_{\sigma_2} \cdot \cdots \cdot E_{\sigma_r})^{-1} = E_{(\sigma_r^{-1})} \cdot E_{(\sigma_{r-1}^{-1})} \cdot \cdots \cdot E_{(\sigma_1^{-1})}.$$

Lemma Let $A \in M_{m\times m}(\mathbb{R})$ have a left inverse B. Then, for <u>any</u> $m\times 1$ matrix Y, $\exists$ an $m\times 1$ matrix X such that $AX = Y$.

Proof Let C be the row reduced echelon matrix obtained by using the row reduction algorithm on A. So, $C = EA$ where E is invertible.

We want to prove that the function
$$M_{m\times 1}(\mathbb{R}) \longrightarrow M_{m\times 1}(\mathbb{R})$$
$$X \longmapsto AX$$
is onto.

Suppose this function is not onto. So, $\exists\, Y \in M_{m\times 1}(\mathbb{R})$ such that there is no X such that $AX = Y$.

Thus, there is no X such that $CX = EY$. Let $EY = Z \in M_{m\times 1}(\mathbb{R})$

C is a row-reduced echelon matrix.

So, as $CX = Z$ has no solutions, C must have a zero row.

Thus, the number of pivots in C is $\leq m-1$.

Now consider the equation $CX=0$.

Suppose $C=(c_{ij})_{i,j}$.

So we are trying to solve

$$\begin{aligned} c_{11}X_1 + \cdots + c_{1m}X_m &= 0 \\ \vdots \qquad\qquad\qquad \vdots \quad &\ \ \vdots \\ c_{m1}X_1 + \cdots + c_{mm}X_m &= 0 \end{aligned}$$

As there are at most $m-1$ pivots, $\exists$ a free variable.

So, this system has infinitely many solutions. In particular, it has a non-zero solution.

So $\exists\ T \in M_{m \times 1}(\mathbb{R})$, $T \neq 0$ such that $CT = 0$

Thus $EAT = 0$

So $E^{-1}EAT = 0 \Rightarrow AT = 0$

As $BA = I_m$, we see that

$T = I_m T = BAT = B0 = 0$ ~ contra.//

An observation Let $1 \leq i \leq m$

Let E_i be the $m \times 1$ matrix which has 1 in row i and 0 elsewhere.

Then, for any $m \times m$ matrix A, we see that

$$AE_i = i^{th} \text{ column of } A.$$

Example

$$\begin{bmatrix} 1 & 0 & 4 & 2 \\ 3 & 2 & -1 & 8 \\ 2 & 4 & 8 & 3 \\ 6 & 7 & 1 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Try out some more examples till you see why this is true.

<u>Lemma</u> Let $J \in M_{m \times m}(\mathbb{R})$ such that $JX = X$ for all $X \in M_{m \times 1}(\mathbb{R})$. Then $J = I_m$.

Proof: $JE_i = E_i$ for $1 \leq i \leq m$.

So, the i-th column of J is E_i

So $J = I_m$. //

Proposition Let $A \in M_{m \times m}(\mathbb{R})$ and let B be its left inverse. Then B is also the right inverse of A.

Proof: As A has a left inverse, for any $Y \in M_{m \times 1}(\mathbb{R})$, $\exists\, X \in M_{m \times 1}(\mathbb{R})$ such that $AX = Y$. So $BAX = BY$, i.e. $X = AY$ (as $BA = I_m$).

So, $(AB)Y = A(BY) = AX = Y$

As this is true for any $Y \in M_{m \times 1}(\mathbb{R})$, it follows that $AB = I_m$.

Thus, B is a right inverse of A. //

Corollary Let $A \in M_{m \times m}(\mathbb{R})$ and let B be a right inverse of A. Then, B is also a left inverse of A.

Proof: A is a left inverse of B. So, by the proposition, A is also a right inverse of B. //

This answers our question

For a square matrix A, the following three statements are equivalent:

(1) A has a left inverse.

(2) A has a right inverse

(3) A is invertible.

The group of invertible matrices

The set of all invertible $m \times m$ matrices is denoted by $GL_m(\mathbb{R})$. Our results show that this is a group under matrix multiplication.

This is called the <u>general linear group of degree m over $\mathbb{R}$</u>.

Calculating the inverse

Suppose A is an invertible matrix. Suppose C is its row reduced echelon form. Thus $C = EA$ where E is a product of elementary matrices.

Thus C is also invertible.

When is a row reduced echelon matrix invertible?

Clearly, C is invertible if $C = I_m$.

What if $C \neq I_m$?

Then, number of pivots $\leq m-1$

So if $C = (c_{ij})_{i,j}$, the following system has some free variables

$$
\begin{aligned}
c_{11}x_1 + \cdots + c_{1m}x_m &= 0 \\
&\vdots \\
c_{m1}x_1 + \cdots + c_{mm}x_m &= 0
\end{aligned}
$$

Thus this system has a non-zero solution T.

So $CT = 0 \implies T = I_m T = C^{-1}CT = 0$

— contra.

Thus C must be equal to I_m.

Thus, a square row reduced echelon matrix is invertible if and only if it is equal to I_m.

Thus, if A is an invertible matrix, its row reduced echelon form is I_m.

Thus, there exist row operations $\sigma_1, \ldots, \sigma_r$ such that

$$E_{\sigma_r} \cdot E_{\sigma_{r-1}} \cdot \cdots E_{\sigma_1} A = I_m.$$

So, $A^{-1} = E_{\sigma_r} \cdot E_{\sigma_{r-1}} \cdots E_{\sigma_1}$.

Thus, we have the following procedure to calculate A^{-1}.

Use the row reduction algorithm to turn A into a row reduced echelon matrix.

Perform the same operations on I_m to get A^{-1}.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{array}{cc} A & I_2 \\ \downarrow & \downarrow \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & 1/2 \end{bmatrix}$