MTH101: Symmetry

Problem Set 1

Problem 1. Let S denote the set $\mathbb{R}\backslash\{-1\}$. We define a binary operation on S by setting $a*b = a+b+ab$. Does S become a group with this binary operation? (Note: $\mathbb{R}\backslash\{-1\}$ means the set of all real numbers except -1.)

Problem 2. Let S and T be two sets such that there exists a $1-1$ correspondence from S to T. Prove that the groups $Perm(S)$ and $Perm(T)$ are isomorphic.

Problem 3. Let G be a group. Let h be an element of G. Define the function $f : G \to G$ by the formula $f(g) = hgh^{-1}$. Prove that f is a group isomorphism.

Problem 4. Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group?

Problem 5. Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

Problem 6. Find an integer x such that $20x \equiv 1 \mod 37$.

Problem 7. Find integers a and b such that $31a + 101b = 1$.

Problem 8. Is the group $U(27)$ cyclic? If the answer to this question is "yes", find all of its generators. If the answer is "no", prove that it is not cyclic.

Problem 9. Let S_n denote the group of permutations of the set $\{1, 2, \ldots, n\}$. Let

$$H = \{\sigma | \sigma \in S_n \text{ such that } \sigma(1) = 1\}.$$

Is H a normal subgroup of S_n?

Problem 10. Does $\mathbb{Z}/100\mathbb{Z}$ have a subgroup of order 20? If the answer is "yes", find all such subgroups. If the answer is "no", explain why not.

Problem 11. Consider the subgroup $\langle\overline{35}\rangle$ of $\mathbb{Z}/50\mathbb{Z}$. Find all generators of this subgroup. (Recall the notation: $\overline{35}$ is our shorthand notation for the coset $35 + 50\mathbb{Z}$ which is an element of the group $\mathbb{Z}/50\mathbb{Z}$.)

Problem 12. Find all generators for the subgroup $\langle\overline{12}, \overline{15}\rangle$ of $\mathbb{Z}/20\mathbb{Z}$.

Problem 13. Let G be a group and let H be a subgroup of G. Let $N(H)$ be defined as

$$N(H) = \{g | g \in G \text{ such that } gHg^{-1} = H\}.$$

1. Prove that $N(H)$ is a subgroup of G.
2. Prove that H is a normal subgroup of $N(H)$.

Problem 14. Let G be a group of order at least 2. Prove that G has an element whose order is a prime number. (Hint: It is enough to prove this for cyclic groups. Do you see why?)

Problem 15. Let G be a group and let S be any subset of G. The *centralizer* of S, denoted by $C(S)$ is defined as

$$C(S) = \{g | g \in G \text{ such that } gs = sg \text{ for all } s \in S\}.$$

Prove that $C(S)$ is a subgroup of G.