Lecture 24

We want to study the following kind of problems.

(A) Given a set S of vectors in a vector space V, determine if they are linearly independent.

(B) Given a set S of vectors in a vector space V, find a basis for $\text{Span}(S)$.

(C) Given a vector space V, find a basis for V (and thus determine $\dim(V)$).

Usually, for a problem of type (C), we begin by finding a spanning set S for V. Thus, we reduce it to a problem of type (B). To solve a problem of type (B), we can first check if it is linearly independent (which is a problem of type (A)). If it is, we are done. If not, we can find a non-trivial linear relation between elements of S. This allows us to find some $v \in S$ such that $v \in \operatorname{Span}(S \setminus \{v\})$.

Now, we can throw v away and repeat the process with $S \setminus \{v\}$. We continue till we end up with a linearly independent set.

So, the key is to solve problems of type (A) first.

So, we focus on such problems first.

Detecting linear dependence/independence.

In a general vector space V, if we are given a set S, to see it it is linearly dependent, we have to try to find a non-trivial solution to the equation

$$\sum_{w \in S} x_w \cdot w = 0$$

(Here, the x_w are variables.)

When S is finite, say $S = \{v_1, \cdots, v_m\}$, this means that we have to find solutions to $x_1 v_1 + \cdots + x_m v_m = \overline{0}$.

How this is to be done may depend on what is known about the space V.

We will focus on $V = \mathbb{R}^n$.

Suppose we are given a set $S = \{v_1, v_2, \cdots, v_m\} \subseteq \mathbb{R}^n$.

We want to determine if they are linearly independent.

If they are linearly dependent, we want to find $x_1, \ldots, x_m \in \mathbb{R}$ such that $\sum_{i=1}^{m} x_i v_i = \overline{0}$ and

Some $x_i \neq 0$.

Let $A \in M_{n\times m}(\mathbb{R})$ be the matrix whose columns are $v_1, v_2, \ldots, v_m$.

Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$.

Then, we want to solve the matrix equation $AX = \overline{0}$ and see rf it has a non-zero solution.

<u>Example</u> Determine if the set

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

is linearly independent.

Solution: We want to find solutions to

$$x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is represented by

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 2 & 4 & 0 \\ 2 & 5 & 1 & 0 \end{array}\right]$$

We solve as usual.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 2 & 4 & 0 \\ 2 & 5 & 1 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3-R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Solution set} = \left\{ \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

Taking $t=1$, we get $x_1 = 2,\ x_2 = -1,\ x_3 = 1$

So the set is not linearly independent. We have the non-trivial linear relation

$$2 \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If we had tried to solve this by Cramer's rule, we would have found that $\det \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 4 \\ 2 & 5 & 1 \end{bmatrix} = 0.$

So this matrix is not invertible and Cramer's rule cannot be used. However, since this is a homogeneous system, it always has a solution. So it will have infinitely many solutions (not invertible square matrix $\Rightarrow$ some free variable in row reduced form, etc). This method does not give a relation. //

The determinant method.

Given a matrix, the dimension of the span of the columns is called the column rank of the matrix.

(Similarly, we can define row rank.)

Column rank can be determined using determinants.

Let $A \in M_{m \times n}(\mathbb{R})$.

Let $k \leq \min(m, n)$

Suppose we pick any k rows and any k columns. They meet in $k \times k$ positions. The $k \times k$ matrix formed like this is called a <u>$k \times k$ minor</u> of A.

Theorem Suppose some $k \times k$ minor of A has non-zero determinant. Then column rank of A is $\geq k$.

Theorem: Column rank of A
= row rank of A

So, we may call this just the **rank** of A.

Given any m vectors in $\mathbb{R}^n$, to determine whether they are linearly independent, we may use the determinant method.

Take the $n \times m$ matrix whose columns are the given vectors.

If there exists an $m \times m$ minor

with non-zero determinant, the vectors are linearly independent.
If <ins>all</ins> $m \times m$ minors have determinant 0, the vectors are linearly dependent.
So, if $m > n$, as there are no $m \times m$ minors at all, the vectors are linearly dependent. (But we already knew that.).

In the last example:

The only 3x3 minor of $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 4 \\ 2 & 5 & 1 \end{bmatrix}$

is the matrix itself. Its determinant is 0. So the vectors are linearly dependent. (But we do not get an explicit relation with this method.).

So the rank is less than 3.

The 2x2 minor $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ has non-zero determinant. So the two vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ are linearly dependent.

So rank is equal to 2.

Example Check if the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

are linearly independent.

We look at the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 4 & -1 & 3 \\ 2 & 9 & 1 \end{bmatrix}$

The 3×3 minor $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 4 & -1 & 3 \end{bmatrix}$ has determinant 0.

However the 3×3 minor

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 2 & 9 & -1 \end{bmatrix}$$ has determinant 24.

So, the matrix has rank ≥ 3.

As column rank $\leq$ no. of columns, we see rank $= 3$.

Also, the three columns are linearly independent.

In this example, the first three rows are not linearly independent (in the space of row vectors).

But row 1, row 2 and row 4 are linearly independent.

Spaces of solutions of a homogeneous system

A homogeneous system of equations can be represented by a matrix equation of the form $AX = \overline{0}$.

The set of solutions is a vector space. We want to find its basis.

Example

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & -1 & 2 & 10 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 \\ 2 & 4 & 0 & -2 & 0 & 8 & 0 \\ 6 & 12 & 2 & 0 & 1 & 23 & 0 \end{array}\right]$$

$$\Big\downarrow \text{ row reduction}$$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & -1 & 0 & 4 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

Solution set:

$$\left\{ \begin{bmatrix} -2t_1 + t_2 - 4t_3 \\ t_1 \\ -3t_2 + 2t_3 \\ t_2 \\ -3t_3 \\ t_3 \end{bmatrix} \;\middle|\; t_1, t_2, t_3 \in \mathbb{R} \right\}$$

The general soln can be written as

$$t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -4 \\ 0 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

So, the space of solutions is the span of the three vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -4 \\ 0 \\ 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

These vectors are linearly independent.

To see this look at the 6×3 matrix:

$$\begin{array}{cl} \left[\begin{array}{ccc} -2 & 1 & -4 \\ 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \\ \leftarrow x_2 \\ \\ \leftarrow x_4 \\ \\ \leftarrow x_6 \end{array} \end{array}$$

The 3×3 minor formed by the rows corresponding to the free variables is I_3.

This is what happens in general.
Number of free variables
= dimension of solution space.

In general, for any matrix A the dimension of the solution space of $AX = \overline{0}$ is called the <u>nullity</u> of A.

For any matrix, the rank and nullity are not changed by row (or column) operations.

It is easy to compute the rank of a row reduced echelon matrix.

Rank = no. of pivots.

(To see this, show that the non-zero rows are linearly independent.

Hint: Focus on the pivots.)

So,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \text{no. of columns of } A.$$

Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ where $A \in M_{m\times n}(\mathbb{R})$.

The i-th column of A is $Ae_i = T(e_i)$.

It is easy to see that

$$\operatorname{Span}(T(e_1), T(e_2) \ldots T(e_n)) = \operatorname{Im}(T).$$

We define $\operatorname{rank}(T) = \dim(\operatorname{Im}(T))$.

So $\operatorname{rank}(T) = \dim(\text{span of columns of } A)$

$$= \operatorname{rank}(A).$$

The kernel of T is the set of solutons of $Ax = \bar{0}$.

So $\dim(\ker(T)) = \operatorname{nullity}(A)$.

Thus

$$\text{rank}(T) + \dim(\ker T) = n$$
$$= \text{dim. of domain of } T.$$

This is called the <u>rank-nullity theorem</u>.