Q:- A rod of mass M and length l is hinged at one end. Neglecting friction determines the minimum angular velocity ω that must be imparted to the rod so that it will swing into a horizontal position (despite the Earth's gravity).

Ans:- From the theorem of conservation of energy we get

$$\frac{1}{2} I \omega_0^2 = mg\frac{l}{2}$$

$I = \frac{1}{3} M l^2$ being the moment of inertia of the rod about the axis of rotation.

Hence $\frac{1}{6} M l^2 \omega_0^2 = Mg\frac{l}{2} \Rightarrow \boxed{\omega_0 = \sqrt{\frac{3g}{l}}}$

Q:- The kinetic energy of a particle moving along a circle of radius R depends on the distance covered s as $T = as^2$, where a is a constant. Find the force acting on the particle as a function of s.

Ans:- Kinetic energy $T = \frac{1}{2} mv^2 = as^2$

$$\Rightarrow v = \sqrt{\frac{2a}{m}}\, s$$

Hence the tangential force acting on the particle is

$$mv\frac{dv}{ds} = m\sqrt{\frac{2a}{m}}\, s \sqrt{\frac{2a}{m}} = 2as$$

The centripetal force, which acts radially is given by

$$\frac{mv^2}{R} = \frac{2as^2}{R}$$

So, the resultant force acting on the particle is $\vec{F} = 2as\left(-\frac{s}{R}\hat{r} + \hat{\theta}\right)$

This has a magnitude of

$$\sqrt{(2as)^2 + \left(\frac{2as^2}{R}\right)^2} = 2as\sqrt{1 - \frac{s^2}{R^2}}$$

As the mass keeps on revolving, the ratio $\frac{s}{R}$ increases without bound. As the number of revolution increases, the velocity becomes increasing closer to the radial direction.

Q:- A car loaded with sand is acted on by a constant horizontal force F. Sand spills at a constant rate α through a hole at the bottom of the car. Write down the differential equation of motion of the car and obtain an expression for the velocity of the car if it starts from rest.

Ans:- Let m be the instantaneous mass of the car and v be its velocity. If in time Δt the mass of sand spilled off the car be Δm and $v + \Delta v$ be the velocity of the car at time $t + \Delta t$, then

$$(m - \Delta m)(v + \Delta v) + \Delta m v - mv = F\Delta t$$

Note that the mass of sand spilled off the car has a velocity v with respect to an inertial observer. Dividing by Δt and going to the limit $\Delta t \to 0$, we obtain the equation of motion of the car and is given by

$$F = m\frac{dv}{dt} = (m_0 - \alpha t)\frac{dv}{dt}$$

where m_0 is the initial mass of the car along with the sand loaded. Integrating and using the condition $v=0$ at $t=0$, we get

$$\int_0^v dv = \int_0^t \frac{F\,dt}{(m_0-\alpha t)} \Rightarrow \boxed{v = \frac{F}{\alpha}\ln\left(\frac{m_0}{m_0-\alpha t}\right)}$$

Q:- Two men, each of mass m, stand on the edge of a stationary buggy of mass M. neglecting friction, find the velocity of the buggy after both men jump off with the same horizontal velocity u relative to the buggy: (a) simultaneously (b) one after the other. In which case will the velocity of the buggy be greater?

Ans:(a) If v_1 be the velocity of the buggy after the men jump off, then

$$2m(u - v_1) = Mv_1 \Rightarrow v_1 = \frac{2m}{M+2m}u$$

(b) If v_1' be the velocity of the buggy after the first man jump off, then

$$m(u - v_1') = (M+m) v_1'$$

$$\Rightarrow \boxed{v_1' = \frac{m}{M+2m} u}$$

If v_2 be the final velocity of the buggy, then $m(u - v_2) = M v_2 - (M+m) v_1'$

$$\Rightarrow M v_2 - (M+m) \frac{m}{M+2m} u$$

Solving for v_2 we get

$$v_2 = \frac{m(2M+3m)}{(M+m)(M+2m)}$$

$$\Rightarrow \frac{v_2}{v_1} = \frac{(2M+3m)}{2(M+m)} = 1 + \frac{m}{2(M+m)}$$

Thus $v_2 > v_1$ i-e, the velocity in the second case will be greater.

Q:- Two masses, m_1 & m_2 are connected by a spring & rests on a floor & μ is the coefficient of friction between the masses and the floor. A varying force slightly larger than the restoring forces is continuously applied to m_1. Find the minimum value of the force F for which the mass m_2 just starts moving.

Ans: When the force F is applied to the mass m_1, it will first move through a certain distance and the condition that the mass m_2 will start moving is given by

$$\mu m_2 g = kx, \quad \text{——— (1)}$$

where k is the spring constant & x is the elongation of the spring. Again from conservation of energy we can write

$$Fx = \mu m_1 g x + \frac{1}{2} k x^2$$

$$\Rightarrow \quad kx = -2\mu m_1 g + 2F \quad \text{——— (2)}$$

From (1) & (2) we get

$$2F - 2\mu m_1 g = \mu m_2 g$$

$$\Rightarrow F = \mu g \left(m_1 + \frac{m_2}{2}\right)$$

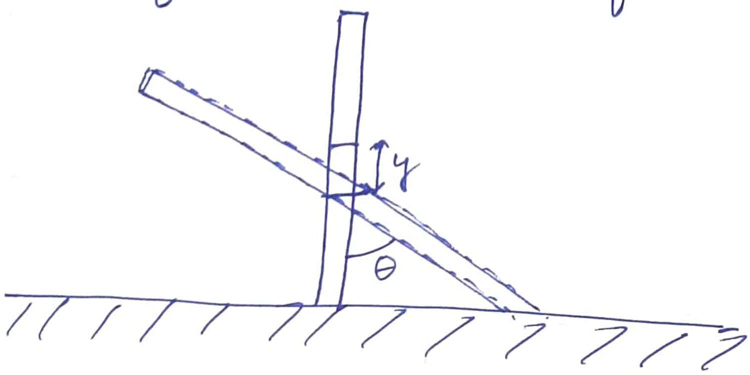
Note that while using conservation of energy given by eqn (2), we have neglected the kinetic energy of the mass m_1. This can be justified by remembering that the force at any stage is only slightly larger than $\mu m_1 g + kx$

and if we write $\epsilon = F - \mu m_1 g - kx$

then $\epsilon = m_1 v \dfrac{dv}{dx}$ or $\epsilon x = \frac{1}{2} m_1 v^2$

As ϵ is assumed to be small, the kinetic energy term can be safely neglected.

Q:- A stick of length l and mass M, initially upright on the frictionless table starts falling. Find the speed of the center of mass as a function of position.



Ans:- The figure shows the instantaneous position of the stick & θ is the angle through which the stick has rotated during this time. Let y be the distance through which the centre of mass has fallen during this period. From conservation of energy we can write

$$Mg\frac{l}{2} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 + mg\left(\frac{l}{2} - y\right)$$

As there is no force other than gravity, the center of mass will always be in the vertical line. Thus the configuration of the stick is completely specified either by y or θ and these related by $y = \frac{l}{2}(1-\cos\theta)$.

Thus $\dot{y} = \frac{l}{2}\sin\theta\,\dot{\theta}$

$$\boxed{\dot{\theta} = \frac{2}{l\sin\theta}\dot{y}}$$

Since $I = \frac{Ml^2}{12}$, we get

$$Mg\frac{l}{2} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}\frac{Ml^2}{12}\left(\frac{2}{l\sin\theta}\dot{y}\right)^2 + mg\left(\frac{l}{2} - y\right)$$

$$\dot{y}^2 = \frac{2gy}{1 + \frac{1}{3\sin^2\theta}} \Rightarrow \dot{y} = \sqrt{\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}}$$

which gives the velocity of the center of mass. If necessary, one can express $\dot{y}$ either in terms of y or θ.

Q:- A table of height h has a small hole at its middle & rests on a horizontal surface. A thin chain of length l & mass M is loosly coiled & placed close to the hole. One end of the Chain is pulled a little through the hole & then released. These is no friction. After what time will the chain reach the floor for

(a) $l = h$, (b) $l < h$ & (c) $l > h$?

Ans:- At any instant during the fall, let x be the length of the hanging part of the Chain and m be its mass. As the part of the chain resting on the table is not moving, equation of motion of the chain considering the change in mass of the moving part as it falls, can be written as

$$\frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

or $$\frac{dv}{dt} = g - \frac{v}{m}\frac{dm}{dt} \quad \text{—— (1)}$$

Now $\frac{m}{x} = \frac{M}{l}$ is the mass per unit length of the chain. so $\boxed{dm = \frac{m}{x}\, dx}$ and

$$\frac{dm}{dt} = \frac{m}{x}\frac{dx}{dt} = \frac{m}{x} V \quad \text{—— (2)}$$

Substituting this value of $\frac{dm}{dt}$ in eqn (1)

$$\frac{dv}{dt} = v\frac{dv}{dx} = g - \frac{v^2}{x}$$

The above eqn can be rearranged

$$2x^2 v\,dv + 2xv^2 dx = 2gx^2 dx$$

$$d(x^2v^2) = \frac{2}{3} g\, d(x^3)$$

Integrating & using the condition $v=0$ when $x=0$

$$v^2x^2 = \frac{2}{3} g x^3 \quad \text{or} \quad v = \sqrt{\frac{2gx}{3}}$$

Note that this is velocity of free fall where the acceleration is $\frac{g}{3}$

Case I) when $l = h$, the first link reaches the floor when the chain runs down from the table. During the time the first link reaches the floor, the chain moves with an acceleration is $\frac{g}{3}$. But as the last link loses contact with the table, the entire chain moves with an acceleration g. The time t_1 for the first link to reach the floor is given by

$$h = \frac{1}{2}\frac{g}{3}t_1^2 \Rightarrow \boxed{t_1 = \sqrt{\frac{6h}{g}}}$$

When the lower end of the chain reaches the floor, the entire chain is vertical & its velocity v_1 is given by

$$v_1^2 = \frac{2g}{3}h \Rightarrow \boxed{v_1 = \sqrt{\frac{2gh}{3}}}$$

Now, consider the last link. When the chain runs out of the table, the last link has the velocity v_1. The time t_2 required to cover a distance h by the last link is given by

$$h = v_1 t_2 + \frac{1}{2}g t_2^2$$

$$t_2^2 + \sqrt{\frac{8h}{3g}}\,t_2 - \frac{2h}{g} = 0$$

Considering the positive root, we get

$$t_2 = \frac{\sqrt{\frac{32h}{3g}} - \sqrt{\frac{8h}{3g}}}{2}$$

$$= \left(\sqrt{8}-\sqrt{2}\right)\sqrt{\frac{h}{3g}} = \sqrt{\frac{2h}{3g}} = \frac{t_1}{3}$$

So the total time taken by the chain to reach the floor is

$$t_1 + t_2 = 4\sqrt{\frac{2h}{3g}} = 4\sqrt{\frac{2l}{3g}}$$

as $h = l$

Case (II) :- when $l < h$, the total time is obviously given by

$$\sqrt{\frac{6l}{g}} + \sqrt{\frac{3h}{3g}}$$

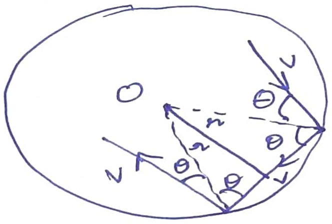
Case III :- when $l > h$, each part of the chain will move with an acceleration is $\frac{g}{3}$ until the last link runs out of the table. The hanging part, when the last link leaves the table, will have a free fall. So, the total time is given by

$$\sqrt{\frac{6l}{g}} + \sqrt{\frac{2h}{3g}}$$

Note that it is tacitly assumed that the links reaching the floor are gently removed so that each part of the chain covers the same distance.

Q:- A spherical container contains N no. of particles each of mass m. The particles move with speed v in random directions & suffer elastic collision with the wall of the container. Find the pressure on the wall of the container.

A:- If a particle, moving with velocity v, be incident on the wall at an angle θ, then it will be reflected in a direction making an angle θ with the normal (see figure).



The change in momentum of the particle is in the direction of the normal & is of the magnitude $2mv\cos\theta$.

The distance traversed by the particle between two successive collisions is $2r\cos\theta$, where r is the radius of the container. So the no. of collision suffered by the particle per second is $\frac{v}{2r\cos\theta}$ & the momentum imparted per second by a single particle to the wall of the container is

$$2mv\cos\theta \frac{v}{2r\cos\theta} = \frac{mv^2}{r}$$

It may be noted that the above expression is independent of θ. For N particles, the momentum imparted to the wall per second is $\frac{Nmv^2}{r}$. As this must be equal to force on the wall, we can write

$$p\,4\pi r^2 = \frac{Nmv^2}{r}$$

where p is the pressure on the wall. We can rewrite the above equation as

$$p\,\frac{4\pi r^3}{3} = \frac{1}{3}Nmv^2, \quad \text{i.e} \quad PV = \frac{1}{3}Nmv^2$$

where $V = \frac{4}{3}\pi r^3$ is the volume of the container. If $n = \frac{N}{V}$ be the no. of particles per unit volume, we can write $p = \frac{1}{3}nmv^2$

This is the familiar expression for the pressure as deducted in the kinetic theory of gases.