Lecture 2 : Symmetries of regular n-gons

Let n be a positive integer. We want to "understand" the symmetries of the regular n-gon.

What does it mean to "understand"?

- We want a complete list of all symmetries.
- We want to calculate compositions of symmetries.

Remark: Measuring angles

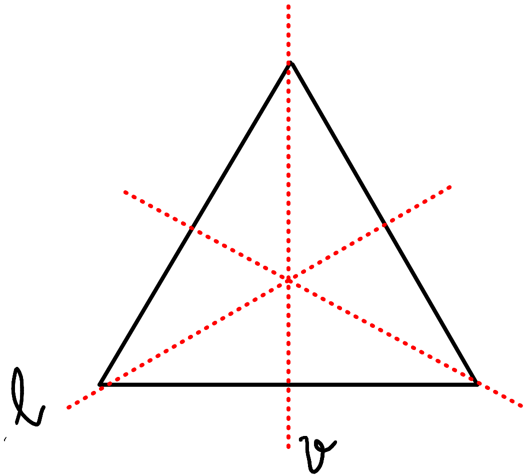
We will express angles in radians.

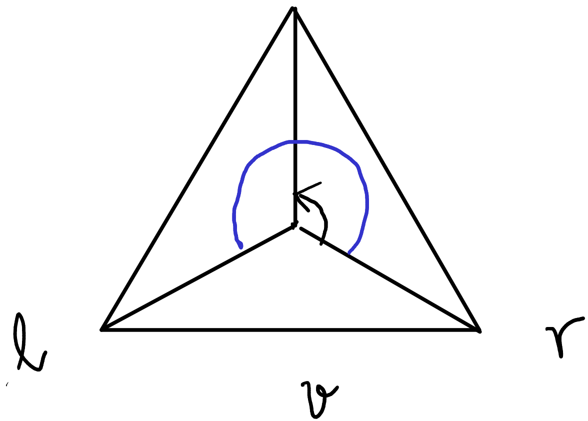
π radians $= 180^\circ$

So, $120^\circ = 2\pi/3$ radians,

$240^\circ = 4\pi/3$ radians, etc.

The case $n=3$





3 reflections in lines v, l, r. Call them τ_v, τ_l, τ_r

3 rotations through 0, $2\pi/3$, $4\pi/3$

Rotation through 0 radians = id.
(identity)

Let ρ = rotation through $2\pi/3$.

Then, $\rho^2 = \rho \circ \rho$ is rotation through $4\pi/3$.

So, the symmetries are
$\{ id, \rho, \rho^2, \tau_v, \tau_l, \tau_r \}$

Compositions

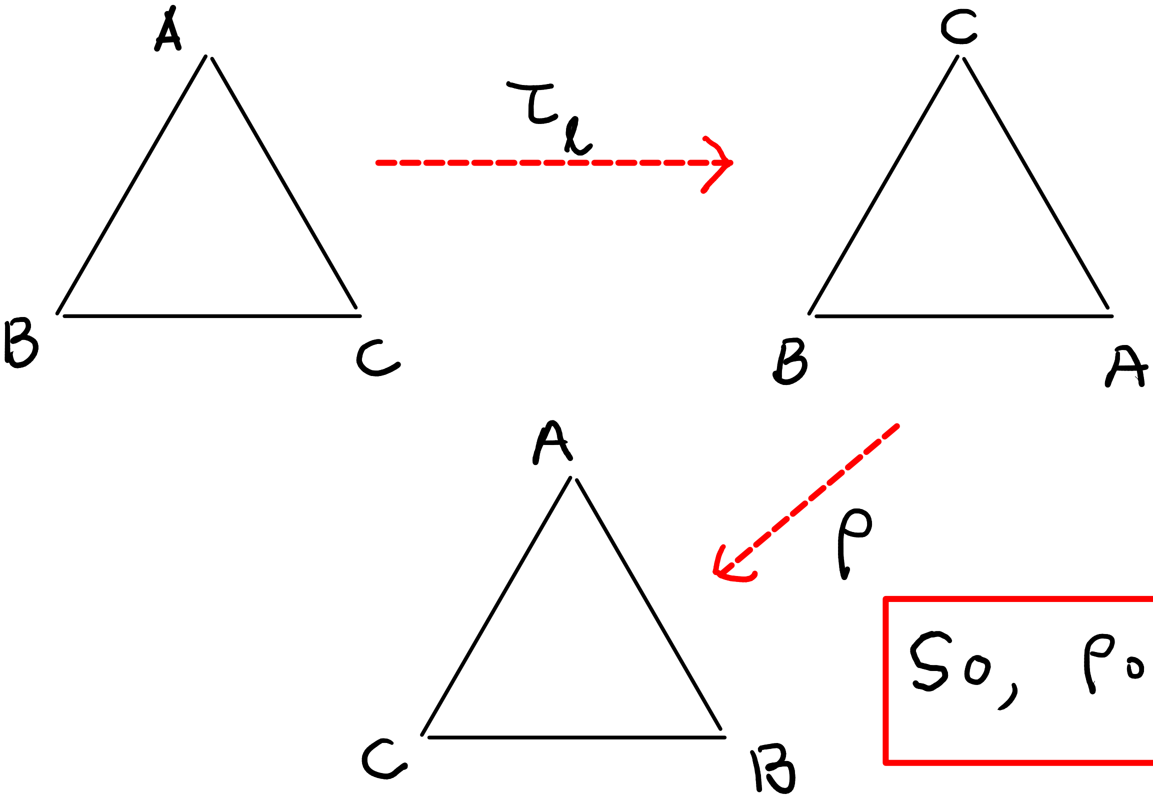
Rotations: $\{ id, \rho, \rho^2 \}$

Clearly, $\rho \circ \rho^2 = \rho^2 \circ \rho = \rho^3$
$= id.$

Reflections: $\{ \tau_v, \tau_\ell, \tau_r \}$

Clearly, $\tau_v^2 = \tau_\ell^2 = \tau_r^2 = id.$

A sample calculation : $\rho \circ \tau_\ell = ?$



$$\boxed{\text{So, } \rho \circ \tau_\ell = \tau_v}$$

A complete table for calculating $x \circ y$

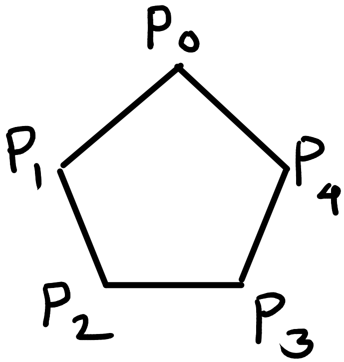
x

		id	ρ	ρ^2	τ_v	τ_ℓ	τ_r
	id	id	ρ	ρ^2	τ_v	τ_ℓ	τ_r
	ρ	ρ	ρ^2	id	τ_ℓ	τ_r	τ_v
y	ρ^2	ρ^2	id	ρ	τ_r	τ_v	τ_ℓ
	τ_v	τ_v	τ_r	τ_ℓ	id	ρ^2	ρ
	τ_ℓ	τ_ℓ	τ_v	τ_r	ρ	id	ρ^2
	τ_r	τ_r	τ_ℓ	τ_v	ρ^2	ρ	id

Objective

- Do a similar calculation for any n.
- Make sure we have found all isometries
- Find some pattern in this table and express it more concisely.

A basic observation



Label vertices as $P_0, \ldots, P_{n-1}$ in anti-clockwise order.

σ – some symmetry.

If $\sigma(P_0) = P_i$, then

$\sigma(P_1) = P_{i+1}$ or P_{i-1}.

Convention:

If $i = 0$, P_{i-1} is understood as P_{n-1}.

If $i = n-1$, P_{i+1} is understood as P_0

In general P_i may also be called P_{n+i} or P_{i-n}.

Case 1

If $\sigma(P_1) = P_{i+1}$, then

$\sigma(P_2) = P_{i+2}$, $\sigma(P_3) = P_{i+3}$, etc.

In fact the image of any vertex is uniquely determined.

$\sigma(P_j) = P_{i+j}$.

So, σ = rotation through $\left(\frac{2\pi i}{n}\right)$.

Observe

In general, rotation preserves "orientation", i.e. anti-clockwise order of labels remains anti-clockwise after rotation.

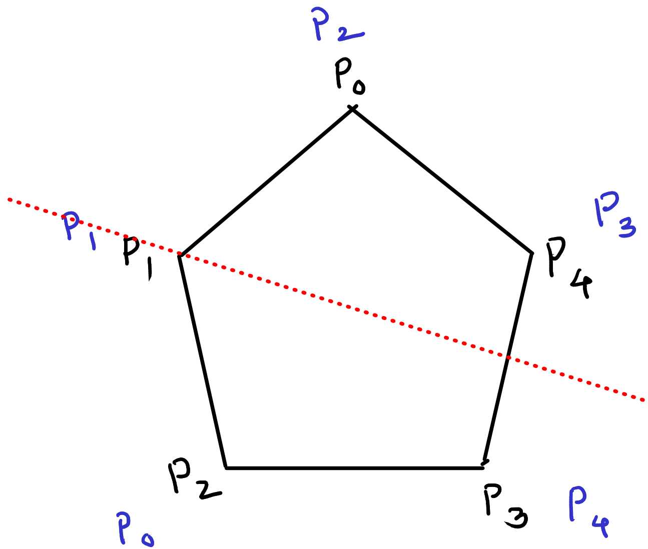
Case 2

If $\sigma(P_1) = P_{i-1}$, $\sigma(P_2) = P_{i-2}$, $\sigma(P_3) = P_{i-3}$, etc.

In general, $\sigma(P_j) = P_{i-j}$

Now the vertex labels run clockwise. So σ cannot be a rotation.

Let l_i be the line joining midpoints of P_0P_i and P_1P_{i-1}



If $P_0 = P_i$, take the first midpt to be P_0.
Similarly, if $P_1 = P_{i-1}$, take the 2nd midpt to be P_1

It is easy to see that the reflection in l_i is a symmetry of the n-gon taking P_0 to P_i, P_1 to P_{i-1} and, in general, P_j to P_{i-j}.

Thus, σ = reflection in l_i

Observe

There are no other symmetries!

Why?

Any symmetry takes P_0 to some P_i. If it takes P_1 to P_{i+1}, it is the rotation through $\left(\frac{2\pi i}{n}\right)$ (anti-clockwise).

If it takes P_1 to P_{i-1}, it must be the reflection in the line ℓ_i defined earlier.

So, there are exactly $2n$ symmetries: <u>n rotations</u> and <u>n reflections</u>

Let $\rho =$ rotation through $\frac{2\pi}{n}$.

Then, $\rho^i =$ rotation through $\frac{2\pi i}{n}$

So, $\{id, \rho, \rho^2, \ldots, \rho^{n-1}\}$ are the n rotations.

Let τ = reflection in l_0

Recall: l_0 = line joining P_0 to midpoint of P_1P_{n-1}

What can we say about $\rho^i\tau$?

τ takes P_j to P_{n-j} for each j.

ρ^i takes P_r to P_{r+i} for each r.

So $\rho^i \tau$ takes P_j to $P_{(n-j)+i}$ for each j.

So, it takes P_0 to $P_{n-0+i} = P_i$.

It takes P_1 to $P_{n-1+i} = P_{i-1}$.

So, we see that $\rho^i \tau$ is the reflection in the line l_i.

So, the n reflections are

$$\{\tau, \rho\tau, \rho^2\tau, \ldots, \rho^{n-1}\tau\}$$

All symmetries:

$$\{id, \rho, \rho^2, \ldots, \rho^{n-1}, \tau, \rho\tau, \rho^2\tau, \ldots, \rho^{n-1}\tau\}$$

What is $\tau\rho^i$?

ρ^i preserves orientation.

τ changes orientation.

So $\tau\rho^i$ changes orientation. So it is some reflection.

Which one is it?

ρ^i takes P_0 to P_i

τ takes P_i to P_{n-i}

So $\tau\rho^i$ takes P_0 to P_{n-i}

So $\tau\rho^i$ is a <u>reflection taking P_0 to P_{n-i}</u>.

So, $\boxed{\tau\rho^i = \rho^{n-i}\tau.}$

Note that $\rho^{n-i} \cdot \rho^{i} = \rho^{n} = \mathrm{id}$.

So, ρ^{n-i} = inverse of ρ^{i}

So, we will write ρ^{-i} instead of ρ^{n-i}.

So, we have the rule

$$\boxed{\tau \rho^{i} = \rho^{-i} \tau}$$

Now we can calculate any composition.

For example,

$$\begin{aligned} \rho^2\tau \circ \rho^3\tau &= \rho^2(\tau\rho^3)\tau \\ &= \rho^2(\rho^{-3}\tau)\tau \\ &= (\rho^2\rho^{-3})(\tau\tau) \\ &= \rho^{-1} = \rho^{n-1}. \end{aligned}$$