Lecture 16

Systems of linear equations:

$$\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \cdots &+ a_{1n}X_n = b_1 \\
a_{21}X_1 + a_{22}X_2 + \cdots &+ a_{2n}X_n = b_2 \\
\vdots \qquad\qquad \ddots \qquad & \qquad\qquad \vdots \\
a_{m1}X_1 + a_{m2}X_2 + \cdots &+ a_{mn}X_n = b_m
\end{aligned}$$

$a_{ij}, b_i \in \mathbb{R}$ for all i,j. X_i are variables.

Example

$m = 2, \; n = 2$

$2x_1 + 3x_2 = 5$ —— (1)

$-x_1 + x_2 = 7$ —— (2)

Add $2 \times$ Eqn (2) to Eqn (1).

$0 + 5x_2 = 19$ —— (3)

$-x_1 + x_2 = 7$

Eqn (3) replaces eqn (1)

Solve eqn (3) for x_2.

$X_2 = 19/5$

Substitute the value of X_2 in eqn (2) and solve for X_1.

$-X_1 + 19/5 = 7$

So $X_1 = 19/5 - 7 = -16/5$

Solution : $\{(-16/5, 19/5)\}$

Example: $m=3$, $n=2$

$x_1 - x_2 = 3$ — (1)

$2x_1 - 3x_2 = 1$ — (2)

$-x_1 + 2x_2 = -1$ — (3)

Add $(-2) \times$ Eqn (1) to Eqn (2).

$0 - x_2 = -5$ — (4) | Replaces eqn (2).

Add eqn (1) to eqn (3).

$0 + x_2 = 2$ — (5) | Replaces eqn (3).

Now we have the system

$$x_1 - x_2 = 3$$
$$-x_2 = -5$$
$$x_2 = 2$$

This system has no solutions.

Solution set $= \phi = \{\ \}$.

Example $m=2, \quad n=3$

$$x_1 + x_2 - x_3 = 4 \quad \text{——} (1)$$

$$2x_1 - x_2 + 3x_3 = 1 \quad \text{——} (2)$$

Add $(-2) \times$ eqn (1) to eqn (2)

$$x_1 + x_2 - x_3 = 4 \quad \text{——} (1)$$

$$0 - 3x_2 + 5x_3 = -7 \quad \text{——} (3)$$

Divide eqn (3) by (-3).

$$X_1 + X_2 - X_3 = 4 \quad \text{—— (1)}$$

$$x_2 - (5/3) X_3 = 7/3 \quad \text{—— (4)}$$

Add $(-1) \times$ eqn (4) to eqn (1).

$$X_1 + 0 + (2/3) X_3 = 5/3 \quad \text{—— (5)}$$

$$X_2 - (5/3) X_3 = 7/3. \quad \text{—— (4)}$$

Choose any value for X_3 and solve for X_1 and X_2.

Suppose we set $x_3 = t$ for some $t \in \mathbb{R}$. Then, we get

$x_1 = 5/3 - 2/3\, t$

and $x_2 = 7/3 + 5/3\, t$.

Solution set:

$$\left\{ \left(5/3 - 2/3\, t,\ 7/3 + 5/3\, t,\ t \right) \,\middle|\, t \in \mathbb{R} \right\}$$

Thus, we see that when we solve a system of linear equations, one of three things could happen:

(1) Unique solution.

(2) No solution.

(3) Family of solutions

Understanding the method

We are given a system (A) of equations. We perform some operations and create a system (B). What is the relationship between the solutions of (A) and (B)?

Example

Consider the system (A) consisting of one equation: $X - 1 = 5$.

Suppose we perform the operation of squaring both sides.

We get the system (B):

$$(X-1)^2 = 25.$$

Solutions of (B) are 6 and -4. But only 6 is a solution of (A).

So, these two systems are not <u>equivalent</u>. Every solution of (A) must satisfy (B), but not the other way around.

But suppose there exist some other operations which allow us to obtain system (A) from system (B). Then, every solution of (B) is a solution of (A). So the systems are equivalent.

Example

System A: $x_1 + x_2 = 5$ —— (1)

$2x_1 - x_2 = 2$ —— (2).

Operation: Replace eqn (1)

eqn(1) + 3 × eqn (2).

$(x_1 + x_2) + 3(2x_1 - x_2) = 5 + 3(2)$

—— (3)

$2x_1 - x_2 = 2$ —— (2).

Notice that eqn (3) is "implied" by eqn (1) and eqn(2). In other words, any solution of eqn (1) and eqn (2) is a solution of eqn (3).

So, every solution of (A) is a solution of (B).

System B : $7x_1 - 2x_2 = 11$ — (3)

$$2x_1 - x_2 = 2 \quad \text{— (2)}$$

Perform the following operation on (B) : Replace eqn (3) by

eqn (3) $-$ $3 \times$ eqn(2)

Let us call the new system as (C).

By the same argument as before, every solution of (B) is a solution of (C).

But (C) is the same as (A).

$$(7x_1 - 2x_2) - 3(2x_1 - x_2) = 11 - 3(2)$$

$$x_1 + x_2 = 5$$

So, systems (A) and (B) are equivalent, i.e. they have the same solution sets.

This is because we obtained (B) from (A) by using a <u>reversible</u> operation.

Not all operations are reversible. For example, if we start with $x - 1 = 5$ and square both sides, we get $(x-1)^2 = 25$. But if we start with $(x-1)^2 = 25$, we cannot deduce $x - 1 = 5$ as square roots are not unique.

From $(x-1)^2 = 25$, we can only get the statement

$x - 1 = 5$ **OR** $x - 1 = -5$

This is because squaring is not a reversible operation.

We will only use reversible operations on systems of linear equations.

They are as follows:

(1) Replace eqn (i) by eqn(i) + $a \times$ eqn(j) for some $a \in \mathbb{R}$.

<u>Inverse</u>: Replace eqn (i) by eqn (i) + $(-a) \times$ eqn (j).

(2) Interchange eqn (i) and eqn (j).

<u>Inverse</u>: Interchange eqn (i) and eqn (j).

(3) Replace eqn (i) by $a \times$ eqn (i) for some $a \neq 0$.

<u>Inverse</u>: Replace eqn (i) by $(1/a) \times$ eqn (i).

Key idea

Consider the system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \ddots \qquad\qquad &\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Suppose $a_{11} \neq 0$.

Then, we can replace eqn (1) by $\left(\frac{1}{a_{11}}\right) \times$ eqn (1) to reduce to a situation where the coefficient of x_1 in eqn (1) is 1.

Now perform the operations

$$\text{eqn } (i) \rightsquigarrow \text{eqn } (i) + (-a_{i1}) \text{ eqn } (1)$$

for $i = 2, 3, \ldots, m$.

This will eliminate x_1 from

eqn(2), ... , eqn(m).

$$
\begin{array}{l}
x_1 + c_{12}x_2 + \cdots\cdots + c_{1n}x_n = d_1 \\
0 \cdot x_1 + c_{22}x_2 + \cdots\cdots + c_{2n}x_n = d_2 \\
\vdots \qquad\qquad \ddots \qquad\qquad \vdots \\
0 \cdot x_1 + c_{m2}x_2 + \cdots\cdots + c_{mn}x_n = d_m
\end{array}
$$

Consider the system consisting of eqn(2), eqn(3), $\cdots$, eqn(m).

This is a smaller system with (n-1) variables and (m-1) equatrons. Solve this smaller system for $x_2, \cdots, x_n$ and substitute in eqn(1). Solve eqn(1) for x_1.

What if $a_{11} = 0$?

Find some i such that $a_{i1} \neq 0$.

Interchange eqn (1) and eqn (i).

If $a_{i1} = 0$ for all i, then

X_1 is not really there in any

equation. Move on to X_2.

How do we solve the smaller system we obtained?

$$
\begin{aligned}
c_{22}x_2 + \cdots\cdots &+ c_{2n}x_n = d_2 \\
\ddots \quad & \qquad\quad \vdots \\
c_{m2}x_2 + \cdots\cdots &+ c_{mn}x_n = d_m
\end{aligned}
$$

Use the same procedure. Start with x_2 instead of x_1.

We will then end up with an even smaller system with $(n-2)$ variables and $(m-2)$ equations.

... and so on.

How does this process end?

(Next time.)