Lecture 21

We now look at another formula to compute the determinant of a matrix.

Given an $n \times n$ matrix $A = (a_{ij})_{i,j}$ let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A.

Expansion by rows

Let $1 \leq k \leq n$.

Then $\det(A) = \sum_{l=1}^{n} (-1)^{k+l} a_{kl} \det(A_{kl})$

Expansion by columns

Let $1 \leq k \leq n$.

Then, $\det(A) = \sum_{l=1}^{n} (-1)^{k+l} a_{lk} \det(A_{lk})$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

We will expand along the first column.

$$A_{11} = \begin{bmatrix} -1 & 5 \\ 6 & 8 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 2 \\ 6 & 8 \end{bmatrix} \quad \text{and}$$

$$A_{31} = \begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix}$$

So,

$$\det(A) = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} -1 & 5 \\ 6 & 8 \end{bmatrix}$$

$$+ (-1)^{2+1} \cdot 4 \cdot \det \begin{bmatrix} 0 & 2 \\ 6 & 8 \end{bmatrix}$$

$$+ (-1)^{3+1} \cdot 2 \cdot \det \begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix}$$

Each of the 2×2 matrices can be calculated similarly.

For example

$$\det\begin{bmatrix} -1 & 5 \\ 6 & 8 \end{bmatrix} = (-1)^{1+1}\cdot(-1)\cdot\det[8]$$
$$+ (-1)^{2+1}\cdot 6\cdot\det[5]$$
$$= -8 - 30 = -38.$$

Similar calculations are done for the other 2x2 matrices.

We get $\det(A) = 1\cdot(-38) - 4\cdot(-8)$
$$+ 2\cdot(-2) = -8$$

Remembering the formula

Choose a row or column along which you want to expand. You form n terms of the type

$$(\pm 1) \cdot (a_{ij}) \, (\det A_{ij})$$

Sign depends on i, j

(i,j) term of A

formed by dropping row i & column j

The signs form a pattern that is easy to remember

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

- The top-left corner has +.
- Signs alternate along each row and column.

Cofactors

The number $(-1)^{i+j} \det(A_{ij})$ is called the **(i,j) - cofactor** of A. Let $c_{ij} = (-1)^{i+j} \det(A_{ij})$.

The matrix $C = (c_{ij})_{i,j}$ is called the **cofactor matrix** of A.

So $\det A = \sum_{l=1}^{n} a_{kl} c_{kl}$, etc.

Properties of determinants

(1) Let B be obtained from A by performing an elementary row operation:

(a) If the operation is $R_i + xR_j$, then $\det(B) = \det(A)$. $(i \neq j)$

(b) If the operation is $R_i \leftrightarrow R_j$ $(i \neq j)$ then $\det(B) = -\det(A)$.

(c) If the operation is xR_i $(x \neq 0)$, then $\det(B) = x\det(A)$.

(2) Similar formulas hold for column operations, which we denote by $C_i + xC_j$, etc.

(3) $\det(I_n) = 1$ for any n.

(4) $\det(AB) = \det(A) \cdot \det(B)$ for any $A, B \in M_{n\times n}(\mathbb{R})$.

Transpose of a matrix

If P is an $m \times n$ matrix, the transpose of P, denoted by P^{tr} is the $n \times m$ matrix whose (i,j)-entry is equal to the (j,i)-entry of P.

e.g. $P = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}$, $P^{tr} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$

Fact If $A \in M_{n \times n}(\mathbb{R})$,

$$\det A^{tr} = \det A.$$

We will omit the proof of this.

One way to prove this is to use the formula we obtained last time.

Remark While calculating determinants, it can be helpful to use row and column operations. If the matrix is brought into row reduced echelon form, the determinant can be easily calculated without effort. Generally we use both techniques.

Example

$$\det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 2^2 & 3^2 & 5^2 \end{bmatrix} = \det\begin{bmatrix} 1 & 1-1 & 1-1 \\ 2 & 3-2 & 5-3 \\ 2^2 & 3^2-2^2 & 5^2-3^2 \end{bmatrix}$$

(by $C_2 - C_1$ and $C_3 - C_1$)

$$= \det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 2^2 & 5 & 16 \end{bmatrix}$$

$$= 1\,(1\cdot 16 - 5\cdot 2) + 0 + 0 \quad \text{(expand by row 1)}$$

$$= 6.$$

Easy observation

Suppose row i and row j of a matrix A are identical.

Then, the operation $R_i + (-1)R_j$ turns the i-th row into a row of zeros.

Expanding along this row, we get $\det(A) = 0$.

(Similar statement holds for columns.)

Let $A = (a_{ij})_{i,j} \in M_{n \times n}(\mathbb{R})$ and let $1 \leq k, l \leq n$, $k \neq l$. As before, let c_{ij} denote the (i,j)-cofactor of A. Let A' be obtained by replacing the k-th column by a copy of the l-th column. Then, A' has two identical columns. So, $\det(A') = 0$.

For any i, the (i,k)-cofactor of A' is the same as the (i,k)-cofactor of A (as A and A' differ only in the k-th column).

The (i,k) entry of A' is a_{il}.

So,

$$0 = \det A' = \sum_{i=1}^{n} a_{il} C_{ik}.$$

A similar argument shows that if $k \neq l$, then

$$\sum_{i=1}^{n} a_{li} c_{ki} = 0.$$

So,

$$\sum_{i=1}^{n} a_{ik} c_{ik} = \sum_{i=1}^{n} a_{ki} c_{ki} = \det A$$

and

$$\sum_{i=1}^{n} a_{il} c_{ik} = \sum_{i=1}^{n} a_{li} c_{ki} = 0$$

if $k \neq l$.

Let us see what this means.

$$\sum_{i=1}^{n} a_{il}\, c_{ik}$$

comes from l-th column of A

comes from k^{th} column of C, i.e. from k^{th} row of $C^{tr.}$

This quantity is $\det A$ if $k = l$ and 0 if $k \neq l$.

This expression is the (k,l)-entry of the matrix product $C^{tr} \cdot A$.

So we see that $C^{tr} \cdot A$ has $\det A$ on the diagonal and 0 elsewhere.

Thus $C^{tr} \cdot A = \det A \cdot I_n$

scalar multiplication

Thus, if $\det(A) \neq 0$, we see that $\left(\frac{1}{\det A} \cdot C^{tr}\right) \cdot A = I_n$.

Thus, we see that if $\det A \neq 0$, then A is invertible, and $A^{-1} = (\det(A))^{-1} \cdot C^{tr}$.

What if $\det A = 0$?

In this case A cannot be invertible.

Indeed, suppose A^{-1} exists.

Then $(\det A)(\det A^{-1}) = \det(AA^{-1}) = 1$

But this is impossible if $\det A = 0$.

— contra.

So we have

<u>Theorem</u> A square matrix A is invertible $\Longleftrightarrow$ $\det A \neq 0$.

If $\det A \neq 0$,

$$A^{-1} = \det(A)^{-1} \cdot C^{tr}$$

where C = cofactor matrix of A.

Example

Suppose the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. So $\det(A) = ad - bc \neq 0$

The matrix of cofactors is

$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$. So, $A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 2^2 & 3^2 & 5^2 \end{bmatrix}$$

Matrix of cofactors $= C = (c_{ij})$

$$c_{11} = 3 \cdot 5^2 - 3^2 \cdot 5 = 75 - 45 = 30$$

$$c_{12} = -(2 \cdot 5^2 - 2^2 \cdot 5) = -30$$

$$c_{13} = 2 \cdot 3^2 - 2^2 \cdot 3 = 6$$

$$c_{21} = -(1 \cdot 5^2 - 3^2 \cdot 1) = -16$$

$$c_{22} = 5^2 - 2^2 = 21, \quad c_{23} = -(3^2 - 2^2) = -5$$

$$C_{31} = 1\cdot5 - 1\cdot3 = 2$$

$$C_{32} = -(1\cdot5 - 1\cdot2) = -3$$

$$C_{33} = (1\cdot3 - 1\cdot2) = 1$$

So $C = \begin{bmatrix} 30 & -30 & 6 \\ -16 & 21 & -5 \\ 2 & -3 & 1 \end{bmatrix}$

$\det A = 6$ (earlier calculation)

So $A^{-1} = \frac{1}{6}\begin{bmatrix} 30 & -16 & 2 \\ -30 & 21 & -3 \\ 6 & -5 & 1 \end{bmatrix}$

Cramer's rule

Suppose we have a system of n linear equations in the n variables $x_1, x_2, \cdots, x_n$.

$$\begin{array}{ccccc} a_{11}x_1 & + \cdots\cdots & + \, a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + \cdots & + \, a_{nn}x_n & = & b_n \end{array}$$

As usual, we write this using matrices.

Let $A = (a_{ij})_{i,j}$

Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

We want to solve the matrix equation $AX = B$.

If A is invertible, we get

$$X = A^{-1}B = \left(\frac{1}{\det A}\right) \cdot C^{tr} \cdot B$$

where $C =$ cofactor matrix of A.

$C^{tr} \cdot B$ is an $n \times 1$ matrix whose $(k, 1)$-entry is equal to

$$\sum_{i=1}^{n} \left((k, i) \text{ entry of } C^{tr} \right) \cdot b_i$$

$$= \sum_{i=1}^{n} c_{ik} \cdot b_i$$

$$= \sum_{i=1}^{n} \left((i, k)\text{-cofactor of } A \right) \cdot b_i$$

This expression is seen to be the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{n1} & \cdots & & b_n & \cdots & a_{nn} \end{bmatrix}$$

k^{th} column.

Expand along the k^{th} column.

Thus we have

<u>Thm (Cramer's rule)</u>

Consider the system of equations

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

Let $A = (a_{ij})_{i,j}$. Let A_k be the matrix obtained by replacing the k^{th} column of A by $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Also, suppose that A is invertible.

Then this system has a unique solution given by $x_k = \dfrac{\det A_k}{\det A}$ for $1 \leq k \leq n$.

Example Let us solve the system

$$2x_1 + 3x_2 - 5x_3 = 2$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$3x_1 - 4x_2 + 2x_3 = -1.$$

We calculate $\det\begin{bmatrix} 2 & 3 & -5 \\ -1 & 2 & -1 \\ 3 & -4 & 2 \end{bmatrix}$.

$$\det\begin{bmatrix} 2 & 3 & -5 \\ -1 & 2 & -1 \\ 3 & -4 & 2 \end{bmatrix} = 2\left(2\cdot 2 - (-1)(-4)\right) - 3\left((-1)\cdot 2 - 3\cdot(-1)\right) + (-5)\left((-1)(-4) - 3.2\right)$$

$$= 2\cdot 0 - 3\cdot 1 + (-5)(-2)$$

$$= 7$$

This is non-zero and so we may use Cramer's rule.

By Cramer's rule,

$$X_1 = \frac{1}{7} \cdot \det \begin{bmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \\ -1 & -4 & 2 \end{bmatrix}$$

$$= \frac{1}{7} \left[2\left(2\cdot 2 - (-4)(-1)\right) - 3\left(1\cdot 2 - (-1)(-1)\right) + (-5)\left(1\cdot(-4) - (-1)\cdot 2\right) \right]$$

$$= \frac{1}{7} \left[2\cdot 0 - 3\cdot 1 + (-5)(-2) \right]$$

$$= 1$$

$$x_2 = \frac{1}{7} \det \begin{bmatrix} 2 & 2 & -5 \\ -1 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$= \frac{1}{7} \Big[2\big(2 - (-1)\cdot(-1)\big) - 2\big((-1)(2) - 3\cdot(-1)\big) + (-5)\big((-1)(-1) - 3\cdot 1\big) \Big]$$

$$= \frac{1}{7} \left[2\cdot 1 - 2\cdot 1 + (-5)(-2) \right]$$

$$= \frac{10}{7}$$

$$x_3 = \frac{1}{7} \det \begin{bmatrix} 2 & 3 & 2 \\ -1 & 2 & 1 \\ 3 & -4 & -1 \end{bmatrix}$$

$$= \frac{1}{7}\Big[2\big(2(-1) - (-4)\cdot 1\big) - 3\big((-1)\cdot(-1) - 3\cdot 1\big) + 2\big((-1)\cdot(-4) - 3\cdot 2\big) \Big]$$

$$= \frac{1}{7}\Big[2(2) - 3\cdot(-2) + 2\cdot(-2) \Big]$$

$$= 6/7.$$

__Remark__ Given a system of n equations in n variables, if the matrix of coefficients is not invertible, Cramer's rule cannot be used. However, this does not mean that the system has no solutions. It can be solved using row reduction. It has infinitely many solutions.