Black Holes in Anti-de Sitter Spacetime

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ABSTRACT: Black holes in anti-de Sitter (AdS) spacetime are known to be thermodynamically stable against radiation, unlike their asymptotically flat counterparts. Moreover, Hawking and Page discovered that they exhibit a first-order phase transition. This was reinterpreted by Witten using the AdS/CFT correspondence as a confinement/deconfinement transition in the quark-gluon plasma. Extending the gauge/gravity correspondence to the fluid/gravity correspondence, it was shown that there is an exact agreement between the thermodynamics and stress-energy tensor of large, rotating AdS black holes and fluids on its conformal boundary. In this essay I review the thermodynamics of AdS black holes, its gauge theory interpretation and discuss a recent work by Bhattacharyya et al., focusing on the $AdS_5 \times S^5$ case.

KEYWORDS: Black holes, Thermal phase transitions, AdS/CFT correspondence.

Contents

1. Introduction

One of the crowning achievements of the Golden Age of General Relativity was the discovery that black holes exhibit thermodynamic properties. A black hole has a natural temperature associated with its surface gravity and the entropy associated with its area is non-decreasing. Such quantities obey relations that are in striking analogy with the classical laws of thermodynamics. In addition, quantum fluctuations allow the black holes to radiate energy and evaporate eventually. Since these properties were discovered, much research has gone into understanding the thermodynamics of black holes. It was noted that a Schwarzschild black hole in an asymptotically flat spacetime has negative specific heat and is thus thermodynamically unstable. After an initial attempt to resolve this problem by putting the black hole in a finite "box" [10], a more natural setting was found in the context of the anti-de Sitter (AdS) space. A Schwarzschild black hole in an asymptotically AdS spacetime has positive specific heat at high temperature and is thermodynamically stable. In 1983, Hawking and Page [11] discovered using path integral methods that Schwarzschild-AdS black holes have negative free energy relative to AdS spacetime at high temperature and exhibit a first-order phase transition. The study of black holes in AdS space was reinvigorated more recently by Maldacena's celebrated AdS/CFT conjecture [16]. He proposed a correspondence between string theories in AdS backgrounds and gauge theories on its conformal boundary. In 1993, Witten [21] interpreted the Hawking-Page phase transition in the gauge theory context as a quark-gluon confinement-deconfinement phase transition. This is an illuminating example of how the study of black holes sheds light on phenomena in other fields. One remarkable application is the fluid/gravity correspondence. Using the gauge/gravity duality, Bhattacharyya et. al. [2] showed that there is an exact correspondence between thermodynamic variables and the stress-energy tensor of large, rotating black holes and fluids on its conformal boundary. These developments have once again put black holes at the centre of theoretical physics. Their properties, via duality principles, can be used to understand and to connect problems from disparate fields. As Strominger puts it: black holes are "the harmonic oscillators of the 21st century" [1].

In this essay, I will begin by reviewing the thermodynamic properties of black holes with a focus on the Schwarzschild-AdS case. I will then give a self-contained treatment of the Hawking-Page phase transition following an introduction to the path integral formalism in quantum gravity. In section 4 I will explain the core ideas of the AdS/CFT correspondence and Witten's interpretation of the Hawking-Page phase transition as a confinement-deconfinement transition in large N gauge theories. The second half of the essay will explore the fluid/gravity correspondence. I will base my discussion on the works of Bhattacharyya et. al., focusing on the $AdS_5 \times S^5$ case. Unless expressly stated, I will assume $c = \hbar = G = k = 1$.

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[1]This is the title from his recent talk at the Institute of Theoretical Physics, Chinese Academy of Sciences.

2. Black Hole Thermodynamics

In the works of Bekenstein and Hawking it was found that black holes have temperature
and entropy

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}, \qquad S_{\mathrm{BH}} = \frac{A}{4}, \tag{2.1}$$

where κ is the surface gravity and A is the area of the black hole. It is essential to note
that temperature and entropy, both quantities associated with the entire black hole, can be
calculated using quantities at its boundary. This naïve observation is a first manifestation
of the holographic principle, which says that information about a system are encoded in
the boundary. In the later sections we will exploit the full power of this principle. For now
we will look at the thermodynamics of specific black holes in detail.

In this section we consider the thermodynamics of Schwarzschild black holes. We
will find that they are thermodynamically unstable in an asymptotically flat spacetime.
We will comment on Hawking's early effort to find thermodynamically stable solutions by
putting the black holes in a box of finite heat capacity. We then discuss black holes in the
AdS context, where there exists thermodynamically stable black holes as the AdS metric
provides a potential wall at asymptotic infinity, thus avoiding the awkward "black hole in
a box" scheme.

2.1 Black hole in asymptotically flat spacetime

The Schwarzschild black hole was the first exact solution to the vacuum Einstein equation
$R_{ij} = 0$. Its metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.2}$$

In fact, the Schwarzschild black hole is the unique static, spherically symmetric solution
of the vacuum Einstein equation. We can easily compute its temperature $T_{\mathrm{H}} = \frac{1}{8\pi M}$. It
is somewhat counter-intuitive that small black holes radiate faster than large black holes.
A Schwarzschild black hole is thermodynamically unstable because its specific heat, $\frac{\partial T_{\mathrm{H}}}{\partial M}$,
is negative. Consider such a black hole in equilibrium with an infinite heat reservoir. A
small positive fluctuation in temperature makes the black hole radiate some mass away,
and the temperature increases, lowering the mass further until the black hole becomes
extremely hot and evaporates. A small negative fluctuation in temperature makes the
black hole absorb more radiation than it radiates, increasing its mass. The black hole cools
off, absorbs mass at a faster rate and grows indefinitely.

2.2 Black hole in a box

To restore stability, Hawking considered the unphysical case of putting the black hole in
a box of finite volume and finite heat capacity [10]. He found that the black hole can be
in stable thermodynamic equilibrium against radiation given that the radiation energy of
the box satisfies $E_{\mathrm{rad}} < \frac{1}{4}M$. We will give a quick derivation of this fact here. The system
is in thermodynamic equilibrium if the total entropy, $S = S_{\mathrm{rad}} + S_{\mathrm{BH}}$ is at a maximum

subject to the constraint of constant total energy, $E = E_{\text{rad}} + M$. This translates into the conditions

1. $d(S - \lambda E) = 0$, where λ is the Lagrange multiplier,

2. $\text{Hess}(S) < 0$.

We find from solving condition (1) that

$$\frac{\partial S_{\text{rad}}}{\partial E_{\text{rad}}} dE_{\text{rad}} + \frac{\partial S_{\text{BH}}}{\partial M} dM - \lambda dE_{\text{rad}} - \lambda dM = 0, \qquad \text{or } \lambda = \frac{\partial S_{\text{rad}}}{\partial E_{\text{rad}}} = \frac{\partial S_{\text{BH}}}{\partial M}. \qquad (2.3)$$

This is just the statement that the black hole is in equilibrium with the radiation: $T_{\text{rad}} = T_{\text{BH}}$. Condition (2) implies that

$$\frac{\partial^2 S_{\text{rad}}}{\partial E_{\text{rad}}^2} + \frac{\partial^2 S_{\text{BH}}}{\partial M^2} < 0, \qquad \text{or } -T_{\text{rad}}^{-2} \frac{\partial T_{\text{rad}}}{\partial E_{\text{rad}}} - T_{\text{BH}}^{-2} \frac{\partial T_{\text{BH}}}{\partial M} < 0. \qquad (2.4)$$

Because the energy radiated by a black body is proportional to the fourth power of temperature, we have

$$\frac{\partial T_{\text{rad}}}{\partial E_{\text{rad}}} = \frac{1}{4} \frac{T_{\text{rad}}}{E_{\text{rad}}}. \qquad (2.5)$$

Hence condition (2) reduces to $E_{\text{rad}} < \frac{1}{4} M$.

2.3 Black hole in a fancy box: the Schwarzschild-AdS solution

We saw that putting a Schwarzschild black hole in a box, albeit completely unphysical, does give a thermodynamically stable solution. Now we shall see a much more natural construction, where the black hole is placed in a spacetime that is no longer asymptotically flat. The anti-de Sitter (AdS) space can be thought to have a potential wall as one approaches the asymptotic infinity. Hence it is a fancier, more natural box. Let us begin with a survey of the AdS space and the Schwarzschild-AdS black hole.

The AdS space is a solution of the Einstein equation with a negative cosmological constant

$$R_{ij} = \Lambda g_{ij}, \qquad \Lambda = -\frac{3}{b^2} < 0. \qquad (2.6)$$

It is simplest to think of it as a submanifold of $\mathbb{R}^{2,n-1}$. Recall that $\mathbb{R}^{2,n-1}$ is equipped with the metric

$$ds^2 = -(dx^0)^2 - (dx^n)^2 + \sum_{i=1}^{n-1} (dx^i)^2. \qquad (2.7)$$

The n-dimensional AdS space, AdS_n, is defined as the set of points

$$\left\{ (x^0, \ldots, x^n) \mid -(x^0)^2 - (x^n)^2 + \sum_{i=1}^{n-1} (x^i)^2 = -b^2 \right\}. \qquad (2.8)$$

Note that by construction the space has an isometry group $\text{SO}(2, n-1)$, analogous to the $\text{SO}(1, n)$ Lorentz group associated with the Minkowski space. This fact will be important

when we establish the gauge/gravity duality where we show that it is the same as the conformal group of the gauge theory.

We begin by finding four sets of coordinates commonly used in describing the AdS space. First consider

$$x_0 = b\cosh\rho\cos\tau, \qquad x_n = b\cosh\rho\sin\tau, \qquad x_i = R\sinh\rho\,\Omega_i, \tag{2.9}$$

where $\tau \in [0, 2\pi)$, $\rho \geq 0$, $i = 1, \ldots, n-1$ and $\sum_i \Omega_i^2 = 1$. Then the metric (2.7) becomes

$$ds^2 = b^2(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_{n-2}^2), \tag{2.10}$$

where $d\Omega_{n-2}^2$ is the metric on S^{n-2}. (τ, ρ, Ω_i) are called the global coordinates because they cover the entire space. Note that AdS_n has topology $S^1 \times \mathbb{R}^{n-1}$ because $ds^2 \approx b^2\left(-d\tau^2 + d\rho^2 + \rho^2 d\Omega_{n-2}^2\right)$ near $\rho = 0$. There is a small caveat that the spacetime contains a closed timelike geodesic represented by the circle S^1. We restore causality by going to the universal cover and unwrap the circle so that $-\infty < \tau < \infty$.

To better see the topology of the spacetime, we conformally compactify AdS by setting $\tan\theta = \cosh\rho$. The metric (2.7) becomes

$$ds^2 = \frac{b^2}{\cos\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\,d\Omega_{n-2}^2\right). \tag{2.11}$$

Note that this metric is conformal to the conformally compactified Minkowski space, i.e., the Einstein static universe $d\widetilde{s}^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\,d\Omega_{n-2}^2$. However, θ takes values between 0 and $\pi/2$, as opposed to between 0 and π for the Minkowski space. We say that the AdS spacetime is conformal to one half of the Einstein static universe. The boundary of the conformally compactified AdS is at $\theta = \pi/2$, where $d\widetilde{s}^2 = -d\tau^2 + d\Omega_{n-2}^2$, which has topology $\mathbb{R} \times S^{n-2}$. The fact that the conformal boundary of AdS_n is identical to the conformally compactified $(n-1)$-dimensional Minkowski spacetime is at the heart of the AdS/CFT correspondence. In section 4 we will give a precise formulation of the correspondence between string theory on AdS_n and gauge theory on its conformal boundary. In section 5 we will study one realization of the correspondence by matching black hole solutions on $AdS_5 \times S^5$ with fluid solutions on its conformal boundary $\mathbb{R} \times S^3$.

Another set of coordinates that will be useful in establishing the gauge/gravity duality are the Poincaré coordinates $(t, \widetilde{x}_i, z)$, $i = 1, \ldots n-1$, defined by

$$x^0 = \frac{1}{2z}\left(z^2 + b^2 + \sum_i(\widetilde{x}^i)^2 - t^2\right), \qquad x^i = \frac{R\widetilde{x}^i}{z}, \tag{2.12}$$

$$x^{n-1} = \frac{1}{2z}\left(z^2 - b^2 + \sum_i(\widetilde{x}^i)^2 - t^2\right), \qquad x^n = \frac{bt}{z}.$$

The metric (2.7) becomes

$$ds^2 = \frac{b^2}{z^2}\left(-dt^2 + \sum_i(d\widetilde{x}^i)^2 + dz^2\right). \tag{2.13}$$

For discussion of Schwarzschild-AdS black holes, it is most useful to work in static coordinates (t, r, θ, ϕ), where $t = b\tau$, $r = b\sinh\rho$. We will specialize in AdS_4 as it captures all the essential thermodynamics and the behavior is similar for higher dimensions. The AdS metric becomes

$$ds^2 = -\left(1 + \frac{r^2}{b^2}\right)dt^2 + \left(1 + \frac{r^2}{b^2}\right)^{-1}dr^2 + r^2 d\Omega^2. \qquad (2.14)$$

Note that the AdS metric is not asymptotically flat. Any metric that asymptotically approaches equation (2.14) is called asymptotically AdS.

One of the important features of the AdS spacetime that motivated its study was that it behaves much like a box – the space has finite energy and has an infinite potential wall at asymptotic infinity. The Tolman law states that the energy measured at asymptotic infinity is red-shifted from the locally measured energy. To see this, consider a particle with four momentum $P^\mu = (-E, p^1, p^2, p^3)$. A static observer at infinity has four velocity $U^\mu = \frac{k^\mu}{\sqrt{-k^2}}$, where $k = \frac{\partial}{\partial t}$. The energy measured by the local observer is

$$E = -g_{\mu\nu}U^\mu P^\nu = \frac{E_\infty}{\sqrt{-k^2}} = \frac{E_\infty}{\sqrt{-g_{00}}}. \qquad (2.15)$$

So E is red-shifted by a factor $\sqrt{-g_{00}}$ from E_∞. For the AdS metric, $-g_{00} = 1 + \frac{r^2}{b^2}$. Thus E is red-shifted to 0 at $r \to \infty$. Since temperature scales as energy, T is red-shifted by the same factor. This reflects the fact that the AdS space has an infinite potential wall at asymptotic infinity.

One might wonder what a black hole analogous to the Schwarzschild black hole might look like in the AdS space. By inspection of equations (2.2) and (2.14), one may be tempted to write down the metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{b^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{r^2}{b^2}\right)^{-1}dr^2 + r^2 d\Omega^2. \qquad (2.16)$$

This metric, known as the Schwarzschild-AdS metric, looks like the Schwarzschild black hole for small r and approaches the AdS space for large r. In fact, the metric satisfies the vacuum Einstein equation with a negative cosmological constant. Moreover, the Schwarzschild-AdS black hole is the unique static, spherically symmetric solution. The black hole has an event horizon at $r = r_+$, the largest root of $V(r) = 1 - \frac{2M}{r} + \frac{r^2}{b^2}$. One may find its temperature by calculating the surface gravity. But a more direct way is to use the Euclidean methods. Perform a Wick rotation $\tau = it$ and look at the metric near the horizon. Let $r = r_+ + \rho^2$, $\quad \rho \ll 1$. To leading order in ρ,

$$V(r) = \frac{b^2\rho^2 + 3r_+^2\rho^2 + 3r_+\rho^4 + \rho^6}{b^2(r_+ + \rho^2)} \approx \frac{b^2 + 3r_+^2}{b^2 r_+}\rho^2. \qquad (2.17)$$

The metric (2.16) looks like

$$ds^2 = \frac{4b^2 r_+}{b^2 + 3r_+^2}\left[\left(\frac{b^2 + 3r_+^2}{2b^2 r_+}\right)^2 \rho^2 d\tau^2 + d\rho^2\right] + r_+^2 d\Omega^2. \qquad (2.18)$$

Note that the term in square brackets resemble the polar coordinates: $ds^2 = r^2 d\theta^2 + dr^2$. If θ has period 2π, then it is the metric of the flat plane, whereas any other value will result in a conical singularity at $r = r_+$. To avoid a conical singularity in equation (2.18), τ needs to be periodic with period $\beta_0 = \frac{4\pi b^2 r_+}{b^2 + 3r_+^2}$. As we shall see soon, it is a general fact that the temperature of the black hole is the inverse of its period, i.e.,

$$T_{\text{BH}} = \beta_0^{-1} = \frac{b^2 + 3r_+^2}{4\pi b^2 r_+}. \tag{2.19}$$
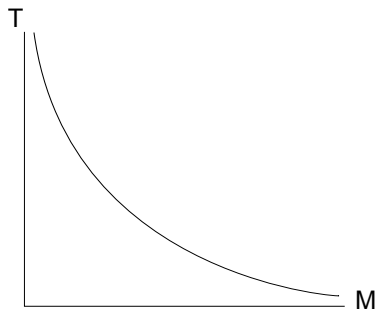
Figure 1: Temperature of a Schwarzschild black hole, plotted against its mass.
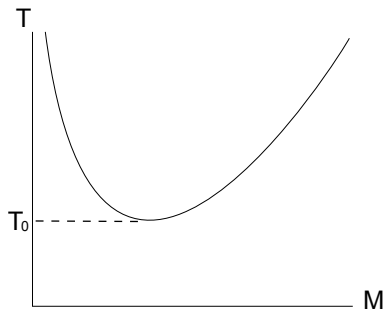
Figure 2: Temperature of a Schwarzschild-AdS black hole, plotted against its mass.

If we plot the temperature against the mass of the black hole (figures 1 and 2), we see that unlike the asymptotically flat case, the Schwarzschild-AdS black hole temperature no longer decreases monotonically with its mass. It attains a minimum at $r_0 = \frac{b}{\sqrt{3}}$, $T_{\text{H}} \equiv T_0 = \frac{\sqrt{3}}{2} \frac{1}{\pi b}$. For $T < T_0$, black holes cannot exist and the space is filled with pure radiation. At any $T > T_0$, there are two black hole solutions at equilibrium. The smaller black hole, represented by the branch with $r < r_0$, has negative specific heat and is thermodynamically unstable. The larger black hole, represented by the branch with $r > r_0$, has positive specific heat and is thermodynamically stable.

3. The Hawking-Page phase transition

In this section, we shall see an even more interesting feature of the Schwarzschild-AdS solution – it exhibits a first-order phase transition. Before jumping into the discussion, we will take a detour to introduce the path integral formalism in quantum gravity. Along the way, we shall define the partition function that can be used to derive various thermodynamic variables, which will prove to be very useful in the remainder of the essay.

3.1 Path integrals in quantum gravity

Let us review the path integral formalism in quantum mechanics. In quantum field theory, the amplitude for a field to propagate from an initial field configuration to a final field configuration is given by the weighed sum over all field configurations, where the weights are determined by the action of the field: $Z = \int \mathcal{D}\phi \, e^{iI[\phi]}$. One needs to be careful about the convergence associated with the oscillatory integral. To circumvent this issue, we perform

a Wick rotation $\tau = it$, thereby rotating the contour of integration of time by $\frac{\pi}{2}$ counter-clockwise. The Wick rotated path integral becomes $Z = \int \mathcal{D}\phi \, e^{-I[\phi]}$, which converges. Now the amplitude can also be written as $Z = \langle \phi_2 | e^{-iH(t_2 - t_1)} | \phi_1 \rangle$. Setting $\phi_1 = \phi_2 = \phi$ and $i(t_2 - t_1) = \tau_2 - \tau_1 = \beta$, we get $Z = \langle \phi | e^{-\beta H} | \phi \rangle$. If we integrate over all ϕ, then

$$Z = \text{Tr} \exp(-\beta H), \tag{3.1}$$

where now the path integral is taken over all fields that are periodic in imaginary time with period β. This is simply the partition function of ϕ at temperature $T = \beta^{-1}$. Hence we are justified in identifying the inverse period of the imaginary time with the temperature of the solution.

Recall that the partition function is defined as $Z = \sum_n e^{-\beta E_n}$. It plays a central role in statistical mechanics because many thermodynamic quantities of interest can be derived from it. For example, the probability that the system is in the n-th energy state is $p_n = \frac{1}{Z} e^{-\beta E_n}$. The expectation energy and the entropy is given by

$$\langle E \rangle = \sum_n E_n e^{-\beta E_n} = -\frac{\partial}{\partial \beta} Z, \qquad S = p_n \log p_n = \beta \langle E \rangle + \log Z. \tag{3.2}$$

In section 5, we will also consider the grand canonical partition function

$$Z_{\text{gc}} = \text{Tr} \exp\left(-\beta(H - \mu_i C_i)\right), \tag{3.3}$$

where μ_i is the chemical potential associated the conserve charge C_i.

The dominant contribution to the path integral comes from the minima of the action where $\delta I = 0$. Note that this is also the classical limit ($\hbar \to 0$) of the path integral that one can derive using the method of stationary phase. Thus we can approximate the integral by $Z \approx e^{-I}$, or $\log Z = -I$. It is useful to define the free energy $F = -T \log Z = TI$. When we are faced with multiple minima, the one with the lowest free energy will dominate the partition function. As we shall see soon, this is the crux of Hawking and Page's argument comparing the likeliness of Schwarzschild-AdS black holes over pure radiation.

In curved spacetime, the path integral is generalized to sum over variations in geometry as well as fields. Consider a field ϕ propagating from an initial configuration ϕ_1 on a hypersurface S_1 with metric g_1 to a final configuration ϕ_2 on a hypersurface S_2 with metric g_2. The path integral is now given by

$$Z = \int \mathcal{D}[g, \phi] e^{iI[\phi]}, \tag{3.4}$$

where the path integral is taken over all fields ϕ and all metrics g with the given initial and final configurations. As in the case of flat spacetime, we perform a Wick rotation to get a convergent path integral. The Euclidean path integral is

$$Z = \int \mathcal{D}[g, \phi] e^{-I[\phi]}. \tag{3.5}$$

Here I is the usual Einstein-Hilbert action with a cosmological constant

$$I = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \, (R - 2\Lambda). \tag{3.6}$$

Because we are dealing with manifolds with boundary, we also need to consider the Hawking-Gibbons-York boundary term, though it will not be important for our discussion below.

3.2 The Hawking-Page phase transition

In their investigations into the thermodynamics of Schwarzschild-AdS black holes, Hawking and Page found that although black holes can be in stable thermal equilibrium with radiation, they are not the preferred state below a certain critical temperature. They discovered that once this temperature is reached, a first-order phase transition occurs and black holes become the preferred state. The idea is to compare the free energy, or equivalently, the actions of the Schwarzschild-AdS metric and the AdS metric. We will find that it is a function of temperature. The phase transition happens when the two actions become equal, and the free energy of one becomes lower than another.

Solutions of equation (2.6) satisfy $R = 4\Lambda$. Hence the action (3.6) reduces to a volume integral $I = \frac{\Lambda}{8\pi} \int d^4x \sqrt{-g}$. It turns out that we need not consider the boundary term because it will cancel out when we take the action difference at asymptotic infinity. The integral is actually infinite if we integrate over all space. We need to regularize the integral by integrating the radial variable only up to some cutoff value $r = K$. For the AdS metric,

$$I_1 = \frac{\Lambda}{8\pi} \int_0^{\beta_1} dt \int_0^K r^2 \, dr \int_{S^2} d\Omega^2 = \frac{\Lambda}{6} \beta_1 K^3. \tag{3.7}$$

For the Schwarzschild-AdS metric,

$$I_0 = \frac{\Lambda}{8\pi} \int_0^{\beta_0} dt \int_{r_+}^K r^2 \, dr \int_{S^2} d\Omega^2 = \frac{\Lambda}{6} \beta_0 (K^3 - r_+^3). \tag{3.8}$$

While β_0 is given by equation (2.19), β_1 can in principle be given any value since the AdS metric need not be periodic in time. We require that the two metrics match at the $r = K$ hypersurface. At $r = K$, the AdS (2.14) and the Schwarzschild-AdS (2.16) metrics read

$$ds^2 = -\left(1 + \frac{K^2}{b^2}\right) dt^2 + K^2 d\Omega^2, \qquad ds^2 = -\left(1 - \frac{2M}{K} + \frac{K^2}{b^2}\right) dt^2 + K^2 d\Omega^2, \tag{3.9}$$

respectively. The requirement that the two metrics match implies, in particular, that the time coordinates have the same period. This gives a condition for β_1:

$$\beta_1 \sqrt{1 + \frac{K^2}{b^2}} = \beta_0 \sqrt{1 - \frac{2M}{K} + \frac{K^2}{b^2}}. \tag{3.10}$$

Thus

$$I = I_0 - I_1 = \frac{\Lambda}{6} \beta_0 \left((K^3 - r_+^3) - \frac{\beta_1}{\beta_0} K^3 \right) = -\frac{\beta_0}{2b^2} \left(K^3 - r_+^3 - \sqrt{1 - \frac{2Mb^2}{b^2 K + K^3}} K^3 \right) \tag{3.11}$$

For large K,

$$I \approx -\frac{\beta_0}{2b^2} \left[K^3 - r_+^3 - K^3 \left(1 - \frac{Mb^2}{b^2 K + K^3} \right) \right] \approx \frac{\beta_0}{2b^2} \left(r_+^3 - Mb^2 \right) = \frac{\pi r_+^2 (b^2 - r_+^2)}{b^2 + 3r_+^2} \tag{3.12}$$

We can compute its expected energy and entropy using equation (3.2):

$$\langle E \rangle = \frac{\partial I}{\partial \beta_0} = M, \qquad S = \beta_0 \langle E \rangle - I = \pi r_+^2 = S_{\text{BH}}, \qquad (3.13)$$

as is expected for a black hole of mass M and radius r_+.

Note that $I = 0$ at $r_+ = b$, $T_{\text{H}} \equiv T_1 = \frac{1}{\pi b}$, $I > 0$ for $r_+ < b$, and $I < 0$ for $r_+ > b$. Thus for $T < T_1$, thermal radiation will dominate the partition, whereas for $T > T_1$, black holes will dominate the partition function. A thermal phase transition occurs at $T = T_1$ in which the preferred state switches to large black holes instead of radiation. This is called the Hawking-Page phase transition. There is another critical temperature $T_2 > T_1$ for which all radiation will inevitably collapse to form a black hole. The phase diagram is given in figure 3.
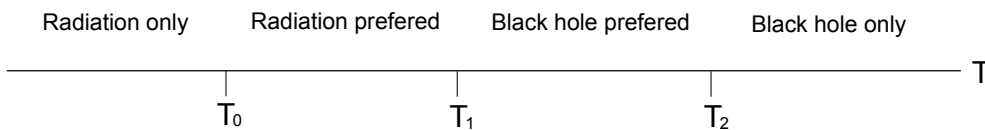
Figure 3: Phase diagram for the Hawking-Page phase transition (not drawn to scale).

For $T < T_0$, black holes cannot exist and the space is dominated by thermal radiation. For $T_0 < T < T_1$, black holes can exist in stable equilibrium with thermal radiation, but is less favored than pure radiation. A black hole can reduce its free energy by evaporating into pure radiation with a tunneling probability $\Gamma \propto e^{-B_0}$ where B_0 is the action difference between the large and the small black hole. For $T_1 < T_2$, black hole is favored over pure radiation. Pure radiation can now reduce its free energy by turning into a black hole with a tunneling probability $\Gamma \propto e^{-B_1}$ where B_1 is the action difference between the small black hole and pure AdS. For $T > T_2$, all radiation must collapse and the universe is dominated by black holes.

We have seen that black holes in AdS space have rich thermodynamics. What is probably more surprising is the discovery that they could be used to understand thermal properties of gauge theories. The bridge connecting the two seemingly different subject is the AdS/CFT correspondence, which we now turn to.

4. The AdS/CFT correspondence

Maldacena's celebrated AdS/CFT conjecture, in its broadest terms, relates string theory on AdS to conformal field theory (CFT) on its boundary. It is a realization of the holographic principle, first introduced in section 2. Its first manifestation is a duality between $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory and type IIB supergravity (SUGRA) theory on $AdS_5 \times S^5$ in the large N limit. In this section we will introduce both sides of the correspondence, formulate the conjecture and give several evidences including symmetry matching, entropy counting and Witten's interpretation of the Hawking-Page phase transition. As this is still a very active area of research where many new territories are being explored by this duality, this discussion will only touch the basics of the correspondence.

4.1 D-branes, p-branes, Dp-branes

We will see how the correspondence between $\mathcal{N} = 4$ SYM and type IIB SUGRA emerges by looking at the near horizon limit of extremal 3-branes. D-branes and p-branes are quite different objects, but are related in an essential way.

p-branes are solutions of supergravity. They are given by the metric

$$ds^2 = H_p^{-1/2}(r)\left(-f(r)dt^2 + \sum_{i=1}^{p}(dx^i)^2\right) + H_p^{1/2}(r)\left(f^{-1}(r)dr^2 + r^2 d\Omega^2_{n-p-2}\right), \quad (4.1)$$

where $f(r) = 1 - \frac{r_0^4}{r^4}$ and $H_p(r) = 1 + \left(\frac{r_p}{r}\right)^{n-p-3}$. The metric has a horizon at $r = r_0$, and extends in the other p dimensions. Thus the solutions describe black holes that extend in p spatial dimensions. We will be mainly interested in extremal $(r_0 \to 0)$ 3-branes in $n = 10$ dimensions. The metric is

$$ds^2 = H^{-1/2}(r)\left(-dt^2 + \sum_{i=1}^{3}(dx^i)^2\right) + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2), \qquad H(r) = 1 + \frac{b^4}{r^4}. \quad (4.2)$$

D-branes, on the other hand, are surfaces on which strings can end with additional boundary conditions. Recall that a string vibrating in spacetime is a map from the parameter space to the bulk spacetime $X : [0,\infty) \times [0,2\pi] \to \mathbb{R}^{1,n-1}$ parametrized by (τ,σ). As the string vibrates, it traces out a 2 dimensional surface in spacetime called the worldsheet. For example, a closed string whose worldsheet satisfies $X(0,\tau) = X(2\pi,\tau)$ traces out a cylinder. The action of the string is given by the Polyakov area functional

$$S = \frac{1}{4\pi l_s^2} \int d\tau d\sigma \sqrt{-g}\, g_{ab}\, \partial^a X^\mu \partial^b X_\mu, \qquad (4.3)$$

where l_s is the length scale of the string. The classical string motion is obtained by minimizing the area of the worldsheet.

For open strings, it is natural to consider the boundary conditions satisfied by their endpoints. These are the Neumann boundary condition $\partial_\mu X_\nu = 0$, and the Dirichlet boundary condition $\delta X_\nu = 0$. In other words, the Dirichlet boundary condition fixes the endpoints of a string. If the string satisfies the Neumann boundary condition for $X^0, \ldots, X^p$ and the Dirichlet boundary condition for $X^{p+1}, \ldots X^n$, then the endpoints of the string are confined to move in a $p + 1$ dimensional hypersurface. This surface is called a Dp-brane, where p indicates the spatial dimension of the brane. A D0-brane is a particle, a D1-brane is a string, a D2-brane is a 2-surface, and so on.

Dp-branes are p-dimensional hypersurfaces on which strings can end that satisfies the Dirichlet boundary condition. Note that using the same letter p for the two different objects p-branes and Dp-branes can cause confusion. However, Polchinski showed that D-branes are the same as extremal p-branes. So we will allow this bit of abuse of notation.

One of the most important properties of D-branes is that they give rise to gauge theories. Consider N incident extremal D3-branes. Each endpoint of an open string can be attached to any of the N branes, so is labeled by the Chan-Paton factor $|i\rangle$. The open string states are described by their two endpoints $|i\rangle, |j\rangle$, so are associated with an $N \times N$

matrix. It is known that the matrix is $U(N)$ and that the open string states live in the adjoint representation of $U(N)$. This corresponds to N^2 massless particle states. In the low energy limit ($l_s \to 0$), this gives rise to a SU(N) Yang-Mills theory.

In 4 dimensions with $\mathcal{N} = 4$ supersymmetry, the theory is $\mathcal{N} = 4$ SU(N) super Yang-Mills. It is a gauge theory with one gauge field, four Weyl fermions, and six real scalars, all in the adjoint representation of the SU(N). Furthermore, it has a vanishing beta function and is a CFT. The theory is controlled by a parameter called the 't Hooft coupling $\lambda \equiv g_{\mathrm{YM}}^2 N$. Soon we shall see why it is so important. On the other side of the duality is type IIB string theory. It contains massless fields such as the graviton and the dilaton Φ and an infinite number of massive string excitations. It is controlled by two parameters: the string length l_s and the string coupling g_s. In the low energy limit $l_s \to 0$, the theory reduces to type IIB SUGRA.

The string coupling g_s is related to the Yang-Mills coupling g_{YM} via

$$4\pi g_s = g_{\mathrm{YM}}^2, \tag{4.4}$$

reflecting the fact that the gauge fields are open string modes on the Dp-branes and two open strings can collide and form a closed string (hence the square).

4.2 The decoupling limit

We mentioned earlier that D-branes are extremal p-branes. This fact lies at the heart of the gauge/gravity duality. The duality was first discovered by taking the large N/ low energy limit of both theories. Now that we know what the players are, let us see exactly how they are dual to each other.

Consider first the gauge theory side. When two open strings on a D-brane collide, they form a closed string. The closed string has no endpoints so is no longer bound to the D-brane. It can now vibrate in the bulk. This implies that string theory with D3 branes can be described by the following action terms:

- S_{bulk}, the action of closed strings in the bulk,

- S_{brane}, the action of open strings attached to D3-branes,

- S_{int}, the action of interactions between the open and closed strings.

In the low energy limit ($l_s \to 0$), $S_{\mathrm{bulk}} \to S_{\mathrm{SUGRA}}$, $S_{\mathrm{brane}} \to S_{\mathrm{SYM}}$. Moreover, since $S_{\mathrm{int}} \approx g_s l_s^4$, $S_{\mathrm{int}} \to 0$. The open and closed string modes decouple. Our theory reduces to

- Free SUGRA in the bulk,

- $\mathcal{N} = 4$ SU(N) SYM on the D3-branes.

Next we use the fact that D-branes are equivalent to extremal p-branes and move to another view point. When p-branes are seen as generating the background field, there are two low energy modes. Far away from the horizon ($r \to \infty$), the metric (4.2) is asymptotically flat. At low energy we have free type IIB SUGRA. Near the horizon $r \to 0$,
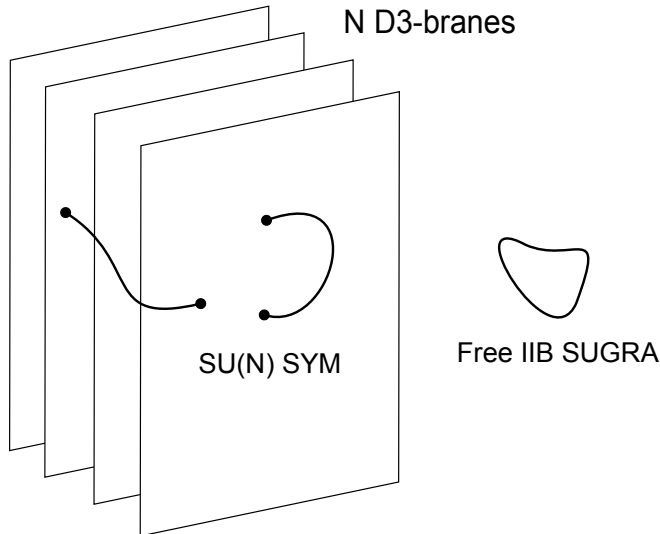
Figure 4: In the low energy limit, open string modes on the D3-brane give rise to $\mathcal{N} = 4\,\mathrm{SU}(N)$ SYM and closed string modes on the bulk give rise to free type IIB SUGRA.

$-g_{00} = \frac{r^2}{b^2}$. By the Tolman law (2.15), the energy measured at infinity is red-shifted to 0 as $r \to 0$. Thus we say that taking the near horizon limit is equivalent to taking the low energy limit. The SUGRA metric (4.2) reduces to

$$ds^2 = \frac{r^2}{b^2}(-dt^2 + dx^2) + \frac{b^2}{r^2}(dr^2 + r^2 d\Omega_5^2). \tag{4.5}$$

Upon the change of coordinates $z = \frac{b^2}{r}$, the metric becomes

$$ds^2 = \frac{b^2}{z^2}(-dt^2 + dx^2 + dz^2) + b^2 d\Omega_5^2. \tag{4.6}$$

which is simply the metric on $AdS_5 \times S^5$ in Poincaré coordinates (2.13). Thus in the near horizon limit, we have another low energy mode – type IIB SUGRA on AdS.

	Gauge (D-branes)	Gravity (p-branes)
$r \to \infty$	Free IIB SUGRA	Free IIB SUGRA
$r \to 0$	$\mathcal{N} = 4\,\mathrm{SU}(N)$ SYM	IIB SUGRA on $AdS_5 \times S^5$

Table 1: Low energy limit of Dp-branes, seeing from the gauge/gravity perspective.

The low energy modes of the two theories are summarized in Table 1. It is striking that both the gauge and gravity ways of looking at extremal D3-branes in the low energy limit give rise to the same free IIB SUGRA in the bulk. But what about the other pair? Maldacena conjectured that they must agree as well. This is the famous AdS/CFT conjecture:

In the large N limit, $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1,3}$ is dual to type IIB SUGRA on $AdS_5 \times S^5$.

4.3 Exploring the gauge/gravity duality

What is meant by the large N limit? To see its relevance, we equate the gravitational tension of the extremal 3-brane to N times the tension of a single D3-brane. This gives us the relation

$$\frac{b^4 \text{Vol}(S^5)}{4\pi G_{10}} = \frac{N}{\sqrt{8G_{10}}}, \tag{4.7}$$

where $G_{10} = 8\pi^6 g_s^2 l_s^8$ is the ten-dimensional gravitational constant. It follows that

$$\frac{b^4}{l_s^4} = 4\pi g_s N = g_{\text{YM}}^2 N = \lambda. \tag{4.8}$$

As we know that SUGRA is the low energy limit of string theory, we need to assume that the radius of curvature b is large compared to the string length l_s. This implies that $4\pi g_s N = g_{\text{YM}}^2 N \gg 1$, i.e., the 't Hooft coupling must be large. Since $S_{int} \to 0$, we must also have $g_s \to 0$. Thus $N \to \infty$.

On the other hand, for perturbation theory to work in the gauge theory, we must have a small 't Hooft coupling. The AdS/CFT correspondence relates gauge theory, which is valid at the weak-coupling limit $\lambda \ll 1$ to supergravity in $AdS_5 \times S^5$, which is valid at the strong-coupling limit $\lambda \gg 1$. This is the reason why it is also called the AdS/CFT *duality*. This is also the reason why it has become a powerful theoretical tool. One can use the AdS/CFT dictionary to translate hard problems (e.g., strongly-coupled systems) in one theory to easy problems (e.g., weakly-coupled system) in another. A stronger version of the conjecture states that the agreement is true for all N. While it is difficult to prove the full conjecture, there is substantial evidence in its favour.

Now that we have stated the AdS/CFT correspondence, in the remainder of this paper we will look for more specific evidences and explore its consequences.

The symmetry groups of the two sides of the correspondence also agree. The global bosonic symmetries of $\mathcal{N} = 4$ SYM are generated by the conformal group SO(2, 4) and the R-symmetry group SU(4) $\cong$ SO(6), the global symmetry rotating the 6 supercharges. Recall that the isometry group for AdS_5 is SO(2, 4) and for S^5 it is SO(6). Therefore we find that the bosonic symmetries of $\mathcal{N} = 4$ SYM is identical to the killing symmetries of the $AdS_5 \times S^5$ background for type IIB supergravity.

The three independent parameters of SO(6) fall into irreducible representations labeled by three numbers (r_1, r_2, r_3), called the weight vectors of SO(6). The Cartan subalgebra of $\mathfrak{so}(6)$ has three commuting elements H_i satisfying $H_i|r_i\rangle = |r_i\rangle$. The generators H_1, H_2, H_3 generate the $U(1)^3$ subgroup of SO(6). In section 5 we shall see several examples of rotating $\mathcal{N} = 4$ SYM plasma with different SO(6) charges, and their gravity duals.

Another evidence for the correspondence comes from counting the entropy. Consider the near-extremal 3-brane metric in Euclidean space

$$ds^2 = \frac{r^2}{b^2}\left(f(r)d\tau^2 + \sum_{i=1}^{3}(dx^i)^2\right) + \frac{b^2}{r^2 f(r)}dr^2 + b^2 d\Omega_5^2. \tag{4.9}$$

Near the horizon $r \approx r_0$, we have

$$\frac{r^2}{b^2}f(r) = \frac{r^4 - r_0^4}{b^2 r^2} = \frac{(r^2 + r_0^2)(r^2 - r_0^2)}{b^2 r^2} = \frac{2(r + r_0)(r - r_0)}{b^2} = \frac{4r_0(r - r_0)}{b^2}. \tag{4.10}$$

Let $r = r_0 + \rho^2$ again. The (τ, r) part of the metric looks like

$$ds^2 = \frac{4r_0}{b^2}(r - r_0)d\tau^2 + \frac{b^2}{4r_0(r - r_0)}dr^2 = \frac{b^2}{r_0}\left(\frac{4r_0^2}{b^4}\rho^2 d\tau^2 + d\rho^2\right). \qquad (4.11)$$

We see that the term in the brackets has the polar coordinate form $ds^2 = r^2 d\theta^2 + dr^2$ again. To avoid a conical singularity at $r = r_0$, we need τ to be periodic with period $\beta = \pi b^2/r_0$. Thus we deduce the Hawking temperature $T_{\mathrm{H}} = r_0/(\pi b^2)$. The eight-dimensional "area" of the black 3-brane is

$$A = \left(\frac{r_0}{b}\right)^3 V_3 b^5 \mathrm{Vol}(S^5) = \pi^6 b^8 V_3 T_{\mathrm{H}}^3, \qquad (4.12)$$

where $(r_0/b)^3 V_3$ is the volume of the $D3$ brane and $b^5 \mathrm{Vol}(S^5)$ is the volume of the 5-sphere of radius b. We wish to compute the entropy of 3-branes. It is given by the Bekenstein-Hawking formula $S_{\mathrm{BH}} = A/4G_{10}$. Using equation (4.8), we can write the gravitational constant as $G_{10} = b^8 \pi^4/2N^2$. So

$$S_{\mathrm{BH}} = \frac{1}{2}\pi^2 N^2 V_3 T_{\mathrm{H}}^3, \qquad (4.13)$$

whereas the entropy of its CFT dual can be calculated using statistical mechanics to be

$$S_{\mathrm{SYM}} = \frac{2}{3}\pi^2 N^2 V_3 T^3. \qquad (4.14)$$

They agree up to a factor of $\frac{3}{4}$. It is remarkable that the entropy calculated from the gravity theory scales exactly like its CFT dual in temperature. The SYM entropy counts the states of gluons, hence the N^2 scaling indicates the presence of N^2 unconfined degrees of freedom. The factor of $\frac{3}{4}$ difference is possibly due to the duality of the two theories. As we have seen, the gravity theory is relevant in the $\lambda \to \infty$ regime whereas the gauge theory is relevant in the $\lambda \to 0$ regime. One might conjecture that the true form of the entropy is given by

$$S = \frac{2}{3}f(\lambda)\pi^2 N^2 V_3 T^3, \qquad (4.15)$$

where $\lim_{\lambda \to 0} f(\lambda) = 1$ and $\lim_{\lambda \to \infty} f(\lambda) = \frac{3}{4}$. $f(\lambda)$ interpolates between the weak coupling and the strong coupling limits.

4.4 The confinement-deconfinement transition

In the Standard Model, Quantum Chromodynamics (QCD) is formulated as a Yang-Mills theory admitting an SU(3) gauge group. It is a well-known feature of QCD that the coupling constant becomes small at small distances and large at large distances. A great amount of energy is required to pull quarks apart whereas when quarks are close together they behave like free particles. This is the idea of asymptotic freedom. Due to this effect, quarks are normally bound together in pairs and triples to form mesons and baryons. Quarks in hadronic states are said to be in confinement. In contrast, at very high temperature, quarks are no longer bound by gluons and enter into a state called deconfinement. They mix freely with other quarks and gluons to form a quark-gluon plasma (QGP). QGP

plays an important role in Big Bang cosmology. It is believed that the very early universe consisted primarily of QGP. As the universe expanded and cooled, a first order phase transition caused the QGP to condense into hadrons. The critical temperature at which hadronization takes place is around 200 MeV. Witten suggested that this thermal phase transition is dual to the Hawking-Page phase transition. We will show how this is done using the correspondence.

The ADS/CFT correspondence has been codified into a "dictionary" that tells the reader how to translate a gravitational calculation to its CFT dual, and vice versa. One of its entries says that the CFT partition function equals the SUGRA partition function, i.e.,

$$Z_{\text{CFT}} = Z_{\text{SUGRA}}. \tag{4.16}$$

Let X be the bulk manifold, and M its boundary. In the approximation of classical SUGRA,

$$Z_{\text{CFT}}(M) = e^{-I_{\text{SUGRA}}(X)}, \tag{4.17}$$

In the large N limit, $I_{\text{SUGRA}}(X) = N^p F(X)$ for some $p > 0$. For $\mathcal{N} = 4$ SYM, $p = 2$. Like in the path integral context, we sum over all geometries X_i with the same boundary manifold M:

$$Z_{CFT}(M) = \sum_i \exp(-N^2 F(X_i)). \tag{4.18}$$

In the large N limit, the partition function will be dominated by the X_i with the smallest $F(X_i)$. Thus we can think of $F(X_i)$ as the free energy of AdS/Schwarzschild-AdS in the gravity setting. Let $F(M) = -\log Z_{\text{CFT}}(M)$ be the CFT free energy. Then

$$F(X_i) = \lim_{N \to \infty} \frac{F(M)}{N^2}. \tag{4.19}$$

As we have seen before, the basic idea of the phase transition is the competition between two (or more) solutions to minimize the free energy/action. For a range of some controlling parameter (e.g., temperature), the solution with a lower action will dominate the path integral. When at a critical parameter value another solution attains a lower action, a phase transition occurs and the new solution will dominate. At low temperature, the color singlet hadrons contribute to free energy of order $N^0 = 1$, reflecting the fact that they are in a confined phase. At high temperature, however, the theory consists of free quarks and gluons in the deconfinement phase. As equation (4.14) shows, it will have N^2 degrees of freedom.

This thermal phase transition in gauge theory resembles the Hawking-Page phase transition we have discussed in section 3. In fact, we can identify the confined phase with pure AdS space and the deconfined phase with the Schwarzschild-AdS black holes. Another motivation for identifying the deconfinement phase with Schwarzschild-AdS black holes is that their specific heat is positive, which is required for any gauge theory.

There is a small caveat here that Witten's work does not strictly follow the AdS/CFT correspondence. The comparison works for $SU(N)$ gauge theory because it exhibits confinement. On the other hand, the $\mathcal{N} = 4$ SYM theory does not have confinement. In fact,

the interpretation of Hawking-Page phase transition as confinement-deconfinement transition is an evidence for a related correspondence – the AdS/QCD correspondence. This is another area of recent interest as it might lead to observable predictions. There have been recent experimental work to create QGP in the laboratory by colliding heavy atomic nuclei, notably at the Relativistic Heavy Ion Collider (RHIC). There is some hope that the AdS/CFT correspondence can shed light into understanding the QGP. For example, one of its successes has been at calculating the shear viscosity of strongly coupled $\mathcal{N} = 4$ SYM plasma from D3-branes [18].

5. The fluid/gravity correspondence

A very active area of research has been to explore beyond the $\mathcal{N} = 4$ SYM/type IIB SUGRA correspondence to establish lesser understood correspondences, taking the original correspondence as a clue. The confinement-deconfinement transition, among other things, are thought as evidence for the AdS/QCD correspondence. Another area that has attracted much attention is the fluid/gravity correspondence. By computing the shear viscosity to entropy density ratio for gravity theories dual to gauge theory, Kovtun et al. [15] conjectured that there is a lower bound for all fluids, including those whose dual gravity theories are not known. Another line of work, by Bhattacharyya et al. [2] examined stress-energy tensors for large rotating AdS black holes and found exact correspondence to equilibrium conformal fluids on the boundary.

Our discussion will follow that of Bhattacharyya et al. We begin with a quick review of conformal fluid dynamics and calculate the thermodynamic quantities, first for fluids on S^3 and then generalize this for S^{n-1}. The focus to fluids on S^{n-1} is important because it is the conformal boundary of AdS_{n+1}, as discussed in section 2. We proceed to establish correspondence with AdS black holes. For uncharged, rotating fluids/black holes, we find that their thermodynamics and stress-energy tensor match exactly in all dimensions upon taking the large event horizon limit and identifying some parameters. For charged, rotating fluids/black holes, we compare several known cases for $\mathcal{N} = 4$ Yang Mills plasma on S^3 with AdS_5 and again find an exact matching of thermodynamic parameters. In the $AdS_5 \times S^5$ case, the correspondence is summarized in table 2.

Rotating $\mathcal{N} = 4$ SYM plasma	Black holes in $AdS_5 \times S^5$
Equal SO(6) charges	Equal R-charges
Equal rotations	Equal rotations
Two equal nonzero SO(6) charges	Two equal large R-charges, one small R-charge
One nonzero SO(6) charge	One nonzero R-charge

Table 2: Correspondence between rotating fluids in gauge theory and black holes in the dual gravity theory.

5.1 Conformal fluid dynamics

The basic equations of fluid dynamics are the conservation of the stress-energy tensor and

the charge currents.

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = 0. \tag{5.1}$$

A conformal fluid is characterized by conformal invariance and extensivity. Recall that a conformal transformation is a diffeomorphism $M \to M$ that preserves the metric up to a conformal factor: $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ for some non-vanishing function Ω on M. Conformal invariance implies that the stress-energy tensor is trace-free. To see this, recall that the stress-energy tensor is defined by $T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\partial S}{\partial g_{\mu\nu}}$. Under a conformal transformation $g_{\mu\nu} \mapsto (1 + \epsilon) g_{\mu\nu}$ such that $\delta g_{\mu\nu} = \epsilon g_{\mu\nu}$, the action must be invariant:

$$\delta S = \frac{\partial S}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = -\frac{1}{2} \epsilon \sqrt{-g} \, T^\mu_\mu = 0. \tag{5.2}$$

Hence $T^\mu_\mu = 0$. If we assume that the fluid is a perfect fluid whose stress-energy tensor is given by $T^{\mu\nu} = (\rho + \mathcal{P}) u^\mu u^\nu + \mathcal{P} g^{\mu\nu}$, then the trace-free condition implies that

$$0 = (\rho + \mathcal{P}) u^\mu u_\mu + \mathcal{P} \delta^\mu_\mu = -(\rho + \mathcal{P}) + n\mathcal{P}, \qquad \text{or } \mathcal{P} = \frac{\rho}{n-1}. \tag{5.3}$$

Define the thermodynamic potential

$$\Phi = \mathcal{E} - \mathcal{T}\mathcal{S} - \mu_i \mathcal{R}_i, \tag{5.4}$$

where $\mathcal{E}, \mathcal{T}, \mathcal{S}, \mu_i, \mathcal{R}_i$ are the internal energy, local temperature, local entropy, local chemical potential and local $\mathcal{R}$-charge of the fluid, respectively. We assume that all thermodynamic quantities, including Φ, are functions of $\mathcal{T}$ and μ_i. By the first law of thermodynamics,

$$d\Phi = d\mathcal{E} - \mathcal{S}d\mathcal{T} - \mathcal{T}d\mathcal{S} - \mathcal{R}_i d\mu_i - \mu_i d\mathcal{R}_i = -\mathcal{S}d\mathcal{T} - \mathcal{P}dV - \mathcal{R}_i d\mu_i. \tag{5.5}$$

Recall that a quantity is extensive if it scales with the volume of space. Φ is an extensive quantity, so $\Phi \propto V$. But V has dimension $(\text{length})^{n-1}$ whereas Φ has dimension $(\text{length})^{-1}$. By dimensional analysis, we must have $\Phi \propto V\mathcal{T}^n$ as $\mathcal{T} = \beta^{-1}$ has dimension $(\text{length})^{-1}$. We define the constant of proportionality h such that

$$\Phi = -V\mathcal{T}^n h(\vec{\nu}), \qquad \nu_i = \frac{\mu_i}{\mathcal{T}}, \tag{5.6}$$

where the dependence on $\vec{\nu}$ is dictated by conformal invariance. We can use Φ to compute the fluid density, the $\mathcal{R}$-charge density, and the entropy density

$$\rho = (n-1)\mathcal{P} = -(n-1)\frac{\partial \Phi}{\partial V} = (n-1)\mathcal{T}^n h(\nu),$$

$$r_i = \frac{1}{V}R_i = -\frac{1}{V}\frac{\partial \Phi}{\partial \mu_i} = -\frac{1}{\mathcal{T}V}\frac{\partial \Phi}{\partial \nu_i} = \mathcal{T}^{n-1} h_i,$$

$$s = \frac{1}{V}\mathcal{S} = -\frac{1}{V}\frac{\partial \Phi}{\partial \mathcal{T}} = \mathcal{T}^{n-1}\left(nh + \mathcal{T}\frac{\partial h}{\partial \mathcal{T}}\right), \tag{5.7}$$

$$= \mathcal{T}^{n-1}\left(nh + \mathcal{T}\frac{\partial \nu_i}{\partial \mathcal{T}}h_i\right) = \mathcal{T}^{n-1}\left(nh - \frac{\mu_i}{\mathcal{T}}h_i\right) = \mathcal{T}^{n-1}(nh - \nu_i h_i),$$

where $h_i = \frac{\partial h}{\partial \nu_i}$.

At the zeroth order, the fluid is described by the perfect fluid solution. Its stress-energy tensor, charge and entropy currents are

$$T^{\mu\nu} = (\rho + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}, \qquad J_i^\mu = r_i u^\mu, \qquad J_S^\mu = s u^\mu. \tag{5.8}$$

It is useful to introduce the projection tensor $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. It projects vectors onto the subspace orthogonal to u^μ, e.g., $P^{\mu\nu}u_\nu = 0$. The stress-energy tensor can be re-written as $T^{\mu\nu} = \rho u^\mu u^\nu + \mathcal{P}P^{\mu\nu}$. The entropy current is conserved in a perfect fluid, as the following calculation shows.

$$0 = -u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho u^\mu) + \mathcal{P}\nabla_\mu u^\mu = \mathcal{T}\nabla_\mu(s u^\mu) + \mu_i \nabla_\mu(R_i u^\mu) = \mathcal{T}\nabla_\mu J_S^\mu, \tag{5.9}$$

where we have used $u^\nu u_\nu = -1$ in the second equality, the first law of thermodynamics in the third equality and the conservation of charge currents in the last equality.

This implies that entropy is produced by dissipation, which occurs at the first subleading order. In the first subleading order, the stress-energy tensor, charge and entropy currents are given by

$$T^{\mu\nu} = -\zeta\vartheta P^{\mu\nu} - 2\eta\sigma^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu, \qquad J_i^\mu = q_i^\mu, \qquad J_S^\mu = \frac{q^\mu - \mu_i q_i^\mu}{\mathcal{T}}, \tag{5.10}$$

where

$$\begin{aligned}
a^\mu &= u^\nu \nabla_\nu u^\mu, \\
\vartheta &= \nabla_\mu u^\mu, \\
\sigma^{\mu\nu} &= \frac{1}{2}(P^{\mu\lambda}\nabla_\lambda u^\nu + P^{\nu\lambda}\nabla_\lambda u^\nu) - \frac{1}{n-1}\vartheta P^{\mu\nu}, \\
q^\mu &= -\kappa P^{\mu\nu}(\partial_\nu \mathcal{T} + a_\nu \mathcal{T}), \\
q_i^\mu &= -D_{ij}P^{\mu\nu}\partial_\nu \nu_i,
\end{aligned} \tag{5.11}$$

are the acceleration, expansion, shear, dissipation and diffusion tensors and $\zeta, \eta, \kappa, D_{ij}$ are parameters known as the bulk viscosity, shear viscosity, thermal conductivity and diffusion coefficients, respectively. As expected, dissipation means that the entropy current is no longer conserved. The change in the entropy current is given in [1] as

$$\mathcal{T}\nabla_\mu J_S^\mu = \frac{1}{\kappa \mathcal{T}}q^\mu q_\mu + \mathcal{T}(D^{-1})^{ij}q_i^\mu q_{j\mu} + \zeta\vartheta^2 + 2\eta\,\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{5.12}$$

We assume that all constants are positive. Since $q^\mu, q_i^\mu, \vartheta$ and $\sigma^{\mu\nu}$ are all spacelike, $\mathcal{T}\nabla_\mu J^\mu$ is nonnegative. This ensures that entropy is non-decreasing hence the second law of thermodynamics is obeyed. In equilibrium, $\nabla_\mu J_S^\mu = 0$, so all these variables must vanish. This then implies that the dissipative part of the stress-energy tensor, charge and entropy currents (5.10) vanish. Therefore dissipative effects do not enter stationary solutions of fluid dynamics.

We will work in $n = 4$ and study conformal fluid on S^3 in detail. In particular, this describes $\mathcal{N} = 4$ SYM plasma on the conformal boundary of black holes in AdS_5, which we shall study later. The thermodynamic quantities we derive will be easily generalized to

S^n. It will be shown that uncharged rotating fluids on S^{n-1} is dual to uncharged rotating black holes on AdS_{n+1}.

In spherical coordinates

$$x^1 = \sin\theta\cos\phi_1, \qquad x^2 = \sin\theta\sin\phi_1, \qquad x^3 = \cos\theta\cos\phi_2, \qquad x^4 = \cos\theta\sin\phi_2 \quad (5.13)$$

The metric and the volume element on $\mathbb{R} \times S^3$ are

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta\,d\phi_1^2 + \cos^2\theta\,d\phi_2^2, \qquad dV = \sqrt{-g}\,d^3x = \sin\theta_1\cos\theta_2\,d\theta\,d\phi_1\,d\phi_2. \tag{5.14}$$

The fact that the expansion and the shear tensor vanish implies that the fluid is purely rotational. By an SO(4) rotation we can choose the rotation to be in the $(1-2)$ and $(3-4)$ planes. The four velocity is given by $u^\mu = \gamma(1, 0, \omega_1, \omega_2)$, where ω_1, ω_2 are the angular velocity in the $(1-2)$ and $(3-4)$ planes, respectively. $\gamma = (1-v^2)^{-1/2}$ is the usual Lorentz factor where v is the tangential velocity given by $v^2 = \omega_1^2\sin^2\theta + \omega_2^2\cos\theta$, and θ is the angle between the two orthogonal planes.

It is straightforward to compute (5.11).

$$\begin{aligned}
a^\mu &= (0, -\partial_\theta\log\gamma, 0, 0), \\
\vartheta &= 0, \\
\sigma^{\mu\nu} &= 0, \\
q^\mu &= -\kappa\gamma\left(0, \partial_\theta\left(\frac{\mathcal{T}}{\gamma}\right), 0, 0\right), \\
q_i^\mu &= -D_{ij}\left(0, \partial_\theta\left(\frac{\mu_i}{\mathcal{T}}\right), 0, 0\right).
\end{aligned} \tag{5.15}$$

Vanishing of q^μ and q_i^μ implies

$$\mathcal{T} = \gamma\tau, \qquad \mu_i = \mathcal{T}\nu_i. \tag{5.16}$$

for some constants τ, ν_i. We can insert (5.16) into (5.7) and obtain the equations of state

$$\rho = 3P = 3\gamma^3\tau^3 h_i, \qquad r_i = \gamma^3\tau^3 h_i, \qquad s = \gamma^3\tau^3(4h - \nu_i h_i). \tag{5.17}$$

It is useful to define

$$A = h(\vec{\nu}), \qquad B = 4h(\vec{\nu}) - \nu_i h_i(\vec{\nu}), \qquad C_i = h_i(\vec{\nu}). \tag{5.18}$$

The stress energy tensor, charge and entropy currents (5.8) become

$$T^{\mu\nu} = \tau^4\gamma^6 h \begin{pmatrix} 3+v^2 & 0 & 4\omega_1 & 4\omega_2 \\ 0 & 1-v^2 & 0 & 0 \\ 4\omega_1 & 0 & 3\omega_1^2 + \csc^2\theta - \omega_2^2\cot^2\theta & 4\omega_1\omega_2 \\ 4\omega_2 & 0 & 4\omega_1\omega_2 & 3\omega_2^2 + \sec^2\theta - \omega_1^2\tan^2\theta \end{pmatrix}, \tag{5.19}$$

$$J_i^\mu = \tau^3\gamma^4 C_i(1, 0, \omega_1, \omega_2), \qquad J_S^\mu = \tau^3\gamma^4 B(1, 0, \omega_1, \omega_2).$$

We know from the conservation law (5.1) that there are conserved charges in our system that are constant in time. The conserved charge associated with a killing vector k^μ, and with the charge and entropy currents are given respectively by

$$Q = \int dV \, g_{\mu\nu} T^{0\mu} k^\nu, \quad Q = \int dV \, J^0. \tag{5.20}$$

Therefore the conserved energy and angular momenta associated with the killing vectors $\partial_t, \partial_{\phi_1}, \partial_{\phi_2}$, and the conserved entropy and R-charges are given by

$$
\begin{aligned}
E &= \int dV \, T^{tt} = \frac{\mathrm{Vol}(S^3)\tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)} \left(\frac{2\omega_1^2}{1-\omega_1^2} + \frac{2\omega_2^2}{1-\omega_2^2} + 3 \right), \\
L_1 &= \int dV \, \sin^2\theta \, T^{t\phi_1} = \frac{\mathrm{Vol}(S^3)\tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)} \left(\frac{2\omega_1}{1-\omega_1^2} \right), \\
L_2 &= \int dV \, \cos^2\theta \, T^{t\phi_2} = \frac{\mathrm{Vol}(S^3)\tau^4 A}{(1-\omega_1^2)(1-\omega_2^2)} \left(\frac{2\omega_2}{1-\omega_2^2} \right), \\
S &= \frac{\mathrm{Vol}(S^3)\tau^3 B}{(1-\omega_1^2)(1-\omega_2^2)}, \\
R_i &= \frac{\mathrm{Vol}(S^3)\tau^3 C_i}{(1-\omega_1^2)(1-\omega_2^2)}.
\end{aligned}
\tag{5.21}
$$

These conserved charges completely describe the thermodynamics of stationary rotating conformal fluids on S^3.

The thermodynamic potential is given by $\Phi = E - TS - \Omega_a L_a - \zeta_i R_i$. It can be verified that the global thermodynamic quantities (T, Ω_a, ζ_i) are related to the local thermodynamic quantities $(\mathcal{T}, \omega_a, \mu_i)$ via

$$T = \tau = \gamma\mathcal{T}, \qquad \Omega_a = \omega_a, \qquad \zeta_i = \tau\nu_i = \gamma\mu_i. \tag{5.22}$$

We can now deduce the grand canonical partition function (3.3)

$$\log Z_{\mathrm{gc}} = -\frac{1}{T}(E - TS - \Omega_i L_i - T\nu_i R_i) = \frac{\mathrm{Vol}(S^3)T^3 h(\vec{\zeta}/T)}{(1-\Omega_1^2)(1-\Omega_2^2)}. \tag{5.23}$$

Note that the form of the partition function is consistent with conformal invariance and extensivity.

These can be generalized to rotating fluids on S^{n-1} in a straightforward fashion:

$$
\begin{aligned}
T^{tt} &= T^n A(n\gamma^{n+2} - \gamma^n), \\
T^{t\phi_a} &= T^n A n \gamma^{n+2} \Omega_a, \\
T^{\phi_a \phi_a} &= T^n A \left[n\gamma^{n+2}\Omega_a^2 + \gamma^n \left(\prod_{b=1}^{a-1} \sec^2\theta \right) \csc^2\theta_a \right], \\
T^{\phi_a \phi_b} &= T^n A n \gamma^{n+2} \Omega_a \Omega_b, \\
T^{\theta_a \theta_a} &= T^n A \gamma^n \left(\prod_{b=1}^{a-1} \sec^2\theta_b \right).
\end{aligned}
\tag{5.24}
$$

The associated conserved charges can be found by integration:

$$E = \frac{\text{Vol}(S^{n-1})T^n h(\vec{\nu})}{\prod_{b=1}^m (1-\Omega_b^2)} \left(\sum_{a=1} \frac{2\Omega_a^2}{1-\Omega_a^2} + n - 1 \right),$$

$$L_a = \frac{\text{Vol}(S^{n-1})T^n h(\vec{\nu})}{\prod_{b=1}^m (1-\Omega_b^2)} \left(\frac{2\Omega_a}{1-\Omega_a^2} \right), \tag{5.25}$$

$$S = \frac{\text{Vol}(S^{n-1})T^{n-1}(nh - \nu_i h(\vec{\nu}))}{\prod_{b=1}^m (1-\Omega_b^2)},$$

$$R_i = \frac{\text{Vol}(S^{n-1})T^{n-1} h_i(\vec{\nu})}{\prod_b (1-\Omega_b^2)}.$$

The grand canonical partition function is

$$\log Z_{\text{gc}} = \frac{\text{Vol}(S^{n-1})T^{n-1} h(\zeta/T)}{\prod_a (1-\Omega_a^2)}. \tag{5.26}$$

5.2 Uncharged rotating conformal fluids v.s. black holes in AdS_{n+1}

In this subsection, we will compute the stress-energy tensor of large, uncharged, rotating black holes in AdS_{n+1} and find that it agrees exactly with that of uncharged, rotating conformal fluids on S^{n-1}.

Fluid   An uncharged fluid has constant $h(\vec{\nu}) = h$. So the conserved charges (5.25) reduce to

$$E = \frac{\text{Vol}(S^{n-1})T^n h}{\prod_{b=1}^m (1-\Omega_b^2)} \left(\sum_{a=1} \frac{2\Omega_a^2}{1-\Omega_a^2} + n - 1 \right),$$

$$L_a = \frac{\text{Vol}(S^{n-1})T^n h}{\prod_{b=1}^m (1-\Omega_b^2)} \left(\frac{2\Omega_a}{1-\Omega_a^2} \right), \tag{5.27}$$

$$S = \frac{\text{Vol}(S^{n-1})T^{n-1} nh}{\prod_{b=1}^m (1-\Omega_b^2)},$$

$$R_i = 0.$$

and the grand canonical partition function (5.26) becomes

$$\log Z_{\text{gc}} = \frac{\text{Vol}(S^{n-1})T^{n-1} h}{\prod_a (1-\Omega_a^2)}. \tag{5.28}$$

The stress-energy tensor (5.24) becomes

$$T^{tt} = hT^n(n\gamma^{n+2} - \gamma^n),$$

$$T^{t\phi_a} = hT^n n\gamma^{n+2}\Omega_a,$$

$$T^{\phi_a\phi_a} = hT^n \left[n\gamma^{n+2}\Omega_a^2 + \gamma^n \left(\prod_{b=1}^{a-1} \sec^2\theta_b \right) \csc^2\theta_a \right], \tag{5.29}$$

$$T^{\phi_a\phi_b} = hT^n n\gamma^{n+2}\Omega_a\Omega_b,$$

$$T^{\theta_a\theta_a} = hT^n\gamma^n \left(\prod_{b=1}^{a-1} \sec^2\theta_b \right).$$

Gravity Uncharged rotating black holes in AdS_{n+1} are parametrized by the horizon radius r_+ and $\lfloor n/2 \rfloor$ independent rotation parameters a_i ([8][9]). We set the radius of the AdS space to unity: $b = 1$. The surface gravity κ is given by

$$
\kappa = \begin{cases} r_+(1+r_+^2)\sum_{i=1}^{m}\frac{1}{r_+^2+a_i^2} - \frac{1}{r_+} & \text{if } n = 2m, \\ r_+(1+r_+^2)\sum_{i=1}^{m}\frac{1}{r_+^2+a_i^2} - \frac{1-r_+^2}{2r_+} & \text{if } n = 2m+1. \end{cases} \tag{5.30}
$$

The above expression simplifies at $r_+ \to \infty$. In the rest of this section, we will consider all quantities in the large r_+ limit.

$$
\kappa = \frac{n}{2}r_+ + \mathcal{O}\left(\frac{1}{r_+}\right), \qquad T = \frac{\kappa}{2\pi} = \frac{n}{4\pi}r_+ + \mathcal{O}\left(\frac{1}{r_+}\right). \tag{5.31}
$$

The mass M is given by

$$
2M = r_+^n + \mathcal{O}\left(r_+^{n-2}\right) = T^n \left(\frac{4\pi}{n}\right)^n + \mathcal{O}\left(T^{n-2}\right). \tag{5.32}
$$

The thermodynamic quantities are given by

$$
\begin{aligned}
\Omega_i &= a_i, \\
E &= \frac{\mathrm{Vol}(S^{n-1})T^n}{16G_{n+1}\prod_{j=1}^{m}(1-a_j^2)}\left(\frac{4\pi}{n}\right)^n\left(\sum_{i=1}^{m}\frac{2a_i^2}{1-a_i^2} + n - 1\right), \\
L_i &= \frac{\mathrm{Vol}(S^{n-1})T^n}{16G_{n+1}\prod_{j=1}^{m}(1-a_j^2)}\left(\frac{4\pi}{n}\right)^n\left(\frac{2a_i}{1-a_i^2}\right), \\
S &= \frac{\mathrm{Vol}(S^{n-1})T^{n-1}n}{16G_{n+1}\prod_{j=1}^{m}(1-a_j^2)}\left(\frac{4\pi}{n}\right)^n, \\
R_i &= 0.
\end{aligned} \tag{5.33}
$$

The stress-energy tensor $\Pi^{\mu\nu}$ are:

$$
\begin{aligned}
\Pi^{tt} &= \frac{m}{8\pi G_{n+1}}(n\gamma^{n+2} - \gamma^n), \\
\Pi^{t\phi_a} &= \frac{m}{8\pi G_{n+1}}n\gamma^{n+2}a_i, \\
\Pi^{\phi_a\phi_a} &= \frac{m}{8\pi G_{n+1}}\left[n\gamma^{n+2}a_i^2 + \gamma^n\left(\prod_{b=1}^{a-1}\sec^2\theta_b\right)\csc^2\theta_a\right], \\
\Pi^{\phi_a\phi_b} &= \frac{m}{8\pi G_{n+1}}n\gamma^{n+2}a_ia_j, \\
\Pi^{\theta_a\theta_a} &= \frac{m}{8\pi G_{n+1}}\gamma^n\left(\prod_{b=1}^{a-1}\sec^2\theta_b\right).
\end{aligned} \tag{5.34}
$$

Comparing the thermodynamic quantities of AdS_{n+1} black holes with that of fluids on S^{n-1} (5.27), we see that they agree exactly if we set

$$
h = \frac{1}{16\pi G_{n+1}}\left(\frac{4\pi}{n}\right)^n. \tag{5.35}
$$

In the large r_+ limit, from (5.32) and (5.35), we have

$$\Omega_i = a_i, \qquad hT^n = \frac{M}{8\pi G_{n+1}}. \tag{5.36}$$

We see that the stress-energy tensor of uncharged rotating black holes in AdS_{n+1} agrees exactly with fluids on its conformal boundary $\mathbb{R} \times S^{n-1}$.

There is another reason to study *large* black holes. In our discussions, we assumed that the fluid is in local thermodynamic equilibrium, even though the locally measured temperature $\mathcal{T}$ and the energy density ρ vary across space. Our assumption is valid if the radius of curvature of the space is large compared to the mean free path, l_{mfp}, of the fluid. For every fluid solution we will examine, it will be the case that the fluid dynamical assumption,

$$l_{\mathrm{mfp}} \ll 1, \tag{5.37}$$

holds if its black hole dual has large event horizon. We shall comment on the validity of fluid dynamics in each of the following examples. First let us check that the correspondence we just established satisfies equation (5.37). The mean free path can be approximated by the viscosity to energy density ratio: $l_{\mathrm{mfp}} \approx \eta/\rho$. Kovtun et al. [15] found a universal value of viscosity to entropy density ratio of $\eta/s = 1/4\pi$ for all fluids with dual black holes. Hence $l_{\mathrm{mfp}} \approx s/4\pi\rho$. For the uncharged rotating fluid (5.27), we find

$$l_{\mathrm{mfp}} \approx \frac{S}{4\pi E} \propto \frac{1}{T}. \tag{5.38}$$

Since T scales as the radius of the black hole event horizon (2.19), we deduce that fluid dynamics is valid for large T, or large r_+.

5.3 Charged rotating conformal fluids v.s. black holes in $AdS_5 \times S^5$

Large black holes in $AdS_5 \times S^5$ appear in a six parameter family, labeled by three SO(6) Cartan charges, two SO(4) rotations and the mass. While we do not know the most general solutions, several families are known. We will compare the thermodynamics of these known black holes with that of the fluid dynamics. Using the AdS/CFT correspondence, one can extract the equation of state of the $\mathcal{N} = 4$ Yang-Mills plasma from non-rotating black holes in $AdS_5 \times S^5$ [19]. It is given by

$$\begin{aligned}
h(\vec{\nu}) &= \frac{P}{T^4} = 2\pi^2 N^2 \frac{\prod(1+k_i)^3}{(2 + \sum_i k_i - \prod_i k_i)^4}, \\
\nu_i &= \frac{\mu_i}{T} = 2\pi \frac{\sqrt{k_i}}{1+k_i} \frac{\prod(1+k_i)}{(2 + \sum_i k_i - \prod_i k_i)} \\
h_i(\vec{\nu}) &= \frac{r_i}{T^3} = 2\pi^2 N^2 \sqrt{k_i} \frac{\prod(1+k_i)^2}{(2 + \sum_i k_i - \prod_i k_i)^3},
\end{aligned} \tag{5.39}$$

where

$$k_i = \frac{4\pi^2 R_i^2}{S^2} \geq 0 \tag{5.40}$$

and satisfies the condition

$$\frac{2 + \sum_i k_i - \prod_i k_i}{\prod_i (1 + k_i)} = \sum_i \frac{1}{1 + k_i} \geq 0. \tag{5.41}$$

The limit $k_i \to \infty$ means that the R_i charge is much greater than the entropy, which is thermodynamically singular. Thus we require $k_i < \infty$. We can go on to find the conserved charges using equation (5.21).

Thus we obtain the grand canonical partition function for $\mathcal{N} = 4$ SYM plasma:

$$\log Z_{\text{gc}} = \frac{2\pi^2 N^2 \text{Vol}(S^3) T^3 \prod (1 + k_i)^3}{(1 - \Omega_1^2)(1 - \Omega_2^2) \left(2 + \sum_i k_i - \prod_i k_i\right)^4}. \tag{5.42}$$

Let $x_i = \frac{1}{1 + k_i}$ and define $\chi = \frac{T}{\sum_i x_i - 1}$. Then the condition on k_i translates into $0 \leq x_i \leq 1$ and $\sum_i x_i \geq 1$. Let us visualize the constraints geometrically. if x_i are the coordinate axis, then the allowed values are those in the box defined by $0 \leq x_i \leq 1$ lying above the plane defined $\sum_i x_i \geq 1$ (figure 5).
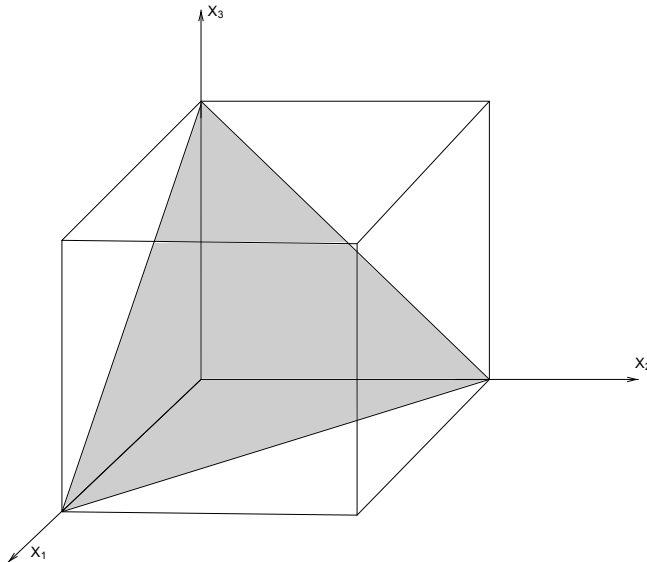
Figure 5: The allowed values of x_i's are those inside the box lying above the triangle $\sum_i x_i = 1$. Those on the plane $x_i = 0$ are the singular extremal values. The interior of the shaded triangle represents the nonsingular extremal values.

The extremal cases are when the points lie on the plane $(\sum_i x_i = 1)$ or on the faces of the box defined by $x_i = 0$. The latter case corresponds to $k_i = \infty$, so is thermodynamically singular. The nonsingular extremal points are the interior of the plane.

From equation (5.40), we see that vanishing of the R_i-charge corresponds to $x_i = 1$. Thus if any of the R_i-charges vanish, then there is no nonsingular extremal point.

If we require that χ be finite, then the condition $\sum_i x_i \to 1$ correspond to $T \to 0$. We shall see that even at zero temperature, the theory is nontrivial. We will be interested in the extremal cases.

Finally, the mean free path of the plasma is

$$l_{\text{mfp}} \approx \frac{S}{4\pi E} \propto \frac{B}{AT} \propto \frac{1}{T}\left(\sum_i x_i - 1\right). \tag{5.43}$$

We will compare fluid dynamical predictions with those coming from the black holes in each of the cases listed in table 2.

5.3.1 Equal SO(6) charges

Fluid   This is the case $k_1 = k_2 = k_3 = k$. Plugging into equation (5.42), we get

$$\log Z_{\text{gc}} = \frac{2\pi^2 N^2 \text{Vol}(S^3)T^3(1+k)^9}{(1-\Omega_1^2)(1-\Omega_2^2)(2+3k-k^3)^4} = \frac{2\pi^2 N^2 \text{Vol}(S^3)T^3(1+k)}{(1-\Omega_1^2)(1-\Omega_2^2)(2-k)^4}. \tag{5.44}$$

The mean free path (5.43) is

$$l_{\text{mfp}} \approx \frac{1}{T}\frac{2-k}{1+k}, \tag{5.45}$$

The extremal limit $\sum_i x_i = 1$ corresponds to $x_i = \frac{1}{3}$, which lies at the centre of the triangle. So is clearly nonsingular.

Gravity   The thermodynamics of black holes with equal SO(6)-charges is given in [3] by

$$\log Z_{\text{gc}} = \frac{2\pi^2 N^2 \text{Vol}(S^3)T^3(1+y^2)}{(1-a_1^2)(1-a_2^2)(2-y^2)^4}, \tag{5.46}$$

which agrees with (5.44) exactly if we set $\Omega_i = a_i$ and $k = y^2$. The temperature is given by

$$T = \frac{r_+}{2\pi}(2-y^2), \tag{5.47}$$

where $r_+ \gg 1$. The fluid mean free path (5.45) can be written as

$$l_{\text{mfp}} \approx \frac{2\pi}{r_+(1+k)}. \tag{5.48}$$

We see that $l_{\text{mfp}} \ll 1$ for large r_+ and that l_{mfp} behaves as $\frac{1}{r_+}$ since $k \leq 2$ is finite.

5.3.2 Equal rotations

Fluid   This is the case $\Omega_1 = \Omega_2 = \Omega$. Plugging into equation (5.42), we get

$$\log Z_{\text{gc}} = \frac{2\pi^2 N^2 \text{Vol}(S^3)T^3 \prod(1+k_i)^3}{(1-\Omega^2)^2 \left(2+\sum_i k_i - \prod_i k_i\right)^4}. \tag{5.49}$$

The mean free path (5.43) is

$$l_{\text{mfp}} \approx \frac{1}{T}\left(\sum_i \frac{1}{1+k_i} - 1\right). \tag{5.50}$$

One obtains nonsingular extremal limit by letting $\sum_i \frac{1}{1+k_i} \to 1$ while keeping k_i finite. In this limit, $l_{\text{mfp}} \ll 1$ so the fluid dynamical assumption is valid.

Gravity   The thermodynamics of black holes with equal rotations is given in [6] by

$$\log Z_{\text{gc}} = \frac{4\pi^2 N^2 T^3 \prod(1+k_i)^3}{(1-a^2)^2 \left(\prod_i H_i\right) \left(\sum_i H_i^{-1} - 1\right)^4}, \tag{5.51}$$

which agrees with (5.49) exactly if we set $\Omega = a$ and $k_i = H_i - 1$. The temperature is given by

$$T = \frac{r_+ \sqrt{1-a^2}}{2\pi} \left(\sum_i H_i^{-1} - 1\right) \prod_i \sqrt{H_i}, \tag{5.52}$$

where $r_+ \gg 1$. The fluid mean free path (5.50) can be written as

$$l_{\text{mfp}} \approx \frac{2\pi}{r_+ \prod_i \sqrt{H_i}}. \tag{5.53}$$

We see that $l_{\text{mfp}} \ll 1$ for large r_+ and that l_{mfp} behaves as $\frac{1}{r_+}$ as long as we do not take the singular limit $H_i \to \infty$.

5.3.3 Two equal nonzero $SO(6)$ charges

Fluid   Assume $k_1 = k_2 = k > 0, k_3 = 0$. Plugging into equation (5.42), we get

$$\log Z_{\text{gc}} = \frac{2\pi^2 N^2 \text{Vol}(S^3) T^3 (1+k)^6}{(1-\Omega_1^2)(1-\Omega_2^2)(2+2k)^4} = \frac{\pi^2 N^2 \text{Vol}(S^3) T^3 (1+k)^2}{8(1-\Omega_1^2)(1-\Omega_2^2)}. \tag{5.54}$$

The extremal limit is $k \to \infty$, which is singular. The mean free path (5.43) is

$$l_{\text{mfp}} \approx \frac{2}{T(1+k)}. \tag{5.55}$$

For nonsingular solutions, the fluid dynamical assumption is valid when T is large.

Gravity   The thermodynamics of black holes with two equal large R-charges and one small R-charge is given in [4] by

$$\log Z_{\text{gc}} = \frac{\pi^2 N^2 \text{Vol}(S^3) T^3 (1+k^2)}{(8-a_1^2)(1-a_2^2)}, \tag{5.56}$$

which agrees with (5.54) exactly if we set $\Omega_i = a_i$. The temperature is given by

$$T = \frac{r_+}{\pi}, \tag{5.57}$$

where $r_+ \gg 1$. The temperature vanishes only if the black hole has a singular horizon at $r_+ = 0$. In this case, the fact that all extremal fluids are singular corresponds to the fact that all extremal black holes are singular. The fluid mean free path (5.55) can be written as

$$l_{\text{mfp}} \approx \frac{2\pi}{r_+(1+k)}. \tag{5.58}$$

We see again that $l_{\text{mfp}} \ll 1$ for large r_+ and that l_{mfp} behaves as $\frac{1}{r_+}$ as long as we do not take the singular limit $k \to \infty$.

5.3.4 One nonzero SO(6) charge

Fluid Assume $k_1 = k_2 = 0, k_3 = k > 0$. Plugging into equation (5.42), we get

$$\log Z_{\text{gc}} = \frac{2\pi^2 N^2 \text{Vol}(S^3) T^3 (1+k)^3}{(1-\Omega_1^2)(1-\Omega_2^2)(2+k)^4}. \tag{5.59}$$

The values of k_i imply $\sum_i x_i \geq 2$, so there is no extremal limit. The limit $k \to \infty$ is again singular. The mean free path (5.43) is

$$l_{\text{mfp}} \approx \frac{1}{T} \frac{2+k}{1+k}. \tag{5.60}$$

For nonsingular solutions, the fluid dynamical assumption is valid when T is large.

Gravity The thermodynamics of black holes with one nonzero R-charge is given in [5] by

$$\log Z_{\text{gc}} = \frac{4\pi^2 N^2 T^3 (1-y^2)}{(1-a_1^2)(1-a_2^2)(2-y^2)}, \tag{5.61}$$

which agrees with (5.59) exactly if we set $\Omega_i = a_i$ and $k = \frac{y^2}{1-y^2}$.

The temperature is given by

$$T = \frac{r_+(2+k)}{2\pi\sqrt{1+k}}, \tag{5.62}$$

where $r_+ \gg 1$. The temperature vanishes only if the black hole is singular with $r_+ \to 0$. So we see that it matches the fluid dynamic fact that there is no nonsingular extremal limit. The fluid mean free path (5.60) can be written as

$$l_{\text{mfp}} \approx \frac{2\pi}{r_+\sqrt{1+k}}. \tag{5.63}$$

We see again that $l_{\text{mfp}} \ll 1$ for large r_+ and that l_{mfp} behaves as $\frac{1}{r_+}$ as long as we do not take the singular limit $k \to \infty$.

6. Conclusion

In this essay we have examined the many facets of anti-de Sitter black holes and their thermodynamics. Not only does the AdS space provide an infinite potential that allows thermodynamically stable black holes to exist, the competition between the pure radiation and black hole free energies gives rise to a first-order phase transition. Perhaps more striking is the the gauge theory interpretation of this, via the AdS/CFT correspondence. The AdS/CFT conjecture establishes a correspondence between string theory in AdS space and gauge theory on its conformal boundary. In this context, pure AdS radiation is interpreted as a confined phase in large N QCD, whereas black holes correspond to a deconfined phase. The Hawking-Page phase transition is then interpreted as a confinement-deconfinement transition. The AdS/CFT correspondence provides a general framework for investigating

theoretical problems. In one of its recent manifestations, it is used to relate gravity solutions to fluid dynamics. We discussed extensively the work of Bhattacharyya et al. relating large, rotating AdS black holes and rotating fluids on their conformal boundary. The correspondence works for $AdS_5 \times S^5$ in all known charged solutions and works for arbitrary dimension in the uncharged case. We see that the interest in black hole thermodynamics is very much alive since it was discovered four decades ago. Harnessed with the power of duality, the study black holes will no doubt lead to many discoveries in the 21st century.

Acknowledgments

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