On the limitations of using quantum mechanics in a fixed background of a Schwarzschild black hole

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Abstract

I. INTRODUCTION

II. CALCULATION

A. Proper time

1. Background spacetime

First we consider the motion of Hawking photons coming from a Schwarzschild metric of the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{1}$$

Consider two Hawking photons coming from a fixed distance, r_e, just above a Schwarzschild black hole separated by a coordinate time interval $\bar{\delta t}_A$ and a corresponding proper time $\bar{\delta\tau}_A$ as shown in figure 1. The two photons propagate through to the observer at r_0 following geodesics. We could potentially solve for the geodesics of the photons to find out the coordinate time of arrival of each photon, however in the absence of additional gravitational fields the coordinate time interval between the photons will not change[1], i.e $\bar{\delta t}_A = \bar{\delta t}_0$, and that is the quantity we need in order to compute the proper time $\bar{\delta\tau}_0$ observed by the observer at r_0,

$$\begin{aligned} \bar{\delta\tau}_0 &= \left(1 - \frac{2M}{r_0}\right)^{\frac{1}{2}} \bar{\delta t}_0 \\ &= \left(1 - \frac{2M}{r_0}\right)^{\frac{1}{2}} \bar{\delta t}_A \approx \bar{\delta t}_A \end{aligned} \tag{2}$$

where in the last step we have assumed r_0 is very large.

2. Introducing perturbations: Double shells

In this section we consider the top part of figure 1. To distinguish the results from the previous section without shell we will not have an overbar in any of the quantities described in this scenario. The same initial conditions for the two Hawking photons are assumed. Using the argument described in the previous section we expect the coordinate time intervals between r_A and r_1 to be equal, $\delta t_A = \delta t_1$. Analogously for the coordinate time interval between r_2 and

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[1] This will prove to be extremely helpful when we look at the calculation in the presence of shells.

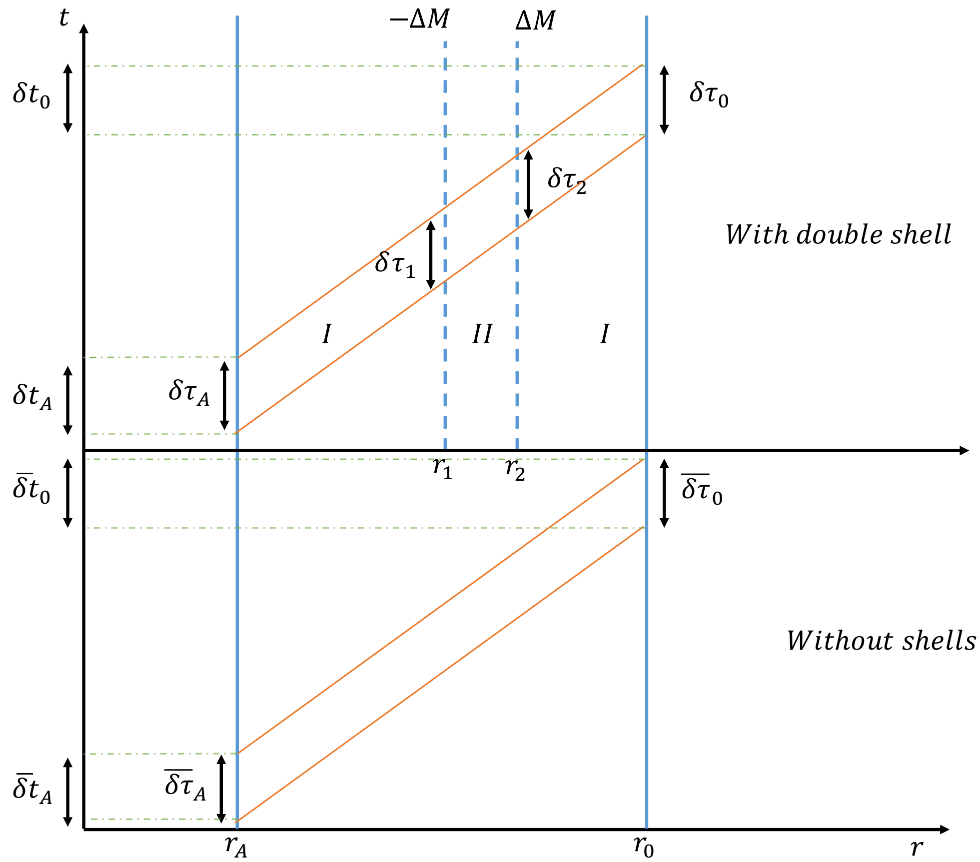

FIG. 1. The two parts of this diagrams describe two different scenarios. The trajectories of the photon are the lines in orange. The trajectory of the observer and the position from which the photons are emitted are represented by solid blue lines. The bottom part is a general case of two Hawking photons coming from a distance r_A from the black hole of mass M and arriving at an observer who is at a distance of r_0. The top part shows two Hawking photons coming from the same distance r_A from a black hole of mass M. Instead of the photons freely propagating through to an observer at r_0, they have to cross two shells, represented by blue dotted line, of equal and opposite mass $-\Delta M, \Delta M$ at r_1, r_2 respectively. The regions I represent a metric with mass M and region II represents a metric with mass $M - \Delta M$.

r_0, $\delta t_2 = \delta t_0$. The new region is between r_2 and r_1 where the metric is different; it is defined by a mass $M - \Delta M$. The coordinate time interval at r_1 will also have a representation in the II metric. Quantities evaluated in the II metric will be denoted by a hat on top. By imposing the physical condition that there cannot be any discontinuities in spacetime we know that the proper times in both the metrics must be the same at r_1, $\delta\tau_1 = \delta\hat{\tau}_1$ and this gives a relation between the coordinate time intervals

$$\delta t_1 = \left(\frac{1 - \frac{2(\Delta M - M)}{r_1})}{1 - \frac{2M}{r_1}} \right)^{\frac{1}{2}} \delta\hat{t}_1. \tag{3}$$

Applying the same condition to the proper times at r_2 gives the analogous relation between the coordinate time intervals at r_2,

$$\delta t_2 = \left(\frac{1 - \frac{2(\Delta M - M)}{r_2})}{1 - \frac{2M}{r_2}} \right)^{\frac{1}{2}} \delta\hat{t}_2. \tag{4}$$

Since $\delta\hat{t}_1$ and $\delta\hat{t}_2$ are both evaluated in the same metric they must be equal and therefore we can combine Eq (3) and Eq (4) to give a relation between δt_1 and δt_2,

$$\delta t_1 = \left(\frac{\left(1 - \frac{2M - \Delta M}{r_1}\right)\left(\frac{1-2M}{r_2}\right)}{\left(1 - \frac{2M}{r_1}\right)\left(1 - \frac{2(M-\Delta M)}{r_2}\right)} \right)^{\frac{1}{2}} \delta t_2$$
$$= \left(\frac{2M - r_2}{2M - r_1}\right)^{\frac{1}{2}} \left(\frac{2(\Delta M - M) + r_1}{2(\Delta M - M) + r_2}\right)^{\frac{1}{2}} \delta t_2. \tag{5}$$

We see that $\delta t_1 = \delta t_2$ in the limit that $r_1 = r_2$ as then the shells will cancel out their masses and we are left with just metric I. By continuing these trajectories forward to the observer at r_0 we know that $\delta t_0 = \delta t_2 \approx \delta \tau_0$.

Without shells we saw that the proper time interval $\bar{\delta\tau_0} \approx \delta t_A$, thus the difference in the proper time intervals with the shells is

$$\delta\tau_0 - \delta\bar{\tau_0} = \delta t_A \left(\left(\frac{2M - r_2}{2M - r_1}\right)^{\frac{1}{2}} \left(\frac{2(\Delta M - M) + r_1}{2(\Delta M - M) + r_2}\right)^{\frac{1}{2}} - 1 \right). \tag{6}$$

As a check we see that when $r_1 = r_2$, $\delta\tau_0 = \delta\bar{\tau_0}$ as expected.

We can explore this relation further by going to a region close to the horizon. If we choose $r_1 = 2M + \epsilon_1$ and $r_2 = 2M + \epsilon_2$ where $\epsilon_{1,2} << 1$, the expression in Eq (6) reduces to

$$\delta\tau_0 - \delta\bar{\tau_0} = \left(\left(\frac{\epsilon_2}{\epsilon_1} \left(\frac{2\Delta M + \epsilon_1}{2\Delta M + \epsilon_2} \right) \right)^{\frac{1}{2}} - 1 \right) \tag{7}$$

B. Frequency of Hawking photons

1. *Background spacetime*

We can do an analogous calculation to see this effect. Instead of looking at the proper time interval between photons we can consider the shift in frequency of single Hawking photon in the same two scenarios as were described in the previous section. We can write the photon four vector at distance r as

$$\bar{k}^\mu(\bar{\omega}_\infty, \bar{r}) = \bar{\omega}_\infty \left(\left(1 - \frac{2M}{\bar{r}_A}\right)^{-1}, \sqrt{1 - \frac{\bar{b}^2}{\bar{r}_A^2}\left(1 - \frac{2M}{\bar{r}_A}\right)}, \frac{\bar{b}}{\bar{r}_A^2}, 0 \right), \tag{8}$$

where $\bar{\omega}_\infty$ is the frequency observed by a stationary observer at infinity and b is the impact parameter of the photon. The general four velocity, $\bar{u}^\mu(\bar{r})$ of a stationary object in this geometry is

$$\bar{u}^\mu(\bar{r}) = \left(\left(1 - \frac{2M}{\bar{r}}\right)^{-\frac{1}{2}}, 0, 0, 0 \right). \tag{9}$$

In the case where there are no shells, $\bar{\omega}_\infty$ is exactly the frequency observed by the observer at $\bar{r}_0$ in the limit that $\bar{r}_0\infty$. In general, the frequency observed at $\bar{r}_0$, $\bar{\omega}_0$, will be redshifted

$$\bar{\omega}_0 = \bar{k}^\mu(\bar{\omega}_\infty, \bar{r}_0)\bar{u}^\nu(\bar{r}_0)\bar{g}_{\mu\nu}(\bar{r}_0)$$
$$= -\bar{\omega}_\infty \left(1 - \frac{2M}{\bar{r}_0}\right)^{-\frac{1}{2}} \tag{10}$$

of course in the limit $\bar{r}_0 \to \infty$ we get $\bar{\omega}_0 = -\bar{\omega}_\infty$.

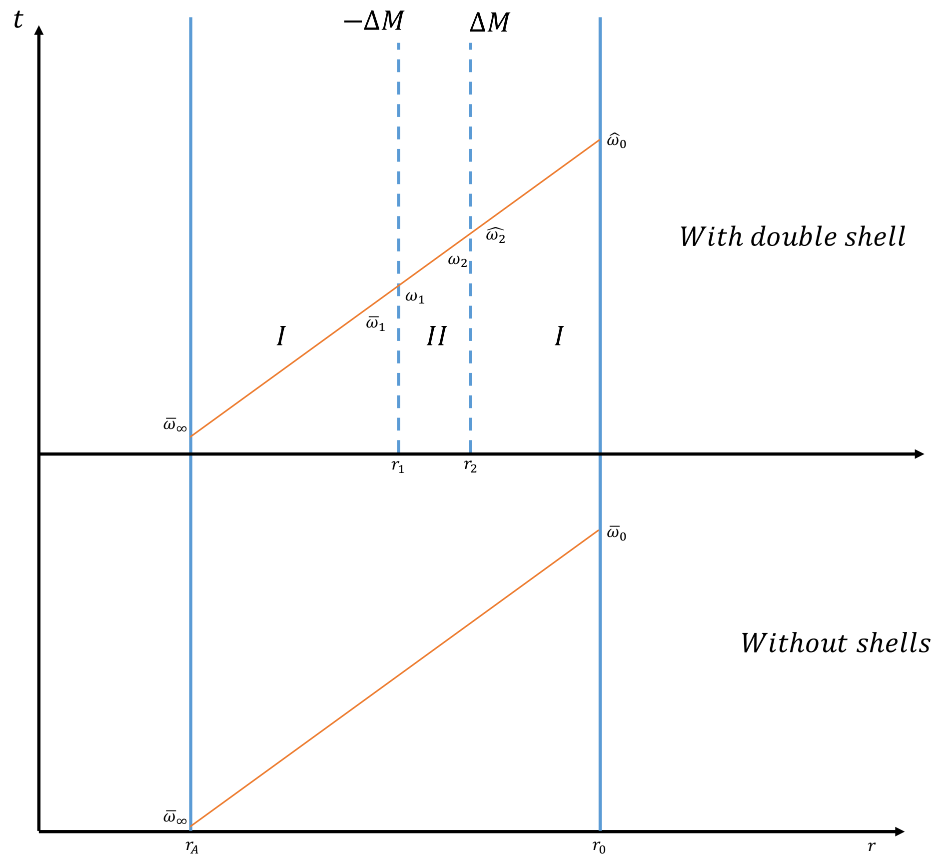

FIG. 2. This diagram follows the same notation as figure 1 except we are looking at one photon represented by the orange line. The frequency at each relevant point is denoted on the diagram. The quantities in the metric I on the left of the shells are denoted with overbars. Quantities in metric II are denoted without any overbars or hats. Quantities in metric I to the right are denoted with hats on top.

2. Introducing perturbations: Double shells

The frequency measured at r_1 in metric I, $\bar{\omega}_1$ is

$$\bar{\omega}_1 = -\bar{\omega}_\infty \left(1 - \frac{2M}{\bar{r}_1}\right)^{-\frac{1}{2}}. \tag{11}$$

In metric II the observed frequency ω_1 is

$$\begin{aligned}\omega_1 &= g_{\mu\nu}(r_1) k^\mu(\omega_\infty, r_1) u^\nu(r_1) \\ &= -\omega_\infty \left(1 - \frac{2(M - \Delta M)}{r_1}\right)^{-\frac{1}{2}}.\end{aligned} \tag{12}$$

Since ω_1 is a physical quantity we expect it to be the same in both in metrics, $\omega_1 = \bar{\omega}_1$ and also we expect $r_1 = \bar{r}_1, r_2 = \bar{r}_2$ (since the radius of the shell corresponds to a physical sphere of radius of r). This gives a relation between ω_∞ and $\bar{\omega}_\infty$

$$\bar{\omega}_\infty = \omega_\infty \left(1 - \frac{2\Delta M}{2(\Delta M - M) + r_1}\right)^{\frac{1}{2}}. \tag{13}$$

At r_2 we can again equate the observed frequencies in both metrics. In this case we define the frequency observed at infinity in metric I on the right side of r_2 as $\hat{\omega}_\infty$, i.e the photon vector in metric I for $r > r_2$ is $\hat{k}$ (But we expect the four velocity of the shell to be $\bar{u}_2$). From $k^\mu(\omega_2, r_2)u^\nu(r_2)g_{\mu\nu}(r_2) = \hat{k}^\mu(\bar{r}_2)\bar{u}^\nu(\bar{r}_2)\bar{g}_{\mu\nu}(\bar{r}_2)$ we have

$$\hat{\omega}_\infty = \omega_\infty \left(1 - \frac{2\Delta M}{2(M-\Delta M)+r_2}\right)^{\frac{1}{2}}. \tag{14}$$

By combining Eq (13) and (14) we get

$$\hat{\omega}_\infty = \bar{\omega}_\infty \left(\frac{(2(\Delta M - M)+r_1)(r_2-2M)}{(r_1-2M)(2(\Delta M - M)+r_2)}\right)^{\frac{1}{2}} \tag{15}$$

The observed frequency at r_0, $\hat{\omega}_0$, is

$$\begin{aligned}\hat{\omega}_0 &= u^\mu(r_0)k^\nu(\hat{\omega}_\infty, r_0)g_{\mu\nu}(r_0)\\ &= -\bar{\omega}_\infty \left(1-\frac{2M}{r_0}\right)^{-\frac{1}{2}} \left(\frac{(2(\Delta M - M)+r_1)(r_2-2M)}{(r_1-2M)(2(\Delta M - M)+r_2)}\right)^{\frac{1}{2}}\end{aligned} \tag{16}$$

In the limit that $r_0 \to \infty$,

$$\hat{\omega}_0 = -\bar{\omega}_\infty \left(\frac{(2(\Delta M - M)+r_1)(r_2-2M)}{(r_1-2M)(2(\Delta M - M)+r_2)}\right)^{\frac{1}{2}} \tag{17}$$

Looking at dynamics close to the horizon,

$$\begin{aligned}r_1 &= 2M + \epsilon_1\\ r_2 &= 2M + \epsilon_2\end{aligned} \tag{18}$$

we get

$$\hat{\omega}_0 = -\bar{\omega}_\infty \left(\frac{\epsilon_2}{\epsilon_1}\left(\frac{2\Delta M + \epsilon_1}{2\Delta M + \epsilon_2}\right)\right)^{\frac{1}{2}} \tag{19}$$