Notes for "Shocks in the Early Universe," by Ue-Li Pen and Neil Turok

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I. EQUATIONS OF MOTION FOR A RELATIVISTIC PERFECT CONFORMAL FLUID

The stress tensor for a perfect fluid is

$$T_f^{\mu\nu} = (p+\rho)u^\mu u^\nu + p g^{\mu\nu}. \tag{1}$$

where $g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ in Minkowski spacetime. The four velocity of the fluid satisfies $g_{\mu\nu}u^\mu u^\nu = -1$.

For a relativistic, conformal-invariant fluid $T^\mu_{\ \mu} = 0$ which implies $p = \frac{1}{3}\rho$, so

$$T^{00} = \frac{4}{3}\rho(u^0)^2 - \frac{1}{3}\rho, \tag{2}$$

$$T^{0i} = \frac{4}{3}\rho u^0 u^i, \tag{3}$$

$$T^{ij} = \frac{4}{3}\rho u^i u^j + \frac{1}{3}\rho\delta^{ij}. \tag{4}$$

It follows from (3) and $g_{\mu\nu}u^\mu u^\nu = -1$ that

$$T^{0k}T^{0k} = (\frac{4}{3}\rho)^2(u^0)^2((u^0)^2 - 1). \tag{5}$$

From this, and equation (2), we find

$$\rho = 2\sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}} - T^{00}, \tag{6}$$

$$\rho(u^0)^2 = \frac{1}{2}\left(T^{00} + \sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}}\right). \tag{7}$$

Hence, (4) becomes

$$T^{ij} = \frac{3}{2}\frac{T^{0i}T^{0j}}{T^{00} + \sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}}} + \frac{1}{3}\delta^{ij}\left(2\sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}} - T^{00}\right), \tag{8}$$

$$\approx \frac{3}{4}\frac{T^{0i}T^{0j}}{T^{00}}\left(1 + \frac{3}{16}\frac{T^{0k}T^{0k}}{(T^{00})^2} + \dots\right) + \frac{1}{3}T^{00}\delta^{ij}\left(1 - \frac{3}{4}\frac{T^{0k}T^{0k}}{(T^{00})^2} - \frac{9}{64}\left(\frac{T^{0k}T^{0k}}{(T^{00})^2}\right)^2 + \dots\right), \tag{9}$$

when $|T^{0k}/T^{00}| \ll 1$.

Then the four evolution equations for the four dynamical quantities T^{00} and T^{0i} are just

$$\partial_0 T^{00} + \partial_i T^{0i} = 0, \tag{10}$$

$$\partial_0 T^{0i} + \partial_i T^{ij} = 0, \tag{11}$$

with T^{ij} given by (8).

In the absence of entropy production, the particle number current $n^\mu \propto \rho^{3/4}u^\mu$ is conserved, $\partial_\mu n^\mu = 0$. The particle number current is expressed in terms of the energy and momentum components of the stress-energy tensor as follows:

$$n^0 = \frac{1}{\sqrt{2}}(\sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}} + T^{00})^{1/2}(2X - T^{00})^{1/4} \approx (T^{00})^{3/4} - \frac{9}{32}\frac{T^{0k}T^{0k}}{(T^{00})^{5/4}} + \dots, \tag{12}$$

$$n^i = \frac{3T^{0i}}{2\sqrt{2}}\frac{(2\sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}} - T^{00})^{1/4}}{(\sqrt{(T^{00})^2 - \frac{3}{4}T^{0k}T^{0k}} + T^{00})^{1/2}} \approx \frac{3}{4}T^{0i}/(T^{00})^{\frac{1}{4}} + \dots \tag{13}$$

For the special case of a 1+1 planar-symmetric flow, the 4-velocity of the fluid $u^\mu = (u^0, u^1, 0, 0)$ so the only nonvanishing stress tensor components are

$$T^{00} = \frac{4}{3}\rho(u^0)^2 - \frac{1}{3}\rho; \qquad T^{01} = \frac{4}{3}\rho u^0 u^1; \qquad T^{11} = \frac{4}{3}\rho(u^1)^2 + \frac{1}{3}\rho; \qquad T^{22} = T^{33} = \frac{1}{3}\rho. \tag{14}$$

Since there is no dependence on the x^2 and x^3 coordinates, the only dynamically important spatial stress is T^{11} which, in this case, reduces to

$$T^{11} = \frac{1}{3}(5T^{00} - 4\sqrt{(T^{00})^2 - \frac{3}{4}(T^{01})^2}) \approx \frac{1}{3}T^{00} + \frac{1}{2}\frac{(T^{01})^2}{T^{00}} + \frac{3}{32}\frac{(T^{01})^4}{(T^{00})^3} + \dots, \tag{15}$$

when $|T^{01}/T^{00}| \ll 1$.

II. RELATIVISTIC HYDRODYNAMICS

The equations of relativistic hydrodynamics follow from the conservation of the stress tensor. For a perfect radiation fluid, with stress tensor $T^{\mu\nu} = \frac{4}{3}\rho u^\mu u^\nu + \frac{1}{3}g^{\mu\nu}$ and $u^\mu = \gamma_v(1, \vec{v})$ with $\gamma_v = (1 - \vec{v}^2)^{-\frac{1}{2}}$, the conservation equations $\partial_\mu T^{\mu\nu} = 0$ imply the Euler equation

$$\rho\big(\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}\big) + \frac{1}{4}(1 - \vec{v}^2)(\vec{\nabla}\rho + \vec{v}\partial_t\rho) = 0 \tag{16}$$

and the conservation of the particle density current nu^μ, with the particle number density $n \propto \rho^{\frac{3}{4}}$:

$$\partial_t\big(\rho^{\frac{3}{4}}\gamma_v\big) + \vec{\nabla}\big(\rho^{\frac{3}{4}}\gamma_v\vec{v}\big) = 0. \tag{17}$$

This last equation is equivalent to the conservation of the entropy current su^μ where s is the entropy density, which is also proportional to $\rho^{\frac{3}{4}}$ in this case.

Defining a "temperature-weighted" particle momentum,

$$\vec{u}_T \equiv \rho^{\frac{1}{4}}\gamma_v\vec{v}, \tag{18}$$

we find (16) and (17) together imply

$$\partial_t\vec{u}_T + \vec{v}\cdot\vec{\nabla}\vec{u}_T = -\frac{1}{4}\rho^{\frac{1}{4}}(1 - \vec{v}^2)^{\frac{1}{2}}\frac{\vec{\nabla}\rho}{\rho}. \tag{19}$$

Using the identity $\vec{v}\cdot\nabla\vec{u}_T = v^j\vec{\nabla}u_e^j - \vec{v}\wedge(\vec{\nabla}\wedge\vec{u}_T)$, and using the definition (18) to express $\vec{v} = \vec{u}_T\,(\vec{u}_T^2 + \rho^{\frac{1}{2}})^{-\frac{1}{2}}$ and $\rho^{\frac{1}{4}}\gamma_v = (\vec{u}_T^2 + \rho^{\frac{1}{2}})^{\frac{1}{2}}$, we can rewrite (19) as

$$\partial_t\vec{u}_T = \vec{v}\wedge(\vec{\nabla}\wedge\vec{u}_T) - \vec{\nabla}(\vec{u}_T^2 + \rho^{\frac{1}{2}})^{\frac{1}{2}}. \tag{20}$$

As a consequence of (20) the vorticity $\vec{\omega} \equiv \vec{\nabla}\wedge\vec{u}_T$ obeys the evolution equation

$$\partial_t\vec{\omega} = \vec{\nabla}\wedge(\vec{v}\wedge\vec{\omega}) \tag{21}$$

which implies that if the vorticity is initially zero it will remain zero for all time. In this circumstance, the fluid motion is termed irrotational and we can write $\vec{u}_T = \vec{\nabla}\phi$ where ϕ is a scalar potential. Then (20) becomes

$$\partial_t\vec{\nabla}\phi = -\vec{\nabla}(a + (\vec{\nabla}\phi)^2)^{\frac{1}{2}}, \tag{22}$$

where we have defined $a \equiv \rho^{\frac{1}{2}}$, and we leave the operator $\vec{\nabla}$ on both sides of the equation to remind ourselves that only the spatial gradient of ϕ is physically meaningful. Finally, from (17) we find that

$$\partial_t a = -\frac{2a + 2(\vec{\nabla}\phi)^2}{3a + 2(\vec{\nabla}\phi)^2}\vec{\nabla}\Big(\frac{a\,\vec{\nabla}\phi}{(a + (\vec{\nabla}\phi)^2)^{\frac{1}{2}}}\Big) \tag{23}$$

Equations (22) and (23) form the basis for the perturbative treatment which follows.

III. PERTURBATIVE EXPANSION FOR VORTICITY-FREE INITIAL CONDITIONS

In this section we study the evolution of initial conditions of the cosmological type, *i.e.*, for which the fluid flow is irrotational. We shall use linear perturbation theory, writing the formal expansions

$$a = \sum_{n=0}^{\infty} a^{(n)}\epsilon^n, \quad \phi = \sum_{n=1}^{\infty} \phi^{(n)}\epsilon^n, \tag{24}$$

in which ϵ is a formal parameter measuring the strength of the initial perturbations, which we shall set equal to unity at the end of the calculation. In perturbation theory, we solve the equations of motion order by order in ϵ. To first order, the perturbations oscillate as acoustic waves of fixed amplitude for all time. However, at the next order in ϵ we shall find secular growth due to wave resonance, leading to the failure of the perturbative expansion after the linearized modes have undergone of order $1/\epsilon$ oscillations.

Substituting (24) into (22) and (23), at linear order in ϵ we obtain

$$\partial_t a^{(1)} = -\frac{2}{3}\vec{\nabla}^2\phi^{(1)}, \qquad \partial_t \phi^{(1)} = -\frac{1}{2}a^{(1)}, \tag{25}$$

and at second order we obtain

$$\partial_t a^{(2)} = -\frac{2}{3}\vec{\nabla}^2\phi^{(2)} - \frac{1}{3}\vec{\nabla}\cdot(a^{(1)}\vec{\nabla}\phi^{(1)}), \qquad \partial_t\phi^{(2)} = -\frac{1}{2}a^{(2)} + \frac{1}{8}(a^{(1)})^2 - \frac{1}{2}(\vec{\nabla}\phi^{(1)})^2, \tag{26}$$

The solutions of interest in cosmology [**?**] involve gravity, but the gravitational effects are only important at early times, when the modes are outside or around the Hubble horizon. As a given mode falls within the horizon, it begins to oscillate and gravitational effects become negligible. (In the radiation epoch the Jeans length is of order the Hubble horizon, so there is no gravitational instability on sub-horizon scales). Once the modes are well inside the horizon, the solutions of interest may be well approximated (see Fig. (**??**) as

$$a^{(1)} = \sum_{\vec{k}} \frac{1}{2}\delta^{(1)}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\cos(kt/\sqrt{3}), \qquad \phi^{(1)} = \sum_{\vec{k}} \delta^{(1)}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\left(-\frac{\sqrt{3}}{4k}\sin(kt/\sqrt{3})\right). \tag{27}$$

We work in a box of comoving volume V with periodic boundary conditions so that the mode sums are discrete.The Fourier amplitudes $\delta^{(1)}_{\vec{k}}$ of the initial fractional density perturbation $\delta(\vec{x}) \equiv \delta\rho(\vec{x})/\rho$, which is a real field, obey $\delta^{(1)}_{\vec{k}} = \delta^{(1)*}_{-\vec{k}}$.

We shall assume the initial perturbations are Gaussian-distributed with an approximately scale-invariant power spectrum:

$$\langle \delta^{(1)}_{\vec{k}}\delta^{(1)}_{\vec{k}'}\rangle = \delta_{\vec{k}+\vec{k}'}\frac{2\pi^2\mathcal{A}_\delta(k)}{Vk^3} \tag{28}$$

where the $2\pi^2$ factor is conventional and the power spectrum amplitude $A_\delta(k)$ (which measures the variance of the field $\delta(\vec{x})$ in real space when it is smoothed on a length scale $2\pi/k$) is a slowly varying function of k. We include volume factors V in order to properly account for dimensions but when we pass to the large V limit and replace the sum $\sum_{\vec{k}}$ by the integral $V\int d^3\vec{k}/(2\pi)^3$, the volume factors disappear in the final result.

The perturbations are determined at second order in ϵ by solving (26) for the Fourier amplitudes: differentiating the equation for $a^{(2)}$ with respect to time and eliminating $\partial_t\phi^{(2)}$, $\partial_t a^{(1)}$ and $\partial_t]phi^{(1)}$ using the corresponding equations of motion, we obtain

$$(\partial_t^2 - \frac{1}{3}\nabla^2)a^{(2)}_{\vec{k}} = \frac{2}{9}\vec{\nabla}\cdot(\vec{\nabla}^2\phi^{(1)}\vec{\nabla}\phi^{(1)}) + \frac{1}{3}\nabla^2((\vec{\nabla}\phi^{(1)})^2). \tag{29}$$

Fourier transforming, this becomes

$$(\partial_t^2 + \frac{1}{3}k^2)a^{(2)}_{\vec{k}} = \sum_{\vec{k}'}\left(\frac{2}{9}\vec{k}\cdot\vec{k}'(\vec{k}-\vec{k}')^2 + \frac{1}{3}k^2(\vec{k}-\vec{k}')\cdot\vec{k}'\right)\phi^{(1)}_{\vec{k}-\vec{k}'}\phi^{(1)}_{\vec{k}'}$$

$$= \sum_{\vec{k}'} \frac{1}{18}(-2A^4 + 3k^4 - 3k^2k'^2 - A^2k^2 + 2A^2k'^2)\phi^{(1)}_{\vec{k}-\vec{k}'}\phi^{(1)}_{\vec{k}'} \equiv \sum_{\vec{k}'} f(\vec{k},\vec{k}')\phi^{(1)}_{\vec{k}-\vec{k}'}\phi^{(1)}_{\vec{k}'} \tag{30}$$

where we define $A \equiv |\vec{k} - \vec{k}'|$ and the function $f(\vec{k},\vec{k}')$ which shall be useful momentarily. Equation (30) is solved using a Green's function which incorporates the correct initial condition:

$$(\partial_t^2 - \frac{1}{3}\nabla^2)a^{(2)}_{\vec{k}} = J_{\vec{k}}(t) \Rightarrow a^{(2)}_{\vec{k}}(t) = \int_0^t dt' \frac{\sqrt{3}}{k}\sin(\frac{k(t-t')}{\sqrt{3}})J_{\vec{k}}(t') \equiv \int_0^t dt' G_k(t,t')J_{\vec{k}}(t'), \tag{31}$$

where the source $J_{\vec{k}}(t)$ is given by the rhs of (30).

IV. TRACING CHARACTERISTICS

The irrotational flow of a conformal relativistic fluid are conveniently described in terms of two scalar quantities: (i) $r \equiv (\rho/\overline{\rho})^{\frac{1}{2}}$, with ρ the energy density and $\overline{\rho}$ its spatial average, and (ii) the velocity potential ϕ defined by $\vec{\nabla}\phi \equiv (\rho/\overline{\rho})^{\frac{1}{4}}\gamma_v\vec{v}$, where $\vec{v}$ is the fluid velocity. In terms of these variables, the full nonlinear fluid equations read:

$$\partial_t\vec{\nabla}\phi = -\vec{\nabla}(r + (\vec{\nabla}\phi)^2)^{\frac{1}{2}}, \qquad \partial_t r = -\frac{2r + 2(\vec{\nabla}\phi)^2}{3r + 2(\vec{\nabla}\phi)^2}\vec{\nabla}(\frac{r\,\vec{\nabla}\phi}{(r + (\vec{\nabla}\phi)^2)^{\frac{1}{2}}}) \tag{32}$$

The cosmological solutions of interest start out, at early times, as a linear superposition of small-amplitude, Gaussian-distributed standing waves with a nearly scale-invariant spectrum. We want to study the effects of nonlinearities upon the evolution of this stochastic background of linearized waves. In particular, we want to see how the linearized waves affect the fluid equations' characteristic surfaces because when these surfaces intersect, shocks form. Therefore we set $r = 1 + r_{sb} + \delta r$, $\phi = \phi_{sb} + \delta\phi$, where r_{sb} and ϕ_{sb} represent the stochastic background of linearized waves and δr and $\delta\phi$ represent small additional disturbances propagating through this background.

From (32), linearizing in both (r_{sb}, ϕ_{sb}) and $(\delta r, \delta\phi)$, we obtain

$$\partial_t\delta\phi + \frac{1}{2}\delta r - \frac{1}{4}r_{sb}\delta r + \vec{\nabla}\phi_{sb}\cdot\vec{\nabla}\delta\phi = 0, \qquad \partial_t\delta r + \frac{2}{3}\vec{\nabla}^2\delta\phi + \frac{1}{3}\vec{\nabla}\cdot(r_{sb}\vec{\nabla}\delta\phi + \delta r\vec{\nabla}\phi_{sb}) = 0. \tag{33}$$

We solve these equations in the stationary phase (or eikonal) approximation, which is to say we assume that the variation of the phase controls the wave fronts. Making the ansatz $\delta\phi = A_\phi e^{i\mathcal{S}}$ and $\delta r = A_r e^{i\mathcal{S}}$, and treating the variation of the amplitudes A_ϕ and A_r as slow, by solving the resulting eigenvalue problem we obtain in leading order

$$\partial_t\mathcal{S} = -\frac{\sqrt{(\vec{\nabla}S)^2}}{\sqrt{3}} - \frac{2}{3}(\vec{\nabla}S\cdot\vec{\nabla}\phi_{sb}). \tag{34}$$

This equation is recognizable as the Hamilton-Jacobi equation for a dynamical system with momentum $\vec{p}(t) = \vec{\nabla}\mathcal{S}(t,\vec{x})$ and Hamiltonian $\mathcal{H}(\vec{p},\vec{x},t) = -\partial_t\mathcal{S}(t,\vec{x})$. The classical action $\mathcal{S}(t,\vec{x}(t))$ is the action calculated on a natural path of the system, *i.e*, a solution of the equations of motion. The Hamiltonian may thus be inferred as

$$\mathcal{H}(\vec{p},\vec{x},t) = -\partial_t\mathcal{S}(t,\vec{x}) = \frac{\sqrt{\vec{p})^2}}{\sqrt{3}} + \frac{2}{3}\vec{p}\cdot\vec{\nabla}\phi_{sb}(t,\vec{x}). \tag{35}$$

The trajectories $\vec{x}(t)$ of this dynamical system describe the stationary phase fronts of small disturbances, *i.e.*, they trace out the characteristics of the fluid equations (32) on the background (r_{sb}, ϕ_{sb}).

The equations of motion read

$$\dot{\vec{x}} = \frac{\vec{n}}{\sqrt{3}} + \frac{2}{3}\vec{\nabla}\phi_{sb}, \qquad \dot{n}_i = -\frac{2}{3}(\partial_i - n_i(n_j\partial_j))(\vec{n}\cdot\vec{\nabla}\phi_{sb}), \tag{36}$$

where $\vec{n} \equiv \vec{p}/|\vec{p}|$ is a unit vector specifying the zeroth order trajectory.

The fact that the Hamiltonian H is homogeneous of degree unity in the momenta has two very important consequences. First, the trajectories $\vec{x}(t)$ depend only on the direction of the momentum, not its magnitude. Second, the value of the classical action $\mathcal{S}(t, vecx(t))$, which is the phase of the wave on the stationary-phase wavefront, is actually a constant independent of time. To see this, note that the classical action is given by $\mathcal{S} = \int dt(\vec{p}\cdot\dot{\vec{x}} - \mathcal{H})$. But from Hamilton's equations, $\dot{\vec{x}} = (\partial\mathcal{H}/\partial\vec{p})$. It follows, since the Hamiltonian is homogeneous of degree unity in $\vec{p}$, that the integrand vanishes. We conclude that in this problem, for propagation on a stochastic background of linearized waves, that when characteristics intersect there are no diffractive or interference phenomena. Hence, one can clearly infer that the intersection of characteristics results in shock formation.

V. DEFLECTION AND INTERSECTION OF CHARACTERISTIC RAYS

We shall describe the evolution of the characteristic rays for the fluid equations by following particles (which we shall call "ray particles," whose trajectories follow these rays. The ray particle trajectories are given by the Hamiltonian equations (36). In particular, let us follow a collection of ray particles which are initially evenly distributed across some infinitesimal volume δV. Each particle has initial coordinates $\vec{q} \in \delta V$. Furthermore, we take all the particle velocities to be initially parallel and pointing in some given direction $\vec{n}^{(0)}$. If we ignore the perturbations, the ray particle trajectories are given by $\vec{x}^{(0)}(t) \equiv \vec{q} + \vec{n}^{(0)}t/\sqrt{3}$, and the infinitesimal volume δV they occupy is simply displaced by $\vec{n}^{(0)}t/\sqrt{3}$. However, to first order in the linear perturbations, the trajectories are deflected: by

$$\vec{x}(t,\vec{q}) = \vec{q} + \vec{n}^{(0)}\frac{t}{\sqrt{3}} + \frac{2}{3}\int_0^t dt'\vec{\nabla}\phi_{sb}(t', \vec{q} + \vec{n}^{(0)}t') + \frac{1}{\sqrt{3}}\int_0^t dt'\delta\vec{n}(t'),$$
$$\text{with} \quad \delta n_i(t') = -\frac{2}{3}\int_0^{t'} dt''(\partial_i - n_i^{(0)}(n_j^{(0)}\partial_j))(\vec{n}^{(0)}\cdot\vec{\nabla}\phi_{sb}(t'', \vec{q} + \vec{n}^{(0)}t'')), \tag{37}$$

The first term describes the movement of the ray due to the motion of the stochastic background it is propagating in. The second describes the deflection of the ray due to gradients which are perpendicular to the ray's unperturbed direction. (Hence, this term is absent in one dimension). As we shall see, in three dimensions and for scale-invariant perturbations, this second term dominates over long times.

We are interested in whether the rays ultimately intersect, because that is the signal for shock formation. At such a convergence, the density of ray particles formally diverges. We are unable to see such a divergence in linear theory, but we can calculate the moment, in linear theory, when the space density of the particles grows by of order unity and when perturbation theory breaks down.

The spatial density of the points is given by number conservation: if the initial number density is $\overline{n}$, a constant, and their final number density at time t is $n(t,\vec{x})$ then particle conservation implies $\overline{n}d^3\vec{q} = n(t,\vec{x})d^3\vec{x}$. We define the fractional overdensity of ray particles $\delta(t,\vec{x})$ by setting $n(t,\vec{x}) =\equiv \overline{n}\,(1+\delta(t,\vec{x})$. From these definitions it follows that

$$1 + \delta(t,\vec{x}) = \det(\frac{\partial x_i}{\partial q_j})^{-1} \approx 1 - \frac{\partial x_i}{\partial q_i} \to \delta(t,\vec{x}) \approx -\vec{\nabla}_q \cdot \vec{x}(t,\vec{q}), \tag{38}$$

as long as the displacements are small. If the displacements grow linearly in time, however, it is reasonable to extrapolate the linear calculation and infer that ray intersection and shock formation become typical once the extrapolated variance of the fractional overdensity $\langle\delta(t,\vec{x})^2\rangle(t)$, taken across the ensemble, becomes of order unity.

The first nontrivial term in (37), due to the fluid motion, contributes as follows:

$$\delta_{fm}(t) \equiv -\frac{2}{3}\int_0^t dt'\vec{\nabla}^2\phi_{sb}(t', \vec{x}^{(0)}(t')),$$
$$\langle\delta_{fm}(t)^2\rangle = \frac{2\pi^2\mathcal{A}}{12}\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{k}\int_0^t dt'\int_0^t dt'' e^{i\vec{k}\cdot\vec{n}^{(0)}(t'-t'')/\sqrt{3}}\sin(kt'/\sqrt{3})\sin(kt''/\sqrt{3}) \tag{39}$$

where we used the ϕ correlator for the stochastic background,

$$\langle\phi_{sb}(t',\vec{x}')\phi_{sb}(t'',\vec{x}'')\rangle = \frac{3}{16}\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2\pi^2\mathcal{A}}{k^5}e^{i\vec{k}\cdot(\vec{x}'-\vec{x}'')}\sin(kt'/\sqrt{3})\sin(kt''/\sqrt{3}). \tag{40}$$

Performing the angular integrations in (40) and setting $x = kt/\sqrt{3}$, $x' = kt'/\sqrt{3}$ and $x'' = kt''/\sqrt{3}$, the remaining integrals become:

$$\langle\delta_{fm}^2\rangle = \frac{\mathcal{A}}{4}\int_0^{k_c}\frac{dk}{k}\int_0^x dx'\int_0^x dx''\frac{\sin(x'-x'')}{(x'-x'')}\sin x'\sin x''$$

$$
\begin{aligned}
&= \frac{\mathcal{A}}{16}\int_0^{k_c}\frac{dk}{k}\big(\gamma - 1 - (\cos(2x)+1)\mathrm{Ci}(2x) + \cos(2x)(1+\gamma+\ln 2) + \ln(2x) + (2x-\sin(2x))\mathrm{Si}(2x)\big)\\
&\approx \frac{\pi\mathcal{A}}{16}\int_1^{x_c} dx \approx \frac{\pi\mathcal{A}\,x_c}{16}, \quad x_c \gg 1
\end{aligned}
\tag{41}
$$

where k_c is the cutoff in k and $x_c = k_c t/sqrt3$. In the exact expression, $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and $\mathrm{Ci}(x) \equiv -\int_x^\infty dx' \cos(x')/x'$ and $\mathrm{Si}(x) \equiv \int_0^x dx' \sin(x')/x'$ are the cosine and sine integrals. The result (44) is physically plausible: it says that the variance in the density of ray particles grows as a random walk in kt, due to encounters with (random-phase) waves of wavenumber k in the stochastic background. The total variance is dominated by the shortest available waves.

The second nontrivial term in (37) encodes the effect of gradients in the fluid velocity on the direction of the particle trajectories:

$$
\delta_g(t) \equiv \frac{2}{3\sqrt{3}}\int_0^t dt'(t-t')\left(\vec{\nabla}^2 - (\vec{n}^{(0)}\cdot\vec{\nabla})^2\right)(\vec{n}^{(0)}\cdot\vec{\nabla}\phi_{sb}(t',\vec{x}^{(0)}(t'))).
\tag{42}
$$

To obtain this expression, we exchanged the orders of the t' and t'' integrations, performed the t' integral and then relabelled t'' as t'. This manipulation makes explicit that $\delta\vec{n}$ receives impulses all along the ray particle path, with an impulse delivered at a time $t-t'$ causing the particle deflection to grow as $t-t'$ later on. This linear growth already suggests that the effect of these impulses might outweigh the effect due to the fluid motion, calculated above.

Before computing the ensemble average, it is interesting to consider the effect of a single Fourier mode in ϕ_{sb}, with wavevector $\vec{k}$. If θ is the angle between $\vec{k}$ and the unperturbed ray direction $\vec{n}^{(0)}$, the effect is proportional to $\cos(\theta)\sin(\theta)^2$. Hence, it is zero for waves whose wavevector is either parallel or perpendicular to the unperturbed trajectory, and maximized for $\cos\theta = \pm 1/\sqrt{3}$, or $\theta \approx 54.7^o$ or 125.3^o.

To compute the contribution to $\langle\delta^2\rangle$, we square δ_g as given in (42), summed over Fourier modes, and take the ensemble average. The angular integral involves

$$
\int_{-1}^1 dz e^{iz\Delta}(1-z^2)^2 z^2 = 16\left(\cos\Delta\left(\frac{90}{\Delta^6} - \frac{9}{\Delta^4}\right) + \sin\Delta\left(-\frac{90}{\Delta^7} + \frac{39}{\Delta^5} - \frac{1}{\Delta^3}\right)\right) \equiv F(\Delta),
\tag{43}
$$

where $z = \cos\theta$ and $\Delta \equiv x' - x''$. The integrals over x' and x'' then yield

$$
\begin{aligned}
&\langle\big(\delta_g(t)\big)^2\rangle = \frac{\mathcal{A}}{8}\int_0^{k_c}\frac{dk}{k}\int_0^x dx'\int_0^x dx''(x-x')(x-x'')\sin x'\sin x''\,F(x'-x'') = \frac{\mathcal{A}}{4}\int_0^{x_c}\frac{dx}{x}G(x),\\
&\text{with}\quad G(x) \equiv \frac{1}{3}x^2 + \frac{73}{6} - 6\gamma - \frac{4}{x^2} + \cos(2x)(-\frac{1}{6} + \frac{4}{x^2}) - \frac{2}{x}\sin(2x) + 6\left(\mathrm{Ci}(2x) - \ln(2x)\right),\\
&\implies \langle\big(\delta_g(t)\big)^2\rangle \approx \frac{\mathcal{A}}{12}\int_1^{x_c} dx x \approx \frac{\mathcal{A}}{24}x_c^2, \quad x_c \gg 1
\end{aligned}
\tag{44}
$$

where $x_c = (k_c t)/\sqrt{3}$, as before.

Since the constant $\mathcal{A}$ is the variance in the radiation density, per log interval in k, a quantity which we also term ϵ^2, we conclude that waves of wavenumber k can be expected to form shocks after theyhave undergone $\approx (\sqrt{6}/\pi)\epsilon^{-1}$ oscillations. From observations of the cosmic microwave background anisotropies, we infer that $\epsilon \approx 6.2\times 10^{-5}$ (cite Planck), we find the modes undergo around 12,600 oscillations before shock formation occurs.

(Mention we can likewise calculate the cross term

$$
\langle 2\delta_{fm}\delta_g\rangle \approx -\mathcal{A}x\frac{\pi}{16}
\tag{45}
$$

but this is subdominant)