Analysis 1: Chapter 3

Set Theory Exercises

Problem (Exercise 3.4.6)**.** *Let X be a set. Show that the collection of all subsets of X is a set.*

Proof. Appealing to the Axiom of Infinity and the Pair Set Axiom we have that $\{0, 1\}$ is a set. By the Power Set Axiom there exists a set containing all functions from X to $\{0, 1\}$ which we shall denote as $\{0, 1\}^X$. Define the property $P(f, S)$ pertaining to each $f \in \{0, 1\}^X$ and any object S to be the statement S is a set and $S = f^{-1}(\{1\})$. There is at most one set S such that $P(f, S)$ is true for each $f \in \{0, 1\}^X$. Thus by the Axiom of Replacement $Q := \{S : S = f^{-1}(\{1\}) \text{ for some } f \in \{0, 1\}^X\}$ exists and is a set. All that remains to show is that every element of Q is a subset of X and every subset of X is contained in Q. Let Z be an arbitrary element of Q, then there exists some function $f \in \{0, 1\}^X$ such that $Z = f^{-1}(\{1\})$ which is indeed a subset of X. Conversely let Z be a subset of X. We shall show that there exists some function $f \in \{0, 1\}^X$ such that $Z = f^{-1}(\{1\})$.

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