Analysis 1: Chapter 5
The Real Numbers

Remark. *A common problem solving technique we'll see again and again is to start with the conclusion and work backwards to a point that we can conclude from our hypothesis. The solution to the problem below was obtained in this manner (try it!).*

Problem. *The sequence $(a_n)_{n=1}^{\infty}$ defined by $a_n \coloneqq \frac{1}{n}$ is a cauchy sequence.*

Proof. Given a positive $\epsilon > 0$ we have that there exists a natural number N such that $\frac{1}{\epsilon} \leq N$ which implies that $\epsilon \geq \frac{1}{N}$ and for all $j, k \geq N$ we have that $|\frac{1}{j} - \frac{1}{k}| \leq \frac{1}{N}$ and the result follows as desired. $\square$

Problem (Exercise 5.1.1)**.** *Every cauchy sequence is bounded.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a cauchy sequence. We have then that $(a_n)_n^{\infty}$ is eventually 1-steady that is there exists a $N \geq 1$ such that for all j, k we have $|a_j - a_k| \leq 1$. We can then split $(a_n)_{n=1}^{\infty}$ into two parts $(a_i')_{i=1}^{N-1}$ and $(b_n')_{n'=N}^{\infty}$. Observe that $(a_i')_{i=1}^{N-1}$ is finite so it is bounded that is there exists some $M \geq 0$ such that $M \geq |a_i|$ for all $1 \leq i \leq N-1$. We also have that $(b_n')_{n'=N}^{\infty}$ is bounded since for $j \geq N$ we have that $|b_j - b_N + b_N| \leq |b_j - b_N| + |b_N| \leq 1 + |b_N|$. Take the max of $1 + |b_N|$ and M and the result follows. $\square$

Problem (Exercise 5.2.1)**.** *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of rational numbers. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent. Show that $(a_n)_{n=1}^{\infty}$ is a cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ is a cauchy sequence.*

Proof. Let ϵ be a positive rational then $\frac{\epsilon}{3}$ is also positive. Suppose that $(a_n)_{n=1}^{\infty}$ is a cauchy sequence then $(a_n)_{n=1}^{\infty}$ is eventually $\frac{\epsilon}{3}$-steady so there exists a positive $N_1 \geq 1$ such that for all $j, k \geq N_1$ we have $|a_j - a_k| \leq \frac{\epsilon}{3}$. Also since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent we have that there exists a positive $N_2 \geq 1$ such that for all $n \geq N_2$ we have $|a_n - b_n| \leq \frac{\epsilon}{3}$. We define $N := max(N_1, N_2)$ and appealing to the triangle inequality we have that for all $j, k \geq N$ that

$$\begin{aligned} |b_j - b_k| &= |b_j - a_j + a_j - b_k + a_k - a_k| \\ &\leq |b_j - a_j| + |a_k - b_k| + |a_j - a_k| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which shows that $(b_n)_{n=1}^{\infty}$ is cauchy as desired. The argument with the roles of $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ swapped shows that $(a_n)_{n=1}^{\infty}$ is bounded if $(b_n)_{n=1}^{\infty}$ is bounded. The result follows. $\square$

Problem (Exercise 5.2.2)**.** *Let ϵ be a positive rational. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are two sequences that are eventually ϵ-close. Then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.*

Proof. We have that there exists an $N \geq 1$ such that for all $n \geq N$ we have $|a_n - b_n| \leq \epsilon$. Suppose that $(a_n)_{n=1}^{\infty}$ is bounded then there exists an $M_1 \geq 0$ such that for all $n \geq 1$ we have $M_1 \geq |a_n|$. The finite sequence $b_1, b_2, b_3, \cdots, b_{N-1}$ is bounded by some $M_2 \geq 0$. Appealing to the triangle inequality we have for all $n \geq N$ that $|b_n| = |b_n - a_N + a_N| \leq |b_n - a_N| + |a_N| \leq \epsilon + M_1$. We then see clearly that $max(\epsilon + M_1, M_2)$ bounds $(b_n)_{n=1}^{\infty}$ thus $(b_n)_{n=1}^{\infty}$ is bounded as desired. The same argument above with the roles of $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ swapped shows that the converse is true. $\square$

Problem (Exercise 5.3.2). *Let $x = LIM_{n\to\infty}a_n$, $x' = LIM_{n\to\infty}a'_n$ and $y = LIM_{n\to\infty}b_n$ be real numbers. Show that xy is real (i.e that $(a_nb_n)_{n=1}^{\infty}$ is cauchy).*

First proof. We have that given a positive ϵ' that for sufficiently large $N \geq 1$ that for all $j, k \geq N$ we have $|a_j - a_k| \leq \epsilon'$ and $|b_j - b_k| \leq \epsilon'$ since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are cauchy. We also have that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are bounded by some common $C > 0$ (see Exercise 5.1.1). Thus we using the triangle inequality we have for all $j.k \geq N$ that

$$\begin{aligned} |a_jb_j - a_kb_k| &= |a_jb_j - a_kbj + a_kbj - a_kb_k| \\ &\leq |b_j||a_j - a_k| + |a_k||b_j - b_k| \\ &\leq C\epsilon' + C\epsilon' \end{aligned}$$

if we are given any positive ϵ we set $\epsilon' := \frac{\epsilon}{2C}$ then the result follows. $\square$

Second proof. We have that given a positive ϵ' and δ' that for sufficiently large $N \geq 1$ that for all $j, k \geq N$ we have $|a_j - a_k| \leq \epsilon'$ and $|b_j - b_k| \leq \delta'$ since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are cauchy. But since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are cauchy they are bounded by a common $C > 0$ We then observe that (see Proposition 4.3.7(h) in the text)

$$\begin{aligned} |a_jb_j - a_kb_k| &\leq |a_j|\epsilon' + |b_k|\delta' + \epsilon'\delta' \\ &\leq C\epsilon' + C\delta' + \epsilon'\delta' \end{aligned}$$

In particular for a given $\epsilon > 0$ we set $\epsilon' := \frac{\epsilon}{3C}$ and $\delta' = min(\frac{\epsilon}{3C}, C)$. Note that $C\epsilon' \leq C\frac{\epsilon}{3C} = \frac{\epsilon}{3}$. Also by the property of the min operator and definition of δ' we have $C\delta' \leq Cmin(\frac{\epsilon}{3C}, C) \leq C\frac{\epsilon}{3C} = \frac{\epsilon}{3}$ and $\epsilon'\delta' \leq \frac{\epsilon}{3C}C = \frac{\epsilon}{3}$. Thus $C\epsilon' + C\delta' + \epsilon'\delta' \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ and the result follows as desired.

$\square$
