Analysis 1: Chapter 7

Series Exercises

Problem (Exercise 7.2.2)**.** *Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if for each $\epsilon > 0$ there exists a natural $N \geq m$ such that for all $p, q \geq N$ we have $|\sum_p^q a_n| \leq \epsilon$.*

Proof. ($\Longleftarrow$) We shall that the sequence of partial sums $S_N := \sum_{n=0}^{N} a_n$ is Cauchy and hence convergent (see Proposition 6.4.18). Let $\epsilon > 0$ then $\frac{\epsilon}{2} > 0$ so by assumption there exists an $N' \geq m$ such that for all $p, q \geq N'$ we have $|\sum_p^q a_n| \leq \frac{\epsilon}{2}$. In particular if we set p to $N'+1$ we have $|\sum_{N'+1}^q a_n| \leq \frac{\epsilon}{2}$ for all $q \geq N'+1$. We can use Lemma 7.1.4(a) to write $S_q = \sum_m^{N'} a_n + \sum_{N'+1}^q a_n$ for all $q \geq N'+1$. From this observe that $|S_q - S_{q'}| = |\sum_{N'+1}^q a_n - \sum_{N'+1}^{q'} a_n| \leq \epsilon$ for all $q, q' \geq N'+1$ from the above discussion and the triangle inequality. Therefore S_N is Cauchy as desired.

($\Longrightarrow$) If the series is convergent then the partial sums sequence S_N is Cauchy. So for a given $\epsilon > 0$ we have that for some natural $N' \geq M$ that $|S_q - S_{q'}| \leq \frac{\epsilon}{2}$ for all $q, q' \geq N'$. In particular if we set q' to N' then we see that using Lemma 7.1.4(a) that $|S_q - S_{N'}| = |\sum_{n=m}^{N'} a_n + \sum_{n=N'+1}^{q} a_n - \sum_{n=m}^{N'} a_n| = |\sum_{N'+1}^q a_n| \leq \frac{\epsilon}{2}$ for all $q \geq N'+1$. Let $p, q \geq N'+1$ and without loss of generality suppose that $q \geq p > N'+1$ then by Lemma 7.1.4(a) we have that $\sum_{n=N'+1}^q a_n = \sum_{n=N'+1}^{p-1} a_n + \sum_p^q a_n$. Thus we see from the above discussion that

$$|\sum_{n=N'+1}^{q} a_n - \sum_{n=N'+1}^{p-1} a_n| = |\sum_{p}^{q} a_n| \leq |\sum_{N'+1}^{q} a_n| + |\sum_{n=N'+1}^{p-1} a_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and this completes the proof. $\square$
