Analysis 1: Chapter 5

The Real Numbers

Remark. *A common problem solving technique we'll see again and again is to start with the conclusion and work backwards to a point that we can conclude from our hypothesis. The solution to the problem below was obtained in this manner (try it!).*

Problem. *The sequence* $(a_n)_{n=1}^{\infty}$ *defined by* $a_n := \frac{1}{n}$ *is a cauchy sequence.*

Proof. Given a positive $\epsilon > 0$ we have that there exists a natural number N such that $\frac{1}{\epsilon} \leq N$ which implies that $\epsilon \geq \frac{1}{N}$ and for all $j, k \geq N$ we have that $|\frac{1}{j} - \frac{1}{k}| \leq \frac{1}{N}$ and the result follows as desired. □

Problem (Exercise 5.1.1). *Every cauchy sequence is bounded.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a cauchy sequence. We have then that $(a_n)_n^{\infty}$ is eventually 1-steady that is there exists a $N \geq 1$ such that for all j, k we have $|a_j - a_k| \leq 1$. We can then split $(a_n)_{n=1}^{\infty}$ into two parts $(a_i')_{i=1}^{N-1}$ and $(b_n')_{n'=N}^{\infty}$. Observe that $(a_i')_{i=1}^{N-1}$ is finite so it is bounded that is there exists some $M \geq 0$ such that $M \geq |a_i|$ for all $1 \leq i \leq N-1$. We also have that $(b_n')_{n'=N}^{\infty}$ is bounded since for $j \geq N$ we have that $|b_j - b_N + b_N| \leq |b_j - b_N| + |b_N| \leq 1 + |b_N|$. Take the max of $1 + |b_N|$ and M and the result follows. $\square$
