

Analysis 1: Chapter 5

The Real Numbers

Remark. *A common problem solving technique we'll see again and again is to start with the conclusion and work backwards to a point that we can conclude from our hypothesis. The solution to the problem below was obtained in this manner (try it!).*

Problem. *The sequence $(a_n)_{n=1}^{\infty}$ defined by $a_n := \frac{1}{n}$ is a cauchy sequence.*

Proof. Given a positive $\epsilon > 0$ we have that there exists a natural number N such that $\frac{1}{\epsilon} \leq N$ which implies that $\epsilon \geq \frac{1}{N}$ and for all $j, k \geq N$ we have that $|\frac{1}{j} - \frac{1}{k}| \leq \frac{1}{N}$ and the result follows as desired. \square

Problem (Exercise 5.1.1). *Every cauchy sequence is bounded.*

Proof. Let $(a_n)_{n=1}^\infty$ be a cauchy sequence. We have then that $(a_n)_n^\infty$ is eventually 1-steady that is there exists a $N \geq 1$ such that for all j, k we have $|a_j - a_k| \leq 1$. We can then split $(a_n)_{n=1}^\infty$ into two parts $(a'_i)_{i=1}^{N-1}$ and $(b'_n)_{n'=N}^\infty$. Observe that $(a'_i)_{i=1}^{N-1}$ is finite so it is bounded that is there exists some $M \geq 0$ such that $M \geq |a_i|$ for all $1 \leq i \leq N - 1$. We also have that $(b'_n)_{n'=N}^\infty$ is bounded since for $j \geq N$ we have that $|b_j - b_N + b_N| \leq |b_j - b_N| + |b_N| \leq 1 + |b_N|$. Take the max of $1 + |b_N|$ and M and the result follows. \square

Problem (Exercise 5.2.1). Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences of rational numbers. Suppose $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent. Show that $(a_n)_{n=1}^\infty$ is a Cauchy sequence if and only if $(b_n)_{n=1}^\infty$ is a Cauchy sequence.

Proof. Let ϵ be a positive rational then $\frac{\epsilon}{3}$ is also positive. Suppose that $(a_n)_{n=1}^\infty$ is a Cauchy sequence then $(a_n)_{n=1}^\infty$ is eventually $\frac{\epsilon}{3}$ -steady so there exists a positive $N_1 \geq 1$ such that for all $j, k \geq N_1$ we have $|a_j - a_k| \leq \frac{\epsilon}{3}$. Also since $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent we have that there exists a positive $N_2 \geq 1$ such that for all $n \geq N_2$ we have $|a_n - b_n| \leq \frac{\epsilon}{3}$. We define $N := \max(N_1, N_2)$ and appealing to the triangle inequality we have that for all $j, k \geq N$ that

$$\begin{aligned} |b_j - b_k| &= |b_j - a_j + a_j - b_k + a_k - a_k| \\ &\leq |b_j - a_j| + |a_k - b_k| + |a_j - a_k| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

which shows that $(b_n)_{n=1}^\infty$ is Cauchy as desired. The argument with the roles of $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ swapped shows that $(a_n)_{n=1}^\infty$ is bounded if $(b_n)_{n=1}^\infty$ is bounded. The result follows. \square

Problem (Exercise 5.2.2). *Let ϵ be a positive rational. Suppose that $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are two sequences that are eventually ϵ -close. Then $(a_n)_{n=1}^\infty$ is bounded if and only if $(b_n)_{n=1}^\infty$ is bounded.*

Proof. We have that there exists an $N \geq 1$ such that for all $n \geq N$ we have $|a_n - b_n| \leq \epsilon$. Suppose that $(a_n)_{n=1}^\infty$ is bounded then there exists an $M_1 \geq 0$ such that for all $n \geq 1$ we have $M_1 \geq |a_n|$. The finite sequence $b_1, b_2, b_3, \dots, b_{N-1}$ is bounded by some $M_2 \geq 0$. Appealing to the triangle inequality we have for all $n \geq N$ that $|b_n| = |b_n - a_n + a_n| \leq |b_n - a_n| + |a_n| \leq \epsilon + M_1$. We then see clearly that $\max(\epsilon + M_1, M_2)$ bounds $(b_n)_{n=1}^\infty$ thus $(b_n)_{n=1}^\infty$ is bounded as desired. The same argument above with the roles of $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ swapped shows that the converse is true. \square
