

Analysis 1: Chapter 3

Set Theory Exercises

Remark. In the text Tao informally gives a way to resolve Russell's paradox. He describes that objects are arranged (at least in set theory) in an hierarchy. At the first rung you have (impure set theory) what are called primitive objects which are not sets. At the next rung you have sets which only contain primitive objects. The rung after that has sets which only contain primitive objects or sets containing only primitive objects and so on. From this you see that if you are on a rung of the hierarchy not at the primitive object level then all sets at that level can only contain objects below it on the hierarchy. So things like a set containing itself can't be true because that set is on the same rung as itself. That is the intuition for the Axiom of Regularity and the problem below.

Problem (Exercise 3.2.2). Let A be a set. Show that $A \notin A$. Moreover show that for any two sets A and B that $B \notin A$ or $A \notin B$ (or both).

Proof. We shall prove the first part that is no set contains itself. If A is the empty set then we are done. Suppose now then that A is non-empty and for contradiction that $A \in A$. Using the singleton set axiom we have the singleton set $\{A\}$ exists. However if $A \in A$ then $A \in \{A\}$ but A is a set that is not disjoint from $\{A\}$. Thus $\{A\}$ is a non-empty set that contains just the set A which is not disjoint from itself. This contradicts the Axiom of Regularity thus for any set A we must have $A \notin A$.

We now prove the second part. Let A and B be two sets. If $A = B$ then from the first part we have that $B \notin A$ and $A \notin B$ and we see the result is true in this case. Moreover if A or B are the empty set then the result follows. Thus we suppose that A and B are two non-empty and non-equal sets. Suppose for contradiction that $A \in B$ and $B \in A$. Using the pair set axiom we can construct the pair set $\{A, B\}$ which is a non-empty set that only contains A and B . That is for all objects z we have $z \in \{A, B\}$ if and only if $z = A$ or $z = B$. In either case we have that $z \cap \{A, B\} \neq \emptyset$ thus $\{A, B\}$ is a non-empty that only contains sets that are not disjoint from itself. This contradicts the Axiom of Regularity thus we must have $A \notin B$ or $B \notin A$. The result follows as desired. \square

Problem (Exercise 3.4.6). *Let X be a set. Show that the collection of all subsets of X is a set.*

Proof. Appealing to the Axiom of Infinity and the Pair Set Axiom we have that $\{0, 1\}$ is a set. By the Power Set Axiom there exists a set containing all functions from X to $\{0, 1\}$ which we shall denote as $\{0, 1\}^X$. Define the property $P(f, S)$ pertaining to each $f \in \{0, 1\}^X$ and any object S to be the statement S is a set and $S = f^{-1}(\{1\})$. There is at most one set S such that $P(f, S)$ is true for each $f \in \{0, 1\}^X$. Thus by the Axiom of Replacement $Q := \{S : S = f^{-1}(\{1\}) \text{ for some } f \in \{0, 1\}^X\}$ exists and is a set. All that remains to show is that every element of Q is a subset of X and every subset of X is contained in Q . Let Z be an arbitrary element of Q , then there exists some function $f \in \{0, 1\}^X$ such that $Z = f^{-1}(\{1\})$ which is indeed a subset of X . Conversely let Z be a subset of X . We shall show that there exists some function $f \in \{0, 1\}^X$ such that $Z = f^{-1}(\{1\})$. For each $x \in X$ define f to be the mapping such that $f(x) := 1$ if $x \in Z$ and $f(x) := 0$ if $x \notin Z$. This is clearly a function from X to $\{0, 1\}$ which has $Z = f^{-1}(\{1\})$ thus $Z \in Q$ and the result follows. □

Problem (Exercise 3.4.7). Let X and Y be sets. A function $f : X' \rightarrow Y'$ from a subset X' of X to a subset Y' of Y is said to be a partial function from X to Y . If X and Y are sets show that the collection of all partial functions from X to Y is a set.

Remark. The idea for the proof we shall give is analogous to a sort of double looping construct procedure in a computer programming language. In particular let 2^X and 2^Y be the collections of all subsets of X and Y respectively which are sets by the previous exercise. Informally, for each element of 2^X we loop through each element of 2^Y creating a set containing all sets of function spaces where a function space is just the set of all functions from some particular element of 2^X to some particular element of 2^Y . Then we use the Axiom of Union to unbox all these sets of function spaces to get the set of all function spaces. Unboxing these sets in the same manner we get the set of all partial functions from X to Y .

Proof. Since X and Y are sets by the previous exercise asserts that the collection of all subsets of X is a set and also that the collection of all subsets of Y is set. Let 2^X and 2^Y denote these sets respectively. For each $X' \in 2^X$ and any object S define the property $P(X', S)$ to be the statement S is a set such that for all objects z , we have

$$z \in S \iff z = Y'^{X'} \text{ for some } Y' \in 2^Y.$$

There is at most one set S such that $P(X', S)$ is true for each $X' \in 2^X$. Thus by the Axiom of Replacement there exists a set $Q := \{S : P(X', S) \text{ is true for some } X' \in 2^X\}$. We observe that Q is a family of sets thus by the axiom of the union the set $\bigcup Q$ exists. This set like Q is also a family of sets. Thus applying the Axiom of Union once again we have that the set $\bigcup(\bigcup Q)$ exists. We claim that every element of $\bigcup(\bigcup Q)$ is a partial function from X to Y and that every partial function from X to Y is contained in $\bigcup(\bigcup Q)$. Let f' be an arbitrary element of $\bigcup(\bigcup Q)$ then $f' \in T$ for some $T \in \bigcup Q$. But then any such T will be contained in H for some set $H \in Q$. However if $H \in Q$ then for some $X' \in 2^X$ we have for every object K that $K \in H$ if and only if $K = Y'^{X'}$ for some $Y' \in 2^Y$. Thus since $T \in H$ we have $T = Y'^{X'}$ for some $Y' \in 2^Y$. So that if $f' \in T$ then f' is a partial function from X to Y . We therefore see that every element of $\bigcup(\bigcup Q)$ is a partial function from X to Y . Conversely let $f' : X' \rightarrow Y'$ be a partial function from X to Y from a subset X' of X to a subset Y' of Y . We shall show that $f' \in \bigcup(\bigcup Q)$. First note that by the Power Set Axiom the set $Y'^{X'}$ exists. Now for each $Y'' \in 2^Y$ let $P'(Y'', S')$ be the statement S' is a set such that $S' = Y''^{X'}$. For each $Y'' \in 2^Y$ there is at most one such set S' thus by the Axiom of Replacement the set $H := \{S' : S' = Y''^{X'} \text{ for some } Y'' \in 2^Y\}$ exists. Now observe that $Y'^{X'} \in H$ and $H \in Q$ which implies that $Y'^{X'} \in \bigcup Q$ but $f' \in Y'^{X'}$ thus $f' \in \bigcup(\bigcup Q)$ and the result follows.

□

Remark. Exercise 3.5.1 in the third edition of the text consists of three parts even though all are listed as a single exercise.

Problem (Exercise 3.5.1 (a)). Let x and y be any two objects (possibly the same). We define $(x, y) := \{\{x\}, \{x, y\}\}$. Which can be shown to be a set that exists using Axiom 3.4 and the Axiom of Union. Show that (x, y) obeys the usual property we would expect of an ordered pair (i.e two ordered pairs are equal if and only if all their components are equal).

Proof. We shall show that for any objects a, a', b and b' that $(a, b) = (a', b') \iff (a = a' \text{ and } b = b')$.
 (\implies) Suppose that $(a, b) = (a', b')$ then by definition we have that $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$. We will first show that $a = a'$. Since $\{a\} \in \{\{a\}, \{a, b\}\}$ we have $\{a\} = \{a'\}$ or $\{a\} = \{a', b'\}$. If $\{a\} = \{a'\}$ then we are done. On the other hand if $\{a\} \neq \{a'\}$ then we must have $\{a\} = \{a', b'\}$ which implies that $\{a\} = \{a'\}$. Therefore in either case $\{a\} = \{a'\}$. Now we show that $\{b\} = \{b'\}$. From our hypothesis we have that $\{a, b\} \in \{\{a'\}, \{a', b'\}\}$ and thus that $\{a, b\} = \{a'\}$ or $\{a, b\} = \{a', b'\}$. If $\{a, b\} = \{a'\}$ then $a = b = a'$. However $\{a', b'\} \in \{\{a\}, \{a, b\}\}$ by our hypothesis so that $\{a', b'\} = \{a\}$ or $\{a', b'\} = \{a, b\}$ and in either case we have that $b' = b$. If on the other hand we have $\{a, b\} \neq \{a'\}$ then we must have $\{a, b\} = \{a', b'\}$ which implies that $b' = a$ or $b = b'$. If $b = b'$ we are done so we consider the case when $b' = a$. Since $\{a, b\} \in \{\{a'\}, \{a', b'\}\}$ we have that $\{a, b\} = \{a'\}$ or $\{a, b\} = \{a', b'\}$ and in either case we have $b = b'$ since $a = a' = b'$.

(\impliedby) The converse is trivial □

Remark. We will appeal to the previous part of this exercise (i.e we will use the result and definition of ordered pair in part a). The proof we will provide is similar to that of Exercise 3.4.7.

Problem (Exercise 3.5.1 (b)). Let X and Y be two sets. Show that the Cartesian Product of X and Y is a set (i.e that a set containing all ordered pairs whose first component is from X and second is from Y exists).

Proof. For each $x \in X$ and any object S define $P(x, S)$ to be the statement S is a set such that for all objects z we have $z \in S \iff z = \{\{x\}, \{x, y\}\}$ for some $y \in Y$. For each $x \in X$ there is at most one set S such that $P(x, S)$ is true. Thus by the Axiom of Replacement there is a set $\{S : P(x, S) \text{ is true for some } x \in X\}$. By the Axiom of Union we have that $Q := \bigcup(\{S : P(x, S) \text{ is true for some } x \in X\})$ is a set. We claim that every element of Q is an ordered pair (x, y) where $x \in X$ and $y \in Y$ and that every such ordered pair is in Q . First suppose that $z \in Q$ then $z \in T$ for some $T \in \{S : P(x, S) \text{ is true for some } x \in X\}$. But then for this T there exists some $x \in X$ such that for every object q we have $q \in T$ if and only if $q = \{\{x\}, \{x, y\}\}$ for some $y \in Y$. Thus we have $z = \{\{x\}, \{x, y\}\}$ for some $x \in X$ and $y \in Y$ but by definition of an ordered pair for this x and y we have $(x, y) = \{\{x\}, \{x, y\}\} = z$ thus every element of Q is an ordered pair (x, y) where $x \in X$ and $y \in Y$. Now suppose that (x', y') is an ordered pair with $x' \in X$ and $y' \in Y$. For each $y \in Y$ and any object q define $P'(y, q)$ to be the statement $q = \{\{x'\}, \{x', y\}\}$. For each $y \in Y$ there is at most one object q such that $P'(y, q)$ is true. Thus by the Axiom of Replacement there is a set $T := \{q : P'(y, q) \text{ is true for some } y \in Y\}$. Observe then that $P(x', T)$ is true thus $T \in \{S : P(x, S) \text{ is true for some } x \in X\}$. Moreover $(x', y') = \{\{x'\}, \{x', y'\}\} \in T$ thus $(x', y') \in Q$. The result follows as desired.

□

Remark. See the text's definition for an ordered pair and observe we do not assert that an ordered pair is a set. However if we take the definition of an ordered set as it is in the exercise below it satisfies the property of what we would expect an ordered pair to have (i.e. that two ordered pairs are equal if and only if their first and second components are the same).

Problem (Exercise 3.5.1 (c)). Show that for any two objects x and y (possibly the same) that $(x, y) := \{x, \{x, y\}\}$ can also be used as an alternative definition for an ordered pair.

Proof. We shall show that for any objects a, a', b and b' that $\{a, \{a, b\}\} = \{a', \{a', b'\}\}$ if and only if $a = a'$ and $b = b'$. (\implies) Suppose that $\{a, \{a, b\}\} = \{a', \{a', b'\}\}$. We shall first show that $a = a'$. Suppose for contradiction that $a \neq a'$ then we must have that $a = \{a', b'\}$ and $a' = \{a, b\}$. However in this situation we will then have that $a \in a'$ and $a' \in a$ contradicting Exercise 3.2.2. Thus our initial assumption was false and we must have $a = a'$. Now we shall show that $b = b'$. By assumption we have $\{a, b\} \in \{a', \{a', b'\}\}$ so that $a' = \{a, b\}$ or $\{a', b'\} = \{a, b\}$. We have that $\{a, b\} \neq a'$ since otherwise we shall have $a = \{a, b\}$ as a direct consequence of the fact $a = a'$ which we have shown already. However then we have that $a \in a$ contradicting the fact that for any set A we must have $A \notin A$ (see Exercise 3.2.2). Therefore $\{a, b\} \neq a'$ but then $\{a, b\} = \{a', b'\}$. We see then that $b = a'$ or $b = b'$. If $b = b'$ we have nothing to show and if that is not the case then $b = a'$ but $b' = a$ and $a = a'$ thus $b = b'$. In either case we have $b = b'$ as desired. (\impliedby) The converse is trivial. □

Remark. Before we provide a solution for Exercise 3.5.2 we will discuss shortly empty functions. A empty function to an arbitrary set X is defined to be a function $f : \emptyset \rightarrow X$ whose domain is the empty set and range is X . In the text Tao states that given any arbitrary set X there is exactly one empty function to X . We will not concern ourselves with the general case but rather with the specific case when X is the empty set. Note that in this case we immediately have that $f : \emptyset \rightarrow \emptyset$ is a function no matter how we define f . Since for each $x \in \emptyset$ we have exactly one $y \in \emptyset$ assigned to x (this is vacuously true). So we do not need to specify what f does. Furthermore any other function from $g : \emptyset \rightarrow \emptyset$ will be equal to $f : \emptyset \rightarrow \emptyset$ since their domains and ranges are the empty set and for each $x \in \emptyset$ we have $f(x) = g(x)$ vacuously. Thus $f : \emptyset \rightarrow \emptyset$ is the unique empty function to the empty set. Note that this f is also surjective (vacuously again) and as you'll see when a definition for an ordered n -tuple is given in Exercise 3.5.2 that this f is the only 0-tuple with respect to that given definition. Thus we are justified in calling f the 0-tuple or the empty tuple which we will denote as $()$.

Problem (Exercise 3.5.2 (a)). Let n be a natural number. Suppose we define an ordered n -tuple to be a surjective function $x : \{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow X$ where X is an arbitrary set (different n -tuples are allowed to have different ranges). We define the i -th component of x denoted by x_i to be $x(i)$, and we also denote x as $(x_i)_{1 \leq i \leq n}$. Show that for two ordered n -tuples x and y that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ if and only if $x(i) = y(i)$ for $1 \leq i \leq n$.

Proof. We first consider the special case when $n = 0$. A 0-tuple is a surjective function $x : \{i \in \mathbb{N} : 1 \leq i \leq 0\} \rightarrow X$. However $\{i \in \mathbb{N} : 1 \leq i \leq 0\} = \emptyset$. Thus x must be an empty function to some set X . x cannot be surjective unless X is the empty set thus any 0-tuple must be an empty function to the empty set. However from our previous discussion we know that there is exactly one function from the empty set to the empty set so we have exactly one 0-tuple. So the result is true when $n = 0$. We now suppose that $n \geq 1$. Let x and y be two ordered n -tuples. (\implies) Suppose that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$. Then since x and y are equal functions they have the same domain $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ and range X such that for each $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$ we have $x(i) = y(i)$. Let i be a natural number between 1 and n , then $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$ thus $x(i) = y(i)$ for all $1 \leq i \leq n$.

(\impliedby) Suppose that x and y are two ordered n -tuples such that $x(i) = y(i)$ for all $1 \leq i \leq n$. Since they both have the same domain $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, we need to show that they have the same range and that they agree for all elements in their shared domain. Let X and X' denote the range of x and y respectively. Suppose that z is an arbitrary element of X , then since x is surjective there is some i in $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ such that $z = x(i)$ however i is an integer between 1 and n thus by assumption we have $x(i) = y(i)$ and thus $z \in X'$. Using the same argument but with the roles of X and X' swapped and the roles of x and y swapped we have that if z is an element of X' then z is also an element of X . Therefore X and X' are equal thus x and y have the same range. Let $i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$ then i is an integer between 1 and n thus by assumption $x(i) = y(i)$. The result follows as desired. \square
