Analysis 1: Chapter 3

Set Theory Exercises

Problem (Exercise 3.4.6)**.** *Let X be a set. Show that the collection of all subsets of X is a set.*

Proof. Appealing to the Axiom of Infinity and the Pair Set Axiom we have that $\{0,1\}$ is a set. By the Power Set Axiom there exists a set containing all functions from X to $\{0,1\}$ which we shall denote as $\{0,1\}^X$. Define the property $P(f,S)$ pertaining to each $f \in \{0,1\}^X$ and any object S to be the statement S is a set and $S = f^{-1}(\{1\})$. There is at most one set S such that $P(f,S)$ is true for each $f \in \{0,1\}^X$. Thus by the Axiom of Replacement $Q \coloneqq \{S : S = f^{-1}(\{1\}) \text{ for some } f \in \{0,1\}^X\}$ exists and is a set. All that remains to show is that every element of Q is a subset of X and every subset of X is contained in Q. Let Z be an arbitrary element of Q, then there exists some function $f \in \{0,1\}^X$ such that $Z = f^{-1}(\{1\})$ which is indeed a subset of X. Conversely let Z be a subset of X. We shall show that there exists some function $f \in \{0,1\}^X$ such that $Z = f^{-1}(\{1\})$. For each $x \in X$ define f to be the mapping such that $f(x) \coloneqq 1$ if $x \in Z$ and $f(x) \coloneqq 0$ if $x \notin Z$. This is clearly a function from X to $\{0,1\}$ which has $Z = f^{-1}(\{1\})$ thus $Z \in Q$ and the result follows.

□

Problem (Exercise 3.4.7)**.** *Let X and Y be sets. A function $f : X' \to Y'$ from a subset X' of X to a subset Y' of Y is said to be a partial function from X to Y. If X and Y are sets show that the collection of all partial functions from X to Y is a set.*

Remark. *The idea for the proof we shall give is analogous to a sort of double looping construct procedure in a computer programming language. In particular let 2^X and 2^Y be the collections of all subsets of X and Y respectively which are sets by the previous exercise. Informally, or each element of 2^X we loop through each element of 2^Y creating a set containing all sets of function spaces where a function space is just the set of all functions from some particular element of 2^X to some particular element of 2^Y. Then we use the Axiom of Union to unbox all these sets function spaces to get a set of all function spaces. Unboxing these sets in the same manner we get the set of all partial functions from X to Y*

Proof. Since X and Y are sets by the previous exercise asserts that the collection of all subsets of X is a set and also that the collection of all subsets of Y is set. Let 2^X and 2^Y denote these sets respectively. For each $X' \in 2^X$ and any object S define the property $P(X', S)$ to be the statement S is a set such that for all objects z, we have

$$z \in S \iff z = Y'^{X'} \text{ for some } Y' \in 2^Y.$$

There is at most one set S such that $P(X', S)$ is true for each $X' \in 2^X$. Thus by the Axiom of Replacement there exists a set $Q := \{S : P(X', S) \text{ is true for some } X' \in 2^X\}$. We observe that Q is a family of sets thus by the axiom of the union the set $\bigcup Q$ exists. This set like Q is also a family of sets. Thus applying the Axiom of Union once again we have that the set $\bigcup(\bigcup Q)$ exists. We claim that every element of $\bigcup(\bigcup Q)$ is a partial function from X to Y and that every partial function from X to Y is contained in $\bigcup(\bigcup Q)$. Let f' be an arbitrary element of $\bigcup(\bigcup Q)$ then $f' \in T$ for some $T \in \bigcup Q$. But then any such T will be contained in H for some set $H \in Q$. However if $H \in Q$ then for some $X' \in 2^X$ we have for every object K that $K \in H$ if and only if $K = Y'^{X'}$ for some $Y' \in 2^Y$. Thus since $T \in H$ we have $T = Y'^{X'}$ for some $X' \in 2^X$ and some $Y' \in 2^Y$. So that if $f' \in T$ then f' is a partial function from X to Y. We therefore see that every element of $\bigcup(\bigcup Q)$ is a partial function from X to Y. Conversely let $f' : X' \to Y'$ be a partial function from X to Y from a subset X' of X to a subset Y' of Y. $\square$
