COMPUTATIONAL FLUID DYNAMICS

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29/09/2024

ME24MTECH 11035

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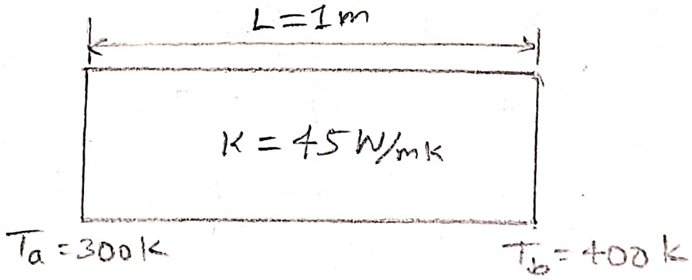
Assignment-1

1D Steady state Heat Conduction equation

(1)

Part 1 Consider a metal rod of length 1 meter with thermal conductivity ($K = 45$ W/m.K) shown in Fig. Both of its ends are maintained at a constant temp of $T_a = 300$ K & $T_b = 400$ K respectively. Assuming that the heat transfer is only taking place along the length of the rod, solve the steady state 1D heat conduction equation numerically & plot steady state contour of the temperature distribution along the length of the rod. Use a second-order central difference scheme to discretize your governing partial differential eqn.



Solution: Analytical results:-

Given, $L = 1$m

$K = 45$ W/m.K

$T_a = 300$ K @ $x = 0$

$T_b = 400$ K @ $x = L$

The eqn of 1D Heat Conduction eqn is

$$\frac{\partial^2 T}{\partial x^2} = 0$$

Tempt is the function of x only

$$\therefore \quad \frac{d^2T}{dx^2} = 0$$

integrate

$$\frac{dT}{dx} = C_1$$

again integrate on both side

$$T = C_1 x + C_2.$$

at $x = 0 \rightarrow T = T_a$

$$T_a = 0 + C_2$$

$$C_2 = T_a$$

$$\therefore \quad T = C_1 x + T_a$$

at $x = L \rightarrow T = T_b$

$$T_b = C_1 L + T_a$$

$$\frac{T_b - T_a}{L} = C_1$$

$\therefore$ The tempt eqn will be.

$$\boxed{T = \left(\frac{T_b - T_a}{L}\right) * x + T_a}$$

→ This is the Analytical soln for the temperature distribution along the metal rod.

→ Now we will try to solve this problem numerically by discretizing the eqn with following:-

1) Gauss-Siedel method
2) Jacobi method
3) TDMA method.

Numerical Solution

→ lets discritize the Governing differential equation which is 1D steady state heat conduct^n equation. with 2^nd order accuracy

→ Discritizing with the help of central difference method.

wkt According to Taylor series of expansion

$$T_{i+1} = T(x+\Delta x) = T_i + \frac{(\Delta x)}{1!}\frac{\partial T}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3} + \cdots$$

$$T_{i-1} = T(x-\Delta x) = T_i + \frac{(-\Delta x)}{1!}\frac{\partial T}{\partial x} + \frac{(-\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2} + \frac{(-\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3} + \cdots$$

add T_{i+1} & T_{i-1}

$$T_{i+1} + T_{i-1} = 2T_i + \cancel{2} \times \frac{\Delta x^2}{\cancel{2}}\frac{\partial^2 T}{\partial x^2} + 2\times\frac{(\Delta x)^4}{4!}\frac{\partial^4 T}{\partial x^4} + \cdots$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + \underbrace{O(\Delta x^2)}_{\text{Order of accuracy 2}}$$

The 1D heat conduct^n eq^n is

$$\frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = 0$$

$$2T_i = T_{i+1} + T_{i-1}$$

$$\boxed{T_i = \frac{T_{i+1} + T_{i-1}}{2}}$$

→ This eq^n is to be solved at each node except the nodes at boundary because of dirichlet boundary condition.

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→ Iterative methods of CFD

1) Gauss - Siedel method.

→ In these iteration methods to solve numerically at first we assume some values & use these methods to improve the solution.

→ In Gauss Siedel method for each iteration we are using the updated value from the previous iterations so this method will converge or give solution with less iterations than Jacobi method.

→ maximum error is taken as the difference b/w previous temp & present tempt.

→ The iterations are done untill the maximum error is less than the given tolerance value.

→ maximum error is taken as maximum of error b/w the perevious iterated tempt & present

→ we are using infinite norm to check the convergence of the given problem.

2) Jacobi method

→ In this method also at first we assume some values of temperature & use this method to improve the results.

→ In Jacobi method we are not using the updated value from the previous iteration to calculate the temperature in the present iteration rather we are storing the calculated values in a different array & we are updating the new values after completing all the calculations at every node.

→ This method will give solution by taking more iterations compared to the Gauss-Siedel method.

→ maximum error is taken as maximum of error between the previous iterated temperature and present one.

→ here also we are using infinite norm to check the convergence of the given problem.

3) TDMA

-> In this method we will convert the $n \times n$ coefficient matrix into tridiagonal matrix.

of the form
$$\begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ c_2 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & c_3 & b_3 & a_3 & 0 & \cdots \\ \vdots & 0 & \ddots & & & \\ \vdots & \vdots & & & \ddots & \end{bmatrix}$$

& the eqn of temp is

$$T_i = \left(\frac{-a_i}{b_i + c_i P_{i-1}} \right) T_{i+1} + \left(\frac{d_i - c_i Q_{i-1}}{b_i + c_i P_{i-1}} \right)$$

$$P_i = \frac{-a_i}{b_i + c_i P_{i-1}} \quad \& \quad Q_i = \frac{d_i - c_i Q_{i-1}}{b_i + c_i \cdot P_{i-1}}$$

at first we will create the Augmented matrix

where $a = -2$ & $b = c = 1$

but the 1st & ~~2nd~~ last ~~row~~ node we don't have to consider because we are taking dirichlet boundary condition

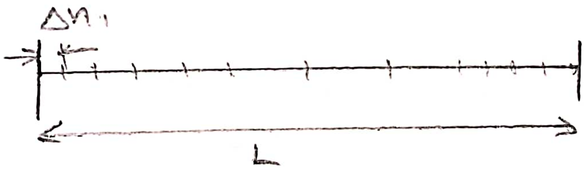
-> In dirichlet boundary condition we should not consider 1st & last node for this calculation

-> So we are taking value 1 in 1st row - 1st element & last row last element.

Non - Uniform grid

→ Untill now we have seen the calculations for the uniform grid, Now lets see how to discretize the governing eqn. for non uniform grid.

→ We are generating fine grids at the corner & coarse grids at the middle. Calculations as follows.



R = increament in Δx

$R = 1.2$ (20% increment preffered)

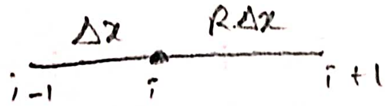
N = number of grids (to be choosen arbitarily)

$$M = \frac{N-1}{2}$$

$$A = \left[\frac{R^{M-1} - 1}{R-1} + \frac{R^{M-1}}{2} \right]$$

$$\Delta x_1 = \frac{L}{2A}$$

→ Now discritize the GDE for non uniform grid.



① $T_{i+1} = T(x + R\Delta x) = T_i + (R\Delta x)\frac{\partial T}{\partial x} + \frac{(R\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2} + \cdots$

② $T_{i-1} = T(x - \Delta x) = T_i + (-\Delta x)\frac{\partial T}{\partial x} + \frac{(-\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2} + \cdots$

→ add ① + R②

$$T_{i+1} + R T_{i-1} = T_i (1+R) + \frac{R^2+R}{2} \Delta x^2 \frac{\partial^2 T}{\partial x^2} + \dots$$

$T = f(x)$ only

$$\frac{d^2 T}{dx^2} = \frac{T_{i+1} + R T_{i-1} - (1+R) T_i}{\frac{(R^2+R)}{2}}$$

the GDE is $\frac{d^2T}{dx^2} = 0$

$$\therefore \quad T_i = \frac{T_{i+1} + R T_{i-1}}{(1+R)}$$

This discritisation is to be applied untill the middle node of the rod

→ we will get finer mesh size at the start & coarser mesh size at the middle

→ Now for 2nd half of the rod

① $T_{i+1} = T\left(x + \frac{\Delta x}{R}\right) = T_i + \left(\frac{\Delta x}{R}\right) \frac{dT}{dx} + \left(\frac{\Delta x}{R}\right)^2 \frac{1}{2!} \frac{d^2T}{dx^2} + \dots$

② $T_{i-1} = T(x - \Delta x) = T_i + (-\Delta x) \frac{dT}{dx} + \frac{(-\Delta x)^2}{2!} \frac{d^2 T}{dx^2} + \dots$

add ① + $\frac{1}{R}$ ②

$$T_{i+1} + \frac{1}{R} T_{i-1} = T_i \left(1 + \frac{1}{R}\right) + \frac{\Delta x^2}{(R^2+R)2} \frac{d^2T}{dx^2} + \dots$$

$$\rightarrow \frac{d^2T}{dx^2} = \frac{T_{i+1} + \frac{1}{R} T_{i-1} - \left(1+\frac{1}{R}\right) T_i}{\frac{\Delta x^2}{(R^2+R)2}}$$

$$\text{W.K.T} \quad \frac{d^2T}{dx^2} = 0$$

$$\therefore \quad T_i \left(\frac{R}{\cancel{R}} + \frac{1}{\cancel{R}}\right) = \frac{R T_{i+1} + T_{i-1}}{\cancel{R}}$$

$$\boxed{T_i = \frac{R T_{i+1} + T_{i-1}}{(R+1)}}$$

→ This equation is to be used from middle node to the final node of the rod to get the Coarse mesh at the middle & fine mesh at the end.

Successive over relaxation method.

→ Successive Over-Relaxation (SOR) is an iterative method to solve linear systems of equations, Particularly useful for large systems arising from discretized partial differential eqns. It is an improvement over the iterative methods, aimed at accelerating convergence.

→ let

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

add & subtract $a_{11}x_1$ on RHS

$$a_{11}x_1 = b_1 - a_{12}x_2 - a_{13}x_3 + a_{11}x_1 - a_{11}x_1$$

$$a_{11}x_1 = \underbrace{b_1 - [a_{11}x_1 + a_{12}x_2 + a_{13}x_3]}_{} + a_{11}x_1$$

This term will be zero when we know the exact solution of x_1 x_2 & x_3

This value we call it as residue.

$$x_1 = \left(\frac{b_1 - [a_{11}x_1 + a_{12}x_2 + a_{13}x_3]}{a_{11}}\right) + x_1$$

initially we are guessing the value of x_1 x_2 & x_3 then we are trying to make the residue zero.

we can write

$$x_1^{(1)} = \alpha\left[\frac{b_1 - (a_{11}x_1 + a_{12}x_2 + a_{13}x_3)}{a_{11}}\right] + x_p^{(0)}$$

The value of α is b/w 0 & 2

→ If our problem's numerical solution is going to diverge then we can use α value b/w 0 to 1 to converge or get the solution ~~by increa~~ which is done by increasing the number of iterations

This method is called <u>under relaxation</u>

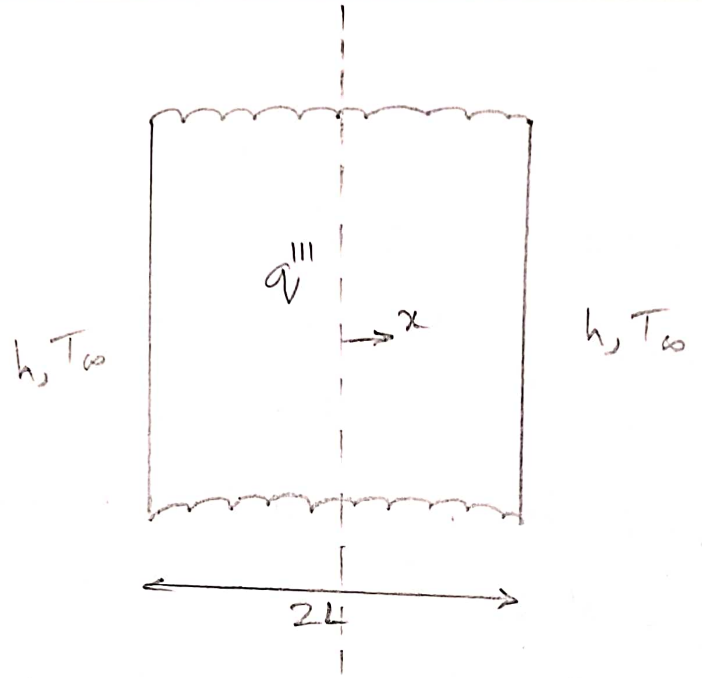
$0 < \alpha < 1$

→ If our problem is going to converge & get solutions but it is taking more number of iterations then we can reduce the no. of iterations or computational time by providing α value between 1 to 2

This method is called <u>over relaxation</u>

$1 < \alpha < 2$

PART-2



Consider a plane wall of thickness 2L with uniformly distributed heat source & therefore, heat is generated at the rate $q''' = 5 \times 10^4$ W/m³. On each exposed surface, the wall is bounded by a circulating fluid of temperature $T_\infty = 25°C$. The convective heat transfer coefficient for both surfaces is $h = 22$ W/m²K. The thermal conductivity of the wall is given as $K = 0.5$ W/mK. Assuming the thickness of the wall as 50 cm & heat transfer is 1-Dimensional, solve for the temperature distribution across the wall.

→ Given:- $2L = 50$ cm

$q''' = 5 \times 10^4$ W/m³

$T_\infty = 25°C$

$h = 22$ W/m² K

$K = 0.5$

1D Heat conduction eqn with steady state heat transfer

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}'''}{k} = 0$$

$$\frac{d^2 T}{d x^2} = -\frac{q_g}{K} \quad \Big| \because T = f(x)$$

integrate on both side

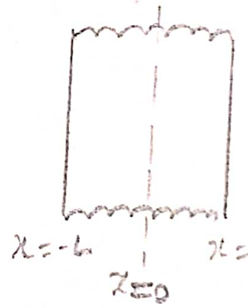
$$\frac{dT}{dx} = -\frac{q_g x}{k} + C_1$$

When $x = 0 \rightarrow \frac{dT}{dx} = 0.$

$$0 = 0 + C_1$$

$$C_1 = 0$$

$$\frac{dT}{dx} = -\frac{q_g x}{K}$$



again integrate on both side

$$T = -\frac{q_g x^2}{2K} + C_2$$

at $x = L, T = T_s$

$$T_s = \frac{-q_g L^2}{2K} + C_2 \quad \Big| \because T_s = \text{Surface tempt.}$$

$$C_2 = T_s + \frac{q_g L^2}{2K}$$

Substitute C_2

$$\text{Then, } T = -\frac{q_g x^2}{2k} + T_s + \frac{q_g L^2}{2k}$$

$$T - T_s = \frac{q_g}{2K}\left[L^2 - x^2\right]$$

$$T - T_s = \frac{q_g L^2}{2k}\left[1 - \frac{x^2}{L^2}\right] \quad \text{—(2)}$$

from the above eqn we can tell that the temperature profile is parabolic

at $x=0$ $T = T_{max}$

$$\therefore \quad T_{max} - T_s = \frac{q_g L^2}{2k} \quad -(3)$$

divide eqn (2) by (3)

$$\frac{T - T_s}{T_{max} - T_s} = \left[1 - \frac{x^2}{L^2}\right]$$

$$\therefore \quad \boxed{T = T_s + (T_{max} - T_s)\left[1 - \frac{x^2}{L^2}\right]}$$

Energy balance at surface to get T_s:

$$\cancel{Q_{in}}^{0} + Q_{gen} - Q_{out} = \cancel{Q_{stored}}^{0}$$

$$q_g \text{ Volume} = hA(T_s - T_\infty) \times 2$$

$$q_g \cancel{A}(\cancel{2}L) = h\cancel{A}(T_s - T_\infty) \times \cancel{2}$$

$$\boxed{T_s = T_\infty + \frac{q_g L}{h}}$$

→ This is the analytical solution for the given problem.

→ Now lets see the numerical solution

→ Before that we need to Discritize the given Governing differential equation.

→ From previous descritization we know that

$$\frac{d^2T}{dx^2} = \frac{T_{i+1} + T_{i-1} - 2T_i}{(\Delta x)^2}$$

the GDE is

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + \frac{q_g}{k} = 0$$

$$K(T_{i+1} - 2T_i + T_{i-1}) + \Delta x^2 q_g = 0$$

$$K(T_{i+1} + T_{i-1}) + \Delta x^2 q_g = 2KT_i$$

$$\boxed{T_i = \frac{T_{i+1} + T_{i-1}}{2} + \frac{\Delta x^2 q_g}{2K}}$$

This is the numerical solution for the given Problem.

This is to be applied b/w the 2 & n-1 th node as we are using dirichlet boundary condition.

→ We can use the following three numerical methods

1) Gauss-Siedel method
2) Jacobi method
3) TDMA method.

→ Both Gauss-Siedel & Jacobi method is to be done as we did in the part 1 only thing is that the eqn at each node to calculate the temperature is different.

→ In TDMA. our eqns will be of matrix form.

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 & \cdots & & & \\ 1 & -2 & 1 & 0 & \cdots & & \\ 0 & 1 & -2 & 1 & \cdots & & \\ \vdots & 0 & 1 & -2 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 1 \end{bmatrix}} \underset{T}{\begin{bmatrix} T_{left} \\ T_2 \\ T_3 \\ \vdots \\ \vdots \\ T_{right} \end{bmatrix}} = \underset{F}{\begin{bmatrix} T_{left} \\ -\frac{q_g \Delta x^2}{k} \\ \vdots \\ \vdots \\ T_{right} \end{bmatrix}}$$

$$T_{i+1} + T_{i-1} - 2T_i = \frac{-q_g \Delta x^2}{k}$$

In TDMA at first F matrix is calculated & with the help of F & A matrix we calculate the P & Q ~~no~~ values at each node.

& from eqn

$$T_i = P_i * T_{i+1} + Q_i$$

Where $P_i = \frac{-a_i}{b_i + c_i P_{i-1}}$ & $Q_i = \frac{F_i - c_i Q_{i-1}}{b_i + c_i P_{i-1}}$

→ As we are using Dirichlet Boundary Condition 1st row 1st element & last row last element is taken as 1 in the A matrix.