Lab2_2024

1 Lab 2: Fourier series

This workbook explores the Fourier series representation of signals. It is shown how symbolic methods can be used to calculate series coefficients, and the accuracy of the time-domain reconstruction for different numbers of coefficients is also considered.

1.1 Background

Any signal $x(t)$ that is periodic with a period T can be written in a Fourier series form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with $\omega_0 = 2\pi/T = \pi$ radians per second. The coefficients satisfy

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt.$$

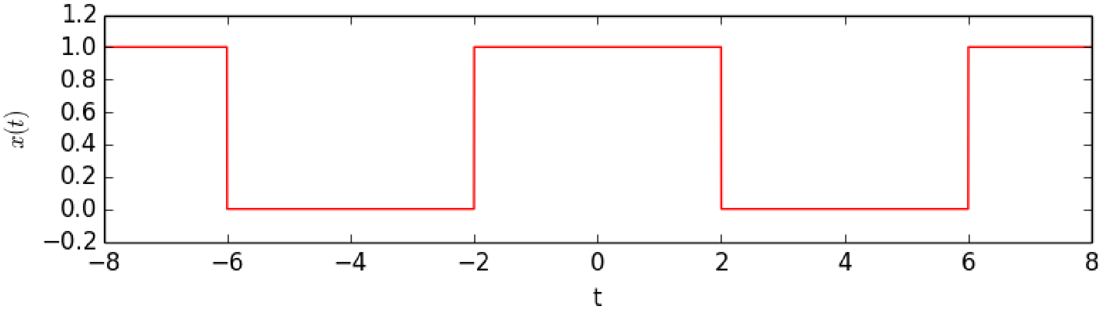
For real signals with $x(t) = x^*(t)$ one can show that $c_{-k} = c_k^*$. Writing in polar form $c_k = |c_k|e^{j\angle c_k}$ the series can be represented trigonometrically as

$$x(t) = c_0 + \sum_{k=1}^{\infty}(c_k e^{jk\omega_0 t} + c_{-k}e^{-jk\omega_0 t}) = c_0 + \sum_{k=1}^{\infty}|c_k|(e^{jk\omega_0 t}e^{j\angle c_k} + e^{-jk\omega_0 t}e^{-j\angle c_k}) = c_0 + \sum_{k=1}^{\infty} 2|c_k|\cos(\omega_0 t + \angle c_k)$$

The coefficient c_k corresponds to a complex exponential with frequency $k\omega_0$. We call the component of the signal with frequency $k\omega_0$ the kth *harmonic*. The first harmonic is also called the *fundamental*.

1.2 Signal definition and analysis

The following periodic signal, $x(t)$, is considered throughout this workbook:



The signal has period $T = 8$ seconds, so $\omega_0 = 2\pi/8 = \pi/4$ rad/second. We can then find the corresponding Fourier series coefficients:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-2}^{2} e^{-jk\omega_0 t}dt = \frac{1}{jk\omega_0 T}[-e^{-jk\omega_0 t}]_{t=-2}^{2} = \frac{2}{k\omega_0 T}\sin(2k\omega_0).$$

Additionally, the DC coefficient $c_0 = 4/8 = 0.5$.

1.3 Fourier series reconstruction

For a given set of coefficients c_k we want to be able to plot the corresponding $x(t)$. The function defined in the cell below takes a set of Fourier series coefficients (for a real signal) `ckv` and a fundamental frequency `omega0`, and then calculates reconstructed values `xv` at the time instants in `tv`.

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

def fsrrec(ckv,omega0,tv):
    """Generate samples from real Fourier series representation
    ckv - 0 to N Fourier series coefficients
    omega0 - fundamental frequency
    tv - input time points
    returns xv - output signal points
    """

    # tv.shape returns the dimensions of the matrix (or, in this case, array) tv
    # np.ones creates an array of the input size populated with 1's
    xv = ckv[0]*np.ones(tv.shape); # Set all values to c0 initially
    for k in range(1,len(ckv)): # Apply the trigonometric Fourier series␣
 ↪equation for k > 0
        xv = xv + 2*np.abs(ckv[k])*np.cos(k*omega0*tv + np.angle(ckv[k]));  #␣
 ↪Update the kth harmonic components

    return(np.real(xv));
# end def
```

The function below does the same as above, but it also creates a plot of the individual harmonic components, i.e. one cosine function for every k.

[2]:
```
def fsrrec_plots(ckv,omega0,tv):
    """The same as fssrec, but also outputs a plot of the individual harmonic␣
 ↪components"""

    xv = ckv[0]*np.ones(tv.shape);
    plt.figure(1)
    plt.plot(tv,np.real(xv), label="k = 0") # Plot the values of xv vs tv
    for k in range(1,len(ckv)):
```

```
        kh = 2*np.abs(ckv[k])*np.cos(k*omega0*tv + np.angle(ckv[k]));  # Create␣
 ↪kth harmonic
        label_str = "k = " + str(k); # Label for the legend
        plt.plot(tv,kh, label=label_str);
        xv = xv + kh; # Add kth harmonic to x

    plt.legend(loc="lower right")
    plt.show() # Show the plot
    return(np.real(xv));
# end def
```

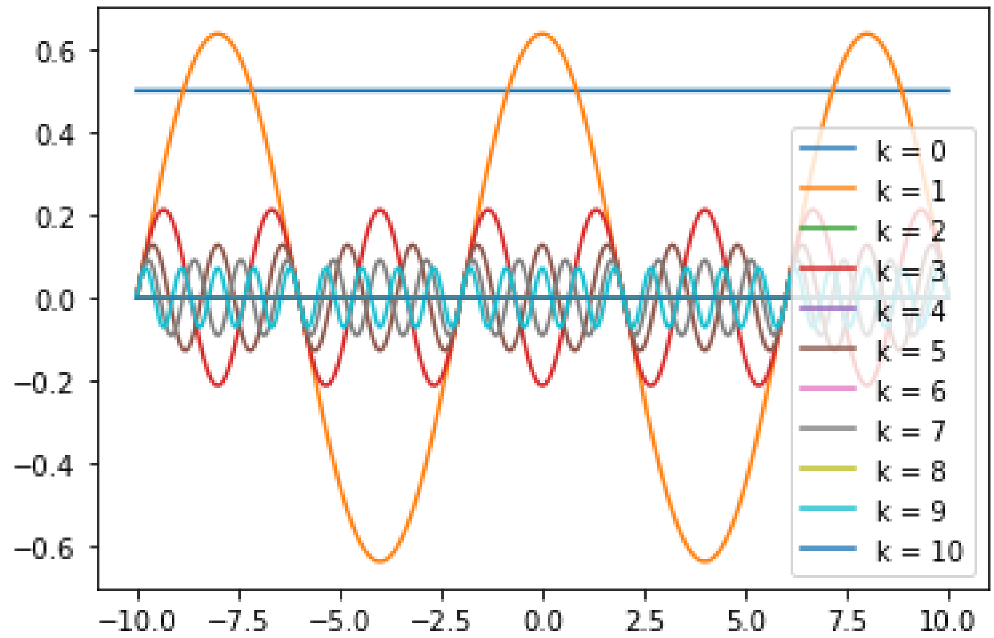
The cell below uses the derived expression for the coefficients of the signal and stores them in the vector `ckv`. The kth element of `ckv` contains the coefficient c_k.

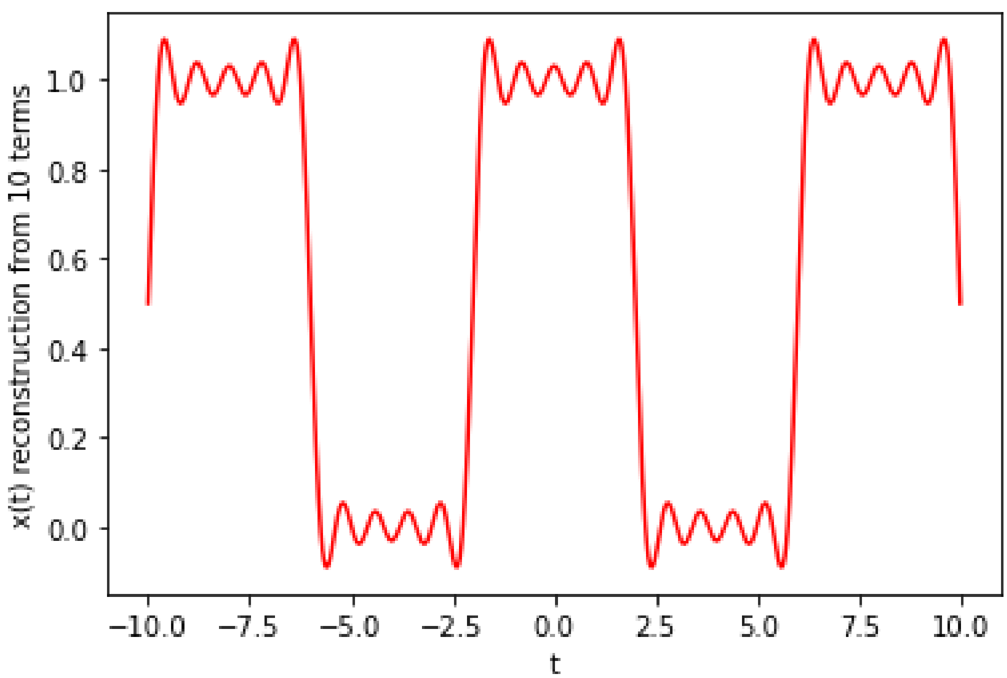
[3]:
```
# Fourier series coefficients for rectangular pulse train
T = 8;  # period
N = 10;  # maximum number of terms
omega0 = 2*np.pi/T;
ckv = np.zeros(N+1, dtype=np.complex64); # Type is complex (has real and␣
 ↪imaginary parts), i.e. I + jQ
# np.zeros is like np.ones but with 0's

for k in range(1,N+1):
    ckv[k] = 2/(k*omega0*T)*np.sin(2*k*omega0); # Set all ck values for k > 0
ckv[0] = 4/8; # Set c0
```

We can use the `fsrrec` function to find the time-domain representation of the signal $x(t)$ using a finite number of terms in the reconstruction. The code below does this and plots the result.

[4]:
```
# Reconstruct from series representation and plot
tv = np.linspace(-10,10,10000); # Set t limits
xv = fsrrec(ckv,omega0,tv); # Create x(t) using fssrec function
fsrrec_plots(ckv,omega0,tv); # Plot individual harmonic components
plt.figure(2) # Create new figure
plt.plot(tv,xv,'r'); # Plot overall x(t) vs t for 10 terms (N)
plt.xlabel('t');  plt.ylabel('x(t) reconstruction from ' + str(N) + ' terms');␣
 ↪# Set axes labels
```





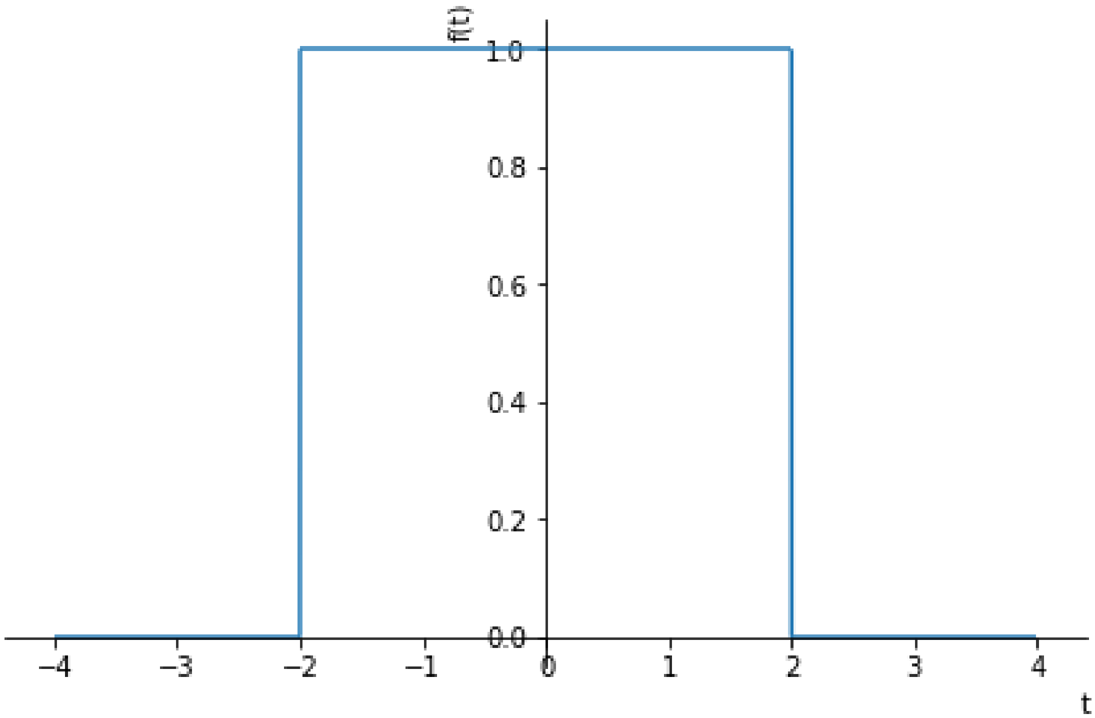
1.4 Finding the coefficients using symbolic math

This section will show how we can calculate the Fourier series coefficients of a signal using symbolic manipulation. The first thing to do is symbolically define the signal. The `Piecewise` function lets you define a signal over different pieces.

[5]:
```
import sympy as sp
from sympy import I
sp.init_printing();  # Pretty printing

t = sp.symbols('t')
x = sp.Piecewise( (0, t<-2), (1, t<2), (0, True));
sp.plot(x, (t,-4,4));  # Plot pulse function between t = -4 and t = 4
#sp.plot(x.subs(t,sp.re(t)), (t,-4,4));  # Use this to plot if the above␣
↪doesn't work (forces t to be real)

#?sp.Piecewise() # Uncomment this if you want Help for the Piecewise function
```



The following cell defines the symbolic integral for computing the coefficients.

[6]:
```
Ts, k, w0 = sp.symbols('Ts k w0');
w0 = 2*sp.pi/Ts;
expt = sp.exp(-I*k*w0*t);
cke = 1/Ts*sp.integrate(x*expt, (t, -Ts/2, Ts/2)); # See ck formula above (in␣
↪Background section)
#cke = sp.integrate(x*expt, (t, -sp.oo, T/2)) - sp.integrate(x*expt, (t, -sp.
↪oo, -T/2));  # Alternative
```

```
ck = cke.subs(Ts,T).doit();  # Set value for period and evaluate
```

We now define a vector `kv` of coefficients of interest, and populate corresponding elements of `ckv` with the coefficient values.

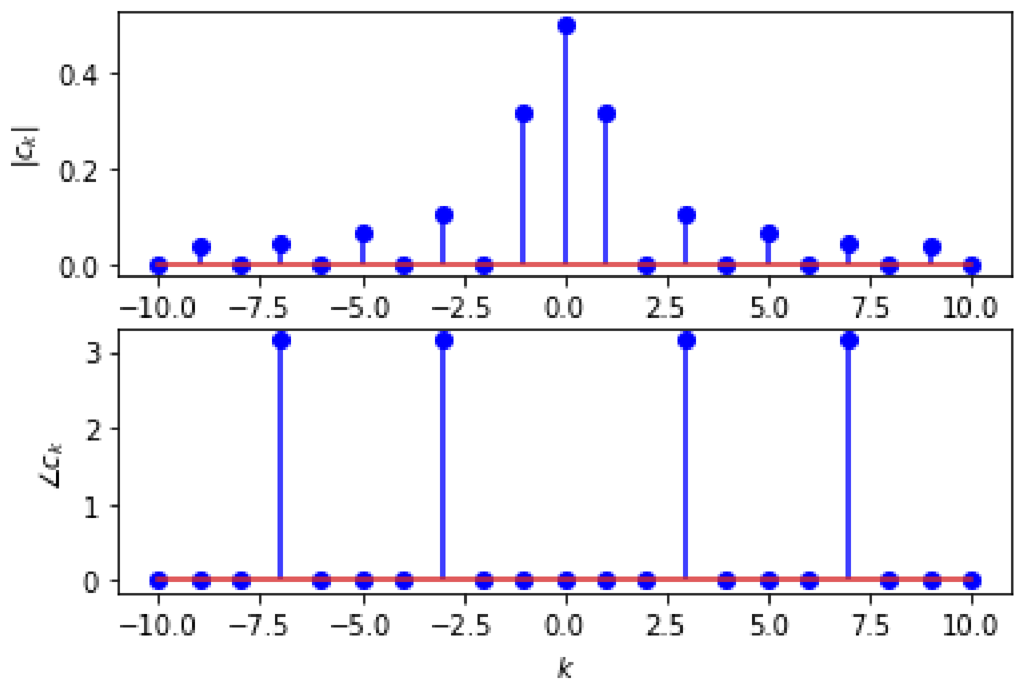
[7]:
```
kv = np.arange(-10,11);  # Coefficients to calculate, 21 points in total
# np.arange returns the integers between the given (start point) and (end point␣
↪- 1), i.e. -10 to +10

ckvs = np.zeros(kv.shape, dtype=np.complex64);  # Corresponding coefficient␣
↪values
for i in range(len(kv)):
    ckvs[i] = ck.subs({k:kv[i]}).evalf();
ckvs
```

[7]:
```
array([ 0.        +0.j,  0.03536776+0.j,  0.        +0.j, -0.04547284+0.j,
        0.        +0.j,  0.06366198+0.j,  0.        +0.j, -0.10610329+0.j,
        0.        +0.j,  0.31830987+0.j,  0.5       +0.j,  0.31830987+0.j,
        0.        +0.j, -0.10610329+0.j,  0.        +0.j,  0.06366198+0.j,
        0.        +0.j, -0.04547284+0.j,  0.        +0.j,  0.03536776+0.j,
        0.        +0.j], dtype=complex64)
```

Now we can plot the frequency-domain representation of the signal $x(t)$ by displaying the value of c_k for each value k of interest. Since c_k can in general be complex we need two plots: one for magnitude and one for phase.

[8]:
```
fh, ax = plt.subplots(2);
ax[0].stem(kv, np.abs(ckvs), 'b', markerfmt='bo');  ax[0].
↪set_ylabel(r'$|c_k|$');
ax[1].stem(kv, np.angle(ckvs), 'b', markerfmt='bo');  ax[1].
↪set_ylabel(r'$\angle c_k$');
plt.xlabel('$k$');
```



We could also have created a lambda function from the symbolic expression. This function takes an array of values for k and calculates c_k directly. NOTE: `lam_ck(0)` may generate a divide-by-zero error depending on your Python version.

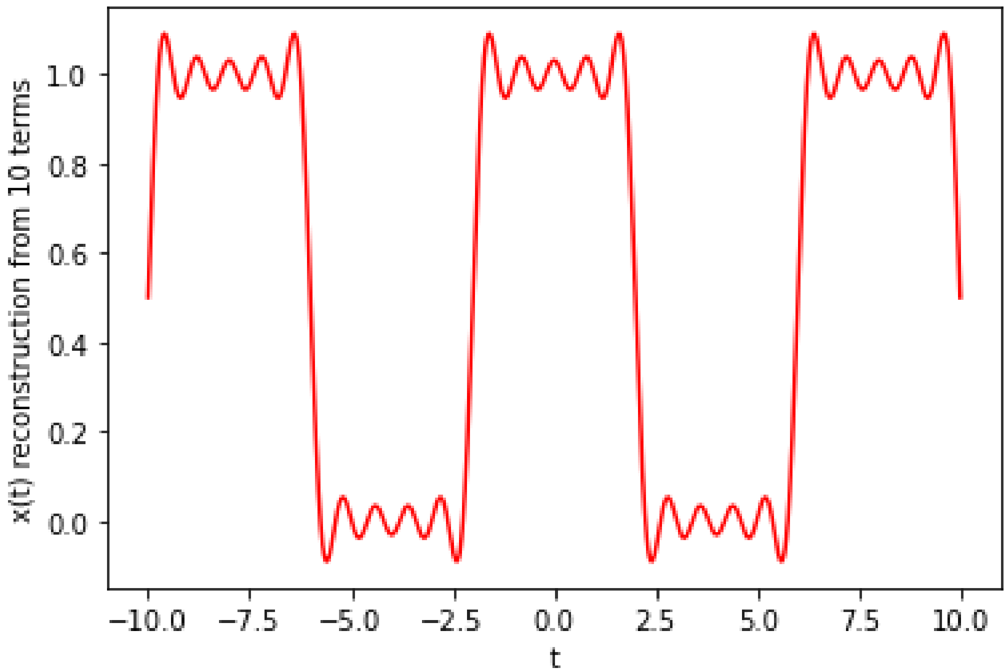
[9]:
```
lam_ck = sp.lambdify(k,ck,modules=['numpy']);
lam_ck(np.array((1,2,3)))
```

[9]:
```
array([ 3.18309886e-01+0.j,  1.94908592e-17+0.j, -1.06103295e-01+0.j])
```

With numerical values for the coefficients, obtained via symbolic computation, we can plot the partial sum for the time-domain reconstruction as before. Recall that our `fsrrec` function only takes the coefficients for non-negative index values.

[10]:
```
kzi = np.where(kv==0)[0][0];  # Index for zero element
ckvsp = ckvs[kzi:];
tv = np.linspace(-10,10,10000);
xv = fsrrec(ckvsp,2*np.pi/T,tv);

fh = plt.figure();
plt.plot(tv,xv,'r');
plt.xlabel('t');  plt.ylabel('x(t) reconstruction from ' + str(len(ckvsp)-1) +␣
 ↪' terms');
```

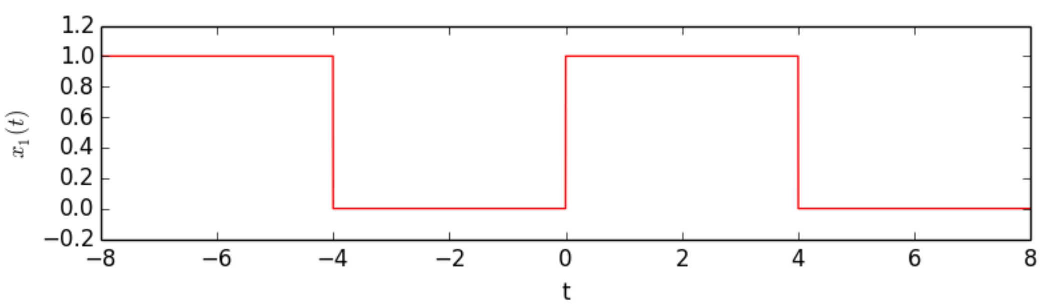


As expected, Figure 7 agrees with Figure 2. This shows how we could use symbolic maths to compute the c_k integral equation, or evaluate the integral manually and then calculate c_k in code for varying k values.

2 Tasks

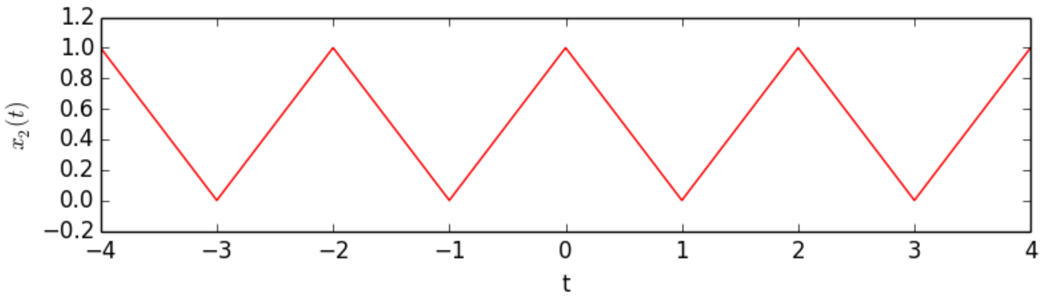
These tasks involve writing code, or modifying existing code, to meet the objectives described.

1. Find and plot the Fourier series frequency-domain representation for the signal $x_1(t)$ below. Show your code and your plots over the range $k = -8, \ldots, 8$. **Do this by evaluating the integral for the coefficients by hand - do not use symbolic processing.** Compare the result with that displayed earlier for $x(t)$ and comment on the changes (if any) in the magnitude and phase. Note that the signals are related in time by $x_1(t) = x(t-2)$. (5 marks)



2. Repeat Task 1 using symbolic processing. Be sure to show your code as well as the resulting plots. (5 marks)

3. Use symbolic processing to find and plot the frequency-domain representation of $x_2(t)$ below over the range $k = -8, \ldots, 8$. Also plot the reconstruction over the range $t = -4$ to $t = 4$ using only components up to and including the 8th harmonic. Comment on the magnitude of the coefficients and the accuracy of the reconstruction. (5 marks)



4. Find and plot the Fourier series frequency-domain representation of $x_3(t)$ below. Show your code and your plots over the range $k = -8, \ldots, 8$. (5 marks)