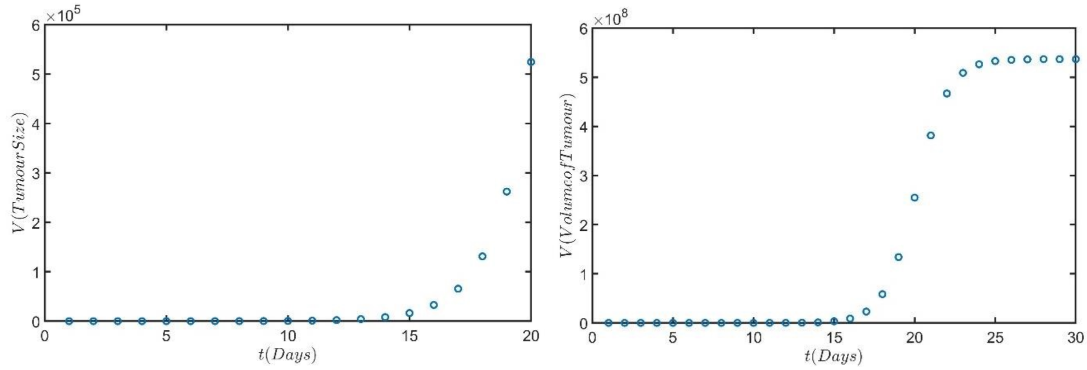
20DS715 : Complex Systems in Engineering, Finance & Biology – Modelling & Analysis

E1 - Date: 05/04/2021

Q1. You are involved in a research which focuses on developing a model for spread of cancer. You are expected to develop a mathematical model for tumour growth. You have been provided with two different sets of data corresponding to two different types of tumours. You are expected to derive mathematical models for the growth rate of each of them. The variation of the size of each type of tumour with time (in days) is shown below. Derive a mathematical model for the growth rate and justify your choice.



Solution:

Looking at the Model graph, we observe that the plot is exponential and does not have a carrying capacity.

Hence the Exponential model can be considered:

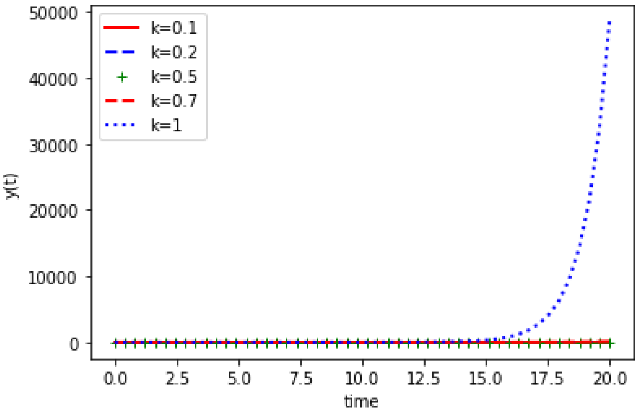
a)Exponential Model

$$\frac{dy}{dt} = ky$$

where

y the Population of the Tumour cell and

k is a Growth Rate



b)Power Law model

Looking at the Model graph, we observe that the plot is exponential

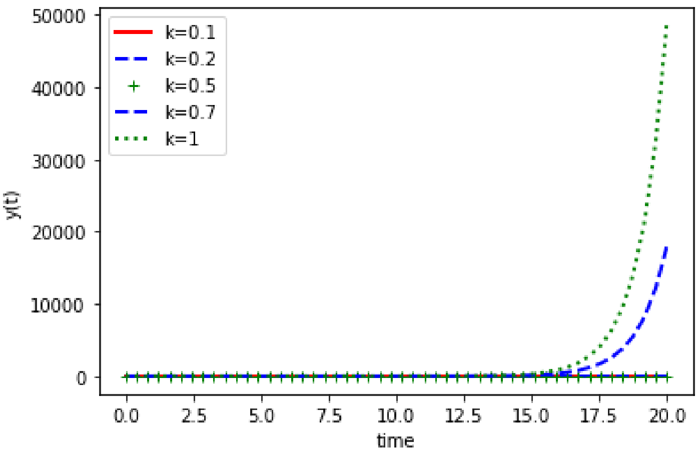
$$\frac{dy}{dt} = ky^a$$

where

y the Population of the Tumour cell k

is a Growth Rate

a is vasculature/immature tumout cell



c)Gompertz model

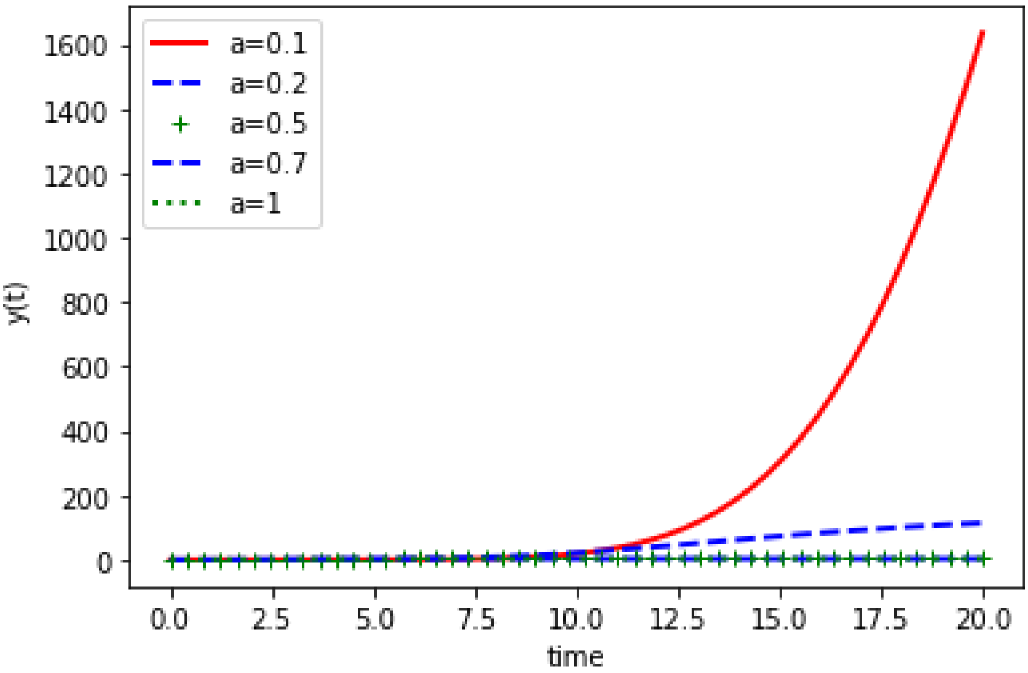
Looking at the Model graph, we observe that the plot is exponential.

$$\frac{dy}{dt} = (a - b * ln(y))y$$

where a is the tumour proliferation rate b is the rate of

exponential decay of the proliferation rate where y the

Population of the Tumour cell



Graph 2.

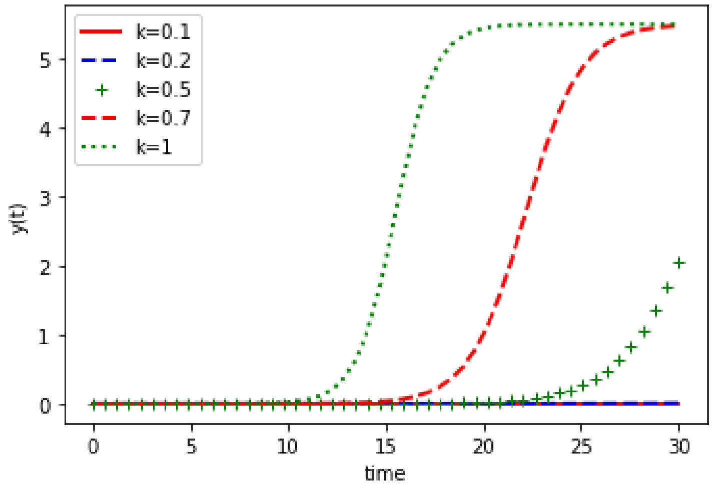
In graph 2, we understand that the exponential growth is but the curve saturates at a point. This point is commonly called a Carrying Capacity. We can use Logistic model and Von Bertalanffy growth model for fitting the tumour data.

a)Logistic Function

$$\frac{dy}{dt} = ky(1 - \frac{y}{K})$$

where,

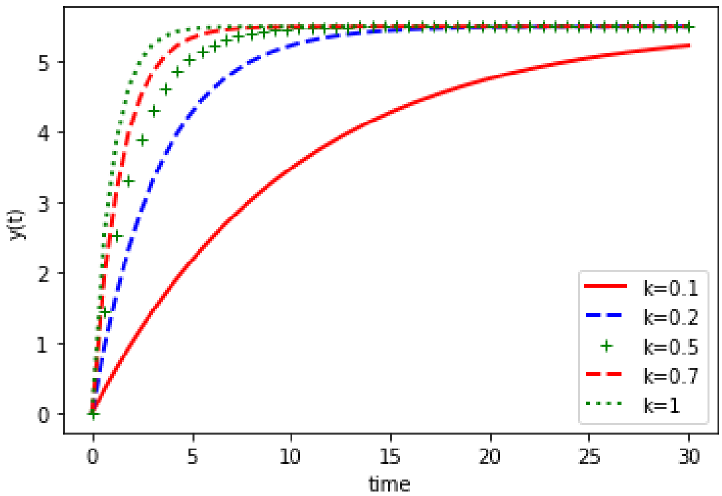
K is the carrying capacity.
y the Population of the Tumour cell. and
k is a Growth rate.



b)Von Bertalanffy growth model

$$\frac{dy}{dt} = k(K - y)$$

where
K is the carrying capacity.
y the Population of the Tumour cell. k
is a Growth rate.



2. Following set of coupled nonlinear one-dimensional differential equations represent the model for a spread of epidemics and the model is popularly known as SIR model. Where S, I and R represent susceptible, infected, and recovered population. β and γ represent

constants specific to a particular epidemic (epidemic of interest). Suppose that you are in the process of developing a mathematical model for the recent pandemic COVID-19.

$$S'(t) = -\beta(t).I(t).S(t)N,$$

$$I'(t) = \beta(t).S(t).I(t)N - \gamma(t).I(t),$$

$$R'(t) = \gamma(t).I(t)$$

Where,

Total Population $N(t) = S(t) + I(t) + R(t)$

β = Transmission rate
γ = Recovery rate

You have the following conditions/information about COVID -19. Can you suggest the modifications to be incorporated in the model (if you think the model needs to be refined) for each of the following observation?

a. It is observed that the rate at which the infection spreads has increased after 3 months of its initial occurrence.

b. Governments implemented complete lockdown after observing the fast spread of the pandemic.

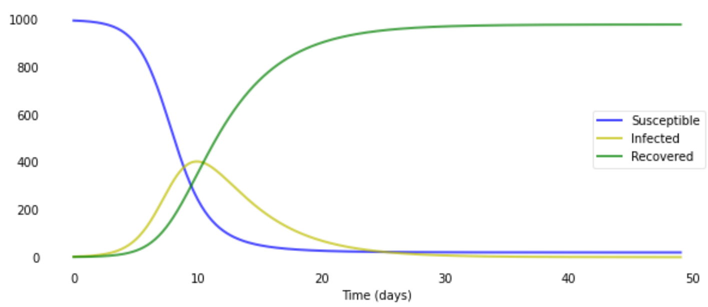
c. A medicine is discovered which will aid in recovery.

d. Scientists discovered vaccines and vaccination drive is in full swing.

e. A new variant of the virus is found which is more virulent (spreads at a higher rate and it takes longer time to recover)

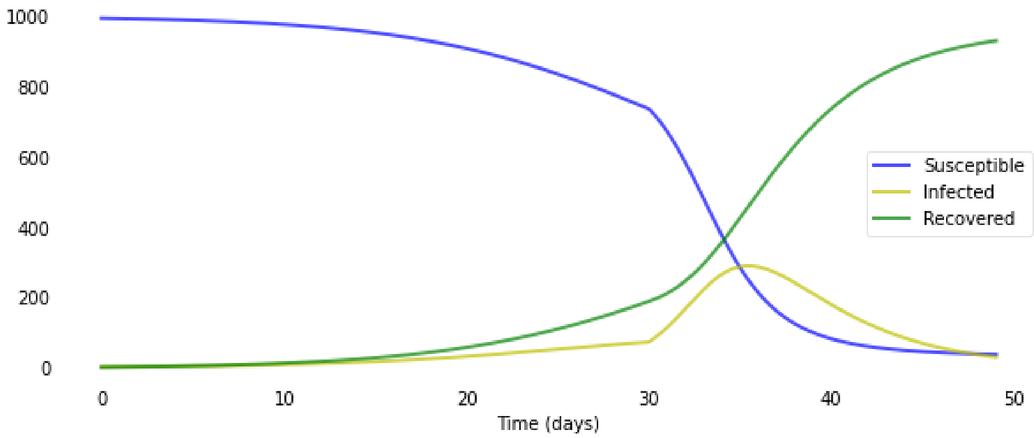
Solution:

The model Inference as plotted in the figure below.



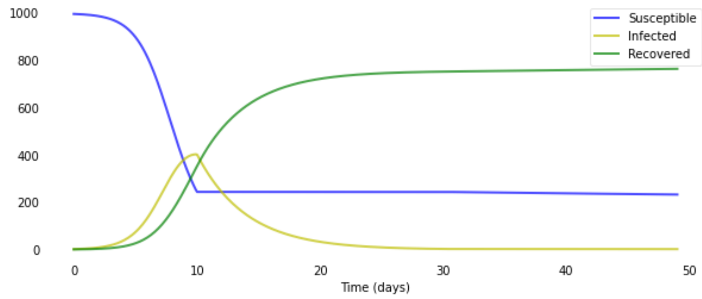
a. **It is observed that the rate at which the infection spreads has increased after 3 months of its initial occurrence.**

Here we assume that when time period t > 30, then β (Transmission rate) increases to 1. From Time period t=1 to 30, β value is 0.4.



b. Governments implemented complete lockdown after observing the fast spread of the pandemic.

This means that the transmission rate will be less. Let's assume **β** = 0 between Day 10 and Day30 Considering the Lockdown



c. A medicine is discovered which will aid in recovery.

After addition of the Medicine (dm) parameter to the existing SIR Model, we get

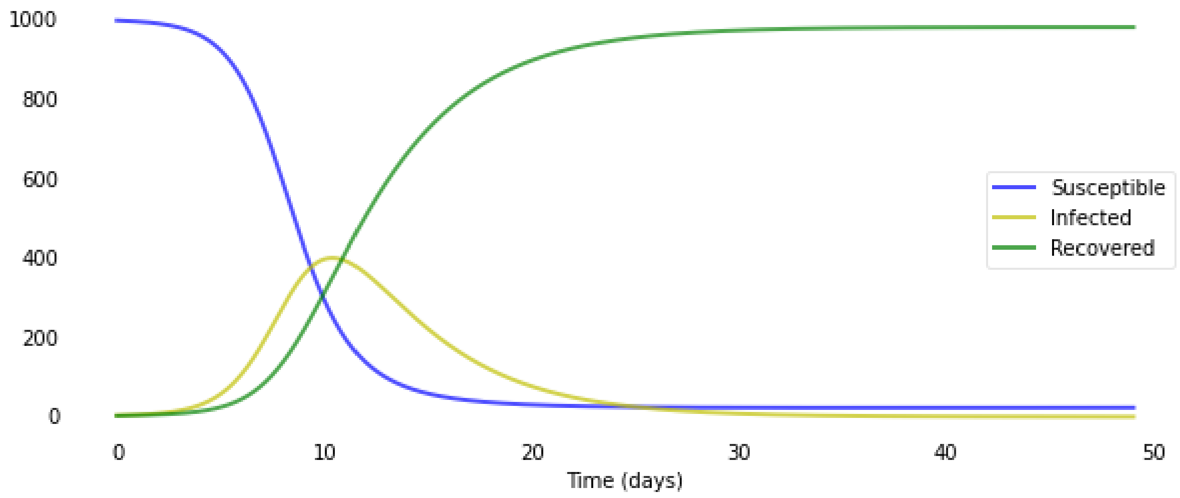
$S'(t) = -\beta(t).I(t).S(t)N$

$I'(t) = \beta(t).S(t).I(t)N - \gamma(t).I(t) - \boldsymbol{dm}$

$R'(t) = \gamma(t).I(t) + \boldsymbol{dm}$

where

Total Population = $N(t)=S(t)+I(t)+R(t)$
β = Transmission rate γ = Recovery
rate dm = Medicine Control



d. Scientists discovered vaccines and vaccination drive is in full swing

After addition of the Medicine (dm), Vaccination (dv) parameter is added to the existing SIR Model, we get

$S'(t)=-\beta\,(t).I(t).\,S(t)N\ -\boldsymbol{dv}$

$I'(t)=\ \beta\,(t).S(t).\,I(t)N-\gamma(t).\,I(t)\ -dm$

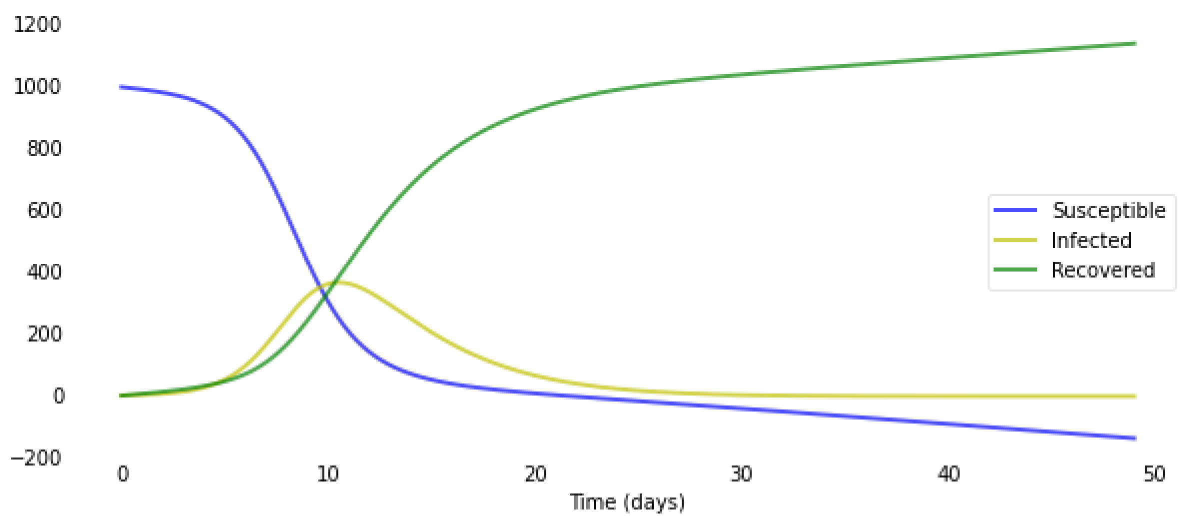
$R'(t)=\gamma(t).I(t)\ +\boldsymbol{dv}\ +dm$ where

Total Population $N(t)=S(t)+I(t)+R(t)$

β = Transmission rate γ = Recovery

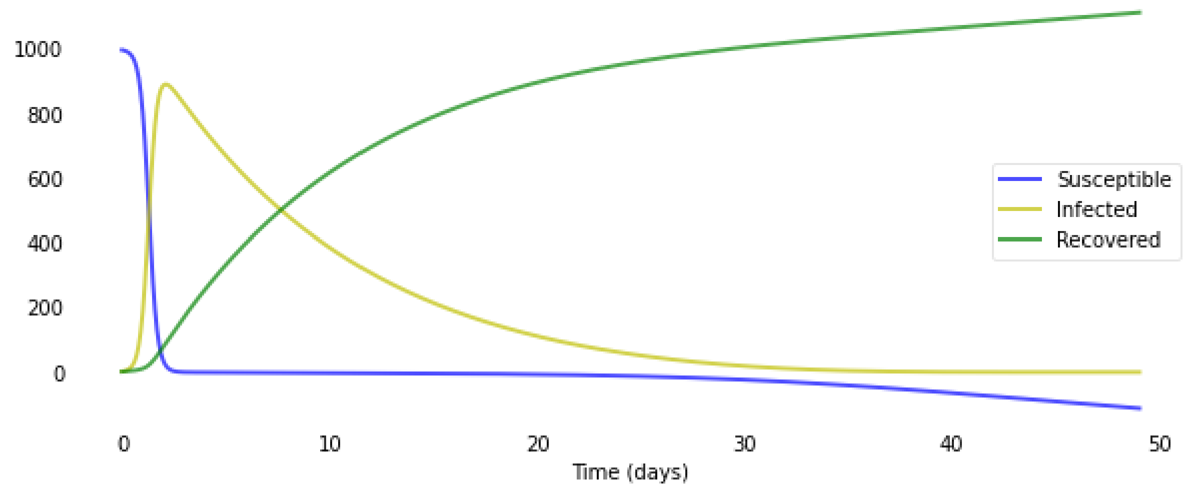
rate dm = Medicine Control dv =

Vaccine Control



e. A new variant of the virus is found which is more virulent (spreads at a higher rate and it takes longer time to recover)

This means that the transmission rate will be more, and the Recovery Rate will be long. Let's assume $\beta = 3$ and Days to recover D = 10days



Q3. You are involved in modelling the sales of a particular commodity There are two major companies A and B which sell this commodity. If the number of units sold by company A (represented by a) increases, it will adversely affect the number of units sold by company B (represented by b). From a market survey it is understood that people prefer the product by company A than that by company B. However, the sales capacity of company A is less than that of company B. Can you suggest a mathematical model that will describe the rate of change of units sold by company A and company B? If you are hired by company B, can you suggest some strategies to wipe-out company A from the market

(strategies based on the mathematical model and not based on any other malicious actions)

Let the commodity be X, the number of units sold by company A be ‘a’ and company B be ‘b’.

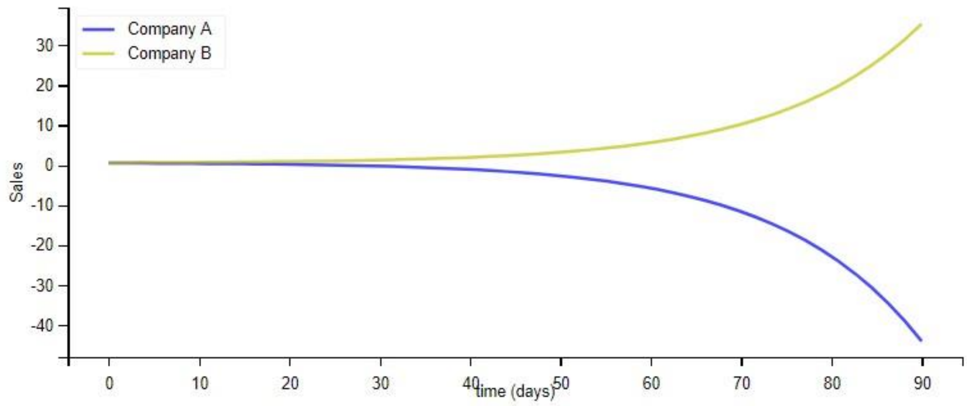
Number of units sold by companies are directly proportional to the Sales capacity(depends on number of sales representatives, number of weeks spent by sales team per year, percentage of time spent selling the commodity etc.)

Let the Sales Capacity of A be K1 and B be K2

Equations of the model are as follows:

$A_t = A_{t-1}$ −K1*a*(B/X)----1

$B_t = B_{t-1}$+K2*b*(A/X)-----2



Q4. You are hired by a firm involved in analysing electoral prospects of political parties. The organisation has given you the first assignment in a particular constituency. In this constituency there are only two major political parties. People vote either for the political party X or for the political party Y. The number of supporters of the political party X is represented by x and that of political party Y is represented by y. From a field survey, you understand that the rate at which the supporters increase is higher for the party Y to begin with. It is obvious that increase in number of supporters of one party will in turn result in decrease of number of supporters of the other political party. Can you arrive at a mathematical model describing the rate of change of number of supporters of the political parties in the chosen constituency ? Suppose that your company is entering in to a contract with the political party X and you are asked to suggest a strategy to completely

eliminate the political party Y. Can you suggest a strategy based on the mathematical model developed ?

Solution:

This case study extends the a predator-prey type model for the interactions between the parties have been considered.

In the prey –predator model ,We can consider the interactions between the two variables, i.e., how one variable influences the other and vice versa. Naturally, there should be the following effects:

• The prey's death rate increases as the predator population increases.

• The predators' growth rate increases as the prey population increases.

In the prey-predator model , total species population 'N' is divided into two classes:

A: Prey Class B: Predator Class

$x_t = x_{t-1} + r_x x_{t-1}(1 - x_{t-1}/K)$ —1, logistic growth model for the prey (x)

$y_t = y_{t-1} + d_y y_{t-1}$ —2, exponential decay model for the predators (y).

r_x is the growth rate of the prey, and d_y is the death rate of the predators ($0 < d_y < 1$).

a. Can you arrive at a mathematical model describing the rate of change of number of supporters of the political parties in the chosen constituency ?

Similarly, Let N be the total population considered in this case study, This total population can be divided into 3 classes:

i) Supporters of Political party X ii) Supporters of Political party Y iii) Mixed class of Voters

As we know that individuals quit the electoral process either due to death or other factors. Whereas individuals enter the electoral process based on the eligibility criteria set for the electoral process. Here we may easily observe that the net shifting of members will be either from party X to party Y or vice versa. Thus, let us put $\theta1-\theta2=\theta$ for the same into consideration.

Hence, the equations of the model are as follows: X=x/N

; Y=y/N ; V=v/N ------1

x denotes existing supporters in party X, whereas y denotes existing supporters in party Y and v denotes existing voters group in V.

We know that X+Y+V=N

Given that party X has minimum supporters initially as per the question hence, campaigns are inversely proportional to the parties' popularity levels based on the existing supporters in that party.

Let k1 be the average number of contacts of members of political party X with voters per unit time, and p1 be the probability of convincement per contact by a voter with a member of party X then the per capita recruitment rate in party X is β1=p1k1.

Similarly, Let k2 be the average number of contacts of members of political party Y with voters per unit time, and p2 be the probability of convincement per contact by a voter with a member of party Y then the per capita recruitment rate in party Y is β2=p2k2.

Supporters of a particular political party increase based on active campaigning , popularity of the party(based on ideologies, policies etc..) and switching from one party to other party on being invited(This is given by α=α1- α2).

Equations of the model are as follows:

$$\frac{dX}{dt} = \beta 1 V\frac{X}{N} - \alpha 1 X\frac{Y}{N} + \alpha 2\, Y\frac{X}{N} - \mu X \text{ -----1}$$

$$\frac{dY}{dt} = \beta 2 V\frac{Y}{N} + \alpha 1 X\frac{Y}{N} - \alpha 2\, Y\frac{X}{N} - \mu Y \text{ -----2}$$

$$\frac{dV}{dt} = \mu N - \beta 1 V\frac{X}{N} - \beta 2 V\frac{Y}{N} - \mu V \text{ -----3}$$

α=α1- α2 Substituting in above equations , will result into

following equations:

$$\frac{dX}{dt} = \beta 1 V\frac{X}{N} - \alpha X\frac{Y}{N} - \mu X \text{ ---4}$$

$$\frac{dY}{dt} = \beta 2 V\frac{Y}{N} + \alpha X\frac{Y}{N} - \mu Y \text{ -----5}$$

WKT , X=x/N ; Y=y/N ; V=v/N

Simplifying,

X+Y+V=N x/N+y/N+v/N=N

dX

$\frac{}{dt}$ = β1vx – αxy - µx ---6

$\frac{dY}{dt}$ = β2vy + αxy — µx -----7

(x+y+v)/N=N => [x+y+v=1] Substituting in above equations we get as follows:

$\frac{dX}{dt}$ = β1(1-x-y)x - αxy -µx ---8

$\frac{dY}{dt}$ = β2(1-x-y)y - αxy -µy -----9

Solving 8 & 9 to attain equilibrium condition.

(β1(1-x-y) - αy -µ)x=0 ----10

(β2(1-x-y) - αx -µ)y=0 -----11

Solving for x and y

x= $\frac{\mu(\beta1-\beta2)-\alpha(\beta2-\mu)}{\alpha(\beta1-\beta2+\alpha)}$ and y= $\frac{\alpha(\beta1-\mu)-\mu(\beta1-\beta2)}{\alpha(\beta1-\beta2+\alpha)}$

Equilibrium exists if $\beta1 > \beta2, Hence$

$\mu(\beta1 - \beta2) - \alpha(\beta2 - \mu)$>0---12

$\alpha(\beta1 - \mu) - \mu(\beta1 - \beta2)$>0---13

From equations 8 and 9 , Jacobian matrix is J for stability is given as follows:

$$J=\begin{pmatrix} \beta1(1-2x-y)-\alpha y-\mu & -\beta1 x-\alpha x \\ -\beta2 y+\alpha y & \beta2(1-x-2y)+\alpha x-\mu \end{pmatrix}$$

b.Suppose that your company is entering in to a contract with the political party X and you are asked to suggest a strategy to completely eliminate the political party Y.

One way to completely eliminate the political party Y is by campaigning which is directly proportional to convincing the voters group to join the party i.e Given that party X has minimum supporters initially as per the question hence, campaigns are inversely proportional to the parties’ popularity levels based on the existing supporters in that party.

Let k1 be the average number of contacts of members of political party X with voters per unit time, and p1 be the probability of convincement per contact by a voter with a member of party X then the per capita recruitment rate in party X is $\beta 1=p1k1$.

Let k2 be the average number of contacts of members of political party Y with voters per unit time, and p2 be the probability of convincement per contact by a voter with a member of party Y then the per capita recruitment rate in party

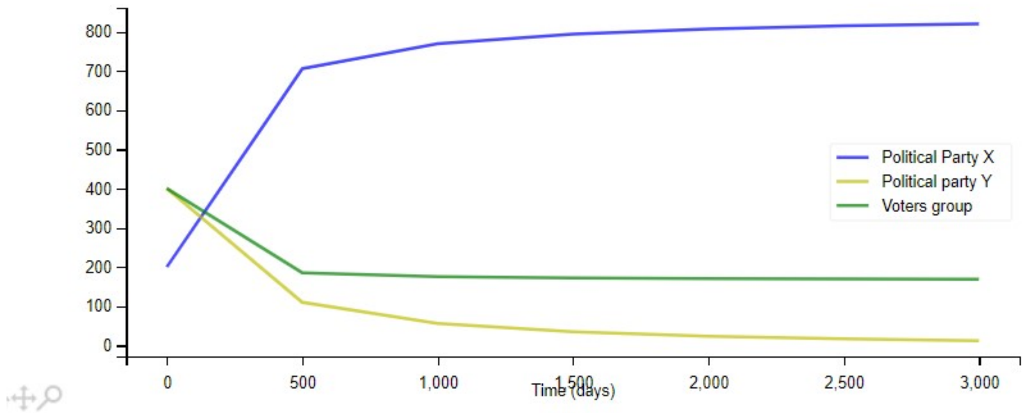
Y is $\beta 2=p2k2$. If $\beta 1 > \beta 2$;

The net rate of switching from X to Y is comparatively less. Hence, Y can be eliminated.

From equation 12

$\mu(\beta 1 - \beta 2) - \alpha(\beta 2 - \mu)>0$

Simplifying, $\alpha < \mu(\beta 1 - \beta 2)/(\beta 2 - \mu)$



5. You have been given an assignment to model the fish population in one of the coastal belts. It is known that the rate of change of the population will depend upon the current population and the resources in the specific coastal area (which is limited). In addition to this, you need to consider the effect of fishing by the fishermen. Fishermen in the area are not aware about the resource limitation and they catch a particular amount of fish everyday. Can you develop a mathematical model that will reflect the effect of constant fishing ? Do you think constant fishing can be allowed? If you do not want to allow constant fishing, can you suggest an alternate? Incorporate this change in the model by altering the term pertinent to fishing. Suppose that the government brings a seasonal ban for fishing (as per the advise from the fisheries scientists), what changes need to made in the model to reflect the new situation.

Solution:

We consider the logistic growth equation to model a fish population with limited to resources and carrying capacity.

Logistic Function

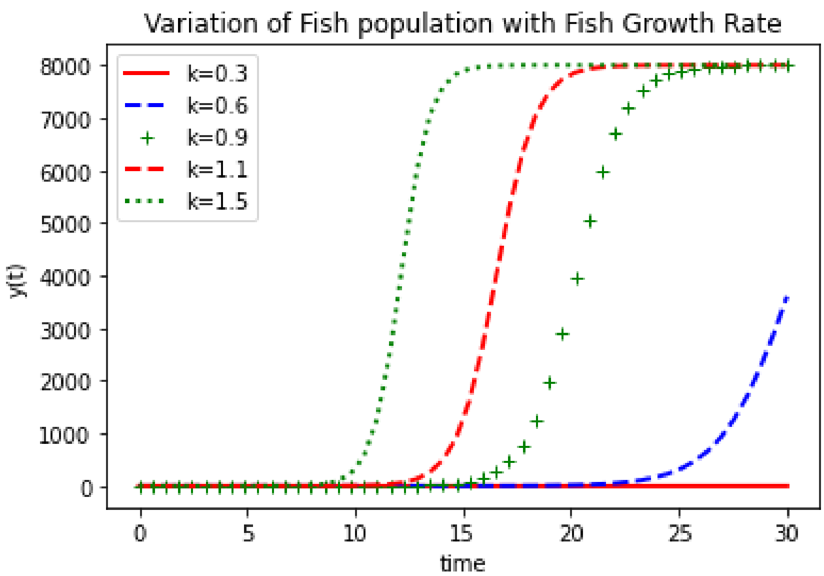
$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right)$$

where

K is the carrying capacity of the Fish

y the Fish Population and *k* is a

Growth Rate

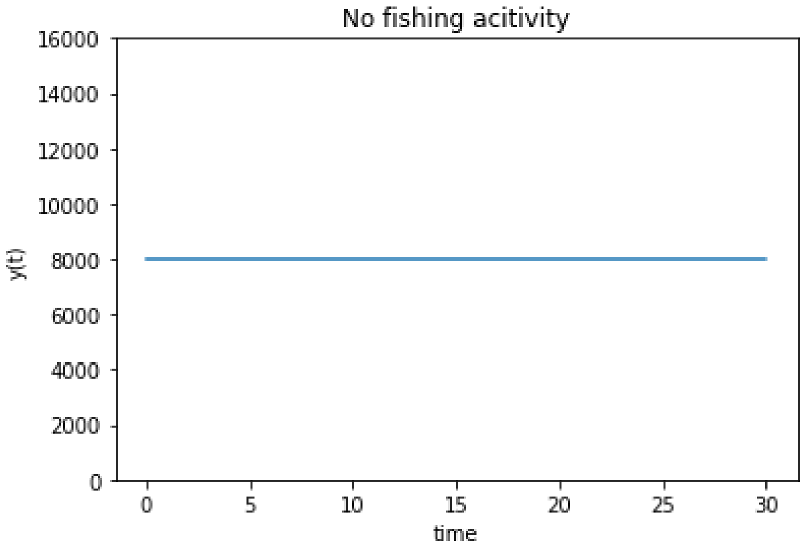


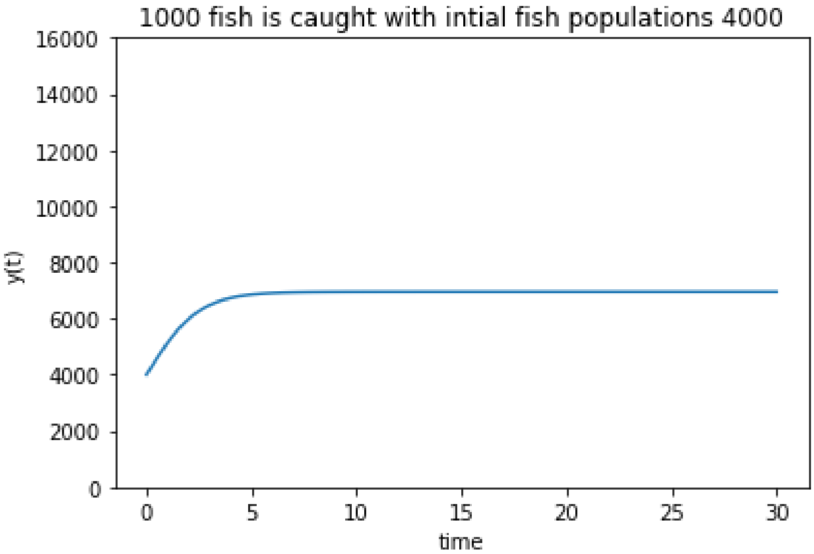
When we consider the effect of fishing by fishermen by a constant amount, we can modify the equation by adding a negative variable *h* that represents fishing rate.

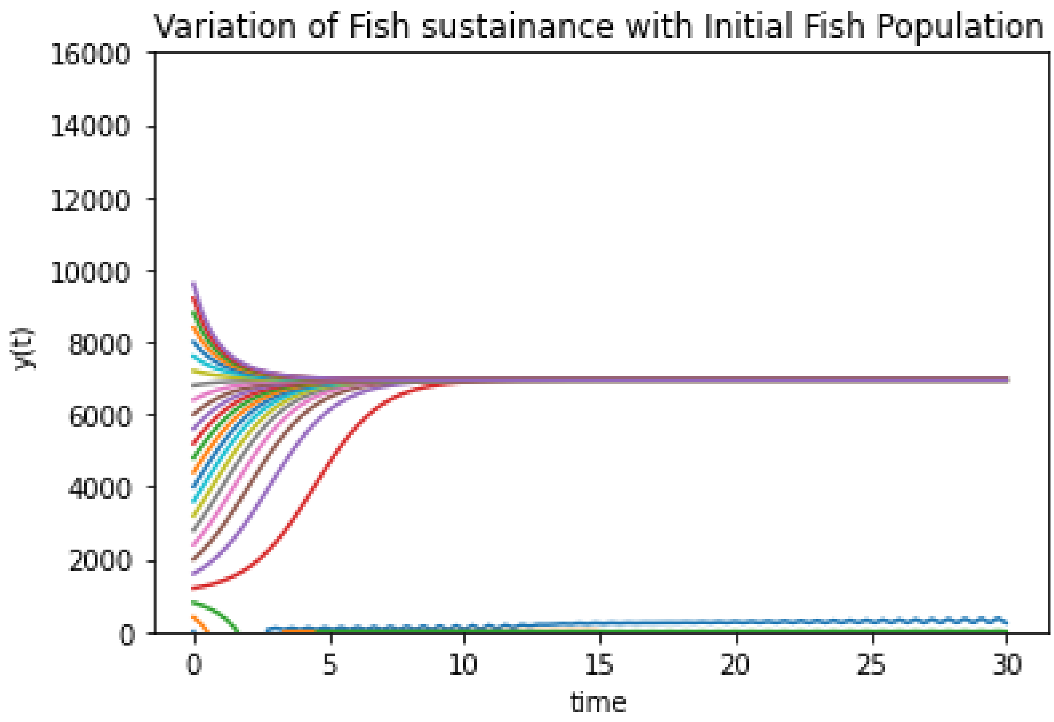
$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right) - \mathbf{h}$$

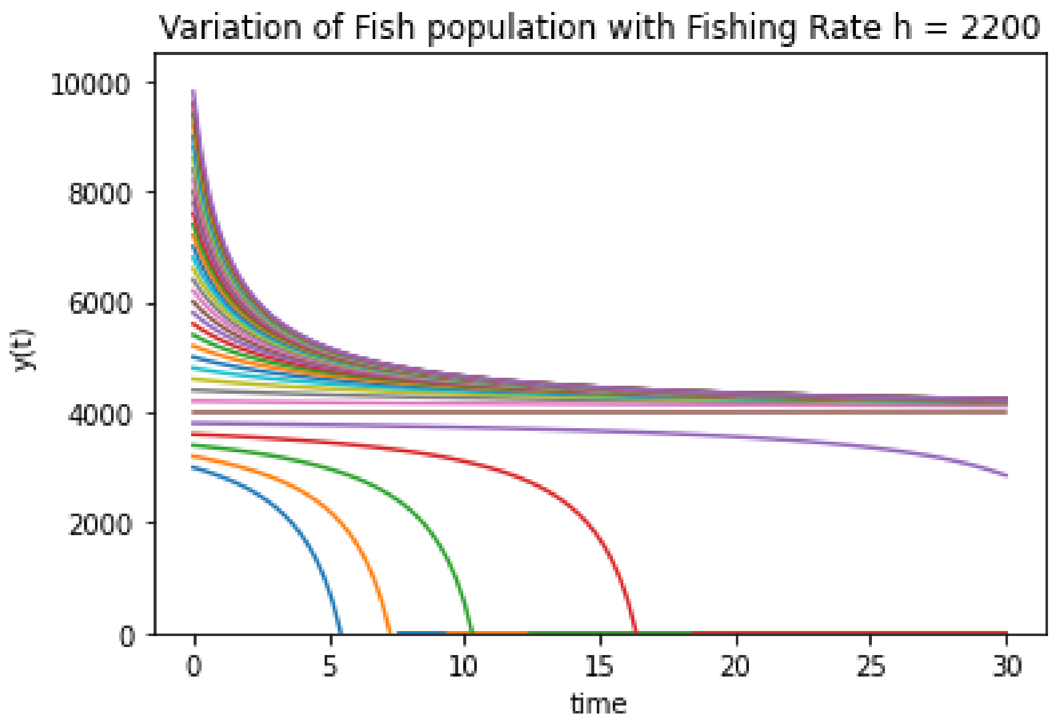
where

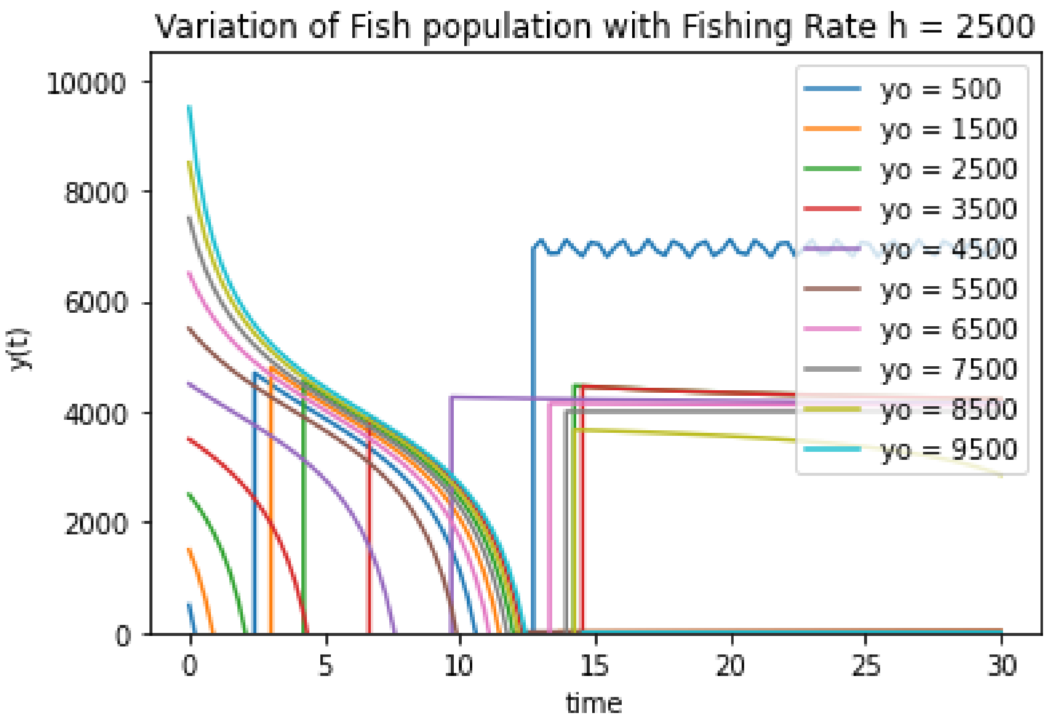
K is the carrying capacity of the Fish
y the Fish Population *k* is a Growth
Rate h is the harvest


No fishing acitivity
16000
14000
12000
10000
8000
6000
4000
2000
0
y(t)
0
5
10
15
20
25
30
time


1000 fish is caught with intial fish populations 4000
16000
14000
12000
10000
8000
6000
4000
2000
0
y(t)
0
5
10
15
20
25
30
time


Variation of Fish sustainance with Initial Fish Population
16000
14000
12000
10000
8000
6000
4000
2000
0
y(t)
0
5
10
15
20
25
30
time


Variation of Fish population with Fishing Rate h = 2200
10000
8000
6000
4000
2000
0
y(t)
0
5
10
15
20
25
30
time



For $h > 2200$, the model shows that the population always goes extinct. This model shows a classic example of a saddle node bifurcation.

$ry(1-yK)-h=0$ $ry(1-yK)=h$

$ry2-rKy+Kh=0$

By quadratic formula, we have the equilibrium solutions as follows, $y=Kr\pm Kr2-4rKh\surd 2r$

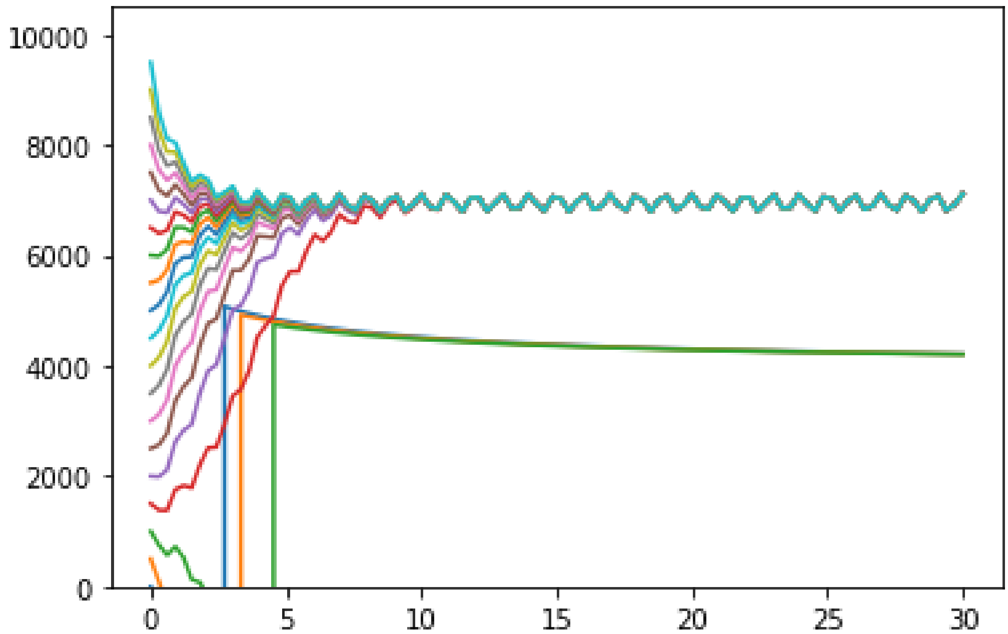
For maximum sustainable harvesting rate, we let the expression under the radical sign equal

zero, as follows $Kr2-4rKh=0$ $h=rK4$ $r=1.1$ $K=8000$ $\boldsymbol{h}=1.1*80004=\mathbf{2200}$

Fishing Cyclic ban

When we consider the effect of fishing by fishermen by a constant amount, we can modify the equation by adding a negative variable *h* that represents fishing rate.

$$\frac{dy}{dt} = ky(1 - yK) - h(1 + sin(2\pi t))$$



Seasonal Ban