Department of Mathematics, IIT Guwahati
MA 622: Galois Theory
Problem Sheet- II
January–May 2025

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1. Let F be a field, and suppose that L_1 and L_2 are field extensions of F contained in some common extension K of F. Then, the composite L_1L_2 of L_1 and L_2 is defined as the subfield of K generated by L_1 and L_2; that is, $L_1L_2 = L_1(L_2) = L_2(L_1)$.
   (a) If L_1 and L_2 are field extensions of F that are contained in a common field, show that L_1L_2 is algebraic over F if and only if both L_1 and L_2 are algebraic over F.
   (b) Let L and M be intermediate fields of the extension $F \subset K$, of finite dimension over F. If $[LM : F] = [L : F][M : F]$, prove that $L \cap M = F$. Further, prove that the converse holds if $[L : F]$ or $[M : F]$ is 2. Give an example where $L \cap M = F$, $[L : F] = [M : F] = 3$, but $[LM : F] < 9$.

2. Let K be a finite dimensional Galois extension of F, and let L and M be two intermediate fields. Prove that:
   (a) $\mathrm{Aut}(K/LM) = \mathrm{Aut}(K/L) \cap \mathrm{Aut}(K/M)$;
   (b) $\mathrm{Aut}(K/L\cap M) = \langle \mathrm{Aut}(K/L) \cup \mathrm{Aut}(K/M)\rangle$, the subgroup generated by $\mathrm{Aut}(K/L) \cup \mathrm{Aut}(K/M)$.
   (c) What conclusion can be drawn if $\mathrm{Aut}(K/L) \cap \mathrm{Aut}(K/M) = \{1\}$.

3. Let L and M be intermediate fields of the extension $F \subset K$. If L is a finite dimensional Galois extension of F, prove that LM is a finite dimensional Galois extension of M and $\mathrm{Aut}(LM/M) \cong \mathrm{Aut}(L/L \cap M)$.

4. Let n and m be positive integers. Prove that:
   (a) $\mathbb{Q}(\zeta_n)\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_\ell)$, where $\ell = \mathrm{lcm}(n, m)$.
   (b) $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_d)$, where $d = \gcd(n, m)$.

5. Let F be a field of characteristic $p > 0$. Let $f(x) = x^p - x - c \in F[x]$. Show that either all roots of $f(x)$ lie in F or $f(x)$ is irreducible in $F[x]$.

6. Let F be a field of characteristic zero and let p be an odd prime. Let $a \in F$ be such $a \neq 0$ and a is not a pth power of any element in F. Show that $f(x) = x^p - a$ is irreducible in $F[x]$. What can you say about the degree of a splitting field of $f(x)$ over F?

7. Let x, y be variables. Let $a, b, c, d \in \mathbb{Z}$ and $n = |ad - bc|$. Show that $L = \mathbb{C}(x, y)$ is a Galois extension of $K = \mathbb{C}(x^a y^b, x^c y^d)$ of degree n. Find $\mathrm{Aut}(L/K)$.

8. Prove that the Galois group of $x^p - 2 \in \mathbb{Q}[x]$, where p is a prime, is isomorphic to the group
$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{F}_p, a \neq 0 \right\}.$$

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