Department of Mathematics, IIT Guwahati
MA 622: Galois Theory
Problem Sheet- I
January–May 2025

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1. Determine the minimal polynomials of $1+i, 2+\sqrt{3}$, and $1+\sqrt[3]{2}+\sqrt[3]{4}$ over $\mathbb{Q}$.

2. Prove that x^3-2 and x^3-3 are irreducible over $\mathbb{Q}(i)$.

3. Let F/K be an algebraic field extension and R be a ring such that $K \subset R \subset F$. Show that R is a field.

4. Let F/K be an extension of degree n.

   (a) For any $a \in F$, prove that the map $T_a : F \to F$ defined by $T_a(x) = ax$ for all $x \in F$, is a linear transformation of the K-vector space F.

   (b) Prove that a is a root of the characteristic polynomial of T_a. Use this procedure to find monic polynomials satisfied by $\sqrt[3]{2}$ and $1+\sqrt[3]{2}+\sqrt[3]{4}$.

5. Prove that -1 is not a sum of squares in the field $\mathbb{Q}(\beta)$, where $\beta = \sqrt[3]{2}\, e^{2\pi i/3}$.

6. Let R be an integral domain containing $\mathbb{C}$. Suppose that R is a finite dimensional $\mathbb{C}$-vector space. Show that $R = \mathbb{C}$.

7. Let K be a field and x be an indeterminate. Let $y = x^3/(x+1)$. Find the minimal polynomial of x over $K(y)$.

8. Find an algebraic extension K of $\mathbb{Q}(x)$ such that the polynomial $f(y) = y^2 - x^3/(x^2+1) \in \mathbb{Q}(x)[y]$ has a root in K.

9. Find degrees of splitting fields over $\mathbb{Q}$ of each of the following polynomials:
   (a) x^3-2 (b) x^4-1 (c) x^4+1 (d) x^6+1 (e) $(x^2+1)(x^3-1)$ and (f) x^6+x^3+1.

10. Find a splitting field of x^3-10 over $\mathbb{Q}(\sqrt{2})$.

11. Let p be a prime. Show that the degree of a splitting field of x^p-2 over $\mathbb{Q}$ is $p(p-1)$.

12. Let $K \subset \mathbb{C}$ be a splitting field of $f(x) = x^3-2$ over $\mathbb{Q}$. Find a complex number z such that $K = \mathbb{Q}(z)$.

13. Let F be a field of characteristic p. Let $f(x) = x^p - x - c \in F[x]$. Show that either all roots of $f(x)$ lie in F or $f(x)$ is irreducible in $F[x]$.

14. Let F be a field of characteristic zero and let p be an odd prime. Let $a \in F^\times$ such that a is not a pth power of any element in F. Show that $x^p - a$ is irreducible over F. What can you say about the degree of its splitting field over F?

15. Let $a \in \mathbb{C}$ and $\sigma_a : \mathbb{C}(x) \to \mathbb{C}(x)$ be the automorphism that substitutes x by $x+a$. Pur $G = \{\sigma_a : a \in \mathbb{C}\}$. Show that the fixed field of G is $\mathbb{C}$, that is, $G' = \mathbb{C}$.

16. Let $\omega = e^{2\pi i/3}$. Define the $\mathbb{C}$-automorphisms σ and τ of $C(x)$ by the equations $\sigma(x) = \omega x$ and $\tau(x) = 1/x$. Show that $\sigma^3 = \tau^2 = id$ and $\tau\sigma = \sigma^{-1}\tau$. Show that the group G of automorphisms generated by σ and τ has order 6 and $G' = \mathbb{C}(x^3 + x^{-3})$.

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