1

a) Consider $f(x) = (x-1-i)(x-1+i) = x^2 - 2x + 2 \in \mathbb{Q}[x]$ by Eisenstein condition and $p = 2$ this is irreducible. in $\mathbb{Q}[x]$.

b) Consider $f(x) = (x-(2+\sqrt{3}))(x-2+\sqrt{3}) = x^2 - 4x + 1$ by Gauss lemma if $f(x)$ can be factorized in $\mathbb{Q}[x]$ so does it in $\mathbb{Z}$ so assume.

$$f(x) = (x-a)(x-b)[\because f(x) \text{ is monic }].$$

So $ab = 1$ and $a + b = 4$ $a, b \in \mathbb{Z}$ this is not possible so irreducible

c) Say $x = 1 + \sqrt[3]{2} + \sqrt[3]{4}$.

so $(x-1) = \sqrt[3]{2} + \sqrt[3]{4}$

$\Rightarrow (x-1)^3 = 2 + 4 + 3 \cdot \sqrt[3]{2} \cdot \sqrt[3]{4}(\sqrt[3]{2} + \sqrt[3]{4})$.

$\Rightarrow (x-1)^3 = 6 + 3 \cdot 2 \cdot (x-1)$

$\Rightarrow x^3 + 3x - 3x^2 - 1 = 6 + 6x - 6$.

$\Rightarrow x^3 - 3x - 3x^2 - 1 = 0$.

So. $(1 + \sqrt[3]{2} + \sqrt[3]{4})$ satisfies the equation $x^3 - 3x - 3x^2 - 1 = 0$.

if this polynomial is reducible then it can be written as (again using gauss lemma)

$$(x-a)(x^2 - bx + c)$$

a , b , c are integers now $ac = 1$ so $a = 1$ or -1 but none of them can satisfy this equation.

2 Consider the field Extension $Q(i, \sqrt[3]{2})/Q$ now $Q(\sqrt[3]{2}) \subseteq Q(i, \sqrt[3]{2})$ also $Q(i) \subseteq Q(i, \sqrt[3]{2})$. Also $x^3 - 2$ is irreducible in Q because of eisenstein criterion. So $[Q(\sqrt[3]{2}): Q] = 3$. also $x^2 + 1$ is irreducible over Q so $[Q(i): Q] = 2$. So $2 \mid [Q(i, \sqrt[3]{2}): Q]$ and $3 \mid [Q(i; \sqrt[3]{2}): Q]$. but $[Q(i, \sqrt[3]{2}): Q] \leqslant [Q[i): Q] \times [Q(\sqrt[3]{2}): Q]$.

$$= 6$$

so $[Q(i, \sqrt[3]{2}): Q] = 6$.

now $[Q(i, \sqrt[3]{2}): Q(i)][Q(i): Q] = 6$

so $[Qi, \sqrt[3]{2}): Q(i)] = 3$

Let $\rho(x)$ be the minimum polynomial of $\sqrt[3]{2}$ over Q (i). $\sqrt[3]{2}$ is a root of $x^3 - 2$

so $f(x) \mid x^3 - 2$ also $p(x)$ has degree 3 . thus $p(x) = x^3 - 2$. So it is irreducible over Q(i).

Similarly $x^3 - 3$ is also not reducible.

4a) F/K degree n

$$T_a: F \to F \text{ given by } T_a(x) = ax$$

$$T_a(x+y) = a(x+y) = ax + ay = T_a(x) + T_a(y)$$

abs $T_a(cx)$ $c \in K$

$$\begin{aligned} &= acx = c \cdot ax \text{ [commutative]} \\ &= cT_a(x) \end{aligned}$$

Thus it is a linear transformation of. K-vector space

3 F/K field extension and R is a ring such that $F \supset R \supset K$ .To prove R is a field we have to prove that. $\forall a \in R$ as $a^{-1} \in R$ as we already know R is a ring with unity . [K is a subfield of R]. $a \in R \Rightarrow a \in F$ which is algebraic over K. So there exists

$$\alpha^n + a_{n-1}x^{n-1} + \cdots a_0 = 0 \; a_i \in K.$$

now $\frac{1}{\alpha} = -\frac{1}{a_0}(\alpha^{n-1} + a_{n-2}\alpha^{n-2} \cdots + a_1) \in K$ thus $\frac{1}{\alpha} \in K$.
5. β satisfies $x^3 - 2$ which is irreducible over Q. Now we know

$$Q(\beta) \cong Q[x]/\langle x^3 - 2 \rangle$$

($\alpha = 2^{1/3}$ is real cube root of 2) in also root of $x^3 - 2$

$$\text{so. } Q(\alpha) \cong Q[x]/\langle x^3 - 2 \rangle \cong Q(\beta)$$

That means *there exist* an isomorphism from $\mu: Q(\beta) \longmapsto Q(\sqrt[3]{2})$
Given it is a ring momorphism 1 goes to 1 so -1 goes to -1 .
now say $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = -1$ in $Q(\beta)$.
how $(\alpha_1^2 + \alpha_2^2 + \cdots \alpha_n^2) = \left(\mu(\alpha_1)\right)^2 + \cdots \mu(\alpha_n)\text{^}2 = 1$
say $\mu(\alpha_i) = \gamma_i$
$\sum \; \gamma_i^2 = -1$ now $Q(\sqrt[3]{2}) \subseteq R$.
so $\gamma_i^2 \geq 0$ thus $\sum \; \gamma_i^2 \geq 0$ so this is impossible. thus there cannot be such α_i ' s.
6. Say any $z^{\neq 0} \in R \notin$ say $1, z, z^2, z^3, \ldots$
as R is finite dimensional then there exist smallest m such that $1, z, z^2, \ldots z^{m-1}$ is linearly independent. let

$$c_0 + c_1 z + c_2 z^2 + \cdots c_{m-1} z^{m-1} = 0$$

now $c_0 \neq 0$ it so $z_*\left(c_1 + c_2 z + \cdots c_{m-1}^{z^{m-2}}\right)$ =0 then $z_1 \neq 0$ integral domain so

$$c_1 + c_2 x + \cdots + c_{m-1} z^{m-1} = 0 \text{ but } m \text{ is least such number}$$

So $z.\left(-\frac{1}{c0}\right) \cdot (c_1 + c_2 x + \cdots + c_{m-1} z^{m-1}) = 1.$
So z has an inverse so R in a field. But R/C is finite dimensional $\forall a \in \mathbb{R}$ consider $\min_a, c(x)$. But min $a_{a,c}(x)$ has all the roots in $\mathbb{C}$ thus $a \in \mathbb{C}$ thus $R = C$.
(7) $y = \frac{x^3}{x+1}$ so $x^3 - xy - y = 0$
in $k(y)[t], p(t) = t^3 - ty - y$ has the root x now.

$$p(t) = y(-1 - t) + t^3$$

$y(-1 - t) + t^3$ is $k(t)[y]$ can written as.
now consider $f(\mathrm{t}))(yg(t) + h(t))$(only way to factorize).
but $\gcd(x^3, x + 1) = 1$ [$\because (x + 1)(x^2 - x + 1) - x^3 = 1$].
thus $f(t) = 1$ and $y(-1 - t) + t^3$ is irreducible in $K[t]$
now $K[t][y] = k[y][t]$as rings)
and $y(-1 - t) + t^3$ has no-non trivial divisor in this ring so $f(t)$ is irreducible over $K[y]$. $K(y)$ is the quotient field of the integral domain $K[y]$. $t^3 + y(-1 -$

t) is monic so using similar argument like gauss lemma for $\mathbb{Z}$ and Q we cen argue $P(t)$ is irreducible over $K(y)$ as well

8

K/Q(x)

$y^2 - \frac{x^3}{x^2+1}$ is irreducible over $Q(x)$

be cause

if $u = \frac{f(x)}{g(x)} \in Q(x)$

. gcd $(f(x), g(x)) = 1$

then $\frac{f(x)^2}{q(x)^2} = \frac{x^3}{x^2+1}$

then $(x^2+1)f(x)^2 = x^3 g\text{^}2(x)$

one size has even degree even otherside $h$as odd.

So $Q(x)[y]/\left\langle\left(y^2 - \frac{x^3}{x^2+1}\right)\right\rangle$ is a field and

consider

$\alpha = y + I$ where $I = \left\langle y^2 - \frac{x^3}{x^2+1}\right\rangle$

$\alpha^2 - \left(\frac{x^3}{x^2+1} + \mathrm{I}\right) = I$ whech is zero of the extension field

9a) $(x^3-2) = \left(x - 2^{1/3}\right)\left(x - 2^{1/3}\omega\right)\left(x - 2^{1/3}\omega^2\right)$.

where ω and its square are the roots of x^2+x+1

So splitting field is $Q\left(2^{1/3}, 2^{1/3}\omega, 2^{1/3}\omega^2\right) = Q(2^3, \omega)$

b) $x^4 - 1 = (x-i)(x+i)(x-1)(x+1)$

Q (i) is the spulting tield.

(c) $x^4 + 1 = R(x^2 - i)(x^2+i)$

$= \left(x - \left(\frac{1+i}{\sqrt{2}}\right)\right)\left(x + \left(\frac{1+i}{\sqrt{2}}\right)\right)\left(x - \left(\frac{1-i}{\sqrt{2}}\right)\right)\left(x + \frac{(1-i)}{\sqrt{2}}\right)$

consider $Q\left(\frac{1+i}{\sqrt{2}}\right)$.

$$x = \frac{1+i}{\sqrt{2}} \ ; \frac{1}{x} = \frac{\sqrt{2}}{1+i} = \frac{\sqrt{2}}{2}(1-i) = \frac{1-i}{\sqrt{2}}$$

So $Q\left(\frac{1+i}{\sqrt{2}}\right)/Q$ is the splitting field we cen show $Q\left(\frac{1+i}{\sqrt{2}}\right) \subseteq Q(i, \sqrt{2})$.

sout $x^4 + 1$ is irreducible thus.

$\left[Q\left[\frac{1+i}{\sqrt{2}}\right] : Q\right] \neq [Q(i, \sqrt{2}) : Q] \leqslant 4$

so $[Q(i, \sqrt{\prime}2) : Q] = 4$ and $Q\left(\frac{1+i}{\sqrt{2}}\right) \subseteq Q(i, \sqrt{2})$ thus $Q(i, \sqrt{2}) = Q\left(\frac{1+i}{\sqrt{2}}\right)$

(d) $x^6 + 1 = (x^2+1)(x^4+x^2+1)$

$$= (x^2+1)(x^2\cos(x^2+1)^2 - x^2)$$
$$(x^2+1)(x^2-x+1)(x^2+x+1)$$

$$= (x+i)(x-i)(x-w)(x-w^2)(x+w)(x+w^2)$$

$\therefore Q(i, \omega)/Q$ is the splitting field extension $\omega \notin Q(i)$,
(e) $(x^2 + 1)(x^3 - 1)$

$$= (x + i)(x - i)(x - 1)(x - w)(x - w^2)$$

again $Q(i, \omega)/Q$ is the sputting field extensicon
(f) $x^6 + x^3 + 1 = 0$ conside $y = x^3$

So $y^2 + y + 1 = 0$
so $y = \omega$ or $y = \omega^v$
so $x^3 = \omega$ or $x^3 = \omega^2$
so $x^3 = e^{\frac{2\pi i}{3}}$ or $x^3 = \frac{4\pi i}{3}$.
So $x \in \left\{e^{\frac{2\pi i}{9}}, e^{\frac{8\pi i}{9}}, e^{\frac{14\pi i}{9}}, e^{\frac{4\pi i}{9}}, e^{\frac{10\pi i}{9}}, e^{\frac{16\pi i}{9}}\right\}$
$Q(\zeta_9)$ where $\zeta_9 = e^{\frac{2\pi i}{9}}$ is the splitting field