1 . $\Rightarrow L_1L_2$ is algebraic over F. $\forall a \in L_1$, $a \in L_1L_2$ *which is algebraic* so There exist a minimum polynomial for a in F. So L_1 is algebraic over F , so is L_2.
$(\Leftrightarrow) a \in L_1L_2 \Rightarrow a \in L_1(a_1, ... a_n)$ *for some* $a_1 ... a_n \in L_2$

so we can write a is a rational function of $a_1, ... a_n$. Let the coefficients of numerator be c_i and denominator be d_i.

So $a \in F\ (c_1 ... c_r, d_1, ..., d_m, a_1 ... a_n) = k$ say where $[F(c_i): F] < \infty, [F(di): F] < \infty$
$[F(a_i): F] < \infty$
so k / F is finite
so K/F is algebraic so any a in this field is algebraic over F.similarly for any element in L1L2 we can prove it to be algebraic over F .

So L_1L_2 is algebraic over F
(b) $E = L \cap M, [L: F] < \infty$ and $[M: F] < \infty$ We know $[LM: F] = [L: F][M: F]$.

$$F \text{ is a subset of } E \text{ so } \quad [L: F] = [L: E][E: F], [M: F] = [M: E][E: F]$$

So. $[LM: F] = [L: E][M: E][E: F]^2$.
also $[LM: F] = [LM: E][E: F]$

$$\leq [L: E][M; E][E: F]$$

So $[M: E][F: E][E: F]^2 \leq [M: E][F: E][E: F]$
So $[E: F] \leq 1$
so $[E: F] = 1$ also F *is subset of* E so $F = L \cap M$
Now if $[L: F] = 2$ or $[M: F] = 2\{wLOg\ [L: F] = 2\}$

now

$$[LM: F] \leq [L: F][M: F].$$

So $[LM: F] \leq 2[M: F]$
so either $[LM: F] = [M: F]$
or $[LM: F] = 2[M: F]$.
$M \subseteq LM$
if $[LM: F] = [M: F]$ then $LM = M. LM \subseteq M$ thus $L \subseteq M$.
so $L \cap M = L \neq F$ (cause $[L: F] = 2$){contradiction }
So

$$[LM: F] = 2[M: F]\ proven$$

$L = Q(2^{1/3})$, $M = Q(2^{1/3}\omega)$, $F = Q$

$$[L: F] = 3; [M: F] = 3 \ , \ LM = Q(\omega, 2^{1/3})$$
$$[LM: F] = 6 < 9$$

2.Claim
(a) Aut $(K/LM) = Aut(K/L) \cap Aut(K/M)$.Take $\sigma \in Aut(K/LM)$
$\Rightarrow \forall a \in LM, \sigma(a) = a$.
now $\forall a \in L \Rightarrow a \in LM \Rightarrow \sigma(a) = a$.

same for $\forall b \in M.\ \sigma(b) = b$.
thus
$\sigma \in Aut(K/L)$ and $\sigma \in Aut(K/M)$.
So $Aut(K/LM) \subseteq Aut(K/L) \cap Aut(K/M)$.
now we can show if $e \in LM$.

$$e \in F\left(a_1 \ldots a_j, b_1 \ldots b_k\right)\ or\ e = \frac{f\left(\bar{a_i}, \bar{b_j}\right)}{g\left(\bar{a_i}, \bar{b_j}\right)}$$

$for\ some\ a_i \in L$ and $b_j \in M$.
if $\sigma\left(a_i\right) = a_i\ and\ \sigma\left(b_j\right) = b_j$
and $\sigma(f) = f, \forall f \in F$
then. $\sigma(e) = e$ as it is a ring homomorphism and e is a rational function of finite number of a's b's and f .
So
Ant $(K/L) \cap Aut(K/M) \subseteq$ Ant (K/LM).
(b)
$H = Aut(k/L)$
$G = Aut(K/M)$

$I = Aut$ (K/LM)

$If\ a \in L \cap M$ then
a is fixed by both H and G and by any combination of them

So

$$a \in \langle H \cup G\rangle'$$

Now say a $\in \langle H \cup G\rangle'$
that means $a \in (H \cup G)'[\ Not\ a\ subgroup\ \ just\ set\ ]\ H \subseteq HUG$.

$$so\ \ (H \cup G)' \subseteq H'$$

so $a \in H'$ similarly $a \in G'$
so $a \in H' \cap G'$
[K: F] is finite galois thus [K: L] is also galois. and so is [K: M] by Fundamental theorem of GT
so $H' = L$ and $G' = M$.
$a \in L \cap M$.

So
$\langle HUG\rangle' = L \cap M$.
again $k/L \cap M$ is galois by FTGT.

So
$\langle H \cup G\rangle = Aut(K/L \cap M)$ (proven)
(c) $Aut(K/L) \cap Aut(K/M) = Aut(K/LM) = \{1\}$

So $(Aut(K/LM))' = \{id\}' = K$
By FTGT $LM = K$

3.
L is galois then L is the splitting field of some set $\{fi\}$ of separable polynomial over F.
Now consider LM/M $f_i \in F[x] \in M[x]$ are Still separable in M . also set $xX = \{$ root of $f_i\}$ $L = F(X)$. $M \subseteq LM$ and $X \subseteq L \subseteq LM$ thus $M(X) \subseteq LM$.
then again $F \subseteq M$ thus

$$L{=}F(X) \subseteq M(X)\ also\ M \subseteq M(X)$$

thus $LM \subseteq M(X)$
So $M(X) = LM$ is the splitting field of S over M so LM/M is galois now.
$F(X)$/F is finite so finitely generated.say $F(x) = F\left(a_1 \dots a_k\right)$. $a_i \in K$
So $M(x)$ can also be written as.

$$M(L) = M(F(x)) \quad = M\left(F\left(a_1 \dots a_k\right)\right) \quad = M\left(a_1 \dots a_k\right)$$

again all a_i's are algebraic over F so algebraic over M, ML/M is finite.
θ: $Aut(LM/M) \longrightarrow Aut(L/F)$
be a map any $r \in Aut(LM/M)$.
let $u \in$ and $f(x)$ be the minimum polynomial over F. then $r(u)$ is a root of $f(x)$ But all root of $f(x)$ is in lee L itself as it is normal so $r(u) \in L$
so $r(L) \subseteq L$
also by similar argument

$$r^{-1}(L) \subseteq L$$

thus $r(L) = L$
So $r|_L$ is an automorphism from $L \to L$
θ: $Aut(LM/M) \longrightarrow Aut(L/F)$

$$\tau \longrightarrow \tau|_L .$$

this is injective because say

$$r|_L = id\ also\ r\ fixes\ M\ then$$

r fixes LM so r is id.

so kernel is {id}. Image is a subgroup of Aut (L/F), L/F is galois so let the Image's fixed Field be E then image $= Aut(L/E)$ if a$\in L \cap M$ then a is fixed by $\sigma|_L$
$as\ \sigma \in Aut(LM/M)\ so\ it\ fixes\ M$. So $a \in E$

So $L \cap M \subseteq E$
say $a \in E$ then $a \in L$ and $\sigma|_L(a) = a\ \forall\sigma \in Aut(KL/L)$. So $V\sigma$ in ($Aut(ML/M)$ $\sigma(a) = a$
so $a \in M$.
so $a \in L \cap M$ so $E \subseteq L \cap M$
so $E = L \cap M$ .Aut(LM/M) = Aut(L/L∩M)

4. $Q(\zeta_n)Q(\zeta_m) = Q(\zeta_l)$ $l = lcm(n, m)$ $n|l$ so $Q(\zeta_n) \subseteq Q(\zeta_l)$
$m|l$ so $Q(\zeta_m) \subseteq Q(\zeta_l)$
$Q(\zeta_n)Q(\zeta_m) \subseteq Q(\zeta_l)$

now

$$\frac{1}{l} = \frac{d}{nm} = \frac{rn+sm}{mn} = \frac{r}{m} + \frac{s}{n} \ for\ some\ (s, r) \in Z^2$$

So $e^{2\pi i/l} \in Q(\zeta_n)Q(\zeta_m)$
so $Q(\zeta_\ell) = Q(\zeta_n)Q(\zeta_m)$
(b) $Q(\zeta_n) \cap Q(\zeta_m) = Q(\zeta_d)$ where

$$d = gcd(n, m)$$

d/n $d|m$,d/n, so

$Q(\zeta_d) \subseteq Q(\zeta_n)$ and $Q(\zeta_d) \subseteq Q(\zeta_m)$
so $Q(\zeta_d) \subseteq Q(\zeta_n) \cap Q(\zeta_m)$.
As we alread prove.

$$[Q(\xi_l): Q(\xi_n)] = [Q(\xi_n)Q(\xi_m) : Q(\xi_m)] = [Q(\xi_n) : Q(\xi_n) \cap Q(\xi_m)].$$

so $\frac{\phi(\ell)}{\phi(m)} = \frac{\phi(n)}{[Q_n \cap Q_m]} \Rightarrow [Q(\xi_n) \cap Q(\xi_m): Q]$

$$= \frac{\phi(n)\phi(m)}{\phi(\rho)} = \phi(d)$$

$Q_m \cap Q_n \subseteq Q_d$

and $[Q_n \cap Q_m: Q] = [Q_d: Q]$
so $Q_d = Q_n \cap Q_m$
5. PAPER 1 Q13

6. PAPER 1 Q14
8.
$P(x) = x^p - 2 \in Q[x]$ splitting field of $P(x)$ over Q is

$$Q(2^{1/p}, \xi_p) = K\ say$$

Let

$$\begin{array}{ccc} \sigma: Q(2^{1/P}) & - & k \\ | & & | \\ id: Q & \longrightarrow & k \end{array}$$

$$min\ polynomial\ stay\ same\ in\ K[x]$$

for any root α of p(x) there exist σ

$$\sigma|_Q = id\ \ and\ \ \sigma\left(2^{1/p}\right) = \alpha.$$

α can be $2^{1/p}\xi_p^k\ 0 \le k <= p - 1.$

t:Q $(2^{1/p},\ \xi_p) - k$

| |

σ: $Q\left(i^{1/p}\right)\cdots \longrightarrow k$

$min_{\zeta_p} = x^{p-1} + x^{p-2} + \cdots + 1 = 0[\ we\ dread\ prove\ this\ \ is\ irreducible\ over\ \ Q\left(2^{1/p}\right)].$

again for any σ fixing Q we have t which can map , ξ_p to any root of min_{ζ_p}

ζ_p can map to any of $\xi, \xi^2, \ldots \xi^p$

$$\left\{2^{1/p} \mapsto \zeta^k 2^{1/p}\ 0 \leqslant k \leqslant p - 1\ ,\ \xi \mapsto \xi^\gamma \mapsto 1 \leqslant \gamma \leqslant p - 1\right\}$$

- list all the elements of Galois group of $x^p - 2 \in Q[x]$.

Consider, the $map\ \theta: Aut(K/Q) \to G.$
σ: $2^{1/p} \longrightarrow \xi^k 2^{1/p}\ and\ \xi \mapsto \xi^j$ is mapped to [[j k][0 1]]

if $0 \leqslant k \leqslant p - 1$, $1 \leqslant j \leqslant p - 1$

This map is injective and surjective (trivally) we have to prove this is a homomorphism

$$\sigma_1 \cdot 2^{1/p} \to \xi^k 2^{1/p} and\ \xi \longrightarrow \xi^j\ ;\ \ \sigma^2: 2^{1/\rho} -> 2^{1/p}\xi^{k_2}\ \ \xi^{'} \to \xi^{j2}$$

$$\sigma_1 \circ \sigma_2\left(2^{1/p}\right) = \sigma_1\left(2^{1/p}\xi^{k_2}\right) = 2^{1/\rho} \cdot \xi^k \cdot \xi^{k_2 j^{\cdot}}\ \sigma_1 o \sigma_2(\xi) = \xi^{j*j2}$$

$$\theta: \sigma_1 \circ \sigma_2 = \left[[j * j2\ ,\ jk_2 + k]\ ,[\ 0\ 1]\right] = [[j\ k][\ 0\ 1\ ]] * \left[[j_2\ k_2][\ 0\ 1]\right]\ (\ proven).$$