10.

$$f(x) = (x^3 - 10) = (x - \sqrt[3]{10})(x - \sqrt[3]{10}\mathrm{w})(x - \sqrt[3]{10}\mathrm{w} * \mathrm{w})$$
in $\mathbb{C}[x]$.

where ω is $e^{\frac{i2\pi}{3}}$
so splitting field over $K = Q(\sqrt{2})$ is $K\left(\omega, \sqrt[3]{10}\right)$.

$K(\omega, \sqrt[3]{10})$ has all the roots $of\ \left(x^3 - 10\right)$ also this is the smallest field which contains all the roots of $f(x)$

$\omega\ is$ complex so $w\ does$ not belong to $Q(\sqrt[3]{10}, \sqrt{2})$ so $Q(\omega, \sqrt[3]{10}, \sqrt{2}) \neq Q(\sqrt[3]{10}, \sqrt{2})$ also $x^2 + x + 1$ is the min polynomial of w over $Q[x]$ so if $m(x)$ is minimum polynomial over $Q(\sqrt[3]{10}, \sqrt{2})$ is in $\deg\left(m(\mathrm{x})\right) \leq 2$ also $\deg(m(x)) \neq 1$ otherwise $\omega\ is$ in $Q(\sqrt[3]{10}, \sqrt{2})$ So $m(x) = x^2 + x + 1$ also we can prove $[Q(\sqrt[3]{10}, \sqrt{2}): Q] = 6$.

So $[Q(\sqrt{2}, \sqrt[3]{10}, \omega): Q] = 12$.
11) p be a prime. Degree of splitting field of $x^p - 2$ There nay be two cases over $\mathbb{Q} \cdot \mathbb{Q}$ say if $P = 2$, $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ over $R[x]$
thus $Q(\sqrt{2})/\ Q$ is a field where $x^2 - 2$ splits alto it is the smallest such field (Containing Q and $\sqrt{2}$ which has degree 2.
$p > 2$ and prime
let ω be ρ^{th} primitive root of unity of 1 in $\mathbb{C}$ sad $e^{2\pi i/p}$ if $p > 2. e^{2\pi i/p}$ is not in Q.
$\left[\because \sin\left(\frac{2Mi}{p}\right) \neq 0\right\}$ thus it is complex].
also $\omega^q \neq 1$;$\forall q$ such that $0 < q < p \in \mathbb{Z}$
So $\omega, \omega^2, \dots \omega^{p-1}, 1$ are distinct element
So $2^{1/p}\omega, 2^{1/\rho}\omega^2 \cdots 2^{\frac{1}{p}}\ \omega^{p-1}$, $2^{1/p}$ also distinct set of elements also $x^p - 2$ has p roots at most [equal if separable] these are precisely all the roots so $Q\left(\omega, 2^{1/p}\right)/Q$ is the splitting field extension
$x^P - 2$ is in irreducible by eisenstein condition .Field extension.

$$\left[Q\left(2^{1/P}\right): Q\right] = p.$$

Now $(x^p - 1) = (x - 1)(x^{p-1} + x^{p-2} \cdots x + 1)$
We claim $= x^{p-1} + x^{p-2} + \cdots x + 1$ is irreducible.
let's say $q(x)$ is reducible
So $q(x + 1)$ also should be reducible

$$q(x+1) = \frac{(x+1)^p - 1}{x} = px + \frac{p(p-1)x^2}{2} + \cdots x^{p-1}$$

now
$p \mid \left(P_{c_r}\right) \forall r < p\ , r \in \mathbb{Z}^+$.
by Eisenstein on P
$q(x + 1)$ is not reducble so $q(x)$ can not be reduced

also we know gcd $(p, p-1) = 1$

$p \left| [Q(\omega, 2^{1/\rho}) : Q] \right.$ and $(\mathrm{p}-1) | [Q\left(\omega, 2^{\frac{1}{\rho}}\right) : Q] \Rightarrow p(p-1) \left| \left[Q\left(\omega, 2^{\frac{1}{\rho}}\right) : Q\right] \right.$

Thus.

$$\text{and } [Q(\omega, 2^{1/P}) : Q] \leq P(P - \mathbb{P})$$

$$\text{So } [Q(\omega, 2^{1/l}) : Q] = \rho(\rho - 1)$$

12. $K\ in\ C\ be$ be a splitting field of $f(x) = x^3 - 2$ over Q. So as we have previously seen

$$K = Q(\omega, 2^{1/3})$$

Consider

$$\alpha = \sqrt[3]{2} + \exp(2\pi i/3)$$
$$3 = (\sqrt[3]{2})^3 + \exp(2\pi i/3)^3$$
$$= \left(\sqrt[3]{2} + \exp\left(\frac{2\pi i}{3}\right)\right) [(\sqrt[3]{2} + e^{2\pi i/3})^2$$
$$-3\sqrt[3]{2}e^{2\pi i/3}]$$
$$= \alpha(\alpha^2 - 3\sqrt[3]{2}e^{2\pi i/3}).$$

So $3\sqrt[3]{2}e^{2\pi i/3} = \alpha^2 - \frac{3}{\alpha}$.
say $x = \sqrt[3]{2}e^{2\pi i/3} \in Q[\alpha]$.
$1 = (\sqrt[3]{2})^3 - (e^{2\pi i/3})^3 = (\sqrt[3]{2} - e^{2\pi i/3})(\alpha^2 - z)$.

now $\alpha \in Q[\alpha], z \in \mathbb{Q}[\alpha]$

So $(\sqrt[3]{2} - e^{2\pi i/3}) \in Q[\alpha]$
so $\sqrt[3]{2} \in Q[\alpha]$ and $e^{2\pi i/3} \in Q[\alpha]$
$\left(\because \frac{\alpha+\gamma}{2}, \frac{\alpha-\gamma}{2}\right]$ So.
$Q(\omega, \sqrt[3]{2}) \subseteq Q[\alpha]$ also $Q[\alpha] \subseteq Q(\omega, \sqrt[3]{2})$

13

$f(x) = x^p - x - c \in F[x]$, p is characteristic of F

$$\mathrm{p}.1 = (\underbrace{1 + 1 \ldots + 1}) = 0$$

Let K be a splitting field of F
If $f(x)$ is reducible in $F[x]$ and $q(x) \mid f(x), q(x) \in F$ and q(x) is irreducible then $0 < \deg q(x) < \deg f(x)$ *and* say α is a root of $q(x)$

if $\alpha^p - \alpha - C = 0$ then $(\alpha+1)^P - (\alpha+1) - C$

$= \alpha^p + 1 - (\alpha+1) - c \left[\begin{array}{l} \because \text{ the characteristics is p so } (\alpha+b)^p \\ \qquad\qquad\qquad = \alpha^p + b\text{^p} \end{array}\right.$

$= \alpha^p - \alpha - c = 0$

First of all by derivative test it is separable

so $\alpha, \alpha+1, \cdots, \alpha+p-1$ are unique roots of f(x) in F(α).
F(α) *is a field where* $f(x)$ splits and it is the smallest such field over F so $[\mathrm{F}(\alpha) : F] = \deg\big(q(x)\big) =$
degree of any splitting field of f(x) *we know all* splitting fields are isomorphic thus all irreducible

divisor of f(x) has same degree . k*($\deg\big(q(x)\big) =$
$\deg f(x)$) where k is the number of irreducible divisors.

as p in prime $\deg(q(x)) = 1$ or p.

So either $q(x) = f(x) \cdot u$ *where* u is some unit in F or $\mathrm{deq}(q(x)) = 1$ for all such divisors.
So either $f(x)$ is irreducible or has all roots.
14. Let F be a field of characteristic zero p be an odd prime $a \in F^x$ such that a is not p^{th} power of any element in F. $p > 2$ is prime f(x) = x^p-a .Let K be the spitting field of $f(x)$ over F thus.

$$f(x) = (x - z_1)(a - z_2) \cdots \big(x - z_p\big)$$

$$and\ {z_i}^p = a\ \forall i \in \{1, \dots, P\}.$$

now.

$$\text{say } f(x) = p(x)q(x) \text{ where neither}$$

$$p(x) = 1 \text{ or } q(x) = 1 \text{ so}$$

in K $p(x) = (x - z_1)(x - z_2) \cdots (x - z_n)$
now

$$z_1 \dots z_n = z \text{ say}$$

$$z^p = z_1^p z_2^p \cdots z_n^p = c \cdot c \dots c = c^n.$$

p is a prime arid $n \le p$ thus. There exists a , b such that

$$ap + bn = 1$$

$$\big((c^a) * (z^b)\big)^p = c^{ap+bn} = c^1 = c.$$

Now $c^a z^b \in F$
but $\exists$ no clement in F whose pth power $= a$. thus we got a contradicton

15. $a \in \mathbb{C}$
$\sigma_a: \mathbb{C}(x) \longrightarrow \mathbb{C}(x)$ which subs x by $x + a$.
$G = \{\sigma_a, a \in \mathbb{C}\}$. To find fixed field of
G any element in $C(x)$

$$= \frac{f(x)}{g(x)} \; f, g \in C(x) \; g \neq 0$$

$$\text{It fixes } \frac{f(x)}{g(x)} \text{ if } \frac{f(x)}{g(x)} = \frac{f(x+a)}{g(x+a)}$$

$$\frac{f(x)}{g(x)} = \frac{f(x+a)}{g(x+a)} \forall a \in \mathbb{C}$$

thus $f(x)g(x+a) - g(x)f(x+a) = 0$
$\forall a \in \mathbb{C}$. now consider $\forall$ fo put $x = b$ for any $b \in x$ such that $g(x) \neq 0$.
then $\frac{f(b)}{g(b)} = \frac{f(b+a)}{g(b+a)} \forall a \in \mathbb{C}$
or $\frac{f(y)}{g(y)} - \frac{f(b)}{g(b)} \forall y \in \mathbb{C}$
thus $\frac{f(y)}{g(y)} = \text{constant} = \frac{f(b)}{g(b)} \in \mathbb{C}$
So $\frac{f(y)}{g(y)} \in \mathbb{C}$
also any element in $\mathbb{C}$ is fixed by this all the automorphis$m$.
$Thus$ we can say $\mathbb{C} \subseteq G'$
and $G' \subseteq C$ so $G' = \mathbb{C}$

16. $\omega = e^{2\pi i/3}$

$$\sigma: C(x) \longmapsto C(x)$$
$$\sigma(x) = wx.$$
$$\sigma|_{\mathbb{e}=id}$$

$$\tau: C(x) -> C(x).$$
$$\tau(x) = \frac{1}{x}$$
$$\Phi r|_{\mathbb{C}} = id$$

ary dement $e \in C(x)$ be $e = \frac{f(x)}{g(x)} f^{(1)'(x)} \in C[x] \; q(a) \neq 0$

$$\tau\left(\tau\left(\frac{a_0+a_1x_1+a_nx^n}{b_0+b_1x+\cdots}\right)\right)=\tau\left(\frac{a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\frac{a_n}{x^n}}{b_0+\frac{b_1}{x}+\frac{b_2}{x^2}+\cdots+\frac{b_m}{x^m}}\right)$$

$$= \frac{a_0 + a_1x + \cdots a_nx^n}{b_0 + b_1x + \cdots + b_mx^m} = \frac{f(\mathrm{x})}{g(x)}$$

$$\tau\sigma\left(\frac{f(x)}{g(x)}\right)=\tau\left(\sigma\left(\frac{f(x)}{g(x)}\right)\right)$$

$$=\tau\left(\sigma\left(\frac{a_0+a_1x+\cdots a_nx^n}{b_0+b_1x+\cdots+b_mx^m}\right)\right.$$

$$=\tau\left(\frac{a_0+a_1\omega x+a_2\omega^2x^2+\cdots+a_n\omega^nx^n}{b_0+b_1\omega x+b_2\omega^2x^2\ldots+b_mx^m\omega^m}\right)$$

$$=\frac{\left(a_0+\frac{a_1\omega}{x}+\frac{a_2\omega^2}{x^2}+\frac{a_n\omega^n}{x\text{^}n}\right)}{\left(b_0+\frac{b_1\omega}{x}+\frac{b_2\omega^2}{x^2}+\cdots+\frac{b^m\omega^m}{x^m}\right).}. \qquad \sigma^{-1}(x)=\frac{x}{\omega}$$

$$\sigma^{-1}\tau\left(\frac{f(a)}{g(x)}\right)$$

$$=\sigma^{-1}\left(\frac{a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\frac{a_n}{x^2}}{b_0+\frac{b_1}{x}+\frac{b_0}{x^2}+\cdots+\frac{b_{nn}}{x_n}}\right).$$

$$=\frac{\left(a_0+\frac{a_1\omega}{x}+\frac{a_2\omega^2}{x^2}+\frac{a_n\omega^m}{xn}\right)}{\left(b_0+\frac{b_1\omega}{x}+\frac{b_2\omega^2}{x^2}+\cdots+\frac{b^m\omega^m}{x^m}\right).}.$$

they are same
$\sigma^3=1$ so $1,\sigma,\sigma^2$ are distinct power of σ
$\sigma^{-1}=v^2$
$\tau^2=1$ So $1,\tau$
$\tau^{-1}=\tau$
any staring $\sigma^{P_1}\tau^{P_2}\sigma^{P_3}\tau^{P_4}\ldots\tau^{P_{2k}}$ wa be

$p_1,p_{2k}\geq 0$ others >0

can be converted to $\sigma^{k_1}\tau^{k_2}$.
for some k_1k_2.
σ^{k_1} is ether $1,\sigma,\sigma^2$ or.
τ^{k_2} is either $1,\tau$.
$\{1,\tau,\sigma,\sigma\tau,\sigma^2,\sigma^2\ \tau\}$ are the elements
we can prove no two element in the list are same by applying them on a simple function like p(x) = x

so there are 6 elements.