1) # Dual Vector Space

* Given a complex space V, the dual space V^* is the set of all linear functionals:

$$V^* = \{ f : V \to \mathbb{C} \mid f \text{ is linear} \}$$

* If $|v\rangle \in V$, then its corresponding dual vector $\langle v| \in V^*$.
* The mapping $V \to V^*$ given by $|v\rangle \mapsto \langle v| = |v\rangle^\dagger$ is conjugate linear.

2) # Tensor Product

↳ way of putting vector spaces together to form larger vector spaces.

$$\left.\begin{array}{l} V : \text{dimension } m \\ W : \text{dimension } n \end{array}\right\} V \otimes W \Rightarrow \underline{\text{dimension } mn}$$

* $(A \otimes B)(|\psi\rangle \otimes |\omega\rangle) = (A|\psi\rangle) \otimes (B|\omega\rangle)$
*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

eg.

$$A \otimes B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{pmatrix}$$

3) # Spectral Decomposition

eigenvector corresponding to λ_i

$$M = \sum_i \lambda_i |i\rangle\langle i|$$

diagonalizable

eigenvalues

In terms of projector $P_i = |i\rangle\langle i|$,

$$\boxed{M = \sum_i \lambda_i P_i}$$

These projectors satisfy the completeness relation $\sum_i P_i = I$, and orthonormality

$P_i P_j = \delta_{ij} P_i$

for operator functions, $\boxed{f(M) = \sum_i f(\lambda_i)\, |i\rangle\langle i|}$

4) Polar Decomposition

$A = UJ = KU$ } J, K are positive operators

A: linear operator; U: unitary operator

$J \equiv \sqrt{A^\dagger A}$, $K \equiv \sqrt{AA^\dagger}$

★ If A is invertible, U is unique!

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5) State Space

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (i.e. a Hilbert Space) known as the state space. The system is completely described by its state vector, which is a unit vector in the system's state space.

$$\boxed{|\psi\rangle = a|0\rangle + b|1\rangle}$$

where $|0\rangle$ and $|1\rangle$ are orthonormal basis for qubit.

$|a|^2 + |b|^2 = 1$

6) Evolution

norm of the state vector is preserved

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation.

$$\boxed{|\psi'\rangle = U|\psi\rangle}$$

state at t_2 — state at t_1

depends only on t_1 and t_2

QM does not tell us which U describes the actual dynamics of the state

e.g.

$$|E\rangle \longrightarrow e^{\frac{-iEt}{\hbar}}|E\rangle$$

energy eigenstates

Postulate 2': Time evolution of the state of a closed quantum system is,

$$\boxed{i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle}$$

Hamiltonian Operator

So,

$$|\psi(t_2)\rangle = e^{\frac{-iH(t_2-t_1)}{\hbar}}|\psi(t_1)\rangle = U(t_1,t_2)|\psi(t_1)\rangle$$

$$\Rightarrow \boxed{U(t_1,t_2) \equiv e^{\frac{-iH(t_2-t_1)}{\hbar}}}$$

Note: Any unitary operator U can be realized as,

$$U = \exp(iK)$$

hermitian operator

7) Quantum Measurement

Postulate 3: They are described by a collection $\{M_m\}$ of measurement operators. The index m refers to the measurement outcomes that may occur.

probability of result m

$$\boxed{p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle}$$

State of the system after measurement

$$\boxed{|\psi_f\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}}}$$

The measurement operators satisfy the completeness relation

$$\boxed{\sum_i M_i^\dagger M_i = I}$$

★ Projective Measurements:

$$M = \sum_m m P_m$$

hermitian operator

Projector onto the eigenspace of M with eigenvalue m.

$$\boxed{\begin{aligned} &\sum_m P_m = I \\ &P_m P_{m'} = \delta_{mm'} P_m \end{aligned}}$$

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

and, $P_m^\dagger P_m = P_m$

$$\underbrace{\left(P_m = P_m^\dagger = P_m^2\right)}_{\text{for projector operators}}$$

$$\Rightarrow \boxed{p(m) = \langle \psi | P_m | \psi \rangle}$$

$$\boxed{|\psi_f\rangle = \frac{P_m |\psi\rangle}{\sqrt{p(m)}}}$$

Expectation / Average Value:

$$E(M) = \sum_m m\, p(m) = \sum_m m \langle \psi | P_m | \psi \rangle$$

$$= \langle \psi | \left(\sum_m m P_m \right) | \psi \rangle$$

$$\boxed{E(M) = \langle \psi | M | \psi \rangle}$$

★ POVM Measurements:

$$E_m = M^\dagger M$$

positive operator

The entire set $\{E_m\}$ is called POVM.

$$\sum_m E_m = I, \quad p(m) = \langle \psi | E_m | \psi \rangle$$

6) <u>Distinguishing Quantum States</u>

Alice $\longrightarrow$ $|\psi_i\rangle$ $\longrightarrow$ Bob

$\{|\psi_i\rangle\}_{i=1}^{N}$ $\quad\hookrightarrow$ uses $M_i = |\psi_i\rangle\langle\psi_i|$ and $M_0 = I - \sum_i M_i$

If $\{|\psi_i\rangle\}$ is orthonormal set, then Bob can get $p_i = \langle\psi_i|M_i^\dagger M_i|\psi_i\rangle = 1 \Rightarrow$ can reliably distinguish the state $|\psi_i\rangle$

But if $\{|\psi_i\rangle\}$ is not orthonormal, $\langle\psi_1|\psi_2\rangle \neq 0$, let's assume, Bob has a measure such that

$$\langle\psi_1|M_1^\dagger M_1|\psi_1\rangle = 1 \text{ and } \boxed{\langle\psi_2|M_2^\dagger M_2|\psi_2\rangle = 1}$$

now,

$$\langle\psi_1|M_2^\dagger M_2|\psi_1\rangle = 0 \text{ and } \therefore M_2|\psi_1\rangle = 0$$

and,

$$M_2|\psi_2\rangle = \alpha \cancelto{0}{M_2|\psi_1\rangle} + \beta M_2|\varphi\rangle = \beta M_2|\varphi\rangle$$

$$\therefore \langle\psi_2|M_2^\dagger M_2|\psi_2\rangle = |\beta|^2 \langle\varphi|M_2^\dagger M_2|\varphi\rangle$$

$$\leq |\beta|^2 \langle\varphi|\underbrace{\textstyle\sum_i M_i^\dagger M_i}_{=I}|\varphi\rangle \leq |\beta|^2 \underline{\underline{<1}}$$

<u>Hence, contradiction!</u>

What if Bob chooses to use a POVM?

let's assume, Alice sends: $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$

In this instance, Bob uses say:

$$E_1 = \frac{\sqrt{2}}{\sqrt{2}+1}|1\rangle\langle 1|, \quad E_2 \equiv \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}$$

$$E_3 = I - E_1 - E_2$$

If Alice sends $|\psi_1\rangle$,

$\langle\psi_1|E_1|\psi_1\rangle = 0 \Rightarrow$ If the result of measurement is E_1 then the state must be $|\psi_2\rangle$

Similarly, if result is E_2, then the state must be $|\psi_1\rangle$

{In case of E_3, nothing can be said!}

Bob does not make a mistake!

9) **Phases:**

★ Global Phase: $\underbrace{e^{i\theta}|\psi\rangle}$ (real θ)

It has the same statistics of measurement as $|\psi\rangle$!

$$\left\{ \langle\psi|M_m^\dagger M_m|\psi\rangle \text{ and } \langle\psi|e^{-i\theta} M_m^\dagger M_m e^{i\theta}|\psi\rangle = \langle\psi|M_m^\dagger M_m|\psi\rangle \right\}$$

Therefore, from an observational point of view these two states are identical!

★ Relative Phase: $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$

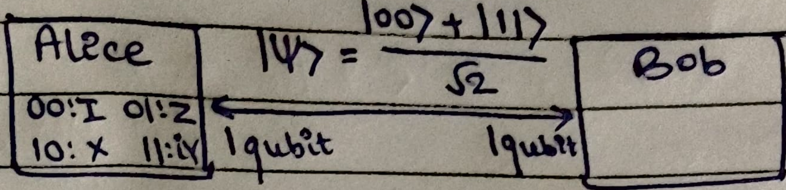
Two amplitudes differ by a relative phase if $\theta \in \mathbb{R}$, $\boxed{a = e^{i\theta} b}$

10) **Composite Systems:**

Postulate 4: If we have systems numbered 1 through n and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \ldots \otimes |\psi_n\rangle$.

Superdense Coding:

fixed entangled State



Alice initially has possesion of first qubit and Bob has the second qubit. By sending the single qubit in her possesion to Bob, Alice can communicate two bits of classical info!

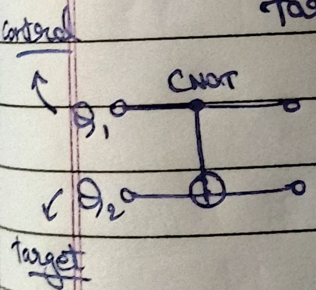
let's say she wants to send $|10\rangle$

$\Rightarrow$ Apply phase flip Z to her qubit.

$$(Z_A \otimes I_B)\left(\frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}\right) = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \doteq |\psi^-\rangle_{AB}$$

Now, Bob will decode this as following:

Apply CNOT gate with A as control, B as target.



$Q_1 = 1$ { flip Q_2

$Q_1 = 0$ { Nothing

$$\Rightarrow \mathrm{CNOT}(|\psi^-\rangle_{AB}) = \frac{|00\rangle - |10\rangle}{\sqrt{2}}$$

$$= \left(\frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}}\right) \otimes |0\rangle_B$$

Now, apply Hadamard gate $\Rightarrow$ (to first qubit)

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$H\left(\frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}}\right) = |1\rangle_A$$

$\Rightarrow \boxed{|10\rangle_{AB}}$ exactly what Alice sent!