Proof of Lorentz Transformation

Sad Baguette

Let us orient our axes S and S', the latter of which is moving at velocity $\vec{u} = u\hat{i}$ so that there is only motion along the x-axes. Also, when the origins coincide, let $t = t' = 0$, and assume that spacetime is homogenous. Let us write:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \quad ...(1)$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \quad ...(2)$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \quad ...(3)$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \quad ...(4)$$

There are only linear terms of x, y, z, and t because anything else would violate the homogeneity argument, meaning that a change in a primed parameter could be dependent on where the change occurs in space or time.

Since there is no relative motion in the y or z directions, they are identical: that is,

$$a_{21} = a_{23} = a_{24} = a_{31} = a_{32} = a_{34} = 0$$

$$a_{22} = a_{33} = a_{44} = 1$$

so we have

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = y$$

$$z' = z$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Then, by symmetry, we assume that t' does not depend on y and z. Otherwise, clocks moving parallel to each other in S would measure time differently (which violates homogeneity). Thus,

$$a_{42} = a_{43} = 0$$

and we have simplified equations (1) and (4) into

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$t' = a_{41}x + a_{44}t$$

However, we also know that having a coordinate $x' = 0$ is identical to having coordinate $x = ut$. Substituting this into the x' relation, we obtain

$$0 = a_{11}(ut) + a_{12}y + a_{13}z + a_{14}t$$

This condition must hold true in all cases, so $a_{12} = a_{13} = 0$ and $ua_{11} = -a_{14}$. So, our relations become

$$x' = a_{11}(x - ut)$$

$$t' = a_{41}x + a_{44}t$$

Now, suppose that when the two coordinate systems overlapped at time 0, we released a spherical wave of light. Its coordinates are given by

$$c^2t^2 = x^2 + y^2 + z^2$$

$$c^2t'^2 = x'^2 + y'^2 + z'^2$$

Substituting the primed relations into these relations, we obtain

$$c^2(a_{41}x + a_{44}t)^2 = [a_{11}(x - ut)]^2 + y^2 + z^2$$

which rearranges into

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 - 2(ua_{11}^2 + c^2a_{41}a_{44})xt = (c^2a_{44}^2 - u^2a_{11}^2)t^2$$

This must agree with the first spherical wave equation, so we have

$$a_{11}^2 - c^2a_{41}^2 = 1$$

$$ua_{11}^2 + c^2 a_{41} a_{44} = 0$$
$$c^2 a_{44}^2 - u^2 a_{11}^2 = c^2$$

At long last, with three equations and three unknowns, we can solve for the Lorentz transform:

$$x' = \frac{x - ut}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(x - ut)$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\left(t - \frac{ux}{c^2}\right)$$