Intuitive Physics

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"The first principle is that you must not fool yourself—and you are the easiest person to fool."
—Richard Feynman

"Never memorize something that you can look up."
—Albert Einstein

"If you want to have good ideas, you must have many ideas."
—Linus Pauling

"I do not feel obliged to believe that the same God who has endowed us with sense, reason, and intellect has intended us to forgo their use."
—Galileo Galilei

"If a cluttered desk is a sign of a cluttered mind, of what, then, is an empty desk a sign?"
—Albert Einstein

"Self-education is, I firmly believe, the only kind of education there is."
—Isaac Asimov

"It is strange that only extraordinary men make the discoveries, which later appear so easy and simple."
—George C. Lichtenberg

"We cannot solve problems with the same thinking we used to create them."
—Albert Einstein

"If your experiment needs statistics, you ought to have done a better experiment."
—Ernest Rutherford

"An American monkey, after getting drunk on brandy, would never touch it again, and thus is much wiser than most men."
—Charles Darwin

"I love deadlines. I love the whooshing noise they make as they go by."
—Douglas Adams

"I'm sure the universe is full of intelligent life. It's just been too intelligent to come here."
—Arthur C. Clarke

"Rockets are cool. There's no getting around that."
—Elon Musk

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Part I
Mechanics

1 Error Propagation

1.1 Types of Error

Error is generally formatted in two equivalent ways:

$$G = 6.67430(15) \times 10^{-11} \ \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

$$G = 6.67430 \pm 0.00015 \times 10^{-11} \ \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

When conducting experiments, a rule of thumb is to keep 1 significant figure on the errors and round the measurement to the same decimal place.

Even with careful measurement, error can occur for different reasons, such as

1. Accuracy of tool: ruler only gives $\pm 0.1 cm$
2. Problem of definition: measuring 'height' is difficult because it is not constant throughout the day

Broadly, there are thus two types of errors:
Random Error: small, unobserved errors—e.g., a measurement is slightly different than what you see

- Independent random variables are special in that increasing the accuracy of one measurement does not affect the others.

Systematic Error: calibration errors (oftentimes)—e.g., your metal ruler shrinks on a cold day

1.2 Estimating Error

Devices that deliver measurements are generally either

1. Digital, where a numerical value is outputted (such as a scale in Q's chemistry class)—for these, the error is the nearest decimal place
2. Analog, where some indicator stops between markings (such as a scale in RN)—for these, the error is half the distance between markings

1.3 Variance

Suppose you took measurements of time that lasted between 8 and 10 seconds. It'd be probably safe to call that 9 ± 1 seconds, but how can we be more precise? This is where **residuals** are useful, defined as the difference between a value and the mean.
$$r = x_i - <x>$$
where $<x>$ denotes the But if we average the residuals, by definition it must be 0—that's not good! So we introduce **variance**, defined as the average of the *squares* of the residuals. Averaging the absolute value of the residuals would theoretically work, but variance has nicer mathematical properties.

$$\text{var}(x) = <(x - <x>)^2>$$

By expanding the expression, it can be shown that

$$\text{var}(x) = <x^2> - <x>^2$$

1.4   Error Propagation

1.4.1   Systematic Error

For independent systematic errors, simply add up the errors. This is because the directionality of the error should be identical; for instance, if your ruler is short by 1 inch, all the errors will add up in the same way, none of them canceling out.

1.5   Independent Errors

Independent variables aren't effected by each other, so

$$P(x|y) = P(x) \ \& \ P(y|x) = P(y)$$

And in general, the probability of two events (variables) is

$$P(x \cap y) = P(x)P(y|x) = P(y)P(x|y)$$

So it follows that for independent variables,

$$P(x \cap y) = P(x)P(y)$$

The average value of the product of both variables is thus

$$<xy> = \int_{xy} xyP(x \cap y) \, d(xy) = \int xP(x) \, dx \int yP(y) \, dy = <x><y>$$

1.5.1   Sums and Differences

Error is defined through standard deviations, so let's consider the variance:

$$\text{var}(x+y) = <(x+y)^2> - <x+y>^2 = \text{var}(x) + \text{var}(y)$$

arriving at the final expression after some algebra. Rewriting, we have

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$$

Thus, errors (standard deviation) add in **quadrature**. More formally, if Q is the sum or difference of several measurements such that

$$Q = (a + b + ... + c) - (x + y + ... + z)$$

then

$$\delta Q = \sqrt{[(\delta a)^2 + (\delta b)^2 + ... + (\delta c)^2] + [(\delta x)^2 + (\delta y)^2 + ... + (\delta z)^2]}$$

Note that δ and σ are interchangeable and equivalent.

Suppose we take N measurements, each with error δx. Then the overall error becomes

$$\delta(Nx) = \sqrt{N}\delta x \rightarrow \delta(<x>) = \frac{\delta(Nx)}{N} = \frac{\delta x}{\sqrt{N}}$$

This is known as the $1/\sqrt{N}$ rule—the error goes inversely with the number of trials/measurements you perform. This rule is also used in radioactive decay; for every N decaying particles observed, the error is $\sqrt{N}$.

Suppose you have several measurements of the same parameter but by different tools, so all the errors are different. How should you average the measurements to minimize the error? Choose the weights so that they are inversely proportional to the squares of the error. For instance, if a ruler has 1/2 the uncertainty of a measuring rod, weigh the ruler 4 times as much in the weighted average.

1.5.2 Products and Quotients

While the absolute error adds in quadrature when measurements are summed, the *relative* error adds in quadrature when measurements are multiplied. If Q is the sum or difference of several measurements such that

$$Q = (a + b + ... + c) - (x + y + ... + z)$$

then

$$\delta Q = \sqrt{\left[(\delta a)^2 + (\delta b)^2 + ... + (\delta c)^2\right] + \left[(\delta x)^2 + (\delta y)^2 + ... + (\delta z)^2\right]}$$

1.5.3 General Functions

Suppose we had two measurements, $\theta = 71°$ and $\theta = 73°$ with $u = \tan\theta$. Without formulas, we might use $\tan 71°$ and $\tan 73°$ to set bounds on the error, subtracting them with the mean. If we generalize this concept to a function $f(x)$, then we have just estimated the error as

$$\delta f = f(x + \delta x) - f(x)$$

$$\delta f = f(x) - f(x - \delta x)$$

But for small δx, we can approximate the following:

$$f(x \pm \delta x) \approx f(x) \pm \frac{df}{dx}\delta x$$

which means that both of these estimates simplify down to

$$\delta f = \frac{df}{dx}\delta x$$

For a multivariate function $f(x, y, z \cdots)$, the error is given by

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2 + \left(\frac{\partial f}{\partial z}\sigma_z\right)^2 \cdots}$$

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2 Kinematics

2.1 Constant Acceleration

For constant acceleration, motion is governed by

$$\Delta x = v_{avg} t$$

$$\Delta x = v_0 + \frac{1}{2}at^2 = v - \frac{1}{2}at^2$$

$$v = v_0 + at$$

$$v^2 = v_0^2 + 2a\Delta x$$

2.2 Variable Acceleration

More generally, though, the relation between velocity and acceleration is given by

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(t)\, dt$$

2.3 Projectile Motion

Consider a particle launched from ground level with velocity $\vec{v}$ at angle ϕ. Then, the speed at any point is given by $v = \sqrt{v_x^2 + v_y^2}$ and angle to the horizontal of $\phi = \tan^{-1}(v_y/v_x)$.

The particle's shape is (approximately, assuming $\vec{g}$ is constant) a parabola given by

$$y = (\tan\phi)x - \left(\frac{g}{2v_x^2}\right)x^2$$

The range of a particle is given by

$$R = \frac{v_0^2 \sin(2\phi_0)}{g}$$

The maximum height of a particle is given by

$$h = \frac{v_0^2 \sin^2\phi_0}{2g}$$

2.4 Kinematics in Polar Coordinates

The unit vectors in polar coordinates are not constant, given by

$$\hat{r} = \cos\theta\hat{i}\sin\theta\hat{j}$$

$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

The derivatives of these unit vectors are

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

Therefore, letting a particle's position be

$$\vec{r} = r\hat{r}$$

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we obtain the velocity and acceleration in polar coordinates,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Here's the intuition for the acceleration equation—consider, what does each term represent?

- Case 1: Radial acceleration. The first term, $\ddot{r}$, refers to the change in radial velocity, while the second term is the centripetal acceleration of the object (which is always radially inwards, hence the negative sign).
- Case 2: Tangential acceleration. The first term, $r\ddot{\theta}$, comes from the change in tangential speed. The second term is known as the Coriolis acceleration, which, contrary to the fictitious nature of the Coriolis force, is actually a very real acceleration.

2.4.1 Translational-Rotational Vector Relationships

By considering circular motion, we can obtain some general relations between vectors:

$$\vec{v} = \vec{\omega} \times \vec{R}$$

$$\vec{a}_T = \vec{\alpha} \times \vec{R}$$

$$\vec{a}_R = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

3 Drag Motion

3.1 Viscous Resistance

For a small object dropped vertically in air at relatively low speeds, the drag force is given by $\vec{D} = -b\vec{v}$. The velocity of such an object is given by

$$v = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)$$

The terminal velocity is reached as t $\to \infty$, given by

$$v_t = \frac{mg}{b}$$

3.2 Drag Equation

For objects dropped at high speeds, the important dependence to remember is that $D \propto v^2$. Specifically, the motion follows the drag equation:

$$D = -\frac{1}{2}\rho v^2 C_d A$$

4   Dynamics

4.1   Newton's Laws

Newtonian Mechanics follows these rules:

1st Law: Principle of Inertia: defines inertial reference frames (in which the principle of inertia holds).

2nd Law: $\vec{F} = m\vec{a}$

3rd Law: All forces are action-reaction pairs.

5 Momentum

5.1 Conservation of Momentum

Conservation of momentum is derived from Newton's 2nd Law, holding true when $\vec{F} = 0$.

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = constant$$

It can be easier to analyze collisions in the CM frame, where total momentum is 0.

5.2 Center Mass

Center mass is defined as

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$

Taking some derivatives yields

$$\vec{v}_{cm} = \frac{\sum m_i \vec{v}_i}{M}$$

and

$$\vec{a}_{cm} = \frac{\sum m_i \vec{a}_i}{M}$$

5.2.1 Triangular Plate

The CM of a triangular plate follows from symmetry; it lies on the intersection of the three medians of the triangle.

5.2.2 Hollow Hemisphere

The CM of an empty half-sphere is given by

$$y_{cm} = \frac{R}{2}$$

where y_{cm} is measured from the center of the sphere.

5.2.3 Solid Hemisphere

The CM of a solid half-sphere is given by

$$y_{cm} = \frac{3R}{8}$$

where y_{cm} is measured from the center of the sphere.

5.3 Elastic Collisions

In linear elastic collisions, the velocities in the CM frame simply reverse. In higher dimensions, the objects' velocities remain anti-parallel, but they may be redirected at an angle.

Working in the lab frame, for initial velocities v_1 and v_2, and CM velocity

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

the final velocities (for a 1D collision) are given by

$$u_1 = 2v_{cm} - v_1$$

$$u_2 = 2v_{cm} - v_2$$

Elastic collisions in 2D should be analyzed in the following way:
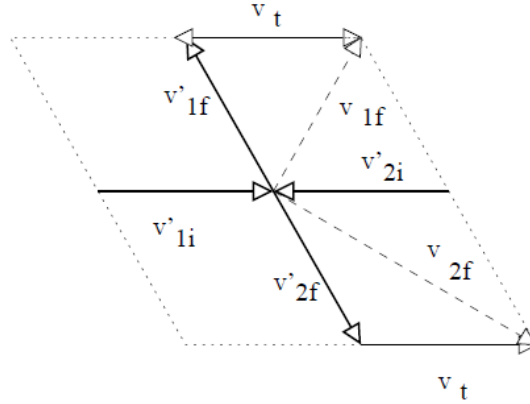
Figure 1: Initial and final velocities in the CM and lab frames

- Perpendicular deflections can be solved by applying the Pythagorean theorem to the momenta, with initial momentum as hypotenuse and final momenta as legs.

- When the masses are identical, it is best to analyze in the CM frame.

As shown in the diagram, primed velocities denote the CM frame and unprimed ones denote the lab frame. Note that the diagonals of a rhombus (and thus the final velocities) must be at right angles.

5.4 Inelastic Collisions

Perfectly inelastic collisions cause objects to stick together.

5.5 Variable Mass Systems

When the mass of a system is changing, such as in rockets, apply

$$\sum F_{ext} + v_{rel}\frac{dm}{dt} = m\frac{dv}{dt}$$

In the special case $\sum F_{ext} = 0$ this becomes

$$v_2 = v_1 + v_{rel}\ln\frac{m_2}{m_1}$$

6 Rotational Dynamics

6.1 Torque

Torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$. Here, $r \sin \phi$ is the *lever arm*. Only net forces contribute to net torque; internal action-reaction pairs cancel out.

Gravitational torque is given by $\vec{\tau} = \vec{r}_{cm} \times M\vec{g}$ for a constant gravitational field.

6.2 Moment of Inertia

Moment of Inertia is defined as $I = \int r^2 \, dm$.

6.2.1 Parallel Axis Theorem

The relation between moments of inertia about an axis through the CM and another parallel axis is given by

$$I_p = I_{cm} + Md^2$$

which holds true for planar and non-planar objects.

6.2.2 Perpendicular Axis Theorem

The relation between moments of inertia about two planar axes and another axis perpendicular to both of them is given by

$$I_z = I_x + I_y$$

which applies only to planar objects. However, non-planar objects still follow some rules:

$$I_x + I_y + I_z = 2 \int r^2 \, dm$$

6.2.3 Radius of Gyration

The radius of gyration is the distance from an axis at which all mass could be concentrated without changing the moment of inertia, given by

$$k = \sqrt{\frac{I}{M}}$$

6.2.4   Common Moments of Inertia

Object	Rotational Axis	Moment of Inertia
Stick	Central Diameter	$I = ML^2/12$
Stick	End Diameter	$I = ML^2/3$
Ring	Central Axis	$I = MR^2$
Ring	Diameter	$I = MR^2/2$
Annular Ring	Central Axis	$I = M(R_1^2 + R_2^2)/2$
Disk	Central Axis	$I = MR^2/2$
Disk	Diameter	$I = MR^2/4$
Solid Sphere	Diameter	$I = 2MR^2/5$
Hollow Sphere	Diameter	$I = 2MR^2/3$
Cylinder	Central Axis	$I = MR^2/2$
Cylinder	Central Diameter	$I = MR^2/4 + ML^2/12$
Cylinder	End Diameter	$I = MR^2/4 + ML^2/3$
Square	Central Axis	$I = ML^2/6$
Square	Midpoints	$I = ML^2/12$
Square	Diagonal	$I = ML^2/12$
Rectangle	Central Axis	$I = M(a^2 + b^2)/12$

6.3   Rotational Motion

6.3.1   Newton's 2nd Law for Rotation

For rotation, Newton found that

$$\tau_z = I\alpha_z$$

which holds in two scenarios. Either 1) origin is inertial frame, or 2) origin is at CM.

6.3.2   Ramp Rolling

For an object with $I = cMR^2$ rolling without slipping down a ramp inclined at angle ϕ, angular acceleration is given by

$$\alpha = \frac{g\sin\phi}{R(c+1)}$$

and acceleration

$$a = \frac{g\sin\phi}{c+1}$$

from which we can find velocity and time at the bottom,

$$v = \sqrt{\frac{2gh}{c+1}}$$

$$t = \sqrt{\frac{2h(c+1)}{g\sin^2\phi}}$$

If we impose the $a = \alpha R$ condition, we find that for no slipping to occur, for a fixed coefficient of (static) friction,

$$\phi \leq \tan^{-1} \frac{\mu(c+1)}{c}$$

or alternatively, for a fixed angle,

$$\mu \geq \frac{c \tan \phi}{c+1}$$

6.4 Angular Momentum

Angular momentum is defined as $\vec{L} = \vec{r} \times \vec{p}$. Here, $r \sin \phi$ is known as the impact parameter. Taking derivatives of this definition yields

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

which again holds true in either 1) inertial frames, or 2) CM frame.

Angular momentum can also be expressed as

$$L_z = I\omega_z$$

which is *not* generally a vector relation, as L and ω may be different in direction. However, they have the same direction for 1) axial symmetry (about axis of rotation), or 2) planar motion (as they both can only be in the z direction).

For a translating and rotating object, the full equation for angular momentum is

$$\vec{L} = \vec{R} \times M\vec{V}_{cm} + \vec{L}_{about\ CM} = \vec{L}_{orbital} + \vec{L}_{spin}$$

6.4.1 Conservation of Angular Momentum

Conservation of angular momentum comes from Newton's 2nd Law for rotation, given by

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \rightarrow \vec{L} = constant$$

6.4.2 Angular Impulse

Angular impulse is the time integral of torque, given by

$$\int \vec{\tau}\, dt = \int d\vec{L} = \Delta\vec{L} = \vec{J}_\phi$$

For rotation about a given axis:

$$\vec{J}_\phi = \int rF\, dt = rF_{avg}\Delta t$$

$$\vec{J}_\phi = \Delta\vec{L} = I\Delta\vec{\omega}$$

where r is the lever arm.

6.4.3 Precession

For a spinning top with spin angular momentum L_s, the angular frequency of precession is given by

$$\omega_p = \frac{Mgr}{L_s}$$

and the vector relationship between these quantities is

$$\vec{\tau} = \vec{\omega_p} \times \vec{L}$$

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7 Kinetic Energy

7.1 Kinetic Energy

Kinetic energy is the energy of motion, defined as

$$K = \frac{1}{2}mv^2$$

7.2 Work

Work involves external forces exerted through the point of application, given by

$$W = \int \vec{F} \cdot d\vec{s} = F_x \Delta x + F_y \Delta y + F_z \Delta z$$

Work is related to kinetic energy through the Work-Energy Theorem, given by

$$W = \Delta K$$

which holds if an object only has kinetic energy.

7.2.1 Work by Spring Forces

For a Hooke's Law force $F = -kx$, the work done by the spring is given by

$$W = \int_{x_i}^{x_f} -kx\, dx = -\frac{1}{2}k(x_f^2 - x_i^2)$$

7.2.2 Two Dimensional Work

For a particle moving along a 2D curve, work is given by

$$W = \int_i^f \vec{F} \cdot d\vec{s} = \int_i^f F \cos\phi\, ds = \int_i^f (F_x\, dx + F_y\, dy)$$

where $\vec{F} = F_x\hat{i} + F_y\hat{j}$, $d\vec{s} = dx\hat{i} + dy\hat{j}$, and ϕ is the angle between the vectors.

7.3 Power

Power is the rate at which work is done, instantaneously given by

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{s}}{dt} = \vec{F} \cdot \vec{v}$$

and thus average power is

$$P_{avg} = \frac{W}{t}$$

7.4 Rotational Kinetic Energy

The energy of rotational motion is given by

$$K = \frac{1}{2}I\omega^2$$

7.5 Rotational Work

Rotational work is given by

$$W = \int \vec{F} \cdot d\vec{s} = \int (F \sin\phi)(r\, d\phi) = \int \tau\, d\phi$$

7.6   Rotational Power

Rotational power is given by

$$P = \frac{dW}{dt} = \tau_z \frac{d\phi}{dt} = \tau_z \omega_z$$

7.7   Energy in Collisions

In a perfectly inelastic collision with one object starting at rest, the ratio of final kinetic energy to initial kinetic energy is given by

$$\frac{K_f}{K_i} = \frac{m_1}{m_1 + m_2}$$

which means the ratio of lost energy to initial energy is

$$\frac{K_{lost}}{K_i} = \frac{m_2}{m_1 + m_2}$$

where m_1 is the mass of the initially moving particle and m_2 is the mass of the particle initially at rest.

8   Potential Energy

8.1   Conservative Forces

A conservative force is defined in two (equivalent) ways. First, work done on a particle as it moves around a closed path (initial and final positions are the same) is 0.

$$\oint \vec{F} \cdot d\vec{s} = \int_a^b \vec{F} \cdot d\vec{s} + \int_b^a \vec{F} \cdot d\vec{s} = 0$$

Second, the work done on a particle is independent of the path it takes.

$$\int_1 \vec{F} \cdot d\vec{s} = \int_2 \vec{F} \cdot d\vec{s}$$

8.2   Potential Energy

All conservative forces have associated potential energy functions defined for them, representing the work needed to assemble a *system* (so objects don't gain PE, systems do!).

$$\Delta U = -W = -\int F \, dx \rightarrow U(x) = U(x_0) - \int_{x_0}^x F(x) \, dx$$

which implies

$$F = -\frac{dU}{dx}$$

Notice that it's *differences* in potential energy that matter.

For gravitational potential energy, we define $U(\infty) = 0$. Thus, the GPE of a system is given by

$$U(x) = 0 - \int_\infty^x -\frac{GMm}{x^2} \, dx = -\frac{GMm}{x}$$

where x is the separation between two gravitationally interacting bodies.

8.3   Conservation of Mechanical Energy

If no external forces do work on a system, mechanical energy is conserved, given by

$$\Delta K + \Delta U = \Delta E_{tot} = 0$$

where the kinetic energy term represents both translation and rotation, given by

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I_{cm}\omega^2$$

For rolling without slipping, this becomes

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)\omega^2 = \frac{1}{2}(c+1)Mv^2$$

In 1D conservative systems, solving for velocity in conservation of energy yields

$$v_x = \pm\sqrt{\frac{2}{m}[E - U(x)]}$$

where points at which $v = 0$ are called *turning points*.

8.4   Types of Equilibrium

Stable equilibrium occurs when force opposes small displacements (restoring force). This occurs at relative minima on the potential energy function.

Unstable equilibrium occurs when force supports small displacements. This occurs at relative maxima on the potential energy function.

Neutral equilibrium occurs when there is no force in either direction for small displacements. This occurs in horizontal regions of the potential energy function.

8.5   General Position Function

Integrating $v = \frac{dx}{dt}$ with the value for v derived above, we obtain

$$t = \int_{x_0}^{x} \frac{1}{\pm\sqrt{\frac{2}{m}[E - U(x)]}} \, dx$$

8.6   3D Conservative Systems

Generalizing our previous result to three dimensions, for position vector $\vec{s} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, we have

$$\Delta U = U(x_f, y_f, z_f) - U(x_i, y_i, z_i) = -\int_{i}^{f} (F_x \, dx + F_y \, dy + F_z \, dz)$$

which implies

$$\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

8.7   Gravitational Potential

Gravitational potential energy is always between a system of masses, but suppose we want to know the effect of one mass on the space around it. This is where the concept of gravitational *potential* comes in handy,

$$V = \frac{U}{m}$$

It is the gravitational parallel of electric potential.

9 Conservation of Energy

9.1 Work Done on a System

We can extend the work-energy theorem to be more general, given by

$$W = \Delta K + \Delta U + \Delta E_{int}$$

where E_{int} is the sum of energies at the microscopic scale, from the kinetic energy of randomly moving particles to the potential energy stored in bonds.

9.2 Frictional Work

Frictional work is *not* given by $W_f = fs$ as sliding friction cannot be treated as acting at a point from an energy viewpoint.
Imagine a book being pulled along a table at constant speed with a rope, given by

$$W_T + W_f = \Delta E_{int}$$

which becomes

$$W_f = -W_T + \Delta E_{int} = -fs + \Delta E_{int}$$

since $T = f$. Thus, since $\Delta E_{int} > 0$ since temperature (and thus internal energy) is increasing, the work done by friction must be *less* than fs.

9.3 Center Mass Energy

For some systems, the point of application does not move but the CM does. These are best analyzed with

$$\int F \, dx_{cm} = \Delta K_{cm}$$

which becomes, for constant forces,

$$F_{ext} x_{cm} = \Delta K_{cm}$$

9.4 First Law of Thermodynamics

Our earlier energy relations did not account for heat (energy in transfer). This is corrected for with

$$\Delta E_{tot} = Q + W \rightarrow \Delta K + \Delta U + \Delta E_{int} = Q + W$$

where Q is the heat transferred into the system.

10   Gravitation

10.1   Newton's Law for Gravitation

For two particles, the gravitational force is given by

$$\vec{F}_{12} = -\frac{Gm_1 m_2}{r_{12}^2}\hat{r}_{12}$$

where $\vec{F}_{12}$ is the force on 1 from 2, and $\hat{r}_{12}$ is the direction to 1 from 2.

10.2   Shell Theorems

There are two shell theorems that govern gravity.

1. A uniform spherical shell attracts other bodies as if its mass were concentrated at its center.

2. A uniform spherical shell has no gravitational field inside of it.

These are special properties of its inverse square nature.

10.3   Gravitational Constant

The gravitational constant is $G = 6.67 \times 10^{-11}\ \frac{Nm^2}{kg^2}$ and is relatively imprecise, as it is measured at extremely small scales with the Torsion Balance.

10.4   Surface Gravity

Near Earth's surface, gravity is given by

$$g_0 = \frac{GM_E}{R_E^2}$$

which of course changes with latitude, as Earth's "radius" is not uniform (it bulges at the equator). The average value of g_0 is about 9.81 m/s^2.
For small heights $h << R_E$ above Earth's atmosphere, we can approximate gravity as

$$g = g_0\left(1 - \frac{2h}{R_E}\right)$$

10.5   Orbit and Escape Speed

The orbital speed of a satellite r from the Earth's center is given by

$$v_{orb} = \sqrt{\frac{GM}{r}}$$

and the escape speed of a particle at that distance is

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

10.6   Orbital Dynamics

Orbits can be simplified as follows;

- Each orbiting body is only influenced by gravity from the central body; the contributions from other bodies are negligible.

- The central body is significantly more massive than the orbiting one, so the latter rotates purely around it.

10.6.1 Kepler's Laws

Kepler experimentally determined 3 laws of orbits, given by

1. *The Law of Orbits*: All planets move in elliptical orbits, with the Sun at a focus.

2. *The Law of Areas*: All planets sweep out equal areas in equal times.

3. *The Law of Periods*: The square of the period of revolution is proportional to the cube of the semi-major axis, given by

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

and the constants disappear if $[T] = years$, $[a] = a.u.$, and $[G] = \frac{a.u.^3}{M_s y^2}$.
Ellipse tidbits to keep in mind:

- The eccentricity is $0 < e < 1$ for an ellipse, and ea is the distance from the center to a focus.

- The prefix *peri-* refers to least, while *apo-* refers to most. Thus, perihelion is Earth's closest distance to the Sun, and aphelion is its farthest, while perigee and apogee replace them for the Moon and Earth.

10.6.2 Geosynchronous Orbit

Geosynchronous orbit occurs when a satellite rotates with same period as Earth, and it occurs at altitude $r = 3.58 \times 10^7 \ m = 22,300 \ mi$ according to Kepler's 3rd Law.

10.7 Orbital Energies

10.7.1 Energy in Orbit

The total energy of an elliptical orbit is given by

$$E = K + U = \frac{GMm}{2a} - \frac{GMm}{a} = -\frac{GMm}{2a}$$

which implies

$$E = -K$$

10.7.2 Vis-Viva

Provided the distance and speed at another point and semi-major axis are known, as we can find the speed at any point or the point corresponding to any speed; this is derived from conservation of energy.

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$$

10.8 Orbital Types

For an object at distance r from the central body, its orbit is governed by:

Velocity	Orbital Type
$0 < v < \sqrt{\frac{GM}{r}}$	Elliptical; r is max distance (apo-)
$v = \sqrt{\frac{GM}{r}}$	Circular; r is radius
$\sqrt{\frac{GM}{r}} < v < \sqrt{\frac{2GM}{r}}$	Elliptical, r is min distance (peri-)
$v = \sqrt{\frac{2GM}{r}}$	Parabolic
$v > \sqrt{\frac{2GM}{r}}$	Hyperbolic

Note that parabolic orbits are special because they have 0 energy.

10.9   Gauss's Law for Gravitation

Using the idea of gravitational fields instead of point masses, Gauss's Law yields

$$\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{en}$$

For an infinite line of mass:
$$g = \frac{2G\lambda}{r}$$

For an infinite plane of mass:
$$g = 2\pi G\sigma$$

For an infinite cylinder of mass:
$$g = 2\pi G\rho r$$

for $r < R$, and
$$g = \frac{2\pi G\rho R^2}{r}$$

for $r > R$.

11 Fluid Statics

There are three types of forces:

1. <u>Compression</u>: pushing force is perpendicular to surface

2. <u>Tension</u>: pulling force is perpendicular to surface

3. <u>Shearing</u>: force is parallel to surface

Solids support all three; liquids can support compression and (to some extent) tension, gases support none.

11.1 Pressure

Pressure is the perpendicular force per unit area, given by

$$P = \frac{F_\perp}{A}$$

and the density is given by

$$\rho = \frac{m}{V}$$

11.2 Bulk Modulus

When we increase pressure, volume must decrease, and this tendency is measured by

$$B = -\frac{\Delta P}{\Delta V/V}$$

where the units are $[B] = [P]$. The modulus of water is $B_w = 2.2 \times 10^9 \ Pa$, meaning it is not very compressible.

11.3 Pressure Variation

The relation of pressure to height in a fluid is given by

$$\frac{dP}{dy} = -\rho g$$

and for incompressible fluids this becomes

$$P_2 - P_1 = -\rho g h$$

Using this, the force due to water on the walls of a pool is given by

$$F = P_{av}A = \frac{\rho g h}{2}A = \frac{\rho g l h^2}{2}$$

for a wall of dimensions $l \times h$. Atmospheric pressure is irrelevant; it equally affects both sides of the wall. And also, we can solve for the density of any fluid if we know another's; that is,

$$\rho_1 g y_1 = \rho_2 g y_2 \rightarrow \rho_2 = \frac{2a}{2a + d}\rho_1$$

where a is the displacement of original fluid from equilibrium, d is the current height differential, and we poured the fluid of ρ_2 into the fluid of ρ_1.

For **atmospheric pressure**: from the ideal gas law $P \propto V^{-1}$ and density yields $\rho \propto V^{-1}$ and thus $P \propto \rho$. Thus, solving the DE involving dP/dy yields

$$P = P_0 e^{-\frac{\rho g}{P_0}h} = P_0 e^{-h/a}$$

where $a = P_0/\rho g$ and h is altitude above Earth's surface.

11.3.1 Raindrop Pressure

Given drop density n, radius r, and contact speed v, the pressure exerted on a horizontal surface is given by

$$P = \frac{4}{3}\pi\rho n v^2 r^3$$

11.4 Pascal's Principle

Pascal's Principle says that for a static fluid, pressure applied is transmitted undiminished equally across the fluid. It holds for both incompressible and (after reaching equilibrium) compressible fluids.

11.5 Archimedes' Principle

For a body in fluid, there is a buoyant (static lift) force coming from pressure differentials, given by

$$B = \rho_{fl} V_d g = \frac{\rho_{fl}}{\rho_{obj}} Mg$$

the latter of which occurring only in full submersion.
Generally, for floating objects,

$$\frac{V_d}{V} = \frac{\rho_{obj}}{\rho_{fl}}$$

11.5.1 Mercury Barometer

Assuming negligible vapor pressure, the mercury barometer lets us measure atmospheric pressure, given by

$$P_{atm} = \rho_m g h$$

and it turns out a column of mercury 760 mm tall is needed, thus defining the Torr.

11.5.2 Open Tube Manometer

The open tube manometer is used to measure gauge pressure, or overpressure, given by

$$P - P_0 = \rho g h$$

11.6 Surface Tension

Surface tension is defined as

$$\gamma = \frac{F}{L}$$

where L is the distance over which the force acts. Manipulating this fraction gives us

$$\gamma = \frac{Fh}{Lh} = \frac{U}{A}$$

where h is the height.

11.6.1 Soap Bubble

The energy stored by a soap bubble is given by

$$U = \gamma(2A) = 8\gamma\pi r^2$$

because there are two *free surfaces* (inside and outside).
How much work is done in expanding the soap bubble by ΔA?

$$W = \gamma \Delta A$$

What is the pressure differential between inside and outside necessary to support the bubble?

$$\Delta P = \frac{4\pi r \gamma}{\pi r^2} = \frac{4\gamma}{r}$$

which comes from considering the bubble as two hemispheres.

12 Fluid Dynamics

12.1 Equations of Continuity

Within a tube of flow (lots of streamlines), conservation of mass dictates

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

which equates the mass flux, reducing into

$$A_1 v_1 = A_2 v_2$$

for incompressible fluids, which is volume flux.
Generally these apply if there are no sources or sinks between the inlet and outlet.

12.2 Bernoulli's Equation

Bernoulli's equation relates flow along a streamline, assuming no losses of mechanical energy.

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

in which P is the static pressure and $\frac{1}{2}\rho v^2$ is the dynamic pressure.

12.2.1 Venturi Meter

The venturi meter measures flow speed through a pipe. Suppose said pipe decreases its diameter, before which fluid of density ρ has speed v_1. Let us attach a barometer with one end at the original diameter and another at the new one, containing fluid of density ρ'. Bernoulli's equation tells us

$$v_1 = A_2 \sqrt{\frac{2(\rho' - \rho)gh}{\rho(A_1^2 - A_2^2)}}$$

where h is the height differential in the barometer.

12.2.2 Pitot Tube

The pitot tube is used to measure flow speed of gas around it, and it works by attaching a manometer to a tube. One side of the manometer will have less static pressure because there is airflow, while the other does not, and thus the speed of the flow can be determined from the height differential:

$$v = \sqrt{\frac{2\rho'gh}{\rho}}$$

12.2.3 Toricelli's Law

Toricelli's law tells us how fast a fluid will pour out of a small opening in a container at depth h below the surface.

$$v = \sqrt{2gh}$$

To find how long it takes for a container to empty, equating volume flux (where A_0 is area at the surface and A is area of the opening) gives us

$$\frac{dh}{dt} = v_0 = \frac{A}{A_0}v$$

and we need some other information to solve for A/A_0 before being able to solve for t.
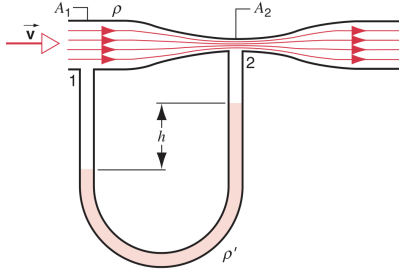
Figure 2: Venturi Meter
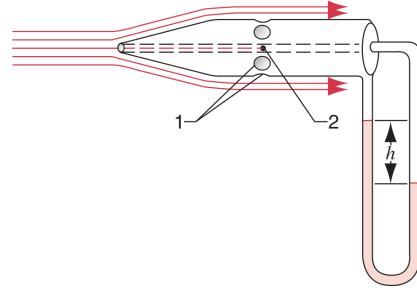
Figure 3: Pitot Tube

12.3 Viscosity

The viscous force is given by

$$F = \eta A \frac{dv}{dy}$$

and the units for viscosity η are Pascal-seconds. In liquids, viscosity results from intermolecular forces that are weakened with temperature (so, hotter means less viscous!). In gases, increased temperature means molecules migrate between layers of flow; however, there are more slow molecules than fast ones due to geometry, so this impedes faster molecules near the center (so, hotter means more viscous!).

12.4 Laminar Flow

At radial distance r from the center of a pipe with radius R, the speed is

$$v = \frac{r\Delta p}{4\eta L}(R^2 - r^2)$$

where L is the length of the pipe and Δp goes from its start to end.
The mass flux is given by

$$Q = \frac{dm}{dt} = \frac{\pi R^4 \rho \Delta p}{8\eta L}$$

the latter of which is known as Poiseuille's Law.

12.5 Turbulent Flow

Flow becomes turbulent above a critical speed, given by

$$v_c = R\frac{\eta}{\rho D}$$

where D is the diameter of flow. For cylinders, the Reynolds number R is 2300.

13   Oscillations

13.1   Simple Harmonic Motion

Simple harmonic motion occurs for sinusoidal motion of constant frequency and amplitude. Most commonly, a block attached to a spring fixed at one end has *equation of motion*

$$\ddot{x} + \frac{k}{m}x = 0$$

which has solution

$$x = x_m \cos(\omega t + \phi)$$

Note that the acceleration is related to position as $a = -\omega^2 x$, which is characteristic of SHM. The angular frequency, period, and frequency are given by

$$\omega = \sqrt{\frac{k}{m}} \to T = 2\pi\sqrt{\frac{m}{k}} \to f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

while amplitude and phase constant ϕ depend on initial conditions.

Energy considerations in our spring-block system yield

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

and solving for velocity v yields

$$v = \pm\sqrt{\frac{k}{m}(x_m^2 - x^2)}$$

13.2   Applications of SHM

13.2.1   Simple Pendulum

The equation of motion is approximately

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

for small displacements. But the full solution has a period of

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{2^2}\sin^2\frac{\theta_m}{2} + \frac{3^2}{2^2 \cdot 4^2}\sin^4\frac{\theta_m}{2} + ...\right)$$

13.2.2   Torsional Pendulum

A torsional pendulum is under restoring torque of $\tau = -\kappa\theta$, so its equation of motion is

$$\ddot{\theta} + \frac{\kappa}{I}\theta = 0$$

which has solution

$$\theta = \theta_m \cos(\omega t + \phi)$$

provided that

$$\omega = \sqrt{\frac{\kappa}{I}} \to T = 2\pi\sqrt{\frac{I}{\kappa}} \to f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$$

13.2.3 Foucault's Pendulum

The period of Foucault's pendulum, which demonstrates Earth's rotation, is given by

$$T = \frac{T_E}{\sin L}$$

where T_E is 24 hours and L is the latitude.

13.2.4 Vertical Spring Oscillation

In a uniform gravitational field, the equation of motion for a vertical spring is

$$\ddot{x} + \frac{k}{m}x - g = 0$$

which has solution

$$x = x_m \cos \omega t + \phi + \frac{mg}{k}$$

meaning that values of ω, f, T, v, a stay the same, but equilibrium shifts by $\frac{mg}{k}$. Moreover, changes in *total* potential energy are the same.

13.3 Damped Harmonic Motion

Consider a spring-mass system subjected to a damping force of $D = -bv$. The equation of motion is

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

which has solution

$$x = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$$

where

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

which means that when $b = 2\sqrt{km}$, we have $\omega' = 0$. This is *critical damping* and it is used to minimize unwanted vibrations so that they decay to 0 as fast as possible.

13.4 Forced Oscillations and Resonance

We can force a system to oscillate at a driving angular frequency ω'' instead of its natural one ω. The equation of motion is

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_m}{m}\cos(\omega'' t)$$

which has solution

$$x = \frac{F_m}{m}\cos(\omega'' t - \beta)$$

where

$$G = \sqrt{m^2(\omega''^2 - \omega^2)^2 + b^2\omega''^2}$$

and

$$\beta = \sin^{-1}\left(\frac{b\omega''}{G}\right)$$

When $\omega'' = \omega$ we have resonance, which gives the maximum displacement. Without damping, this would be infinitely large.

13.5 Two-Body Oscillations

Suppose masses of m_2 and m_1 are connected by a spring, with the former on the left and positive rightwards. The stretch is then $x = x_1 - x_2 - l$, and Newton's second law yields

$$m_1 \ddot{x}_1 = -kx$$

$$m_2 \ddot{x}_2 = kx$$

Linearly combining with weights of m_2 and $-m_1$, respectively, yields

$$\frac{m_1 m_2}{m_1 + m_2}(\ddot{x}_1 - \ddot{x}_2) = -kx$$

where the reduced mass is

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

With $\ddot{x} = \ddot{x}_1 - \ddot{x}_2$ this becomes

$$\ddot{x} + \frac{k}{m}x = 0$$

and it can also be shown that the kinetic energy is given by

$$K = \frac{1}{2}mv^2$$

where m is reduced mass and $v = v_1 - v_2$ is relative velocity.

13.6 Springs

13.6.1 Parallel Springs

For a parallel spring combination, the effective spring constant is

$$k = k_1 + k_2 + ... + k_n$$

13.6.2 Series Springs

For a series spring combination the effective spring constant is

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + ... + \frac{1}{k_n}$$

13.6.3 Delta-Star

To go from Delta (triangular) to Star configuration, apply

$$k_a = \frac{k_{AB}k_{BC} + k_{BC}k_{CA} + k_{CA}k_{AB}}{k_{BC}}$$

$$k_b = \frac{k_{AB}k_{BC} + k_{BC}k_{CA} + k_{CA}k_{AB}}{k_{CA}}$$

$$k_c = \frac{k_{AB}k_{BC} + k_{BC}k_{CA} + k_{CA}k_{AB}}{k_{AB}}$$

In the special case $k_{AB} = k_{BC} = k_{CA} = k$ then $k_a = k_b = k_c = 3k$. More generally, we have

$$k_{star} = \frac{1}{k_{opp}} \sum k_{pairs}$$

13.6.4   Star-Delta

To go from the Star to Delta (Triangular) Configuration, apply

$$k_{AB} = \frac{k_a k_b}{k_a + k_b + k_c}$$

$$k_{BC} = \frac{k_b k_c}{k_a + k_b + k_c}$$

$$k_{CA} = \frac{k_c k_a}{k_a + k_b + k_c}$$

More generally, we have

$$k_{delta} = \frac{\prod k_{adj}}{\sum k_{star}}$$

14 Waves

14.1 Traveling Waves

The most general formula for a wave is given by

$$y(x,t) = f(x - vt)$$

For a particular point on the shape of the wave to remain at constant height as it travels, we must have

$$x - vt = constant \rightarrow \frac{dx}{dt} = v$$

the latter of which is known as the phase velocity. v always is a positive constant, so for waves traveling in the negative x direction, we have

$$y(x,t) = f(x + vt)$$

Ultimately, if we let x be constant and vary t, we see how a point on the wave moves through time. If we let t be constant and consider the heights at various positions x, we have a "snapshot" of the wave at a moment in time.

14.1.1 Sinusoidal Waves

The equation for a sinusoidal wave is given by

$$y(x,t) = y_m \sin(kx - \omega t)$$

where $k = 2\pi/\lambda$. Moreover, the ω and k are related as

$$v = \frac{\lambda}{T} = \frac{\omega}{k}$$

The transverse speed of a particle is thus given by

$$\frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$$

and its acceleration as

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y_m \sin(kx - \omega t)$$

In the previous equations we have assumed that $y(0,0) = 0$; the more general equation is

$$y(x,t) = y_m \sin(kx - \omega t - \phi)$$

where $kx - \omega t - \phi$ is the phase and ϕ is the phase constant, which moves a wave forwards or backwards in space or time. This can be demonstrated by writing

$$y(x,t) = y_m \sin\left[k\left(x - \frac{\phi}{k}\right) - \omega t\right]$$

$$y(x,t) = y_m \sin\left[kx - \omega\left(t + \frac{\phi}{\omega}\right)\right]$$

which indicate that introducing a phase constant $\phi > 0$ implies moving forwards in space or backwards in time. Both of these bring the new wave *ahead* of the one with $\phi = 0$.

14.1.2 Wave Speed

The speed of a wave is independent of frequency and wavelength (if it is dependent, the medium is dispersive). It is given by

$$v = \sqrt{\frac{T}{\mu}}$$

Because it relies on small angle approximation, this only holds for relatively small transverse displacements.

The speed in a medium is dependent on its properties, as shown above. Thus, the wavelength and frequency change inversely while speed remains constant like $v = \lambda f = \lambda' f'$.

However, if a medium changes, the *frequency* remains constant while speed and wavelength change, as $f = v/\lambda = v'/\lambda'$.

14.1.3 Group Speed and Dispersion

In a real wave, which may not preserve its shape like sinusoidal waves or pulses, we must use the *group speed*, which is the speed at which energy or information travels. This occurs when a medium is *dispersive*: component waves (Fourier Analysis) travel at different speeds.

14.2 The Wave Equation

The general form of waves on a string is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

It can be shown that $y(x,t) = f(x \pm vt)$ is a solution to this equation.

The general form of a spherically symmetry system is given by

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial y}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

which has solution

$$y(r,t) = \frac{A}{r} \sin(kr - \omega t)$$

14.3 Energy in Wave Motion

Energy is generally *not* constant for a certain mass element of string: waves transmit energy in their direction of propagation. Kinetic energy at a point is given by

$$dK = \frac{1}{2}(\mu dx)[-\omega y_m \cos(kx - \omega t)]^2$$

and thus

$$\frac{dK}{dt} = \frac{1}{2}\mu\omega^2 y_m^2 v^2 \cos^2(kx - \omega t)$$

while potential energy is given by the work done by tension,

$$dU = F(dl - dx) = \frac{1}{2}Fdx \left(\frac{\partial y}{\partial x} \right)^2$$

and thus

$$\frac{dU}{dt} = \frac{1}{2}Fvk^2 y_m^2 \cos^2(kx - \omega t)$$

Note that $dK + dU$ is not constant - it is 0 at the peak, and they both increase on the way down. Energy is thus being transmitted. Moreover, substituting $F = \mu v^2$ and $kv = \omega$ into the second equation reveals that $dK = dU$.

14.3.1 Power and Intensity

The power is given by

$$P = \frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = \mu\omega^2 y_m^2 v \cos^2(kx - \omega t)$$

Thus average power is

$$P_{av} = \frac{1}{2}\mu\omega^2 y_m^2 v$$

over a large number of cycles, assuming no dissipation. The dependence of average power on the square of amplitude and frequency is a general characteristic of waves.

It is more effective to describe some wavefronts (e.g. spherical) in terms of intensity, not power, given by

$$I = \frac{P_{av}}{A}$$

where A is the area of the wavefront. It is better to use intensity because while energy in each wavefront is the same, it is being spread over a larger area as it propagates.

For spherical or circular waves, amplitude is not constant (but *intensity*, not power!, is still proportional to its square). Thus, when you double the distance from a spherical light source, the intensity is multiplied by $1/4$ and the amplitude is multiplied by $1/2$.

For cylindrical waves, the area of dissipation is the lateral surface area of the cylinder, which grows linearly with distance. Thus, intensity is inversely proportional to distance.

Intensity can also be written in terms of the energy density,

$$I = uv$$

where u is the energy density.

14.4 The Principle of Superposition

The principle of superposition says that when several waves combine at a point, the resultant displacement is the sum of all the individual ones; that is,

$$y(x, t) = y_1(x, t) + y_2(x, t) + \ldots$$

and it holds true for mechanical waves in elastic media when the restoring force follows Hooke's law (that is, it varies linearly with displacement).

14.4.1 Fourier Analysis

All periodic waves can be broken down into a linear combination of sinusoidal component waves of (potentially) different frequencies and amplitudes.

For a waveform at a particular time with wavelength λ, the displacement is given by

$$y(x) = A_0 + A_1 \sin kx + A_2 \sin 2kx + \ldots + B_1 \cos kx + B_2 \cos 2kx + \ldots$$

A waveform may change its shape if the speeds of these component waves are different or if the medium is dissipative, causing energy loss. Dissipation is often related to speed, so it is waves at high particle speeds (high frequencies, since $u_y \propto f$) that are affected the most.

14.5 Interference of Waves

Interference is when two waves combine at a point. Let the two waves be

$$y_1(x,t) = y_m \sin(kx - \omega t - \phi_1)$$

$$y_2(x,t) = y_m \sin(kx - \omega t - \phi_2)$$

so that their interference produces a wave

$$y(x,t) = [2y_m \cos(\Delta\phi/2)] \sin(kx - \omega t - \phi')$$

where $\phi' = (\phi_1 + \phi_2)/2$ and $\Delta\phi$ is the phase difference. This result comes from the sum to product trigonometric formulae. When $\Delta\phi$ is 0, the waves undergo *constructive interference*. When $\Delta\phi$ is 180°, the waves undergo *destructive interference*.

More generally, should the waves have differing amplitudes y_{m1} and y_{m2}, this becomes

$$y(x,t) = \sqrt{y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos\Delta\phi}\,\sin(kx - \omega t - \phi')$$

where

$$\phi' = \sin^{-1}\left[\frac{y_{m1}\sin\phi_1 + y_{m2}\sin\phi_2}{\sqrt{y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos\Delta\phi}}\right]$$

14.6 Standing Waves

When two waves of equal amplitude and frequency are moving in opposite directions, a standing wave is created. Let the two waves be

$$y_1(x,t) = y_m \sin(kx - \omega t)$$

$$y_2(x,t) = y_m \sin(kx + \omega t)$$

so that their resultant is

$$y(x,t) = [2y_m \sin kx] \cos \omega t$$

which represents the general equation of a standing wave. Points of 0 displacement are called nodes, while points of maximum displacement are antinodes. Also, note that amplitude is a function of x for a standing wave; it is not the same for all points!

Therefore, the antinodes are at maximum displacement,

$$kx = \left(n + \frac{1}{2}\right)\pi \rightarrow x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}$$

and the nodes are at minimum displacement,

$$kx = n\pi \rightarrow x = n\frac{\lambda}{2}$$

for non-negative integers $n = 0,\ 1,\ 2...$

Nodes are always at rest, so energy cannot flow past them: energy remains standing in the string.

14.6.1 Standing Wave Ratio

Consider an incident wave of amplitude A_i as it is reflected from a boundary with amplitude A_r. The superposition of these creates a standing wave, with maximum and minimum resultant amplitude given by

$$\text{SWR} = \frac{A_i + A_r}{A_i - A_r} = \frac{A_{max}}{A_{min}}$$

and the percent reflection, defined as ratio of average power in reflected wave to that in the incident wave (times 100), is given by

$$\%\text{ reflection} = \frac{(\text{SWR} - 1)^2}{(\text{SWR} + 1)^2} \times 100$$

14.6.2 Standing Waves on a String

We can create standing waves by plucking a string. It may have a triangular shape when we release it, but the higher frequency terms from Fourier Analysis damp out quickly, leaving only the lowest possible frequency. By holding the nodes and plucking at the antinodes, we can create standing waves.

For a string of length L, with n loops, the wavelength is given by

$$L = n\frac{\lambda}{2} \rightarrow \lambda = \frac{2L}{n}$$

because the spacing between nodes is $\lambda/2$. Also, from $v = \lambda f$ we can find

$$f = \frac{nv}{2L}$$

Why does this string have infinite frequencies of oscillation, but the block-spring system in SHM only has 1? Well, a lumped system of N elements has N different oscillating frequencies. While the block and spring have only one way to store potential and kinetic energy, the string has an infinite number of ways.

14.6.3 Resonance on Standing Waves

When you shake a string,

- If the hand is out of phase ($\omega'' \neq \omega$), you do work on rope and rope does work on you

- If the hand is in phase ($\omega'' \approx \omega$), you do work on rope

The driving frequency is not exactly natural frequency at resonance due to damping. Also, the apparent nodes are not true nodes again because energy must be flowing past the nodes to counteract losses from damping.

14.7 Wave Reflection

14.7.1 At a Boundary

Consider a pulse reflecting off a fixed boundary. The rope exerts an upwards force on the wall as the pulse arrives, so the wall exerts an equal but opposite downwards force. Thus, the reflection is flipped upside down by this force, meaning the pulse changes phase by $180°$.

Consider a pulse reflecting off a free end. The pulse causes the free end to shoot up, exerting an upwards force just like it would if the pulse kept traveling. Thus, the reflected pulse is in phase with the incident one.

14.7.2 At a Change in Medium

Consider two strings of different mass density, μ_1 and μ_2, tied together. This time, a wave that travels from 1 to 2 is partially transmitted and partially reflected.

- If $\mu_2 > \mu_1$: reflected wave is shifted $180°$, $v_1 > v_2$, and $\lambda_1 > \lambda_2$.

- If $\mu_2 < \mu_1$: reflected wave stays in phase, $v_1 < v_2$, and $\lambda_1 < \lambda_2$.

In both cases, energy is transmitted from string 1 to 2. However, the best way to minimize energy transfer is by making $\mu_2 << \mu_1$, in which case the boundary is very similar to the free end discussed above. Note that frequencies are the same to avoid discontinuity.

All our discussion of transverse ways have so far lied in one plane; this is known as being *plane polarized.*

15 Sound Waves

Sound waves usually refer to the frequency range between 20 Hz and 20,000 Hz, the typical ranges of human hearing.

15.1 Traveling Sound Waves

A simple way to generate sound waves is to have a piston compress and rareify the air in front of it.

$$\Delta\rho(x,t) = \Delta\rho_m \sin(kx - \omega t)$$

$$\Delta p(x,t) = \Delta p_m \sin(kx - \omega t)$$

We can relate these equations with

$$\Delta\rho_m = \rho_0 \frac{\Delta p_m}{B}$$

where B is the bulk modulus.

Note that internal energy increases due to particle motion caused by sound. In fluids, this energy flows rather slowly, so it is approximately *adiabatic* (no heat transfer). If energy does flow non-negligibly, it is *isothermal* (constant temperature). The appropriate bulk modulus must be used in each case.

15.2 Displacement from Sound Waves

Expressing sound as a displacement wave results in

$$s(x,t) = s_m \cos(kx - \omega t)$$

where

$$s_m = \frac{\Delta\rho_m}{k\rho_0} = \frac{\Delta p_m}{kB}$$

Thus, the longitudinal velocity of particles is given by

$$u(x,t) = \frac{\partial s}{\partial t} = u_m \cos(kx - \omega t)$$

where

$$u_m = \omega s_m = \frac{\omega\Delta\rho_m}{k\rho_0} = \frac{\omega\Delta p_m}{kB}$$

It is generally preferable to describe sound waves with pressure functions rather than displacement functions. Differences arise because pressure adds as scalars while displacement as vectors. Also, pressure changes are what're detected by ears and microphones.

15.3 The Speed of Sound

The speed of sound in a medium is given by

$$v = \sqrt{\frac{B}{\rho_0}}$$

In gases, the Bulk modulus can be written as

$$B = \gamma p_0$$

where γ is the specific heat ratio. For air, $\gamma = 1.4$ and $\rho_0 = 1.21 \ kg/m^3$.

In air, the speed of sound as a function of temperature is given by

$$v = 331 + 0.6T \ m/s$$

and under typical conditions (room temperature), $v = 343 \ m/s$.

15.4   Power and Intensity of Sound Waves

The power of a sound wave is given by

$$P = \frac{Av(\Delta p_m)^2}{B} \sin^2(kx - \omega t)$$

and so the average power over a large number of cycles is

$$P_{av} = \frac{Av(\Delta p_m)^2}{2B} = \frac{A(\Delta p_m)^2}{2\rho v}$$

which means that

$$P_{av} \propto (\Delta p_m)^2 \propto f^2 s_m^2$$

Intensity is given by

$$I = \frac{(\Delta p_m)^2}{2\rho v} = 2\pi^2 \rho v f^2 s_m^2$$

In terms of intensity, we have

$$\Delta p_m = \sqrt{2I\rho v}$$

$$s_m = \sqrt{\frac{I}{2\pi^2 \rho f^2 v}}$$

Human ears respond logarithmically to sound of increasing intensity, so it makes sense to define *sound level* as

$$\beta = 10 \log \frac{I}{I_0}$$

where the reference intensity is $I_0 = 10^{-12}$ W (for midrange frequencies of about 1000 Hz). Relatively, we can also write

$$\beta_2 - \beta_1 = 10 \log \frac{I_2}{I_1}$$

The threshold of pain is around 120 dB. I find this very... interesting.

15.5   Interference of Sound Waves

Two sound waves interfering at a point follow the principle of superposition, so their pressure disturbances add. The type of intefereence depends on the phase difference, given by

$$\frac{\Delta \phi}{2\pi} = \frac{\Delta L}{\lambda}$$

where the path difference is $\Delta L = |r_1 - r_2|$.

Therefore, constructive interference (maximum intensity) occurs at

$$\Delta L = m\lambda$$

where $m = 0, 1, 2, \ldots$.

And thus destructive interference (minimum intensity) occurs at

$$\Delta L = \left(m + \frac{1}{2}\right)\lambda$$

where $m = 0, 1, 2, \ldots$.

If speakers emit a mixture of many different wavelengths (real music consists of many!), some points could have destructive interference for one wavelength bubt constructive interference for another.

15.6   Standing Longitudinal Waves

Standing waves occur when sound either reflects off of an open or closed end.

- If the end of the tube is open, there is a rarefaction (pressure node) as there is nothing for air to bounce off of. There is thus a $180°$ change in phase.

- If the end of the tube is closed, there is a compression (pressure antinode) as air bounces off the wall. Thus, there is no change in phase.

For a open-open tube, each loop is $\lambda/2$, so we can solve for wavelength as

$$\lambda_n = \frac{2L}{n}$$

so frequency is given by

$$f_n = \frac{nv}{2L}$$

where $n = 1, 2, 3 ...$
For an open-closed tube, each half-loop is $\lambda/4$, so we can solve for wavelength as

$$\lambda_n = \frac{4L}{n}$$

so frequency is given by

$$f_n = \frac{nv}{4L}$$

where $n = 1, 3, 5 ...$

15.7   Vibrating Systems

There are a large (maybe infinite!) number of ways that a distributed system like air can vibrate. Suppose it's able to vibrate at frequencies

$$f_1, \ f_2, \ f_3 \ ...$$

in ascending order. The lowest frequency is known as the *fundamental* and the subsequent ones are known as *overtones*, with the second frequency being the first overtone.

When the overtones are integer multiples of the fundamental,

$$f_n = nf_1$$

then they are known as *harmonics*, with the fundamental being the first harmonic.

In an open-open tube, the relation between overtones and harmonics is given by

$$H = O + 1$$

and in an open-closed tube, the relation is given by

$$H = 2O + 1$$

because open-closed tubes can only handle odd harmonics.

15.8 Beats

When two sounds have nearly the same frequency, they take on a simple form. Consider two sound waves with identical pressure amplitude at constant x, given by

$$\Delta p_1(t) = \Delta p_m \sin \omega_1 t$$

$$\Delta p_2(t) = \Delta p_m \sin \omega_2 t$$

so their resultant is given by

$$\Delta p(t) = \left[2\Delta p_m \cos\left(\frac{\omega_1 - \omega_2}{2}\right) t \right] \sin\left(\frac{\omega_1 + \omega_2}{2}\right) t$$

The first factor is the 'amplitude' of the envelope that contains the sinusoidal variation of the second factor, so we can rewrite this as

$$\Delta p(t) = [2\Delta p_m \cos \omega_{amp} t] \sin \omega_{av} t$$

When the two frequencies are close, ω_{amp} is small and the amplitude fluctuates slowly. Also, the rapid fluctuation within the envelope is approximately that of either individual wave.

Maximum intensity occurs twice in an envelope, when $\cos \omega_{amp} t = \pm 1$. Thus, we can define

$$\omega_{beat} = 2\omega_{amp} = |\omega_1 - \omega_2|$$

or in terms of frequency,

$$f_{beat} = |f_1 - f_2|$$

15.9 Doppler Effect

When an observer moves, she hears more/less wavelengths in some time, so we have

$$f' = \frac{v \pm v_o}{v} f$$

where positive implies observer going towards source, and negative implies away.

When a source moves, the observer hears shortened/elongated wavelengths, so we have

$$f' = \frac{v}{v \mp v_s} f$$

where negative implies source going towards observer, and negative implies away.

Thus, combining them, if both the source and observer are moving:

$$f' = \frac{v \pm v_o}{v \mp v_s} f$$

Note that for reflected waves, the 'observer' becomes the 'source' after reflection, so there may be two Doppler shifts. Also, all speeds must be taken with respect to the medium: Doppler shift must be analyzed in the frame in which the medium is at rest.

15.9.1 Effects at High Speeds

When v_o or v_s become comparable to v, the prior formulae may not apply because the restoring force of air may not be proportional to displacement anymore in the medium, among many other complications.

Ultimately, when the source moves faster than the phase speed of the wave in that medium, a *Mach cone* is formed, with angle given by

$$\sin \theta = \frac{v}{v_s} = \frac{1}{M}$$

where $M = v_s/v$ is the Mach number. Note that v_s is the speed of the source, not of sound. And a sonic boom comes from the Mach cone hitting a surface, not from directly breaking the sound barrier.

16　Special Relativity

16.1　Troubles with Classical Physics

16.1.1　Troubles with Time

The pion, a particle that can be created in an accelerator, has a much longer lifetime (before decay) than expected.

16.1.2　Troubles with Length

From the pion's point of view, less distance has been traveled compared to what we see in the lab, due to the time difference between the frames.

16.1.3　Troubles with Speed

Cause and effect can be violated by Newtonian physics. Consider a pitcher who throws a baseball faster than the speed of light. From the catcher's point of view, he catches the ball before it even leaves the pitcher's hand!

16.1.4　Troubles with Energy

An electron and positron move towards each other at low speeds (classically, $K \approx 0$), annihilating at contact. Radiation is released after the collision, thus increasing the internal energy of the system. The final (internal) energy is obviously greater than the initial (kinetic) energy by classical analysis, which violates conservation of energy.

16.2　Postulates of Relativity

Einstein proposed two postulates of relativity:

1. The Principle of relativity: *The laws of physics are the same in all inertial reference frames.*

2. The Principle of the constancy of the speed of light: *The speed of light in free space has the same value c in all inertial reference frames.*

The speed of light is $c = 3.00 \times 10^8$ m/s in a vacuum. It may *appear* to travel slower in other media, but it is really just interacting with particles in the medium.

16.3　Consequences of the Postulates

16.3.1　Time Dilation

In a particular frame, the proper time Δt_0 is measured by a clock at rest relative to the frame. However, a clock outside the frame measures time as

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}}$$

where u is the speed of the moving frame. It can be summed up with this phrase: "moving clocks run slow."

16.3.2　Length Contraction

In a particular frame, the proper (or rest) length L_0 is measured by an observer at rest relative to the frame. However, an observer outside the frame measures the length as

$$L = L_0 \sqrt{1 - u^2/c^2}$$

where u is the speed of the moving frame. It can be summed up with this phrase: "moving objects grow shorter."

16.3.3 Relativistic Velocity Addition

A reference frame moves at u relative to an outside observer, and an object moves at speed v_0 (along the same axis) as viewed from that frame. The observer sees said object moving at

$$v = \frac{v_0 + u}{1 + uv_0/c^2}$$

Notice that for $v_0 = c$, $v = c$ as well. Light moves at the same speed regardless of frame.

16.4 Lorentz Transformation

Consider two frames, S and S', with S' moving at velocity $\vec{u} = u\hat{i}$ relative to S. An event has coordinates x, y, z, t in S and x', y', z', t' in S'. Then, we can relate them as:

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut)$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

where $\gamma > 1$ is the Lorentz factor and $y' = y$ and $z' = z$. Moreover, it is convenient to introduce the speed parameter $\beta = u/c$. The relation between β and γ is

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

The inverse Lorentz transformations describe how to go from an event in S' to one in S, derived by replacing u with $-u$.

On the contrary, the interval transformations describe how length changes:

$$\Delta x' = \frac{\Delta x - u\Delta t}{\sqrt{1 - u^2/c^2}} = \gamma(\Delta x - u\Delta t)$$

$$\Delta t' = \frac{\Delta t - u\Delta x/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(\Delta t - u\Delta x/c^2)$$

and once again, $\Delta y' = \Delta y$ and $\Delta z' = \Delta z$.

16.5 Transformation of Velocities

The primed velocities are given by

$$v_x' = \frac{\Delta x'}{\Delta t'} = \frac{v_x - u}{1 - uv_x/c^2}$$

$$v_y' = \frac{\Delta y'}{\Delta t'} = \frac{v_y}{\gamma(1 - uv_x/c^2)}$$

$$v_z' = \frac{\Delta z'}{\Delta t'} = \frac{v_z}{\gamma(1 - uv_x/c^2)}$$

Note that because of time dilation, even with no relative motion in the y and z directions, their respective velocities are still different. The inverse transformations are found in the same way for coordinates.

16.6 Consequences of the Lorentz Transform

16.6.1 Consequences with Time

1. *The relativity of simultaneity*: If two observers are in relative motion, they do not agree on whether two events at different locations are simultaneous.

2. *The Doppler shift*: Because of Einstein's postulates, motion relative to a medium is an invalid concept in relativity, so only relative velocities matter in calculating the Doppler effect with light. Moreover, there is a transverse Doppler effect when a source or object move relative to each other, caused by time dilation.

3. *The twin paradox*: if a twin stays on a stationary Earth and another goes off on an intergalactic space journey, when the latter gets back, who will be older? Isn't velocity all relative, so each person sees the other as younger? Well, no, because the spaceship twin has to accelerate and decelerate, so the trip is not symmetric.

16.6.2 Consequences with Length

The Lorentz transform tells us that measurements of length are meaningless unless done simultaneously — it does no good to measure the coordinates of two ends of a rod, for instance, at different times.

16.7 Relativistic Momentum

Momentum is still conserved, but what is momentum?

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

which can be split into components. The units of momenta on such tiny scales are often measured in MeV/c, which is given by the conversion

$$1 \text{ kg} \cdot \text{m/s} = 1.875 \times 10^{21} \text{ MeV/c}$$

16.8 Relativistic Energy

Total relativistic energy is given by

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

and rest energy is given by

$$E_0 = mc^2$$

and kinetic energy is given by

$$K = E - E_0 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

The rest energies of common particles are:

- Proton: $m_p c^2 = 938$ MeV.

- Neutron: $m_n c^2 = 940$ MeV.

- Electron; $m_e c^2 = 0.511$ MeV.

However, we must not only rethink our conceptions of momentum and energy, but mass as well. For a particle at rest, a change in energy implies a change in mass:

$$\Delta m = \frac{\Delta E}{c^2}$$

Raising the temperature (and thus internal energy) also increases mass in a similar fashion.

16.9 Conservation of Total Relativistic Energy

Total relativistic energy is conserved in an isolated system of particles. And, the energy is related to momentum by the following:

$$E^2 = (pc)^2 + (mc^2)^2$$

Additionally, the relation between β and E_0 and K is

$$\beta = \sqrt{1 - \left(\frac{E_0}{K + E_0}\right)^2}$$

17 Temperature

17.1 Temperature

Consider two systems separated by a wall. If the wall does not allow heat transfer between the systems, it is adiabatic, or thermally insulating. But if the wall allows heat transfer, it is diathermic, or thermally conducting. Now consider two systems are in thermal contact, but there is no heat transfer, they are in thermal equilibrium. Thus, we can state a postulate known as the **zeroth law of thermodynamics**:

If systems A and B are in thermal equilibrium with third system C, then A and B are in thermal equilibrium with each other.

The zeroth law allows us to establish *temperature*: when two systems are in thermal equilibrium, we say that they have the same temperature. In practical physics, system C is referred to as a thermometer. We can restate the zeroth law in terms of temperature:

Temperature is a scalar property of all thermodynamic systems in equilibrium. Two systems are in thermal equilibrium iff their temperatures are equal.

17.2 Temperature Scales

There are three scales: Kelvin, Celsius (formerly centigrade), and Fahrenheit. The Kelvin scale is defined as follows:

- 0 K is absolute zero, and 273.16 K is the triple point of water.

- 0° C is the freezing point of water; 100° its boiling point. However, this definition has been swapped for one based on a conversion from the Kelvin scale, changing them to 0.00° C and 99.975°.

- 32° F is the freezing point of water; 212° F its boiling point. Again, this definition is historical only.

The relations between the scales are as follows:

$$T_F = \left(\frac{9 \text{ F}°}{5 \text{ C}°} \right) T_C + 32°\text{F}$$

$$T_C = \left(\frac{1 \text{ C}°}{1 \text{ K}} \right) T_K - 273.15°\text{C}$$

and the relation between Fahrenheit and Kelvin can be found from substitution. Thus, 9 F° = 5 C°. Notice how the degree symbol comes after the scale symbol when describing temperature intervals.

17.3 Thermometers

In principle, any object with a property that varies with temperature can be used as a thermometer. These properties may include gas pressure, resistance of a wire, length of a metal strip, color of a lamp filament, and more. However, the rules for these "private thermometers" is as follows:

- *A device-sensative temperature scale is defined only for that property; it does not necessarily agree with other private scales. However, all thermometers agree on the triple point of water (by definition).*

However, these thermometers are still useful as secondary standards for measuring temperature.

Let's analyze thermometers based on properties that vary linearly with temperature; that is,

$$T^*(X) = (273.16 \text{ K}) \frac{X}{X_{tr}}$$

where T^* is the private temperature and X is the property used to measure it. This relation must hold because at at triple point, the temperature must be 273.16 K no matter the scale.

The Constant-Volume Gas Thermometer is officially used for measuring temperatures because it is independent of the type of material (gas) used.
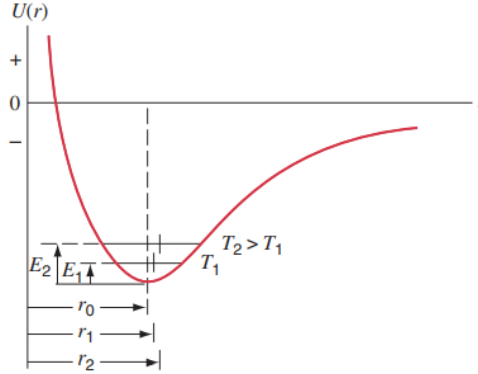
Figure 4: Potential energy vs separation curve

17.4   Thermal Expansion

Hot things make objects grow larger (as ladies know very well). The linear change in a solid is given by

$$\Delta L = \alpha L \Delta T$$

where α $[°C^{-1}]$ is the coefficient of linear expansion (representing the fractional change in length per unit temperature), on the order of $10^{-5} \sim 10^{-6}$. Strictly speaking, this coefficient does depend on the reference (initial) temperature, but generally these effects are negligible. However, the more accurate equation is

$$L \approx L_0 \left[1 + \int_{T_0}^{T} \alpha(T)dT \right]$$

For isotropic solids, the percent change in length for a given temperature change is the same for all lines in the solid, meaning that every this change is the same for all diagonals, lengths, widths, etc. Thus, it can also be shown that to a very good approximation, the changes in area and volume are

$$\Delta A = 2\alpha A \Delta T$$

$$\Delta V = 3\alpha V \Delta T$$

For liquids, linear expansion is a bit meaningless due to their capability of flowing easily, but volume is still relevant:

$$\Delta V = \beta V \Delta T$$

where β is coefficient of volume expansion. It is relatively independent of temperature and large (usually over an order of magnitude more than solids' linear coefficients). Knowing this, we can also relate changes in temperature with density:

$$\Delta \rho = -\beta \rho \Delta T$$

Of course, water is weird. Above 4°C, $\beta > 0$ but is not constant. As temperature lowers from 4°C to 0°C, though, $\beta < 0$ and water expands with decreasing temperature. Thus, the densest water is at 4°C, which is why the bottoms of lakes (super dense!) freeze later than the surfaces.

17.4.1   Microscopic Basis of Thermal Expansion

As shown in the diagram, there is an asymmetry in the potential energy function. For distances less than equilibrium, PE increases rapidly due to strong repulsive forces. However, for distances greater than equilibrium, weaker attractive forces take over, causing PE to rise more slowly.

1. At some vibrational energy, the separation r goes from maximum to minimum. The asymmetry, however, keeps average separation greater than the equilibrium separation.

2. At larger separations, the kinetic energy is lower, which causes it to contribute more to the time average.

Since vibrational energy increases with temperature, so does average separation.

17.5 Newton's Law of Cooling

If the temperature difference between an object and its surroundings ($\Delta T = T_{obj} - T_{sur}$) is not too great, the rate of cooling or warming is approximately proportional to their temperature difference:

$$\frac{d\Delta T}{dt} = -A(\Delta T)$$

where A is a constant. The negative appears because the object will always "try" to decrease the temperature difference. This is known as *Newton's law of cooling*.

If at time $t = 0$ the temperature difference is ΔT_0, it is

$$\Delta T = \Delta T_0 e^{-At}$$

some time later.

17.6 Ideal Gases

The results of the Constant-Volume Gas Thermometer suggests that at low enough densities, their properties converge. This gives us the concept of an *ideal gas*, which exists at low pressures and high temperatures. Three experimental results have been observed:

1. *Boyle's Law*: PV = constant

2. *Charles' Law*: V/T = constant

3. *Gay Lussac's Law*: P/T = constant

4. *Avogadro's Law*: V/n = constant

where n is the number of moles. Together, they make up the this following relation:

$$pV = NkT$$

where $N = nN_A$ is the number of molecules and $k = 1.38 \times 10^{-23}$ J/K is the Boltzmann Constant, and $N_A = 6.022 \times 10^{23}$ mol/molecule is Avogadro's constant.

$$pV = nRT$$

where n is the number of moles and $R = 8.31$ J/mol $\cdot$ K is the molar gas constant.

18   Molecular Properties of Gases

18.1   Brownian Motion

18.2   Brownian Motion

Brownian motion is characterized by the erratic movement of molecules at an atomic level. A particle suspended in a fluid is bombarded on all sides by atoms of the fluid; on average, it will experience no net movement, but due to randomness there will be fluctuations around its original position, given by

$$[(\Delta x)^2]_{av} = \frac{RT}{3\pi \eta a N_A} \Delta t$$

if the bombarded particle is of radius a. Known as *random walk* patterns, these motions supported the atomic theory of matter.

18.3   Properties of Ideal Gases

1. Large number of molecules—makes the mean more constant
2. Molecules are of negligible volume—implies that the volume of the container is the free volume that the gas can move through
3. No intermolecular forces—electromagnetic, quantum, etc.
4. Collisions are elastic and of negligible duration—implies kinetic energy is constant and potential energy of molecules is negligible

18.4   Molecular View of Pressure

Consider one particle bouncing off one wall of a cuboidal box. The force delivered to the particle is

$$F_x = \frac{2mv_x}{2L/v_x}$$

Extending this reasoning to all particles,

$$p = \frac{F_x}{A} = \frac{Nm}{L^3} \frac{(v_{1x}^2 + v_{2x}^2 + \cdots + v_{Nx}^2)}{N}$$

Then,

$$p = \frac{1}{3}\rho(v^2)_{av}$$

because $v^2 = v_1^2 + v_2^2 + v_3^2$, with no preference among the three directions. Finally, solving for the root-mean-square speed yields

$$v_{rms} = \frac{3p}{\rho}$$

18.5   Mean Free Path

The mean free path is the average distance a molecule travels between collisions. Consider if a molecule doubles in radius while all others become particles. Then, as the particle travels through a cylindrical volume in time t,

$$\lambda = \frac{L_{cyl}}{N_{cyl}} = \frac{v_{av}t}{N \cdot \frac{\pi d^2 v_{rel}t}{V}} = \frac{kT}{\pi d^2 p}$$

The relative speed of two molecules is used in the denominator when counting the effective number of particles in its path. It can easily be seen that on average, when two particles are traveling at $90°$ to each other, the relative speed is $v\sqrt{2}$. In real gases, the ability to conduct heat, viscosity, and rate of diffusion from high to low concentration are all proportional to the mean free path.

18.6 Maxwell's Speed Distribution

Maxwell's distribution of a (large) number of molecules in a gas) gives

$$N(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

The product $N(v)dv$ gives the number of molecules within the speed distribution dv. The total number of molecules N is given by

$$N = \int_0^\infty N(v)\ dv$$

18.7 Consequences of the Speed Distribution

18.7.1 Most Probable Speed

This quantity is the maximum value of N(v), given by:

$$v_p = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}}$$

derived from setting $dN(v)/dv = 0$.

18.7.2 Average Speed

The average speed is

$$v_{av} = \frac{1}{N}\int_0^\infty v N(v)\ dv = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}$$

18.7.3 Root-Mean Square Speed

The RMS speed is the root of the average square speed, or

$$v_{rms} = \sqrt{\frac{1}{N}\int_0^\infty v^2 N(v)\ dv} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}$$

18.7.4 Average Translational Kinetic Energy

The average translational kinetic energy per molecule is given by

$$K_{trans} = \frac{1}{2}m(v^2)_{av} = \frac{3}{2}kT$$

18.7.5 The Ideal Gas Law

We can use the Ideal Gas Law to rewrite the RMS speed,

$$v_{rms}^2 = \frac{3p}{\rho} = \frac{3RT}{M}$$

Note that the third expression comes directly from the first by substituting $\rho = \frac{\Sigma m}{V}$.

18.8 Experimental Verification of Maxwell Speed Distribution

Molecules are sent from a slit in an oven through a rotating cylinder with grooves, so their speed can be found as

$$\frac{L}{v} = \frac{\phi}{\omega}$$

where ϕ is the angular displacement between the start and end of a helical groove. The rotating cylinder is thus known as a *velocity selector* because only certain velocities will make it through the grooves without colliding with the walls. Adjusting this for a lot of speeds can be used to plot the Maxwell speed distribution.

The distribution of speeds from the cylinder $N(v)$ is proportional to v^3, not v^2, because higher speeds escape the oven slit and bombard the cylinder with higher frequency, in the proportion v_2/v_1.

18.9 Maxwell-Boltzmann Energy Distribution

The Maxwell-Boltzmann energy distribution is applicable when molecules only have translational kinetic energy. It is based on the following argument: $N(v)dv = N(E)dE$, which yields

$$N(E) = \frac{2N}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT}$$

The factor $e^{-E/kT}$ is the *Boltzmann factor* and is a general feature of the Maxwell-Boltzmann Energy Distribution, no matter the form of energy E. It gives a rough estimate of the relative probability for a particle to have an energy E in particles at temperature T. Similary, the total number of molecules N is

$$N = \int_0^\infty N(E) \, dE$$

Note that unlike the speed distribution, the mass has no effect. This is because when mass doubles, for instance, speed goes down by the square root of that factor.

18.10 Equations of State for Real Gases

Real gases are not ideal at any density and depart more and more for higher densities.

18.10.1 Virial Expansion

The Virial expansion adjusts the right side of the Ideal Gas Law:

$$pV = nRT \left[1 + B_1 \frac{n}{V} + B_2 \left(\frac{n}{V} \right)^2 + \cdots \right]$$

The constants B_1, B_2, $\cdots$ are called *Virial coefficients* and grow successively smaller as the series progresses. As $n/V \to 0$, the expansion simplifies down to the Ideal Gas Law.

18.10.2 Van der Waals Equation

Van der Waals' equation of state adjusts the left side of the Ideal Gas Law:

$$\left(p - \frac{an^2}{V^2} \right)(V - nb) = nRT$$

1. Pressure Correction: intermolecular forces are exerted, decreasing the pressure needed by walls. The force is proportional to number of molecules n/V, and another factor of n/V because 5 times as many molecules means 5 times the correction.

2. Volume Correction: molecules aren't particles. Let one particle be a sphere of double radius and all other be particles; we get:
$$b = \frac{V_{cor}}{n} = \frac{\frac{1}{2} \left(4\pi d^3 \right) N}{n} = \frac{2}{3} \pi d^3 N_A$$

19   First Law of Thermodynamics

19.1   Heat Transfer

Heat is defined as energy that flows from one system to another due to a temperature difference between them. It is a transfer of energy, not a state function, and there are three mechanisms through which it occurs.

19.1.1   Conduction

Thermal conduction is heat transfer through physical contact. On the molecular level, when atoms are in contact, they are able to transfer their higher kinetic energies to those nearby with lower energies.

Consider a rectangular slab of concrete, which has one face at temperature $T + \Delta T$ and another at T. Empirically, the heat transfer is

1. $\propto A$. More area means more atoms can transfer their higher kinetic energies across the slab.

2. $\propto 1/\Delta x$. More separation means that it takes longer (more collisions) for atoms to transfer their higher kinetic energies across the slab.

3. $\propto \Delta T$. More temperature differential means that each atom has more kinetic energy to transfer.

In summary, we can write the heat as
$$H = kA\frac{\Delta T}{\Delta x}$$

in which k is the *thermal conductivity* of the material, in units of $\mathrm{W/m \cdot K}$. Oftentimes, materials are also rated by their *thermal resistance* or *R-value*, defined as

$$R = \frac{L}{k}$$

where L is the thickness. In terms of the temperature gradient, we can rewrite

$$H = -kA\frac{dT}{dx}$$

where x is defined positive in the direction of heat flow, and the negative is present to keep H positive. And for materials in series with the same cross sectional area, we can write

$$H = \frac{A\Delta T}{\Sigma R_n}$$

because the heat flow through the two materials and the temperature at their boundary must be the same.

19.1.2   Convection

Convection occurs when a fluid is in contact with an object of higher temperature. Portions of the fluid near the object heat up and thus expand (in most cases), so they become less dense and rise.

19.1.3   Radiation

Radiation, which is composed of electromagnetic waves, is emitted by all objects due to their temperature (and absorbed as well).

19.2   First Law of Thermodynamics

The first law is expressed as,

$$Q + W = \Delta E_{int}$$

where W is mechanical work (done on the system). Thermodynamic work would see W replaced with $-W$. Generally, this assumes that the system only contains internal energy; if changes in kinetic, potential, or other forms of energy took place, they would need to be included on the RHS.

Consider a process connecting the initial and final equilibrium states, i and f. While Q and W may differ, experiment shows that ΔE_{int} remains the same no matter what. This means that internal energy is a *state function*.

19.3   Heat Capacity and Specific Heats

An object's temperature increases as heat is transferred to it. However, different materials and different conditions affect the relationship between ΔT and Q, which we measure as *heat capacity*:

$$C = \frac{Q}{\Delta T}$$

And the heat capacity per unit mass of the body is the *specific heat capacity* is given by

$$c = \frac{Q}{m\Delta T}$$

While heat capacity is a property of an object, specific heat capacity is a property of a substance. Both are dependent on temperature and pressure, so to find the amount of heat necessary to change from one temperature to another, we would need to valuate

$$Q = \sum_{n=1}^{N} mc_n\Delta T_n = m \int_{T_i}^{T_f} c \, dT$$

Note that in order to properly define c, we must specify how the heat is added, such as at constant pressure (c_v), constant pressure (c_p), or otherwise.

19.3.1   Heats of Transformation

When a substance is at a phase change temperature, adding or removing heat may not cause a temperature change. Instead, the heat goes towards transforming it from one phase to another, related by

$$Q = Lm$$

where L is known as the *heat of fusion* for solid-liquid changes and *heat of vaporization* for liquid-gas changes, with the latter being generally higher. More broadly, they are referred to as the *heats of transformation* or *latent heats*.

19.3.2   Heat Capacities of Solids

If we multiply specific heat c by the molar mass M, we obtain the molar heat capacity,

$$C = \frac{Q}{n\Delta T}$$

Dulong and Petit first noticed that nearly all solids have molar heat capacities close to 25 J/mol · K. This is because we are in effect measuring the heat capacity per atom, which should be approximately the same even for different materials. However, other factors such as temperature and pressure affect molar heat capacity, so the Dulong-Petit value is only approached by high temperatures.

19.4 Work Done by an Ideal Gas

Consider a piston of area A moving upwards in a container full of ideal gas. The force is opposite the direction of motion (downwards), so the work done *on* the gas is negative:

$$W = \int -pA \, dx = \int -p \, dV$$

Thus we often plot pressure-volume (pV) diagrams to analyze the work, where the negative of the area under the curve is the work done. Note that it does depend on the path taken. For closed processes, the magnitude of net work done on the gas is the enclosed area, and the sign is given by something like a right-hand rule.

19.4.1 Work Done at Constant Volume

At constant volume, $\Delta V = 0$ so no work is done.

19.4.2 Work Done at Constant Pressure

At constant pressure,

$$W = -p \int_{V_i}^{V_f} dV = -p(V_f - V_i)$$

19.4.3 Work Done at Constant Temperature

At constant temperature,

$$W = -\int_{V_i}^{V_f} \frac{nRT}{V} \, dV = -nRT \ln \frac{V_f}{V_i}$$

This is known as *isothermal* and the curve on a pV diagram is known as an *isotherm*.

19.4.4 Work Done in Thermal Isolation

An adiabatic process is defined to be a process in thermal isolation, and its path follows a curve

$$pV^\gamma = \text{constant}$$

where $\gamma > 1$ is the dimensionless *ratio of specific heats*, so an adiabatic curve is steeper than an isothermal curve where they intersect.

By substituting $p = p_i V_i^\gamma / V^\gamma$ and integrating, adiabatic work can be expressed as

$$W = \frac{p_i V_i}{\gamma - 1}\left[\left(\frac{V_i}{V_f}\right)^{\gamma-1} - 1\right] = \frac{1}{\gamma - 1}(p_f V_f - p_i V_i)$$

19.4.5 Bulk Modulus for Adiabatic Process

In the differential limit,

$$B = -V\frac{dp}{dV}$$

By taking the differential of $pV^\gamma = \text{constant}$, it can be shown that

$$B = \gamma p$$

And, since the speed of sound in such a gas is

$$v = \frac{B}{\rho} = \frac{\gamma RT}{M}$$

19.5 Internal Energy of an Ideal Gas

The *degrees of freedom* of a molecule is how many ways it can absorb energy; for instance, a diatomic gas has 5 d.o.f. since its energy is

$$K = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 + \frac{1}{2}I\omega_x^2 + \frac{1}{2}I\omega_y^2$$

Maxwell's *equipartition of energy theorem* then states that each degree of freedom yields an average energy per particle of $\frac{1}{2}kT$ when the number of molecules is large. Therefore, for a gas with N molecules,

$$E_{int} = \frac{3}{2}NkT = \frac{3}{2}nRT \quad \text{(monatomic)}$$

$$E_{int} = \frac{5}{2}NkT = \frac{5}{2}nRT \quad \text{(diatomic)}$$

$$E_{int} = \frac{6}{2}NkT = 3nRT \quad \text{(polyatomic)}$$

A corollary of the theorem is thus that internal energy only depends on temperature.

19.5.1 Molar Heat Capacity of Solids

In solids, atoms are stuck in a lattice can thus can oscillate with kinetic energy in three d.o.f., but they also have intermolecular interactions with potential energy in three more d.o.f., for a total of 6 d.o.f. Thus, since a solid does not do work,

$$Q = \Delta E_{int} = 3nR\Delta T$$

The molar heat capacity is

$$C = \frac{Q}{n\Delta T} = 3R$$

Note that $C \approx 25$ J/mol $\cdot$ K is the Dulong-Petit value!

19.6 Heat Capacities of an Ideal Gas

The measured heat capacity of a substance depends on how the heat is added to it. Let us analyze some common cases.

19.6.1 Molar Heat Capacity at Constant Volume

There is no work done at constant volume, so

$$C_V = \frac{Q}{n\Delta T} = \frac{f}{2}$$

where f is the number of degrees of freedom. Therefore,

1. Monatomic: $C_V = 3/2R = 12.5$ J/mol $\cdot$ K

2. Diatomic: $C_V = 5/2R = 20.8$ J/mol $\cdot$ K

3. Polyatomic: $C_V = 3R = 24.9$ J/mol $\cdot$ K

19.6.2 Molar Heat Capacity at Constant Pressure

At constant pressure,

$$Q = \Delta E_{int} - W = \frac{f}{2}nR\Delta T + p\Delta V = \frac{f+2}{2}nR\Delta T$$

which implies

$$C_p = \frac{Q}{n\Delta T} = \frac{f+2}{2}R$$

Therefore,

1. Monatomic: $C_p = \frac{5}{2}R = 20.8$ J/mol $\cdot$ K

2. Diatomic: $C_p = \frac{7}{2}R = 29.1$ J/mol $\cdot$ K

3. Monatomic: $C_p = 4R = 33.3$ J/mol $\cdot$ K

19.6.3 Ratio of Specific Heats

The ratio of molar heat capacities is defined as

$$\gamma = \frac{C_p}{C_V} = \frac{f+2}{f}$$

But this is equivalent to the ratio of specific heats (or specific heat ratio) because C and c are related by a factor of M only. Therefore,

1. Monatomic: $\gamma = \frac{5}{3} = 1.67$

2. Diatomic: $\gamma = \frac{7}{5} = 1.4$

3. Polyatomic: $\gamma = \frac{8}{6} = 1.33$

19.7 Applications of the First Law of Thermodynamics

19.7.1 Adiabatic Processes

Here we will show that pV^γ is constant for adiabatic processes. Our goal is to obtain an integratable equation involving dp and dV.

Finding dV:
Assuming that the process is carried out slowly (so pressure is well-defined and volume is constant),

$$pdV = -dW = -dE_{int} = -nC_V dT$$

Finding dV: And taking a differential of $pV = nRT$ yields

$$V dp = nRdT - pdV = nRdT + dW = nC_p dT$$

Solving: Combining the results yields

$$\frac{V dp}{pdV} = -\frac{C_p}{C_V} = -\gamma$$

Integrating,

$$p_i V_i^\gamma = p_f V_f^\gamma$$

But i and f are arbitrarily chosen, so we can rewrite as

$$pV^\gamma = \text{constant}$$

We can also rewrite in terms of **temperature**:

$$p_i V_i (V_i)^{\gamma-1} = p_f V_f^{\gamma-1}$$

Using the ideal gas law and solving,

$$TV^{\gamma-1} = \text{constant}$$

19.7.2   Isothermal Processes

In isothermal (constant temperature) processes,

$$Q + W = 0$$

19.7.3   Isometric Processes

In isometric (constant volume) processes,
$$Q = \Delta E_{int}$$

19.7.4   Cyclical Processes

In cyclical (same initial and final state) processes,

$$Q + W = 0$$

because internal energy is a state function.

19.7.5   Free Expansion

Consider a thermally isolated container divided in two halves, one of which contains gas. When the barrier between the halves is dropped, the gas will expand to fill the entire container, given by

$$Q = W = \Delta E_{int} = 0$$

This is a *non-equilibrium process*: while the initial and final states have well-defined pressures and volumes (and thus temperatures), intermediate states do not. Thus, we cannot plot the entire process on a pV diagram.

20 Second Law of Thermodynamics

20.1 One Way Processes

The *Entropy Principle* states that in an <u>irreversible</u> process, entropy always increases. The reverse of certain processes would have $\Delta S < 0$ if they occurred, violating the entropy principle. Therefore, they do not.

There are two approaches to analyzing entropy; one is macroscopic (involving heat and temperature) and the other is microscopic (involving the number theory of particles).

20.2 Definining Entropy Change

For reversible (ideal) processes, entropy change is defined as

$$\Delta S = \int_i^f \frac{dQ}{T}$$

and if a process is isothermal, this becomes

$$\Delta S = \frac{Q}{T}$$

20.3 Entropy as a State Property

Taking a differential of the First Law of Thermodynamics,

$$dQ = dE_{int} - dW = nC_V \, dT + p \, dV$$

and dividing by T,

$$dS = \frac{dQ}{T} = nC_V \frac{dT}{T} + nR \frac{dV}{V}$$

Integrating,

$$\Delta S = nC_V \ln \frac{T_f}{T_i} + nR \ln \frac{V_f}{V_i}$$

Therefore, since temperature and volume are state properties, so is entropy.

20.4 Entropy Change for Irreversible Processes

The following method is used to calculate the entropy of an irreversible process:

1. Find a reversible process that connects the same initial and final states.

2. Calculate ΔS for the equivalent reversible process. This entropy change is the same as that of the irreversible one.

Example 1: Consider a thermally isolated bucket of water. A stone of potential energy U falls into it, causing the temperature of the final system to be greater than before. We would calculate ΔS by instead imagining that a heater at the bottom of the water increased the temperature by transferring heat $Q = U$ to the water.

Example 2: Consider a free expansion of a gas within a thermally isolated container. For the equivalent reversible process, imagine instead that the gas is enclosed by a container with a piston that rises to isothermally increase the volume.

20.5 The Second Law of Thermodynamics

The second law of thermodynamics states that within a closed system, $\Delta S \geq 0$.

The $\Delta S = 0$ applies to reversible processes and $\Delta S > 0$ applies to irreversible processes.
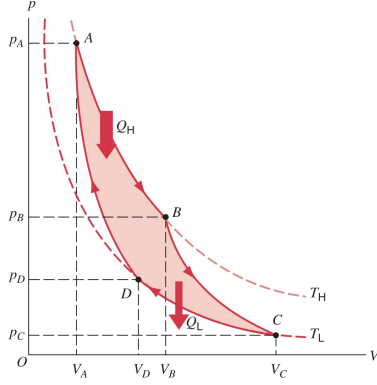
Figure 5: pV Carnot Plot
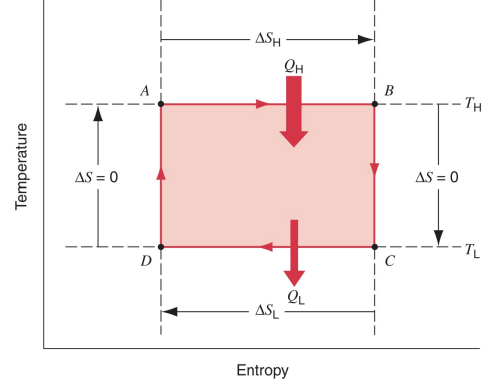
Figure 6: TS Carnot Plot

20.6 Entropy and the Performance of Engines

A *heat engine* is a device that converts heat to work, with a *working substance* that operates in a cycle.

20.6.1 Carnot Engine

The Carnot Engine is an engine with an Ideal Gas as the working substance and completely reversible processes. There are four steps to a Carnot cycle:

1. Isothermal expansion from V_A to V_B, as heat is transferred to the container via the hot reservoir at T_H.

2. Adiabatic expansion from V_B to V_C, as the container is placed on an insulating slab.

3. Isothermal compression from V_C to V_D, as heat is transferred from the container via the cold reservoir at T_L.

4. Adiabatic compression from V_D to V_A, as the container is placed on an insulating slab.

Consider Figure 5. During expansions AB and BC, the gas is doing positive work *on* the system. But in CD and DA, the gas is doing negative work *on* the system. With engines, it is more convenient to analyze work done by the gas, not on the gas.

Consider Figure 6. During AB, the entropy of the gas increases, but during CD, the entropy of the gas decreases. Processes BC and DA are *isentropic*, or constant entropy, processes.

20.6.2 Efficiency of a Carnot Engine

The efficiency is defined as

$$\epsilon = \frac{|W|}{|Q_H|}$$

In essence, it is the energy you get relative to the energy you pay for. Since E_{int} of the working substance is a state function, it does not change in one cycle, so $|Q_H| - |Q_L| - |W| = 0$. Therefore,

$$\epsilon = \frac{|Q_H| - |Q_L|}{|Q_H|} = 1 - \frac{|Q_L|}{|Q_H|}$$

However, from Figure 6, we can see that

$$\Delta S = \frac{|Q_H|}{T_H} - \frac{|Q_L|}{T_L} = 0$$

Using this result, the efficiency becomes

$$\epsilon = 1 - \frac{T_L}{T_H}$$

Necessarily, the efficiency is less than 100% since $T_L < T_H$. Real engines are all less efficient than the Carnot engine.

20.6.3 Stirling Engine

The Stirling Engine involves two two isothermal heat transfers, just like a Carnot engine, but they are instead connected by isometric (constant volume) processes, rather than adiabatic ones. Its ideal efficiency will be lower than that of a Carnot cycle, and real Stirling engines will have lower efficiencies yet.

20.7 Entropy and the Performance of Refrigerators

A *refrigerator* is a device takes in work in order to transfer heat from a lower temperature reservoir to a higher temperature reservoir.

- A household refrigerator uses electrical work to keep its insides cool by transferring heat out.

- An air conditioner keeps a house cool by transferring heat out of the home.

Assuming that the processes are reversible, we have an ideal refrigerator. However, this refrigerator is simply a Carnot engine running backwards, so we call it a Carnot refrigerator.

In a refrigerator, the measure of efficiency is the *coefficient of performance* given by

$$K = \frac{|Q_L|}{|W|}$$

Like a Carnot engine, we can rewrite it as

$$K = \frac{|Q_L|}{|Q_H| - |Q_L|} = \frac{T_L}{T_H - T_L}$$

Note that refrigerators are more effective when the temperatures are close together.

Can we construct a perfect refrigerator with $W = 0$ and $K \to \infty$? No, because then entropy change in a cycle would be

$$\Delta S = -\frac{|Q|}{T_L} + \frac{|Q|}{T_H} < 0$$

which violates the second law of thermodynamics.

20.8 The Second Law Revisited

The Second Law of Thermodynamics can be expressed in three ways for closed systems,

1. $\Delta S \geq 0$, or entropy never decreases.

2. $\epsilon < 100\%$, or there are no perfect engines.

3. $K < \infty$, or there are no perfect refrigerators.

While these statements seem different, they are all identical. In deriving the workings of engines and refrigerators, we have seen that statements 2 and 3 result from statement 1. We can also see that statement 2 results from statement 3 by linking up a perfect engine to a Carnot refrigerator; the net effect is to have a perfect refrigerator (a violation of statement 3!). Therefore if statement 3 is true, statement 2 must be too.

20.9 A Statistical View of Entropy

Consider a box, which we will divide into halves to place molecules in. Suppose there are n particles, and we want to split them into a *configuration* of k and $n - k$ portions between the halves. An independent configuration is known as a *microstate*, and the number of ways to arrive at it is the *multiplicity* w, given by

$$w = \frac{n!}{k!(n - k)!}$$

In essence, we are "chooosing" k particles from n total. There are a total of 2^n equally probable microstates, but note that each configuration is not probable, rather being skewed towards those with high multiplicities.

There are **two important trends** here.

1. The entropy of a system is the *sum* of its subsystems' entropies.

2. The probability of a system is the *product* of its subsystems' probabilities.

Logarithms take products and return sums, so it is logical that they are involved. The actual relationship is

$$S = k \ln w$$

which is *Boltzmann's entropy equation*, with $k = 1.38 \times 10^{-23}$ being the Boltzmann constant. w is often large enough to result in overflow errors, so we instead use *Stirling's Approximation*,

$$\ln N! \approx N \ln N - N$$

20.10 Entropy and Disorder

Entropy is often cast as the amount of disorder in a system, but for this definition to have significance, it must be carefully defined (on a case-to-case basis). One example of this is swirling a cup of tea.

- While swirling, the tea is in uniform angular motion, so there are less possible speeds (and thus few microstates).

- After swirling, the tea is at rest, so there are many more possible speeds due to random motion (and thus many more microstates).

Thus, a swirled cup of tea has more 'disorder' and entropy and one at rest.

Part II
Electromagnetism

21   Electric Charges and Forces

21.1   Electric Charge

Charge is a scalar quantity created by the transfer of electrons. However, it is a *derived unit*, defined as the ampere-second because it is very difficult to measure the force between two static charges.

21.2   Atomic Structure

Electric charge is quantized, meaning that all charges are multiples of the elementary charge e:

$$e = 1.602 \times 10^{-19} \text{ C}$$

which means that number of electrons per Coulomb is

$$N = 6.242 \times 10^{18}$$

All particles such as electrons and protons have *antiparticles*, which have the same mass but opposite charge. Specifically, they are the positron and antiproton for these ones.

Ordinary nuclei have Z protons, known as the atomic number, so that the charge of the nucleus is $+Ze$. Electrons make the atom neutral overall, but ions can be formed by adding and removing electrons.

While electrons are *fundamental* particles since they have no substructure, protons and neutrons are believed to be made up of more elementary entities known as quarks. An up quark has $q = +\frac{2}{3}e$ and a down quark has $q = -\frac{1}{3}e$, so a proton has 2 up quarks and 1 down quark, while a neutron has 1 up quark and 2 down quarks.

21.3   Conductors and Insulators

Conductors are materials in which electrons can move relatively freely; their electrons do not belong to specific atoms but rather flow readily. Examples include tap water and the human body. *Insulators*, contrarily, are materials in which electrons can hardly flow at all. Examples include glass, plastics, and many crystalline materials such as NaCl.

When an object has a source from which to obtain or expel electrons, it is known as being *grounded*, and as a result there is no net charge. The Earth is a common source and is considered to possess an infinite supply of electrons.

Isolated atoms of conducting materials have loosely bound electrons known as *conduction electrons*. They can easily move through the material, leaving fixed positive ions in their wake.

Polarization allows insulators to be affected by electric forces, too. Consider a group of electrons near a block of wood; the electrons in the wood are slightly repelled and the protons are slightly attracted, creating a net charge gradient in the wood.

21.4 Charging by Contact and Induction

Consider a neutral copper rod put in contact with a positively charged glass rod. Electrons will move from copper to glass in order to neutralize the glass, but only at the point of contact. By wiping the electrons to other areas, the entire rod can be neutralized. This is *charging by contact.* -

Now consider the polarized copper rod in the presence of the positive glass, connected to the ground by a string. Electrons flow from the ground to the positive end of the copper to neutralize it, and if we disconnect the string, the copper rod thus retains net negative charge. This is *charging by induction.*

21.5 Coulomb's Law

Coulomb's Law gives the force between two point charges,

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

where $\epsilon_0 = 8.854 \times 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}$. In vector form, this becomes

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

where 12 denotes the force on 1 by 2 and the direction to 1 from 2.
By breaking up distributions of charge into small elements that are 'point charges', electric forces can be found.

21.6 Shells of Charge

Consider a thin spherical shell of charge. We have

1. There is no electric force exerted on a point charge located anywhere inside the shell.

2. The shell can be treated as a point charge regarding forces on other charges outside the shell.

21.7 Conservation of Charge

No exceptions have ever been found to conservation of charge. Particles at the subatomic level must follow these rules.

- Hydrogen ion: H (one proton)

- Deuterium: ^{2}H (one proton, one neutron); 'heavy hydrogen'

22　Electric Field

22.1　Electric Field

Electric field is defined as

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

where q_0 is a test charge at some location in space. Thus, for a point charge,

$$E = \frac{1}{4\pi\epsilon_0} \frac{|q|}{r^2}$$

22.2　Electric Dipole

An electric dipole is a configuration of two opposite charges, $+q$ and $-q$, separated by distance d. The electric dipole moment is then

$$\vec{p} = q\vec{d}$$

where the vector d is drawn from the negative to positive charge. (Note that this is opposite of the convention used in chemistry.) Generally, the field of a dipole is

$$\vec{E} = \vec{E_+} + \vec{E_-}$$

On a perpendicular bisector of the line joining the charges, the field varies as

$$E = \frac{1}{4\pi\epsilon_0} \frac{p}{x^3}$$

in the direction of $-\hat{d}$ for $x >> d$. On the axis of the dipole,

$$E = \frac{1}{4\pi\epsilon_0} \frac{p}{z^3}$$

in the direction of $\hat{k}$ for $z >> d$.

The quadrupole, octupole, and other higher-order moments also exist. For a quadrupole, $E \propto r^{-4}$ and for an octupole, $E \propto r^{-6}$.

22.3　Electric Field Lines

Field lines emanate from positive charges and end on negative charges. The number of electric field lines per unit cross-sectional area is proportional to the strength of the field at that point.

22.4　Measuring Elementary Charge

The earliest measurements of the quantized elementary charge were conducted by Robert Millikan as follows: oil drops of charge q (assumed negative) were allowed to fall in a chamber, eventually reaching terminal speed v. Then, a downwards field $\vec{E}$ would be applied, sending electrons upwards until a new terminal speed v' was reached. Thus,

$$m\vec{g} - b\vec{v} = 0$$
$$q\vec{E} + m\vec{g} - b\vec{v'} = 0$$

Measurements of v and v' would then allow q to be determined. Millikan discovered that all charges q were multiples of an elementary charge e; that is,

$$q = ne \quad n = 0, \pm1, \pm2, \pm3 \cdots$$

22.5 A Dipole in an Electric Field

Suppose a dipole is placed in an external electric field, with $\vec{d}$ at an angle θ to $\vec{E}$. The torque on the dipole would be

$$\tau = 2F\left(\frac{d}{2}\sin\theta\right) = pd\sin\theta$$

in vector form,

$$\vec{\tau} = \vec{p} \times \vec{E}$$

The work done by the external field is

$$W = \int_{\theta_0}^{\theta} \vec{\tau} \cdot d\vec{\theta} = pE(\cos\theta - \cos\theta_0)$$

because the torque always tends to restore the dipole into alignment with the field, so the dot product is negative.

The potential energy, allowing the reference angle to be $\theta_0 = 90°$, is

$$U = -\vec{p} \cdot \vec{E}$$

22.6 Nuclear Model of the Atom

One way to test the structure of an atom is by shooting a beam of positively charged particles towards it. By letting each projectile be significantly less massive than the atom but more massive than an electron, we ensure that only the electric force from the nucleus has a significant effect on the deflection of the beam.

Suppose the horizontally-launched projectiles pass very close to the surface of the nucleus. Then, the maximum transverse velocity deflection is

$$\Delta v = \frac{F}{m}\Delta t = \frac{1}{m}\left(\frac{Qq}{4\pi\epsilon_0 R^2}\right)\left(\frac{2R}{v}\right)$$

The maximum deflection angle would thus be

$$\theta = \frac{\Delta v}{v}$$

Ernest Rutherford first conducted this experiment with a stream of alpha particles. While many particles were deflected only marginally, some were deflected by angles greater than $90°$, something impossible under the Thompson's Plum-Pudding Model (where electrons and protons were uniformly distributed throughout a sphere). He used this to develop the concept of a nucleus.

23 Gauss' Law

23.1 Flux and Lines of Field

Electric flux through a closed surface is always 0 unless there are sources of charge within the surface. In other words, Gauss' Law can be applied to closed surfaces even if there are external fields present.

23.2 Gauss' Law

If we construct a *Gaussian surface* through space, Gauss' Law states that

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Or, in differential form,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Notice that if a sphere is chosen as the Gaussian surface around a point charge q, Gauss' Law reduces to Coulomb's Law.

23.3 Applications of Guass' Law

For an infinite line of charge,

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

For an infinite plane of charge,

$$E = \frac{\sigma}{2\epsilon_0}$$

23.4 Shell Theorems

From Gauss' Law, two shell theorems can be deduced:

1. A uniform spherical shell of charge has no electric field in its interior.
2. A uniform spherical shell of charge has an electric field in its exterior that behaves as if all the charge were concentrated at its center.

A result of this is that within a uniform sphere of charge, which can be thought of as a succession of shells,

$$E = \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3}$$

23.5 Gauss' Law and Conductors

If charges were placed arbitrarily on a conductor, they would all redistribute themselves on its outer surface. This is because charges tend towards electrostatic equilibrium; if the interior had an electric field, the conduction electrons would experience a force and current would be observed. (Consider a uniform sphere of charge: because $E \neq 0$ inside the conductor, charges inside would not be at equilibrium.) Alternatively, equilibrium minimizes the potential energy of the charges.

23.6 Charge on Interior Surfaces

Suppose a conductor has a cavity inside of it, creating an "outer" surface on its interior. Does charge also move to that surface? Gauss' Law tells us no, because then there would be an electric field in the rest of the conductor's interior.

If some charge is magically moved inside the cavity, what happens? Again, because there cannot be electric field in the conductor's interior, charges will shift from the outermost surface to the interior surface—just enough to neutralize the electric flux.

24 Electric Potential

24.1 Potential Energy

Because the electrostatic force is conservative, it has a potential energy function given by

$$\Delta U = -\int_a^b \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \, dr = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{b} - \frac{1}{a}\right)$$

Letting $a = \infty$ and $b = r$ and, by convention, allowing $U = 0$ at $r = \infty$, we have

$$U = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$$

24.2 Electric Potential

The potential energy is proportional to the test charge; to hell with that, so we define *electric potential* as

$$\Delta V = \frac{\Delta U}{q_0}$$

Voltage is used interchangeably with electric potential, and it is measured in *volts*, or joules per coulomb. Rearranging, we can also write

$$\Delta U = q\Delta V$$

Electric potential and electric field can be also connected via

$$\Delta V = -\frac{W_{ab}}{q_0} = -\int_a^b \vec{E} \cdot d\vec{s}$$

Suppose we have a charge q and a test charge q_0. We want to know how much work per unit charge is needed to move q_0 from a to b on the line joining the charges. This is

$$V_b - V_a = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{b} - \frac{1}{a}\right)$$

24.3 Potential Due to an Electric Dipole

The work per unit charge necessary to bring a test charge from infinity to a distance r from the center of a dipole is given by

$$V = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_+} + \frac{-q}{r_-}\right) \approx \frac{1}{4\pi\epsilon_0} \frac{p\cos\theta}{r^2}$$

when $r >> d$, using the approximation $r_- - r_+ \approx d\cos\theta$ and $r_- r_+ \approx r^2$.

24.4 Electric Potential of Continuous Charge Distributions

For a uniform line of charge, the potential is approximately

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{y}$$

for points along its perpendicular bisector on the y-axis.

For a uniform ring of charge,

$$V = \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda R}{\sqrt{R^2 + z^2}}$$

for points along its axis of symmetry.

For a uniform disk of charge, by integrating the potential of rings of infinitesimal thickness,

$$V = \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2 + z^2} - |z|\right)$$

24.5   Calculating Field from Potential

Consider if you nudge a particle of charge q in some direction. The external work you do is

$$W = q\Delta V = -qE\Delta s \rightarrow E = -\frac{\Delta V}{\Delta s}$$

because the force you apply is opposite the force from the electric field. More generally,

$$\vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$$

24.6   Equipotential Surfaces

On an equipotential surface, where V is constant, the electric field must do 0 work as a charge travels along the surface. Any given charge distribution may have an entire *family* of charge distributions.

By this logic, the electric field must be entirely perpendicular to the equipotential surface; otherwise, work would be done along it.

24.7   Potential of a Charged Conductor

Consider the surface of a charged conductor. It is an equipotential because, by symmetry, all electric field lines emanate in a (perpendicular) radial direction to it. However, because the electric field is 0 inside the conductor, the potential must not change on the interior as well. Thus, we can conclude that the *entire* conductor is an equipotential surface with

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

24.8   Corona Discharge

Electric breakdown occurs when the electric field at some location exceeds the *dielectric strength* of the fluid (air) around it. A specific form of breakdown is corona discharge, caused by sharp, pointy areas due to their high charge density.

Qualitatively, consider two (distant) conducting spheres connected by a wire. They must be at equipotential, so

$$\frac{V_2}{V_1} = \frac{\sigma_2 R_2}{\sigma_1 R_1} = 1 \rightarrow \sigma \propto \frac{1}{R}$$

At regions where the radius of curvature is very small (i.e., very sharp regions), the charge density is super duper high. And because $E \propto \sigma$ the electric field is very large as well, possibly even exceeding the dielectric strength of air and leading to corona discharge.

24.9   Electrostatic Accelerator

Consider two concentric conducting spheres. If charge can be placed on the inner sphere consistently (perhaps through a small hole in the outer sphere), then a wire can repeatedly be connected between the spheres so that the charge flows to the outer surface, thereby increasing potential without limit (theoretically). Robert Van de Graaff utilized this principal to invent the electrostatic accelerator, known as a Van de Graaff today, with a moving belt that sprayed charge onto a conducting sphere.

25 Electrical Properties of Materials

25.1 Types of Materials

The conductivity of a material correponds the number of *conduction electrons* it has.

- In **conductors**, each atom may give up one or two electrons
- In **semiconductors**, 1 in $10^{1}0$ to $10^{1}2$ atoms may give up an electron
- In **insulators**, electrons are not free to move
- In **superconductors**, there is absolutely no resistance to the flow of electrons

However, insulators can behave more like conductors if a strong enough electric field is applied (known as *breakdown*).

Polar materials have permanent dipole moments. Electric fields can align the dipoles of molecules; in *ferroelectric* materials, the dpioles stay aligned even after the field is removed.

25.2 Conductor in an Electric Field

If an electric field is applied to a conductor, *current* may begin to flow:

$$i = \frac{dq}{dt}$$

with units of amperes. While it has both magnitude and direction, current is not a vector... but current density is!

$$j = \frac{i}{A} \rightarrow i = \int \vec{j} \cdot d\vec{A}$$

25.3 Current Density and Drift Speed

When current flows, electrons drift through at a constant speed, even in the presence of an electric field, because they constantly crash into the lattice of ions and lose energy. Consider current moving through a segment of a conductor of electron density n and length L,

$$j = \frac{q}{At} = \frac{enAL}{A(L/v_d)} = env_d$$

In vector form,

$$\vec{j} = -en\vec{v}_d$$

25.4 Ohmic Materials

Because electric field and current density are both proportional to the drift velocity, it stands to reason that they are proportional to each other:

$$\vec{j} = \sigma \vec{E} \longleftrightarrow \vec{E} = \rho \vec{j}$$

where σ $[\frac{S}{m}]$ (siemens per meter) is the conductivity and ρ $[\Omega m]$ (Ohm meters) is the resistivity. The siemen and ohm are defined as

$$1 \text{ siemen} = 1 \frac{\text{ampere}}{\text{volt}}$$

$$1 \text{ ohm} = 1 \frac{\text{volt}}{\text{ampere}}$$

In some materials, the resistivity varies with the applied electric field. But in **Ohmic** materials, the resistivity is independent of the external field. Many homogeneous materials (including conducting metals) obey Ohm's Law for a range of strengths of the applied field.

Rearranging the definition of resistivity,

$$\rho = \frac{E}{j} = \frac{\Delta V/L}{i/A} = \frac{\Delta V}{i}\frac{A}{L}$$

The resistance is defined as $R = \Delta V/i$, or

$$R = \rho\frac{L}{A}$$

Thus, we can rephrase Ohm's Law as: the resistance of an object is independent of its applied electric field. While ordinary resistors are ohmic, semiconducting devices such as diodes and transistors tend not to be.

In general, finding the resistance of weird objects often involves applying an electric field across it. Then, $\Delta V = \int \vec{E} \cdot d\vec{s}$ and $i = \int \vec{j} \cdot d\vec{A}$ can be measured, allowing resistance R to be solved for.

25.5   Current and Heat Analogy

We can rewrite the definition of resistance as

$$i = \frac{\Delta V}{R} = \frac{\Delta V}{\rho\Delta x/A} = \sigma A\frac{\Delta V}{\Delta x}$$

In differential form,

$$\frac{dq}{dt} = -\sigma A\frac{dV}{dx}$$

which is analagous to the heat flow equation,

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}$$

Pure metals share more than a nice mathematical analogy; both heat and electricity are conducted via their free electrons.

25.6   Temperature Variance of Resistivity

The relation between resistivity and temperature is linear,

$$\Delta\rho = \rho_0\alpha_{av}\Delta T$$

Rearranging and taking the limit as the two points converge,

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT}$$

The value α is known as the temperature coefficient of resistivity.

25.7   Ohm's Law: A Microscopic View

The drift speed in a conducting material is

$$v_d = at = -\frac{eE\tau}{m}$$

where $\tau = \lambda/v_{av}$ is the mean time between collisions. But $v_d = E/\rho en$ too, so we can rearrange as

$$\rho = \frac{m}{ne^2\tau}$$

Thus, conductivity is proportional to the electron density, which checks out intuitively.

25.8   Insulators in Electric Fields

Consider an electric field E_0 applied to an insulating material. The polarization of atoms will lead to an induced field E' that is proportional to the applied field, so the net field is $E = E_0 - E' = \frac{1}{\kappa}E_0$ where κ is the dielectric constant. Insulating materials are also known as *dielectric materials*, and the electric field at which they break down are known as *dielectric strengths*.

Because electric fields are inversely proportional to both ϵ_0 and κ, we define the *permittivty* of a material as

$$\epsilon = \kappa\epsilon_0$$

and thus, ϵ_0 is known as the permittivity of free space.

26 Capacitance

26.1 Capacitors

Capacitors are devices that store energy in electric fields. The voltage difference between a capacitor's plates is always proportional to the charge on each plate; that is,

$$q = C\Delta V$$

The constant of proportionality is the *capacitance*, in units of farads (=coulombs/volt). The permittivity can free space can conveniently be expressed in these units as $\epsilon_0 = 8.854 \; pF/m$.

The surefire way to calculate capacitance is by integrating $\Delta V = \int \vec{E} \cdot d\vec{s}$ and applying $C = q/\Delta V$. This is how the following special cases will be solved.

Parallel Plate Capacitor: the capacitance is

$$C = \frac{A\epsilon_0}{d}$$

Spherical Capacitor: the capacitance is

$$C = 4\pi\epsilon_0 \frac{ab}{b - a}$$

where the two plates are of radii a and b.

Cylindrical Capacitor: the capacitance is

$$C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)}$$

26.2 Capacitors in Series and Parallel

Capacitors in parallel have the same voltage drop between their plates. The actual physical appearance of the setup is irrelevant, though most commonly they will be drawn parallel to each other in a circuit. The equivalent capacitance is then

$$C_{eq} = C_1 + C_2 + \cdots + C_N$$

Capacitors in series have the same charge buildup on their plates. The equivalent capacitance is

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}}$$

26.3 Energy Stored in an Electric Field

The energy stored in an electric field between two capacitor plates is the work required to move the charge q from one plate to another (in effect, charging the capacitor). This can be found by integrating

$$dU = \Delta V dq = \frac{q}{C} dq$$

The result is

$$U = \frac{q^2}{2C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}q\Delta V$$

The energy density is thus

$$u = \frac{U}{Ad} = \frac{1}{2}\epsilon_0 E^2$$

26.4   Capacitor with Dielectric

The effect of a dielectric on a capacitor depends on whether the battery is connected or not.

Suppose the battery remains connected as a dielectric slides in between the capacitor plates. The potential difference remains constant, but because the electric field needs to increase, there must be more charge on the surfaces (and thus capacitance).

$$E' = \frac{E}{\kappa} = \frac{q}{\kappa \epsilon_0 A} \rightarrow q' = \kappa q_0$$

$$C' = \frac{q'}{\Delta V} = \kappa C$$

Although Faraday originally showed this relation for parallel plate capacitors, any capacitor's capacitance will increase by a factor κ when a dielectric material is slid in between with the battery connected.

Suppose the battery is disconnected. Now, as the dielectric slides in, the charge on each plate must remain constant.

$$E' = \frac{E}{\kappa} \rightarrow \Delta V' = \frac{\Delta V}{\kappa}$$

$$C' = \frac{q}{\Delta V'} = \kappa C$$

26.5   Dielectrics and Guass' Law

Consider the electric field within a capacitor with a dielectric (that fills the entire space between the plates). From Gauss' Law,

$$\int \vec{E} \cdot d\vec{A} = \frac{q - q'}{\epsilon_0}$$

where q' is the magnitude of the induced electric field caused by the polarization of the insulator. Because

$$E_0 - E' = \frac{E_0}{\kappa} \rightarrow \frac{q - q'}{A\epsilon_0} = \frac{q}{A\kappa\epsilon_0}$$

it can be solved that $q' = q(1 - 1/\kappa)$ which yields, upon substitution back into Gauss' Law,

$$\int \kappa \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Note that q is the free charge only; induced charge is accounted for by the factor of κ.

27 DC Circuits

27.1 Kirchhoff's Laws

Because charge must be conserved, at any junction, the total input current must equal the total output current. This is known as Kirchhoff's Current Law, or KCL.

Because energy must be conserved, around a closed loop, sum of all electric potentials must be 0. This is known as Kirchhoff's Voltage Law, or KVL.

27.2 Electromotive Force

The emf is defined as the work per unit charge done in a circuit by any component, or

$$\varepsilon = \frac{dW}{dq}$$

It is analagous to a pump in fluid flow, which maintains a pressure difference between two points. Note that the emf does not supply charge to a circuit; it simply increases pumps energy into it by (usually) elevating electrons to a lower potential.

The emf can also be expressed as

$$\varepsilon = \oint \frac{\vec{F}}{q} \cdot d\vec{s}$$

Generally, $\vec{F}/q$ may not equal $\vec{E}$ because the work done might be mechanical, chemical, thermal, or magnetic in nature.

Oftentimes, the emf is provided by a battery which has internal resistance. Because the current is $i = \varepsilon/(R + r)$, where r is the internal resistance and R is the resistance of the rest of the circuit, the effective voltage provided by the battery is

$$\Delta V = \varepsilon - ir = \varepsilon \frac{R}{R + r}$$

27.3 Voltage Divider

A common way to obtain a lower voltage is by connecting two resistors in series and putting an output path between them. Suppose the resistors are R_1 and R_2, with the divider coming after the first resistor. Then

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$$

27.4 Resistors in Series and Parallel

Resistors in parallel share the same voltage drop across them, no matter the physical appearance of their arrangement. The equivalent resistance is then

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}}$$

Resistors in series share the same current through them. The equivalent resistance is

$$R_{eq} = R_1 + R_2 + \cdots + R_N$$

Probably one of the most useful techniques for solving resistors problems is that equipotential points can be connected by a wire without changing the circuit. This is because no charge will "flow" between two regions of same potential, but connecting them with a wire may allow us to cordon off a part of the circuit that we can apply series/parallel resistor rules to. Symmetry considerations are also supremely useful for circuits in general; for instance, two points are at equipotential if they are on opposite sides of a symmetrical circuit.

27.5   Energy Transfers in a Circuit

The power delivered by a battery is

$$P_{emf} = \frac{dW}{dq}\frac{dq}{dt} = \varepsilon i$$

For a resistor, the power dissipated is

$$P = i\Delta V_R = \frac{\Delta V_R^2}{R} = i^2 R$$

where ΔV_R is the voltage across the resistor. This energy transfer to a resistor is known as *Joule heating*.

27.6   RC Circuits

By applying KVL to an RC circuit, we obtain a differential equation that can be solved for current and capacitor charge as a function of time. While charging the capacitor, we have

$$\varepsilon - iR - \frac{q}{C} = 0$$

which yields, after substituting $i = dq/dt$,

$$q = C\varepsilon(1 - e^{-t/RC})$$

$$i = \frac{\varepsilon}{R}e^{-t/RC}$$

where the *capacitive time constant* of the circuit is $\tau_C = RC$.
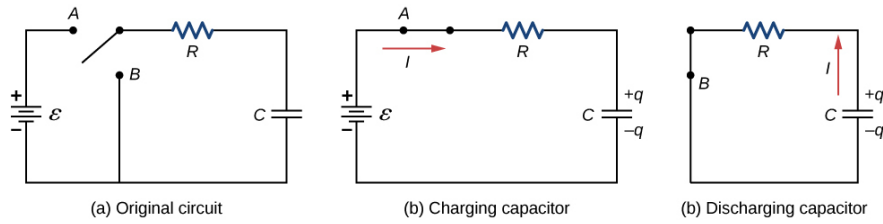

Figure 7: Charging and discharging an RC circuit

While discharging the capacitor, we have

$$\frac{q}{C} - iR = 0$$

which yields

$$q = q_0 e^{-t/\tau_C}$$

$$i = -\frac{q_0}{RC}e^{-t/\tau_C}$$

If the capacitor was fully charged at $q_0 = \epsilon C$, then the current function reduces down to

$$i = -\frac{\varepsilon}{R}e^{-t/\tau_C}$$

28 Units, Conversions, Constants

28.1 Some Physical Constants

Speed of light in vacuum	c	3.00×10^8 m/s
Newtonian gravitational constant	G	6.67×10^{-11} N $\cdot$ m^2/kg^2
Avogadro constant	N_A	6.02×10^{23} mol^{-1}
Molar gas constant	R	8.31 J/mol $\cdot$ K
Mass-energy relation	c^2	8.99×10^{16} J/kg
		931.5 MeV/u
Permittivity of Free Space	ε_0	8.85×10^{-12} F/m
Permeability of Free Space	μ_0	$4\pi \times 10^{-7}$ H/m
		1.26×10^{-6} H/m
Planck constant	h	6.63×10^{-34} J $\cdot$ s
Boltzman constant	k	1.38×10^{-23} J/K
		8.62×10^{-5} eV/K
Elementary charge	e	1.60×10^{-19} C
Electron mass	m_e	9.11×10^{-31} kg
Electron rest energy	$m_e c^2$	511.0 keV
Proton mass	m_p	1.67×10^{-27} kg
Proton rest energy	$m_p c^2$	938.3 MeV
Bohr radius	a_0	5.29×10^{-11} m
Bohr magneton	μ_B	9.27×10^{-24} J/T
		5.79×10^{-5} eV/T

28.2 Some Conversion Factors

28.2.1 Mass

1 kg $= 6.02 \times 10^{26} u$
1 u $= 1.66 \times 10^{-27}$ kg
1 slug $= 14.6$ kg

28.2.2 Length

1 m $= 39.4$ in. $= 3.28$ ft
1 mi $= 1.61$ km $= 5280$ ft
1 in. $= 2.54$ cm
1 light-year $= 3.26$ parsec $= 9.46 \times 10^{15}$ m
1 Å $= 0.1$ nm $= 100$ pm $= 10^{-10}$ m

28.2.3 Time

1 d $= 86,400$ s
1 y $= 365.25$ d $= 3.16 \times 10^7$ s

28.2.4 Volume

1 L $= 1000$ cm^3 $= 10^{-3}$ m^3 $= 1.06$ quart
1 gal (U.S.) $= 231$ in.^{3} $= 3.79$ L

28.2.5 Angular measure

1 rad $= 57.3° = 0.159$ rev
1 grad $= 0.9° = 0.0157$ rad

28.2.6 Speed

$1 \text{ m/s} = 3.6 \text{ km/h} = 3.28 \text{ ft/s} = 2.24 \text{ mi/h}$

28.2.7 Force and Pressure

$1 \text{ N} = 10^5 \text{ dyne} = 0.225 \text{ lb}$
$1 \text{ Pa} = 10 \text{ dyne/cm}^2 = 1.45 \times 10^{-4} \text{ lb/in.}^2$

28.2.8 Energy and Power

$1 \text{ J} = 10^7 \text{ erg} = 0.239 \text{ cal} = 0.738 \text{ ft} \cdot \text{lb}$
$1 \text{ kW} \cdot \text{h} = 3.6 \times 10^6 \text{ J}$
$1 \text{ cal} = 4.184 \text{ J}$
$1 \text{ eV} = 1.50 \times 10^{-19} \text{ J}$
$1 \text{ BTU} = 1055 \text{ J}$
$1 \text{ hp} = 746 \text{ W} = 550 \text{ ft} \cdot \text{lb/s}$

28.2.9 Electricity and Magnetism

$1 \text{ T} = 10^4 \text{ gauss}$

28.3 Some Physical Properties

28.3.1 Air (dry, $20°$ C, 1 atm)

Density	1.21 kg/m^3
Specific heat capacity at constant pressure	$1010 \text{ J/kg} \cdot \text{K}$
Ratio fo specific heat capacities	1.40
Speed of sound	343 m/s
Electrical breakdown strength	$3 \times 10^6 \text{ V/m}$
Effective molar mass	0.0289 kg/mol

28.3.2 Water

Density	1000 kg/m^3
Speed of sound	1460 m/s
Specific heat capacity at constant pressure	$4190 \text{ J/kg} \cdot \text{K}$
Heat of fusion (0 °C)	333 kJ/kg
Heat of vaporization (100 °C)	2260 kJ/kg
Index of refraction ($\lambda = 589$nm)	1.33
Molar mass	0.0180 kg/mol

28.3.3 Earth

Mass	$5.98 \times 10^{24} \text{ kg}$
Mean radius	$6.37 \times 10^6 \text{ km}$
Standard atmosphere	$1.01 \times 10^5 \text{ Pa}$
Radius of the geosynchronous orbit	$42,200 \text{ km}$
Escape speed	11.2 km/s
Magnetic dipole moment	$8.0 \times 10^{22} \text{ A} \cdot \text{m}^2$
Mean electric field at surface	150 V/m, down