How many ways can you gerrymander a chessboard?

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1. Introduction

A hot topics in politics recently is gerrymandering. Governments seek to divide up the population into voting districts, and individual parties can gain a large advantage by clever redistricting. In the interest of studying how to redistrict fairly, we consider counting the number of ways to divide up a grid into connected regions. This is the subject of many OEIS sequences, and the results of this paper compute many new terms for the OEIS. In particular, see A167238 - A167265 in [2]. The special case of dividing the $3 \times n$ grid into 2 connected regions was solved by Don Knuth, who posed it as a problem in the American Mathematical Monthly [1].

2. Finite State Automaton

Consider fixing the number of rows in the grid, r, as well as the number of regions to divide the grid into, q. We now can sequentially look at the columns, and determine whether (up until the columns read so far) the grid has been divided divided into q connected regions. To do this we must keep track of a finite amount of information, which we will store in the state of a finite state automaton.

For us, a column will consist of a string of length r, where the letters in the string are integers from 1 to q, indicating to which region the corresponding square will be assigned to. A state will store the following information:

1. The contents of the most recent column
2. For each square in the column, a separate connectivity label. Connectivity labels will only be shared by squares of the same region, and only when those squares are connected in the grid formed by the columns read so far.
3. For each region that is not present in the most recent column, whether it has not yet appeared in the previous columns, and must appear in a future column, before an accept state can be reached.

The number of states grows quite rapidly with the number of rows and parts. Here is a table:

	q=2	q=3	q=4	q=5	q=6	q=7	q=8
r=2	6	24	80	240	672	1792	4608
r=3	12	66	272	960	3072	9184	26112
r=4	28	210	1064	4400	16032	53536	167680
r=5	74	756	4688	22600	93312	346528	1190912
r=6	208	2946	22424	125960	588432	2425696	
r=7	610	12216	114464	749120			
r=8	1836	53034	614456				

If we just read a column, and are in a certain state, and now read a new column, we should either end up in a new state or REJECT. If we do not REJECT, then this transition has a nonzero weight. The weight is a monomial in the variables z_i for i from 1 to q. The exponent of z_i indicates how many squares were assigned to region i in the column that corresponds to this transition.

We now form a transition matrix. The rows and columns of the matrix will be indexed by the states, and the a, b entry will be the sum of the weights of the possible transitions from state a to state b. Call this matrix $M_{r,q}$.

We must now add a start state and an accept state. The start state can transition into any state as long as the state does not assign non-adjacent squares to the same connectivity label. The final state receives dummy transitions of weight 1 from all states that have each region started and fully connected.

Now suppose we are interested in the number of ways to divide the $r \times k$ grid into q connected regions. We raise $M_{r,q}$ to the power of $(k+1)$, and look at the entry indexed by the transition from the start state to the final state. This entry will be a polynomial in the z_i variables, and the coefficient of $z_1^{t_1} z_2^{t_2} ... z_q^{t^q}$ will give the number of ways to divide the $r \times k$ grid into q connected regions, such that region 1 has area t_1, region 2 has area t_2, etc.

3. References

[1] D. E. Knuth (Proposer) and Editors (Solver), Balanced tilings of a rectangle with three rows, Problem 11929, Amer. Math. Monthly, 125 (2018), 566-568.

[2] The On-Line Encyclopedia of Integer Sequences, founded in 1964 by N. J. A. Sloane. `http://oeis.org/`