Lecture 8

February 13, 2019

Anders Sundheim
asundheim@wisc.edu

Line Integrals

C^1 and piecewise C^1 paths

Defintion

$\alpha : [a, b] \to \mathbb{R}^n$ is a path
And if α is continuous on $[a, b]$ we say it is a continuous path
If α is a continuous path and $\alpha'(t)$ is continuous in (a, b) then we say α is a C^1 path
If α is continuous and we can find $a_0 = a < a_1 < \cdots < a_k$ and $a_k < b = a_{k+1}$ such that
$\alpha'(t)$ is continuous in (a_i, a_{i+1}) for all $0 \leq i \leq k$, then we say that α is a piecewise C^1 path

Definition of path integrals

1 Let $\alpha : [a, b] \to \mathbb{R}^n$ be a C^1 path
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field

$$\text{Define } \int f \cdot dx = \int_a^b f(\alpha(t)) \cdot \alpha'(t) dt$$

In a more explicit way,

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_n(t))$$

$$f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$$

$$\Rightarrow \int f \cdot dx = \int_a^b f(\alpha(t)) \cdot \alpha'(t)dt = \int_a^b \sum_{k=1}^{n} f_k(\alpha(t))\alpha_k'(t))dt$$

2 If α is a piecewise C^1, then we define

$$\int f \cdot dx = \int_{a_0}^{a_1} f(\alpha(t)) \cdot \alpha'(t)dt + \cdots + \int_{a_k}^{a_{k+1}} f \cdot dt$$

3 If α is a closed curve, that is $\alpha(a) = \alpha(b)$, then we also write

$$\int f \cdot d\alpha = \oint f d\alpha$$

Note that there are many ways to parameterize a path/curve in $\mathbb{R}^n$

Concern

Do all parameterizations of the same path yield some line integral?

$$\int f \cdot d\alpha = \int f \cdot d\beta$$

Proof

Proof. Let $\alpha : [a, b] \to \mathbb{R}^n$ is C^1, $t \mapsto \alpha(t) \in \mathbb{R}^n$
Let $\beta : [c, d] \to \mathbb{R}^n$ such that $s \mapsto \beta(s) = \alpha(u(s))$ for $c \leq s \leq d$
Where $u : [c, d] \to [a, b]$ is C^1, is increasing, and $u(c) = a, u(d) = b$
$\int f \cdot d\beta = \int_c^d f(\beta(s)) \cdot \beta'(s)ds = \int_c^d f(\beta(s)) \cdot \alpha'(u(s))u'(s)ds$
Let $t = u(s) \Rightarrow dt = u'(s)ds \to= \int_a^b f(\alpha(t)) \cdot \alpha'(t)dt$

□