Lecture 12

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Anders Sundheim
asundheim@wisc.edu

Last Time

Theorem

Let $U \subset \mathbb{R}^n$ be open and connected
Let $f : U \to \mathbb{R}^n$ be a continuous vector field. The following are equivalent
(i) $f = \nabla\Psi$ for some $\Psi \cdot U \to \mathbb{R}$, $\Psi \in C^1$
(ii) Line integrals of f are independent of paths
(iii) $\oint_C f \cdot d\alpha = 0$ for every piecewise C^1 closed path

Proof of (iii)⇒(ii)

Proof. To prove (ii), let's take two curves C_1, C_2 with some end points

$$C_1 : \alpha[0,t] \to U : \alpha(0) = x, \alpha(t) = y$$
$$C_1 : \beta : [0,s] \to U : \beta(0) = x, \beta(s) = y$$

Need to show: $\int f \cdot d\alpha = \int f \cdot d\beta$
Let γ to be a parameterization of C_2 in the reverse direction, that is

$$\gamma : [0,s] \to U : \gamma(r) = \beta(s-r) \text{ for } 0 \leq r \leq s$$

Then $C = C_1 \cup C_2 = \alpha \cup \gamma$ is a closed path

$$\text{By (iii)} \int f \cdot d\alpha + \int f \cdot d\gamma - \int f \cdot d\beta = 0$$

$$\Rightarrow \int f \cdot d\alpha = \int f \cdot d\beta$$

$\square$

Remark

1 For f to be a gradient vector field, we really need to know that path integrals are <u>independent of paths</u>

2 <u>Not all vector fields are gradient vector fields</u>

Example

In general, $f : U \to \mathbb{R}^n$

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

is a continuous vector field. There is no relation between $f_i(x)$ and $f_j(x)$
For f to be a gradient vector field

$$f(x) = \nabla\Psi(x) \text{ where } \Psi : U \to \mathbb{R} \text{ is } C^1$$

$$(f_1(x), f_2(x), \dots, f_n(x)) = (\Psi_{x_1}(x), \Psi_{x_2}(x), \dots, \Psi_{x_n}(x))$$

$$\begin{cases} f_i(x) = \Psi_{x_i}(x) \\ f_j(x) = \Psi_{x_j}(x) \end{cases} \implies \text{There is a strong relation between} f_i \text{ and } f_j$$

Ψ is called a potential function

Lemma

Assume $f = \nabla\Psi$, and $\Psi \in C^2$. Then:

$$(f_i)_{x_j} = (f_j)_{x_i}$$

Proof

Proof. $f_i = \Psi_{x_i} \Rightarrow (f_i)_{x_j} = \Psi_{x_i x_j}$
$f_j = \Psi_{x_j} \Rightarrow (f_j)_{x_i} = \Psi_{x_j x_i}$
Since $\Psi \in C^2, \Psi_{x_i x_j}, \Psi_{x_j x_i}$ are continuous in U, and we can switch order of differentiation, that is $\Psi_{x_i x_j} = \Psi_{x_j x_i}$
$\Rightarrow (f_i)_{x_j} = (f_j)_{x_i}$ □

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$f(x, y) = (x, xy) : \begin{cases} f_1(x, y) = x \\ f_2(x, y) = xy \end{cases}$$

1st way: Assume that f is a gradient vector field, that
$(f_1)_y = (f_2)_x$
$\Rightarrow 0 = y \otimes$
$\to f$ is not a gradient vector field
2nd way: If f were a gradient vector field, we would have

$$f(x, y) = (x, x \cdot y) = (\Psi_x, \Psi_y)$$
$$\begin{cases} \Psi_x = x \\ \Psi_y = xy \end{cases} \quad \Rightarrow \begin{cases} \Psi(x, y) = \frac{x^2}{2} + C(y) \\ \Psi(x, y) = \frac{xy^2}{2} + C(x) \end{cases} \quad \to \bigotimes$$

Question

If $f : U \to \mathbb{R}^n$ is a C^1 vector field, and

$$(f_i)_{x_j} = (f_j)_{x_i} \text{ for all } 1 \leq i, j \leq n$$

then do we have that f is a gradient vector field?

Answer

1 No in general (give one counterexample)
2 Yes if U is convex

(U is convex if for $x, y \in U \Rightarrow [xy] \subset U$)

Lemma

Let $f : \mathbb{R}^2 \backslash \{\vec{0}\} \to \mathbb{R}^2$ s.t.

$$f(x,y) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

Then $(f_1)_y = (f_2)_x$, but f is NOT a gradient vector field

$$f_1(x,y) = -\frac{y}{x^2+y^2}, f_2(x,y) = \frac{x}{x^2+y^2}$$

Compare:

$$\begin{aligned}
(f_1)_y &= -\frac{1}{x^2+y^2} - y(-1)(x^2+y^2)^{-2} \\
&= -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} \\
&= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}$$

DIY: $(f_2)_x = \frac{y^2-x^2}{(x^2+y^2)^2}$
However, f is not a gradient vector field

$$f(\alpha(\theta)) = f(\cos(\theta), \sin(\theta)) = (-\sin(\theta), \cos(\theta))$$

$$\int f \cdot d\alpha = \int_0^{2\pi} (-\sin(\theta), \cos(\theta)) \cdot (-\sin(\theta), \cos(\theta)) d\theta = 2\pi \neq 0$$