Lecture 23

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Change of variables in a multiple integral

$$\int_S f(x)\,dx = \int \cdots \int_S f(x_1, x_2, \ldots, x_n)\,dx_1\,dx_2\,\ldots\,dx_n : S \subset \mathbb{R}^n$$

$$\begin{cases} x_1 = X_1(u_1, u_2, \ldots, u_n) \\ x_2 = X_2(u_1, u_2, \ldots, u_n) \\ \vdots \\ x_n = X_n(u_1, u_2, \ldots, u_n) \end{cases} \Rightarrow (u_1, \ldots, u_n) \mapsto (x_1, \ldots, x_n), u \mapsto x$$

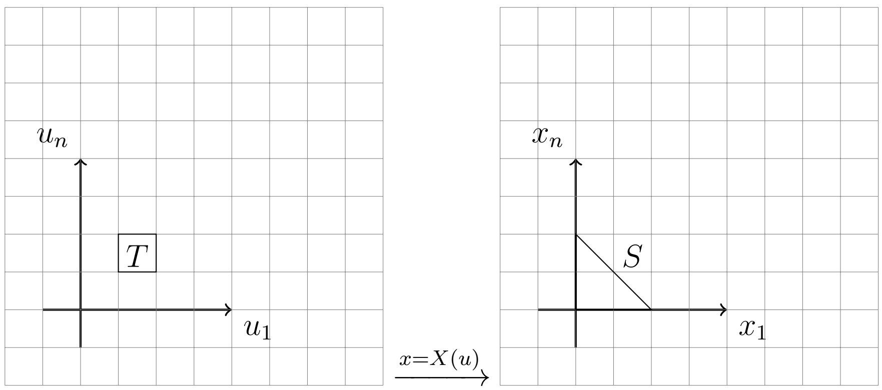
$$x = X(u) : \begin{cases} X(u) \text{ continuous and differentiable} \\ u \mapsto X(u) = x \text{ is one-to-one} \end{cases}$$

Jacobian Matrix

$$DX(u) = \begin{pmatrix} \frac{dX_1}{du_1} & \cdots & \frac{dX_1}{du_n} \\ \vdots & & \vdots \\ \frac{dX_n}{du_1} & \cdots & \frac{dX_n}{du_n} \end{pmatrix}$$

Jacobian determinant: $J(u) = \det[DX(u)] \neq 0$

$$\int_S f(x)\,dx \quad S = X(T)$$



$$\int_S f(x)\,dx = \int_T f(X(u))|J(u)|\,du$$

Cylindrical coordinates $\mathbb{R}^3\{x, y, z\}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow (r,\theta,z) \mapsto (x,y,z) : \begin{cases} 0 < r < +\infty \\ 0 \le \theta \le 2\pi \\ z \in \mathbb{R} \end{cases}$$

$$J(r,\theta,z) = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{dz} \\ \frac{dy}{dr} & \frac{dy}{d\theta} & \frac{dy}{dz} \\ \frac{dz}{dr} & \frac{dz}{d\theta} & \frac{dz}{dz} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \iiint_T (r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz$$

Example

Cylinder of radius R and height H

$$\begin{cases} 0 < r < R \\ 0 \le \theta \le 2\pi \\ 0 \le z \le H \end{cases} \Rightarrow \iiint_T f = \int_0^H \int_0^{2\pi} \int_0^R f$$

Spherical Coordinates in $\mathbb{R}^3\{x, y, z\}$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases} \Rightarrow (\rho, \varphi, \theta) \mapsto (x, y, z) : \begin{cases} 0 < \rho < +\infty \\ 0 < \varphi < \pi \\ 0 \leq \theta < 2\pi \end{cases}$$

$r = \rho \sin\varphi$

$$J(\rho, \varphi, \theta) = \begin{vmatrix} \frac{dx}{d\rho} & \frac{dx}{d\varphi} & \frac{dx}{d\theta} \\ \frac{dy}{d\rho} & \frac{dy}{d\varphi} & \frac{dy}{d\theta} \\ \frac{dz}{d\rho} & \frac{dz}{d\varphi} & \frac{dz}{d\theta} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2 \sin\varphi$$

$$\iiint_S f(x, y, z)\, dx\, dy\, dz = \iiint_T f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\sin\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$