Lecture 19

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Greene's Theorem

Setting

Let C be a piecewise C^1 closed Jordan Curve, that is,
C is not self-intersecting. C encloses a region $S \subset \mathbb{R}^2$

Theorem

[Greene's Theorem] Let $P, Q :\to \mathbb{R}$ be
C^1 functions(real valued). Then,

$$(*) \quad \oint_C P\,dx + Q\,dy = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy$$

Observations

1 In the theorem, there are 2 functions P and Q that are not related
$\Rightarrow$ To prove (*), it's equivalent to show 2 simpler claims.

$$\begin{cases} (**)\ \oint_C P\,dx = \iint_S -\frac{dP}{dy}\,dx\,dy \\ (***)\ \oint_C Q\,dy = \iint_S \frac{dQ}{dx}\,dx\,dy \end{cases}$$

(or, you can let $P \equiv 0, Q \equiv 0$ in (*))
(**) and (***) are more or less the same
2 Let's prove (**). $\oint_C P\,dx = \iint_S -\frac{dP}{dy}\,dx\,dy$
Key pt. Divide S into simple domains (in x), and we'll show that for each simple domain S_k:

$$\oint_{C_k} P\,dx = \iint_{S_k} -\frac{dP}{dy}\,dx\,dy$$

(C_k is the corresponding boundary of S_k).
Afterwards, adding all these together in k to get:

Proof of (**) for simple domains (in x)

Proof.

$$S = \big\{(x,y) : a \leq x \leq b, \varphi(x) \leq y \leq Psi(x)\big\}$$

We now compute easily:

$$\iint_S -\frac{dP}{dy}\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\Psi(x)} -\frac{dP}{dy}\,dy\right) dx$$
$$= \int_a^b (P(x,\varphi(x)) - P(x,\Psi(x)))\,dx$$

Let's compute $\oint_C P\,dx$:
C is broken into 4 pieces as following.
I $\alpha(x) = (x, \varphi(x)) : a \leq x \leq b, dx = dx$

$$\int_\alpha = \int_a^b P(x,\varphi(x))\,dx$$

II $\beta(s) = (b,s) : \varphi(x) \leq s \leq \Phi(x), dx = 0$ as it's constant

$$\int_\beta P\,dx = 0$$

Similarly, $\int_\delta P\,dx = 0$.
III $\gamma(x) = (x, \Psi(x)) : a \leq x \leq b, dx = x$

$$\int_\gamma P\,dx = \int_a^b P(x,\Psi(x))\,dx = -\int_a^b P(x,\Psi(x))\,dx$$

Sum all up,

$$\oint_C P\,dx = \int_a^b \big(P(x,\varphi(x)) - P(x,\Psi(x))\big)\,dx$$

$\square$

Remark

To prove the other identity:

$$\oint_C Q\,dy = \iint_S \frac{dQ}{dx}\,dx\,dy, \text{ we simply break } S \text{ into simple domains in } y$$

Example

Compute work done by force field

$$f(x,y) = (y+3x, 2y-x)$$

in a moving particle counter-clockwise once around the ellipse

$$4x^2 + y^2 = 4$$

<u>Earlier</u>: Parameterize C:

$$\alpha(s) = (\cos(s), 2\sin(s)), 0 \le s \le 2\pi$$

$$\text{Work done} = \int f \cdot d\alpha = \int_0^{2\pi} f(\alpha(s)) \cdot \alpha'(s)\,ds$$

$$= \int_0^{2\pi} (2\sin(s) + 3\cos(s), 4\sin(s) - \cos(s)) \cdot (-\sin(s), 2\cos(s))\,ds$$

$$= \int_0^{2\pi} (-2\sin^2(s) - 3\cos(s)\sin(s) + 8\sin(s)\cos(s) - 2\cos^2(s))\,ds$$

$$= \int_0^{2\pi} (-2 + 5\sin(s)\cos(s))\,ds = \dots$$

The other way: $P(x,y) = y + 3x, Q(x,y) = 2y - x$
Work done $= \oint_C P\,dx + Q\,dy$

$$= \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right)$$

$$= \iint_S (-2)\,dx\,dy$$