Lecture 10

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Recap

Curve C has a parameterization

$$\alpha : [a, b] \to \mathbb{R}^n$$

$$t :\mapsto \alpha(t), \text{ which is } C^1$$

Arc Length

$$s(t) = \int_a^t |\alpha'(s)| ds$$

$$s'(t) = |\alpha'(t)|$$

Line Integral w.r.t. arc length

$$\varphi : \mathbb{R}^n \to \mathbb{R} \text{ is continuous}$$

$$\int_C \varphi ds = \int_a^b \varphi(\alpha(t)) s'(t) dt$$

$$\int_C \varphi ds = \int_a^b \varphi(\alpha(t)) |\alpha'(t)| dt$$

Quick Applications

C is a wire, and $\alpha : [a,b] \to \mathbb{R}^n$ is one parameterization of C

1 Length of $C = L(\alpha) = L(C) = \int\limits_a^b |\alpha'(t)| dt$

2 Mass of C: assume that at $\alpha(t)$, mass density of C is $\varphi(\alpha(t))$ (per unit length)

Mass of $C = M = \int_C \varphi ds = \int\limits_a^b \varphi(\alpha(t))|\alpha'(t)|dt$

3 Center of mass: assume our wire C is in $\mathbb{R}^3$
center of mass of C is $(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^3$ s.t.

$$\begin{cases} M\bar{x} = \int_C x\varphi ds \\ M\bar{x} = \int_C y\varphi ds \\ M\bar{x} = \int_C z\varphi ds \end{cases}$$

Example

C is a helix shape
$\alpha(t) = (a\cos(t), a\sin(t), bt) : t \in [0, 2\pi]$
Mass density at $(x,y,z) \in \mathbb{R}^3$ is

$$\varphi(x,y,z) = x^2 + y^2 + z^2$$

Mass $M = \int_C \varphi ds$

$$= \int_0^{2\pi} \varphi(\alpha(t))|\alpha'(t)|dt$$

$$\begin{aligned} \varphi(\alpha(t)) &= \varphi(a\cos(t), a\sin(t), bt) \\ &= (a\cos(t))^2 + (a\sin(t))^2 + (bt)^2 \\ &= a^2 + b^2t^2 \end{aligned}$$

$$\alpha'(t) = (-a\sin(t), a\cos(t), b)$$

$$|\alpha'(t)| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + b^2} = \sqrt{a^2 + b^2}$$

$$M = \int_0^{2\pi} (a^2 + b^2t^2)\sqrt{a^2+b^2}dt = \sqrt{a^2+b^2}(2\pi a^2 + \frac{8\pi^3b^2}{3})$$

4 : Moment of inertia about a line L
C is a curve in $\mathbb{R}^3$
L is a given line in $\mathbb{R}^3$
For each point $(x, y, z) \in C$,
let $\delta(x, y, z)$ be the distance from (x, y, z) to L
Moment of inertia about L is

$$I_L = \int_C \delta(x, y, z)^2 \varphi ds$$

Second Fundamental Theorem of calculus for Line Integrals

Let $U \subset \mathbb{R}^n$ be an open set

Define Connectedness of U

We say U is connected(path-connected)
if for any given two points $x, y \in U$
we can find a piecewise C^1 path
  $\alpha : [a, b] \to U$ s.t.
  $\alpha(a) = x$ and $\alpha(b) = y$
Ex. of non-connectedness
$U = U_1 \cup U_2 : U_1, U_2$ distinct sets

Theorem

Let $U \subset \mathbb{R}^n$ be open and connected
Let $\Psi : U \to \mathbb{R}$ be a C^1 real valued function
Let $f : U \to \mathbb{R}^n$ be a vector field s.t.

$f(x) = \nabla\Psi(x)$: gradient vector field of Ψ

Then, for any $x, y \in U$ being connected by any piecewise C^1 path $\alpha : [a, b] \to U$, $\alpha(a) = x, \alpha(b) = y$

$$\boxed{\int f \cdot d\alpha = \Psi(y) - \Psi(x)}$$

Proof

Proof. Assume α is C^1

$$\int f \cdot d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t) dt$$

$$= \int_a^b \nabla\Psi(\alpha(t)) \cdot \alpha'(t) dt$$

$$= \int_a^b \frac{d}{dt}(\Psi(\alpha(t))) dt = \Psi(\alpha(b)) - \Psi(\alpha(a)) = \Psi(y) - \Psi(x)$$

□

Corollary

$$\oint f \cdot d\alpha = 0 \text{ under the assumption of our theorem}$$

Remark

If f is not a gradient vector field, then in general,

$$\oint f \cdot \alpha \neq 0$$

Remark

How do we know if f is a gradient vector field or not?
In $\mathbb{R}^2$, if everything is nice, and $f(x) = \nabla\Psi(x)$
Then $f(x) = (f_1(x), f_2(x)) = (\Psi_{x_1}(x), \Psi_{x_2}(x))$

$$\begin{cases} f_1(x) = \Psi_{x_1}(x) \\ f_2(x) = \Psi_{x_2}(x) \end{cases} \Rightarrow \begin{cases} (f_1)_{x_2} = \Psi_{x_1 x_2}(x) \\ (f_2)_{x_1} = \Psi_{x_2 x_1}(x) \end{cases}$$

So $\boxed{(f_1)_{x_2} = (f_2)_{x_1}}$