Lecture 9

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Anders Sundheim
asundheim@wisc.edu

Last Time

Path integral does not depend on parameterization
More precisely,

$\alpha : [a, b] \to \mathbb{R}^n$ is a C^1 path

$\beta : [c, d] \to \mathbb{R}^n$

$\beta(s) = \alpha(u(s))$ for a C^1 function

$u : [c, d] \to [a, b]$ s.t. u increasing, $u(c) = a, u(d) = b$

Remark

If we fix the direction of α
Then $\int f \cdot d\alpha = -\int f \cdot d\gamma$

Path integral on work done

A particle has a path $\alpha(t) : a \leq t \leq b$
We're given a vector field

$f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous

Q: Work done by the vector field f on the path of this particle?
At any $t \in (a,b), \alpha'(t)$ is a tangent vector to the path
$\rightarrow$ Component $f(\alpha(t)) \cdot \alpha'(t)$
$\rightarrow$ Work done:
= total effective amount of force
$= \int_a^b f(\alpha(t)) \cdot \alpha'(t)dt = \int f \cdot dx$

Remark

$\alpha'(t)$ needs not to be a unit tangent vector, and in fact,
it's size can be large(if one goes fast) or small (if one goes slow)

Length of a path

Let $\alpha : [a,b] \rightarrow \mathbb{R}^n$ be a C^1-path
What is the length of the path?

Theorem

Length of our path is

$$L(\alpha) = \int_a^b |\alpha'(t)|dt$$

Here $|\alpha'(t)|$ is the Euclidean lenth of vector $\alpha'(t) \in \mathbb{R}$

Proof

Proof. Divide $[a,b]$ into k intervals
$[a,b] = [a,a_1]U \ldots U[a_{k-1},b]$

$$
\begin{aligned}
L(\alpha) &= \sum \text{ length of these } k \text{ line segments} \\
&= \sum_{i=1}^{k} |\alpha(a_i) - \alpha(a_{i-1})| \\
&= \sum_{i=1}^{k} |\alpha'(a_{i-1})(a_i - a_{i-1})| + \text{ error} \\
&\approx \sum_{i=1}^{k} |\alpha'(a_{i-1})(a_i - a_{i-1})| \\
&\xrightarrow[\text{length of subintvervals to 0}]{k\to\infty} \int_a^b |\alpha'(t)| dt
\end{aligned}
$$

$\square$

Example

Length of the unit circle in $\mathbb{R}^2$

$$
\begin{aligned}
&\alpha : [0, 2\pi] \to \mathbb{R}^2 \\
&\theta \mapsto \alpha(\theta) = (\cos(\theta), \sin(\theta))
\end{aligned}
$$

We know $\alpha'(\theta) = (-\sin(\theta), \cos(\theta))$

$$
L(\alpha) = \int_0^{2\pi} |\alpha'(\theta) d\theta = \int_0^{2\pi} 1 d\theta = 2\pi
$$

Arc Length

how much have we traveled up to time t

$$
\alpha[a, b] \to \mathbb{R}^n \text{ is } C^1 \text{ - path}
$$

Arc length $s(t)$ for $a \leq t \leq b$

$$
S(t) = \int_a^t |\alpha'(r)| dr
$$

$S(t)$ measures the distance α travels from time a to time t
Clearly, $S'(t) = |\alpha'(t)|$
So a fun corollary is the following

Corollary

If $s(t) = t - a$ for all $a \leq t \leq b$, then the velocity $|\alpha'(t)|$ is always 1, and $\alpha'(t)$ is the unit tangent vector to the path

Proof

Proof. If $s(t) = t - a \Rightarrow S'(t) = 1$ always
$\Rightarrow S'(t) = |\alpha'(t)| = 1$ $\square$

Remark

You can always paramerize your curve with unit constant velocity

Line integral with respect to arc length

$$\begin{cases} \alpha : [a,b] \to \mathbb{R}^n \text{ is } C^1 \text{ - curve} \\ \varphi : \mathbb{R}^n \to \mathbb{R} : \text{ is a real-valued function} \end{cases}$$

$$\int_C \varphi ds = \int_a^b \varphi(\alpha(t)) \cdot s(t) dt$$

$\boxed{s'(t) = |\alpha'(t)}$

$\boxed{\varphi(\alpha(t)) \text{ is the value of } \varphi \text{ on curve } C}$

$\boxed{s'(t) \text{ is the arc length } s(t)}$

Theorem

$\alpha : [a, b] \to \mathbb{R}^n$ is C^1 curve

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} \text{ unit tangent vector to the path at } \alpha(t)$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field
Define $\varphi(\alpha(t)) = f(\alpha(t)) \cdot T(t)$
Then: $\boxed{\int f \cdot d\alpha = \int_C \varphi ds}$

Proof

Proof.

$$\int f \cdot dx = \int_a^b f(\alpha(t)) \cdot \alpha'(t)dt = \int_a^b f(\alpha(t)) \cdot T(t) \cdot |\alpha'(t)|dt = \int_C \varphi ds$$

□