Lecture 20

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Change of variables for double integrals

Example

Recall one example for single variable integral

$$\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta \sin\theta d\theta \rightarrow \text{change of variable } \begin{cases} u = \cos\theta \quad \theta : 0 \rightarrow \frac{\pi}{2} \\ du = u'\,d\theta = -\sin\theta\,d\theta \quad \theta : 1 \rightarrow 0 \end{cases}$$

$$= \int_1^0 (1-u^2)(-du) = \int_0^1 (1-u^2)\,du = u - \frac{u^3}{3}\Big]_0^1 = \boxed{\tfrac{2}{3}}$$

Given variables $(x, y) \in S$

$$\iint_S f(x,y)\,dx\,dy$$

$$(u, v) \mapsto (x, y)$$

New variables $(u, v) \in T$

$$\text{Here, } \begin{cases} x = \overline{X}(u,v) \\ y = \overline{Y}(u,v) \end{cases}$$

Theorem

Change of variables:
Assume $(\overline{X}, \overline{Y}) : T \to S$

$$(u, v) \mapsto (\overline{X}(u, v), \overline{Y}(u, v) = (x, y)$$

$$\text{and} \begin{cases} \text{the map is one-to-one} \\ \frac{d\overline{X}}{du}, \frac{d\overline{X}}{dv}, \frac{d\overline{Y}}{du}, \frac{d\overline{Y}}{dv} \text{ are continuous} \\ \text{Jacobian determinant} = \det J(u, v) = \begin{vmatrix} \frac{d\overline{X}}{du} & \frac{d\overline{Y}}{du} \\ \frac{d\overline{X}}{dv} & \frac{d\overline{Y}}{dv} \end{vmatrix} \neq 0 \text{ always} \end{cases}$$

$$\text{Then, } \iint_S f(x, y)\, dx\, dy = \iint_T f(\overline{X}(u, v), \overline{Y}(u, v)) |J(u, v)|\, du\, dv$$

Theorem

Let $S \subset \mathbb{R}^2$ be an open set such that it's boundary, c, is a piecewise C^1 Jordan curve,
and c encloses exactly S. Let $P, Q : S \to \mathbb{R}$ be C^1 real-valued functions such that $\frac{dP}{dy}(x, y) = \frac{dQ}{dx}(x, y)$ for all $(x, y) \in S$.
Then $f(x, y) = (P(x, y), Q(x, y))$ is a gradient vector field(that is, $f = \nabla\Psi$, for some potential function $\Psi : S \to \mathbb{R}$)
Rmk.: 1 c encloses exactly S is very important
2 Key point again: if $f = \nabla\Psi$, then $\int f \cdot d\alpha = \Psi(\alpha(b)) - \Psi(\alpha(a))$

Proof. Fix $z \in S$
For any $x \in S$, we can connect z to x by line segments parallel to the axes
Take any such path α, connects $z \to x : (\alpha(a) = z, \alpha(b) = x)$
Define $\Psi(x) = \int f \cdot d\alpha = \int_\alpha P\, dx + Q\, dy$
Issues:
1 Is φ well-defined?$\Rightarrow$ we need to show that $\varphi(x)$ does not depend on path α
Call γ boundary of A: $\oint_\gamma f \cdot d\alpha = 0 \Rightarrow \int f \cdot d\alpha = \int f\, dB$
$\Rightarrow \varphi$ is well-defined
2 We need to check $\nabla\varphi = (P, Q) \to$ we have checked this before.

$\square$