Lecture 7

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Theorem 1

Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be a closed box
Let $f : R \to \mathbb{R}$ be a continuous function.
Then $\boxed{\text{max of } R \text{ in } f \text{ and min of } R \text{ in } f \text{ exist}}$

Remarks

1. In single variable calculus,
   If $f : [a, b] \to R$ continuous, then
   $\min\limits_{[a,b]} f$ and $\max\limits_{[a,b]} f$ exist
2. If we replace by a non-closed interval,
   then the conclusion might fail.
   Ex. $f(x) = \frac{1}{x}$ for $x \in (0, 1]$
   max f does not exist

Proof for $n = 2$

Proof. We claim f is bounded in $R = [a_1, b_1] \times [a_2, b_2]$
Proof by contradiction, assume f is not bounded from above in R
(that is f takes values going to $+\infty$ on R)
We divide the boxes in an inductive manner as following

- First, divide R into 4 equal sized boxes since f is not bounded from above in R, f must not be bounded from above in one sub-box $R_1, R_2 = [a_1^1, b_1^1] \times [a_2^1, b_2^1]$
- Then take R_1, divide it into 4 equal sub-boxes and repeat as above$\Rightarrow$we find R_2
- Keep doing so indefinitely, we find boxes

$$\begin{cases} R \supset R_1 \supset R_2 \supset \cdots \supset R_k \\ R_k = [a_1^k, b_1^k] \times [a_2^k, b_2^k], b_1^k - a_1^k = \frac{b_1 - a_1}{2^k} \\ f \text{ is not bounded from above in any of } R_k \end{cases}$$

Observations

1

$$\{a_1^k\} \text{ increasing in } k$$
$$\{b_1^k\} \text{ decreasing in } k$$

2

$$b_1^k - a_1^k = \frac{b_1 - a_1}{2^k} \xrightarrow{k \to \infty} 0$$

$\Rightarrow \{a_1^k\}, \{b_1^k\}$ converge to the same limit,

$$\lim_{k\to\infty} a_1^k = \lim_{k\to\infty} b_1^k = \bar{a}$$

Similarly,

$$\lim_{k\to\infty} a_2^k = \lim_{k\to\infty} b_2^k = \bar{b}$$

$$\Rightarrow \bigcap_{k\in\mathbb{N}} R_k = \big\{(\bar{a}, \bar{b})\big\}$$

In words, $\{R_k\}$ shrinks to exactly one point, $(\bar{a}, \bar{b})$, which contradicts as $f\big((\bar{a}, \bar{b})\big) \in \mathbb{R}$, and is not $+\infty$

Now that we have f is bounded in R, we show $\max\limits_R f$ exists

as f is bounded in $R \Rightarrow \boxed{\sup\limits_R f, \inf\limits_R f \text{ exist}}$

$\sup\limits_V$ is a generalization of $\max\limits_V$

(in fact, if $\max\limits_V$ exists, it's clear that $\sup\limits_V = \max\limits_V$)

However, in many cases, $\max\limits_V$ does not exist

$$\text{Ex. } V = (0,1) \subset \mathbb{R}, \min_V, \max_V \text{ DNE}, \inf_V = 0, \sup_V = 1$$

$$W = \Big\{\frac{1}{n} : n \in \mathbb{N}\Big\} \subset \mathbb{R}, \max_W = \sup_W = 1, \min_W \text{ DNE }, \inf_W = 0$$

What remains now is to show that $\max\limits_R f$ exists, and equals $\sup\limits_R f$

$$\sup_R f = M \in \mathbb{R}$$

Repeat exactly all step 1, and note

$$\sup_{R_k} f = M$$

Remember that $\{R_k\}$ shrinks to exactly one point,

$$\boxed{(\bar{a}, \bar{b}) \Rightarrow f\big((\bar{a}, \bar{b})\big) = M}$$

$\square$

Theorem 2

$R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n, f : R \to \mathbb{R}$ is continuous
Then for any fixed $\epsilon > 0$, we can divide R into 2^m sub-boxes
$R_1, R_2, ..., R_{2^m}$ so that

$$\boxed{\max_{R_k} - \min_{R_k} < \epsilon \forall k}$$

(f is uniformly continuous on R)