Lecture 17

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Last Time

1 $\boxed{\text{Double integral} = \text{repeated integral}}$

2 $Q = [a,b] \times [c,d] \subset \mathbb{R}^2$ If $f : Q \to \mathbb{R}$ is continuous,

then $\displaystyle\iint_Q f = \iint_Q f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$

3 It's important to noting that for $\iint_Q f$ to exist,
we do not always need that f is continuous.
In fact, if f is piecewise continuous on Q, that is, f is continuous
fon various smaller subdomains, and is discontinuous on the boundaries,
things are still fine.

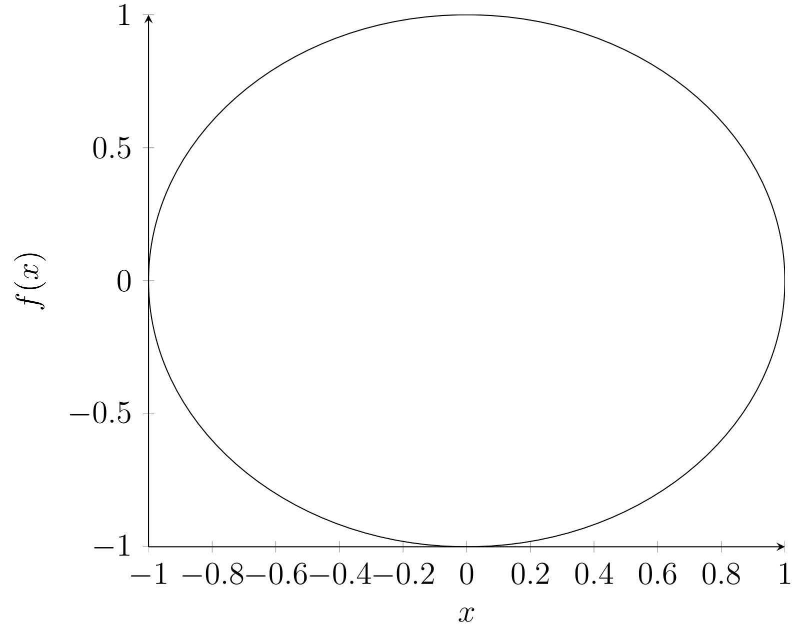
Simple domain in $\mathbb{R}^2$

$$S = \left\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, \varphi(x) \leq y \leq \Psi(x)\right\}$$

For given continuous functions $\varphi, \Psi : [a,b] \to \mathbb{R}$

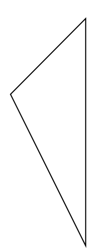
Examples of S

Unit Circle:



$$S_1 = \left\{(x,y) : -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\right\}$$

Triangle with vertices $(0,0),(1,1),(1,-2)$



$$S_2 = \left\{(x,y) : 0 \leq x \leq 1, -2x \leq y \leq x\right\}$$

Let's focus on $S = \left\{(x,y) : a \leq x \leq b, \varphi(x) \leq y \leq \Psi(x)\right\}$
Let's assume $f \to \mathbb{R}$ is continuous.

Theorem

We have:

$$\iint_S f(x,y)\,dx\,dy = \iint_Q \widetilde{f}(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\Psi(x)} f(x,y)\,dy \right) dx$$

Proof. Just need to go from second to third term

$$\iint_Q \widetilde{f}(x,y)\,dx\,dy = \int_a^b \left(\int_c^d \widetilde{f}(x,y)\,dy \right) dx = \int_a^b \left(\int_{\varphi(x)}^{\Psi(x)} f(x,y)\,dy \right) dx$$

□

The other kind of simple domains is :

$$T = \left\{ (x,y) \in \mathbb{R}^2 : c \le y \le d, g(y) \le x \le h(y) \right\}$$

for g, h are continuous functions: $[c,d] \to \mathbb{R}$
Same constructions as earlier
Then, for $f : T \to \mathbb{R}$ continuous, then

$$\iint_T f(x,y)\,dx\,dy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$

Remarks

1 Sometimes, S is simple both ways
Ex: $S = \overline{B}(0,1) = \left\{ (x,y) : x^2 + y^2 \le 1 \right\}$

$$\begin{aligned} S &= \left\{ (x,y) : -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \right\} \\ &= \left\{ (x,y) : -1 \le y \le 1, -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \right\} \end{aligned}$$

2 A complicated domain can be divided into simple domains

Two simple applications

$$S = \left\{ (x,y) : a \le x \le b, \varphi(x) \le y \le \Psi(x) \right\}$$

<u>Area of S</u>

$$\begin{aligned}
|S| &= \text{Area of } S = \iint_S 1\,dx\,dy \\
&= \int_a^b \left(\int_{\varphi(x)}^{\Psi(x)} 1\,dy \right) dx = \int_a^b (\Psi(x) - \varphi(x))\,dx
\end{aligned}$$

<u>Ex</u>:

$$|S| = \int_0^1 (x^2 - (-x^3))\,dx = \int_0^1 (x^2 + x^3)\,dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in S, 0 \le z \le f(x,y) \right\}$$

Volume of $D = |D| = \iint_S f(x,y)\,dx\,dy$