Lecture 22

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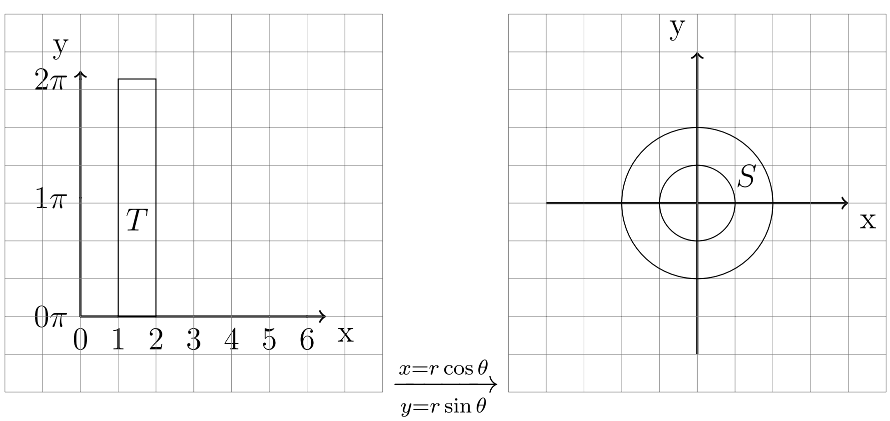
Change of Variables

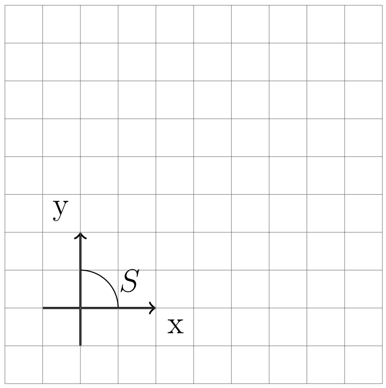
$$\iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta$$

Ex. 1

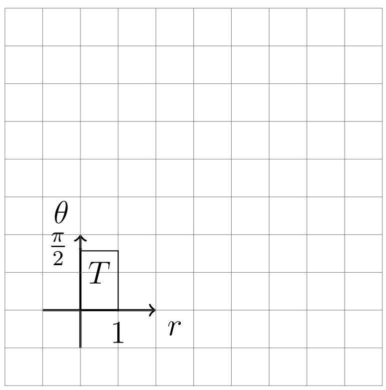


Ex. 2

$$\iint_S \sqrt{1-x^2-y^2}\,dx\,dy$$



Sln. Use change of variables using polar coordinates



$$T = [0,1] \times [0, \frac{\pi}{2}]$$

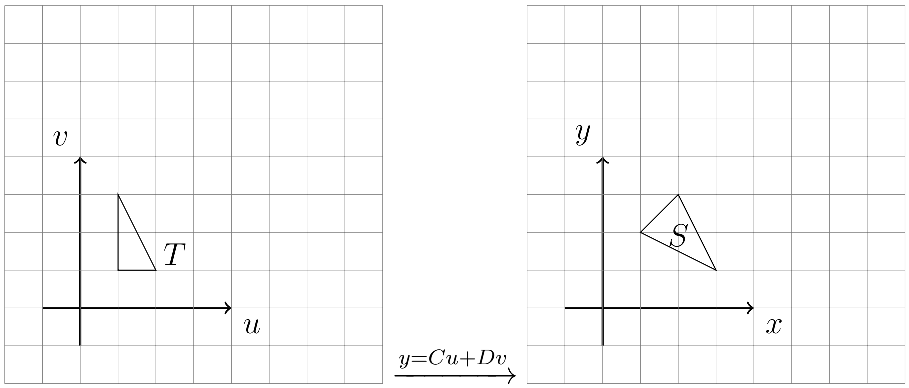
$$\iint_S \sqrt{1 - x^2 - y^2}\, dx\, dy = \iint_T \sqrt{1 - r^2\cos^2\theta - r^2\sin^2\theta} r\, dr\, d\theta = \iint_T \sqrt{1-r^2} r\, dr\, d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} d\theta \right) \left(\int_0^1 \sqrt{1-r^2} r\, dr \right)$$

$$= \frac{\pi}{2} \int_0^1 \frac{1-r^2}{r}\, dr \qquad \begin{cases} u = 1 - r^2 \\ du = -2r\, dr \end{cases}$$

$$= \frac{-\pi}{4} \int_1^0 \sqrt{u}\, du = \frac{\pi}{4} \int_0^1 \sqrt{u}\, du = \frac{\pi}{4} \frac{2}{3} u^{\frac{3}{2}} \Big]_0^1 = \frac{2\pi}{12} = \frac{\pi}{6}$$

Linear Transformations in Two Dimensions



$$\xrightarrow[x=Au+Bv]{y=Cu+Dv}$$

Q: $\iint_S f(x,y)\,dx\,dy$
S: a polygon in 2D
We need to find what is $J(u,v)$?

$$J(u,v) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix} = AD - BC$$

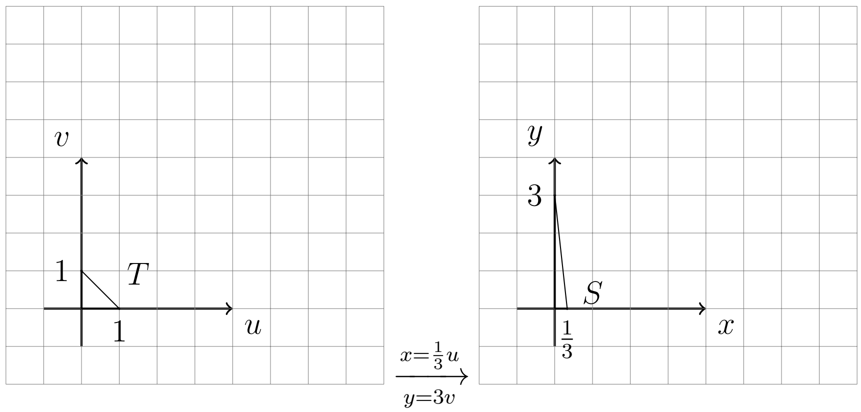
<u>Condition we need</u> $AD - BC \neq 0 \to$ the linear map
$(u,v) \mapsto (x,y)$ is 1-to-1
Change of variables formula:

$$\iint_S f(x,y)\,dx\,dy = \iint_T f(Au+Bv, Cu+Dv)|AD-BC|\,du\,dv$$

$$|AD-BC| \iint_T f(Au+Bv, Cu+Dv)\,du\,dv$$

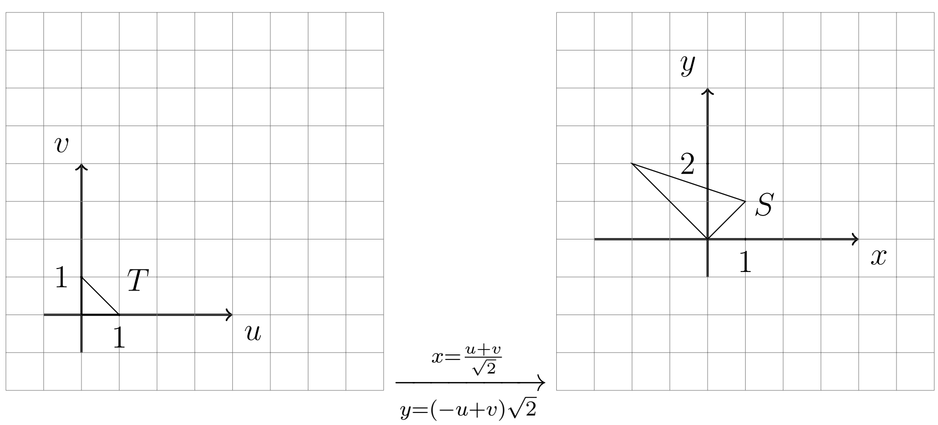
Ex. 1

Let $f = 1$, then we have:
Area of $S = |AD - BC| \cdot$(Area of T)



$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{vmatrix} = 1$$

Ex. 2

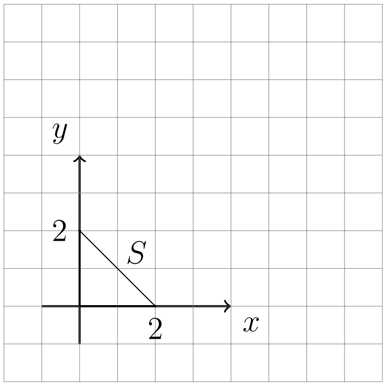


$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{vmatrix} = 2$$

$$|AD - BC| = \frac{\text{Area of } S}{\text{Area of } T}$$

Ex. 3

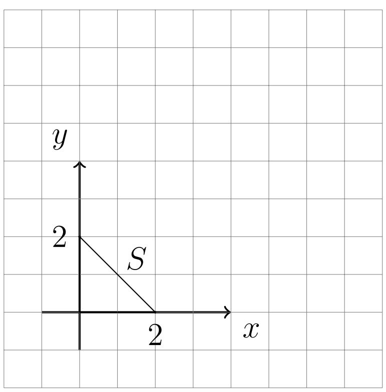
$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy$$

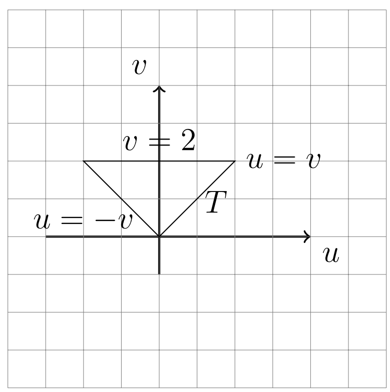


$$\begin{cases} u = y - x \\ u = y + x \end{cases} \quad \begin{vmatrix} A & C \\ B & D \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\Rightarrow \begin{cases} x = \frac{v-u}{2} \\ y = \frac{u+v}{2} \end{cases}$$

$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy = \iint_T e^{\frac{u}{v}} \frac{1}{2}\, du\, dv$$





$$= \int_0^2 \left(\int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}}\, du \right) dv = \int_0^2 \left(\frac{1}{2} v e^{\frac{u}{v}} \Big]_{u=-v}^{u=v} \right) dv$$

$$= \frac{1}{2} \int_0^2 v(e - \frac{1}{e})\, dv = e - \frac{1}{e}$$