Lecture 2

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1 Saddle Points and Critical Points

1.1 Remark

Local max/min at y and differentiability at $y \Rightarrow \nabla f(y) = \vec{0}$

1.2 Example 1

$f : \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto f(x, y) = xy$
We have $\nabla f(x, y) = (\frac{df}{dx}, \frac{df}{dy}) = (y, x)$
Critical points of $f : \nabla f(x, y) = (y, x) = \vec{0} = (0, 0)$
Only one critical point at the origin $(0, 0)$, and $\vec{0}$ is not a local max or min

Proof. Pick any $r > 0$, look at $B(\vec{0}, r)$
here $f(\vec{0}) = 0$, pick $x_1 = (\frac{r}{2}, \frac{r}{2})$, $x_2 = (\frac{r}{2}, \frac{-r}{2})$
Then x_1, $x_2 \in B(\vec{0}, r)$, $f(x_1) = \frac{r^2}{4}$, $f(x_2) = \frac{-r^2}{4}$
and $x_2 < \vec{0} < x_1$ thus it is not a local min or max $\square$

1.3 Saddle Point

y is a saddle point of f if it is a critical point but not a local max or local min

1.4 Clear characterization

y is a saddle point if $\nabla f(y) = \vec{0}$, and for any $r > 0$
we can find x_1, $x_2 \in B(y, r)$ and $f(x_1) > f(y) > f(x_2)$

2 Finding global min/max

2.1 Example 2

$g : [-1, 3] \times [-1, 3] \to \mathbb{R}$
$(x, y) \mapsto g(x, y) = (x - 1)y$
Finding the global max of g, find all critical points inside

$$\nabla g(x, y) = (g_x, g_y) = (y, x - 1)$$

$$\nabla g(x, y) = \vec{0} \Rightarrow (y, x - 1) = (0, 0) \Rightarrow (x, y) = (1, 0)$$

On the edges

$$x = -1, -1 \leq y \leq 3,$$

$$g(x, y) = -2y$$

min value of $g = -6$, max value of $g = 2$
... for $x = 3, -1 \leq y \leq 3$, etc.

$$g(1, 0) = 0$$

Conclusion: global max of $g = 6$ at $(x, y) = (3, 3)$

3 Characterization of critical points

3.1 Single variable calculus

$f(x) = x^2$, local min at $x = 0$, $f''(y) \geq 0$
$f(x) = -x^2$, local max at $x = 0$, $f''(y) \leq 0$
$f(x) = x^3$, neither at $x = 0$, $f''(y) = 0$

3.2 Analog of critical points

Idea: use Taylor expansion around y
In single variable:
$f : \mathbb{R} \to \mathbb{R}$ is C^2(twice differentiable)
$f(y+h) = f(y) + f'(y)h + \frac{1}{2}f''(y)h^2 + \omega(h)|h|^2$ where $\lim_{h\to 0} \omega(h) = 0$
If $f'(y) = 0$, then $f(y+h) = f(y) + \frac{1}{2}f''(y)h^2 + \omega(h)|h|^2$
If $f''(y) > 0 \to y$ is a local min
If $f''(y) < 0 \to y$ is a local max
If $f''(y) = 0 \to$ inconclusive

3.3 In n dimensions

This is the same for $f : B(y, r) \to \mathbb{R}$ where $B(y, r) \subset \mathbb{R}^n$
Fix a direction $e \in \mathbb{R}^n, |e| = 1$
Define $\varphi : (-r, r) \to \mathbb{R}$
$\quad t \mapsto \varphi(t) = f(y + te)$
By Taylor expansion,

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{1}{2}\varphi''(0)t^2 + \omega(t)t^2$$

$$\varphi(t) = f'(y + te, e) = \nabla f(y + te)e$$

$$\varphi''(t) = f''(y + te, e, e) = eH(y + te)e^T$$

here $H(y + te)$ is the Hessian of f at $y + te$

$$= \begin{pmatrix} \frac{d^2f}{dx_1^2} & \cdots & \frac{d^2f}{dx_n dx_1} \\ \vdots & & \vdots \\ \frac{d^2f}{dx_1 dx_n} & \cdots & \frac{d^2f}{dx_n^2} \end{pmatrix}$$

$$\varphi'(t) = \sum_{i=1}^{n} f_{x_i}(x+te)e_i$$

$$\varphi''(t) = \sum_{i=1,j=1}^{n} f_{x_i x_j}(x+te)e_i e_j$$