Lecture 20

March 11, 2019

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Greene's Theorem Pt. 2

Ex. 1

$$f(x, y) = (y + 3x, 2y - x)$$

Work done by this force field in a moving particle around $4x^2 + y^2 = 4$
Sln. by Greene's Theorem:
Set $P(x, y) = y + 3x$, $Q(x, y) = 2y - x$

$$\oint_C f \cdot dx = \oint_C P\,dx + Q\,dy$$

$$= \iint_S (\frac{dQ}{dx} - \frac{dP}{dy})\,dx\,dy$$

$$= \iint_S (-2)\,dx\,dy$$

$$= -2 \cdot \text{Area}(S)$$

$$\text{Area}(S) = \iint_S 1\,dx\,dy = \int_{-1}^{1} \left(\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} 1\,dy \right) dx$$

$$= \int_{-1}^{1} 2\sqrt{4 - 4x^2}\,dx = \int_{-1}^{1} 4\sqrt{1 - x^2}\,dx = 2\int_{-1}^{1} 2\sqrt{1 - x^2}\,dx$$

$$2 \cdot \text{area unit disc} = 2 \cdot (\pi \cdot 1^2) = 2\pi$$

Another way

$$4\int_{-1}^{1}\sqrt{1-x^2}\,dx = 8\int_{0}^{1}\sqrt{1-x^2}\,dx$$

$$x = \sin(\theta), 0 \leq \theta \leq \frac{\pi}{2}$$

$$\sqrt{1-x^2} = \cos(\theta)$$

$$= 8\int_0^{\frac{\pi}{2}} \cos(\theta)\cdot\cos(\theta)\,d\theta = 8\int_0^{\frac{\pi}{2}} \cos^2(\theta)\,d\theta = 8\int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\,d\theta = 2\pi$$

Ex. 2

Let C be the boundary of $[0,1]^2 \subset \mathbb{R}^2$
Let $f(x,y) = (5-xy-y^2, x^2-2xy)$
Compute the work done by a particle moving around C once counter-clockwise
Work done $= \oint_C f\cdot d\alpha$
Here, S is a square, and hence, computing integrals in S is much simpler.
Sln. By Greene's Theorem:

$$P(x,y) = 5-xy-y^2, Q(x,y) = x^2-2xy$$

$$\iint_S (2x-2y-(-x-2y))\,dx\,dy = \iint_S 3x\,dx\,dy$$

$$= \left(\int_0^1 3x\,dx\right)\left(\int_0^1 1\,dy\right) = \boxed{\tfrac{3}{2}}$$

Remark (Interpretation of Green's thm.)

Given $S \subset \mathbb{R}^2$, and C is its boundary
To measure total change of a quantity inside S = To measure total change on C
Total change inside $S = \iint_S (\text{rate of change})\,dx\,dy$
= Total change on $C = \oint_C P\,dx + Q\,dy$

Necessary and sufficient conditions for a vector field to be a gradient vector field in 2D

Recall: $f(x,y) = (P(x,y), Q(x,y)) : S \to \mathbb{R}^2$ is a given vector field
Recall: f is a gradient vector field if:

$$f(x,y) = \nabla\Psi(x,y) = (\Psi_x, \Psi_y)$$

In this case, $\oint f \cdot \alpha = 0$

$$\int f \cdot d\beta = \Psi(\beta(b)) - \Psi(\beta(a))$$

Claim 1: If f is a gradient vector field

$$\Rightarrow \frac{dP}{dy} = dQdx$$

Proof.

$$f = (P, Q) = (\Psi_x, \Psi_y)$$

$$\Rightarrow \begin{cases} P = \Psi_x \\ Q = \Psi_y \end{cases} \Rightarrow P_y = Q_x = \Psi_{xy}$$

□

Claim 2: In general, $\frac{dP}{dy} = \frac{dQ}{dx} \not\Rightarrow f$ is a gradient vector field

$$\underline{\text{Earlier}} : \begin{cases} \frac{dP}{dy} = \frac{dQ}{dx} \\ S \text{ is convex} \end{cases} \Rightarrow f \text{ is a gradient vector field}$$

Let's now recall Green's thm.
For any closed Jordan curve α in S

$$\oint f\, d\alpha = \oint_\alpha P\, dx + Q\, dy = \iint_A (\frac{dQ}{dx} - \frac{dP}{dy}\, dx\, dy$$

Clearly, if $\frac{dQ}{dx} = \frac{dP}{dy}$, then we always have $\oint f \cdot d\alpha = 0$

Theorem

Assume $f(x, y) = (P(x, y), Q(x, y))$ with P, Q are C^1 and $\frac{dQ}{dx} = \frac{dP}{dy}$ always.
Assume the boundary of S is C is a closed, piecewise C^1 Jordan curve, and C encloses S.
Then f is a gradient vector field.