Lecture 18

March 6, 2019

Anders Sundheim
asundheim@wisc.edu

Applications

Let $S \subset \mathbb{R}^2$ represents a shape of a thin plate
For $(x, y) \in S$ let $f(x, y)$ be the mass density at (x, y) of S,
we assume $f \geq 0$
1 Total mass of S:

$$m(S) = \iint_S f(x, y)\, dx\, dy$$

2 Average mass density:

$$\frac{m(s)}{|S|} = \frac{\iint_S f(x, y)\, dx\, dy}{\iint_S 1\, dx\, dy}$$

Sometimes, when we don't say anything about mass density, then we really mean that mass density is constant, that is, $f(x, y) \equiv c > 0$
3 Center of mass: $(\overline{x}, \overline{y}) \in \mathbb{R}^2$ such that

$$\begin{cases} \overline{x} = \frac{\iint_S x\, f(x,y)\, dx\, dy}{m(s)} = \frac{\iint_S x\, f(x,y)\, dx\, dy}{\iint_S f(x,y)\, dx\, dy} \\ \overline{y} = \frac{\iint_S y\, f(x,y)\, dx\, dy}{m(s)} = \dots \end{cases}$$

Sometimes, centroid of S is $(\overline{x}, \overline{y})$
4 Moment of inertia:

$$
\begin{aligned}
I_L &= \iint_S d(x,y)^2 f(x,y)\,dx\,dy \\
&= \iint_S \delta(x,y)^2 f(x,y)\,dx\,dy
\end{aligned}
$$

If we rotate S around the x-axis,

$$I_x = \iint_S y^2 f(x,y)\,dx\,dy$$

Similarly,

$$I_y = \iint_S x^2 f(x,y)\,dx\,dy$$

Example

Find the centroid of S which is determined by $y = 0$, $y = \sin(x)$, $x = 0$,$x = \pi$
Solution: Let's say mass density is constant $c > 0$
Note S is symmetric about $x = \frac{\pi}{2}$,
we actually get right away that $\overline{x} = \frac{\pi}{2}$
Let's check:

$$
\begin{aligned}
\overline{x} &= \frac{\iint_S x \cdot c \, dx \, dy}{\iint_S c \, dx \, dy} \\
&= \frac{\iint_S x \, dx \, dy}{\iint_S 1 \, dx \, dy} \\
&= \frac{\int_0^\pi \left(\int_0^{\sin(x)} x \, dy \right) dx}{\int_0^\pi \left(\int_0^{\sin(x)} 1 \, dy \right) dx} \\
&= \frac{\int_0^\pi x \sin(x) \, dx}{\int_0^\pi \sin(x) \, dx} \\
&= \frac{1}{2} \int_0^\pi x \sin(x) \, dx \\
&= \frac{1}{2} \left(uv \Big|_0^\pi - \int_0^\pi v \, du \right) \\
&= \frac{1}{2} \pi + \int_0^\pi \cos(x) \, dx \\
&= \frac{1}{2} \pi
\end{aligned}
$$

For $\begin{cases} u = x \\ dv = \sin(x) \, dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = -\cos(x) \end{cases}$

$$
\begin{aligned}
\overline{y} &= \frac{\iint_S y\, c\, dx\, dy}{\iint_S c\, dx\, dy} \\
&= \frac{\iint_S y\, dx\, dy}{\iint_S 1\, dx\, dy = 2} \\
&= \frac{1}{2} \int_0^{\pi} \left(\int_0^{\sin(x)} y\, dy \right) dx \\
&= \frac{1}{2} \int_0^{\pi} \frac{\sin^2(x)}{2}\, dx \\
&= \frac{1}{8} \int_0^{\pi} (1 - \cos(2x))\, dx \\
&= \frac{\pi}{8}
\end{aligned}
$$

Greene's Theorem

Jordan Curve

<u>Jordan Curve</u>: closed, piecewise C^1, not self intersected curve in $\mathbb{R}^2$

Greene's Theorem

<u>Greene's Theorem</u>:
$P, Q : \overline{S} \to \mathbb{R}$ are C^1.
Then,

$$
\oint_C P\, dx + Q\, dy = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\, dy
$$