Lecture 15

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Claim 1

$S, T \neq \emptyset$
Let $s(x) = -M$ for all $x \in Q$, then,
s is a step function, $s \leq f$ on Q
$\to s \in S \to S \neq \emptyset$
Similarly, for $t(x) = M$ for $x \in Q, t \in T \to T \neq \emptyset$

Claim 2

For $s \in S, t \in T$, then $\iint_Q s(x,y)\,dx\,dy \leq \iint_Q t(x,y)\,dx\,dy$
By definition, $s \leq f \leq t \Rightarrow s \leq t$ on Q

Claim 3

$$A = \left\{ \iint_Q s : s \in S \right\} \subset \mathbb{R}$$

$$B = \left\{ \iint_Q t : t \in T \right\} \subset \mathbb{R}$$

$$\text{Last time, } \int_0^1 x\,dx = \frac{1}{2}$$

$$\frac{k-1}{2k} \in A, \frac{k+1}{2k} \in B$$

Def. We say that f is integrable on $Q\left(\iint_Q \text{ exists} \right)$
if we can find a number $c \in \mathbb{R}$ s.t.
for any $\varepsilon > 0$, there are functions $s \in S$, $t \in T$ s.t.

$$c - \varepsilon \leq \iint_Q s \leq \iint_Q t \leq c + \varepsilon$$

A and B meet at c

Theorem

$\iint_Q f$ exists $\Leftrightarrow$ for each $\varepsilon > 0$, we can find

$$s \in S, t \in T \text{ s.t. } 0 \leq \iint_Q t - \iint_Q s \leq \varepsilon$$

Proof. If $\iint_Q f = c$, then for $\varepsilon > 0$, we can find $s \in S$, $t \in T$

$$c - \frac{\varepsilon}{2} \leq \iint_Q s \leq \iint_Q t \leq c + \frac{\varepsilon}{2}$$

$$\Rightarrow \iint_Q t - \iint_Q s \leq \varepsilon$$

□

Example 1

$$\iint_0^1 f(x)dx \text{ exists or not if } f(x) = \begin{cases} 0 & x \in Q \\ 1 & x \notin Q \end{cases}$$

1 For a step function $s \in S$, that is

$$s \leq f \Rightarrow \boxed{s \leq 0}$$

$$\int_0^1 s(x)\, dx \leq 0 \text{ for all } s \in S$$

2 For $t \in T$, that is,

$$t \geq f \Rightarrow \boxed{t \geq 1}$$

$$\int_0^1 t(x)\,dx \geq 1 \text{ for all } t \in T$$

$$\Rightarrow \int_0^1 f(x)\,dx \text{ DNE}$$

Example 2

$$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$$

<u>Claim</u>

$$\int_0^1 \quad \text{exists, equals } 0$$

Proof. Zero function is in S, of course $\int_0^1 0\,dx = 0$

$$\text{for any } h > 0, \text{ set } t(x) = \begin{cases} 1 & 0 \leq x \leq h \\ 0 & h < x \leq 1 \end{cases}$$

Then $t \in T$, and $\int_0^1 t(x)dx = h$
Let $h \to 0^+$ to conclude

□

Theorem: Double integrals and repeated integrals

Let $Q = [a,b] \times [c,d]$, $f : Q \to \mathbb{R}$ be integrable on Q
Assume for each $x \in [a,b]$, $\int_c^d f(x,y)\,dy$ exists

$$\text{Then, } \iint f = \iint_Q f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Example 3

$$\iint_{[0,1]^2} xy\,dx\,dy = \int_0^1 \left(\int_0^1 xy\,dx \right) dy$$

$$= \int_0^1 y \left(\int_0^1 x\,dx \right) dy = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$