Lecture 11

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Last Time

$$\int f \cdot d\alpha = \int \nabla\Psi \cdot d\alpha = \Psi(y) - \Psi(x)$$

$$\oint f \cdot d\alpha = \oint \nabla \cdot d\alpha = 0$$

Example

Compute path integral with $f(x_1, x_2) = (x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}^2$
and $\alpha(t) = (t^5 \cos^{2019}(t), t^{2020}), 0 \leq t \leq 1$
Note: $f(x_1, x_2) = (x_1, x_2) = \nabla\Psi(x_1, x_2)$, where

$$\Psi(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$$

Then, $\displaystyle\int f \cdot d\alpha = \int \nabla\Psi \cdot d\alpha = \Psi(\alpha(1)) - \Psi(\alpha(0)) =$

$$\Psi(\cos^{2019}(1), 1) - \Psi(0, 0) = \frac{\cos^{4058}(1)}{2} + \frac{1}{2}$$

1st Fundamental Theorem of Line Integrals

Let $U \subset \mathbb{R}^n$ be <u>open</u> and <u>connected</u>
Let $f : U \to \mathbb{R}^n$ be a continuous vector field
(If path integrals $\int f \cdot d\alpha$ are independent of the paths,
then f is a gradient vector field)
Assume path integrals $\int f \cdot d\alpha$ are independent of paths.
Denote by $\Phi(x) = \int f \cdot d\alpha$ for x connects z to x for z fixed in U
Then, $\boxed{f(x) = \nabla\Psi(x)}$

Proof

Proof. We'll show that at every $x \in U$

$$f(x) = (f_1(x), \ldots, f_n(x)) = (\Psi_{x_1}(x), \ldots, \Psi_{x_n}(x))$$

Let's just show $f_1(x) = \Psi_{x_1}(x)$
Assume $x \in U$ open, we can find $r > 0$ s.t. $B(x, r) \subset U$
For $|s| < r$, we'll compare values of $\Psi(x + se_1)$ with $\Psi(x)$
Define $\gamma : [0, s] \to U$ s.t. $\gamma(r) = x + re_1$

$$\text{Then, } \Psi(x + se_1) - \Psi(x) = (\int f \cdot d\alpha + \int f \cdot d\gamma) - \int f \cdot d\alpha$$

$$\Rightarrow \Psi(x + se_1) - \Psi(x) = \int f \cdot d\gamma = \int_0^s f(x + re_1) \cdot \gamma'(r) dr$$

$$= \int_0^s f(x + re_1) \cdot e_1 dr = \int_0^s f_1(x + re_1) dr$$

$$\text{We get } \frac{\Psi(x + se_1) - \Psi(x)}{s} = \frac{1}{s} \int f_1(x + re_1) dr$$

Let $s \to 0$, we get

$$\lim_{s \to 0} \frac{1}{s} \int_0^s f_1(x + re_1) dr = f_1(x)$$

$$\Rightarrow \boxed{\Psi_{x_1}(x) = f_1(x)}$$

$\square$

Fun Fact

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function

Then, $\boxed{\lim_{s \to 0} \frac{1}{s} \int\limits_{x}^{x+s} g(r)dr = g(x)}$

Proof

Proof. Define: $G(y) = \int\limits_{x}^{y} g(r)dr \to G'(y) = g(y)$

LHS $\int\limits_{x}^{x+s} g(r)dr = \frac{G(x+s)-G(x)}{s}$

$G'(y) = g(z)$ for some z in $[x, x+s]$

$$\lim_{s \to 0} \frac{1}{s} \int_{x}^{x+s} g(r)dr = g(x) \text{ as } z \to x, s \to 0$$

□

Corollary

If $\oint f \cdot d\alpha = 0$ for all closed paths α, then

f is a gradient vector field