Lecture 1

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1 Minimums, Maximums, Saddle Points

1.1 Example

$U \subset \mathbb{R}^n$ or $U = \mathbb{R}$, U open
$f : U \mapsto \mathbb{R}$
$f : \mathbb{R}^2 \mapsto \mathbb{R}$
$(x, y) \in \mathbb{R}^2 \mapsto f(x, y) = x^2 - y^2$
No global max
$f(x, y) \to \infty$ for y fixed, $x \to \infty$
$(0, 0)$ min of $x \mapsto f(x, 0)$ and max of $y \mapsto f(0, y)$

1.2 Global Max

$y \in U$ is global max if $f(y) \geq f(x) \forall x \in U$

1.3 Global Min

$z \in U$ is a global min if $f(z) \leq f(x) \forall x \in U$

1.4 Local Min

$y \in U$ is a local max of f if there is $r > 0$ such that $B(y, r) \subset U$ and $f(y) \leq f(x) \forall x \in B(y, r)$

1.5 Critical Point

$y \in U$ is a critical point of f if there is $r > 0$ such that $B(y,r) \subset U$, f is differentiable in $B(y,r)$, and $\nabla f(y) = \vec{O}$

1.6 Gradient

$$\begin{aligned}\nabla f(y) &= (\tfrac{df}{dx_1}(y), \tfrac{df}{dx_2}(y), ... \tfrac{df}{dx_n}(y)) \\ &= (f_{x_1}(y), f_{x_2}(y), ... f_{x_n}(y)) \\ &= (f'(y, e_1), f'(y, e_2), ... f'(y, e_n))\end{aligned}$$

1.7 Lemma

If y is a local minimum or maximum of f and f is differentiable in $B(y,r)$, then $\nabla f(y) = \vec{O}$

1.8 Proof

For $e_1, s \in (-r, r)$

$s \mapsto P(s) = f(y + se_1)$

then P has a maximum at $s = 0$

so $P^{'}(0) = 0$

$$\begin{aligned}P^{'}(s) &= f'(y + se_1, e_1) \\ &= \tfrac{df}{dx_1}(y + se_1)\end{aligned}$$

$P'(0) = 0 = \frac{df}{dx_1}(y)$

similarly, $\frac{df}{dx_k}(y) = 0$

$\Rightarrow \nabla f(y) = \vec{O}$