Lecture 13

February 25, 2019

Anders Sundheim
asundheim@wisc.edu

Theorem

If U is convex, and $(f_j)_{x_i}$ in U for all i, j
then f is a gradient vector field

Proof

Proof. Lets just do it for $n = 2$, that is, in two dimensions
Fix $z \in U$. Define

$$\Psi(x) = \int_{[zx]}^{r} f \cdot d\alpha$$

$$\alpha(t) = z + t(x - z) \text{ for } 0 \leq t \leq 1$$

$$\begin{aligned}\Psi(\alpha) &= \int_0^1 f(\alpha(t)) \cdot \alpha'(t) dt \\ &= \int_0^1 f(z + t(x - z)) \cdot (x - z) dt\end{aligned}$$

Claim: $\nabla\Psi = f$
Lets just show: $\Psi_{x_1} = f_1$

$$
\begin{aligned}
\Psi_{x_1}(x) &= \frac{d}{dx_1}\Big(\int_0^1 f(z+t(x-z))\cdot(x-z)dt\Big) \\
&= \frac{d}{dx_1}\Big(\int_0^1 f_1(z+t(x-z))\cdot(x_1-z_1)+f_2(z+t(x-z))\cdot(x_2-z_2)dt\Big) \\
&= \int_0^1 t(f_1)_{x_1}(\sim)(x_1-z_1)+f_1(\sim)+tf_2(f_2)_{x_1}(\sim)\cdot(x_2-z_2)dt \\
&= \int_0^1 t\nabla f_1(\sim)\cdot(x-z)+f_1(\sim)dt \\
&= \int_0^1 \frac{d}{dt}(tf_1(z+t(x-z))) = f_1(z+x-z) = f_1(x)
\end{aligned}
$$

$\square$