Lecture 14

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Multiple Integrals

Setting

$$Q = [a, b] \times [c, d] \subset \mathbb{R}^2$$

is a given rectangle

$f : Q \to \mathbb{R}$ is a real valued function

which is bounded, that is, there is $M > 0$ s.t.

$|f(x)| \leq M$ for all $x \in Q$

Object of interest

$$\iint_Q f(x, y)\, dx\, dy \text{ or } \iint_Q f$$

A few notions

1 Partitions of Q

$P_1 = \{x_0, x_1, \ldots, x_m\}$ is called a partition of $[a, b]$ if we have

$$x_0 = a < x_1 < \cdots < x_{m-1} < x_m = b$$

$P_2 = \{y_0, y_1, \ldots, y_k\}$ is called a partition of $[c, d]$ if we have

$$y_0 = c < y_1 < \cdots < y_{k-1} < y_k = d$$

Then $P_1 \times P_2$ creates a partition of our rectangle Q, that is,
Q is divided into $m \cdot k$ subrectangles Q_{ij}, where

$$\boxed{Q_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]}$$

2 Let $P_1 \times P_2$ be a partition of Q,
and $\widetilde{P_1} \times \widetilde{P_2}$ be another partition of Q

$$\text{If } \begin{cases} P_1 \subset \widetilde{P_1} \\ P_2 \subset \widetilde{P_2} \end{cases} \quad \text{we say } \widetilde{P_1} \times \widetilde{P_2} \text{ is a finer partition than } P_1 \times P_2$$

3 Step functions in Q
Let $P_1 \times P_2$ be a partition of Q
<u>Def.</u> we say that $f : Q \to \mathbb{R}$ is a step function if

$$f(x, y) = C_{ij} \text{ for all } (x, y) \in Q_{ij}$$

(that is, f takes a constant value in each Q_{ij})
For this f, $f(x, y) = C_{ij}$ for $x \in Q_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
<u>we define</u>

$$\iint_Q f = \iint_Q f(x, y)\, dx\, dy = \sum_{i=1}^{m} \sum_{j=1}^{k} C_{ij} \cdot \text{area}(Q_{ij})$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{k} C_{ij} (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$$

Initial construction of $\iint_Q f$
given $f : Q \to \mathbb{R}$ bounded

$$S = \{g : Q \to \mathbb{R} : g \text{ is a step function, } g \le f \text{ on } Q$$
$$T = \{h : Q \to \mathbb{R} : h \text{ is a step function, } f \le h \text{ on } Q$$

functions in S underestimate f
functions in T overestimate f

Example

$f : [0, 1] \to \mathbb{R}$ with

$$f(x) = x$$

Claim: $\int_0^1 x \cdot dx = \frac{1}{2}$

Proof. Lets consider a partition

$$\left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\right\}$$

and we'll let $k \to \infty$
For this P^k, define $g(x) = \frac{i}{k}$ for $\frac{i}{k} \leq x < \frac{i+1}{k}$
so $g \in S$, and

$$\begin{aligned} \int_0^1 g(x)dx &= \sum_{i=0}^{k+1} \frac{i}{k} \cdot \text{length of subinterval} \\ &= \sum_{i=0}^{k-1} \frac{i}{k} \cdot \frac{1}{k} = \frac{1}{k^2} \frac{k(k-1)}{2} = \frac{k-1}{2k} \end{aligned}$$

$$h(x) = \frac{i+1}{k} \text{ for } \frac{1}{k} \leq x < \frac{i+1}{k}$$

$$\text{so } h \in T, \int_0^1 h(x)dx = \sum_{i=0}^{k+1} \frac{i+1}{k} \cdot \frac{1}{k} = \frac{1}{k^2} \frac{k(k+1)}{2} = \frac{k+1}{2k}$$

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