Lecture 16

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Last time

$$\iint_Q f(x,y)\,dx\,dy = \int_a^b \Big(\int_c^d f(x,y)\,dy \Big)\,dx$$
$$\iint_Q f(x,y)\,dx\,dy = \int_c^d \Big(\int_a^b f(x,y)\,dx \Big)\,dy$$

Proof

(double integral = repeated integral)

Proof. Firstly, if we have g is a step function, then surely

$$\iint_Q g(x,y)\,dx\,dy = \int_a^b \Big(\int_c^d g(x,y)\,dy \Big)\,dx$$

By induxtion, we actually reduce this to just that $g(x) = \alpha$ on Q

$$\iint_Q \alpha\,dx\,dy = \alpha(b-a)(d-c)$$
$$\int_a^b \Big(\int_c^d \alpha\,dy \Big)\,dx = \int_a^b \alpha(d-c)\,dx = \alpha(d-c)(b-a)$$

Secondly, for any $\varepsilon > 0$, we can find $s \in S, t \in T$:

$$\alpha - \varepsilon \leq \iint_Q s \leq \iint_Q f = \alpha \leq \iint_Q t \leq \alpha + \varepsilon$$

Of course $s(x,y) \leq f(x,y) \leq t(x,y)$
$Q = [a,b] \times [c,d] \subset \mathbb{R}^2, f : Q \to \mathbb{R}$ is bounded

$$\Rightarrow \int_c^d s(x,y)\,dy \leq \int_c^d f(x,y)\,dy \leq \int_c^d t(x,y)\,dy$$

$$\begin{aligned}\Rightarrow \int_a^b \Big(\int_c^d s(x,y)\,dy \Big)\,dx &\leq \int_a^b \Big(\int_c^d f(x,y)\,dy \Big)\,dx \\ &\leq \int_a^b \Big(\int_c^d t(x,y)\,dy \Big)\,dx\end{aligned}$$

$$\alpha - \varepsilon \leq \iint_Q s \leq \int_a^b \Big(\int_c^d f(x,y)\,dy \Big)\,dx \leq \iint_Q t \leq \alpha + \varepsilon$$

Let $\varepsilon \to 0$, $\displaystyle\int_a^b \Big(\int_c^d f(x,y)\,dy \Big)\,dx = \alpha$

$\square$

Theorem

Let $f : Q \to \mathbb{R}$ be continuous. Then all integrals earlier exist and

$$\iint_Q f = \int_a^b \left(\int_c^d f(x,y)\,dy \right)\,dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right)\,dy$$

To be proved later

geometric interpretation of $\displaystyle\iint_Q f = \int_a^b \left(\int_c^d f(x,y)\,dy \right)\,dx$

Let's assume f is continuous and $f \geq 0$

$$\iint_Q f\,dx\,dy = V = \text{ volume of } S$$

Each slide has area

$$\begin{aligned}
&= \int_c^d f(x,y)\,dy \\
V &= \int_a^b \Big(\text{Area}\Big)\,dx \\
&= \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx
\end{aligned}$$

Two examples

$Q = [a,b] \times [c,d] \subset \mathbb{R}^2$, $f : Q \to \mathbb{R}$ is bounded

1 $f(x,y) = g(x)h(y)$, where

$$g : [a,b] \to \mathbb{R}, h : [c,d] \to \mathbb{R} \text{ continuous}$$

$$\begin{aligned}
\text{Then } \iint_Q f &= \int_a^b \left(\int_c^d g(x)h(y)\,dy\right) dx \\
&= \int_a^b g(x)\left(\int_c^d h(y)\,dy\right) dx \\
&= \left[\int_a^b g(x)\,dx\right] \cdot \left[\int_c^d h(y)\,dy\right]
\end{aligned}$$

$$Q = [0,1] \times [0,2]$$

$$\iint_Q xe^y\,dx\,dy = \left[\int_0^1 x\,dx\right] \cdot \left[\int_0^2 e^y\,dy\right] = \frac{e^2-1}{2}$$

2 $f(x,y) = \sin(x+y)$ in $Q = [0, \frac{\pi}{2}]^2$

$$\begin{aligned}
&\iint_Q \sin(x+y)\,dx\,dy = \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \sin(x+y)\,dx\right) dy \\
&= \int_0^{\frac{\pi}{2}} \left(-\cos(x+y\Big]_{x=0}^{x=\frac{\pi}{2}}\right) dy = \int_0^{\frac{\pi}{2}} \left(-\cos(\frac{\pi}{2}+y) + \cos(y)\right) dy \\
&= \left(-\sin(\frac{\pi}{2}+y) + \sin(y)\right)\Big]_{y=0}^{y=\frac{\pi}{2}} = 1 + 1 = 2
\end{aligned}$$

Sketch of proof of theorem

Proof. <u>Last theorem in Chapter 9</u>
For a fixed $\varepsilon > 0$, we can find
a partition $P_1 \times P_2$ of Q
such that in each subrectangle Q_ij:
$\boxed{0 \leq \max_{Q_{ij}} f - \min_{Q_{ij}} f \leq \varepsilon}$ Define $s \in S$ as following
$s(x, y) = \min_{Q_{ij}} f$ for $(x, y) \in Q_{ij}$
for all $1 \leq i \leq m$, $1 \leq j \leq k$
Clearly $s \leq f$
Define $t \in T$:
$t(x, y) = \max_{Q_{ij}} f$ for $(x, y) \in Q_{ij}$
$\boxed{s \leq f \leq t}$
Since $0 \leq \max_{Q_{ij}} f - \min_{Q_{ij}} f \leq \varepsilon \Rightarrow 0 \leq t(x, y) - s(x, y) \leq \varepsilon$ in Q_{ij}

$$\Rightarrow 0 \leq t(x, y) - s(x, y) \leq \varepsilon \text{ for } (x, y) \in Q$$

$$\Rightarrow 0 \leq \iint_Q t - \iint_Q s \leq \iint_Q \varepsilon = \varepsilon(b - a)(d - c) \to 0 \text{ as } \varepsilon \to 0$$

□