Lecture 6

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Anders Sundheim
asundheim@wisc.edu

Lagrange Multipliers

Theorem: Lagrange Multiplier

At a local max/min of f on S, say $x_0 \in S$, we have
$\nabla f(x_0) = \lambda \nabla g(x_0)$ for some $\lambda \in \mathbb{R}$

Proof. Lets asume f has a local min $x_0 \in S$
Lets take any nice/smooth curve $\delta : (-r, r) \to S$, with $\delta(0) = x_0$. We have 2 facts.
1 $g(\delta(s)) = 0 \ \forall \ s \in (-r, r)$
2 $s \mapsto f(\delta(s))$ has a local min at $s = 0$

For 1 :

$$g(\delta(s)) = 0$$
$$\Rightarrow \frac{d}{ds} g(\delta(s)) = \nabla g(\delta(s)) \cdot \delta'(s) = 0$$

In particular, at $s = 0$,

$$\nabla g(\delta(0)) \cdot \delta'(0) = \boxed{\nabla g(x_0) \cdot \delta'(0) = 0}$$

$\Rightarrow \nabla g(x_0) \perp \delta'(0)$ for any tangent vector $\delta'(0)$
$\Rightarrow$ geometrically,

$$\boxed{\nabla g(x_0) \text{ is a normal vector to surface } S \text{ at } x_0}$$

For 2 :
$s \mapsto \varphi(s) = f(\delta(s))$ has a local min at $s = 0$
that means that $\varphi'(0) = 0$.
And same as before,

$$\varphi'(0) = \nabla f(\delta(0)) \cdot \delta'(0) = \nabla f(x_0) \cdot \delta'(0) = 0$$

so $\nabla f(x_0)$ is another normal vector to S at x_0 $\square$

Example 1

$$f, g : \mathbb{R}^3 \to \mathbb{R}$$

Find the maximum of $f(x) = x_1^2$
subject to $g(x) = x_1^2 + x_2^2 + x_3^2 - 4 = 0$

$$\nabla f(x0 = \lambda \nabla g(x) \text{ for } x \in S)$$

$$(2x_1, 0, 0) = \lambda(2x_1, 2x_2, 2x_3)$$

There are two cases:

$$\begin{cases} \lambda = 0 : x_1 = 0, x_2^2 + x_3^2 = 4 \\ \qquad \Rightarrow (0, x_2, x_3) \text{ where } x_2^2 + x_3^2 = 4 \\ \lambda \neq 0 : x_2 = x_3 = 0 \\ \qquad \Rightarrow x_1 = \pm 2 \end{cases}$$

$$\begin{aligned} \text{We have } & f(0, x_2, x_3) = 0 \\ & f(\pm 2, 0, 0) = 4 \end{aligned}$$

By comparing, we get the maximum of $f = 4$

Example 2

In $\mathbb{R}^2$, find the minimum distance from the origin to $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 = 1\right\}$
Find the minimum of $f(x) = x_1^2 + x_2^2$
subject to $g(x) = x_1 x_2 - 1 = 0$

$$\nabla f(x) = \lambda \nabla g(x) \text{ for } x \in S$$

$$\iff (2x_1, 2x_2) = \lambda(x_2, x_1)$$

In this case, we have $\lambda \neq 0$

$$\begin{cases} 2x_1 = \lambda x_2 \\ 2x_2 = \lambda x_1 \end{cases} \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

$\boxed{\text{If } \lambda = 2, x_1 = x_2}$