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Riemann Surfaces ①

Notation : $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ ($\mathbb{N}$ "$=\{1,2,\dots\}$")

Region in $\mathbb{C}$ = Connected Open Subset

Analytic = Holomorphic

Reference : Complex Functions : An algebraic and geometric viewpoint (by Jones and Singerman)

__Example__

$z^{1/n}$, $n \in \mathbb{N}$, $n > 1$

- The function $g : \mathbb{C} \to \mathbb{C}$, $g(z) = z^n$, g is holomorphic
- Question : Can we define the inverse of g ?
- More precisely, $\exists$? holomorphic $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = w \Leftrightarrow w^n = z \quad \forall w, z \in \mathbb{C}$
- Answer : No, because g is not 1-1 (one could try to define f as a multi-valued function)
- Question : $\exists$? holomorphic $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^n = z$ $\forall z \in \mathbb{C}$
- Answer : No, because if $\exists f$, $\Rightarrow f(0) = 0$, and $n f(0)^{n-1} f'(0) = 1$, a contradiction
- Question : Can we define f on some surface subset of $\mathbb{C}$?
- Answer : Yes. Let $U = \{ re^{i\theta} \mid r \in \mathbb{R}_{>0},\ 0 < \theta < 2\pi \}$
- Define $f : U \to \mathbb{C}$ by $f(re^{\theta i}) = r^{1/n} e^{\theta/n \, i}$
- f continuous $\Rightarrow f$ is holomorphic because $f(z)^n = z \ \forall z \in U$

Example

$\log z$

- The function $g : \mathbb{C} \to \mathbb{C}$ is defined by $g(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$
- g is holomorphic
- Question: Can we define the inverse of g?

More precisely, $\exists$? holomorphic $f : \mathbb{C} \to \mathbb{C}$ such that $e^{f(z)} = z$ $\forall z \in$

- Answer: No, otherwise $e^{f(0)} = 0$ which is not possible.
- But we can define f on $U = \{re^{i\theta} \mid r \in \mathbb{R}_{>0},\ 0 < \theta < 2\pi\}$

by $f(re^{i\theta}) = \ln(r) + \theta i$

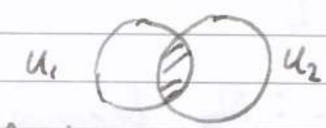
- f is continuous and $e^{f(z)} = z$ $\forall z \in U \Rightarrow f$ is holomorphic.

Remark

In both examples, f cannot be extended to $\mathbb{C}$, even as a continuous function.

Definition

A function element is of the form (U, f) where U is a region and $f : U \to \mathbb{C}$ is holomorphic.



Assume that (U_1, f_1) and (U_2, f_2) are function elements, such that $U_1 \cap U_2 \neq \emptyset$ and $f_1 = f_2$ on $U_1 \cap U_2$.

Then we say that (U_2, f_2) is a direct analytic continuation of $(U_1, f_1$

(Note, f_2 is uniquely determined by f_2).

Example: $U_1 = D(0,1)$, $f_1(z) = \sum_{n=0}^{\infty} z^n$,

($D(a,r)$ is the open disc with centre a and radius r)

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Riemann Surfaces ①

(U_1, f_1) is a function element.

- Put $U_2 = \mathbb{C} \setminus \{1\}$, $f_2(z) = \frac{1}{1-z}$
- f_1 is the power series expansion of f_2 near $0 \Rightarrow f_2 = f_1$ on $U_1 = U_1 \cap U_2$
- (U_2, f_2) is a direct analytic continuation of (U_1, f_1)

<u>Definition</u>

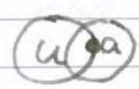
A function element (V, g) is an analytic continuation of a function element (U, f) if $\exists$ function elements $(U_i, f_i)_{1 \le i \le n}$ such that $(U_1, f_1) = (U, f)$, $(U_n, f_n) = (V, g)$

and (U_{i+1}, f_{i+1}) is a direct analytic continuation of (U_i, f_i) $\forall 1 \le i \le n$

<u>Remark</u>

Analytic continuation defines an equivalence relation on the set of function elements. An equivalence class is called a <u>complete holomorphic function</u>

<u>Definition</u>



(U, f) a function element. A point $a \in \partial U$ is called <u>regular</u> if $\exists$ a direct analytic continuation (V, g) such that $a \in V$

Otherwise a is called <u>singular</u>.

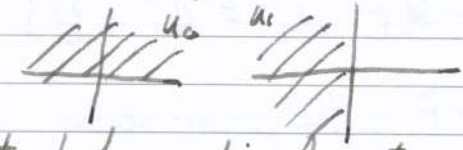
If every point of ∂U is singular we say that ∂U is the natural boundary of (U, f).

<u>Example</u>

Let $U_k = \{re^{i\theta} \mid r \in \mathbb{R}_{>0}, \frac{k\pi}{2} < \theta < \pi + \frac{k\pi}{2}\}$, $n \in \mathbb{N}$, $0 \le k \le 4n$

Let $f_k : U_k \to \mathbb{C}$ be defined by $f_k(re^{i\theta}) = r^{\frac{1}{n}} e^{i\theta/n}$

(U_k, f_k) is a function element and (U_{k+1}, f_{k+1}) is a direct analytic continuation of (U_k, f_k).



All the (U_k, f_k) determine a complete holomorphic function.

If $n > 1$, we have $f_0 \neq f_4$ although $U_0 = U_4$.

However, $f_0 = f_{4n}$. (Imagine the U_i looking like a spiral staircase.)

By putting the U_i together (as explained) we get a space $\mathcal{F}$ and a function $f : \mathcal{F} \to \mathbb{C}$.

Example

U_k as above, $k \in \mathbb{Z}$. Define $f_k : U_k \to \mathbb{C}$ by $f(re^{i\theta}) = \ln(r) + \theta i$

(U_{k+1}, f_{k+1}) is a direct analytic continuation of (U_k, f_k) $\forall k \in \mathbb{Z}$

So all of the (U_k, f_k) determine a complete holomorphic function.

Remark

There are many other ways to construct Riemann Surfaces.

- Quotient Spaces : Consider $\underbrace{\mathbb{Z} + \mathbb{Z}i}_{=\Lambda}$, a subgroup of $\mathbb{C}$

We can form $\mathbb{C}/\Lambda$, a group but also a topological space.
Locally, $\mathbb{C}/\Lambda$ looks like open discs. Actually, $\mathbb{C}/\Lambda$ also has a holomorphic structure.

- Algebraic Curves : Take polynomials $p(z, w)$. Consider
$\{(a,b) \in \mathbb{C}^2 \mid p(a,b) = 0\}$

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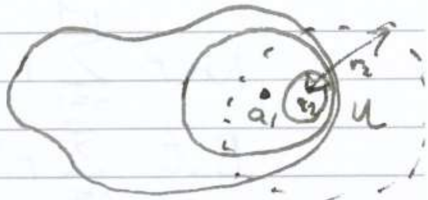


Remark

Analytic continuation using power series:

Suppose that (U, f) is a function element. Pick $a_1 \in U$. Let $U_1 = D(a_1, r_1)$ be the largest open disc with centre a_1 and contained within U. Then $\exists$ a power-series f_1 which represents f on U_1. Then (U_1, f_1) is a direct analytic continuation of (U, f).

Choose $a_2 \in U_1$ such that $a_2 \neq a_1$. Let $U_2 = D(a_2, r_2')$ be the largest open disc with centre at a_2 such that $U_2 \subseteq U_1$.

$\exists$ a power series f_2 on U_2 which represents f_1. Then (f_2, U_2) is a direct analytic continuation of (U_1, f_1).

Let r_2 be the radius of convergence of f_2. Then (U_3, f_3) is a direct analytic continuation of (U_2, f_2) where $U_3 = D(a_2, r_2)$ and f_3 is the extension of f_2 to U_3. Continue this process.

Example

Let $U = D(0, 1)$, $f(z) = \sum_{n=1}^{\infty} z^{n!}$. We show that ∂U is the natural boundary of (U, f) i.e. every point of ∂U is a singular point of (U, f).

It is enough to show that if $a = e^{2\pi \frac{p}{q} i} \in \partial U$, $p, q \in \mathbb{Z}$, $q \neq 0$, then a is a singular point of (U, f) (since such points are dense in ∂U). We will calculate

$$\lim_{r \to 1} f(r e^{2\pi \frac{p}{q} i}) = \lim_{r \to 1} \sum_{n=1}^{\infty} r^{n!} e^{2\pi \frac{p}{q} i n!} = \lim_{r \to 1} \left(\sum_{n=1}^{q-1} r^{n!} e^{2\pi \frac{p}{q} n! i} + \sum_{n=q}^{\infty} r^{n!} \right.$$

Now $\sum_{n=1}^{q-1} r^{n!} e^{2\pi \frac{p}{q} n! i}$ is bounded independently of r.

But $\sum_{n=q}^{\infty} r^{n!}$ is not bounded: $\forall L \in \mathbb{Z}$, $\exists q' \gg q$ and $\exists r$ such that $\sum_{n=q}^{\infty} r^{n!} > \sum_{n=q}^{q'} r^{n!} \geq (q'-q) r^{q'!} > L$

$\Rightarrow \sum_{n=q}^{\infty} r^{n!}$ diverges $\Rightarrow f(z)$ has no limits near a.

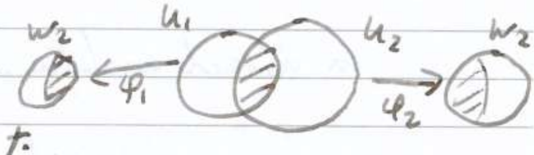
$\Rightarrow \nexists$ an analytic continuation of (U, f) near a.

<u>Riemann Surfaces</u>

A (topological) surface is a Hausdorff topological space X such that locally, X 'looks like' open sets in $\mathbb{C}$. More precisely, we have a covering $X = \bigcup_i U_i$, of open sets, and we have homeomorphisms $\varphi_i : U_i \to W_i \subseteq \mathbb{C}$, W_i open in $\mathbb{C}$.

We call (U_i, φ_i) a <u>chart</u> or a coordinate neighbourhood.

$\mathcal{A} = \{(U_i, \varphi_i)\}$ is called an <u>atlas</u>.



If $U_i \cap U_j \neq \emptyset$, we call the function

$$\begin{array}{ccc} \varphi_i(U_i \cap U_j) & \xrightarrow{\varphi_i \varphi_j^{-1}} & \varphi_j(U_i \cap U_j) \\ \cap\!\!\! | & & \cap\!\!\! | \\ W_i & & W_j \end{array}$$

a <u>transition function</u>.

We say that $\mathcal{A}$ is <u>analytic</u> or <u>holomorphic</u> if all the transition functions are holomorphic.

Two atlases $\mathcal{A}$, $\mathcal{B}$ are equivalent if $\mathcal{A} \cup \mathcal{B}$ is also holomorphic.

An equivalence class is then called a <u>complex structure</u>.

A surface with a complex structure is called a <u>Riemann Surface</u>.

(Informally, a Riemann Surface is a surface on which we can talk about holomorphic functions).

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Example

Suppose $X \subseteq \mathbb{C}$ is an open set. Take $U = X$, $W = X$, $\varphi : U \to W$ the identity. $\mathcal{A} = \{(U, \varphi)\}$ is a holomorphic atlas on $X \Rightarrow$ it defines a complex structure on $X \Rightarrow X$ is a Riemann surface

Example

Assume that X is a Riemann surface with a complex structure $\mathcal{A} = \{(U_i, \varphi_i)\}$. Let $Y \subseteq X$ be an open set. Then Y is a Riemann surface in the natural way, that is, by the complex structure $\mathcal{B} = \{(Y \cap U_i, \varphi_i|_{Y \cap U_i})\}$

$S^2 \subseteq \mathbb{R}^3$

Example (Riemann Sphere)

Let $X = \mathbb{C} \cup \{\infty\}$, with ∞ an extra point.

The open sets $U \subseteq X$ are of the following forms:

$$\left.\begin{array}{l} \text{i) } U \subseteq \mathbb{C} \text{ open in } \mathbb{C} \\ \text{ii) } \infty \in U \text{ and } X \setminus U \text{ is compact in } \mathbb{C} \end{array}\right\} \text{One point compactification of } \mathbb{C}$$

Let $U_1 = \mathbb{C}$, $U_2 = X \setminus \{0\}$

$\varphi_1 : U_1 \xrightarrow{id} W_1 = \mathbb{C}$, $\varphi_2 : U_2 \to W = \mathbb{C}$, $z \mapsto 1/z$, $\infty \mapsto 0$

Then $\mathcal{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is an atlas.

We have $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ and the functions

$\varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$, $w \mapsto 1/w$

$\varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$, $w \mapsto 1/w$ holomorphic on $\mathbb{C} \setminus \{0\}$

$\Rightarrow \mathcal{A}$ is holomorphic. Note that X is compact.

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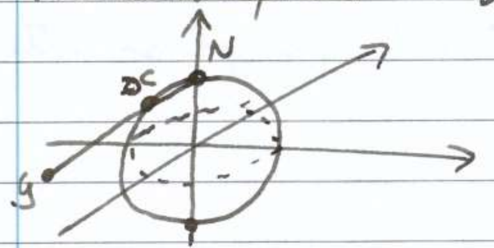
Riemann Surfaces ③

Example

Riemann Sphere $\mathbb{C} \cup \{\infty\}$, stereographic projection



$x \mapsto y$ gives a map $S^2 \setminus \{N\} \to \mathbb{C}$

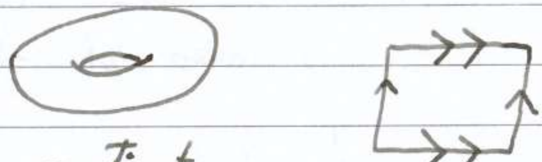
This is a 1-1 correspondence and actually a homeomorphism.

If $N \in U \subseteq S^2$ then $f(U \setminus \{N\})$ = complement of some compact set in

By putting $f(N) = \infty$, we get a 1-1 correspondence $S^2 \to X$ (a homeomorph

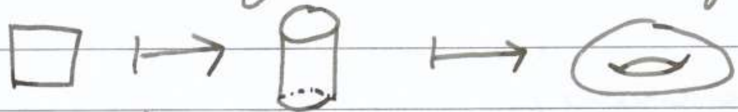
The Riemann Sphere is also denoted $\mathbb{CP}^1$, $\mathbb{P}^1\mathbb{C}$.

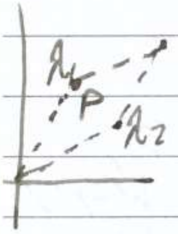
Example (Complex Tori)



A torus is simply $S^1 \times S^1$.

We could also define the torus as a quotient.





A lattice $\Lambda \subseteq \mathbb{C}$ is a subgroup generated by two $\mathbb{R}$-linearly independent $\lambda_1, \lambda_2 \in \mathbb{C}$: $\Lambda = \{m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z}\}$

We put $X = \mathbb{C}/\Lambda$. Topologically, X is just the torus, because X is topologically the same as the quotient of P by identifying the opposite sides. X also has an algebraic structure.

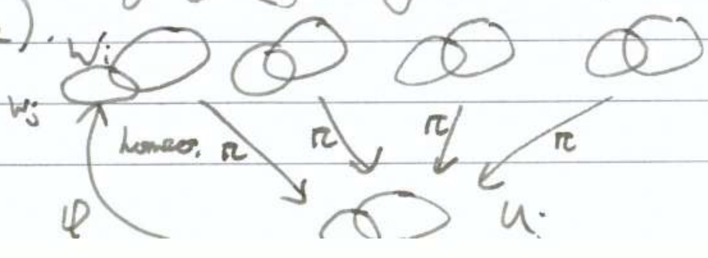
X is a surface:

First note that we have a quotient map $\mathbb{C} \xrightarrow{\pi} X$, $z \mapsto [z]$

X carries the quotient topology.

$U \subseteq X$ is open $\iff$ $\pi^{-1}(X) \subseteq \mathbb{C}$ is open.

For each point $x \in X$, choose a small neighbourhood U of x such tha $\pi^{-1}(U)$ is a disjoint union of infinitely many exact copies of U (in a topological sense).



Pick one copy (say W) and let $\varphi : U \to W$ be the inverse of π
So, (U, φ) is a chart near x.
If we do the same for every point in X, we get an atlas.
$\mathcal{A} = \{ (U_i, \varphi_i) \}$
So X is a topological surface.

X is a Riemann Surface: We must show that $\mathcal{A}$ is holomorphic.
The transition functions are of the following form:
if $U_i \cap U_j \neq \emptyset$, $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ holomorphic
$z \mapsto z + \alpha$ for some $\alpha \in \Lambda$
$\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ are both mapped to $U_i \cap U_j$ homeomorphically ($\pi(z_1) = \pi(z_2) \Leftrightarrow z_1 - z_2 \in \Lambda$)

So $\mathcal{A}$ is holomorphic.

Example (Algebraic Curves)
Recall the Implicit Function Theorem:
Let $p(z, u) \in \mathbb{C}[z, u]$, and assume $\frac{\partial p}{\partial u}(a, b) \neq 0$, then $\exists$ open sets $W, V \subseteq \mathbb{C}$ and a unique holomorphic $\psi : W \to V$ such that for $(a, b) \in W \times V$, $p(a, b) = 0 \Leftrightarrow b = \psi(a)$.

Informally, in some neighbourhood $\mathbb{C}^2$, the second coordinate of solutions of P is a function of the first coordinate.

Definition
Pick $p(z, u) \in \mathbb{C}[z, u]$, non-zero, irreducible.
Then, the set $X = \{ (a, b) \in \mathbb{C}^2 \mid p(a, b) = 0 \}$ is called an (affine) algebraic curve.

We say that $(a, b) \in X$ is smooth if $\frac{\partial p}{\partial z}(a, b) \neq 0$ or $\frac{\partial p}{\partial u}(a, b) \neq 0$. Otherwise we say that (a, b) is singular.

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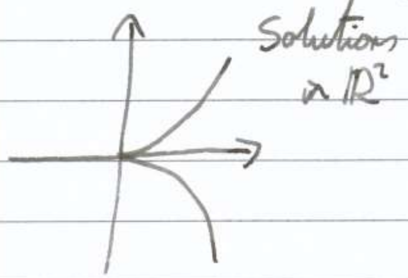
Example

$p(z, u) = u^2 - z^3$

$\frac{\partial p}{\partial z} = -3z^2$, $\frac{\partial p}{\partial u} = 2u$

If $\frac{\partial p}{\partial z}(a,b) = -3a^2 = 0$, $\frac{\partial p}{\partial u}(a,b) = 2b = 0 \Rightarrow (a,b) = (0,$

So $(0,0)$ is the only singular point of the curve defined by P.



$P(z,u) = z^2 + u^2 - 1$

$\frac{\partial p}{\partial z} = 2z$, $\frac{\partial p}{\partial u} = 2u$, if $\frac{\partial p}{\partial z}(a,b) = \frac{\partial p}{\partial u}(a,b) = 0$,

then $2a = 2b = 0 \Rightarrow (a,b) = (0,0)$

But $P(0,0) \neq 0 \Rightarrow$ Every point is smooth.

Example

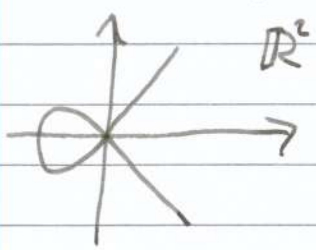
$p(z,u) = u^2 - z^3 - z^2$

$\frac{\partial p}{\partial z} = -3z^2 - 2z$, $\frac{\partial p}{\partial u} = 2u$

$\frac{\partial p}{\partial z}(a,b) = \frac{\partial p}{\partial u}(a,b) = 0$

$\Rightarrow b = 0, \ a = 0, -\frac{2}{3}$

But ~~p(0~~ $p(-\frac{2}{3}, 0) \neq 0 \Rightarrow (0,0)$ is the only singular point



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Examples of Riemann Surfaces

Let $p \in \mathbb{C}[z,u]$ be a non-constant irreducible polynomial and $X = \{(a,b) \in \mathbb{C}^2 \mid p(a,b) = 0\}$ the associated algebraic curve.

X is smooth if $\frac{\partial p}{\partial z}\Big|_{(a,b)} \neq 0$ or $\frac{\partial p}{\partial u}\Big|_{(a,b)} \neq 0 \quad \forall (a,b) \in X$

Theorem

A smooth algebraic curve is (naturally) a Riemann Surface.

Proof

Let p, X be as above. $\mathbb{C}^2$ has the product topology induced by the usual topology on $\mathbb{C}$. Then $X \subseteq \mathbb{C}^2$ inherits a topology from $\mathbb{C}^2$.

Pick $(a,b) \in X$, and WLOG assume that $\frac{\partial p}{\partial z}(a,b) \neq 0$. Then, by the Implicit Function Theorem $\exists\, V, W \subseteq \mathbb{C}$, open, and a holomorphism $\psi: W \to V$ such that $(a,b) \in V \times W$ and for any $(a',b') \in V \times W$, $(a',b') \in X \Leftrightarrow a' = \psi(b')$.

Put $U = V \times W \cap X$, and define $\varphi: U \to W$ by $\varphi((a',b')) = b'$. φ is a homeomorphism: its inverse is given by $b' \mapsto (\psi(b'), b')$. (U, φ) is then a chart near (a,b).

On the other hand, if $\frac{\partial p}{\partial z}(a,b) = 0$, then $\frac{\partial p}{\partial u}(a,b) \neq 0$.

In this case, we can define a chart (U, φ) in a similar way.

By the Implicit Function Theorem, $\exists\, V, W \subseteq \mathbb{C}$, open, and a holomorphic $\psi: V \to W$ such that $(a,b) \in V \times W$ and for any $(a',b') \in V \times W$, $(a',b') \in X \Leftrightarrow b' = \psi(a')$

Again, put $U = V \times W \cap X$ and define $U \to V$ by

$\varphi(a', b') = a'$, so (U, φ) is a chart.

Carrying out this process for all $(a,b) \in X$ determines an atlas $\mathcal{A} = \{(U_i, \varphi_i$

We will show that $\mathcal{A}$ is a holomorphic atlas.

We need to show that the transition functions $\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \varphi_i^{-1}} \varphi_j(U_i \cap U_j$

Assume that (U_i, φ_i), (U_j, φ_j) are defined near (a_i, b_i), (a_j, b_j)

If $\frac{\partial p}{\partial z}(a_i, b_i) \neq 0 \neq \frac{\partial p}{\partial z}(a_j, b_j)$, then the transition function is simply the identity ($\Rightarrow$ holomorphic).

If $\frac{\partial p}{\partial z}(a_i, b_i) = 0 = \frac{\partial p}{\partial z}(a_j, b_j)$, then again, the transition function is the identity ($\Rightarrow$ holomorphic)

If $\frac{\partial p}{\partial z}(a_i, b_i) \neq 0 = \frac{\partial p}{\partial z}(a_j, b_j)$ then the transition function is simply ψ_i (produced by the Implicit Function Theorem, holomorphic)

Finally, if $\frac{\partial p}{\partial z}(a_i, b_i) = 0 \neq \frac{\partial p}{\partial z}(a_j, b_j)$ then the transition function is again φ_i (holomorphic).

$\therefore$ $\mathcal{A}$ defines a complex structure on X, and X is a Riemann Surfac

<u>Remark</u> (Topology of Algebraic Curves)

Let p, X be as above. It is easy to draw a picture for the solutions of p in $\mathbb{R}^2$.

X can be compactified into a new, compact Riemann Surface Y.

It is well known that every compact Riemann Surface looks topologically like

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Then $X = Y \setminus \{\text{finite set}\}$

The number of holes in Y can be determined by the equation p.

There is also a close relation between Topology and Arithmetic (Weil Conjecture).

Example

Let $p = u - z^n$, $n \in \mathbb{N}$, and X its algebraic curve.

X is smooth because $\frac{\partial p}{\partial u}(a,b) = 1 \quad \forall (a,b) \in X$.

Define $f: X \to \mathbb{C}$ by $f(a,b) = b$.

Let $b \in \mathbb{C}$. $f^{-1}(b) = \{(0,0)\}$ if $b = 0$.

$f^{-1}(b) =$ exactly n distinct points $= \{(a,b) \mid a^n = b\}$ if $b \neq 0$.

We could consider the inverse of f as the multi-valued function $b \mapsto f^{-1}(b)$.

Recall the start of the course where we tried to define $z^{1/n}$. X is essentially the Riemann Surface obtained by glueing open sets in the analytic continuation of $z^{1/n}$.

Example (Elliptic Curves)

Let $p = u^2 - z^3 + z$ and X its algebraic curve.

X is smooth: $\frac{\partial p}{\partial z} = -3z^2 + 1$ and $\frac{\partial p}{\partial u} = 2u$

Such curves are called Elliptic Curves (needs more precise formulation

Such curves are extremely important in number theory.

We can compactify X into Y by adding one point.

Topologically, Y looks like a complex torus. 

$Y = \mathbb{C}/\Lambda$ for some lattice Λ.

<u>Remark</u> (Non-Examinable)

Let P_1, P_2 be non-constant irreducible polynomials and X_1, X_2 their algebraic curves. $X_1 = X_2 \Rightarrow P_2 = aP_1$ for some $a \in \mathbb{C}$.

(But we could have $X_1 \overset{\text{biholomorphic}}{\simeq} X_2$, but even $\deg P_1 \neq \deg P_2$)

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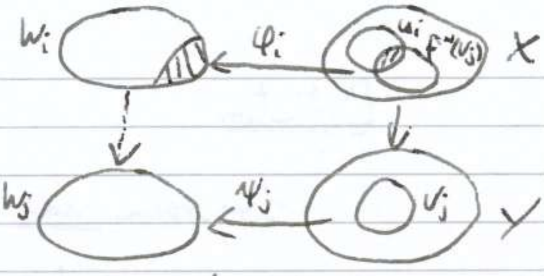


Holomorphic Maps

Definition

Let X, Y be Riemann Surfaces with given complex structures $\mathcal{A} = \{(U_i, \varphi_i)\}$ on X, $\mathcal{B} = \{(V_j, \psi_j)\}$ on Y.

A continuous map $f: X \to Y$ is called holomorphic if for each i, j, the map $\psi_j f \varphi_i^{-1}$ is holomorphic.

The map $\psi_j f \varphi_i^{-1}$ sends $\varphi_i(U_i \cap f^{-1}(V_j))$ to W_α.

Example

1. Suppose that $U \subseteq \mathbb{C}$ is open and $f: U \to \mathbb{C}$ is holomorphic in the sense of complex analysis. $\mathcal{A} = \{(U, \overset{\varphi}{\text{''}} id)\}$, $\mathcal{B} = \{(V \overset{\mathbb{C}}{=}, \psi = id)\}$

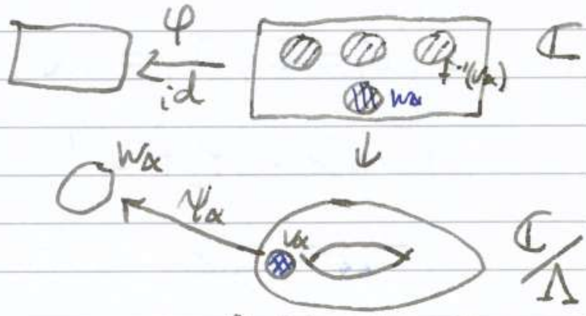
   Then $\psi f \varphi = f \Rightarrow$ holomorphic in the sense of Riemann Surfaces

2. Let $\Lambda \subseteq \mathbb{C}$ be a lattice and let $f: \mathbb{C} \to \mathbb{C}/\Lambda$ be the quotient map, $f(z) = [z]$. A complex torus

   $\mathcal{A} = \{(U = \mathbb{C}, \varphi = id)\}$, $\mathcal{B} = \{(V_\alpha, \psi_\alpha)\}$

   where $\mathcal{B}$ is the complex structure defined in earlier lectures.

   f is continuous by definition of f.



   Pick (V_α, ψ_α)

   The map $\psi_\alpha f \varphi^{-1} : f^{-1} V_\alpha \to W_\alpha$

   is given by a translation on each component of $f^{-1}(V_\alpha)$. In particular these are all holomorphic. So f is holomorphic.

__Definition__

A holomorphic __map__ $f: X \to Y$ is called __biholomorphic__, or a __conformal equivalence__ if f is a homeomorphism and f^{-1} is also holomorphic.

__Definition__

A holomorphic map $f: X \to \mathbb{C}$ is called a holomorphic __function__ on a Riemann Surface X. A holomorphic map $f: X \to \underbrace{\mathbb{C} \cup \{\infty\}}_{\text{Riemann Sphere}}$ is called a __meromorphic__ function on X.

__Example__

Let $U \subseteq \mathbb{C}$ an open set, g a meromorphic function on U in the sense of Complex Analysis. g determines (uniquely) a holomorphic map $U \xrightarrow{f} \mathbb{C} \cup \{\infty\}$. Define

$$f(z) = \begin{cases} g(z) & \text{if } z \text{ is not a pole of } g. \\ \infty & \text{if } z \text{ is a pole of } g. \end{cases}$$

f is continuous (exercise).

$$\mathcal{A} = \{ (U, \underbrace{\varphi = id}_{\text{on } U}) \} \quad , \mathcal{B} = \{ (V_1, \Psi_1), (V_2, \Psi_2) \}$$

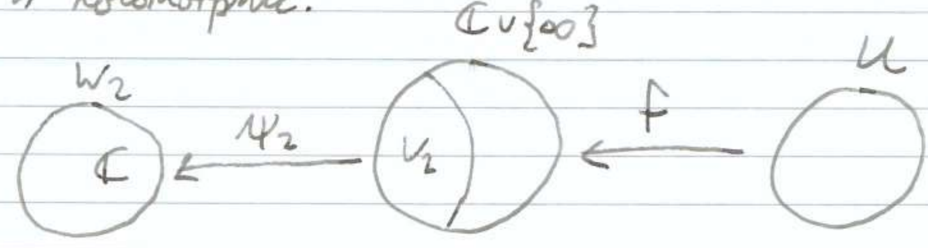
Consider $\Psi_1 f \varphi^{-1} : \underbrace{f^{-1} V_1}_{U \setminus \text{poles of } g} \to W_1 = \mathbb{C}$, $z \mapsto g(z)$

$(V_1 = \mathbb{C}, \Psi_1 = id, W_1 = \mathbb{C})$ $\Rightarrow \Psi_1 f \varphi^{-1}$ is holomorphic

Consider $\Psi_2 f \varphi^{-1} : f^{-1} V_2 = U \setminus \text{zeroes of } g \to W_2 = \mathbb{C}$

$z \mapsto \frac{1}{g(z)}$ $\Rightarrow$ holomorphic.

So f is holomorphic.



31/01/13 Riemann Surfaces 5

Theorem (Open Mapping)

Suppose that $f: X \to Y$ is a non-constant holomorphic map between connected Riemann Surfaces. If $U \subseteq X$ is open then $f(U) \subseteq Y$ is open.

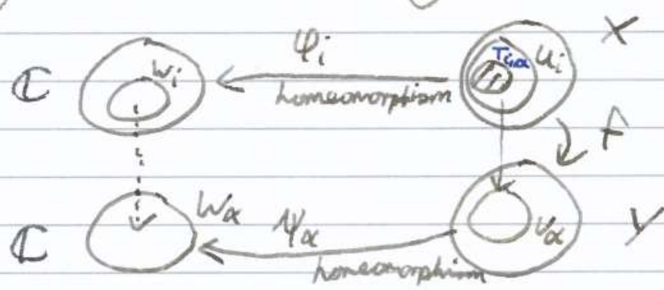
Proof

Let $\mathcal{A} = \{(U_i, \varphi_i)\}$ on X and $\mathcal{B} = \{(V_\alpha, \psi_\alpha)\}$ on Y be complex structures. Let $T_{i,\alpha} = U \cap U_i \cap f^{-1}(V_\alpha)$, open in X.

Now, $f(U) = \bigcup_{i,\alpha} f(T_{i,\alpha})$. It is enough to show that each $f(T_{i,\alpha})$ is open in Y

Since φ_i, ψ_α are homeomorphisms, it is enough to show that the image of $\varphi_i(T_{i,\alpha})$ is open in W_α under the map $\psi_\alpha f \varphi_i^{-1}$

Since f is holomorphic, $\psi_\alpha f \varphi_i^{-1}$ is holomorphic. The desired property of $\varphi_i(T_{i,\alpha})$ follows from the Open Mapping Theorem in classic Complex Analysis.



Corollary

Suppose that X is a connected, compact Riemann Surface. Any holomorphic map $f: X \to \mathbb{C}$ is constant.

Proof

Assume f non-constant.

By the Open Mapping Theorem, $f(X)$ is open $\subseteq \mathbb{C}$.

Since X is compact, $f(X)$ is also compact ($f(X)$ closed in $\mathbb{C}$, $f(X) \neq \mathbb{C}$)

This is impossible because $\mathbb{C}$ is a connected topological space (because $f(X)$ should be a connected component of $\mathbb{C}$).

<u>Example</u>

Let $P \in \mathbb{C}[z, w]$ be a non-zero irreducible polynomial and X its algebraic curve. $X \subseteq \mathbb{C}^2$. Define $f: X \to \mathbb{C}$, $f(a,b) = a$.

We show that f is holomorphic:

f is continuous (exercise). $\mathcal{A} = \{(U_i, \varphi_i)\}$ on X, $\mathcal{B} = \{(\overset{\mathbb{C}}{\overset{\shortparallel}{V}}, \overset{id}{\overset{\shortparallel}{\psi}})\}$ on $\mathbb{C}$

The function $\psi f \varphi_i^{-1}$ is given as

$$\begin{cases} a \mapsto a & \text{if } \varphi_i : U_i \to W_i \text{ is given by } \varphi_i(a,b) = a \\ b \mapsto \mu(b) & \text{if } \varphi_i : U_i \to W_i \text{ is given by } \varphi_i(a,b) = b \\ \text{where } \mu \text{ is given by the Implicit Function Theorem.} & \end{cases}$$

(Caution: Different μ! Remember that the inverse of φ_i is given by $b \mapsto (\mu(b), b)$).

05/02/13

Riemann Surfaces ⑥

Local Representation of Holomorphic Maps

Theorem

Suppose that $f: X \to Y$ is a non-constant map between Riemann Surfaces. Then, locally, f is represented by $z \mapsto z^n$ for some $n \in \mathbb{N}$. More precisely, for every $x \in X$ $(y = f(x))$, we have

$$\begin{tikzcd} D(0,1) \arrow[d, "g"'] & U \ni x \arrow[l, "\sigma"'] \arrow[d, "f"] \\ D(0,1) & V \ni y \arrow[l, "\tau"'] \end{tikzcd}$$

$\tau f = g \sigma$, $g(z) = z^n$, $\sigma(x) = 0$

$\tau(y) = 0$, σ, τ biholomorphic, U, V open

Proof

Let $\mathcal{A} = \{(U_i, \varphi_i)\}$ on X, $\mathcal{B} = \{(V_\alpha, \psi_\alpha)\}$ on Y be holomorphic atlases defining complex structures. $\exists i, \alpha$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "\psi_i"] \\ \mathbb{C} \arrow[r, "h"] & \mathbb{C} \end{tikzcd}$$

$x \in U_i$, $y \in V_\alpha$ $(x \in U_i \cap f^{-1} V_\alpha)$

It is enough to prove the local representation statement for h which is defined on $\varphi_i(U_i \cap f^{-1} V_\alpha)$ as $\psi_\alpha f \varphi_i^{-1}$.

Since f is holomorphic, h is holomorphic by definition.

We replace W_α and $\varphi_i(U_i \cap f^{-1} V_\alpha)$ by open discs $D(x', r)$, $D(y', s)$. By a simple change of variables, we can assume that $x' = 0$, $y' = 0$. Let n be the order of vanishing of h at 0. We can write $h(z) = z^n p(z)$ where p is holomorphic and $p(0) \neq 0$. After replacing r with a smaller number if necessary we can assume that $P(D(0, r)) \subseteq \{te^{i\theta} \mid \frac{k\pi}{2} < \theta < \pi + \frac{k\pi}{2}\} = H$ where $t \in \mathbb{R}_{>0}$. Recall that we have a holomorphic function

$\alpha : H \to \mathbb{C}$ given by ~~composing~~ $\alpha(te^{i\theta}) = t^{1/n} e^{i\theta/n}$. Composing p, α gives a holomorphic function $q : D(0,r) \to \mathbb{C}$ such that $p(z) = q(z)^n$. Next we write $h \stackrel{\text{def}}{=} z^n p(z) = z^n (q(z))^n = (zq(z))^n$.
Now $zq(z)$ vanishes at 0 but $(zq)'(0) \neq 0$ ($\Rightarrow$ locally, zq defines a biholomorphic map).
↳ or $h(z)$ would have higher vanishing order

Again, maybe after taking a smaller r, we can say that $zq(z)$ maps $D(0,r)$ onto some open set $A \subseteq \mathbb{C}$ such that $D(0,r) \xrightarrow{zq(z)} A$ is biholomorphic.

Then we have a commutative diagram

$$\begin{tikzcd} D(0,r) \arrow[r, "\text{biholomorphic}", "zq"'] \arrow[d, "h"] & A \arrow[dl, "\omega^n"] \\ D(0,s) & \end{tikzcd}$$

So it is enough to focus on $A \to D(0,s)$, $\omega \mapsto \omega^n$

We could replace A by some open disc, denoted again by $D(0,r')$. First, we could replace s with r'^n. Finally, note that the diagram is commutative:

$$\begin{tikzcd} D(0,1) \arrow[d, "z \mapsto z^n"'] & D(0,r') \arrow[l, "\text{biholomorphic}"', "w/r'"] \arrow[d] \\ D(0,1) & D(0,r'^n) \arrow[l, "\text{biholomorphic}"', "u/r'^n"] \end{tikzcd}$$

$\square$

<u>Definition</u>

Let $f : X \to Y$ be a non-constant holomorphic map between Riemann Surfaces. Pick $x \in X$. We define the <u>branching order</u> of f at x, denoted $\nu_f(x)$, to be $= n$, when f is locally, near x, represented by $z \mapsto z^n$. If $n > 1$, we call x a <u>ramification point</u> of f and call $f(x)$ a <u>branch point</u> of f.

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Example

1. Let $f : D(0,1) \to D(0,1)$ be given by $f(z) = z^n$. By definition, $v_f(0) = n$. But $v_f(x) = 1$ for any $x \neq 0$ because $f'(x) \neq 0$.

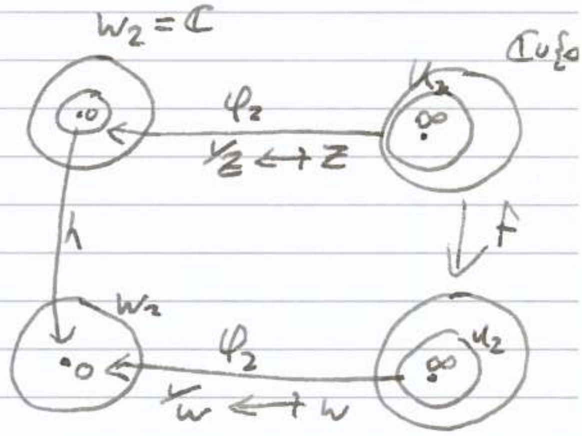
2. Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be the function $f(z) = \begin{cases} z^4 + 2z^2 + 1 & z \neq \infty \\ \infty & z = \infty \end{cases}$

We will see that f is holomorphic. We calculate branching orders. f is holomorphic on $\mathbb{C}$ and the ramification points of f in $\mathbb{C}$ are given by the roots of $f'(z) = 4z^3 + 4z = 4z(z^2+1)$.

So, the points are $0, i, -i$. Now:

$v_f(0) = 2$, since $f''(0) \neq 0$. Similarly, $v_f(i) = v_f(-i) = 2$.

It remains to treat f near ∞.



$\mathcal{A} = \{ (\underset{\mathbb{C}}{U_1}, \underset{id}{\varphi_1}), (\underset{(\mathbb{C}\setminus\{0\})\cup\{\infty\}}{U_2}, \underset{1/z}{\varphi_2}) \}$

h is given (near 0) by the formula

$h(z) = \frac{1}{f(1/z)} = \frac{z^4}{1 + 2z^2 + z^4}$

Since $\frac{1}{1+2z^2+z^4}$ does not vanish at 0,

$v_h(0) = v_f(\infty) = 4$

Example

[illegible]

[illegible]

[illegible]

we will see that f is holomorphic [illegible]

f is holomorphic on U and the configuration [illegible]

[illegible]

[illegible]

[illegible] Similarly, [illegible]

[illegible] ∞

$A = \{$ [illegible] $\}$

[illegible]

[illegible]

[illegible]

$V_A(\infty) =$ [illegible]

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Riemann Surfaces 7

Definition

A <u>rational function</u> is of the form p/q where p, q are polynomials in $\mathbb{C}[z]$, where we assume that p, q are never 0 at the same time. Thus, rational functions p_1/q_1, p_2/q_2 are the same if $p_1 q_2 = p_2 q_1$

Definition

Suppose that p/q is a rational function. We can define a map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ as follows: first we can assume that p, q have no common factors (every polynomial is a product of polynomials of degree 1.

$$f(a) = \begin{cases} p(a)/q(a) & a \neq \infty,\ q(a) \neq 0 \\ \infty & a \neq \infty,\ q(a) = 0 \\ \lim\limits_{|z| \to \infty} \dfrac{p(z)}{q(z)} & a = \infty \end{cases}$$

Theorem

There is a 1-1 correspondence $\{\text{rational functions}\} \leftrightarrow \{\text{holomorphic maps } \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}\}$

Proof

Let p/q be a rational function and $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ the associated map. We can assume that p, q have no common factors. We may also assume that f is not constant (otherwise f is trivially a holomorphic map).

Remember the atlas $\mathcal{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ which gives the complex structure on $\mathbb{C} \cup \{\infty\}$. We need to look at the following functions

$\varphi_1 f \varphi_1^{-1} : \varphi_1(U_1 \cap f^{-1} U_1) \to W_1$ given by $\dfrac{p(z)}{q(z)}$

$\varphi_2 f \varphi_1^{-1} : \varphi_1(U_1 \cap f^{-1} U_2) \to W_2$ given by $\dfrac{q(z)}{p(z)}$

$\varphi_1 f \varphi_2^{-1} : \varphi_2(U_2 \cap f^{-1} U_1) \to W_1$ given by $\dfrac{p(1/z)}{q(1/z)}$

$\varphi_2 f \varphi_2^{-1} : \varphi_2(U_2 \cap f^{-1}U_2) \to W_2$ given by $\frac{q(\frac{1}{z})}{p(\frac{1}{z})}$

Then, it is easy to see that all of these functions are holomorphic. So f is a holomorphic map.

Conversely, assume that we are given a holomorphic map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. If f is constant then it is more or less trivial to see that f is some rational function. So assume that f is not constant.

Since $\mathbb{C} \cup \{\infty\}$ is compact, $f^{-1}\{0, \infty\}$ is finite.

Since f is holomorphic the function $g := \varphi_1 f \varphi_1^{-1} : \varphi_1(\overset{\mathbb{C}\setminus f^{-1}\{\infty\}}{\overset{\|}{U_1}} \cap f^{-1}U_1) \to \overset{\mathbb{C}}{\overset{\|}{W_1}}$ is holomorphic. We show that g extends to $\mathbb{C}$ as a meromorphic function.

Suppose that $f^{-1}\{\infty\} \cap U_1 = \{a_1, \dots, a_n\}$. It is enough to show that $\frac{1}{g}$ is holomorphic near each a_i.

The function $\varphi_2 f \varphi_1^{-1} : \varphi_1(U_1 \cap f^{-1}U_2) \to W_2$ is holomorphic because f is holomorphic. In fact, $\varphi_2 f \varphi_1^{-1} = \frac{1}{g}$ on $\varphi_1(U_1 \cap f^{-1}U_1 \cap f^{-1}U_2)$.

This means that $\frac{1}{g}$ is holomorphic near each a_i $(\in \varphi_1(U_1 \cap f^{-1}U_2))$

Therefore, g extends to $\mathbb{C}$ as a meromorphic function.

$\exists\, m_i \in \mathbb{N}$ such that $h(z) := (z-a_1)^{m_1} \cdots (z-a_n)^{m_n} g(z)$ is a holomorphic function. So, we can write h as a power series $h(z) = \sum_{i=0}^{\infty} c_i z^i$, $c_i \in \mathbb{C}$.

Our goal is to show that $h(z)$ is a polynomial. This in turn follows from showing that $h(\frac{1}{z})$ is meromorphic.

(note that $h(1/z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$)

In fact, it is enough to show that $g(\frac{1}{z})$ is meromorphic on $\mathbb{C}$.
Again, it is enough to show that $\frac{1}{g(\frac{1}{z})}$ is a meromorphic function.
Since f is holomorphic, the function $\varphi_2 f \varphi_2^{-1} : \varphi_2(U_2 \cap f^{-1}U_2) \to W_2$ is holomorphic. This function is the same as $\frac{1}{g(\frac{1}{z})}$ on $\varphi_2(U_1 \cap U_2 \cap f^{-1}U_1 \cap f^{-1}U_2)$.
On the other hand, $\varphi_2 f \varphi_2^{-1}$ extends to $\mathbb{C}$ as a meromorphic function (by the same reasoning that we applied to $\varphi_1 f \varphi_1^{-1}$).
So, $\frac{1}{g(\frac{1}{z})}$ extends to $\mathbb{C}$ as a meromorphic function. Hence, $h(\frac{1}{z})$ is meromorphic on $\mathbb{C}$. So $h(\frac{1}{z}) = \sum_{i=0}^{\infty} c_i \frac{1}{z^i}$ is meromorphic.
Thus, for all but finitely many i, $c_i = 0$. So $h(z)$ is a polynomial and $g(z) = \frac{h(z)}{(z-a_1)^{m_1} \cdots (z-a_n)^{m_n}}$ is a rational function.
The associated holomorphic map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is f. $\square$

Example

Let p/q be a rational function, and $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ the associated holomorphic map. Assume that p, q have no common factors, and that f is not constant.

$$\deg(p/q) := \max\{\deg(p), \deg(q)\}$$

Assume that $\deg(p/q) = 1$. Then, $\deg(p), \deg(q) \leq 1$ with equality for at least one of them. We can write

$$\frac{p(z)}{q(z)} = \frac{az+b}{cz+d} \overset{\text{notation}}{=} A \cdot z \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since p, q have no common factor, $\det A \neq 0$.

This implies that f is 1-1 because
$\frac{p(z)}{q(z)} = A \cdot z = A \cdot w = \frac{p(w)}{q(w)} \Rightarrow A^{-1}Az = A^{-1}Aw, \ z = w$
Thus f is biholomorphic because by the Local Representation Theorem f is locally given by $z \mapsto z$.
Conversely, assume that f is biholomorphic. ~~By the previous theorem f is the~~ We show that $\deg(p/q) = 1$. We can write
$p(z) = a_n z^n + \ldots + a_0, \ q(z) = b_m z^m + \ldots + b_0$ where
$a_n \neq 0 \neq b_m$. If $\deg(p/q) > 1$, then $n > 1$ or $m > 1$.
Then $f^{-1}\{\alpha\}$ has more than one point for some $\alpha \in \mathbb{C}$. But this contradicts our assumption that f is biholomorphic.

Riemann Surfaces ⑧

Theorem

Let $F: X \to Y$ be a non-constant holomorphic map between compact, connected Riemann Surfaces.
Then the function $d: Y \to \mathbb{Z}$, defined by $d(y) = \sum_{f(x)=y} v_f(x)$ is constant, i.e. $d(y)$ is independent of y. We call the value of d the degree of f, denoted by $\deg(f)$.

Proof

Since X, Y are compact and connected, $F(X) = Y$.
(X open, Open Mapping $\Rightarrow f(X)$ open, since f non-constant, X compact $\Rightarrow f(x)$ compact. Riemann Surfaces are Hausdorff. Compact subspace of a Hausdorff space closed $\Rightarrow f(x)$ closed. $Y = f(x) \cup Y \setminus f(x)$, connectivity $\Rightarrow Y = f(x)$)
Pick $y \in Y$. Again by compactness, $f^{-1}(y)$ is finite, say $\{x_1, \dots, x_l\}$
(Let X, Y have complex structures $\{U_i, \varphi_i\}, \{V_\alpha, \Psi_\alpha\}$. For each i, α, $\Psi_\alpha f \varphi_i^{-1}$ is holomorphic. We can assume WLOG that φ_i, Ψ_α are centred on 0, and $\Psi_\alpha f \varphi_i^{-1}$ has an isolated zero at 0 since it is non-constant. So points x such that $f(x) = y$ are isolated. $f^{-1}(y)$ is also closed, and compact under the subspace topology since X is compact.
So $f^{-1}(y)$ is finite since each $x \in f^{-1}(y)$ can be separated from the others by an open set, so that x is open in $f^{-1}(y)$, and finitely many such x cover $f^{-1}(y)$)

By the local representation theorem, ~~$f^{-1}(y)$ is finite, say $\{x_1$,~~ for each x_i, $\exists$ open sets $x_i \in U_i \subseteq X$, $y \in V_i \subseteq Y$ such that $f(U_i) = V_i$ and $f|_{U_i}$ is represented by $z \mapsto z^{n_i}$.

If we choose U_i small enough, we can make sure that $U_i \cap U_j = \emptyset$ for $i \neq j$. We will show that there exists an open $y \in V' \subseteq \bigcap_{i=1}^{l} V_i$, such that $f^{-1}(V) \subseteq \bigcup_{i=1}^{l} U_i$.

To see this, let $S = X \setminus \bigcup_{i=1}^{l} U_i$ which is compact. So $f(S)$ is compact, hence closed. (Y Hausdorff)

By our previous assumptions, $y \notin f(S)$. So $V_1 \cap V_2 \cap \dots \cap V_l \cap (Y \setminus f$ is open and it contains y. Take any open set V' inside this set which contains y. Then $f^{-1}(V') \cap S = \emptyset \Rightarrow f^{-1}(V') \subseteq \bigcup_{i=1}^{l} U_i$

We show that d is constant on V'. By definition,

$d(y) = \sum_{i=1}^{l} v_f(x_i)$.

On the other hand, $v_f(x_i) = n_i$, so $d(y) = \sum_{i=1}^{l} n_i$

Now, if $y' \in V$, $y' \neq y$, then $f^{-1}\{y'\} \cap U_i$ has precisely n_i points and the branching order of f at such points is exactly 1.

So $d(y') = \#$ points in $f^{-1}\{y'\} = \sum_{i=1}^{l} n_i$.

Thus, d is constant on V'.

Apply the same argument near all points y of Y. Then, by compactness, $\exists$ open sets $V_1', \dots, V_m'$ such that $Y = \bigcup_{i=1}^{m} V_i$.

If $V_j' \cap V_k' \neq \emptyset$, then d takes the same values on V_j' and V_k'.

Riemann Surfaces ⑧

So if d is not constant, then Y can be written as a finite union of finitely many disjoint open sets, contradicting connectedness of Y $\square$

<u>Example</u>

Suppose that p/q is a rational function on the Riemann Sphere, and assume that $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is the corresponding holomorphic map. In addition, assume that f is not constant.

We will show that $\deg(f) = \deg(p/q)$.

Remove all common factors, then $\deg(p/q) = \max\{\deg(p), \deg(q)\}$.

By the theorem, $\deg(f) = \sum_{f(x)=0} V_f(x)$

Assume that $f^{-1}\{0\} \cap \mathbb{C} = \{a_1, \dots, a_l\}$

So $a_1, \dots, a_l$ are the zeros of p. $V_p(a_i)$ = order of vanishing of p at a_i.

i) Assume $\deg p \geq \deg q$. In this case $\infty \notin f^{-1}\{0\}$.

So $\deg(f) = \sum_{i=1}^{l} V_f(a_i) = \deg p$

ii) Now assume instead that $\deg p < \deg q$. Then $\infty \in f^{-1}\{0\}$.

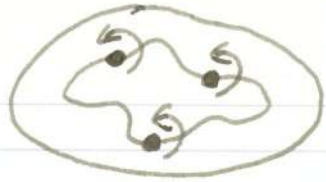
We calculate $V_f(\infty)$ by looking at the appropriate coordinate charts.

This is the same as the vanishing order of $\varphi_1 f \varphi_2^{-1}$ at 0.

$$\varphi_1 f \varphi_2^{-1} = \frac{p(1/z)}{q(1/z)} = z^{\deg q - \deg p} \frac{p(1/z)\, z^{\deg p}}{q(1/z)\, z^{\deg q}} \qquad (*)$$

Since $(*)$ does not vanish at 0, $V_{\varphi_1 f \varphi_2^{-1}}(0) = V_f(\infty) = \deg q - \deg p$

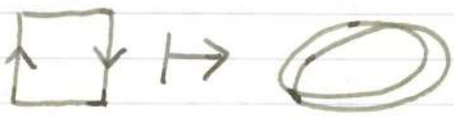
Finally, $\deg(f) = \sum_{i=1}^{l} V_f(a_i) + V_f(\infty) = \deg q$



Topology of Surfaces (Informal)

A (topological) surface X is orientable if, when we choose an orientation near $x \in X$, and travel along any loop, the orientation has not changed when we return to x.

Examples

i)  Möbius Strip, non-orientable

ii)  Klein Bottle

Fact

Every Riemann Surface is orientable.

Theorem

Every compact, connected, orientable surface looks like "a donut with g holes". 

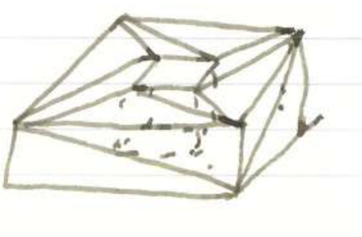
Triangulation

Triangulation of a compact, connected surface is covering it by "shapes which look like triangles". We assume that edges meet in vertices, and that vertices are the starting or ending of some edge.

Fact

Every compact, connected, orientable surface has a triangulation (evident from the theorem above).

Example

 $\underset{\text{homeomorphic}}{\cong}$  etc

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Riemann Surfaces 9

The Euler Characteristic

Remember a triangulation of a compact, connected Riemann Surface.

We have finitely many vertices, edges, and faces.

A refinement of a triangulation is adding new vertices, edges, and fac[es] so that the result is also a triangulation.

Definition

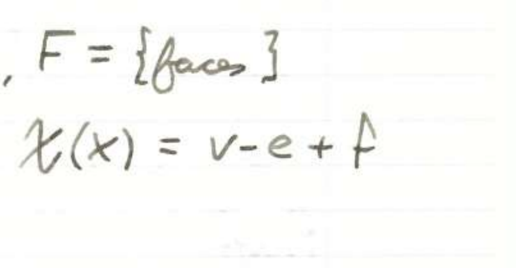
Assume that $V = \{\text{vertices}\}$, $E = \{\text{edges}\}$, $F = \{\text{faces}\}$

The Euler Characteristic of X is defined as $\chi(X) = v - e + f$ where $v = |V|$, $e = |E|$, $f = |F|$.

Remark

$\chi(X)$ is independent of the triangulation. This follows from the fact that given two triangulations, they can be refined to one common triangulation (apply induction)

Example



1. $X = \mathbb{C} \cup \{\infty\}$ = the Riemann Sphere

   X is homeomorphic to the boundary of a tetrahedron.

   π defines a triangulation on X. We have 4 vertices, 6 edges, 4 faces

   Then $\chi(X) = 4 - 6 + 4 = 2$

2. Let X be a complex torus.

   As in the picture, π defines a triangulation of X, and a careful calculation shows that $\chi(X) = 0$.

Definition

We define the genus of X as $\text{genus}(X) = 1 - \frac{1}{2}\chi(X)$

$X = \mathbb{C} \cup \{\infty\} \Rightarrow \text{genus}(X) = 0$, $X = $ complex torus $\Rightarrow \text{genus}(X) =$

Fact

$\text{genus}(X) = \#$ holes in X

Theorem

Let $f: X \to Y$ be a non-constant holomorphic map between compact, connected Riemann Surfaces. Let $d = \deg(f)$. Then we have

$$\chi(X) = d\,\chi(Y) - \sum_{x \in X} (v_f(x) - 1)$$

$$\text{genus}(X) = d(\text{genus}(Y) - 1) + 1 + \frac{1}{2} \sum_{x \in X} (v_f(x) - 1)$$

Proof

Pick a triangulation on Y given by V, E, F. After a refinement of the triangulation, if necessary, we can assume that we have:

$V = \{y_1, \dots, y_m\}$, $f^{-1} y_i = \{x_{j,i}\}_j$

$\exists$ open $W_i \subseteq Y$ such that $y_i \in W_i$

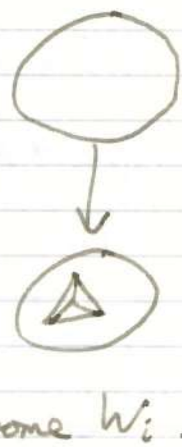
$f^{-1} W_i =$ disjoint union of $U_{j,i}$

$x_{j,i} \in U_{j,i}$

Recall the proofs of the theorem on the degree of a holomorphic map.

$f: U_{j,i} \to W_i$ is "represented by $z \mapsto z^{n_{j,i}}$

$$\begin{tikzcd}
z \arrow[d, mapsto] & D(0,1) \arrow[d] & \tilde{U}_{j,i} \arrow[l, "\text{bihol.}"'] \arrow[d] \arrow[r, hook, "\text{open}"'] & U_{j,i} \\
z^{n_{j,i}} & D(0,1) & \tilde{W}_i \arrow[l, "\text{bihol.}"] \arrow[r, hook, "\text{open}"'] & W_i
\end{tikzcd}$$



X

$f^{-1}(V)$ contains all the ramification points of X.

Each edge (similarly, each face) is contained in some W_i.

We will define a triangulation on X.

$V' = \{\text{vertices on } X\} = f^{-1} V$.

By the theorem on the degree of a holomorphic map, for each $y \in$

$d = \sum_{x \in f^{-1}\{y\}} v_f(x) \Rightarrow$ # points in $f^{-1}\{y\} = d - \sum_{x \in f^{-1}\{y\}} (v_f(x) - 1)$

If $f^{-1}\{y\}$ contains no ramification, then # points in $f^{-1}\{y\} = d$.

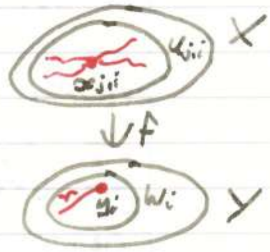
Applying the formula to the points in V, we get

points in $V' = d$ (# points in V) $- \sum_{x \in X} (v_f(x) - 1)$

Now we define edges on X.

Pick $y_i \in V$, and an edge $\gamma \in E$ such that γ starts or ends at y_i.

We could assume that $\gamma \subseteq W_i$.



The inverse $f^{-1}\gamma$ consists of $n_{j,i} = v_f(x_{j,i})$

edges sharing the vertex $x_{j,i}$, inside $U_{j,i}$.

In total, $f^{-1}\gamma$ gives $\sum_j n_{j,i} = d$

Similarly, for each face δ, we get d faces in $f^{-1}\delta$ on X.

Putting all these edges and faces together, we get a triangulation on

using V', E', F'. Thus, we have

$\chi(X) =$ (# elements in V') $-$ (# elements of E') $+$ (# elements of F')

$= d$(# elements of V) $- \sum_{x \in X} (v_f(x) - 1) - d$ (# elements of E)

$\quad + d$ (# elements of F)

$= d\,\chi(Y) - \sum_{x \in X} (v_f(x) - 1)$

This gives the formula for the genus.

These two formulae are called the <u>Riemann-Hurwitz formulae</u>

<u>Example</u>

Let $f: X = \mathbb{C} \cup \{\infty\} \to X$ be a non-constant holomorphic

map of degree $d > 1$.

We can show that f has at least one ramification point.

We know that $\chi(X) = 2$. Riemann-Hurwitz ~~ge~~ gives

$$2 = 2d - \sum_{x \in X} (v_f(x) - 1) \qquad (*)$$

Since $d > 1$, $\sum_{x \in X} (v_f(x) - 1) \neq 0 \Rightarrow \exists x \in X$ such that $v_f(x) >$

$\Rightarrow x$ is a ramification point.

We can actually show that f has at least two ramification points. Let r be the number of such points. For any such point x,

$v_f(x) - 1 \leq d - 1$.

$\sum_{x \in X} (v_f(x) - 1) \leq r(d-1)$.

$(*) \Rightarrow \quad r(d-1) \geq 2(d-1) \Rightarrow r \geq 2$

Riemann Surfaces ⑩

Analytic Continuation and the space of Germs

Definition

Let X, Y be connected Riemann Surfaces. A function element (U, f) (of X into Y) consisting of a connected open set $U \subseteq X$ and a holomorphic map $f: U \to Y$. A direct analytic continuation is another function element (V, g) such that $U \cap V \neq \emptyset$ and $f = g$ on $U \cap V$. An analytic continuation of (U, f), say (V, g), is obtained by a sequence of direct analytic continuations: $\exists$ function elements (U_i, f_i), $i = 1, 2, \dots, n$ such that $(U, f) = (U_1, f_1)$, $(V, g) = (U_n, f_n)$, and (U_{i+1}, f_{i+1}) is a direct analytic continuation of (U_i, f_i) for $1 \leq i < n$.

Analytic continuation defines an equivalence relation on the set of function element. A complete function element (of X into Y) is one of the equivalence classes.

Definition

Let X, Y be connected Riemann Surfaces, $x \in X$. We say that (U, f), (V, g) are equivalent near x if $x \in U \cap V$ and $f = g$ near x, i.e. $\exists$ open $W \ni x$, $W \subseteq U \cap V$ such that $f|_W = g|_W$.

This again defines an equivalence relation. The class of (U, f) is denoted by $[x, f]$. Such a class is called a germ. The set of all such germs is denoted by G, and called the space of germs (of X into Y).

Theorem

G is a Riemann Surface in a natural way.

Proof

We first define a topology on G in a natural way, specifying a base

Recall

A base for a topology on a set T is a family $\{U_i\}$ of subsets of T such that $\phi \in \{U_i\}$,
$U_1, U_2 \in \{U_i\}$, $t \in U_i \cap U_j \Rightarrow \exists U_3 \in \{U_i\}$ such that
$t \in U_3 \subseteq U_1 \cap U_2$, and $T = \bigcup_i U_i$

The base consists of all sets $[U, f]$ when (U, f) is a function element. $[U, f] = \{[x, f] \mid x \in U\}$

This gives a base:

- It contains ϕ as an element
- $G = \bigcup_{(U,f)} [U, f]$
- If $[x, f] \in [U, f] \cap [V, g] \Rightarrow g$ is also defined near x, and $\exists W \ni x$ such that $f|_W = g|_W$
  $\Rightarrow [x, f] \in [W, f] \subseteq [U, f] \cap [V, g]$

This defines a topology on G.

In fact, this topology is Hausdorff. Pick $[x, f] \neq [y, g] \in G$

We have two cases.

Basic Set X is a Riemann Surface $\therefore$ Hausdorff

i) $x \neq y$. Then, $\exists U, V$ such that $x \in U$, $y \in V$, $U \cap V = \phi$
$[x, f] \in [U, f]$, $[y, g] \in [V, g]$.

So, $[U,f] \cap [V,g] = \emptyset$, giving the Hausdorff property.

ii) $x = y$. This means that f, g are not equal on any neighbourhood of $x = y$. If W is any connected? open set such that $x \in W$, and f, g are defined on W, then $[W,f] \cap [W,g] = \emptyset$. Since $[x,f] \in [W,f]$, $[y,g] \in [W,g] \Rightarrow$ Hausdorff property

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Next, we define a map $\pi : G \to X$ by $\pi([x,f]) = x$.

We show that π is continuous. Pick any open set $V \subseteq X$ and pick $[x,f] \in \pi^{-1}V$. Here $x \in V$ and f is defined on some open $W \ni x$, $W \subseteq V$. In particular, $[W,f] \subseteq \pi^{-1}V$. Moreover, $[x,f] \in [W,f]$. In other words, every point of $\pi^{-1}V$ has an open neighbourhood contained in $\pi^{-1}V$. This implies that $\pi^{-1}V$ is open $\Rightarrow$ π continuous.

In fact, $\pi|_{[U,f]}\ [U,f] \to U$ is a homeomorphism for any (U,f). It is clear that $\pi|$ is 1-1. It is enough to show that $\pi|$ (open) = open.

If $[W,f] \subseteq [U,f]$ $\Rightarrow \pi|_{[U,f]}([W,f]) = W$, open.

Since the sets $[W,f]$ generate the topology on $[U,f]$, i.e. every open set $[U,f]$ is a union of such open sets, we deduce that $\pi|$(open) = open $\Rightarrow$ $\pi|$ is a homeomorphism (note that $\pi|^{-1}$ is also continuous).

Finally, we show that G has a natural complex structure.

Let $\mathcal{A} = \{(U_i, \varphi_i)\}$ be a holomorphic atlas defining the complex structure of X. We define $\mathcal{B} = \{([V_\alpha, f_\alpha], \Psi_\alpha)\}$ when $\forall \alpha$, (V_α, f_α) is a function element (of X into Y), and $\exists i$ such that $V_\alpha \subseteq U_i$ and $\Psi_\alpha = \varphi_i \pi$

$$[V_\alpha, f_\alpha] \xrightarrow{\pi|} V_\alpha \subseteq U_i \xrightarrow{\varphi_i} W_i \subseteq \mathbb{C}$$

This is holomorphic because the transition functions of $\mathcal{B}$ are restrictions of the transition functions of $\mathcal{A}$.

<u>Remark</u>

π is holomorphic with respect to the complex structures on G, X.

<u>Theorem</u>

Let X, Y be connected Riemann Surfaces. Then, $\exists$ 1-1 correspondence

$$\left\{\begin{array}{c}\text{complete holomorphic functions}\\ X \to Y\end{array}\right\} \leftrightarrow \{\text{Connected components of } G\}$$

<u>Proof</u>

For any function element (U, f), define $\mathcal{C}_{(U,f)} = \bigcup_{\substack{(V,g) \text{ is a continuation}\\ \text{of } (U,f)}} [V, g]$

This defines an open set $\mathcal{C}_{(U,f)} \subseteq G$, but it is also connected.

If we pick $[x, f], [y, g] \in \mathcal{C}_{(U,f)} \Rightarrow \exists$ a sequence of (U_i, f_i) such that $(U, f) = (U_1, f_1)$, $(U_n, f_n) = (V, g)$, and (U_{i+1}, f_{i+1}) is a direct analytic continuation of (U_i, f_i),

each U_i open, connected, $U_i \cap U_{i+1} \neq \emptyset$

$x \in U$, $y \in V$. The <u>union of all U_i is a connected open set</u>; similarly, the ~~sequence~~ union $\bigcup [U_i, f_i]$ is a connected open set containing $[x, f]$, $[y, g] \Rightarrow \mathcal{C}_{(U,f)}$ is connected.

By definition, if (V, g) is an analytic continuation of (U, f), then $\mathcal{C}_{(U,f)} = \mathcal{C}_{(V,g)}$. Also, clearly $G = \bigcup_{(U,f)} \mathcal{C}_{(U,f)}$

Finally, it is enough to show that if (V, g) is <u>not</u> an analytic continuation of (U, f)

$\Rightarrow \mathcal{C}_{(U,f)} \cap \mathcal{C}_{(V,g)} = \emptyset$

If not, $\exists x \in X$, and some open, connected $W \ni x$ such that $[W, h] \subseteq \mathcal{C}_{(U,f)} \cap \mathcal{C}_{(V,g)}$ and such that (W, h) is an analytic continuation of both (U, f), (V, g) $\divideontimes$

So the sets $\mathcal{C}_{(U,f)}$ correspond in a 1-1 fashion to the connected components of G.

Riemann Surfaces (11)

Covering Spaces and the Monodromy Theorem

Definition

1. A surjective map, holomorphic, between connected Riemann Surfaces, is a branched covering $f : X \to Y$
2. A continuous surjective map $f : X \to Y$ between connected topological surfaces is a covering in the sense of (complex) analysis if $\forall x \in X$, $\exists$ open $U \subseteq X$, $U \ni x$ such that $f|_U : U \to f(U)$ is a homeomorphism.
3. A continuous surjective map $f : X \to Y$ between connected topological surfaces is called a covering in the sense of topology if $\forall y \in Y$, $\exists$ open $V \subseteq Y$, $V \ni y$ such that f restricted to any connected component of $f^{-1}(V)$ gives a homeomorphism onto V. ("$f^{-1}(V)$ is a disjoint union of copies of V.

Example

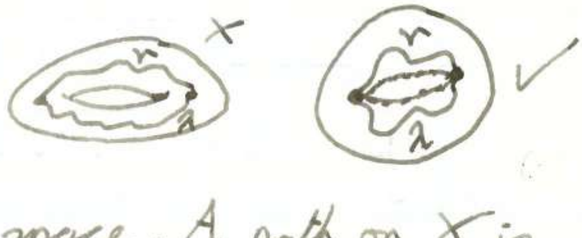
a) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$. This is a branched covering. f is not a covering in the sense of analysis or topology.

b) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $f(z) = z^n$. Then f is a covering in the sense of topology.

c) $f : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{0\}$, $f(z) = z^n$, $n > 1$. This f is then a covering in the sense of analysis but not the sense of topology.

d) Suppose that $\Lambda \subseteq \mathbb{C}$ is a lattice and $f : \mathbb{C} \to \mathbb{C} \setminus \Lambda$ is the quotient map. f is a cover in the sense of topology.

a) Suppose that $h : X \to Y$ is a (non-)constant holomorphic map between connected Riemann Surfaces. Let (U, f) be a function element (of X into Y). Let F be the connected component of the space of germs $\mathcal{G}$ and $\pi : F \to X$ the covering map. If $W = \pi(F)$, then $\pi : F \to W$ is a covering in the sense of complex analysis. (Remember that π is locally a homeomorphism).



<u>Definition</u> (Paths and Homotopy)

Suppose that X is a topological space. A <u>path</u> on X is a continuous map $\gamma : [0,1] \to X$. Two paths γ, λ are <u>homotopic</u> if:

i) $\gamma(0) = \lambda(0)$, $\gamma(1) = \lambda(1)$

ii) $\exists$ continuous $H : [0,1] \times [0,1] \to X$ such that

$$H(s,1) = \gamma(s) \ \forall s \qquad H(s,0) = \lambda(s) \ \forall s$$

$$H(0,t) = \gamma(0) = \lambda(0) \ \forall t \qquad H(1,t) = \gamma(1) = \lambda(1) \ \forall t$$

H is called the homotopy between γ, λ.

We say that X is <u>simply connected</u> if any two paths γ, λ with $\gamma(0) = \lambda(0)$, $\gamma(1) = \lambda(1)$ are homotopic (i.e. any closed path γ, $\gamma(0) = \gamma(1)$ is homotopic to the <u>constant</u> path λ)

<u>Examples</u>

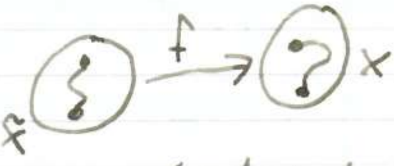
a) Any open disc $\subseteq \mathbb{C}$ is simply connected.

b) $\mathbb{C}$ is simply connected.

c) The Riemann Sphere is simply connected.

d) A punctured open disc is not simply connected.

e) A complex torus is not simply connected.

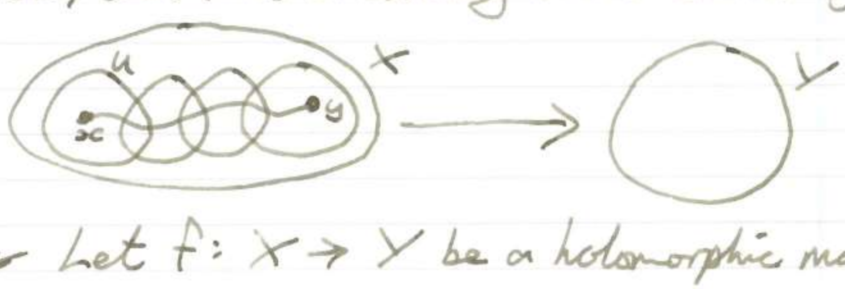


<u>Definition</u>

Let $f: \tilde{X} \to X$ be a continuous map between two topological spaces.
Suppose that γ is a path on X, $f(\tilde{x}) = x$, $\gamma(0) = x$.
We say that γ can be <u>lifted</u> to $\tilde{X}$ to a path starting at $\tilde{x}$ if $\exists$ a path $\tilde{\gamma}$ on $\tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{x}$, $\gamma = f \circ \tilde{\gamma}$
Now assume that γ, λ are two homotopic paths starting at X (given by H). We say that this homotopy can be lifted to $\tilde{X}$ starting at $\tilde{x}$ if γ, λ can be lifted to $\tilde{\gamma}, \tilde{\lambda}$, starting at $\tilde{x}$, if $\exists$ a homotopy $\tilde{H}$ between $\tilde{\gamma}, \tilde{\lambda}$ such that $H = f \circ \tilde{H}$.

<u>Fact</u>

Let $f: \tilde{X} \to X$ be a covering in the sense of complex analysis, $x \in X$, $f(\tilde{x}) = x$. If every path on X starting at x can be lifted to $\tilde{x}$, then f is a covering in the sense of topology.



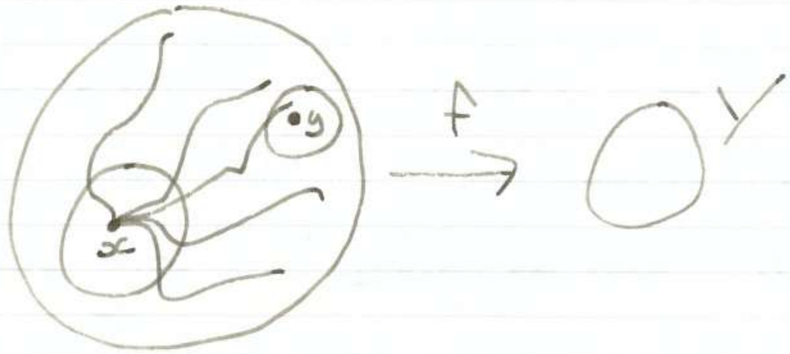
<u>Definition</u>

~~Let X be a conne~~ Let $f: X \to Y$ be a holomorphic map between connected Riemann Surfaces, (U, g) a function element, $x \in U$
Suppose that γ is a path on X starting at x. We say that (U, g) can be analytically continued along γ if γ can be lifted to a path in $\mathcal{G}$ starting at $[x, g]$.

space of Germs

Suppose that (U, G) can be analytically continued along every path starting at x.

$\Rightarrow$ any analytic continuation gives the same germ near y.



Riemann Surfaces (12)

Theorem (Monodromy)

Assume that $\pi : \tilde{X} \to X$ is a covering in the sense of topology, and $\pi(\tilde{x}) = x$. Then

i) Every path γ starting at x can be lifted uniquely to a path starting at $\tilde{x} \in \tilde{X}$

ii) If γ, λ are two homotopic paths starting at x (homotopy given by H) then γ, λ, H can be lifted, uniquely to $\tilde{X}$ at $\tilde{x}$
$(\tilde{\gamma}(0) = \tilde{\lambda}(0) = \tilde{x})$

(try to prove this. The main point of note is the compactness of $[0,1]$, $[0,1]$

Theorem

X, Y connected Riemann Surfaces. (U, f) is a function element $X \to Y$.
Let $x \in U$. Assume that (U,f) can be analytically continued along every path starting at x. Then:

i) If γ, λ are homotopic paths starting at x, then the germs of the analytic continuation of (U,f) along γ, λ are the same. $\gamma(1) = \lambda(1)$

ii) If X is simply-connected, then $\pi : F \to X$ is biholomorphic.
Here, F is the connected component of G corresponding to (U, f).

Proof

i) Remember that π is a covering in the sense of analysis onto some open subset of X. Since X is connected, and since it is a (topological) surface, it is path-connected i.e. every two points can be connected by a path. So, ~~suppose~~ $\forall x' \in X$, $\exists \alpha$, a path such that $\alpha(0) = x$, $\alpha(1) = x'$. By our assumptions, α can be lifted to $F \Rightarrow \exists [x', g] \in F$ mapping to x'. Thus, π is surjective.

Again, since every path can be lifted to F, π is a covering in the sense of topology (recall this from the previous lecture).

So, by the monodromy theorem, γ, λ can be lifted to homotopic paths $\tilde{\gamma}, \tilde{\lambda}$ starting at $[x, f]$. By definition of homotopy, $\tilde{\gamma}(1) = \tilde{\lambda}(1)$. This implies i).

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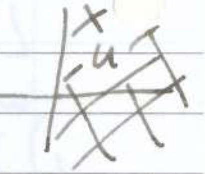
Since $\pi : G \to X$ is already holomorphic

ii) It is enough to show that π is injective. Assume not, i.e. $\exists x'$ and two distinct points $[x', g], [x', h] \in \pi^{-1}\{x'\}$. Since F is path connected, $\exists$ paths $\tilde{\gamma}, \tilde{\lambda}$ on F such that $\tilde{\gamma}(0) = \tilde{\lambda}(0) = [x, f]$, and $\tilde{\gamma}(1) = [x', g]$, $\tilde{\lambda}(1) = [x', h]$

Define $\gamma = \pi\tilde{\gamma}$, $\lambda = \pi\tilde{\lambda}$. Since X is simply connected, γ, λ are homotopic. By the Monodromy Theorem, γ, λ are lifted to the same path $\tilde{\gamma}, \tilde{\lambda}$, and we get a homotopy between $\tilde{\gamma}, \tilde{\lambda}$. Thus, $\tilde{\gamma}(1) = \tilde{\lambda}(1)$ ※ Therefore π is 1-1 $\Rightarrow$ π biholomorphic

Example

Let $X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C}$. Next, define connected (U, f):



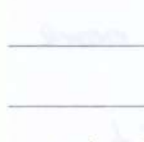
$U = \{re^{i\theta} \mid r \in \mathbb{R}_{>0}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$

$h : U \to \mathbb{C}$, $re^{i\theta} \mapsto r^{\frac{1}{2}} e^{\theta/2\, i}$

Remember that h is holomorphic. Let $f(re^{i\theta}) = h(1 + h(re^{i\theta}))$ $(h(z) = \sqrt{z}$, $f(z) = \sqrt{1+\sqrt{z}}\,)$. Obviously, (U, f) is a function element $X \to Y$.

(U, f) has the important property that $z - (f(z)^2 - 1)^2 = 0$ $(*)$ $\forall z \in U$. Any analytic continuation of (U, f) satisfies $(*)$ (Sheet 1). We will focus on this property.

$g_1(a_1) = g_2(a_2)$



Assume that $(V_1, g_1), (V_2, g_2)$ are function elements satisfying $(*)$ i.e. $\forall z \in V_i$, $z - (g_i(z)^2 - 1)^2 = 0$, $i = 1, 2$
Assume also that $\exists a_1 \in V_1, a_2 \in V_2$ such that $g_1(a_1) = g_2(a_2)$. By $(*)$, $a_1 = a_2$. For similar reasons, $g_1 = g_2$ in some open neighbourhood of $a_1 = a_2$. Thus $[a_1, g_1] = [a_2, g_2]$

So, if (V, g) is a function element satisfying $(*)$, and if $a \in V$, then $[a, g]$ is uniquely determined by $g(a)$.

Next, we construct lots of function elements satisfying $(*)$.

Riemann Surfaces (12)

Put $p(z, u) = z - (u^2 - 1)^2$

$L = \{(a, b) \in \mathbb{C}^2 \mid p(a, b) = 0\}$

If $(a, b) \in L \setminus \{(0,1), (0,-1), (1,0)\}$ then $\frac{\partial p}{\partial u}(a, b) \neq 0$.

By the implicit function theorem, $\exists$ open $V \subseteq \mathbb{C} \setminus \{0\}$, open $W \subseteq \mathbb{C} \setminus \{0, 1, -1\}$ and a unique holomorphic map

i) $g: V \to W$ such that $(a, b) \in V \times W$ and if $(a', b') \in V \times W$ then $((a', b') \in L \Leftrightarrow b' = g(a'))$. Then (V, g) is a function element satisfying $(*)$.

Next, we show that such (V, g) are analytic continuations of (U, f). Pick a path γ starting at $(1, \sqrt{2})$ ($\Leftrightarrow [1, f]$) and ending at (a, b). $\gamma \subseteq L \setminus \{(0,1), (0,-1), (1,0)\}$.

ii) Since "γ is compact", $\exists (a_1, b_1), \ldots, (a_n, b_n)$ such that $(a_1, b_1) = (1, \sqrt{2})$, $(a_n, b_n) = (a, b)$, and function elements (V_i, g_i) constructed (as above) for (a_i, b_i). We could assume that the V_i are all open discs.

The property $(*)$ shows that if $(V_i \times W_i) \cap (V_j \times W_j) \neq \emptyset$, $g_i = g_j$ on $V_i \cap V_j$ because V_i, V_j are open discs. We could choose (a_i, b_i) such that all the $V_i \times W_i$ cover the whole of (the image of) γ.

Maybe after re-indexing the (a_i, b_i), we can assume that (V_i, g_i) is a direct analytic continuation of (V_{i-1}, g_{i-1})

$\Rightarrow (V, g)$ is an analytic continuation of (U, f).

Atlases:

We have a diagram:

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, "\varphi"'] & \mathbb{C} & \\
L\setminus\{(0,1),(0,-1),(1,0)\} \arrow[rrr, leftrightarrow, "\varphi"] \arrow[drr, "\mu"'] & & & F \arrow[dl, "\pi"] \arrow[ul, "\pi"'] \\
& & X &
\end{tikzcd}$$

$$(a,b) \longleftrightarrow [a,g], \qquad (a,b) \mapsto a, \qquad [a,g] \mapsto a$$

φ maps to F because of ii)

It is easy to ~~show~~ see that φ is $\underline{1\text{-}1}$ and that it is biholomorphic. implicit function theorem, i) $\varphi : L\setminus\{(0,1),(0,-1),(1,0)\} \to \mathbb{C}$ holomorphic; look back at definitions of algebraic curves.

A little more work shows that $\pi^{-1}\{\frac{1}{2}\}$ has 4 elements but $\pi^{-1}\{1\}$ has only two elements

$\Rightarrow$ π is not a covering in the sense of topology.

Riemann Surfaces (13)

Complex Tori and Elliptic Functions

Theorem

Suppose that Λ_1, Λ_2 are lattices. Then we have
$\mathbb{C}/\Lambda_1$, $\mathbb{C}/\Lambda$ biholomorphic $\Leftrightarrow \Lambda_2 = a\Lambda_1$, $a \in \mathbb{C}, a \neq 0$

Proof

First, assume that $\Lambda_2 = a\Lambda_1$, $a \neq 0$. The map $\mathbb{C} \to \mathbb{C}$, $z \mapsto az$, (biholomorphic) gives a map $\rho : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ defined by $\rho([z]) = [az]$. The map ρ is holomorphic because $\mathbb{C} \to \mathbb{C}/\Lambda_1$, $\mathbb{C} \to \mathbb{C}/\Lambda_2$ are locally biholomorphic.
We could similarly define $\mathbb{C}/\Lambda_2 \xrightarrow{\rho^{-1}} \mathbb{C}/\Lambda_1$ by $\rho^{-1}([w]) = [\frac{w}{a}]$.
ρ^{-1} is also biholomorphic and the inverse of ρ. So ρ is biholomorphic.
Conversely, assume that we are given a biholomorphic map
$f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$. We will construct a diagram: so that $f\pi_1$ is a cover (topology)

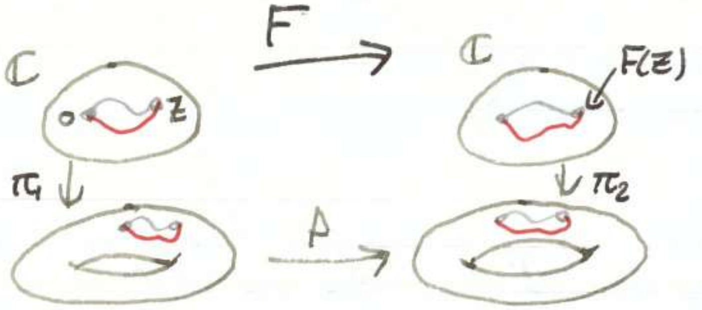
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow[\text{to be defined}]{F} & \mathbb{C} \\ {\scriptstyle\text{projection}}\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathbb{C}/\Lambda_1 & \xrightarrow{f} & \mathbb{C}/\Lambda_2 \end{array}$$

We define F as follows.

i) Pick $b \in \mathbb{C}$ such that $f\pi_1(0) = \pi_2(b)$

ii) For any $z \in \mathbb{C}$, pick a path γ in $\mathbb{C}$ such that $\gamma(0) = 0$, $\gamma(1) = z$.

By the Monodromy Theorem, the path $f\pi_1\gamma$ $\in \mathbb{C}/\Lambda_2$ can be lifted (through π_2) to a path λ such that $\lambda(0) = b$. Put $F(z) = \lambda(1)$.
The Monodromy Theorem implies that $F(z)$ does not depend on the choice of γ: we are using the fact that $\mathbb{C}$ is simply connected.



We can choose a connected open set $\mathcal{U} \ni z$ such that $\mathcal{U} \to \pi_1(\mathcal{U})$ is biholomorphic

We can also make sure that π_2^{-1} of $f\pi_1\mathcal{U}$ is a <u>disjoint</u> union of homeomorphic copies of $f\pi_1(\mathcal{U})$.

From the definition of F, it is easy to see that $F(\mathcal{U})$ is one of the copies in the above disjoint union. Thus $F(\mathcal{U}) \to f\pi_1(\mathcal{U})$ is biholomorphic. The opposite diagram shows that F is locally biholomorphic $\Rightarrow F$ is also biholomorphic

$$\begin{tikzcd} \mathcal{U} \arrow[r, "F"] \arrow[d, "\pi_1"'] & F(\mathcal{U}) \arrow[d, "\pi_2"] \\ \pi_1(\mathcal{U}) \arrow[r] & f\pi_1(\mathcal{U}) \end{tikzcd}$$

(note that we can also define the inverse of F in a similar way).

Fact: F biholomorphic $\Rightarrow F \in \mathrm{Aut}(\mathbb{C}) \Rightarrow \exists a \in \mathbb{C},\ a \neq 0$ such that $F(z) = az + b$ (Sheet 2, Question 8)

Now since any two elements of Λ_1 are mapped to the same point in $\mathbb{C}/\Lambda_1$, $F(\lambda) - F(0) \in \Lambda_2$ for each $\lambda \in \Lambda_1$. $F(\lambda) - F(0) = a\lambda$. Thus $a\Lambda_1 \subseteq \Lambda_2$. Arguing as above using f^{-1} gives $\Lambda_1 \supseteq \frac{1}{a}\Lambda_2$, so $a\Lambda_1 \supseteq \Lambda_2$.

Thus $a\Lambda_1 = \Lambda_2$ $\square$

<u>Definition</u>

any function!

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a function. We say that λ is a period of f if $f(z+\lambda) = f(z)\ \forall z$.

We denote $\Lambda_f = \{\text{periods of } f\}$, a group.

Theorem

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be non-constant, holomorphic.

Then Λ_f is a discrete subgroup of $\mathbb{C}$.

Proof

Λ_f is a subgroup:

$0 \in \Lambda_f$, $\lambda \in \Lambda_f \Rightarrow -\lambda \in \Lambda_f$ as $f(z-\lambda) = f(z-\lambda+\lambda) = f(z)\ \forall z$

$\lambda_1, \lambda_2 \in \Lambda_f \Rightarrow \lambda_1 + \lambda_2 \in \Lambda_f$ similarly

Λ_f is discrete, i.e. Λ_f has no accumulation points in $\mathbb{C}$:

Assume not. Then $\exists \lambda_1, \lambda_2, \ldots \in \Lambda_f$ such that $\lambda = \lim_{n\to\infty} \lambda_n$, $\lambda \in \mathbb{C}$. Since λ_n are periods, $f(0) = f(\lambda_1) = f(\lambda_2) = \ldots$

This is not possible, otherwise $f(z) - f(0)$ would have infinitely many zeroes in some compact set $\subseteq \mathbb{C}$.

$\Rightarrow \Lambda_f$ discrete $\square$

Fact: (Sheet 3, Question 9)

Λ_f as in the theorem is of one of the forms

$$\begin{cases} 0 \\ \mathbb{Z}\lambda\ , & 0 \neq \lambda \in \mathbb{C} \\ \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 & \lambda_1, \lambda_2 \in \mathbb{C} \text{ are } \mathbb{R}\text{-linearly independent} \end{cases}$$

With f, Λ_f as in the theorem, assume that we are given a lattice Λ. If $\Lambda_f \cong \mathbb{Z}$, we say that f is simply periodic.
If $\Lambda_f \cong \mathbb{Z} \oplus \mathbb{Z}$, $\Lambda \subseteq \Lambda_f$ then we say that f is *elliptic* (or doubly periodic) with respect to Λ.

Theorem

Let Λ be a lattice. Then, we have a 1-1 correspondence between {elliptic functions wrt Λ} $\Leftrightarrow$ {holomorphic maps $\mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$}

Proof (See Riemann-Roch?)

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be elliptic with respect to Λ. This induces a map $g : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$, $g([z]) = f(z)$.

g is also holomorphic: If $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the quotient map, then for each $z \in \mathbb{C}$, $\exists$ open $U \ni z$ such that $U \to \pi(U)$ is biholomorphic. $\Rightarrow$ g is holomorphic on $\pi(U)$ $\Rightarrow$ g is holomorphic everywhere.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\text{biholomorphic}"'] & \mathbb{C} \cup \{\infty\} \\ \pi(U) \arrow[ur, "g"'] & \end{tikzcd}$$

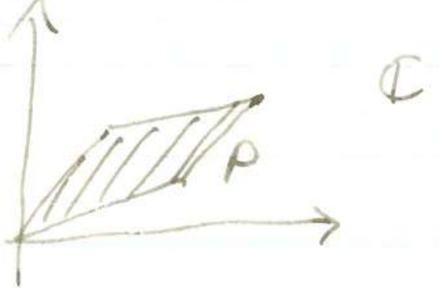
Conversely, assume that we are given a holomorphic map $g : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$. Let $f = g\pi$ which is holomorphic and elliptic with respect to Λ. $\square$

Assume that f is elliptic with respect to Λ. We define the <u>order</u> of f to be the <u>degree</u> of the corresponding map $g : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$.

Suppose that Λ is a lattice and $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the quotient map. Assume $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$

Let P be the parallelogram determined by $0, \lambda_1, \lambda_2, \lambda_1 + \lambda_2$

Now $\pi(P) = \mathbb{C}/\Lambda$. Actually, π is 1-1 on the interior of P. Any translation of P has the same properties.



$$\begin{tikzcd} \mathbb{C} \arrow[r, "\pi"] \arrow[dr, "f"'] & \mathbb{C}/\Lambda \arrow[d, "g"] \\ & \mathbb{C} \cup \{\infty\} \end{tikzcd}$$

Elliptic functions redefined

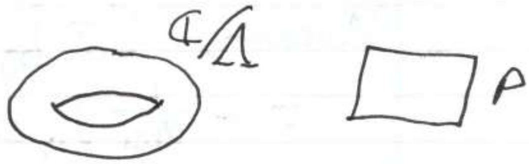
Let Λ be a lattice. An elliptic function f with respect to Λ, is a holomorphic map $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ such that $f(z+\lambda) = f(z)$ $\forall \lambda \in \Lambda$. We allow f constant or non-constant.

Lemma

Let Λ be a lattice and f an elliptic function with respect to Λ. If $f^{-1}\{\infty\} = \emptyset$ then f is constant.

Proof

Let $g : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$ be the holomorphic map associated to f. By assumption, $g^{-1}\{\infty\} = \emptyset$. Since $\mathbb{C}/\Lambda$ is compact, any holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}$ is constant. Thus, g, and therefore f, is a constant function.



Setup

Remember that for a lattice Λ, we can choose a parallelogram $P \subseteq \mathbb{C}$ such that $P \to \mathbb{C}/\Lambda$ is surjective and 1-1 in the interior of P. Now we assume that f is elliptic with respect to Λ. We can assume that $f^{-1}\{0, \infty\} \cap \partial P = \emptyset$ (using the fact that $f^{-1}\{0, \infty\}$ is discrete in $\mathbb{C}$).

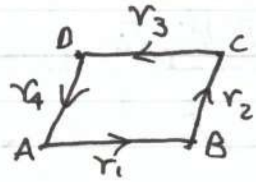
Theorem

With the notation and assumptions in the setup, we have

i) $\sum_{z \in P} \mathrm{res}_z(f) = 0$

ii) If $f^{-1}\{0\} \cap P = \{a_1, \dots, a_k\}$, $f^{-1}\{\infty\} \cap P = \{b_1, \dots, b_l\}$ then $\sum_{i=1}^{k} v_f(a_i)\, a_i - \sum_{j=1}^{l} v_f(b_j)\, b_j \in \Lambda$

and $\sum_{i=1}^{k} v_f(a_i) - \sum_{j=1}^{l} v_f(b_j) = 0$



$B - A = \lambda$
$D - A = \lambda$
$\Lambda = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$

Proof

i) Remember that $\sum_{z \in P} \operatorname{res}_z(f) = \int_{\partial P} f(z)\,dz$

Now $\int_{\partial P} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \dots + \int_{\gamma_4} f(z)\,dz$

Since f is elliptic wrt Λ, $\int_{\gamma_3} f(z)\,dz = -\int_{\gamma_1} f(z+\lambda_2)\,d(z+\lambda_2)$

$= -\int_{\gamma_1} f(z)\,dz$.

Similarly, $\int_{\gamma_2} f(z)\,dz = -\int_{\gamma_4} f(z)\,dz$

$\Rightarrow \int_{\partial P} f(z)\,dz = 0 \Rightarrow \sum_{z \in P} \operatorname{res}_z(f) = 0$

ii) By Question 2, Sheet 1, we have

$$\frac{1}{2\pi i}\int_{\partial P} z\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{k} v_f(a_i)\,a_i - \sum_{j=1}^{l} v_f(b_j)\,b_j$$

As before $\frac{1}{2\pi i}\int_{\partial P} z\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma_1} z\frac{f'(z)}{f(z)}\,dz + \dots + \frac{1}{2\pi i}\int_{\gamma_4} z\frac{f'(z)}{f(z)}\,d$

Moreover, $\frac{1}{2\pi i}\int_{\gamma_3} z\frac{f'(z)}{f(z)}\,dz = -\frac{1}{2\pi i}\int_{\gamma_1}(z+\lambda_2)\frac{f'(z+\lambda_2)}{f(z+\lambda_2)}\,d(z+\lambda$

$$= -\frac{1}{2\pi i}\int_{\gamma_1} z\frac{f'(z)}{f(z)}\,dz - \lambda_2\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)}\,dz$$

(Note that $f'(z)$ is also elliptic ~~over~~ wrt Λ.

Similarly $\frac{1}{2\pi i}\int_{\gamma_2} z\frac{f'(z)}{f(z)}\,dz = -\frac{1}{2\pi i}\int_{\gamma_4} z\frac{f'(z)}{f(z)}\,dz - \lambda_1\frac{1}{2\pi i}\int_{\gamma_4}\frac{f'(z)}{f(z)}\,dz$

$\Rightarrow \frac{1}{2\pi i}\int_{\partial P} z\frac{f'(z)}{f(z)}\,dz = -\lambda_2\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)}\,dz - \lambda_1\frac{1}{2\pi i}\int_{\gamma_4}\frac{f'(z)}{f(z)}\,dz$

$= m\lambda_2 + n\lambda_1$ (for some $m, n \in \mathbb{Z}$) $\in \Lambda$.

Finally, again by Question 2, Sheet 1, we have

$$\frac{1}{2\pi i}\int_{\partial P}\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{k} v_f(a_i) - \sum_{j=1}^{l} v_f(b_j)$$

Since $\frac{f'}{f}$ is elliptic wrt Λ, by i),

$$\frac{1}{2\pi i}\int_{\partial P}\frac{f'(z)}{f(z)}\,dz = 0$$

$\square$

Remark

Let Λ be a lattice. Then, $\exists a \in \mathbb{C}, a \neq 0$ such that $a\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda$, $\mathrm{Im}(\lambda) > 0$. Remember that $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/a\Lambda$. So we can replace Λ by $a\Lambda$, and assume from now on that $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \cdot \lambda$, $\mathrm{Im}(\lambda) > 0$.

Definition (Theta Function)

Let $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \cdot \lambda$, $\mathrm{Im}(\lambda) > 0$. Then we define the theta function as $\cancel{\Theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i(\frac{1}{2}n^2\lambda + nz)}}$

$$\Theta(z) = \sum_{n=-\infty}^{\infty} \exp\left(2\pi i\left(\tfrac{1}{2}n^2\lambda + nz\right)\right)$$

Theorem

With the above notation, we have

i) Θ is holomorphic on $\mathbb{C}$.

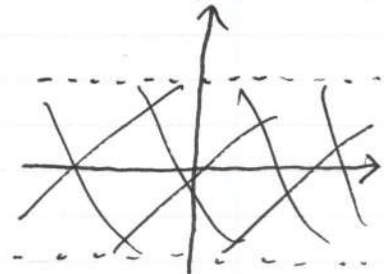
ii) $\Theta(-z) = \Theta(z)$

iii) $\Theta(z+1) = \Theta(z)$, $\Theta(z+\lambda) = e^{2\pi i(-\frac{\lambda}{2} - z)}\Theta(z)$

iv) If we choose a parallelogram P for Λ such that $\Theta^{-1}\{0\} \cap \partial P = \emptyset$ then Θ has a single zero in P with order 1.

v) $\Theta\left(\frac{\lambda}{2} + \frac{1}{2}\right) = 0$

S_R



Proof

i) We use the Weierstrass-M-test. Let $S_R = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < R\}$

Then, if $\lambda = a + bi$, $z = x + yi$, then

$$\left|e^{2\pi i(\frac{1}{2}n^2\lambda + nz)}\right| = e^{-\pi n^2 b - 2\pi n y} \leq e^{-\pi n^2 b + 2\pi |n| R} \text{ for } z \in S_R$$

Now, it is not difficult to show that $\sum_{n=-\infty}^{\infty} e^{-\pi n^2 b + 2\pi |n| R}$ is convergent $\Rightarrow$ M-test shows that Θ is holomorphic on S_R $\Rightarrow$ Θ holomorphic on $\mathbb{C}$.

ii) This is clear.

iii) $\Theta(z+1) = \Theta(z)$ is again clear. $\Theta(z+\lambda) = \sum_{n=-\infty}^{\infty} e^{2\pi i\left(\frac{1}{2}n^2\lambda + n\lambda + nz\right)}$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi i\left(\frac{1}{2}(n+1)^2\lambda + (n+1)z - z - \lambda/2\right)}$$

$$= e^{2\pi i(-z - \frac{\lambda}{2})}\,\Theta(z)$$

iv) # zeroes of Θ calculated with order $= \frac{1}{2\pi i}\int_{\partial P} \frac{\Theta'(z)}{\Theta(z)}\,dz$

$$= \frac{1}{2\pi i}\int_{r_1} \frac{\Theta'(z)}{\Theta(z)}\,dz + \dots + \frac{1}{2\pi i}\int_{r_4} \frac{\Theta'(z)}{\Theta(z)}\,dz$$

From iii) we have $\frac{\Theta'(z+\lambda)}{\Theta(z+\lambda)} = -2\pi i + \frac{\Theta'(z)}{\Theta(z)}$. This allows you to compare $\int_{r_1}$, $\int_{r_3}$ and $\int_{r_2}$, $\int_{r_4}$. At the end, we see that $\frac{1}{2\pi i}\int_{\partial P} \frac{\Theta'(z)}{\Theta(z)}\,dz = 1$.

v) Put $h(z) = \Theta(z - \frac{1}{2} - \frac{\lambda}{2})$.

Then $h(z)\, e^{2\pi i(z+\frac{1}{2})} = \Theta(z - \frac{1}{2} + \frac{\lambda}{2})$ (use ii), iii))

$$= \Theta(z + \tfrac{1}{2} + \tfrac{\lambda}{2}) = \Theta(-z - \tfrac{1}{2} - \tfrac{\lambda}{2}) = h(-z)$$

Finally $h(0)\,e^{\pi i} = h(0) \Rightarrow 2h(0) = 0$, $h(0) = 0$

$\Rightarrow \Theta(\frac{1}{2} + \frac{\lambda}{2}) = 0$

Elliptic Functions

Theorem

Let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda$ be a lattice, $Im(\lambda) > 0$. Let P be a parallelogram for Λ. Let $a_1, \dots, a_k, b_1, \dots, b_l$ be in the interior of P and let $m_1, \dots, m_k, n_1, \dots, n_l \in \mathbb{N}$ such that

i) $\sum_{i=1}^{k} m_i - \sum_{j=1}^{l} n_j = 0$ ii) $\sum_{i=1}^{k} m_i a_i - \sum_{j=1}^{l} n_j b_j \in \Lambda$

Then, $\exists$ an elliptic function with respect to Λ such that

$P \cap f^{-1}\{0\} = \{a_1, \dots, a_k\}$, $P \cap f^{-1}\{\infty\} = \{b_1, \dots, b_l\}$ and $v_f(a_i) = m_i$, $v_f(b_j) = n_j$

Proof

Let $h(z) = \Theta(z - \frac{1}{2} - \frac{\lambda}{2})$. Let $g(z) = \dfrac{h(z-a_1)^{m_1} \dots h(z-a_k)^{m_k}}{h(z-b_1)^{n_1} \dots h(z-b_l)^{n_l}}$

Then $g(z+1) = g(z)$ because 1 is a period of Θ.

By properties of Θ, we have $g(z+\lambda) = \exp\left(2\pi i x \left(\sum_{i=1}^{k} m_i a_i - \sum_{j=1}^{l} n_j b_j\right)\right) g(z)$

$= e^{2\pi i r\lambda} g(z)$ for some $r \in \mathbb{Z}$.

Let $f(z) = e^{2\pi i(-rz)} g(z)$

Now $f(z+1) = f(z)$, $f(z+\lambda) = e^{2\pi i(-rz)} e^{2\pi i(-r\lambda)} g(z+\lambda)$

$= e^{2\pi i(-rz)} g(z) = f(z)$.

So $f(z)$ is elliptic with respect to Λ, and the other properties follow from the definition of f. $\square$

We will now apply this theorem when $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda$, $Im(\lambda) > 0$

P a parallelogram for Λ, with $a = -\frac{1}{2} - \frac{\lambda}{2}$, $b = 0$, $m = n = 2$. Choose P so that a, b are in the interior of P.

By the ~~function~~ theorem, the function $f(z) = e^{2\pi i z} \dfrac{\Theta(z)^2}{\Theta(z - \frac{1}{2} - \frac{\lambda}{2})^2}$

is elliptic with respect to Λ and $f^{-1}\{0\} \cap P = \{a\}$, $f^{-1}\{\infty\} \cap P = \{b\}$, $V_f(a) = V_f(b) = 2$.

Then, we choose $\alpha, \beta \in \mathbb{C}$ such that $\alpha f(z) + \beta$ has a Laurent series expansion near 0 of the form $\frac{1}{z^2} + c_{-1}\frac{1}{z} + 0 + c_1 z + \ldots$

Now we define $P(z) = \alpha f(z) + \beta$, called the Weierstrass P-function.

The function $P(z)$ is still elliptic with respect to Λ and has order 2 as an elliptic function i.e. the corresponding holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$ has degree 2.

<u>Theorem</u> (Properties of $P(z)$)

With notation and assumptions as above:

i) $P(-z) = P(z)$

ii) $P'(z)$ is also elliptic with respect to Λ of order 3

iii) $P'(z)$ vanishes at $\frac{1}{2}, \frac{\lambda}{2}, \frac{1}{2} + \frac{\lambda}{2}$ with branching order $= 1$.

iv) $P(z)$ satisfies $p'(z)^2 - 4p(z)^3 + up(z) + v = 0$ for some $u, v \in \mathbb{C}$ to be determined.

<u>Proof</u>

$$\text{i) } p(-z) = \alpha e^{2\pi i(-z)} \frac{\Theta(-z)^2}{\Theta(-z-\frac{1}{2}-\frac{\lambda}{2})^2} + \beta \overset{\text{by properties of } \Theta}{=} \alpha \frac{e^{2\pi i(-z)}\,\Theta(z)^2}{e^{4\pi i(-z-\frac{1}{2})}\Theta(z-\frac{1}{2}-\frac{\lambda}{2})^2} + \beta$$

$$= \alpha \frac{e^{2\pi i(z)}\Theta(z)^2}{\Theta(z-\frac{1}{2}-\frac{\lambda}{2})^2} + \beta = P(z)$$

ii) Suppose that h is an elliptic function with respect to Λ.

$h: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ holomorphic $\Rightarrow h': \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ holomorphic

Let $a \in \mathbb{C}$. $a \in h^{-1}\{\infty\} \Rightarrow a \in h'^{-1}\{\infty\} \Rightarrow a+1, a+\lambda \in h^{-1}\{\infty\}$

$\Rightarrow a+1, a+\lambda \in h'^{-1}\{\infty\}$

If $a \notin h^{-1}\{\infty\} \Rightarrow h'(a+\omega) = \lim\limits_{z \to a+\omega} \dfrac{h(z) - h(a+\omega)}{z - a - \omega}$

$= \lim\limits_{z \to a} \dfrac{h(z+\omega) - h(a+\omega)}{z+\omega - a - \omega} = \lim\limits_{z \to a} \dfrac{h(z) - h(a)}{z - a} = h'(a)$

$\Rightarrow h'$ is elliptic with respect to Λ, so in particular, $\wp'(z)$ is.

By definition $\wp^{-1}\{\infty\} \cap P = \{\text{one point}\}$ and the branching order of $\wp$ at this point is 2. But $\wp'^{-1}\{\infty\} \cap P = \wp^{-1}\{\infty\} \cap P$ and has branching order $3 \Rightarrow$ a similar statement holds for the corresponding holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$ i.e. the latter holomorphic map sends only one point to ∞ and it has branching order 3

$\Rightarrow \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$ has degree 3 (we could also apply theorems of previous lectures).

iii) Since $\wp(-z) = \wp(z)$, $\wp'(-z) = -\wp'(z)$ elliptic

So $\wp'(\frac{1}{2}) = -\wp'(-\frac{1}{2})$. However $\wp'(\frac{1}{2}) = \wp'(\frac{1}{2} - 1) = \wp'(-\frac{1}{2})$

Thus $\wp'(\frac{1}{2}) = 0$. Similarly, we can show that $\wp'(\frac{\lambda}{2}) = 0$ and $\wp'(\frac{1}{2} + \frac{\lambda}{2}) = 0$.

Since $\wp'(z)$ is elliptic of order 3, $\wp'(z)$ has branching order 1 at $\frac{1}{2}, \frac{\lambda}{2}, \frac{1}{2} + \frac{\lambda}{2}$ because these points belong to the interior of some appropriate parallelogram P.

iv) From $\wp(-z) = \wp(z)$, $\wp(z)$ has the Laurent expansion

$\frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \dots$ near 0 $= \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + z^6 g_1(z)$

We can then calculate

$$\wp'(z) = -\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + z^5 g_2(z)$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8c_2}{z^2} - 16c_4 + z^2 g_3(z)$$

$g_i(z)$ holomorphic near 0

$$4\wp(z)^3 = \frac{4}{z^6} + \frac{12c_2}{z^2} + 12c_4 + z^2 \overbrace{g_4(z)}^{\text{holomorphic near } 0}$$

Thus, $A(z) := \wp'(z)^2 - 4\wp(z)^3 + u\wp(z) + v = z^2 g_5(z)$

$\Rightarrow A(z)$ is holomorphic near 0. On the other hand, $A(z)$ is elliptic with respect to Λ. Moreover, by construction,

$A^{-1}\{\infty\} = \emptyset \quad \Rightarrow A(z)$ is constant

But $A(0) = 0 \quad \Rightarrow A(z) \equiv 0$

The Uniformisation Theorem

Remark

Suppose that $\pi: Y \to X$ is a covering in the sense of topology, where X is a connected Riemann Surface and Y is a connected topological surface. We can define a unique complex structure on Y so that Y becomes a Riemann Surface and π a holomorphic map. (Sheet 3, Question 2, typo: $\pi = f$)

Remark

i) Now assume that Y is a connected Riemann Surface. Let $G \leq \mathrm{Aut}(Y)$. G acts on Y naturally. $g \in G$ acts on $y \in Y$ by $g(y)$. Let $X := Y/G$ and $\pi: Y \to X$ the quotient map.
Here $\pi(y) = \pi(y') \iff \exists g \in G$ such that $g(y) = y'$.
We say that G acts <u>properly discontinously</u> if X is a topological surface and π a covering in the sense of topology, i.e. $\forall y \in Y$, $\exists$ open $U \ni y$ such that $g_1(U) \neq g_2(U)$ if $g_1 \neq g_2 \in G$.

In this case, we can define a complex structure on X so that π is holomorphic (defined similarly to the case of complex tori).

ii) Suppose that X is a connected Riemann Surface. Then there is a universal cover $\pi: \tilde{X} \to X$. Here, $\tilde{X}$ is a simply connected Riemann Surface and π is a covering in the sense of topology (π is holomorphic).

Theorem (Uniformisation)

Suppose that X is a connected Riemann Surface and $\pi : \tilde{X} \to X$ a universal cover. Then, $\tilde{X}$ is biholomorphic to one of :

i) the Riemann Sphere, $\mathbb{C} \cup \{\infty\}$

ii) the Complex Plane, $\mathbb{C}$

iii) the Open Unit Disc, $D(0,1)$

Moreover, $\exists$ a subgroup $G \leq \mathrm{Aut}(\tilde{X})$ such that $X \xrightarrow{\text{biholomorphic}} \tilde{X}/G$ where G acts on $\tilde{X}$ properly discontinuously.

Corollary

Suppose that $\pi : \tilde{X} \to X$ is as in the theorem. Assume that X is compact and simply connected. Then π is biholomorphic.

Proof

We know that π is surjective. Moreover, π is 1-1 ; assume that $\pi(\tilde{x}_1) = \pi(\tilde{x}_2)$. Choose a path $\gamma : \tilde{x}_1$ to $\tilde{x}_2$. $\pi\gamma$ is homotopic to the constant path because X is simply connected. By the Monodromy Theorem, γ is homotopic to the constant path at $\tilde{x}_1$ $\Rightarrow \gamma(0) = \gamma(1) = \tilde{x}_2 \Rightarrow \pi$ is 1-1 $\Rightarrow \pi$ is biholomorphic $\square$

Then, by uniformisation, X is biholomorphic to $\tilde{X}$ and to the Riemann Sphere (the only compact choice of the three).

Corollary

$\pi : \tilde{X} \to X$ as above. If $\tilde{X}$ is the Riemann Sphere, then π is biholomorphic and X is biholomorphically equivalent to the Riemann Sphere.

Proof

$\tilde{X}$ compact $\Rightarrow X$ compact. We know that $\chi(\tilde{X}) = 2$. By the Riemann-Hurwitz Formula, $2 = \chi(\tilde{X}) = \deg(\pi) \cdot \chi(X)$ (there are no ramification points)

$\Rightarrow \chi(X) > 0 \underset{\text{by topology}}{\Rightarrow} X = \mathbb{C} \cup \{\infty\}$, the Riemann Sphere

$\Rightarrow \deg \pi = 1 \Rightarrow \pi$ is biholomorphic.

Remark

The above corollaries immediately imply that the sphere carries only one complex structure.

Example

Let $\pi : \tilde{X} \to X$ be as in the theorem. Assume $\tilde{X} = \mathbb{C}$. Then, X is biholomorphic to either $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, or a complex torus.

By the theorem, $\exists$ a subgroup $G \leq \mathrm{Aut}(\mathbb{C})$ such that $X = \tilde{X}/G$, where G acts on $\mathbb{C}$ properly discontinuously. Recall that $\mathrm{Aut}(\mathbb{C}) = \{az+b \mid a, b \in \mathbb{C}, a \neq 0\}$

i) If $G = 0$ then $X = \tilde{X}/G = \tilde{X} = \mathbb{C} \Rightarrow X$ biholomorphic to $\mathbb{C}$

ii) If $0 \neq g(z) = az + b \in G$, then $a = 1$, otherwise $g\left(\frac{b}{1-a}\right) = \frac{b}{1-a}$ which contradicts the fact that G should act properly discontinuously.

Then we can define a homomorphism $\varphi : G \to \mathbb{C}$, $g(z) = z + b \mapsto b$.

Again, since G acts properly discontinuously, $\mathrm{Im}(\varphi)$ has no accumulation points. So, $\mathrm{Im}(\varphi)$ is a discrete subgroup of $\mathbb{C}$.

So $\mathrm{Im}(\varphi) = 0$, $\mathrm{Im}(\varphi) \cong \mathbb{Z}$, or $\mathrm{Im}(\varphi) \cong \mathbb{Z} \oplus \mathbb{Z}$.

If $\mathrm{Im}(\varphi) \cong \mathbb{Z} \oplus \mathbb{Z}$ then $\mathrm{Im}(\varphi)$ is a lattice in $\mathbb{C}$. Thus, X is biholomorphic to $\mathbb{C}/\mathrm{Im}(\varphi)$ = Complex Torus

Finally, assume that $\mathrm{Im}(\varphi) \cong \mathbb{Z}$. So $\exists\ 0 \neq \lambda \in \mathbb{C}$ such that $\mathrm{Im}(\varphi) = \lambda\mathbb{Z}$. We have the holomorphic map $\mathbb{C} \to \mathbb{C}^*$ defined by $z \mapsto \exp\left(\frac{2\pi i}{\lambda} z\right)$. This induces a holomorphic map $h: \mathbb{C}/\lambda\mathbb{Z} \to \mathbb{C}^*$, which is biholomorphic:

h is surjective, and in fact 1-1, because if $h([z]) = h([z'])$ then $z - z' \in \lambda\mathbb{Z} \Rightarrow [z] = [z'] \Rightarrow h$ biholomorphic.

Now $X = \tilde{X}/G$ is biholomorphic to $\mathbb{C}/\lambda\mathbb{Z}$ and in turn to $\mathbb{C}^*$.

<u>Theorem</u> (Picard)

Suppose $f: \mathbb{C} \to \mathbb{C}$ is holomorphic, and assume that $f(\mathbb{C}) \subseteq \mathbb{C} \setminus \{a, b\}$. Then f is constant.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[dr, "f"'] & D(0,1) \arrow[d, "\pi"] \\ & X = \mathbb{C}\setminus\{a,b\} \end{tikzcd}$$

<u>Proof</u>

By the results of this lecture, if $\pi: \tilde{X} \to X = \mathbb{C}\setminus\{a,b\}$ is the universal cover $\Rightarrow \tilde{X} = D(0,1)$.

We define $F: \mathbb{C} \to D(0,1)$, holomorphic, such that $f = \pi F$

Pick $b \in D(0,1)$ such that $f(0) = \pi(b)$. For any $z \in \mathbb{C}$, choose a path γ such that $\gamma(0) = 0$, $\gamma(1) = z$, and lift $f\gamma$ to a path α starting at b. Put $F(z) = \alpha(1)$. $\mathbb{C}$ simply-connected $\Rightarrow F(z)$ independent of choice of γ. π locally biholomorphic $\Rightarrow F$ holomorphic.

Now apply Liouville's theorem: F constant $\Rightarrow f$ constant.