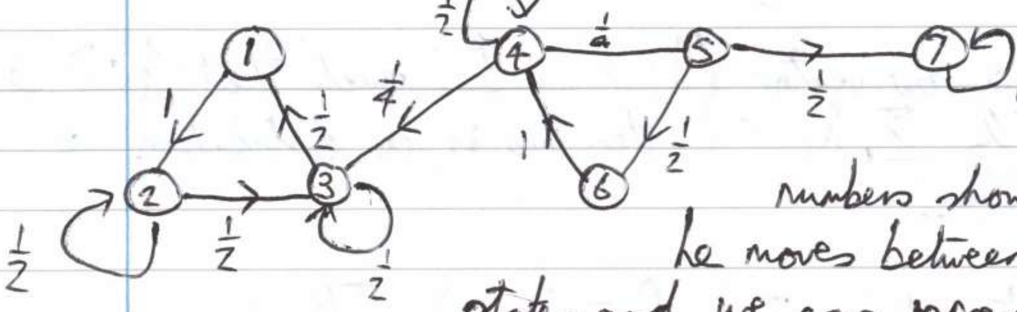
Markov Chains (1)

1. Definitions, Basic Properties and the Transition Matrix

1.1 An Example, and Interesting Questions



A frog hops between 7 lily-pads. The numbers show the probabilities with which he moves between pads. There are 7 states and we can represent this diagram as a matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The element p_{ij} is the probability that when starting in state i, the next jump takes us to state j. We would like to know where the frog goes, how long it takes to get there, and what happens in the long run.

a) Starting in state 1, what is the probability of being in state 1 after n steps? ($p_{11}^{(n)}$).

$p_{11}^{(3)} = \frac{1}{4}$, $p_{11}^{(5)} = \frac{3}{16}$, $p_{11}^{(n)} \to \frac{1}{5}$ as $n \to \infty$

b) Starting in state 4, what is the probability of reaching state 2?

c) Starting at 4, how many steps on average until reaching 3 or 7 $\frac{11}{3}$

d) Starting at 2, what is the long term proportion of time spent in 3? $\frac{2}{5}$

1.2 Definitions

Let I be a countable set (e.g. $\mathbb{N}$). I is called the state space and each $i \in I$ is called a state

We work in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Ω a set of outcomes, $\mathcal{F}$ a set of subsets of Ω, and for $A \in \mathcal{F}$,

$p(A)$ is the probability of A occuring. Recall that a random variable X is a function $X : \Omega \to I$.

A <u>measure</u> is a row vector $(\lambda_i : i \in I)$ such that $\lambda_i \geq 0$ for all i. If $\sum_i \lambda_i = 1$ then this is a <u>distribution</u> or <u>probability measure</u>.

We have a <u>transition matrix</u> $P = (p_{ij})$. This is a <u>stochastic matrix</u> so that $p_{ij} \geq 0 \ \forall i, j \in I$ and $\sum_{j \in I} p_{ij} = 1$ (i.e. each row of P is a distribution over I).

We begin with an <u>initial distribution</u> over I, specified by $\lambda_i = p(X_0 = i)$

<u>Definition 1·2</u>

We say that $(X_n)_{n \geq 0}$ is a Markov Chain with initial distribution λ and transition matrix P, if $\forall n \geq 0$, and $i_0, i_1, \ldots, i_{n+1} \in I$:

i) $p(X_0 = i_0) = \lambda_{i_0}$

ii) $p(X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n) = p(X_{n+1} = i_{n+1} \mid X_n = i_n) = p_{i_n i_{n+1}}$

Then we say that $(X_n)_{n \geq 0}$ is Markov(λ, P)

<u>Theorem 1·3</u>

$(X_n)_{n \geq 0}$ is Markov(λ, P) if and only if $\forall n \geq 0$ and $i_0, i_1, \ldots, i_n \in I$

$$p(X_0 = i_0, \ldots, X_n = i_n) = \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$$

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Proof

Assuming 1·2:

$p(X_0=i_0,\ldots,X_n=i_n) = p(X_n=i_n \mid X_0=i_0,\ldots,X_{n-1}=i_{n-1})\,p(X_0=i_0,\ldots,X_{n-1}=i_{n-1})$

$= p(X_n=i_n \mid X_{n-1}=i_{n-1})\,p(X_0=i_0,\ldots,X_{n-1}=i_{n-1})$

$= \ldots = p_{i_{n-1}i_n} \cdots p_{i_0 i_1} \lambda_{i_0}$

Now suppose that 1·3 is true:

Sum over $i_1, i_2, \ldots, i_n$ to get $p(X_0=i_0)=\lambda_{i_0}$

$p(X_{n+1}=i_{n+1} \mid X_0=i_0,\ldots,X_n=i_n) = \dfrac{p(X_{n+1}=i_{n+1},\ldots,X_0=i_0)}{p(X_n=i_n,\ldots,X_0=i_0)}$

$= (\lambda_{i_0} p_{i_0 i_1} \cdots p_{i_n i_{n+1}}) / (\lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}) = p_{i_n i_{n+1}}$

1·3 Where do Markov Chains come from?

We might take $U_1, U_2, \ldots$ as some i.i.d r.v's taking values in a set E, and a function $F: I \times E \Rightarrow I$.
Take $i \in I$, set $X_0 = i$ and define $X_{n+1} = F(X_n, U_{n+1}), n \geq 0$
Then $(X_n)_{n\geq 0}$ is Markov(δ_i, P), where $p_{ij} = P(F(i,U)=j)$.

1·4 How do we Simulate them?

Use a computer to simulate $U_1, U_2, \ldots$ as i.i.d $U[0,1]$.

$U \in \left[\sum_{k=1}^{j-1} p_{ik}, \sum_{k=1}^{j} p_{ik}\right) \Rightarrow X_{n+1} = j$

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1·5 The n-step Transition Matrix

Let $p_i(A) = p(A \mid X_0 = i)$, and $p_i(X_1 = j) = p_{ij}$

We will treat the initial distribution λ as a row vector.

$p(X_1 = j) = \sum_i \lambda_i p_i(X_1 = j) = \sum_i \lambda_i p_{ij}$

Similarly:

$p_i(X_2 = j) = \sum p_{ik} p_{kj} = (P^2)_{ij} = P^{(2)}_{ij}$

$p(X_2 = j) = (\lambda^k P^2)_j$

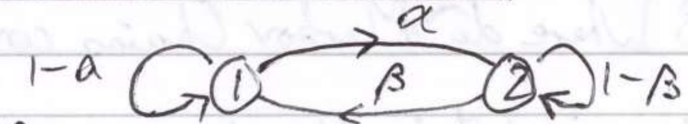
$p_i(X_n = j) = (P^n)_{ij} = P^{(n)}_{ij}$

Chapman-Kolomogorov Equations:

$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)} p_{kj}^{(m)} \qquad \forall n, m \geq 0$

1·6 $P^{(n)}$ for a Two-State Markov Chain

Example 1·4



The eigenvalues of P are 1 and $1-\alpha-\beta$.

$P = U \begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{pmatrix} U^{-1}$, $P^n = U \begin{pmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{pmatrix} U^{-1}$

$p_{11}^{(n)} = A + B(1-\alpha-\beta)^n$

Using $p_{11}^{(0)} = 1 = A + B$ and

$p_{11}^{(1)} = 1 - \alpha = A + B(1-\alpha-\beta)$

we obtain $\quad p_{11}^{(n)} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n$

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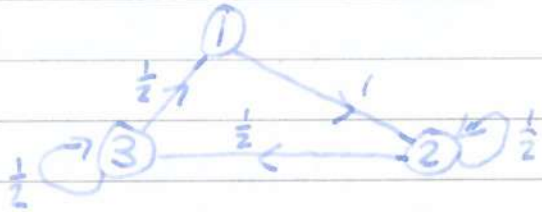
Calculation of an n-step transition matrix, class structure, absorption, irreducibility

2.1 A 3 State Markov Chain



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

We look for eigenvalues μ_i of P, since $p_{ij}^{(n)} = A\mu_1^n + \dots + C\mu_k^n$

$$\det(P - xI) = \begin{vmatrix} -x & 1 & 0 \\ 0 & \frac{1}{2}-x & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2}-x \end{vmatrix} = 0 \Rightarrow \frac{1}{4}(x-1)(4x^2-1) = 0$$

$\mu_i = 1, \frac{1}{2}i, -\frac{1}{2}i$

$$p_{11}^{(n)} = A + B\left(\tfrac{1}{2}i\right)^n + C\left(-\tfrac{1}{2}i\right)^n, \quad \left(\pm\tfrac{i}{2}\right)^n = \left(\tfrac{1}{2}\right)^n\left(\cos\tfrac{n\pi}{2} \pm i\sin\tfrac{n\pi}{2}\right)$$

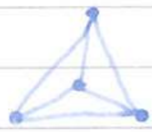
$$\Rightarrow p_{11}^{(n)} = A + \left(\tfrac{1}{2}\right)^n\left[B'\cos\tfrac{n\pi}{2} + C'\sin\tfrac{n\pi}{2}\right]$$

$p_{11}^{(0)} = 1,\ p_{11}^{(1)} = 0,\ p_{11}^{(2)} = 0$

$$\Rightarrow p_{11}^{(n)} = \frac{1}{5} + \left(\tfrac{1}{2}\right)^n\left[\tfrac{4}{5}\cos\tfrac{n\pi}{2} - \tfrac{2}{5}\sin\tfrac{n\pi}{2}\right],\ p_{11}^{(n)} \to \frac{1}{5}$$

If some eigenvalue μ has multiplicity k, then there will be a term of the form $(b_0 + b_1 n + \dots + b_{k-1}n^{k-1})\mu^n$. Some μ_i are complex, but they come in pairs, so $p_{ij}^{(n)}$ can be written in terms of sines and cosines.

2.2 Use of Symmetry



Tetrahedron

Consider a random walk on K_4

$$P = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$. Eigenvalues of P are $1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}$

$$p_{11}^{(n)} = A + \left(-\tfrac{1}{3}\right)^n(a + bn + cn^2)$$

For $j \neq i$, observe that $p_{ij}^{(n)} = \frac{1}{3}\left(1 - p_{ii}^{(n)}\right)$

$$p_{11}^{(n)} = \sum_{j\neq 1} p_{1j}\,p_{j1}^{(n-1)} = \sum_{j\neq 1}\frac{1}{3}\cdot\frac{1}{3}\left(1 - p_{11}^{(n-1)}\right) = \frac{1}{3}\left(1 - p_{11}^{(n-1)}\right)$$

$$\Rightarrow p_{11}^{(n)} = A + B\left(-\tfrac{1}{3}\right)^n,\quad A = \tfrac{1}{3}(1-A) \Rightarrow A = \tfrac{1}{4}$$

$$p_{11}^{(n)} = \frac{1}{4} + \frac{3}{4}\left(-\tfrac{1}{3}\right)^n$$

2.3 Markov Property

Theorem 2.3 (Markov Property)

Let $(X_n)_{n\geq 0}$ be Markov (λ, P). The conditional distribution on $X_m = i$, $(X_{n+m})_{n\geq 0}$ is Markov (δ_i, P) and is independent of $X_0, X_1, \ldots, X_m$.

Proof

We must show that for any event A determined by $X_0, \ldots, X_m$ we have $P(\{x_m = i_m, \ldots, x_{m+n} = i_{m+n}\} \cap A \mid x_m = i)$
$$= \delta_{i i_m} p_{i_m i_{m+1}} \cdots p_{i_{m+n-1} i_{m+n}} P(A \mid X_m = i)$$

Let $A = \{x_0 = i_0, \ldots, x_m = i_m\}$ (i.e. suppose A has an elementary form)
We must show that $P(X_0 = i_0, \ldots, X_{m+n} = i_{m+n}, i = i_m) / P(X_m = i)$
$$= \delta_{i i_m} p_{i_m i_{m+1}} \cdots p_{i_{m+n-1} i_{m+n}}$$
This is true using Theorem 1.3, and then by definition, and any A can be written as the union of some elementary events.

2.4 Class Structure

The idea is to decompose a Markov Chain into smaller pieces which are easy to understand. We say that i leads to j, writing $i \to j$, if $P_i(X_n = j$ for some $n \geq 0) > 0$. i and j communicate if $i \to j$ and $j \to i$.

Theorem 2.4

For distinct states i and j, the following are equivalent:

i) $i \to j$

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ii) $p_{i i_1} p_{i_1 i_2} \dots p_{i_{n-1} j} > 0$ for some states $i_0, i_1, \dots, i_n$ where $i_0 = i$, $i_n = j$

iii) $p_{ij}^{(n)} > 0$ for some $n \geqslant 0$.

<u>Proof</u> (i) $\Leftrightarrow$ (iii)

$$p_{ij}^{(n)} \leqslant P_i(X_n = j \text{ for some } n = 0) \leqslant \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

$$P\left(\bigcup_n B_n\right) \leqslant \sum P(B_i), \quad B_n = \{X_n = j\}, \text{ so } (i) \Rightarrow (iii)$$

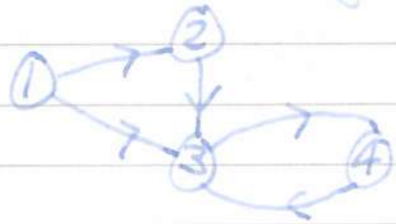
$$p_{ij}^{(n)} = \sum_{i_1, \dots, i_{n-1}} p_{i i_1} p_{i_1 i_2} \dots p_{i_{n-1} j}, \text{ so } (ii) \Leftrightarrow (iii)$$

<u>Closed Classes</u>

Clearly $i \Leftrightarrow i$, and $i \Leftrightarrow j \Leftrightarrow j \Leftrightarrow i$. Also, $i \Leftrightarrow j$, $j \Leftrightarrow k \Rightarrow i \Leftrightarrow k$. Therefore $\Leftrightarrow$ is an equivalence relation.

A class is <u>closed</u> if $i \in C$, $i \to j \Rightarrow j \in C$

A state is <u>absorbing</u> if $\{i\}$ is a closed class.



Here, $\{3, 4\}$ is a closed class.

<u>2.6 Irreducibility</u>

A Markov Chain, or matrix P, is irreducible when the whole state space I is a single closed class.

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Hitting Probabilities and Mean Hitting Times

3.1 Absorption Probabilities and mean hitting times

$P = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}$ $\quad P_1(\text{hit } 2) = \sum_{n=1}^{\infty} P(\text{hit } 2 \text{ at time } n) = \sum_{n=1}^{\infty} (1-p)^{n-1} p =$

$E_1(\text{hit } 2) = \sum_{n=1}^{\infty} n\, P(\text{hit } 2 \text{ at time } n) = \sum_{n=1}^{\infty} n(1-p)^{n-1} p$

$= -p \frac{d}{dp} \sum_{n=0}^{\infty} (1-p)^n = \frac{1}{p}$

OR: $h = P_1(\text{hit } 2)$, $k = E_1(\text{time to hit } 2)$, $h = (1-p)h + p \Rightarrow h = 1$

$k = 1 + (1-p)k + p \cdot 0 \Rightarrow k = \frac{1}{p}$

3.2 Calculation

$A \subseteq I$. $H^A(\omega) = \inf\{n \geq 0 : x_n(\omega) \in A\}$

$(X_n)_{n\geq 0}$ is a Markov Chain with transition matrix P.

$H^A(\omega) = \infty$ if we are taking the infimum of an empty set.

$h_i^A = p_i(H^A < \infty)$

$k_i^A = E_i(H^A) = \sum_{n<\infty} n\, p_i(H^A = n) + \infty\, p(H^A = \infty)$

$h_i^A = p_i(\text{hit } A)$ $\quad k_i^A = E_i(\text{time to hit } A)$

Example 3.2

$h_i = p_i(\text{hit } 3)$, $k_i = E_i(\text{time to hit } \{0,3\})$

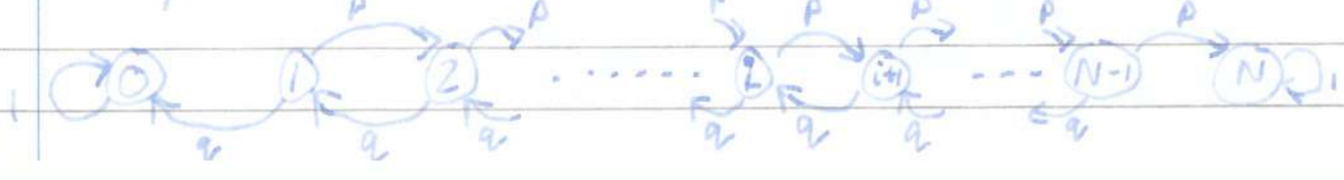
$h_0 = 0$, $h_1 = \frac{1}{2}h_0 + \frac{1}{2}h_2$, $h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3$, $h_3 = 1$

$\Rightarrow h_1 = \frac{1}{3}$, $h_2 = \frac{2}{3}$

$k_0 = 0$, $k_3 = 0$, $k_1 = 1 + \frac{1}{2}k_0 + \frac{1}{2}k_2$, $k_2 = 1 + \frac{1}{2}k_1 + \frac{1}{2}k_3$

$\Rightarrow k_1 = k_2 = 2$

Example 3.3



$h_i = P_i(\text{hit } 0)$, $h_0 = 1$, $h_N = 0$

$h_i = q h_{i-1} + p h_{i+1}$, $i = 1, 2, \ldots, N-1$
$px^2 - x + q = 0 = (x-1)(px - q)$, roots $1, \frac{q}{p}$

$$h_i = A + B\left(\frac{q}{p}\right)^i = \frac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N} \text{ for } p \neq q$$

If $p = q$, $h_i = A + Bi = 1 - \frac{i}{N}$ and $k_i = E_i(\text{hit}\{0, N\}) = i(N-i)$

3.3 Absorption Probabilities are the minimal solutions to Right Hand Equations

Theorem 3.4

The vector of hitting probabilities $h^A = (h_i^A \mid i \in I)$ is the minimal solution to the linear equations:
$h_i^A = 1$ for $i \in A$ and $h_i^A = \sum_j p_{ij} h_j^A$ for $i \notin A$.
(minimal: if $x = (x_i)_{i \in I}$ is a non-ve solution then $x_i \geq h_i^A$)

Proof

If $X_0 = i \in A$, then clearly $h_i^A = 1$.
If $X_0 \notin A$ then $H_i^A \geq 1$, $h_i^A = p_i(H^A < \infty) = \sum_j p_i(H^A < \infty, X_1 = j)$
$= \sum_j p_i(H^A < \infty \mid X_1 = j) p(X_1 = j) = \sum_j p_{ij} h_j^A$

Suppose x solves 3.1. Take $i \notin A$.
$x_i = \sum p_{ij} x_j = \sum_{j \in A} p_{ij} x_j + \sum_{j \notin A} p_{ij} x_j$
$= \sum_{j \in A} p_{ij} x_j + \sum_{j \notin A} p_{ij} \left(\sum_{k \in A} p_{jk} x_k + \sum_{k \notin A} p_{jk} x_k \right)$

$= p_i(X_1 \in A) + p_i(X_1 \notin A, X_2 \in A) + \sum_{j, k \notin A} p_{ij} p_{jk} x_k$
By repeated substitution:
$x_i = p_i(X_1 \in A) + p_i(X_1 \notin A, X_2 \in A) + \ldots + p_i(X_1, X_2 \ldots X_{n-1} \notin A, X_n \in A)$
$+ \sum_{j_1, \ldots, j_n \notin A} p_{ij_1} p_{j_1 j_2} \ldots p_{j_{n-1} j_n} x_{j_n}$
$x_{j_n} \geq 0$.

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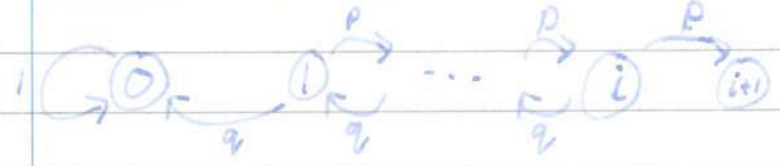
Hence $x_i \geq P_i(H^A \leq n)$
Let $n \to \infty$, $x_i \geq P_i(H^A < \infty) = h_i^A$

3.4 Gambler's Ruin



$p+q=1$, $p_{00}=1$, $p_{i\,i-1}=q$, $p_{i\,i+1}=p$, 0 otherwise

$h_i = P_i(\text{hit } 0)$
$h_0 = 1$, $h_i = p h_{i+1} + q h_{i-1}$, $x^2 p - x + q = (x-1)(xp-q) = 0$
We have roots 1 and q/p, so for $p \neq q$, $h_i = A + B\left(\frac{q}{p}\right)^i$

$p<q$, $0 \leq h_i \leq 1$, $\left(\frac{q}{p}\right)^i \to \infty$ as $i \to \infty \Rightarrow B = 0$, $h_i = 1\ \forall i$
$p>q$, $h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right)$ since $h_0 = 1 \Rightarrow A + B = 1$

h_i is the minimal non-negative solution to the right hand equation

$\Rightarrow h_i = \left(\frac{q}{p}\right)^i$, $A = 0$

$p=q$, $h_i = A + Bi$, $B = 0$, $h_0 = 1 \Rightarrow h_i = 1\ \forall i$.

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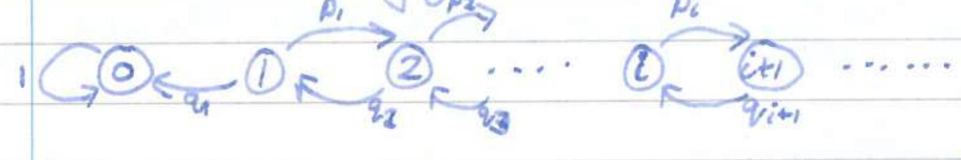
Survival Probability for Birth-Death Chains, Stopping Times, and the Strong Markov Property

4.1 Survival Probability for Birth-Death Chains



$0 < p_i = 1 - q_i < 1$

$h_i = P_i(\text{hit } 0)$, $h_0 = 1$, $h_i = p_i h_{i+1} + q_i h_{i-1} = p_i h_i + q_i h_i$

$p_i (h_i - h_{i+1}) = q_i (h_{i-1} - h_i)$

$u_i = h_{i-1} - h_i$, $p_i u_{i+1} = q_i u_i$

$u_{i+1} = \frac{q_i}{p_i} u_i = \left(\frac{q_i q_{i-1} \cdots q_1}{p_i p_{i-1} \cdots p_1}\right) u_1 = \gamma_i u_1$

$u_1 + u_2 + \dots + u_i = h_0 - h_i = 1 - h_i = (1 + \gamma_1 + \dots + \gamma_{i-1}) u_1$

$h_i = 1 - (1 + \gamma_1 + \dots + \gamma_{i-1}) A$ (where $A = u_1$)

$\sum_{i=0}^{\infty} \gamma_i = \infty \Rightarrow A = 0$, $h_i = 1$

$\sum_{i=0}^{\infty} \gamma_i < \infty \Rightarrow A = \left(\sum_{i=0}^{\infty} \gamma_i\right)^{-1}$, $h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j}$

Mean Hitting Times are solutions to Right Hand Equations

Theorem 4.2

The vector of mean hitting times $(k_i^A : i \in I) = k^A$ is the minimal non-negative solution to the system of right hand equations $k_i^A = 0$ for $i \in A$, $k_i^A = 1 + \sum_j p_{ij} k_j^A$ for $i \notin A$.

Proof

If $X_0 = i \in A$ then $H^A = 0$ so $k_i^A = 0$. If $X_0 = i \notin A$ then $H^A \geq 1$. $E(H^A | X_1 = j) = 1 + E_j(H^A) = 1 + k_j^A$

$k_i^A = E_i(H^A) = E_i\left(\sum_{j \in I} H^A 1_{\{X_1 = j\}}\right) = \sum_{j \in I} E_i(H^A 1_{\{X_1 = j\}})$

$= \sum_j E_i(H_A | X_1 = j) P_i(X_1 = j) = 1 + \sum_j p_{ij} k_j^A$

Suppose $y = (y_i : i \in I)$ satisfies 4·1 and is ≥ 0.
$k_i^A = y_i = 0$ for $i \in A$.
$i \notin A$, $y_i = 1 + \sum_{j \notin A} p_{ij} y_j + \cancel{\sum_{j \in A} p_{ij} y_j} = 0$

$y_i = 1 + \sum_{j \notin A} p_i (1 + \sum_{k \notin A} p_{jk} y_k) = p_i(H^A \geq 1) + p_i(H^A \geq 2) + \sum_{j,k \notin A} p_{ij} p_{jk} y_k$

By repeated substitution:
$y_i = p_i(H^A \geq 1) + p_i(H^A \geq 2) + \ldots + p_i(H^A \geq n) + \sum_{j_1, j_2, \ldots j_n \notin A} p_{ij_1} p_{j_1 j_2} \ldots p_{j_{n-1} j_n} y_{j_n}$
$\geq \sum_{k=1}^{n} p_i(H^A \geq k) \to E_i(H^A)$ as $n \to \infty$.
This is because for integer valued random variables, $E(X) = \sum_{k=1}^{\infty} p(X \geq k)$

4·3 Stopping Times

1. $H_i = \inf\{n : X_n = i\}$ is a stopping time.

2. $H_i + 1$ is a stopping time, but $H_i - 1$ is not.

3. $L_i = \sup\{n : X_n = i\}$ is not a stopping time.

A random variable $T : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a stopping time if $\{T = n\}$ depends only on $X_0, \ldots, X_n$.

4·4 Strong Markov Property

Theorem 4·4

Let $(X_n)_{n \geq 0}$ be Markov (λ, P) and let T be a stopping time. Then conditional on $T < \infty$ and $X_T = i$, $(X_{T+n})_{n \geq 0}$ is Markov (δ_i, P) and is independent of $X_0, \ldots, X_T$.

<u>Markov Chains ④</u>

<u>Proof</u> (Non-Examinable)

Let B be an event determined by $X_0, \dots, X_T$, and then $B \cap \{T = m\}$ is determined by $X_0, \dots, X_m$ and so by the Markov Property:

$$p(\{X_T = j_0, X_{T+1} = j_1, \dots, X_{T+n} = j_n\} \cap B \cap \{T = m\} \cap \{X_T = i\})$$
$$= p_i(\{X_0 = j_0, \dots, X_n = j_n\}) \, p(B \cap \{T = m\} \cap \{X_T = i\})$$

Sum over $m = 0, 1, 2, \dots$, and divide by $p(T < \infty, X_T = i)$

$$p(\{X_T = j_0, \dots, X_{T+n} = j_n\} \cap B \mid T < \infty, X_T = i)$$
$$= p(\{X_0 = j_0, \dots, X_n = j_n\}) \, p(B \mid T < \infty, X_T = i) \qquad \square$$

$$\circlearrowright 0 \xleftarrow{q} 1 \xrightarrow{p} 2 \ \dots \qquad h_i = p_i(\text{hit } 0)$$

$h_1 = ph_2 + q$, $\quad h_2 = ph_3 + q h_1 = h_1^2$

since $p_2(\text{hit } 0) = p_2(\text{hit } 1)\, p_1(\text{hit } 0) = p_1(\text{hit } 0)^2$

$h_1 = ph_1^2 + q \Rightarrow h_1 = \min\{1, \frac{q}{p}\}$

Suppose $q > p$. $k_i = E_i(\text{time to hit } 0)$

$k_1 = 1 + pk_2 + q \cdot 0$

$k_2 = 2k_1$, $\quad k_1 = 1 + 2pk_1 \Rightarrow k_1 = \frac{1}{q-p}$

Markov Chains (5)

Recurrence and Transience

Let $(X_n)_{n\geq 0}$ be a Markov Chain with transition matrix P.
$H_i = \inf\{n \geq 0 : X_n = i\}$ = hitting time of state i
$T_i = \inf\{n \geq 1 : X_n = i\}$ = first passage time to state i
$H_i = T_i$ unless we start in state i, $\{X_0 = i\}$

$V_i = \sum_{n=0}^{\infty} 1_{\{X_n = i\}}$ = number of visits to i.
$f_i = P_i(T_i < \infty)$ = return probability of i.
$m_i = E_i(T_i)$ = mean return time to i.
A state i is called recurrent if $P_i(V_i = \infty) = 1$, and otherwise, i is transient

5.2 Equivalence of recurrence and certainty of return

Lemma 5.1

For all $k \geq 0$, $P_i(V_i \geq k+1) = (f_i)^k$

Proof

By induction. This is true for $k=0$. Assume true for $k-1$.
$P(V_i \geq k+1) = \underbrace{P_i(V_i \geq k+1 \mid V_i \geq k)}_{= P(T_i < \infty)} P_i(V_i \geq k) = f_i (f_i)^{k-1}$

□

Theorem 5.2

We have the dichotomy: i is recurrent $\Leftrightarrow f_i = 1$, i is transient $\Leftrightarrow f_i <$

Proof

$P_i(V_i < \infty) = P_i\left(\bigcup_{k=1}^{\infty} \{V_i = k\}\right) = \sum_{k=1}^{\infty} P_i(V_i = k)$
$= \sum_{k=1}^{\infty} f^{k-1}(1 - f_i) = \begin{cases} 0, & f_i = 1 \\ 1, & f_i < 1 \end{cases}$

1

i transient $\Rightarrow f_i < 1$

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = E_i(V_i) = \sum_{r=0}^{\infty} p_i(V_i > r) = \sum_{r=0}^{\infty} (f_i)^r = \frac{1}{1-f_i} < \infty$$

5.4 Recurrence as a class property

Theorem 5.4

Let C be a communicating class. Then either all states in C are transient, or all are recurrent.

Proof

Take $i, j \in C$. Suppose i is transient. Because C is a communicating class, $\exists n, m : p_{ij}^{(n)} > 0$, $p_{ji}^{(m)} > 0$.

$$p_{ii}^{(n+m+r)} \geq p_{ij}^{(n)} p_{jj}^{(r)} p_{ji}^{(m)}$$

$$\sum_{r=0}^{\infty} p_{jj}^{(r)} \leq \frac{1}{p_{ij}^{(n)} p_{ji}^{(m)}} \sum_{r=0}^{\infty} p_{ii}^{(n+m+r)} < \infty \text{ since } i \text{ is transient.}$$

$\Rightarrow j$ is transient $\square$

5.5 Relation with closed classes

Theorem 5.6

Every recurrent class is closed.

Proof

Let C be a class which is not closed, $i \in C$, $j \notin C$. Then suppose $p_i(X_m = j) > 0$ for some m.

$$p_i(V_i = \infty) = \sum_k p_i(V_i = \infty \mid X_m = k)\, p(X_m = k) < 1$$

since $p_i(V_i = \infty \mid X_m = j) = 0$, $p_i(X_m = j) > 0$

$\Rightarrow i$ is transient.

Markov Chains (5)

Theorem 5.7

Every finite closed class is recurrent.

Proof

Let C be such a class. Pick any initial condition $p(X_0 = i) = \lambda_i$.

$\sum_{i \in C} V_i = \infty \Rightarrow V_i = \infty$ for some i.

$1 = p\left(\cup_{i \in C} \{V_i = \infty\} \right) \leq \sum_{i \in C} p(V_i = \infty)$

$\Rightarrow p(V_i = \infty) > 0$ for at least some i.

$0 < p(V_i = \infty) = p(H_i < \infty)\, p(V_i = \infty)$ since $p_i(V_i = \infty)$ is equal to 0 or 1.

$\Rightarrow p(H_i < \infty) > 0 \Rightarrow p_i(V_i = \infty) = 1$, i is recurrent.

Theorem 5.8

Suppose P is irreducible and recurrent. Then, $\forall j \in I$, we have $p(T_j < \infty) = 1$.

Proof

$p(T_j < \infty) = \sum_{i \in I} p(X_0 = i)\, p_i(T_j < \infty)$, so it suffices to prove $p_i(T_j < \infty) = 1 \ \forall i \in I$.

$p_{ji}^{(m)} > 0$ for some m.

$$1 = p_j(X_n = j \text{ for infinitely many } n)$$
$$\leq p_j(X_n = j \text{ for some } n \geq m+1)$$
$$= \sum_k p_j(X_n = j \text{ for some } n \geq m+1 \mid X_m = k)\, p_j(X_m = k)$$
$$= \sum_k p_k(T_j < \infty)\, p_{jk}^{(m)}$$

$k = i$. The fact that $p_{ji}^{(m)} > 0 \Rightarrow p_i(T_j < \infty) = 1$

$\textcircled{*}$ $\sum_k p_{jk}^{(m)} = 1$

Markov Chains (6)
Random Walks in One, two and three dimensions

6.1 Simple Random Walks on $\mathbb{Z}$

$$\cdots \quad -2 \quad -1 \quad 0 \quad 1 \underset{q}{\overset{p}{\rightleftarrows}} 2 \quad 3 \quad \cdots \qquad 0 < p = 1-q < 1$$

$f_0 = p_0(\text{return to } 0) = p_0(T_0 < \infty)$
$h_i = p_i(\text{hit } 0)$

$f_0 = q h_{-1} + p h_1$, $h_1 = q h_0 + p h_2 = q + p h_1^2$

Roots : $h_1 = \min(1, q/p)$
$p = q \Rightarrow h_1 = 1 \Rightarrow f_0 = 1 \Rightarrow$ Recurrence
$q < p \Rightarrow h_1 = q/p < 1 \Rightarrow f_0 < 1 \Rightarrow$ Transience

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} : \begin{cases} = \infty & \text{as } i \text{ is recurrent} \\ < \infty & \text{transient} \end{cases}$$

$p_{00}^{(2n+1)} = 0$, since you can't return in an odd number of steps.
n steps must be taken to the right, and n to the left.

$p^n q^n = p(\text{taking a given path})$, $p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$
$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} = A\sqrt{n}\left(\frac{n}{e}\right)^n$

$$p_{00}^{(2n)} = \frac{(2n)!}{(n!)^2}(pq)^n \sim \frac{(4pq)^n}{A\sqrt{\frac{n}{2}}} \quad \text{as } n \to \infty$$

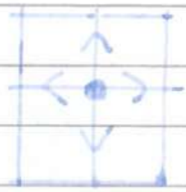
If $p = q = \frac{1}{2}$, $4pq = 1$, $p_{00}^{(2n)} \geq \cancel{\binom{2n}{n}} \frac{1}{2A\sqrt{n}}$
$\Rightarrow \sum_{n=1}^{\infty} p_{00}^{(2n)} = \infty$, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges
$\therefore$ The random walk is recurrent.

~~If~~
If $p \neq q$, $4pq < 1$.

$$\sum_{n=N}^{\infty} p_{00}^{(2n)} \leq \frac{1}{A}\sum_{n=N}^{\infty} r^n < \infty$$
$\therefore$ The random walk is transient.

6.2 Simple Symmetric Random Walk on $\mathbb{Z}^2$

This can be considered as two random walks on $\mathbb{Z}$, specifically along the lines $y = x$ and $y = -x$.

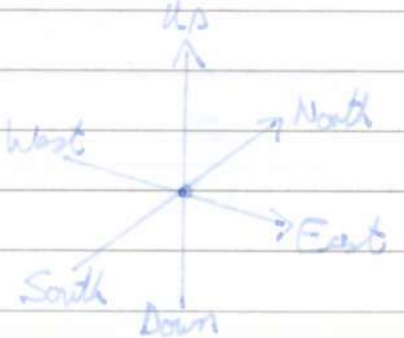


$$p_{ij} = \begin{cases} \frac{1}{4} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{00}^{(2n)} = \left(\binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right) \sim \frac{2}{A^2 n} \Rightarrow \sum_{n=N}^{\infty} p_{00}^{(2n)} = \infty$$

$\therefore$ This random walk is recurrent.

6.3 Simple, Symmetric Random Walk on $\mathbb{Z}^3$



$$p_{ij} = \begin{cases} \frac{1}{6} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$i + j + k = n$

$$p_{00}^{(2n)} = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \left(\frac{(2n)!}{(i!\,j!\,k!)^2} \right) \left(\frac{1}{6}\right)^{2n}$$

$$= \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \binom{n}{i\ j\ k}^2 \left(\frac{1}{3}\right)^{2n}$$

Observe that $\sum \binom{n}{i\ j\ k} \left(\frac{1}{3}\right)^n = 1$, since $\sum \binom{n}{i\ j\ k} = 3^n$, the total number of ways to put n balls in 3 boxes.

$$\binom{n}{i\ j\ k} = \frac{n!}{i!\,j!\,k!} \leq \frac{n!}{(m!)^3} = \binom{n}{m\ m\ m}, \text{ where } n = 3m$$

$$p_{00}^{(2n)} \leq \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \binom{n}{m\ m\ m} \left(\frac{1}{3}\right)^n \sim \frac{1}{2A^3} \left(\frac{6}{n}\right)^{\frac{3}{2}} \text{ as } n \to \infty$$

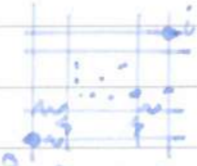
$$p_{00}^{(6m)} \geq p_{00}^{(6m-2)} \left(\frac{1}{6}\right)^2 \geq p_{00}^{(6m-4)} \left(\frac{1}{6}\right)^4$$

Since we know that $\sum \frac{1}{n^{3/2}} < \infty$, $\sum_{n=1}^{\infty} p_{00}^{(2n)} < \infty$

So this random walk is transient.

Markov Chains (6)

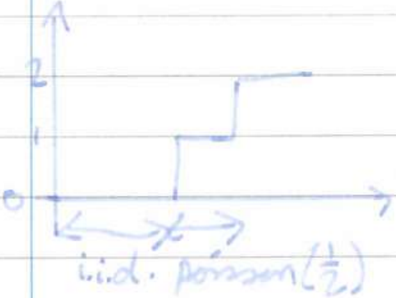
In $\mathbb{Z}^3$, $f_0 \approx 0.34$, and in $\mathbb{Z}^4$, $f_0 \approx 0.19$.



This can be used to show that for networks of 1Ω resistors, the resistance between 0 and $i \to \infty$ as $i \to \infty$, for 1 and 2D lattices

* A "Continuous" Version of a Random walk on $\mathbb{Z}^3$ *

Let $X^+(t), X^-(t), Y^+(t), Y^-(t), Z^+(t), Z^-(t)$ be six independent stochastic processes, identically distributed.



$$p(X^+(t) = X^-(t)) = \sum_{i=0}^{\infty} \left(\frac{(\frac{1}{2}t)^i}{i!} e^{-\frac{1}{2}t} \right)^2$$

$$= I_0(t) e^{-t} \sim (2\pi t)^{-\frac{1}{2}}$$

where I_0 is a modified Bessel function of the first kind, order 0.

$$\sum_{n=0}^{\infty} p_{00}^{(n)} = 3 \int_0^{\infty} p_{00}(t)\, dt < \infty \text{, transient}$$

27/10/11 Markov Chains 7

7 Invariant Distributions

7·1 Examples of invariant distributions

$\lambda = (\lambda_i : i \in I)$ is a <u>measure</u> if all $\lambda_i \geq 0$ and is a <u>distribution</u> if $\sum_{i \in I} \lambda_i = 1$.

λ is an <u>invariant measure</u> if $\lambda^T = \lambda^T P$ and is an <u>invariant distribution</u> if moreover $\sum_{i \in I} \lambda_i = 1$

<u>Example 7·1</u>

$P = \begin{pmatrix} 3/4 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\quad \lambda = (\frac{2}{3} \;\; \frac{1}{3})$, $\lambda P = \lambda$ so λ is an invariant distribution.

<u>Invariant distributions</u> are also called equilibrium, stationary and steady-state distributions.

- Do invariant measures/distributions exist? Are they unique?
- What does an invariant distribution tell us?
- How do we calculate an invariant distribution?

7·2 Notation

$T_i = \inf\{n \geq 1 : X_n = i\}$ = <u>first passage time</u> to i

$m_i = E_i(T_i)$ = <u>mean return time</u> to i.

$V_j(n) = \sum_{k=0}^{n-1} 1_{\{X_k = j\}}$ = # visits to j before time n

$V_j^i = V_j(T_i)$ = # visits to j before first return to i.

$\gamma_j^i = E_i(V_j^i)$ = expected # visits to j before return to i

$V_i^i = 1$, $E(V_i^i) = \gamma_i^i = 1$

7·3 What does an invariant measure tell us?

Suppose π is an invariant distribution. $\sum_i \pi_i = 1$, $\pi = \pi P$

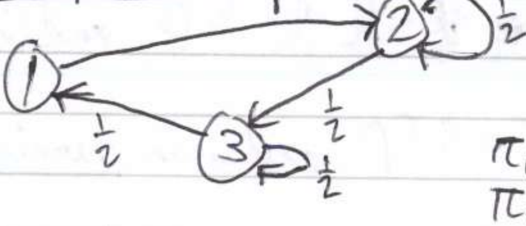
$m_i = \frac{1}{\pi_i}$. $\gamma_j^i = \frac{\pi_j}{\pi_i} = \frac{\lambda_j}{\lambda_i}$ (if λ is an invariant measure)

$P\left(\frac{V_j(n)}{n} \to \pi_j \text{ as } n \to \infty\right) = 1$

$\nearrow \pi = \pi P$

7.4 Invariant Distribution as the solution to Left Hand Equations

Example 7.3



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\pi = (\pi_1, \pi_2, \pi_3) \qquad \left.\begin{aligned} \pi_1 &= \tfrac{1}{2}\pi_3 \\ \pi_2 &= \pi_1 + \tfrac{1}{2}\pi_2 \\ \pi_3 &= \tfrac{1}{2}\pi_2 + \tfrac{1}{2}\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right\} \Rightarrow \pi = \left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}\right)$$

$r^2 = \frac{1}{2} \qquad p_{11}^{(n)} = a + \left(\frac{1}{2}\right)^n \left(b \cos \frac{n\pi}{2} + c \sin \frac{n\pi}{2}\right)$

$p_{11}^{(n)} \to \pi_1 = \frac{1}{5}$ as $n \to \infty \Rightarrow a = \frac{1}{5}$

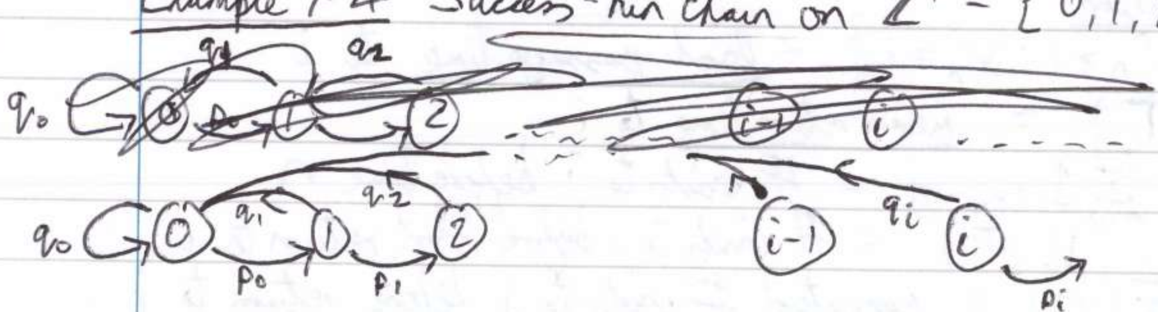
Sometimes we can find π from detailed balance equations

$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall\, i, j \in I$

Sum on j: $\pi_i = \sum_j \pi_j p_{ji}$ detailed balance equations $\Rightarrow$ left hand equations

$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \pi_1 \alpha = \pi_2 \beta \quad \Rightarrow \pi_1 = \frac{\beta}{\alpha+\beta},\ \pi_2 = \frac{\alpha}{\alpha+\beta}$

Example 7.4 Success-run chain on $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$



$p_{i\,i+1} = p_i, \ p_{i0} = q_i = 1 - p_i$

$\pi = \pi P \qquad \pi_0 = \sum_{i=0}^{\infty} \pi_i q_i \Rightarrow p_0 \pi_0 = \sum_{i=1}^{\infty} \pi_i q_i$

$\pi_i = \pi_{i-1} p_{i-1}, \quad i \geq 1$

$\quad = \dots = p_{i-1} p_{i-2} \cdots p_0 \pi_0 \qquad \pi_i = r_i \pi_0$

$r_0 = 1, \ r_i = p_0 p_1 \cdots p_{i-1}$

Choose $p_i \to 1$ sufficiently fast that $r = \prod_{i=0}^{\infty} p_i > 0 \quad \left(\equiv \sum_{i=0}^{\infty} q_i < \infty\right)$

$q_i \pi_i = (1 - p_i) r_i \pi_0 = (r_i - r_{i+1}) \pi_0$

$\pi_0 = \sum_{i=0}^{\infty} \pi_i q_i = \lim_{n \to \infty} \sum_{i=0}^{n} \pi_i q_i = \lim_{n \to \infty} \sum_{i=0}^{n} (r_i - r_{i+1}) \pi_0$

$= \lim_{n \to \infty} (r_0 - r_{n+1}) \pi_0 = (1 - r) \pi_0, \ r > 0 \Rightarrow \pi_0 = 0$

$\nexists$ an invariant distribution $\qquad \Rightarrow \pi_i = 0 \ \forall i$

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Markov Chains (7)

If $r=0$, $\prod_{i=0}^{\infty} p_i = 0$ then we do have an invariant distributio
$\pi_i = \pi_0 r_i$

$$\pi_i = \frac{r_i}{\sum_{j=0}^{\infty} r_j}, \quad \pi_0 = \frac{1}{\sum_{j=0}^{\infty} r_j} \quad \text{if } \sum_{j=0}^{\infty} r_j < \infty$$

7.5 Stationary Distributions

Theorem 7.3 Let $(X_n)_{n\geq 0}$ be Markov(λ, P) where λ is an invariant distribution for P. Then $(X_{n+m})_{n\geq 0}$ is also Markov(λ, P)

Proof

$$P(X_m = i) = (\lambda P^{(m)})_i = (\lambda P^m)_i = \lambda_i = P(X_0 = i)$$

Clearly, after m the transition matrix remains P.

7.6 Equilibrium Distributions

Theorem 7.6 Let I be finite. Suppose that for some $i \in I$ $p_{ij}^{(n)} \to \pi_j \ \forall j \in I$. Then $\pi = (\pi_j : j \in I)$ is an invariant distribution.

Proof.

$$\sum_j \pi_j = \sum_j \lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} \underbrace{\sum_j p_{ij}^{(n)}}_{=1} = 1$$

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} \sum_{k\in I} p_{ik}^{(n-1)} p_{kj} = \sum_{k\in I} \lim_{n\to\infty} p_{ik}^{(n)} p_{kj} = \sum \pi_k p_{kj}$$

$\Rightarrow \pi$ satisfies the Left Hand Equations, and is in invariant distribution

lim and $\sum$ can be swapped because I is finite.

(Compare what happened with the random walk on $\mathbb{Z}$).

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Markov Chains (8)

8 Existence and uniqueness of the invariant distribution, mean return Time, positive and null recurrence

8·1 Existence and uniqueness up to a constant multiple

$\lambda = \lambda P$. Does λ exist? Is it unique

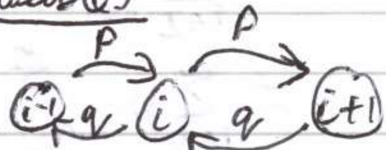
Example 8·1 (Non Uniqueness)

$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda = \lambda P \quad \forall \lambda$

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\quad \lambda = (1, 0, 0)$, $\lambda = (0, \frac{1}{2}, \frac{1}{2})$

P is not irreducible.

Example 8·2 (non-uniqueness when irreducible)



asymmetric random walk on $\mathbb{Z}$

Left hand equation $\quad \pi_i = \pi_{i-1} p + \pi_{i+1} q$

$\pi_i = 1 \quad \forall i$ is a solution

$\pi_i = (p/q)^i \quad \forall i$ is a solution

But (for $p \neq q$) this Markov Chain is transient.

Theorem 8·3 Suppose P is irreducible and $\lambda \geq 0$, $\lambda = \lambda P$.
Then $\lambda \equiv 0$, or $(0 < \lambda_i < \infty \ \forall i)$ or $\lambda \equiv \infty$

Proof:

$\lambda = \lambda P = \lambda P^n$. Given i, j, $\exists n$ such that $p_{ij}^{(n)} > 0$

$$\lambda_j = \sum_k \lambda_k p_{kj}^{(n)} \geq \lambda_i p_{ij}^{(n)}$$

$\lambda_j < \infty \Rightarrow \lambda_i < \infty$, and $\lambda_i > 0 \Rightarrow \lambda_j > 0 \quad \square$

$P\underline{1} = \underline{1}$, $\lambda P = \lambda 1$

$$\gamma_i^k = E_k \sum_{k=0}^{T_k - 1} 1_{\{x_n = i\}}$$

1

Theorem 8.4 (Existence of an invariant measure)
Let P be irreducible and recurrent. Then:

i) $\gamma_k^k = 1$

ii) $\gamma^k = (\gamma_i^k : i \in I)$ satisfies $\gamma^k P = \gamma^k$

iii) $0 < \gamma_i^k < \infty \ \forall i$

Proof

i) is Obvious.

ii) $P_k(X_{n-1} = i \text{ and } X_n = j \text{ and } n \le T_k)$ ← depends only on $X_0, \dots, X_{n-1}$

$= P_k(X_{n-1}, n \le T_k) p_{ij}$

since $P(T_k < \infty) = 1 = f_k$

$$\gamma_j^k = E_k \sum_{n=1}^{T_k} 1_{\{X_n = j\}} = E_k \sum_{1}^{\infty} 1_{\{X_n = j, n \le T_k\}}$$

$$= \sum_{i \in I} \sum_{n=1}^{\infty} P_k(X_{n-1} = i, X_n = j, n \le T_k)$$

$$= \sum_{i \in I} p_{ij} \sum_{n=1}^{\infty} P_k(X_{n-1} = i \text{ and } n \le T_k)$$

$$= \sum_{i \in I} p_{ij} E_k \sum_{n=1}^{\infty} 1_{\{X_{n-1} = i\} \cap \{n \le T_k\}}$$

$$= \sum_{i \in I} p_{ij} E_k \sum_{n=0}^{\infty} 1_{\{X_n = i, n \le T_k - 1\}}$$

$$= \sum_{i \in I} p_{ij} E_k \sum_{n=0}^{T_k - 1} 1_{\{X_n = i\}} = \sum_i p_{ij} \gamma_i^k$$

iii) $\gamma_k^k = 1 \in (0, \infty) \Rightarrow \gamma_i^k \in (0, \infty) \ \forall i$ By Theorem 8.3

$\exists m, n$ such that $p_{ik}^{(n)}, p_{ki}^{(m)} > 0$

$\gamma_i^k \ge \gamma_k^k p_{ki}^{(m)} > 0$

$1 = \gamma_k^k \ge \gamma_i^k p_{ik}^{(n)} \Rightarrow \gamma_i^k < \infty$

1/11/11

Markov Chains (8)

Theorem 8.5 (Uniqueness of an Invariant Measure)

Let P be irreducible and let $\lambda = \lambda P$ with $\lambda_k = 1$. Then $\lambda \geq \gamma^w$. If in addition P is recurrent then $\lambda = \gamma^k$. $\gamma_k^k = 1$.

Proof

$$\lambda_j = \sum_{i_0 \in I} \lambda_{i_0} p_{i_0 j} = p_{kj} + \sum_{i_0 \neq k} \lambda_{i_0} p_{i_0 j}$$

$$= p_{kj} + \sum_{i_0 \neq k} p_{k i_0} p_{i_0 j} + \sum_{i_0, i_1 \neq k} \lambda_{i_1} p_{i_1 i_0} p_{i_0 j}$$

$$= \dots$$

$$= p_{kj} + \sum_{i_0 \neq k} p_{k i_0} p_{i_0 j} + \dots + \sum_{i_0, i_1, \dots, i_n \neq k} p_{k i_{n-1}} p_{i_{n-1} i_{n-2}} \cdots p_{i_1 i_0} p_{i_0 j}$$

$$+ \sum_{i_0, \dots, i_n \neq k} \lambda_{i_n} p_{i_n i_{n-1}} \cdots p_{i_0 j}$$

$$\geq P_k(X_1 = j, T_k \geq 1) + P_k(X_2 = j, T_2 \geq 2) + \dots$$

$$+ P_k(X_n = j, T_k \geq n) \to \gamma_j^k$$

$\therefore \lambda_j \geq \gamma_j^k$

Suppose λ is also an invariant measure.

$$\mu = \lambda - \gamma^k \geq 0$$

$$\mu_k = \lambda_k - \gamma_k^k = 1 - 1 = 0$$

Theorem 8.3

But for an invariant measure either $\mu \equiv 0$, $\mu \equiv \infty$ or $0 < \mu_i < \infty \ \forall i$. $\Rightarrow \mu = 0$, $\lambda \equiv \gamma^k$ $\square$

8.2 Mean return time and positive and null recurrence

$m_i = E_i(T_i)$ mean return time

If $m_i < \infty$ we say i is positive recurrent, otherwise i is null recurrent. ($m_i = \infty$)

$P_i(T_i < \infty) = 1$

For any irreducible Markov Chain, either:

- P is invariant
- P is null recurrent, $m_i = \infty \ \forall i$
- P is positive recurrent, $m_i < \infty \ \forall i$ and $\exists$ an invariant distribution π with $\pi = \pi P$. $\sum \pi_i = 1$

8.3 Random Surfer

Google Page Rank Algorithm:

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Markov Chains (9)

9 Convergence to equilibrium for ergodic chains

9.1 Equivalence of positive recurrence and existence of an invariant measure

Recall Theorem 7.6. If $p_{ij}^{(n)} \to \pi_j$ as $n \to \infty$, $\forall i,j$, then π is an invariant distribution

Theorem 9.1 Let P be irreducible. Then the following are equivalent:

i) Every state is positive recurrent

ii) Some state is positive recurrent $m_i = E_i(T_i) < \infty$

iii) P has an invariant distribution, π.

Moreover, when iii) holds we have $m_i = \frac{1}{\pi_i}$ $\forall i$

Proof

Obviously i) $\Rightarrow$ ii)

ii) $\Rightarrow$ iii) i recurrent $\Rightarrow P$ is recurrent

γ^i is an invariant, $\sum_{j \in I} \gamma_j^i = m_i < \infty$

$\pi_j = \gamma_j^i / m_i$ is an invariant distribution.

iii) $\Rightarrow$ i) $\sum_{i \in I} \pi_i = 1 \Rightarrow \pi_k > 0$ for some k

$\lambda_i = \pi_i / \pi_k$ is an invariant measure

$\lambda_k = 1 \Rightarrow \lambda \geq \gamma^k$ (by Theorem 8.5)

$$m_k = \sum_i \gamma_i^k \leq \sum_i \pi_i / \pi_k = 1/\pi_k < \infty$$

$\Rightarrow k$ is positive recurrent.

But now Theorem 8.5 tells us that $\lambda = \gamma^k$ since P is recurrent

$\Rightarrow m_k = 1/\pi_k$

Example 9.2 Symmetric walk on $\mathbb{Z}$

$\lambda_i = 1$ is an invariant measure.

$\sum \lambda_i = 1 \Rightarrow \nexists$ an invariant distribution

$\Rightarrow$ This random walk is null recurrent. $m_0 = \infty$

Example 9.3 Existence of an invariant measure does not imply recurrence.

The random walk on $\mathbb{Z}^3$ has measure $\lambda_i = 1$ $\forall i$

$$\lambda_{(i,j,k)} = \frac{1}{6}\lambda_{(i+1,j,k)} + \cdots + \frac{1}{6}\lambda_{(i,j,k-1)}$$

Example 9.4 Aperiodic chains

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^3 = P$$

$$p_{11}^{(n)} = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \quad \text{so } p_{11}^{(n)} \not\to \text{limit}$$

We say i is aperiodic if $p_{ii}^{(n)} > 0 \ \forall n$ sufficiently large

Lemma 9.5

A state I is aperiodic if $\exists\, n_1, \ldots, n_k \geq 1$ with no common factor such that $p_{ii}^{(n_j)} > 0 \ \forall j = 1, \ldots, k$

Proof

For all sufficiently large n, we can write

$n = a_1 n_1 + \ldots + a_k n_k$ for some non-negative integers $a_1, \ldots, a_k$.

$$p_{ii}^{(n)} = \underbrace{p_{ii}^{(n_1)} \cdots p_{ii}^{(n_1)}}_{a_1 \text{ times}} \underbrace{p_{ii}^{(n_2)} \cdots p_{ii}^{(n_2)}}_{a_2 \text{ times}} \cdots \underbrace{p_{ii}^{(n_k)} \cdots p_{ii}^{(n_k)}}_{a_k \text{ times}} > 0$$

Lemma 9.6

Suppose P is irreducible and has an aperiodic state i. Then for all states j, k, $p_{jk}^{(n)} > 0 \ \forall$ sufficiently large n (In particular, $p_{ji}^{(n)} = 0$)

Proof

$$p_{jk}^{(r+n+s)} \geq \underbrace{p_{ji}^{(r)}}_{>0 \text{ for } r} \underbrace{p_{ii}^{(n)}}_{>0 \text{ for large enough } n} \underbrace{p_{ik}^{(s)}}_{>0 \text{ for } s} > 0 \quad \forall \text{ large enough } n$$

A Markov Chain is regular if $P^n > 0$ for some n.

A Markov Chain is ergodic if it is irreducible, positive recurrent, and aperiodic.

$p_{ij}^{(n)} \to \pi_j \quad \forall i, j$

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Markov Chains (9)

9.3 Convergence to equilibrium for ergodic chains and *proof by coupling*

Theorem 9.8

Let P be the transition matrix of an ergodic Markov Chain with invariant distribution π. Then for any initial distribution $P(X_n = j) \Rightarrow \pi_j$ as $n \to \infty$ $\forall j$, $p_{ij}^{(n)} \to \pi_j$ as $n \to \infty$ $\forall i, j$

Proof

Let $(X_n)_{n\geq 0}$, $(Y_n)_{n\geq 0}$ be independent Markov Chains with initial distributions λ, π. Markov(λ, P), markov(π, P)

Pick some state $b \in I$. $T = \inf\{n \geq 1 : X_n = Y_n = b\}$

Step 1 $p(T < \infty) = 1$

$W_n = (X_n, Y_n)$

Initial distribution of $(W_n)_{n\geq 0}$ is $P(W_0 = (i,j)) = \lambda_i \pi_j$

$\tilde{P}_{(ik)(jl)} = p_{ij} p_{kl}$, $\tilde{P}^{(n)}_{(ik)(jl)} = p_{ij}^{(n)} p_{kl}^{(n)} > 0$

for all sufficiently large $n \Rightarrow \tilde{P}$ is irreducible.

$\tilde{P}$ has invariant distribution $\tilde{\pi}_{(i,j)} = \pi_i \pi_j$

$\tilde{\pi}\tilde{P} = \tilde{\pi}$ (easy to check). So by Theorem 9.1 all states are positive recurrent $\Rightarrow p(T < \infty) = 1$

Step 2 Y_n has initial distribution π.

$p(Y_n = i) = \pi_i$ $\forall n$

$$p(Y_n = i) = \pi_i = \underbrace{p(Y_n = i, n \geq T)}_{\to\ \pi_i} + \underbrace{p(Y_n = i, n < T)}_{\to\ 0\ \text{as } n \to \infty,\ \leq p(T = \infty)}$$

$$p(X_n = i) = p(X_n = i, n \geq T) + p(X_n = i, n < T)$$

$$p(X_n = i, n \geq T) = p(Y_n = i, n \geq T) \to \pi_i$$

$$p(X_n = i, n < T) \leq p(n < T) \to p(T = \infty) =$$

$\Rightarrow p(X_n = i) \to \pi_i$ as $n \to \infty$

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$X_0 = 1$. If Y_n starts at 2, they never couple.

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Markov Chains (10)

10 Long run proportion of time spent in a given state

$V_i(n) = \sum_{k=0}^{n-1} 1_{\{X_k = i\}}$

Theorem 10·2 (Ergodic Theorem)

Let P be irreducible and λ be any initial distribution. Suppose $(X_n)_{n\geq 0}$ is Markov(λ, P). Then

$P\left(\frac{V_i(n)}{n} \to \frac{1}{m_i} \text{ as } n \to \infty\right) = 1$ with $m_i = E_i(T_i)$

Moreover, in the positive recurrent case, for any bounded function $f: I \to \mathbb{R}$

$$P\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to \bar{f} \text{ as } n \to \infty\right) = 1$$

where $\bar{f} = \sum_i \pi_i f(i)$ and π is the invariant distribution.

Proof

If P is transient, V_i is finite. $\frac{V_i(n)}{n} \leq \frac{V_i(\infty)}{n} \to 0$ as $n \to \infty$

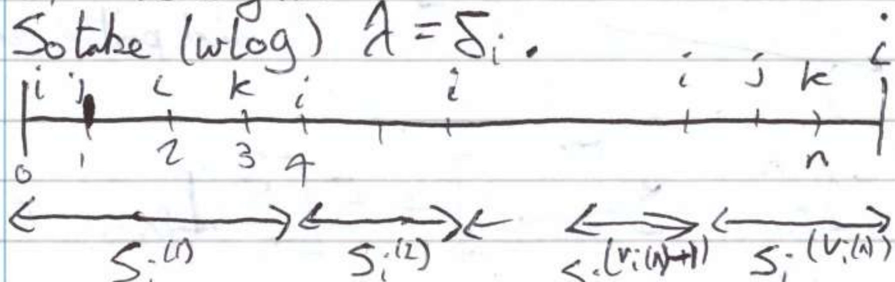
$m_i = \infty$ $\qquad$ "$\frac{1}{m_i}$"

Suppose P is recurrent. Pick a state i.

$T = T_i$, $P(T < \infty) = 1$

$(X_n)_{n\geq 0}$, $(X_{n+T})_{n \geq 0}$ have the same proportion of time spent in state i, in the long run.

So take (wlog) $\lambda = \delta_i$.



$T_i^{(0)} = 0$, $T_i^{(1)} = T_i = \inf\{n \geq 1, X_n = i\}$

$T_i^{(r+1)} = \inf\{n \geq T_i^{(r)} + 1, X_n = i\}$

$(r+1)^{th}$ passage to state i

$S_i^{(r)} = T_i^{(r)} - T_i^{(r-1)}$ $\qquad T_i^{(r)} < \infty$

$\quad = 0$ $\qquad$ otherwise

$S_i^{(1)} + \ldots + S_i^{(V_i(n)-1)} \leq n-1$

time of last visit to i before n

$S_i^{(1)} + \ldots + S_i^{(V_i(n))} \geq n$

time of last visit to i after $n-1$

$$\Rightarrow \frac{S_i^{(1)} + \ldots + S_i^{(V_i(n)-1)}}{V_i(n)} \leq \frac{n}{V_i(n)} \leq \frac{S_i^{(1)} + \ldots + S_i^{(V_i(n))}}{V_i(n)}$$

$$P\left(\frac{S_i^{(1)} + \dots + S_i^{(n)}}{n} \to E S_i^{(1)} = m_i \text{ as } n \to \infty\right) = 1$$
$$P(V_i(n) \to \infty \text{ as } n \to \infty) = 1$$
$$\Rightarrow P\left(\frac{n}{V_i(n)} \to m_i \text{ as } n \to \infty\right) = 1$$
$$P\left(\frac{V_i(n)}{n} \to \frac{1}{m_i} \text{ as } n \to \infty\right) = 1$$

Example $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $p_{11}^{(n)} > 0$ iff $2 \mid n$

$\pi = (\tfrac{1}{2}, \tfrac{1}{2})$

$$P\left(\frac{V_1(n)}{n} \to \frac{1}{2} = \pi_1 \text{ as } n \to \infty\right) = 1 \qquad \pi = (\tfrac{1}{2}, \tfrac{1}{2})$$

$m_i = 2$, $X_n = 1, 2, 1, 2, \dots$

$$\frac{\sum_{i=0}^{n-1} p_{ij}^{(n)}}{n} = \frac{E \sum_{0}^{n-1} 1_{\{X_n = j\}}}{n} \to \pi_j \quad L^1 \text{ convergence.}$$

*10.2 ~~Kenrys~~ Kemeny's Constant and the Random Target Lemma *

$m_{ij} = E_i(T_j)$ $\quad m_{ii} = m_i$

$(X_n)_{n \ge 0}$ is Markov$(\cancel{\delta_i}, P)$

$(Y_n)_{n \ge 0}$ is Markov(π, P) invariant distribution

(We assume that our Markov Chain is irreducible and positive recurrent)

$$Z_{ij} = E\left[\sum_{n=0}^{\infty} (1_{\{X_n = j\}} - 1_{\{Y_n = j\}})\right]$$
$$\sum_{n=0}^{\infty} (p_{ij}^{(n)} - \pi_j)$$

$$Z_{ij} = E\left[\sum_{n=0}^{T_j - 1} (1_{\{X_n = j\}} - \overset{\pi_j}{\cancel{1_{\{Y_n = j\}}}})\right] + E\left[\sum_{n=T_j}^{\infty} (1_{\{X_n = j\}} - \pi_j)\right]$$
$$= \delta_{ij} - \pi_j E_i(T_j) + Z_{jj}$$

Put $i = j \Rightarrow m_j = E_j(T_j) = \frac{1}{\pi_j}$ $\qquad m_{ij} = \frac{Z_{jj} - Z_{ij}}{\pi_j}$

$X_0 = i$. Pick j according to π.

$$k_i = \sum_j \pi_j E_i(T_j) = \sum_j \cancel{\pi_j} \frac{(Z_{jj} - \cancel{Z_{ij}})}{\cancel{\pi_j}} = \sum_j Z_{jj} = k$$

Which does not depend on i.

Kemeny's constant

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Markov Chains (10)

Lemma 10·3 (random target lemma)

Starting in state i, the expected time to hit state j that is chosen according to the invariant distribution is $\sum_j \pi_j E_i(T_j) = k$
(a constant not depending on i)

Proof

$k_i = \cancel{1} + \sum p_{ij} k_j - \cancel{\pi_i m_i}$ ⟵ Since $E_i(T_i) = 0$ if $i = j$
when we begin in the target state

$k = (k_1, \ldots, k_N)$, $k = Pk \Rightarrow k$ must be a constant vector

$k = (k, \ldots, k)$

$\Pi = \begin{pmatrix} -\pi- \\ -\pi- \end{pmatrix}$ has all rows equal to π. $P^n \to \Pi$

$\Pi P = P\Pi = \Pi^2 = \Pi \Rightarrow (P-\Pi)^k = P^k - \Pi$

$$\bar{Z} = Z + \Pi = I + (P - \Pi) + (P^2 - \Pi) + \ldots$$
$$= I + (P-\Pi) + (P-\Pi)^2 + \ldots$$
$$= (I - (P - \Pi))^{-1}$$

$k = tr(Z) = trace(\bar{Z} - \Pi) = trace(\bar{Z}) - 1$

$\bar{Z}^{-1} = I - (P - \Pi)$ ⟵ has eigenvalues $1, 1-\lambda_2, \ldots, 1-\lambda_N$
where $1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of P.

$$\Rightarrow k = \sum_{i=2}^{N} \frac{1}{1-\lambda_i}$$

e.g. $P = \begin{pmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{pmatrix}$, eigenvalues $1, 1-\alpha-\beta$

$$k = \frac{1}{\alpha+\beta}$$

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Markov Chains (11)

Time reversal, detailed balance, reversability, random walk on a graph

11.1 Time reversal

Theorem 11.1: Let P be irreducible with invariant distribution π. Suppose $(X_n)_{n\geq 0}$ is Markov (π, P) and set $Y_n = X_{N-n}$, $n=0,\dots$
Then $(Y_n)_{0\leq n\leq N}$ is Markov $(\pi, \hat{P})$

$$\pi_j \tilde{p}_{ji} = \pi_i p_{ij}$$

Proof

Check $\hat{P}$ is a stochastic matrix: $\sum_i \hat{p}_{ji} = \frac{1}{\pi_j}\sum_i \pi_i p_{ij} = \frac{\pi_j}{\pi_j} = 1$

Check $\hat{P}$ has invariant distribution π:

$$\sum_j \pi_j \hat{p}_{ji} = \sum_j \pi_i p_{ij} = \pi_i$$

$$p(Y_0 = i_0, Y_1 = i_1, \dots, Y_N = i_N)$$
$$= p(X_0 = i_N, X_1 = i_{N-1}, \dots, X_N = i_0)$$
$$= \pi_{i_N} p_{i_N i_{N-1}} \dots p_{i_1 i_0} = \pi_{i_0} \hat{p}_{i_0 i_1} \dots \hat{p}_{i_{N-1} i_N}$$

by Theorem 1.3 we have (Y_n) is Markov $(\pi, \hat{P})$

$$p_{i_0 i_1} > 0, \dots, p_{i_{n-1} i_n} > 0$$

$$\frac{\pi_{i_0}}{\pi_{i_n}} p_{i_0 i_1} \dots p_{i_{n-1} i_n} = \hat{p}_{i_n i_{n-1}} \dots p_{i_1 i_0} \qquad \square$$

11.2 Detailed Balance

A stochastic matrix P and a measure λ are said to be in detailed balance if $\lambda_i p_{ij} = \lambda_j p_{ji} \ \forall i,j$.

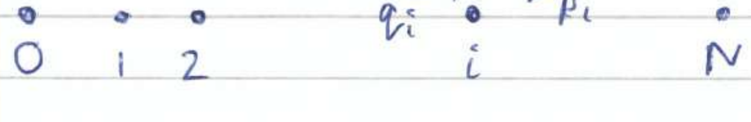
Lemma 11.2 If P and λ are in detailed balance then λ is an invariant measure for P.

$$\sum_i \lambda_i p_{ij} = \sum_i \lambda_j p_{ji} = \lambda_j \qquad \square$$

Examples 11.3

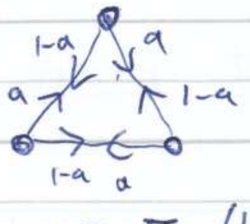
a) $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ $\qquad \pi_1 \times \frac{1}{2} = \pi_2 \times \frac{2}{3} \Rightarrow \pi_1 = \frac{4}{7}, \ \pi_2 = \frac{3}{7}$

b) Walk on $0, 1, 2, \dots, N$



Detailed balance

$$\pi_i p_i = \pi_{i+1} q_{i+1}$$
$$\Rightarrow \pi_i = \frac{p_{i-1} \dots p_0}{q_i \dots q_1} \pi_0$$

c) 

$$P = \begin{pmatrix} 0 & a & 1-a \\ 1-a & 0 & a \\ a & 1-a & 0 \end{pmatrix}$$

$\pi_1 a = \pi_2 (1-a)$ No solution unless $a = \frac{1}{2}$
$\pi_2 a = \pi_3 (1-a)$
$\pi_3 a = \pi_1 (1-a)$

11.3 Reversibility

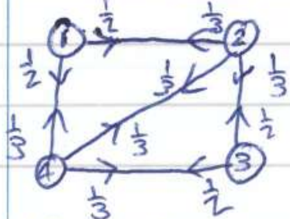
Let $(X_n)_{n \geq 0}$ be Markov (λ, P) with P irreducible. We say that $(X_n)_{n \geq 0}$ is reversible if $\forall N \geq 1$
$(X_{N-n})_{0 \leq n \leq N}$ is also Markov (λ, P)

Theorem 11.4 Let P be an irreducible stochastic matrix and let λ be a distribution. Suppose that $(X_n)_{n \geq 0}$ is Markov (λ, P). Then the following are equivalent.
a) $(X_n)_{n \geq 0}$ is reversible
b) P and λ are in detailed balance

Proof
a) $\hat{p}_{ij} = \frac{\lambda_j}{\lambda_i} p_{ji} = p_{ij} \iff$ b) $\forall i, j$
(and λ is an invariant distribution
X_n has the same distribution as X_0)

11.4 Random walks on a graph

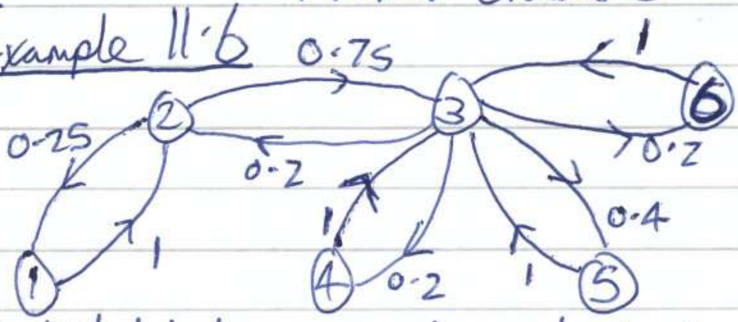
$G = (V, E)$



$p_{ij} = \frac{1}{v_i}$ if $(i,j) \in E$
$(v_i$ is the "valence" or "degree" of vertex i)
$p_{ij} = 0$ if $(i,j) \notin E$
$v_i p_{ij} = 1 = v_j p_{ji} \quad \forall (i,j) \in E$
So P and v are in detailed balance, $v = (v_1, \ldots, v_4)$
$\Rightarrow \pi_i = v_i / \sum_j v_j$ so $\sum_i \pi_i = 1$, $v = (2, 3, 2, 3)$
$\pi = (\frac{1}{5}, \frac{3}{10}, \frac{1}{5}, \frac{3}{10})$

10/11/11 Markov Chains (11)

Example 11·6



All nodes are connected, no cycles

Detailed balance can be solved for any graph that is a $\boxed{\text{tree}}$

$\pi_i p_{ij} = \pi_j p_{ji}$

$\pi_1 p_{12} p_{23} p_{35} = \pi_5 p_{53} p_{32} p_{21}$

(N.B. need only consider direct path $1 \to 5$ as any other path generates terms on both sides which cancel)

Example 11·7 A random knight makes each permissible move with equal probability. If it starts on a corner, how long on average does it take to return?

$\sum_{i=1}^{64} V_i = 336$. $\pi_{corner} = \frac{2}{336} = \frac{1}{168}$

$M_{corner} = \frac{1}{\pi_{corner}} = 168$

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Markov Chains (12)

12 Concluding Problems and Recommendations for further study

$(X_n)_{n\geq 0}$ is Markov(λ, P)

$P = (p_{ij})$, $P^{(n)} = (p_{ij}^{(n)})$

h_i^A, k_i^A — absorption, RHEs

T_i, H_i, f_i — recurrence / transience

V_i, $V_i(n)$ — invariants, LHEs

λ, π

γ_i^k

m_i — mean recurrence time

12.1 Reversability and the Ehrenfest Urn model

Example 12.1

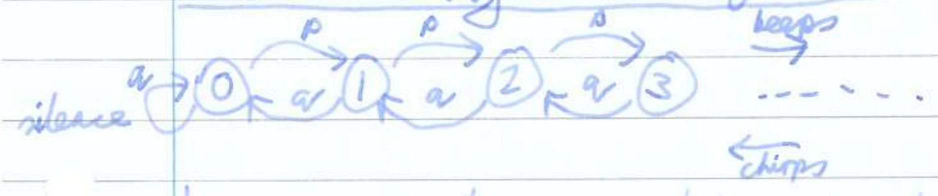
| i balls | | N−i balls |

$$0 \xrightarrow{1} 1 \; \cdots \; (i-1) \xleftarrow{\frac{i}{N}} i \xrightarrow{1-\frac{i}{N}} (i+1) \qquad \xleftarrow{1} N$$

Solved detailed balance equation : $\pi_i = \frac{1}{2^N}\binom{N}{i}$, $i = 0, 1, \ldots, N$

$i = \frac{N}{2}$, $\pi_{\frac{N}{2}} \sim \frac{1}{\sqrt{\pi \frac{N}{2}}}$, $m_{\frac{N}{2}} = \sqrt{\pi \frac{N}{2}}$

12.2 Reversability and the Reflected Random Walk



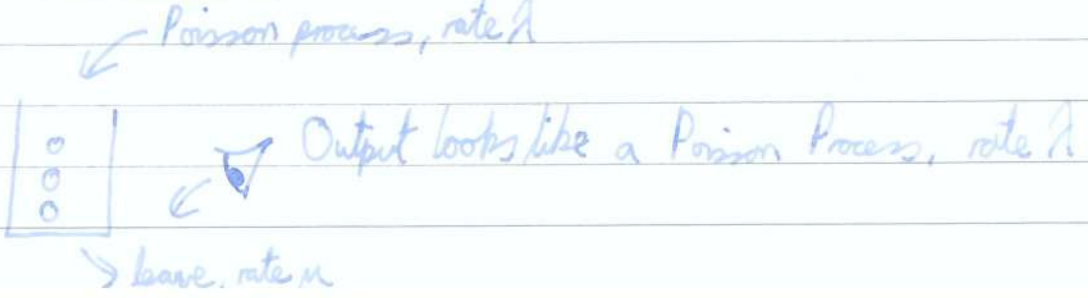
beeps occur as times separated as geometric random variables with parameter p

The mean time between beeps is $1/p$.

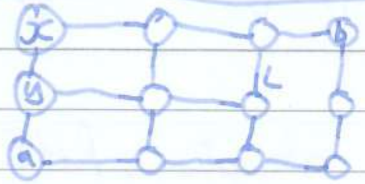
Similarly, the mean chirp time is $\frac{1}{q}$, for $i > 0$. But if $i = 0$, this is $\frac{1}{p} + \frac{1}{q}$.

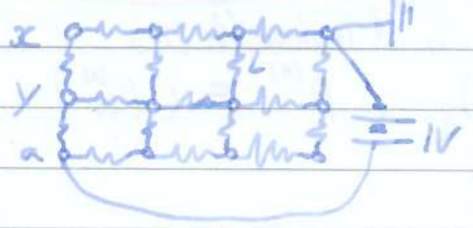
But this Markov Chain is reversible, so the chirps and beeps processes must be statistically identical.

Burke's Theorem M/M/1 Queue



12·4 Random Walks and Electrical Networks





$h_x = p_x(H_a < H_b) \qquad h_a = 1, \; h_b = 0$

$h_x = \sum_y p_{xy} h_y = \sum_y \frac{1}{d_x} h_y \qquad x \neq a, b$

$i_{xy} = \frac{\text{voltage drop from } x \text{ to } y}{\text{resistance between } x \text{ and } y} = \frac{V_x - V_y}{1}$ (Ohm's Law)

$0 = \sum_y i_{xy} = \sum_y (V_x - V_y) \Rightarrow V_x = \frac{1}{d_x} \sum_y V_y$

$V_a = 1, \quad V_b = 1$ (Kirchoff's Law)

$\Rightarrow h = V$ since they satisfy the same equations

(Harmonic functions $V = PV$, $h = Ph$)

What is $h_a = P_a(T_a < H_b)$? How does this change if we remove an edge of the graph, say L.

$\sum_y i_{ay} = \sum_y (V_a - V_y) = d_a\left(1 - \sum_y p_{ay} V_y\right)$

$= d_a\left(1 - P_a(T_a < H_b)\right)$

Removing edge L, or increasing it's resistance from 1 to ∞ must increase the resistance between a and b, hence reduce $\sum_y i_{xy}$ and thus increase $P_a(T_a < H_b)$

Rayleigh's Monotonicity Law

So we have a correspondence that on a random walk on a graph is recurrent iff an electrical network has infinite resistance between the origin and a boundary point.