Assignment 0 Solutions

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1 Problem 1

Let us denote each club as an n dimensional vector $|v\rangle$ where each component is an element of the vector space $\mathbb{F}_2$ with depending on whether that person is part of that club(1) or not(0). Take the set of these vectors V, then this set forms a vector space. Also for any two vectors $|v\rangle$ and $|w\rangle$ in V since there are an even number of common elements, $(|v\rangle, |w\rangle) = 0$. Now consider the set of all possible subsets $\mathbb{F}_2^n$. By definition of $V^\perp$, every element of V is also an element of $V^\perp$ so V is a subspace of $V^\perp$. Finally by rank-nullity theorem,

$$2\text{dim}V \leq \text{dim}V + \text{dim}V^\perp = \text{dim}\mathbb{F}_2^n = 2^n$$

So, the maximum possible number of clubs can't exceed $2^{n/2}$.

For oddtown again we have $(|v\rangle, |w\rangle) = 0$ but this time it is true only for $|v\rangle \neq |w\rangle$. This time the vectors in V are linearly independent because,
If $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$, then $v_i \cdot (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) = 0 \implies \lambda_i v_i \cdot v_i \implies \lambda_i = 0$. This implies that the maximum number of elements in this set is n.

2 Problem 2

The first part is trivial, $x^*Ay = x^*A^*y \implies x^*Ax = x^*A^*x$
For the second part notice that, $x^*Ay = \frac{1}{4}((x+y)^*A(x+y) - (x-y)^*A(x-y) - i(x+iy)^*A(x+iy) + i(x-iy)^*A(x-iy))$. Therefore we can always write x and y in terms of some vector z such that $x^*Ay = z^*Az$ $\blacksquare$

3 Problem 3

Proposition: For two square matrices A and B with spectral decomposition

$$A = \sum_i \lambda_i |v_i\rangle \quad \text{and} \quad B = \sum_j \mu_j |w_j\rangle,$$

$|v_i\rangle \otimes |w_j\rangle$ is an eigenvector of the matrix $A \otimes B$ with eigenvalue $\lambda_i \mu_j$.

Proof:

$$\begin{aligned} A \otimes B(|v_i\rangle \otimes |w_j\rangle) &= A(|v_i\rangle) \otimes B(|w_j\rangle) \\ &= \lambda_i |v_i\rangle \otimes \mu_j |w_j\rangle \\ &= \lambda_i \mu_j |v_i\rangle \otimes |w_j\rangle \end{aligned}$$

Now notice that the three parts can be rewritten as

$$\begin{pmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} & 1 \cdot \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} & 0 \cdot \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \otimes \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$$

Notice that the eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are 1 and -1 and that of $\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$ are $\frac{7 \pm \sqrt{57}}{2}$ so the eigenvalues of (a) and (b) will be $\pm \frac{7 \pm \sqrt{57}}{2}$ and that of (c) will be $\pm \frac{52 \pm 7\sqrt{57}}{2}$, -2 and -2.

4 Problem 4

According to Wikipedia any norm $p : X \to \mathbb{R}$ must satisfy the following conditions:

1. Non-negativity: $p(x) \geq 0$ for all $x \in X$, and $p(x) = 0$ if and only if x is the zero vector.

2. Homogeneity: $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and all scalars α.

3. Triangle inequality: $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Clearly the given norm $\sqrt{\langle x|x \rangle}$ satisfies these conditions because,

1. $\sqrt{\langle x|x \rangle} = \sqrt{\sum_i |x_i^2|} \geq 0$ with $\sqrt{\langle x|x \rangle} = 0$ only if $|x_i| = 0 \ \forall \ i$

2. $\sqrt{\langle \alpha x|\alpha x \rangle} = \sqrt{\alpha^* \alpha \langle x|x \rangle} = |\alpha| \sqrt{\langle x|x \rangle}$

3. $\sqrt{\langle x+y|x+y\rangle} = \sqrt{\langle x|x\rangle + \langle x|y\rangle + \langle y|x\rangle + \langle y|y\rangle} \leq \sqrt{\langle x|x\rangle + 2\sqrt{\langle x|x\rangle\langle y|y\rangle} + \langle y|y\rangle} = \sqrt{\langle x|x\rangle} + \sqrt{\langle y|y\rangle}$, where the second condition holds due to Cauchy-Schwarz Inequality.

Similarly, the metric $d(x,y) = |x-y|$ is valid because it satisfies the properties,

1. $d(x,x) = |x-x| = 0$

2. $d(x,y) = |x-y| > 0$ if $x \neq y$

3. $d(x,y) = |x-y| = |y-x| = d(y,x)$

4. $d(x,y) = |x-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$

The space of continuous functions defined on $[-1,1]$ form a vector space because, addition and scalar multiplication are closed, associative and distributive, there exist the "0" element and the "1" scalar, and there also exist additive inverses of all functions