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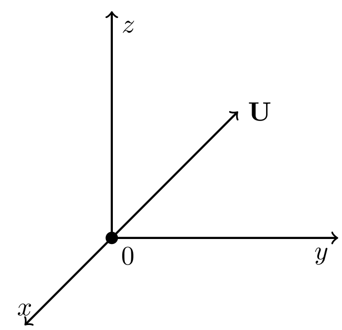
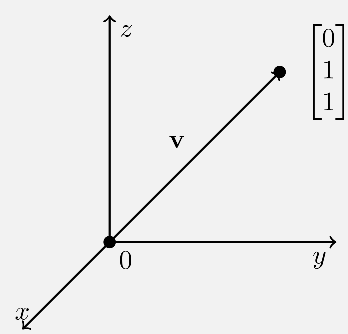


Figure 1: A vector in three-dimensional space. Starting at fixed point 0, the vector v represents the point **U**.

EXAMPLE



Notice that the 1st value in the vector (representing x), $v \in \mathbb{R}$ is 0 so the vector stays on the z and y axis.

Definition 1.3 (Commutative Property). Matrix multiplication is commutative. For Arithmetic operations of matrices, the commutative property is applied to $A + B = B + A$ as it would be applied to $5 + 7 = 7 + 5$ or $5 \times 7 = 7 \times 5$. This is illustrated by the following example

EXAMPLE Perform matrix addition for the following matrices:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix}$$

To add two matrices, simply add the corresponding elements:

$$A + B = \begin{bmatrix} 2+5 & 3+(-2) \\ (-1)+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -1 & 5 \end{bmatrix}$$

By the commutative property:

$$B + A = \begin{bmatrix} 5+2 & (-2)+3 \\ 0+(-1) & 1+4 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -1 & 5 \end{bmatrix}$$

So, the sum of matrices A and B is:

$$A + B = \begin{bmatrix} 7 & 1 \\ -1 & 5 \end{bmatrix} = B + A$$

This example demonstrates the commutative property in 2-dimensions. The same property applies to tensors, where the difference is that the addition is performed on multiple matrices, not just 2. In quantum computing, operations on qubits are represented by matrices called quantum gates.

Definition (Distributive): Matrix multiplication distributes over addition. For matrices A, B, and C, the distributive property is expressed as $A \times (B + C) = A \times B + A \times C$, analogous to the distributive property of real numbers, $a \times (b+c) = a \times b + a \times c$. This property is illustrated by the following example.

EXAMPLE Perform matrix multiplication and addition for the following matrices:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix} \quad \text{, and} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

To illustrate the distributive property, first, compute $B + C$:

$$B + C = \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 3 & -3 \end{bmatrix}$$

Now, compute $A \times (B + C)$ and $A \times B + A \times C$:

$$A \times (B + C) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 6 & 0 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 15 & -6 \\ 18 & -9 \end{bmatrix}$$

$$A \times B + A \times C = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 15 & -6 \\ 18 & -9 \end{bmatrix}$$

Thus, $A \times (B + C) = A \times B + A \times C$, demonstrating the distributive property of matrix multiplication over addition.

Linearity: Matrix operations exhibit linearity, which is characterized by two properties: additivity and homogeneity.

- **Additivity**: For matrices A, B, and C, and a scalar k, the additivity property states that

$A + (B + C) = (A + B) + C$, similar to the associative property of addition in arithmetic. This property ensures that matrix addition is associative.

- **Homogeneity**: For a matrix A and scalars k and m, the homogeneity property states that $k \times (m \times A) = (k \times m) \times A$, similar to the associative property of multiplication in arithmetic. This property ensures that scalar multiplication is associative with respect to real numbers.

These properties are illustrated by the following example.

EXAMPLE Consider the matrix A and scalars k and m:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \quad , \quad k = 3 \quad , \text{and} \quad m = 2$$

To demonstrate linearity, first, compute $k \times (m \times A)$ and $(k \times m) \times A$:

$$k \times (m \times A) = 3 \times (2 \times \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}) = 3 \times \begin{bmatrix} 4 & 6 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ -6 & 24 \end{bmatrix}$$

$$(k \times m) \times A = (3 \times 2) \times \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = 6 \times \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ -6 & 24 \end{bmatrix}$$

Now, compute $A + (B + C)$ and $(A + B) + C$:

$$A + (B + C) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \left(\begin{bmatrix} 6 & 0 \\ 3 & -3 \end{bmatrix} \right) = \begin{bmatrix} 8 & 3 \\ 2 & 1 \end{bmatrix}$$

$$(A + B) + C = \left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix} \right) + \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 2 & 1 \end{bmatrix}$$

Thus, $k \times (m \times A) = (k \times m) \times A$ and $A + (B + C) = (A + B) + C$, demonstrating the linearity of matrix operations.

0.3 Linear Operation in Hilbert Space

0.3.1 Dirac's Notation (Bra-ket)

- **Ket** is the word used for the column vector: $|x\rangle = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$

- **Bra** is the word used for the row vector(**Conjugate Transpose**): $\langle x| = [x_1^*, x_2^*, \ldots, x_n^*] = (|x\rangle)^* \in \mathbb{C}^n$

- **Inner Product**: The inner product of two quantum state vectors $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$ is denoted as $\langle \mathbf{x}|\mathbf{y}\rangle$, and it is given by:

  $$\langle \mathbf{x}|\mathbf{y}\rangle = \langle \psi|\phi\rangle = \psi_1^* \phi_1 + \psi_2^* \phi_2$$

  where ψ_i and ϕ_i are the components of the vectors $|\psi\rangle$ and $|\phi\rangle$, respectively. Here's an example with vectors:

  $$|\psi\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\langle\psi|\phi\rangle = \begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Projects $|\phi\rangle$ onto $|\psi\rangle$

- **Outer Product**: Denoted as $|\mathbf{x}\rangle\langle\mathbf{y}|$.

$$|\psi\rangle\langle\phi| = \begin{bmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_m\end{bmatrix}\begin{bmatrix}\phi_1^* & \phi_2^* & \cdots & \phi_n^*\end{bmatrix}$$

Here's an example with vectors:

$$|\psi\rangle = \begin{pmatrix}1\\ i\end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

The outer product $|\psi\rangle \otimes |\phi\rangle$ is calculated as follows:

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix}1\\ i\end{pmatrix} \otimes \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix}1\cdot\frac{1}{\sqrt{2}} & 1\cdot\frac{1}{\sqrt{2}}\\ i\cdot\frac{1}{\sqrt{2}} & i\cdot\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}$$

- The **Norm** is defined by: $\|\mathbf{x}\| = \sqrt{\langle\mathbf{x}|\mathbf{x}\rangle} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$
- **Orthonomal Set**: $B = \{|b_i\rangle, i \in I\}$
  $\langle b_i|b_i\rangle = \partial_{ij} = \begin{cases}1, & \text{if } i = j\\ 0, & \text{if } i \neq j\end{cases}$
- **Hilbert Space H**: Complete inner-product vector, i.e. the vector is in a finite-dimensional vector space.
  - Infinite-dimensional-completeness is not guaranteed.

0.4 Matrix Definitions and Operations

Definition 1.1 (Matrix). A real-valued (m, n) matrix A with $m, n \in \mathbb{N}$ is an $m \cdot n$-tuple of elements a_{ij}, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$, arranged in a rectangular pattern with m rows and n columns:

$$A = \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$$

Matrix addition is an operation performed on two matrices of the same size. Given matrices $A \in \mathbb{R}^{m\times n}$

and $B \in \mathbb{R}^{m \times n}$, adding the matrices together involves adding their corresponding elements (Element-Wise):

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix multiplication is performed between two matrices, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times k}$, and results in $C = AB \in \mathbb{R}^{m \times k}$. The product C is calculated by taking the dot product of each row of matrix A with each column of matrix B. We sum the products of the multiplication to obtain an entry in the corresponding row i and column j of C.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

- a_{ik} is the element at row i and column k of matrix A,
- b_{kj} is the element at row k and column j of matrix B,
- n is the number of columns in matrix A (which is also the number of rows in matrix B).

such that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \in \mathbb{R}^{3 \times 2}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{bmatrix}$$

Definition 1.2 (Identity Matrix). A matrix in $\mathbb{R}^{n \times n}$ that contains only 1's on its diagonal entries.

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition 1.3 (Linear Independence). In a linearly independent matrix, each column holds the property that it cannot be expressed by performing linear combinations of the other columns. i.e. if a matrix has 3 columns, you cannot multiply columns 2 or 3 by any value $c \in \mathbb{R}$ such that the columns would be the same as column 1 and vice versa. It also holds that adding two columns together with the ability to scale them also cannot equal another column in the matrix.
The formal definition states: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V over a field F is said to be linearly independent if the only scalars $c_1, c_2, \ldots, c_n$ satisfying the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

the only solution to this should be $c_1 = c_2 = \ldots = c_n = 0$. In other words, the only way to obtain the zero vector $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is by setting all the scalar coefficients to zero.

0.5 Tensors

Definition 1.4 (Tensors) A Tensor is a multidimensional array. Image a matrix like a flat sheet of paper, and a tensor is when you stack more than 1 onto it.

Kronecker product of two tensors

$$T_1 \otimes T_2 = \begin{bmatrix} t_{11} \cdot T_2 & t_{12} \cdot T_2 & \cdots & t_{1n} \cdot T_2 \\ t_{21} \cdot T_2 & t_{22} \cdot T_2 & \cdots & t_{2n} \cdot T_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} \cdot T_2 & t_{m2} \cdot T_2 & \cdots & t_{mn} \cdot T_2 \end{bmatrix} \tag{1}$$

$T_1 \otimes T_2 = T_2 \otimes T_1$